PLANAR ORTHOGONAL POLYNOMIALS AND TWO DIMENSIONAL COULOMB GASES

by
Iván Parra

Dissertation
Submitted in Partial Fulfillment of the Requirements for the
the doctoral degree “Doctor rerum naturalium”

Fakultät für Physik,
Universität Bielefeld
July 31, 2020

Iván Parra for the doctoral degree "Doctor rerum naturalium", to be presented on September 29, 2020, at Universität Bielefeld.

Abstract

In this thesis we focus on the relation between random matrix theory and orthogonal polynomial theory in the complex plane. It is well known that even if the entries of a random matrix are independent, the eigenvalues will be highly correlated. This correlation, which is a pairwise logarithmic repulsion between the eigenvalues, leads one to think that the eigenvalues of a random matrix behave like particles in a Coulomb gas, since the logarithmic repulsion is the Coulomb interaction in two dimensions.

We consider the case when the particles are confined to an ellipse in the plane. At inverse temperature $\beta = 2$, we introduce new families of exactly solvable two-dimensional Coulomb gases for a fixed and finite number of particles N. We find, in the analysis of local fluctuations in the weak non-Hermiticity limit – as $N \to \infty$ – of the correlation functions, old and new universality classes. This is achieved by showing that certain subfamilies of Jacobi polynomials extend to orthogonality relations over a weighted ellipse in the plane.

ADVISOR: Prof. Dr. Gernot Akemann.

Publications

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[2] G. Akemann, T. Nagao, I. Parra, G. Vernizzi. *Gegenbauer and other planar orthogonal polynomials on an ellipse in the complex plane.* Constr. Approx. (2020).

[3] I. Gonzales, I. Kondrashuk, E. A. Notte-Cuello, I. Parra. *Multi-fold contour integrals of certain ratios of Euler gamma functions from Feynman diagrams: orthogonality of triangles.* Anal. Math. Phys. 8 (2018) 589-602.

[4] I. Gonzales, B. A. Kniehl, I. Kondrashuk, E. A. Notte-Cuello, I. Parra, M. Rojas-Medar. *Explicit calculation of multi-fold contour integrals of certain ratios of Euler gamma functions. Part 1.* Nucl. Phys. B 925 (2017) 607-614.

Contents

1. Introduction

Random Matrix Theory (RMT) was originally conceived in mathematical Statistics by J. Wishart and in Physics by E. Wigner and F. Dyson. In the 1950's, Wigner introduced ensembles of symmetric real random matrices as well as complex Hermitian $N \times N$ matrices with statistically independent entries (including Gaussian ensembles) as a theoretical model of statistical behaviour of energy levels. His idea, roughly speaking, is to replace the Hamiltonian of the quantum system – such as a heavy nucleus, this was indeed one of the first applications of RMT – which is an operator similar to an infinite size matrix and complicated to diagonalize numerically, by a random matrix of size $N \times N$ ($N >> 1$), whose entries are taken randomly from a known distribution, and that has the same symmetries as the original Hamiltonian. The problem is to get information on the behaviour of its eigenvalues (levels). Many of his works on this theme are collected in the work of Porter [5]. Dyson, in the early 1960s [6], classified the "very classic families" (nowadays) of random matrices, i.e. Gaussian Orthogonal Ensemble (GOE), Gaussian Symplectic Ensemble (GSE) and Gaussian Unitary Ensemble (GUE). Dyson has shown that these three classic ensembles mimicing the symmetries of the Hamiltonian of a system, the first two corresponding to the cases when the Hamiltonian commutes with the time reversal operator (known to be anti-unitary). If there are no anti-unitary symmetries the Hamiltonian is Hermitian and this corresponds to the GUE.

Since its introduction, RMT is motivated to a large extent by practical experimental problems. Today, successful applications of real eigenvalue statistics can be found in many fields, such as Quantum Chromodynamics (QCD), two-dimensional (2D) Quantum Gravity, 2D String Theory (see [7] and references therein). Not only physics has been enriched with the applicability of RMT, the Circular Unitary Ensemble (CUE) has been extensively studied in [8], [7, chap. 1] in connection with Number Theory. We refer to [9] for a review on RMT where both the theoretical aspects, and the application of the theory has been discussed.

Despite the fact that operators having real eigenvalues are the main interest in physic, in 1965 Ginibre has started the study of Gaussian random matrices without symmetry constraint (whose entries are real, complex or quaternion random variables) as a mathematical extension of Hermitian random matrix theory. Due the fact that their eigenvalues may lie anywhere on the complex plane, no physical applications, in particular in quantum physic, were evident at that time. However, Ginibre has expressed –end of first paragraph p. 440 [10]–

Apart from the intrinsic interest of the problem, one may hope that the methods and results will provide further insight in the cases of physical interest or suggest as yet lacking applications.

For an overview on complex non-Hermitian Ensembles, including the three Ginibre ensembles and their elliptic deformations we refer to [11, chap. 18]. Nowadays, eigenvalue statistics in the complex plane, have a wide range of interesting applications, perhaps the most well known occurs in statistical mechanics and quantum mechanics, for instance, as a two-dimensional Coulomb gas. Here, the Coulombic nature is manifested by the pairwise *logarithmic repulsion* between the particles in the gas. Futhermore, this two-dimensional Coulomb gas turns out to be directly related to the Laughlin wave function in the fractional quantum Hall effect – which will be discussed later.

Additional applications appear in QCD with chemical potential [12] (we refer to the lecture notes [13] for a comprehensive review). In QCD, one is interested in eigenvalue statistics of Dirac-type operators, an enthralling situation happens in the addition of a chemical potential, while allows the number of the corresponding particles to fluctuate, the global symmetry of the Dirac operator breaks down. It turns out to no longer be an anti-Hermitian operator and becomes complex non-Hermitian, therefore the need of complex eigenvalue statistics arises.

Another interesting application occurs in resonances in Chaotic Scattering, for instance, in the presence of open channels [14]. Under suitable assumptions, this turns out to be modeled by eigenvalues statistics of truncated unitary matrices, while the eigenvalue of unitary matrices lie on the unit circle, once we consider the top left square truncation of this unitary matrix, the symmetry once again breaks down and the eigenvalues may lie anywhere inside the unit disk. Further, complex eigenvalue statistics apply to Quantum Information [15], Financial Mathematics [16] and Wireless Communications [17], Neural Networks [18], and we refer to [11] for a guideline about current applications of random matrix theory.

In order to make the motivation of this thesis more precise, let us start with an example: The Complex Ginibre Ensemble.
It is defined on the space of complex $N \times N$ matrices with independent, identically distributed complex Gaussian entries. In his celebrated paper *Statistical Ensembles of Complex, Quaternion, and Real Matrices* Ginibre has found that the Joint Probability Density Function (jpdf for short) of complex eigenvalues of such ensemble of matrices, with $Q(z) = |z|^2$ and $\beta = 2$, is given by

$$P_\beta^Q(z_1, \cdots, z_N) = \frac{1}{Z_N^\beta(Q)} \exp\left(-\frac{\beta}{2}\sum_{i=1}^{N} Q(z_i)\right) \prod_{1\le i<j\le N} |z_j - z_i|^\beta. \tag{1.1}$$

Here, $\prod_{j>i}^N (z_j - z_i) = \Delta(z)$ is the Vandermonde determinant and the constant $Z_N^\beta(Q)$ called in physics partition function, it is the normalization constant that makes P_β^Q a probability measure (dA stands for planar Lebesgue measure):

$$Z_N^\beta(Q) \quad = \quad \int_{\mathbb{C}^N} \exp\left[-\beta\left(\frac{1}{2}\sum_i Q(z_i) - \sum_{i<j} \log|z_j - z_i|\right)\right] \prod_{i=1}^{N} dA(z_i). \tag{1.2}$$

Even though the integral in (1.2) converges for any $\beta \ge 0$, its value is only known for $\beta = 2$. In contrast, in the GUE, where the integral is taken on the real line, this partition function (1.2) is known for any value of beta, thanks to a result provided by

Selberg [19, 20] who evaluated a more general version of the integral (1.2) on the real line.

The Slater determinant is known to be an expression that describes the wave function of a multi-fermionic system. An example of this is the fractional quantum Hall effect [21], where a charged particle interacts with an external magnetic field, and in the case of a system with N non-interacting particles with Fermi statistics in the lowest Landau level, the Slater determinant is an orthogonal basis of wave functions [22, 15.2.2]

$$\psi_{l_1,\ldots,l_N}(z_1,\ldots,z_N) = \det_{i,j}[\phi_{l_j}(z_i)], \quad \phi_l(z) = \frac{1}{\sqrt{\pi}} z^l e^{-\frac{|z|^2}{2}}. \tag{1.3}$$

Here, $\phi_l(z)$ – known as the spin-orbital – is an orthogonal complete set of states in the lowest Landau level and $l = 0, 1, \ldots$ can be interpreted as the angular momentum eigenvalues. The state with lowest total angular momentum corresponds to the choice $l_i = i - 1$, in which case the Slater determinant reduces up to a factor to the weighted Vandermonde determinant

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right)\Delta(z). \tag{1.4}$$

A particular interesting case is that related to the ground state of the fractional quantum Hall effect, proposed by Laughlin in [23], where the quantum wave function at odd fractional filling for the values $\nu = \frac{1}{2s+1}$, $s = 0, 1, \ldots$ takes the form

$$\psi_s(z_1,\ldots,z_N) = \frac{1}{\pi^{N/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right)\Delta(z)^{2s+1}, \tag{1.5}$$

and the nomalization of the Laughlin's wave functions

$$Z_N(2s+1) = \langle \psi_s, \psi_s \rangle = \int \exp\left(-\sum_{i=1}^{N}|z_i|^2\right)\left(\Delta(z)\overline{\Delta(z)}\right)^{2s+1}\prod_{i=1}^{N} dA(z_i). \tag{1.6}$$

which, up to a constant factor, correspond to the choice $\beta = 2(2s+1)$ in (1.2). $Z_N(1)$ may be identified as the normalization of the "densest state" (filling fraction 1). This is the only treatable case for arbitrary N (up two cases with 2 or 3 particles, for wich the Selberg-type integral can be computed [21]) due to the orthogonal polynomials technique, we will come back to this shortly.

The jpdf (1.1) coincides with the Boltzmann factor form $e^{-\beta\mathcal{E}_Q}$ for a 2D log-gas system at special value of the inverse temperature $\beta = 2$ and suitable background charges, with total potential energy

$$\mathcal{E}_Q(z) = \sum_i Q(z_i) - \sum_{i\neq j}\log|z_j - z_i|. \tag{1.7}$$

The first term in (1.7), the *external field* or potential, represents a harmonic attraction towards the origin, and the second is a pairwise *logarithmic repulsion* between the particles in the gas, which is the Coulomb interaction in 2D.

For $\beta = 2$, in the early 60s M. Gaudin and M. Mehta [24, 25] introduced the use of orthogonal polynomials (OP) to the study of eigenvalue statistics, showing that the k-point correlation function defined by Dyson 1962, which describes the probability density to find – in principle real – k eigenvalues around each of the points $z_1, \ldots, z_k$ while the positions of the remaining eigenvalues are unobserved:

$$\rho_N(z_1, \ldots, z_k) = \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} P_2^Q(z_1, \cdots, z_N) \prod_{i=k+1}^{N} dA(z_i), \tag{1.8}$$

which can be written in a determinantal formula

$$\rho_N(z_1, \ldots, z_k) = \det_{1 \le i,j \le k} \left[K_N(z_i, z_j) \right], \tag{1.9}$$

with its correlation kernel

$$K_N(z, \bar{w}) = \exp[-(Q(z) + Q(w))/2] \sum_{n=0}^{N-1} P_n(z) \overline{P_n(w)}. \tag{1.10}$$

Here, $P_n(z)$ are orthonormal polynomials with respect to the complex normal distribution $\exp[-Q(z)]$, $Q(z) = |z|^2$, also referred to a weight function. In particular when $k = 1$, one gets the level density, also known as spectral density

$$\rho_N(z) = K_N(z, \bar{z}) = \exp[-Q(z)] \sum_{n=0}^{N-1} |P_n(z)|^2. \tag{1.11}$$

Note that the one-point correlation function corresponds to the density of eigenvalues $d(z) = \sum_i \delta(z - z_i)$ overaged over the ensemble distribution (1.1), $\rho_N(z) = \langle d(z) \rangle$, so if we set $n_E = \{$ number of the eigenvalues on the region $E\}$, then the expected number of eigenvalues in E is given by

$$\langle n_E \rangle = \int_E \rho_N(z) dA(z). \tag{1.12}$$

In particular $\langle n_{\mathbb{C}} \rangle = N$, which is verified from the orthogonality relations of P_n.

It is well known that for complex Ginibre-type Ensembles at $\beta = 2$ – where a more general potential Q is allowed – the study of eigenvalues statistics leads to a determinantal point process (1.9), while for real or quaternion Ginibre ensembles it leads to Pfaffian processes. In the last case a very useful tool has been introduced in the literature [26, chap. 15], the so-called skew-orthogonal polynomials. In this thesis we will not deal with this topic, but rather focus on orthogonal polynomials and their connection with random matrices ($\beta = 2$).

The Coulomb gas approach [22] allows to borrow potential theory techniques and get for the prediction of the leading asymptotic form of the spectral density

$$\rho_N(z) = \begin{cases} \frac{1}{\pi} & |z| < N^{1/2} \\ \\ 0 & \text{otherwise.} \end{cases} \qquad (N >> 1). \tag{1.13}$$

This means that the so-called global density $\rho(z)$ obeys the following limit formula known as the circular law [27, 28]

$$\rho_b(z) := \lim_{N\to\infty} \rho_N(\sqrt{N}\, z) = \frac{1}{\pi} \mathbb{1}_{\mathbb{D}}(z) \ , \tag{1.14}$$

where $\mathbb{1}_{\mathbb{D}}$ is the characteristic function on the unit disk in the complex plane, meaning that on average most of the eigenvalues are uniformly distributed within the unit disk. We refer to the two-dimensional eigenvalue support described by (1.14) as the the *Droplet* (also called the *Bulk* of the spectrum) and we refer to its boundary as the *Edge* of the spectrum.

The eigenvalue density (1.11) in the transitional region at $p + \sqrt{N}z$, $p \in \partial\mathbb{D}$ on the edge of the spectrum, abruptly crosses over from $\rho(z) = 1/\pi$ at $|z| < 1$ to $\rho(z) = 0$ at $|z| > 1$. The *crossover* is described by the local fluctuations of the density of eigenvalues (1.11) which, in the N-large limit, is given in terms of the complementary error function [29]

$$\rho_e(z) := \lim_{N\to\infty} \rho_N(p + \sqrt{N}\, z) = \frac{1}{2\pi} \operatorname{erfc}(\sqrt{2}\, \mathrm{Re}(z)). \tag{1.15}$$

When a more general potential $NQ(z)$ is allowed – with suitable condition of "*admissibility*" for Q – it is known [30, 31], that under a proper scaling limit $s(N, z)$, the spectral density (1.11) satisfies

$$\rho_b(z) := \lim_{N\to\infty} \frac{1}{|s'(N, z)|^2} \rho_N(s(N, z)) = \sigma_Q(z) \mathbb{1}_S(z) dA(z), \tag{1.16}$$

where S is a two-dimensional compact set on the complex plane (the droplet) and $d\mu(z) = \sigma_Q(z)\mathbb{1}_S(z)dA(z)$ minimizes the energy functional

$$\mathcal{E}_Q(\mu) = \int\int \frac{1}{\log|z - w|} d\mu(z)d\mu(w) + \int Q(z)d\mu(z). \tag{1.17}$$

For example, when the potential $Q(z)$ is given by

$$Q(z) = \frac{1}{1-\tau^2}|z|^2 - \frac{\tau}{1-\tau^2} \mathrm{Re}(z^2), \quad 0 < \tau < 1. \tag{1.18}$$

the droplet S coincides with a standard ellipse of parameters $a = 1 + \tau$ and $b = 1 - \tau$, and instead of having the circular law, we have Girko's elliptic law [32]. Here, a fascinating phenomena occurs, called *universality*, as it has been observed that when we modify the potential, the droplet has changed from a disk to an ellipse, but the local statistics near to the edge of the ellipse, are again given by the complementary error function (1.15) (see [33]). It was shown more recently in [34] that for a quite general potential Q the local fluctuations on the edge of the spectrum are given by the

complementary error function in the N-large limit matrix size.
Not only this phenomenon occurs at the boundary of the droplet, also it have been shown in [35] with a potential of the form (1.18) that local statistics, around any point $p \in E^\circ$, and under an appropriate scaling $r(p, N, z)$ limit, the correlation kernel (1.10) satisfies the following limit

$$c(N)K_N(r(p,N,z), \overline{r(p,N,w)}) \to G(z,w) = \frac{1}{\pi} \exp\left(-\frac{|z|^2 + |w|^2}{2} + z\overline{w}\right). \tag{1.19}$$

The expression in the right hand side of (1.19) is the so-called Ginibre Kernel, it is the same for the potential $|z|^2$.
In order to understand this *universality* phenomenon, several extensions have been made of the Ginibre Ensembles, we have mentioned before, elliptic deformations that also apply to their chiral companions. Also more general potentials in the plane have been considered, such as the normal matrix model [36] and it has been found that the complex eigenvalue statistics (1.15) and (1.19) provided by the Ginibre ensembles appear to be *universal*.

The correlation kernel (1.10) tells us that the statistics of complex eigenvalues are governed by the associated planar orthogonal polynomials (and its asymptotics). For example, eigenvalues statistics of the complex elliptic Ginibre ensemble [37] are linked to holomorphic Hermite polynomials, orthogonal on the complex plane. Likewise, the chiral companion of this ensemble [12] leads to a kernel of holomorphic Laguerre polynomials. The asymptotic behavior of planar orthogonal polynomials with respect to exponentially varing measure $e^{-mQ(z)}dA(z)$ have been the main ingredient, in [34], to proof universality of the complex eigenvalues statistics on the edge of the spectrum for a large class of potentials.
In analogy with the ensembles GUE, Laguerre Unitary Ensemble (LUE), Jacobi Ensemble having associated Hermite, Laguerre and Jacobi polynomials, respectively, the statistics of the real eigenvalues gives the well-known Sine, Airy, and Bessel kernels, that are universal in the bulk, soft-edge, and hard-edge scaling limits. *The investigation of planar OP in a bounded region E of the complex plane would lead to a new type of universal kernel and it could, perhaps help to understand existing results.*

The theory of orthogonal polynomials and related kernel functions on the real line has been developed by many mathematicians starting with the special OP of Legendre, Jacobi, Gegenbauer, Chebyshev, Hermite and Laguerre. These polynomials, are considered nowadays, the very classic orthogonal polynomials. See Szegő for references [38]. Hermite polynomials were studied extensively by Laplace in connection with probability theory. The Hermite differential equation may be identified as the stationary one-dimensional Schrödinger equation for the quantum harmonic oscillator, the Hermite polynomials being an appropiate basis that span the oscilation modes as ladder operators, with a Rodriguez-like rule obeying the Hermite recurrence relations. Another classic example of the application of these orthogonal polynomials in quantum mechanic are the associated Legendre polynomials (which can be rewritten via the ordinary Legendre polynomials) being the solution of the Schrödinger equation for a static Coulomb potential that may be interpreted as the (attractive) potential for the nucleus of an atom, that exerts on the charge that orbits it, forming an Hydrogen-like atom.

The study of more general orthogonal polynomials in weighted L^2-spaces on the line is associated with the names of Markov, Stieltjes, Szegő, Chebyshev, Bernstein, among others. The theory of orthogonal polynomials on the unit circle is almost completely the creation of one person, Gabor Szegő. He, also, studied intensively the case of (holomorphic) orthogonal polynomials in $L^2(\Gamma, ds)$, where ds is arc length measure, Γ is a real-analytically smooth Jordan curve in the complex plane. The pioneers in the study of the asymtotics behaviour of (holomorphic) orthogonal polynomials in $L^2(E, dA)$ on the simply connected bounded domain E with real-analytic boundary curve Γ, were Carlemann and Suetin [39]. Suetin extended the result by Carleman to domains whose boundary has a lower degree of smoothness, and the case when a weight function is present. Holomorphic orthogonal polynomials in $L^2(\Gamma, ds)$, are called Szegő polynomials. Holomorphic orthogonal polynomials in $L^2(E, dA)$, where dA is Lebesgue area measure on bounded domain E, are called Bergman polynomials.

This thesis concerns itself with the question *whether further classical orthogonal polynomials on the real line also form a set of orthogonal polynomials on a two dimensional domain in the complex plane*. The planar OP would extend the class of exactly solvable 2D Coulomb gases and bring further insight to understand universality. The Gram-Schmidt construction of orthogonal polynomials on any subset of the real line and in the complex plane is completely analogous. The fact that the orthogonal polynomials on the real line always satisfy a three-step recurrence relation is special. Conversely, Favard's theorem reads that, if a sequence of polynomials satisfy a suitable three-term recurrence relation, then there is a distribution function such that these polynomials became orthogonal. Recovering the integration measure by knowing the coefficients of expansion from the three-terms recurrence relations is known as an inverse problem see [40, chap. 5] and in many cases it is possible to carry this out on the real line.

In the complex plane we do not have these tools. Lempert [41] (1976a) showed that we cannot expect any finite term recurrence for orthogonal polynomials on a bounded domain in the complex plane in general. It was shown much more recently if a sequence of (holomorphic) orthogonal polynomials in $L^2(E, dA)$ satisfy a finite term recurrence relation on a bounded domain E with regular enough boundary implies that the domain E is an ellipse and the size of the recursion is three [42, 43, 44], with corresponding Bergman polynomials: the Chevyshev polynomials of the second Kind [45]. This limits our search to elliptic domains as our polynomials originating from the real line do have a three-step recurrence. We note, however, that the aforementioned results mentioned above only apply to unweighted domains. For the Chebyshev polynomials of first, third and fourth kind, the weight function on the ellipse is no longer flat [46].

This thesis is mainly based on two published papers [1, 2] and includes some unpublished results. The content of this thesis is organized as follows: Section 2 is standard material related to special functions and explicit representation of classical OP on the real line, it contains an additional Lemma 2.6 on combinations of hypergeometric functions and powers type integral in which we present our own proof. Section 3.1 contains standard material on planar OP and the part 3.2 is new where we provide an extension of the multiple Hermite polynomials to a planar orthogonality. Section 4 and 5 contain our main results on weighted Bergman spaces of an ellipse, in particular in theorem 4.5, we show that the classical Gegenbauer or ultraspherical polynomials $C_n^{(1+\alpha)}(z)$, for $\alpha > -1$, provide a family of planar orthogonal polynomials on the interior of an ellipse parametrised by $Q(z) := A|z|^2 - B\,\mathrm{Re}(z^2) < 1$, with $A > B > 0$ and weight function $(1 - Q(z))^{\alpha}$. Additionally, based on a particular quadratic transformation of the ellipse that fixes the focal points, we find a subfamily of Jacobi polynomials $P_n^{\alpha+\frac{1}{2},\pm\frac{1}{2}}$ to be orthogonal on a weighted ellipse in theorems 5.6 and 5.8. These findings establish as a corollary an alternative proof to the all four Chebyshev polynomials from [46] that we show in Section 5.3.

Finally in Section 6, at inverse temperature $\beta = 2$, we introduce and solve two (new) 2D, static one-component Coulomb gases. In Section 6.1 we describe the local scale regime to be consider in Section 6.2 and 6.3. In these last two sections we present our main results regarding to the asymptotic analysis in the weak non-Hermiticity limit of the correlation kernel induced by the Gegenbauer polynomials, the weak non-Hermiticity parameter s (to be specified later) allows to interpolate our findings between old and new universality classes. In Section 6.4 we present the analysis for the non-symmetric case. We conclude this thesis with the Section 7, containing summary and outlook.

2. Orthogonal Polynomials

2.1. **Special Functions.** The gamma and beta functions has several representations, but the two most important, found by Euler, represent these as integrals of the form

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad \mathrm{Re}(z) > 0. \tag{2.1}$$

$$B(z, w) = \int_0^1 x^{z-1} (1-x)^{w-1} dx, \quad \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0. \tag{2.2}$$

They are related through

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{2.3}$$

The functional relation

$$\Gamma(z+1) = z\Gamma(z), \tag{2.4}$$

extends the gamma function to a meromorphic function with poles at $z = 0, -1, \ldots$ and also extends $B(z, w)$ to a meromorphic function of z and w. The gamma function satisfies

$$\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)/\sqrt{\pi}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{2.5}$$

known as duplication and reflection formulas.

The Pochhammer symbol, also known as shifted factorial, is

$$(z)_n = z(z+1)\cdots(z+n-1) \quad n > 0, (z)_0 = 1. \tag{2.6}$$

The functional relation (2.4), gives

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \tag{2.7}$$

and clearly

$$\frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n \sim z^n, \quad as\, z \to \infty. \tag{2.8}$$

Some useful identities are

$$(z)_m (z+m)_n = (z)_{m+n}, \quad (z)_{n-k} = \frac{(z)_n (-1)^k}{(-z-n+1)_k}. \tag{2.9}$$

Note that (2.7) extend the Pochhammer symbol to any complex number n, providing that $z + n$ is not a pole of the gamma function.

The Gauß' hypergeometric function (Ghf) is

$$F(\alpha, \beta, \gamma, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}\,, \tag{2.10}$$

in which α, β and γ are the function parameters and z is the variable of the Ghf. By the ratio test the Ghf is analytic in the unit disc, provided that γ is neither a negative integer nor zero.

The following theorem is an important integral representation of the Ghf due to Euler.

Theorem 2.1 (Euler)**.** *Let $\alpha, \beta, \gamma \in \mathbb{C}$, such that $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, then*

$$F(\alpha, \beta, \gamma, z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha}dx. \tag{2.11}$$

in the x plane cut along the real axis from 1 to ∞. Here it is understood that $\arg(x) = \arg(1-x) = 0$ and $(1-zx)^{-\alpha}$ has its principal value.

If one of the parameters in the numerator of the Ghf is a negative integer, say $-n$, then the series (2.10) becomes a finite sum, $0 \leq k \leq n$. This follows directly from (2.9) setting $n = k$ and $z = -n$ we get $(-n)_k = (-1)^k(n-k+1)_k$ and we obtain

$$F(-n, \beta, \gamma, z) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\beta)_k}{(\gamma)_k} z^k. \tag{2.12}$$

Proposition 2.2. *The hypergeometric polynomial defined in* (2.12) *satisfies the following reflection formula*

$$F(-n, b, c, z) = \frac{(c-b)_n}{(c)_n} F(-n, b, b-c-n+1, 1-z). \tag{2.13}$$

Proof.

$$\begin{aligned} F(-n, b, c, z) &= \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} \frac{(b)_\ell}{(c)_\ell} z^\ell \\ &= \sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} \frac{(b)_\ell}{(c)_\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k (1-z)^k \\ &= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \binom{\ell}{k} (-1)^{\ell+k} \frac{(b)_\ell}{(c)_\ell} (1-z)^k \\ &= \sum_{k=0}^{n} \sum_{\ell=0}^{n-k} \binom{n}{k} \binom{n-k}{\ell} (-1)^\ell \frac{(b)_{\ell+k}}{(c)_{\ell+k}} (1-z)^k, \end{aligned} \tag{2.14}$$

we note that

$$\frac{(b)_{\ell+k}}{(c)_{\ell+k}} = \frac{B(b+k+\ell, c-b)}{B(b, c-b)}\,. \tag{2.15}$$

using the integral representation of the beta function (2.3) allows us to rewrite the sum over ℓ as

$$\sum_{\ell=0}^{n-k}\binom{n-k}{\ell}(-1)^\ell\frac{(b)_{\ell+k}}{(c)_{\ell+k}} = \frac{1}{B(b,c-b)}\sum_{\ell=0}^{n-k}\binom{n-k}{\ell}(-1)^\ell\int_0^1 x^{b+k+\ell-1}(1-x)^{c-b-1}dx$$
$$= \frac{B(b+k,c-b+n-k)}{B(b,c-b)}. \quad (2.16)$$

Inserting this last result in (2.14) and using (2.9) to change the $-k$ sign, the expected result is obtained. □

Corollary 2.3 (Chu-Vandermonde)**.**

$$F(-n,b,c,1) = \frac{(c-b)_n}{(c)_n}. \quad (2.17)$$

The following proposition and corollary can be found in standard books on integration formulas [47]. We have decided to provide the poof for these integrals for completeness of this text and also because we have observed a result, lemma 2.6, presented at the end of this section that will be extremely useful in our calculations.

We will use the following two integrals; the first (2.18) is the so called Mellin-Barnes transform and the second (2.19) is known as the first Barnes' lemma [48]. Barnes' contour integrals appear naturally in the context of loop calculations in quantum field theory [3, 4, 49].

$$\frac{1}{(1+x)^\alpha} = \frac{1}{\Gamma(\alpha)}\int_{c-i\infty}^{c+i\infty} dz\, x^z\Gamma(-z)\Gamma(z+\alpha). \quad (2.18)$$

The right poles of gamma functions of the type $\Gamma(p-z)$ at $z=p,p+1,\ldots$ must lie to the right of the path of integration , whereas the left poles of gamma functions $\Gamma(q+z)$ at $z=-q,-q-1,\ldots$ lie to the left of it. If $\operatorname{Re}\alpha>0$, the path of integration can be chosen as a straight line in the strip $-\operatorname{Re}\alpha<\operatorname{Re}z<0$, otherwise we deform the contour with the above specifications. For $x<1$ the Mellin-Barnes integration contour can be closed to the right, and the series of residues at $z=0,1,\ldots$ reproduces the Taylor expansion of the left-hand side of (2.18), for $x<1$. When $x>1$, the contour may be closed to the left and the series of residues at $z=-\alpha,-\alpha-1,\ldots$ give us the expansion for $x>1$.

$$\int_{c-i\infty}^{c+i\infty} dz\,\Gamma(\alpha+z)\,\Gamma(\beta+z)\,\Gamma(\gamma-z)\,\Gamma(\delta-z) = \frac{\Gamma(\alpha+\gamma)\,\Gamma(\alpha+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}. \quad (2.19)$$

Proposition 2.4. *Let* $\alpha,\beta,\gamma,\rho\in\mathbb{C}$ *such that* $\operatorname{Re}\rho>0,\operatorname{Re}\gamma>0,\operatorname{Re}(\gamma+\rho-\alpha-\beta)>0$, *then*

$$\int_0^1 dt\, t^{\gamma-1}(1-t)^{\rho-1}\,F(\alpha,\beta,\gamma,t) = \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\,\Gamma(\gamma+\rho-\beta)}. \quad (2.20)$$

Proof. We use the Euler integral representation (2.11) of the Gauß' hypergeometric function followed by a change of variable $x \to 1-x$ and then we use the Mellin-Barnes transform (2.18) to obtain

$$\begin{aligned}
&\int_0^1 dt\, t^{\gamma-1}(1-t)^{\rho-1}\, F(\alpha,\beta,\gamma,t) \\
=\;& \frac{1}{\Gamma(\alpha)B(\beta,\gamma-\beta)} \int_0^1 dt\, t^{\gamma-1}(1-t)^{\rho-\alpha-1} \int_0^1 dx x^{\gamma-\beta-1}(1-x)^{\beta-1} \\
&\times \int_{c-i\infty}^{c+i\infty} dz \left(\frac{tx}{1-t}\right)^z \Gamma(-z)\Gamma(z+\alpha), \qquad (2.21)
\end{aligned}$$

noting that

$$\int_{c-i\infty}^{c+i\infty} dz \left(\frac{tx}{1-t}\right)^z \Gamma(-z)\Gamma(z+\alpha) = i\int_{-\infty}^{\infty} dy \left(\frac{tx}{1-t}\right)^{c+iy} \Gamma(-c-iy)\Gamma(c+iy+\alpha). \qquad (2.22)$$

Due to the asymptotic behavior of the gamma function,

$$|\Gamma(\sigma+i\tau)| \sim_{|\tau|\to\infty} \sqrt{\pi}e^{-|\tau|\pi/2}|\tau|^{\sigma-1/2}, \quad \sigma,\tau\in\mathbb{R}\,. \qquad (2.23)$$

it follows that the integral in the right hand side of (2.21) is absolutely convergent, so we have

$$\begin{aligned}
&\int_0^1 dt\, t^{\gamma-1}(1-t)^{\rho-1}\, F(\alpha,\beta,\gamma,t) \\
=\;& \frac{1}{\Gamma(\alpha)B(\beta,\gamma-\beta)} \int_{c-i\infty}^{c+i\infty} dz\, \Gamma(-z)\Gamma(z+\alpha)B(z+\gamma,\rho-\alpha-z)B(\beta,\gamma-\beta+z) \\
=\;& \frac{\Gamma(\beta)}{\Gamma(\alpha)B(\beta,\gamma-\beta)\Gamma(\gamma+\rho-\alpha)} \int_{c-i\infty}^{c+i\infty} dz\, \Gamma(\alpha+z)\Gamma(\gamma-\beta+z)\Gamma(-z)\Gamma(\rho-\alpha-z)\,. \qquad (2.24)
\end{aligned}$$

and the proposition follow by the first Barnes' lemma. □

Corollary 2.5. *Let n be a non negative integer, $\beta,\gamma,\rho\in\mathbb{C}$ such that $\operatorname{Re}\rho>0$ and $\operatorname{Re}(\beta-\gamma)>n-1$, then*

$$\int_0^1 t^{\rho-1}(1-t)^{\beta-\gamma-n}\, F(-n,\beta;\gamma;t)\,\mathrm{d}t = \frac{\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\beta-\gamma+1)\Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n)\,\Gamma(\beta-\gamma+\rho+1)\Gamma(\gamma-\rho)}\,. \qquad (2.25)$$

Proof. First we apply the result obtained in proposition 2.2 and then we can apply the change of variable $t \to 1-t$ to obtain

$$\begin{aligned}
&\int_0^1 t^{\rho-1}(1-t)^{\beta-\gamma-n}\, F(-n,\beta;\gamma;t)\,\mathrm{d}t \\
=\;& \frac{(\gamma-\beta)_n}{(\gamma)_n}\int_0^1 t^{\beta-\gamma-n}(1-t)^{\rho-1}\, F(-n,\beta;\beta-\gamma-n+1;t)\,\mathrm{d}t \\
=\;& \frac{(\gamma-\beta)_n(1+\beta-\gamma)_{-n}\Gamma(1+\beta-\gamma)\Gamma(\rho)}{(\gamma)_n(1+\rho-\gamma)_{-n}\Gamma(\beta-\gamma+\rho+1)}\;. \qquad (2.26)
\end{aligned}$$

In the second step we have used (2.20). Using (2.9) to change the $-n$ sign, we arrive in the right hand side of (2.25). □

The following lemma is a consequence of the previous corollary. This lemma will be useful in Section 4.1, also this lemma provides an elementary poof for the orthogonality relations of the Jacobi polynomials on the real line and gives the exact value of its norms in a simple way.

Lemma 2.6. *Let n be a non negative integer, $\beta,\gamma,\rho\in\mathbb{C}$ such that* $\operatorname{Re}\rho>0$ *and* $\operatorname{Re}(\beta-\gamma)>n-1$. *Assume $\gamma-\rho=-k\in\mathbb{Z}_{\le 0}$, then*

$$\int_0^1 t^{\rho-1}(1-t)^{\beta-\gamma-n}\, F(-n,\beta;\gamma;t)\,\mathrm{d}t = \begin{cases} \frac{(-1)^n\Gamma(\gamma)\,\Gamma(\rho)\,\Gamma(\beta-\gamma+1)\Gamma(1+n)}{\Gamma(\gamma+n)\,\Gamma(\beta+n+1)} & k=n \\ 0 & k<n. \end{cases} \qquad (2.27)$$

Proof. Let us introduce a regularising parameter $\varepsilon>0$. We have then

$$\left|x^{\rho+\varepsilon-1}(1-x)^{\beta-\gamma-n}F(-n,\beta,\gamma,x)\right| \le C_F x^{\rho-1}(1-x)^{\beta-\gamma-n}\;,\quad x\in[0,1]\;,$$

for some constant C_F. Since $x^{\rho-1}(1-x)^{\beta-\gamma-n}\in L^1([0,1])$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\int_0^1 t^{\rho-1}(1-t)^{\beta-\gamma-n}\, F(-n,\beta;\gamma;t)\,dt \\
&=\lim_{\varepsilon\to 0}\int_0^1 t^{\rho+\varepsilon-1}(1-t)^{\beta-\gamma-n}\, F(-n,\beta;\gamma;t)\,dt \\
&=\lim_{\varepsilon\to 0}\frac{\Gamma(\gamma)\,\Gamma(\rho+\varepsilon)\,\Gamma(\beta-\gamma+1)\Gamma(\gamma-\rho-\varepsilon+n)}{\Gamma(\gamma+n)\,\Gamma(\beta-\gamma+\rho+\varepsilon+1)\Gamma(\gamma-\rho-\varepsilon)} \qquad (2.28)\\
&=\lim_{\varepsilon\to 0}\frac{\Gamma(\gamma)\,\Gamma(\rho+\varepsilon)\,\Gamma(\beta-\gamma+1)\Gamma(n-k-\varepsilon)}{\Gamma(\gamma+n)\,\Gamma(\beta+k+\varepsilon+1)\Gamma(-k-\varepsilon)} \\
&=\lim_{\varepsilon\to 0}\frac{(-1)^{k+1}\Gamma(\gamma)\,\Gamma(\rho+\varepsilon)\,\Gamma(\beta-\gamma+1)\Gamma(1+k+\varepsilon)}{\pi\Gamma(\gamma+n)\,\Gamma(\beta+k+\varepsilon+1)} \\
&\qquad\times\Gamma(n-k-\varepsilon)\sin(\pi\varepsilon)\;.
\end{aligned}$$

In the second step we have used the integral (2.25) and in the next step Euler's reflection formula (2.5). Finally, the limit

$$\lim_{\varepsilon\to 0}\Gamma(n-k-\varepsilon)\sin(\pi\varepsilon)=\begin{cases}-\pi & k=n\\ 0 & k<n.\end{cases} \tag{2.29}$$

establishes the lemma.

□

2.2. **Orthogonal polynomials on the real line.** In this section we introduce basic properties of the general orthogonal polynomials (OP) over the real line $\mathbb{R}$, for a short and very instructive note we cite [50] and [38, 51, 52, 53] for a complete treatise. We will consider $(\mathbb{R},\mu)$ as measurable space equipped with a positive (finite)-Borel measure μ with infinite support, for which

$$m_n := \int x^n d\mu(x) < \infty, \text{ for all}\, n\geq 0\ . \tag{2.30}$$

A unique sequence of polynomials

$$p_n(x)=\gamma_n x^n+\dots,\qquad \gamma_n>0, \tag{2.31}$$

can be constructed using the Gram-Schmidt process, that form an orthonormal system in $L^2(d\mu)$, that is

$$(p_n,p_m)=\int p_n(x)p_m(x)d\mu(x)=h_n\delta_{n,m}. \tag{2.32}$$

Remark 2.7. The p_n's are called the orthogonal polynomials, γ_n is the leading coefficient, h_n is the norm, and

$$\frac{p_n(x)}{\gamma_n},\ \frac{p_n(x)}{\sqrt{h_n}}\ . \tag{2.33}$$

are called the monic orthogonal polynomial and the orthonormal polynomial, respectively.

The moments m_n determine the polynomials p_n. In terms of them one can write up explicit determinant formulae:

Let $n\geq 0$, the Hankel matrix (also known as Gram matrix) is given by

$$H_n=\begin{pmatrix} m_0 & m_1 & \cdots & m_n\\ m_1 & m_2 & \cdots & m_{n+1}\\ \vdots & \vdots & \ddots & \vdots\\ m_n & m_{n+1} & \cdots & m_{2n}\end{pmatrix}. \tag{2.34}$$

For any vector $v^T=(\alpha_0,\dots,\alpha_n)\in\mathbb{R}^{n+1}\setminus\{0\}$ we have the relation,

$$v^T H_n v=\int\left(\alpha_n x^n+\cdots+\alpha_0\right)^2 d\mu(x)>0. \tag{2.35}$$

This implies that the Henkel Matrix is positive definite, consequently, the corresponding Hankel determinants,

$$\Delta_n = \det H_n > 0,\ n \geq 0, \tag{2.36}$$

are all strictly positive.

Theorem 2.8. *The Orthonormal polynomials* $\{p_n\}$ *are given by*

$$p_n(x) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}. \tag{2.37}$$

Proof. By expanding the determinant along the last row, we have

$$(p_n, x^k) = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_{n+1} & \cdots & m_{2n-1} \\ m_k & m_{k+1} & \cdots & m_{n+k} \end{vmatrix} = 0 \quad \text{for } k = 0, 1, \ldots, n-1 \tag{2.38}$$

and

$$(p_n, x^n) = \sqrt{\frac{\Delta_n}{\Delta_{n-1}}}. \tag{2.39}$$

Since

$$p_n(x) = \sqrt{\frac{\Delta_{n-1}}{\Delta_n}} x^n + \cdots, \tag{2.40}$$

and due to the linearity of the inner product, the theorem is complete. □

The representation of the OP in terms of the Hankel matrices is very useful for the theoretical point of view (as we will see in the section 3.2), but not very useful for the actual computation of the OP since it involves the evaluations of determinants. However, for OP on the real line there are a much more efficient way to compute them, the so-called *three term recurrence relation.*

Theorem 2.9. *The orthonormal polynomials* $\{p_n\}$ *on the real line satisfy a three term recurrence relation*

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \tag{2.41}$$

with initial condition $p_0 = 1$ *and* $p_{-1} = 0$.

Proof. Since $xp_n(x)$ is a polynomial of degree $n+1$, so we can expand this polynomial in terms of the first $n+2$ orthonormal polynomials

$$xp_n(x) = \sum_{\ell=0}^{n+1} c_{\ell,n} p_\ell(x), \tag{2.42}$$

where the Fourier coefficients $c_{\ell,n}$ are given by

$$\begin{aligned} c_{\ell,n} &= \int [xp_n(x)]\, p_\ell(x) d\mu(x) \\ &= \int p_n(x)\, [xp_\ell(x)]\, d\mu(x) \\ &= 0, \quad \text{for } \ell+1<n. \end{aligned} \tag{2.43}$$

Because xp_ℓ is a polynomial of degree $\ell+1$, by orthogonality the Fourier coefficient $c_{\ell,n}$ vanishes for $\ell+1<n$. Therefore, the Fourier serie (2.42) only contains three terms, that is when $\ell = n-1$, $\ell = n$ and $\ell = n+1$. And the three term recurrence relation follows by taking $a_{n+1} = c_{n+1,n}$ and $b_n = c_{n,n}$. Note that $c_{n-1,n} = c_{n,n-1} = a_n$. □

An immediate consequence of the theorem 2.9 is the Christoffel-Darboux identity:

Corollary 2.10 (Christoffel-Darboux kernel)**.**

$$\sum_{k=0}^{n-1} p_k(x)p_k(y) = a_n \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}. \tag{2.44}$$

2.3. **Jacobi polynomials.** The most important OP on the real line are the very classical orthogonal polynomials. They are charaterized by a second order differential equation. The Jacobi polynomials, usually denoted by $P_n^{\alpha,\beta}(x)$, are orthogonal with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$, on $[-1,1]$. They are the solution of the second order differential equation,

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+1)x)y' = -n(n+\alpha+\beta+1)y\ . \tag{2.45}$$

Using Frobenius' method (or performing the change of variable $x \to 1-2x$, to transform the differential equation in the Gauß' hypergeometric equation) one can obtain an explicit representation of these polynomials,

$$P_n^{\alpha,\beta}(x) = \frac{(1+\alpha)_n}{n!} F\left(-n, n+\alpha+\beta+1, 1+\alpha, \frac{1-x}{2}\right)\ . \tag{2.46}$$

An immediate consequence of this representation is

$$P_n^{\alpha,\beta}(1) = \frac{(1+\alpha)_n}{n!}. \tag{2.47}$$

Note, if we apply the reflection formula (2.13), we obtain

$$P_n^{\alpha,\beta}(x) = \frac{(1+\beta)_n(-1)^n}{n!} F\left(-n, n+\alpha+\beta+1, 1+\beta, \frac{1+x}{2}\right), \tag{2.48}$$

from which follows

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x)\ . \tag{2.49}$$

Jacobi polynomials will play an important role throughout this text, the norm of these polynomials is given in the following theorem

Theorem 2.11. *Let $\alpha, \beta > -1$. The Jacobi polynomials satisfy the following orthogonality condition*

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x)(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}\delta_{n,m}. \tag{2.50}$$

Proof. Without restriction we can assume $m \geq n$

$$\begin{aligned} &\int_{-1}^{1} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x)(1-x)^\alpha(1+x)^\beta dx \\ &= 2^{\alpha+\beta+1}\int_0^1 P_n^{\alpha,\beta}(1-2x) P_m^{\alpha,\beta}(1-2x) x^\alpha (1-x)^\beta dx \\ &= 2^{\alpha+\beta+1}\frac{(1+\alpha)_n}{n!}\frac{(1+\alpha)_m}{m!}\sum_{k=0}^{n}(-1)^n\binom{n}{k}\frac{(1+\alpha+\beta+n)_k}{(1+\alpha)_k} \\ &\qquad \times \int_0^1 x^{\alpha+k}(1-x)^\beta F(-m, m+\alpha+\beta+1, 1+\alpha, x)dx. \end{aligned} \tag{2.51}$$

in the first step we have changed variable $x \to 1-2x$, in the second step we have used the explicit representation (2.46) of $P_n^{\alpha,\beta}(x)$ together with (2.12). Note that $-k = 1+\alpha-(1+\alpha+k)$, by Lemma 2.6 the last integral vanish when $k < m$ and contributes to the sum only when $k = m$, that is when $m = n$.

□

Some special cases are

$$T_n(x) = \frac{1}{P_n^{(-1/2,-1/2)}(1)} P_n^{(1/2,1/2)}(x), \tag{2.52}$$

$$U_n(x) = \frac{1+n}{P_n^{(1/2,1/2)}(1)} P_n^{(1/2,1/2)}(x), \tag{2.53}$$

$$V_n(x) = \frac{1+2n}{P_n^{(1/2,-1/2)}(1)} P_n^{(1/2,-1/2)}(x), \tag{2.54}$$

$$W_n(x) = \frac{1}{P_n^{(-1/2,1/2)}(1)} P_n^{(-1/2,1/2)}(x)\,, \tag{2.55}$$

known as Chebyshev polynomials of the first kind, second kind, third kind and fourth kind, respectively. When $\alpha = \beta \to \alpha + \frac{1}{2}$, the polynomials

$$C_n^{(\alpha+1)}(x) = \frac{(2(\alpha+1))_n}{(\alpha+3/2)_n} P_n^{(\alpha+1/2,\alpha+1/2)}(x)\,, \tag{2.56}$$

are known as symmetric Jacobi, Ultraspherical polynomials and also as Gegenbauer polynomials. Gegenbauer polynomials may be expressed by Jacobi polynomials with α or $\beta = \pm 1/2$, [38, Thm. 4.1]

$$C_{2n}^{(1+\alpha)}(x) = \frac{(1+\alpha)_n}{(1/2)_n} P_n^{\alpha+1/2,-1/2}(2x^2-1), \tag{2.57}$$

$$C_{2n+1}^{(1+\alpha)}(x) = \frac{(1+\alpha)_{n+1}}{(1/2)_{n+1}}\, x\, P_n^{\alpha+1/2,1/2}(2x^2-1), \tag{2.58}$$

As a consequence of these important relations, known as quadratic transformations, together with (2.46), Gegenbauer polynomials may be expressed in terms of the Gauß' hypergeometric function

$$C_{2n}^{\alpha}(x) = \frac{(-1)^n(\alpha)_n}{n!} F(-n, n+\alpha, 1/2, x^2), \tag{2.59}$$

$$C_{2n+1}^{\alpha}(x) = \frac{(-1)^n(\alpha)_{n+1}}{n!}\, 2x\, F(-n, n+\alpha+1, 3/2, x^2), \tag{2.60}$$

The corresponding three-term recurrence relation for Gegenbauer polynomials is

$$zC_n^{(1+\alpha)}(z) = \frac{n+1}{2(n+\alpha+1)} C_{n+1}^{(1+\alpha)}(z) + \frac{n+2\alpha+1}{2(n+\alpha+1)} C_{n-1}^{(1+\alpha)}(z), \quad n = 1, 2, 3, \ldots . \tag{2.61}$$

2.4. **Hermite and Laguerre polynomials.** The Hermite polynomials, denoted by $H_n(x)$, are orthogonal with respect to the normal distribution $\exp[-x^2]$ over the real line. The orthogonality relations are

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}\, dx = \sqrt{\pi}2^n\, n!\delta_{n,m}. \tag{2.62}$$

The Hermite polynomials also can be defined in terms of its generating function [51, chap. 6]

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{2xt-x^2}, \tag{2.63}$$

which follows using the fact that the normal distribution it is essentially its own Fourier transform.

Laguerre polynomials $L_n^{\alpha}(x)$, are orthogonal with respect to the Gamma distribution $x^{\alpha}e^{-x}$, $\alpha > -1$. Their orthogonality relations are

$$\int_{\mathbb{R}^+} L_n^{\alpha}(x)L_m^{\alpha}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m}. \tag{2.64}$$

The Hermite and Laguerre polynomials are limits of Jacobi polynomials. There are several ways to obtain these limits, one can use for example:

$$\sqrt{\alpha}\int_{-1}^{1}(1-x^2)^{\alpha}dx \;\to\; \int_{\mathbb{R}} e^{-x^2}dx, \quad as\,\alpha\to\infty. \tag{2.65}$$

$$\frac{\beta^{\alpha+1}}{2^{\alpha+\beta+1}}\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}dx \;\to\; \int_{\mathbb{R}^+} x^{\alpha}e^{-x}dx, \quad as\,\beta\to\infty. \tag{2.66}$$

The limit (2.65) tells us, up to a suitable scaling, Hermite polynomials are followed as a limit of Gegenbauer polynomials, this limit reads

$$\lim_{\alpha\to\infty}\alpha^{-n/2}C_n^{\alpha}(x/\sqrt{\alpha}) = \frac{H_n(x)}{n!}\,. \tag{2.67}$$

Using (2.8), the term $\alpha^{-n/2}$ can be obtained by the asymptotic form $2^n(\alpha)_n/n! \sim 2^n\alpha^n/n!$, $\alpha\to\infty$ of the Gegenbauer norm.

Similarly, the limit (2.66) tells us, Laguerre polynomials can be obtained as a limit of Jacobi polynomials,

$$\lim_{\beta\to\infty} P_n^{\alpha,\beta}(1-2x/\beta) = L_n^{\alpha}(x)\,. \tag{2.68}$$

This can be seen using for example the hypergeometric representation (2.46) of Jacobi polynomials together with (2.12) and (2.8).

One can derive the properties of Laguerre and Hermite polynomials from those of Jacobi polynomials. However, it is usually easier to deal with these polynomials directly.

2.5. **Multiple orthogonal polynomials type II.**

Definition 2.12 (see [54])**.** A polynomial $P_n(x)$ is called a multiple orthogonal polynomial (MOP) of a vector index

$$n = (n_1,\dots,n_p)\in\mathbb{N}^p,$$

with respect to a vector of positive Borel measures, supported on the real line

$$\mu = (\mu_1,\dots,\mu_p),\ \operatorname{supp}\mu_i\in\mathbb{R},\ i=1,\dots,p\,,$$

if it satisfies the following conditions:

- $\deg P_n \le |n| := \sum n_i$.
- $\int P_n(x)x^k\,d\mu_i(x) = 0$, $k=0,\dots,n_i-1$ and $i=1,\dots,p$.

Remark 2.13. When $p=1$ the MOP becomes the standard OP, i.e

- $\deg P_n = n$.
- $\int P_n(x)x^k\,d\mu(x) = 0$, $k=0,\dots,n-1$.

Remark 2.14. The notion of MOP can be generalized if we consider the non-Hermitian complex orthogonality with respect to a complex value vector function

$$f(z) = (f_1(z),\dots,f_p(z)),$$

on some contours Γ_i on the complex plane $\mathbb{C}$.

Definition 2.15. A polynomial $P_n(z)$ is called MOP if

- $\deg P_n \leq |n|$.
- $\displaystyle\int_{\Gamma_i} P_n(x)x^k\, f_i(z)dz = 0$, $k = 0, \dots, n_i - 1$ and $i = 1, \dots, p$.

Let $\alpha = (\alpha_1, \dots, \alpha_p)$, the Multiple Hermite polynomials (on the real line) $\{H_n^\alpha(x)\}$ of index $n = (n_1, \cdots, n_p)$, satisfy

$$\int_{\mathbb{R}} H_n^\alpha(x)x^k\, w_i(x)dx = 0,\ k = 0, \dots, n_i - 1,\ i = 1, \dots, p, \tag{2.69}$$

where

$$w_i(x) = \exp\left(\frac{\delta}{2}x^2 + \alpha_i x\right); \tag{2.70}$$

w_i are the Hermite weights with $\delta < 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$.

Theorem 2.16 (See [55], theorem 2.1)**.** *Let $\{H_n^\alpha(x)\}$ be the multiple Hermite polynomials defined by the equation* (2.69)*. Then the generating functon is given by*

$$\sum_{n_1,\dots,n_p=0}^{\infty} H_n^\alpha(x)\frac{t_1^{n_1}\cdots t_p^{n_p}}{n_1!\cdots n_p!} = \exp\left(\delta x \sum_{i=1}^p t_i + \frac{\delta}{2}\left(\sum_{i=1}^p t_i\right)^2 + \sum_{i=1}^p \alpha_i t_i\right). \tag{2.71}$$

3. Orthogonal polynomials on the Complex plane

Let μ be a positive Borel measure on the complex plane, with an infinite number of points in its support, for which

$$m_{nm} = \int z^n \bar{z}^m d\mu(z) < \infty, \quad n, m \in \mathbb{N}\,. \tag{3.1}$$

By Gram-Schmidt process one can construct a unique sequence of polynomials

$$p_n(z) = \gamma_n z^n + \dots, \quad \gamma_n > 0, \tag{3.2}$$

that form an orthonormal system in $L^2(d\mu)$,

$$\langle p_n, p_m\rangle_\mu = \int p_n(z)\overline{p_m(z)}d\mu(z) = \delta_{nm}. \tag{3.3}$$

Like in the Section 2.2 one can show that the Gram Matrix

$$G_n = \begin{pmatrix} m_{00} & m_{10} & \cdots & m_{n0} \\ m_{01} & m_{11} & \cdots & m_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{0n} & m_{1n} & \cdots & m_{nn} \end{pmatrix}. \tag{3.4}$$

is positive definite, and the corresponding Gram determinants,

$$\Delta_n = \det G_n > 0,\ n \geq 0, \tag{3.5}$$

are all strictly positive.

Remark 3.1. If μ is supported on the real line then

$$m_{ij} = \int x^{i+j} d\mu(x) =: \alpha_{i+j} \tag{3.6}$$

thus, $\Delta_n = |\alpha_{i+j}|_{i,j=0}^n$ is a Hankel determinant.

Remark 3.2. If μ is supported on the unit circle then

$$m_{ij} = \int z^i \overline{z}^j d\mu(z) = \int z^{i-j} d\mu(z) =: \beta_{i-j} \tag{3.7}$$

so, $\Delta_n = |\beta_{i-j}|_{i,j=0}^n$ is a Toeplitz determinant.

In these two very important cases the orthogonal polynomials have many special properties that are missing in the general theory.

An alternative representation to Gram-Schmidt that allows to construct orthogonal polynomials, is the Heine formula, see [38]. For a given domain $D \subseteq \mathbb{C}$ in the complex plane, a non-negative weight function $w(z)$, and normalised area measure $\mathrm{d}A$ on D such that all moments exist, we define the following expectation value:

$$\langle\, \mathcal{O}\, \rangle_{N,w} \ = \mathcal{Z}_N^{\ -1} \int_{D^N} \mathcal{O}\, |\Delta_N(z)|^2 \prod_{i=1}^N w(z_i) \mathrm{d}A(z_i)\ , \tag{3.8}$$

where $\mathcal{O}$ depends on $z_{i=1,\dots,N} \in \mathbb{C}$. Here, $\Delta_N(z) = \prod_{j>i}^N (z_j - z_i)$ is the Vandermonde determinant, and $\mathcal{Z}_N$ is a normalisation constant that ensures $\langle\, 1\, \rangle_{N,w} = 1$. The expectation value can be thought of resulting from the joint density of complex eigenvalues of a complex non-Hermitian random matrix ensemble, such as the elliptic Ginibre ensemble. The Heine formula then states that the orthogonal polynomials of degree N in monic normalisation, $\tilde{p}_N(z) = z^N + \dots$, are given by

$$\tilde{p}_N(z) = \left\langle \prod_{i=1}^N (z - z_i) \right\rangle_{N,w} \ . \tag{3.9}$$

That is, they are given by the expectation value of a single characteristic polynomial. Denoting the squared norms of the monic polynomials by $\tilde{h}_N$, we have from (3.3)

$$\int_D \tilde{p}_n(z) \overline{\tilde{p}_m(z)} w(z)\, \mathrm{d}A(z) = \delta_{n,m} \tilde{h}_n\ . \tag{3.10}$$

It is well known (see e.g. [26]) that the normalisation constant in (3.8) can be expressed in terms of these norms as

$$\mathcal{Z}_N = \int_{D^N} |\Delta_N(z)|^2 \prod_{i=1}^N w(z_i) \mathrm{d}A(z_i) = N! \prod_{j=0}^{N-1} \tilde{h}_j\ . \tag{3.11}$$

The following theorem proved in [56], generalises Christoffel's Theorem for polynomials on $\mathbb{R}$:

Theorem 3.3. *Let $\{v_i;\, i = 1, \ldots, K\}$ and $\{u_i;\, i = 1, \ldots, L\}$ be two sets of complex numbers which are pairwise distinct among each set. Without loss of generality we assume $K \geq L \geq 0$, where the empty set is permitted. Then, the following statement holds*[1]*:*

$$\left\langle \prod_{k=1}^{N} \left[\prod_{i=1}^{K} (v_i - z_k) \prod_{j=1}^{L} (\bar{u}_j - \bar{z}_k) \right] \right\rangle_{N,w} = \frac{\prod_{i=N}^{N+K-1} \tilde{h}_i^{\frac{1}{2}} \, \prod_{j=N}^{N+L-1} \tilde{h}_j^{\frac{1}{2}}}{\Delta_K(v)\, \Delta_L(\bar{u})} \det_{1 \leq l,m \leq K} [\, \mathcal{B}(v_l, \bar{u}_m)\,] \,, \tag{3.12}$$

with matrix

$$\mathcal{B}(v_l, \bar{u}_m) \equiv \begin{cases} \kappa_{N+L}(v_l, \overline{u_m}) := \sum_{i=0}^{N+L-1} p_i(v_l) \overline{p_i(u_m)} & \text{for} \quad m = 1, \ldots, L \\ p_{N+m-1}(v_l) & \text{for} \quad m = L+1, \ldots, K \end{cases} . \tag{3.13}$$

The monic polynomials $\tilde{p}_n(z)$ are orthogonal w.r.t $w(z)$, with squared norms $\tilde{h}_n$ and $p_n(z) = \tilde{p}_n(z)/\sqrt{\tilde{h}_n}$.

One may wonder if there are measures supported on the complex plane such that the very classical polynomials satisfy an orthogonality relation with respect to a hermitian inner product of the form (3.3). For Hermite and Laguerre polynomials, the measures supported on the entire complex plane are know. The results are fairly recent see [57] for Hermite polynomials and they give rise to orthonormal bases in Bagmann-like Hilbert space. The orthogonality relations for Holomorphic Laguerre polynomials was shown in [58] and they appear in the study of analytic continuation for functions defined on the positive half-line. The case of Gegenbauer polynomials and some subfamilies of Jacobi polynomials are new and will be presented in Section 4 and 5.

3.1. Holomorphic Hermite polynomials.

Lemma 3.4 (Hermite addition formula)**.** *Let $v, w \in \mathbb{C}$, we define $(v, w) := \sum_{i=1}^{d} v_i w_i$. Then, the Hermite polynomials satisfy the following addition formula*

$$H_n\left(\frac{(v,w)}{\sqrt{(w,w)}}\right) = \frac{n!}{(w,w)^{n/2}} \sum_{m_1+\cdots+m_d=n} \frac{w_1^{m_1} \cdots w_d^{m_d}}{m_1! \cdots m_d!} H_{m_1}(v_1) \cdots H_{m_d}(v_d), \tag{3.14}$$

in particular, when $d = 2$, $a > b > 0$, $v = (x/a, y/b) \in \mathbb{R}^2$ and $w = (a, ib)$ we have

$$H_n\left(\frac{z}{c}\right) = \frac{n!}{c^n} \sum_{k=0}^{n} \frac{a^{n-k}(ib)^k}{(n-k)!k!} H_{n-k}\left(\frac{x}{a}\right) H_k\left(\frac{y}{b}\right), \tag{3.15}$$

where $z = x + iy$ and $c = \sqrt{a^2 - b^2}$.

[1]The empty products are understood in the following sense: $\Delta_0(x) = \Delta_1(x) = 1$ and $\prod_{i=N}^{M \leq N-1} h_i = 1$.

We will provide a proof for the Lemma 3.4 in Section 3.7 together with an extension of this formula to the so-called multiple Hermite polynomials.

The following theorem extends the orthogonality relation for the Hermite polynomials to the complex plane, it was first proven in [57] by van Eindhoven and Meyers. The proof that we will present here is not the proof offered in [57]. However, as it was pointed out by the referee of this article, [57, eq. (1.4)] the proof for the Theorem 3.5 also follows from the relation (3.15), so, we will use this approach.

Theorem 3.5 (van Eindhoven-Meyers)**.** *Let $A > B > 0$ and $Q(z) = A|z|^2 - B\operatorname{Re} z^2$, then the Hermite polynomials satisfy the following orthogonality relations*

$$\int_{\mathbb{C}} H_n\left(\frac{z}{c}\right)\overline{H_m\left(\frac{z}{c}\right)}e^{-Q(z)}dA(z) = n!\left(2\frac{A}{B}\right)^n\delta_{n,m}, \tag{3.16}$$

wehere $c = \sqrt{\frac{2B}{A^2-B^2}}$ and $dA(z)$ in the normalized planar Lebesgue measure.

Proof. Let $a = \frac{1}{\sqrt{A-B}}$, $b = \frac{1}{\sqrt{A+B}}$ and apply the relation (3.15) to obtain

$$\begin{aligned}
&\int_{\mathbb{C}} H_n\left(\frac{z}{c}\right)\overline{H_m\left(\frac{z}{c}\right)}e^{-Q(z)}dA(z) \\
= \;& \frac{n!m!}{\pi c^{n+m}}\sum_{k=0}^{n}\sum_{k'=0}^{m}\frac{a^{n-k}(ib)^k}{(n-k)!k!}\frac{a^{m-k'}(-ib)^{k'}}{(m-k')!k'!}\int_{\mathbb{R}} H_{n-k}\left(\frac{x}{a}\right)H_{m-k'}\left(\frac{x}{a}\right)e^{-(x/a)^2}dx/a \\
& \qquad\qquad \times\int_{\mathbb{R}} H_k\left(\frac{y}{b}\right)H_{k'}\left(\frac{y}{b}\right)e^{-(x/b)^2}dy/b \\
= \;& \frac{n!m!2^n}{c^{n+m}}\sum_{k=0}^{n}\sum_{k'=0}^{m}\frac{a^{n+m-k-k'}(ib)^k(-ib)^{k'}}{(m-k')!k'!}\delta_{n-k,m-k'}\delta_{k,k'} \\
= \;& \frac{n!m!2^n}{c^{n+m}}\sum_{k=0}^{\wedge(n,m)}\frac{a^{n+m-2k}(b^2)^k}{(m-k)!k!}\delta_{n-k,m-k} \\
= \;& \frac{n!2^n\delta_{n,m}}{c^{2n}}\sum_{k=0}^{n}\binom{n}{k}a^{2(n-k)}(b^2)^k \\
= \;& n!\left(2\frac{a^2+b^2}{a^2-b^2}\right)^n\delta_{n,m}.
\end{aligned}$$

In the second step we have used the orthogonality relation (2.4) and in the third step we have performed the sum over k' and the theorem follows using the definition of a, b. ☐

During this proof, Theorem 3.5, we have noticed that the Hermite polynomials satisfy a non-Hermitian orthogonality relation on the complex plane, that is, without the need to conjugate the second factor, see [59, eq. (2.4)] for a different proof,

$$\int_{\mathbb{C}} H_n\left(\frac{z}{c}\right) H_m\left(\frac{z}{c}\right) e^{-Q(z)} dA(z) = n! 2^n \delta_{n,m}, \tag{3.17}$$

and by the change of variable $y \to -y$ on the imaginary part the same result, with both factors conjugated, is obtained.

Furthermore, we also have noticed that the previous theorem can be extended to an orthogonality relation for Hermite polynomials in several variables on the complex plane. For instance:

Let $a_1 > b_1 > 0$, $a_2 > b_2 > 0$, such that $a_1^2 - b_1^2 > a_2^2 - b_2^2$ and let $Q_i(z) = A_i|z|^2 - B_i \operatorname{Re} z^2$, $i = 1, 2$, with

$$A_i = \frac{a_i^2 + b_i^2}{2a_i^2 b_i^2}, \quad B_i = \frac{a_i^2 - b_i^2}{2a_i^2 b_i^2},\ i = 1, 2. \tag{3.18}$$

Theorem 3.6. *With the above notation, the Hermite polynomials satisfy the following orthogonality condition.*

$$\begin{aligned} &\int_{\mathbb{C}^2} H_n\left(\frac{z_1 + iz_2}{c}\right) \overline{H_m\left(\frac{z_1 + iz_2}{c}\right)} e^{-Q_1(z_1) - Q_2(z_2)} dA(z_1) dA(z_2) \\ = \;& n! 2^n \left(\frac{a_1^2 + b_1^2 + a_2^2 + b_2^2}{a_1^2 - b_1^2 - a_2^2 + b_2^2}\right)^n \delta_{n,m}, \end{aligned} \tag{3.19}$$

where $c^2 = a_1^2 - b_1^2 - a_2^2 + b_2^2$.

We have not been able to find this result in the literature.

3.2. **Planar multiple Hermite polynomials.**

Definition 3.7. A polynomial $P_n(z)$ is called a planar MOP (PMOP) of a vector index

$$n = (n_1, \ldots, n_p) \in \mathbb{N}^p,$$

with respect to complex value vector (weight) function,

$$w(z) = (w_1(z), \ldots, w_p(z)),$$

on some domains G_i in the complex plane $\mathbb{C}$, if it satisfies the following conditions

- $\deg P_n \leq |n| := \sum n_i$.
- $\int_{G_i} P_n(z) \bar{z}^k w_i(z) dA(z) = 0$, $k = 0, \ldots, n_i - 1$ and $i = 1, \ldots, p$.

Remark 3.8. When $p = 1$ and $w(z)$ is a positive weight function, the PMOP becomes the standard planar OP, i.e

- $\deg P_n = n$
- $\int P_n(z) \bar{z}^k w(z) dA(z) = 0$, $k = 0, \ldots, n - 1$.

Following Bleher and Kuijlaars [60] we define the induced PMOP by the Elliptic Ginibre Ensembre with a “kind” of external field as follows,

Let $A > B > 0$, $Q(z) = A|z|^2 - B\,\mathrm{Re}(z^2)$, $w_i \in \mathbb{C}$ with $w_i \neq w_j$ if $i \neq j$

$$P_n(z) = \frac{1}{C_n}\int \prod_{i=1}^{n}(z - z_1)\Delta(z)\prod_{i=1}^{n} e^{-(Q(z_i)-\mathrm{Im}(w_i z_i))}dA(z_i), \tag{3.20}$$

where $\Delta_n(z)$ is the $n \times n$ Vandermonde determinant and

$$dA(z) := \frac{\sqrt{A^2 - B^2}}{\pi}dxdy.$$

Since

$$\prod_{i=1}^{n}(z - z_1)\Delta_n(z) = \Delta_{n+1}(z, z_{n+1} = z), \tag{3.21}$$

we can bring P_n in a determinantal form

$$P_n(z) = \frac{1}{C_n}\begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{nn} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{3.22}$$

with

$$m_{jk} := \int z^k e^{-(Q(z)-\mathrm{Im}(w_j z))}dA(z).$$

In order to make $P_n(z)$ a monic polynomial we choose C_n as

$$C_n = \begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{n,n-1} \end{vmatrix}. \tag{3.23}$$

Proposition 3.9. *Consider C_n defined above, then*

$$C_n = (2i)^{\frac{n(n-1)}{2}}\prod_{i=1}^{n} e^{h(w_i)}\Delta(\alpha\bar{w} - \beta w), \tag{3.24}$$

where $h(w) = \alpha|w|^2 - \beta\,\mathrm{Re}(w^2)$ and

$$\alpha = \frac{1}{4}\frac{A}{A^2 - B^2}, \quad \beta = \frac{1}{4}\frac{B}{A^2 - B^2}.$$

Proof. Let $w \in \mathbb{C}$, by performing the Gaussian integrals, is easy to see

$$\int e^{-(Q(z)-\mathrm{Im}(wz))}dA(z) = e^{h(w)}, \tag{3.25}$$

we have then

$$\begin{aligned} m_{jk} &= (2i)^k \partial_{w_j}^k m_{j0} \\ &= (2i)^k \partial_{w_j}^k e^{h(w_j)} \\ &= (2i)^k q_k(\alpha \bar{w}_j - \beta w_j) e^{h(w_j)}, \end{aligned} \tag{3.26}$$

with $q_k(x)$ some monic polynomial of degree k.

By taking out side $e^{h(w_j)}$ in each row of the determinant (3.23) and the factor $2i$ in each column of (3.23) and using the fact that the remaining determinant is invariant under column transformation, we arrive in (3.24).

□

Using (3.22) we see that $P_n(z)$ satisfy

$$\int P_n(z) e^{-(Q(z)-\mathrm{Im}(w_j z))} dA(z) = 0, \quad \text{for } j = 1, \ldots, n, \tag{3.27}$$

and by letting $P_n(z) = z^n + p_{n-1} z^{n-1} + \ldots + p_0$, (3.27) can be written as

$$m_{jn} + \sum_{\ell=0}^{n-1} p_\ell m_{j\ell} = 0, \quad \text{for } j = 1, \ldots, n. \tag{3.28}$$

and, clearly, this can be brought to a matrix form

$$Mp = -m. \tag{3.29}$$

With $p = (p_0, \ldots, p_{n-1})^T$, $M = (m_{jk})_{j=1,\ldots,n; k=0,\ldots,n-1}$ and $m = (m_{jn})_{j=1,\ldots,n}$

By Proposition 3.9, $\det M \neq 0$. Thus *equations* (3.27) *uniquely determine the monic polynomial* P_n.

From now on we will refer to the complex numbers w_i, appearing in the Definition (3.20) of P_n, as the complex eigenvalues of $W \in \mathbb{C}^{n \times n}$ and we would like to study the case where multiple w_i's are allowed. First we observe that

$$P_n(z) = \frac{1}{n! C_n} \int \prod_{i=1}^n (z - z_1) \Delta(z) \det_{1 \le i,j \le n} \left[e^{\mathrm{Im}(w_i z_j)} \right] \prod_{i=1}^n e^{-Q(z_i)} dA(z_i). \tag{3.30}$$

Now, we proceed as in proposition 2.2 in [60].

Proposition 3.10. *Suppose* W *has distinct eigenvalues* w_i, $i = 1, \ldots, p$ *with respective multiplicities* n_i *so that* $n_1 + \cdots + n_p = n$. *Let* $n^{(i)} = n_1 + \cdots + n_i$ *and* $n^{(0)} = 0$. *Define*

$$g_j(z) = \bar{z}^{d_j - 1} e^{-(Q(z) - \mathrm{Im}(w_i z))}, \qquad j = 1, \ldots, n,$$

where $i = i_j$ *is such that* $n^{(i-1)} < j \le n^{(i)}$ *and* $d_j = j - n^{(i-1)}$. *Then the following holds:*

(a) *P_n is given by*

$$P_n(z) = \frac{1}{K_n} \int \prod_{j=1}^{n} (z - z_j) \Delta(z) \prod_{j=1}^{n} g_j(z_j) dA(z_j). \tag{3.31}$$

where

$$K_n = \frac{(2i\alpha)^{\frac{n(n-1)}{2}} \prod_{j=1}^{p} e^{n_j h(w_j)} \Delta_0(\alpha \bar{w} - \beta w) \prod_{i=1}^{p} \prod_{k=1}^{n_i - 1} k!}{\prod_{j<k\le p} \alpha^{n_j n_k} \prod_{j=1}^{n} \left(\frac{i}{2}\right)^{d_j - 1}} \tag{3.32}$$

with

$$\Delta_0(x) := \prod_{1\le i<j\le p} (x_j - x_i)^{n_i n_j}$$

(b) *Let*

$$m_{jk} = \int_{\mathbb{C}} z^k g_j(z) dA(z).$$

Then we have the determinantal formula

$$P_n(z) = \frac{1}{K_n} \begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{nn} \\ 1 & z & \cdots & z^n \end{vmatrix}. \tag{3.33}$$

(c) *For $i = 1, \ldots, p$,*

$$\int_{\mathbb{C}} P_n(x) \bar{z}^j e^{-(Q(z) - \operatorname{Im}(w_i z))} dA(z) = 0, \qquad j = 0, \ldots, n_i - 1, \tag{3.34}$$

and these equations uniquely determine the monic polynomial P_n.

Remark 3.11. Let

$$A = \frac{a^2 + b^2}{2a^2 b^2},\ B = \frac{a^2 - b^2}{2a^2 b^2};\ \ a > b > 0 \tag{3.35}$$

then

$$dA(z) = \frac{1}{\pi ab} d\operatorname{Re}(z) d\operatorname{Im}(z), \quad Q(z) - \operatorname{Im}(w_i z) = \left(\frac{\operatorname{Re} z}{a}\right)^2 + \left(\frac{\operatorname{Im} z}{b}\right)^2 - \operatorname{Im}(w_i z). \tag{3.36}$$

The change of variables for the imaginary part $\operatorname{Im}(z) \to b \operatorname{Im}(z)$, together with (3.36), allows us to take the limit $b \to 0$ on (3.20) and we recover the GUE with an external source, see [61] and references therein.

Let us see one explicit representation for P_n. Let $p = 2$ in the Definition 3.7 and

$$\beta = \left(\frac{w}{c}\alpha_1, \frac{w}{c}\alpha_2\right),\ w = a + ib. \tag{3.37}$$

Proposition 3.12. *The multiple Hermite polynomials $H^{\beta}_{(n,m)}(z/c)$ satisfy the following conditions*

$$\begin{aligned} \int_{\mathbb{C}} H^{\beta}_{(n,m)}\left(\frac{z}{c}\right) \bar{z}^k \, e^{-Q_1(z)} dA(z) &= 0, \quad k = 0, \dots, n-1\,, & (3.38) \\ \int_{\mathbb{C}} H^{\beta}_{(n,m)}\left(\frac{z}{c}\right) \bar{z}^k \, e^{-Q_2(z)} dA(z) &= 0, \quad k = 0, \dots, m-1\,, & (3.39) \end{aligned}$$

where the Hermite weight function is given by

$$\exp\left(-Q_i(z)\right); \quad Q_i(z) := Q(z) - \alpha_i \sqrt{A^2 - B^2}\, \mathrm{Im}(zw), \tag{3.40}$$

and

$$A = \frac{a^2 + b^2}{2a^2 b^2}, \; B = \frac{a^2 - b^2}{2a^2 b^2}; \; c^2 = a^2 - b^2 \;\; a > b > 0. \tag{3.41}$$

Proof. Let $p = 2$ in the Theorem 2.16

$$\sum_{n,m=0}^{\infty} H^{\alpha}_{n}(x) \frac{t_1^n \, t_2^m}{n! \, m!} = \exp\left(\frac{\delta}{2}(t_1^2 + t_2^2) + \alpha_1 t_1 + \alpha_2 t_2 + \delta x (t_1 + t_2) + \delta t_1 t_2\right). \tag{3.42}$$

Now let $x, a \in \mathbb{C}^d$, we define $(a, b) := \sum_i a_i b_i$, and $(a, 1) := \sum_i a_i$ and let

$$\beta = \left(\frac{(a,1)}{\sqrt{(a,a)}}\alpha_1, \frac{(a,1)}{\sqrt{(a,a)}}\alpha_2\right). \tag{3.43}$$

By letting

$$x \mapsto \frac{(x,a)}{\sqrt{(a,a)}}, \quad t_i \mapsto \sqrt{(a,a)}\, t_i, \tag{3.44}$$

in (3.42) we obtain

$$\begin{aligned} &\sum_{n,m=0}^{\infty} H^{\beta}_{(n,m)}\left(\frac{(a,x)}{\sqrt{(a,a)}}\right) (a,a)^{\frac{n+m}{2}} \frac{t_1^n \, t_2^m}{n! \, m!} \\ =\; &\exp\left(\frac{\delta}{2}(a,a)(t_1^2 + t_2^2) + \alpha_1 (a,1) t_1 + \alpha_2 (a,1) t_2 + \delta (a,x)(t_1 + t_2) + \delta (a,a) t_1 t_2\right) \\ =\; &\prod_{i=1}^{d} \sum_{n_i, m_i = 0}^{\infty} H^{\alpha}_{(n_i, m_i)}(x_i) a_i^{n_i + m_i} \frac{t_1^{n_i} \, t_2^{m_i}}{n_i! \, m_i!} \\ =\; &\sum_{n,m=0}^{\infty} \left(\sum_{\substack{n_1 + \cdots + n_d = n \\ m_1 + \cdots + m_d = m}} \frac{a_1^{n_1 + m_1} \cdots a_d^{n_d + m_d}}{n_1! m_1! \cdots n_d! m_d!} H^{\alpha}_{(n_1, m_1)}(x_1) \cdots H^{\alpha}_{(n_d, m_d)}(x_d) \right) t_1^n t_2^m \,. \end{aligned}$$

Therefore

$$H^{\beta}_{(n,m)}\left(\frac{(a,x)}{\sqrt{(a,a)}}\right) = \frac{n!m!}{(a,a)^{\frac{n+m}{2}}} \sum_{\substack{n_1+\cdots+n_d=n \\ m_1+\cdots+m_d=m}} \frac{a_1^{n_1+m_1}\cdots a_d^{n_d+m_d}}{n_1!m_1!\cdots n_d!m_d!} H^{\alpha}_{(n_1,m_1)}(x_1)\cdots H^{\alpha}_{(n_d,m_d)}(x_d). \tag{3.45}$$

In particular, for $d = 2$, $a = (a, ib)$, $x = (x/a, y/b)$, we have $(a, x) = x + iy =: z$, $(a, a)^{\frac{1}{2}} = \sqrt{a^2 - b^2} =: c$ and $(a, 1) = a + ib =: w$ then

$$H^{\beta}_{(n,m)}\left(\frac{z}{c}\right) = \frac{n!m!}{c^{n+m}} \sum_{\substack{0\le k\le n \\ 0\le \ell\le m}} \frac{a^{n+m-k-\ell}(ib)^{k+\ell}}{(n-k)!k!(m-\ell)\ell!} H^{\alpha}_{(n-k,m-\ell)}\left(\frac{x}{a}\right) H^{\alpha}_{(k,\ell)}\left(\frac{y}{b}\right) \tag{3.46}$$

For $\delta = -2$, the identity (3.46) tells us (3.38) and (3.39) holds true with the weight function

$$\exp\left(-Q_i(z)\right); \quad Q_i(z) := Q(z) - \alpha_i\sqrt{A^2 - B^2}\,\mathrm{Im}(zw). \tag{3.47}$$

□

4. The weighted Bergman space $\mathbf{A}^{\mathbf{p}}_{\alpha}$ of the ellipse

The theory of Hilbert spaces of analytic functions in planar domains (and in higher-dimensional complex space), was developed by Stefan Bergman [62], His work focused on spaces of analytic functions $A^2(E, dA)$ that are square-integrable over the given domain E with respect to Lebesgue area measure dA (or volume measure), it relying to a large extent on a reproducing kernel that became known as the Bergman kernel function. When attention was later directed to the spaces of analytic functions A^p that are p-integrables over a given domain with respect to the Lebesgue area measure, it was natural to call them Bergman spaces.

Historically Bergman's space theory on the unit disk has attracted the attention of many mathematicians, H. Hedenmalm, P. Duren, H.S. Shapiro, A. L. Shields, among others. One of the main reasons is that the Bergman spaces A^p of the unit disc contain the Hardy spaces H^p. A function f analytic in the unit disk $\mathbb{D}$ is said to belong to the Hardy space H^p, if the integrals $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ remain bounded as $r \to 1$. However, the Bergman spaces are in many respects much more complicated than their Hardy space cousins. For instance, the invariant subspaces need not be singly generated as they are for the Hardy space [63]. It has been pointed out in the literature (see [64, 65] and references therein) that one particular reason for studying the invariant subspaces of $A^2(\mathbb{D})$ is that the general invariant subspace conjecture in Hilbert space reduces to a special question about invariant subspaces of $A^2(\mathbb{D})$.

In [64] Hedenmalm et al. have studied intensively the weighted Bergman space of the unit disk $\mathbb{D}$, where the weight function has the form $(1 - |z|^2)^\alpha$. Because the measure is rotationally invariant, the associated Bergman polynomials are monomials z^n. These polynomials play an important role in theory, since in general the Bergman Kernel function is given by the infinite sum over the orthonormalized Bergman polynomials, in this case

$$K_\alpha(z, w) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n!\Gamma(2 + \alpha)} (z\overline{w})^n = \frac{1}{(1 - z\overline{w})^{\alpha+2}}. \tag{4.1}$$

For more general domains (up to annulus and lemniscates) the explicit form of this kernel is not known, and the Bergman polynomials are not known either. This can be seen as a second motivation to study Bergman polynomials, also called planar polynomials.

In what follows, we will introduce the weighted Bergman space of the ellipse. To begin, let $a > b > 0$, the equation of the ellipse centered at the origin is

$$\mathbb{R}^2 \ni (x, y) : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4.2}$$

Due to the standard identification $\mathbb{C} \cong \mathbb{R}^2$, we have an equivalent representation

$$\mathbb{C} \ni z = x + iy : \quad A|z|^2 - B\,\mathrm{Re}(z^2) = 1, \tag{4.3}$$

with

$$A = \frac{a^2+b^2}{2a^2b^2}, \quad B = \frac{a^2-b^2}{2a^2b^2}. \tag{4.4}$$

Therefore, the function

$$Q(z) = A|z|^2 - B\,\mathrm{Re}(z^2), \quad z \in \mathbb{C}, \tag{4.5}$$

provides an explicit parametrisation of the interior of an ellipse E:

$$E \;=\; \{z \in \mathbb{C} : Q(z) < 1\}\ . \tag{4.6}$$

For $0 < p < \infty$ and $-1 < \alpha < \infty$, we will denote by $A^p_\alpha := A^p_\alpha(E) \subseteq L^p(E, \mathrm{d}A_\alpha)$ the (weighted) Bergman space of the ellipse E, i.e. the subspace of analytic functions in $L^p(E, \mathrm{d}A_\alpha)$ with finite p-norm. Here,

$$\mathrm{d}A_\alpha(z) \;=\; \frac{(1+\alpha)}{\pi ab}(1-Q(z))^\alpha \mathrm{d}A(z), \tag{4.7}$$

and dA is the planar Lebesque measure, H(E) stand for the space of analytic function in E:

$$A^p_\alpha = \left\{ f \in H(E) : \int_E |f(z)|^p dA_\alpha(z) < \infty \right\}. \tag{4.8}$$

For $1 \le p < \infty$ the associated L^p-norm is definded by

$$||f||_{p,\alpha} = \left(\int_E |f(z)|^p \,\mathrm{d}A_\alpha(z) \right)^{1/p}, \tag{4.9}$$

and for $0 < p < 1$ the corresponding metric is given by

$$d(f,g) \;=\; \int_E |f(z) - g(z)|^p \,\mathrm{d}A_\alpha(z)\ . \tag{4.10}$$

Note, when $a = b$ then the ellipse become a disk of radius a and the measure (4.7) reduces back to [64].

In this section we show that the Bergman space A^p_α is a Banach space when $1 \le p < \infty$, and a complete metric space when $0 < p < 1$. The proof is quite standard and follows the lines of Corollary 1.12 and Proposition 1.13 in [66].

Proposition 4.1. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$, and K be a compact subset of E, with positive minimum distance to ∂E. Then, there is a positive constant C such that*

$$\sup_K |f(z)|^p \le C\|f\|^p_{p,\alpha}\ ,$$

for all $f \in A^p_\alpha$.

Proof. Let $t \in E$ and $0 < r < \mathrm{dist}(t, \partial E) =: d$ be arbitrary. We define the smaller ellipse

$$E_r \;=\; \{z \in \mathbb{C} : Q_r(z) := (\mathrm{Re}\, z)^2/(a - r/2)^2 + (\mathrm{Im}\, z)^2/(b - r/2)^2 \le 1\}\ , \tag{4.11}$$

and suppose that there is a point $z_0 \in B(t, r/2) \setminus E_r$:

$$\{|z_0 - w| \, : w \in \partial E\} \subseteq \{|z - w| \, : z \in B(t, r/2); w \in \partial E\} \; . \tag{4.12}$$

Taking the infimum on both sides of (4.12), we obtain

$$\operatorname{dist}(B(t, r/2), \partial E) \le \operatorname{dist}(z_0, \partial E) \; . \tag{4.13}$$

But (4.13) implies that $d - r/2 \le r/2$, therefore $B(t, r/2) \subseteq E_r$. In consequence we obtain

$$\sup_{z \in B(t, r/2)} h(z) \le \sup_{z \in E_r} h(z) \le h(z_*) =: c(r) \; , \quad z_* \in \partial E_r \; . \tag{4.14}$$

It is easy to see that $0 < c(r) < 1$, and it can be computed explicitly by introducing a Lagrange multiplier, for example.

Thus, given $f \in A^p_\alpha$, $B(t, \varepsilon) \subseteq E$ with positive minimum distance to the boundary ∂E, i.e. $0 < r < \operatorname{dist}(B(t, \varepsilon), \partial E)$, we can find another positive constant $C > 0$ such that

$$\begin{aligned} |f(z)|^p &\le \frac{4}{\pi r^2} \int_{B(z, r/2)} |f(w)|^p \, \mathrm{d}A(w) \\ &\le C \int_{B(z, r/2)} |f(w)|^p \, \mathrm{d}A_\alpha(w) \\ &\le C \int_E |f(w)|^p \, \mathrm{d}A_\alpha(w) \\ &= C \|f\|^p_{p,\alpha} \qquad \text{for } z \in B(t, \varepsilon) \; . \end{aligned} \tag{4.15}$$

In the first step we have used the subharmonicity of $|f|^p$. In the second step the upper bound is trivial for negative $-1 < \alpha < 0$, due to $0 \le h(z)$, whereas for positive $\alpha > 0$ we have used the estimate from (4.14). □

One immediate consequence of Proposition 4.1 is that any Cauchy sequence $\{f_n\} \in A^p_\alpha$ is locally bounded, and so by Montel's Theorem it constitutes a normal family. Thus, some subsequence converges locally uniformly in E, to a function in A^p_α, and we have

Corollary 4.2. *For every $0 < p < \infty$, $-1 < \alpha < \infty$, the weighted Bergman space A^p_α is closed in $L^p(E, dA_\alpha)$.*

Before to proceed with the proof of corollary (4.2) we will recall some definitions and standard theorems in measure theory [67] and complex analysis [68].

Let (X, μ) be a measure space, we say that a property holds $\pmod{\mu}$ (or μ-almost everywhere) if it holds on a set $X \setminus N$, where $\mu(N) = 0$. Convergence $f_n \to f \pmod{\mu}$ means that there is a set $N \subset X$ such that $\mu(N) = 0$ and

$$(\forall x \in X \setminus N) \;\; : \;\; f_n \to f. \tag{4.16}$$

Clearly uniform convergence implies convergence modulo measure, $(N = \emptyset)$.

A sequence of finite measurable function $\{f_n\}$ is called convergent in measure to a measurable function f if

$$\lim_{n\to\infty} \mu(\{|f_n - f| \geq \sigma\}) \;=\; 0, \tag{4.17}$$

for any $\sigma > 0$.

To denote convergence in measure, we write $f_n \xrightarrow{\mu} f$.

Remark convergence $f_n \to f$ in the norm of L^p implies convergence in measure $f_n \xrightarrow{\mu} f$. This follows from the fact that, for any $\sigma > 0$, we have

$$\begin{aligned} \int_X |f_n - f|^p \mathrm{d}\mu &\geq \int_{\{|f_n - f| \geq \sigma\}} |f_n - f|^p \mathrm{d}\mu \\ &\geq \sigma^p \mu\{|f_n - f| \geq \sigma\} \end{aligned} \tag{4.18}$$

Theorem 4.3 (Riesz)**.** *Assume that a sequence $\{f_n\}$ of finite measurable functions converge in measure to a function f. Then, one can indicate a subsequence $\{f_{n_k}\}$ of this sequence such that* $\lim_{k\to\infty} f_{n_k} = f \pmod{\mu}$.

Theorem 4.4 (Montel)**.** *A family $\mathscr{F}$ in $H(E)$ is normal if and only if $\mathscr{F}$ is locally bounded.*

$\mathscr{F} \subset C(E, \mathbb{C})$ is normal means that each sequence in $\mathscr{F}$ has a subsequence which converges to a function f in $C(E, \mathbb{C})$. $H(E)$ is closed in $C(E, \mathbb{C})$.

Based on the above Theorems, let us continue with the proof of Corollary 4.2.

Proof. Let $\{f_n\}$ be a Cauchy sequence in A^p_α and $f \in L^p(E, dA_\alpha)$ such that $\int |f_n - f|^p \mathrm{d}A_\alpha \to 0$ as $n \to \infty$.

And suppose that $B(t, r) \subseteq E$ and let $0 < \rho < \mathrm{d}ist(B(t, r), \partial E)$. By the preceding proposition 4.1 there is a positive constant c such that

$$|f_n(z) - f_m(z)|^p \leq c\|f_n - f_m\|^p_{p,\alpha}, \tag{4.19}$$

for all m, n and $|z - t| \leq r$. Thus $\{f_n\}$ is a uniformly Cauchy sequence on any closed disk K in E. Now since $\{f_n\}$ is a Cauchy sequence, given $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that

$$|f_n(z) - f_m(z)| \;<\; \varepsilon^{1/p} \text{ for } n, m \geq N, \forall z \in K, \tag{4.20}$$

taking $m = N$, we have

$$|f_n(z)| \;<\; |f_N(z)| + \varepsilon^{1/p} \text{ for } n \geq N. \tag{4.21}$$

Then, the inequality $(a+b)^p \leq k(a^p+b^p)$ where a and b are arbitrary nonnegative numbers, and k is a constant depending on p, yields.

$$|f_n(z)|^p \quad < \quad k(|f_N(z)|^p + \varepsilon) \;\text{ for } n \geq N, \tag{4.22}$$

and choosing $M = \max\{c\|f_1\|^p_{p,\alpha}, ..., c\|f_{N-1}\|^p_{p,\alpha}, ck\|f_N\|^p_{p,\alpha} + k\varepsilon\}$ we have

$$|f_n(z)|^p \quad \leq \quad M \;\text{ for all } n, \forall z \in K, \tag{4.23}$$

i.e $\{f_n\}$ is locally bounded.

And so, by Montel's theorem there is an analytic function g on E such that $f_n(z) \to g(z)$ uniformly on compact subsets of E.

Since $\int |f_n - f|^p \mathrm{d}A_\alpha \to 0$, this implies $f_n \overset{\mu}{\to} f$. And by Riesz' theorem there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f \pmod{\mu}$, but $\{f_n\}$ is Cauchy sequence, that means $f_n \to f \pmod{\mu}$. Using uniqueness of the limit in measure, i.e if $f_n \overset{\mu}{\to} f$ and $f_n \overset{\mu}{\to} g$, then $f = g \pmod{\mu}$ implies that $f \in A^p$. □

4.1. **Holomorphic Gegenbauer polynomials.** For $p \geq 1$ it follows from Corollary 4.2 that the Bergman space is a Banach space, and in particular for $p = 2$ a Hilbert space with the inner product defined as

$$\langle f, g\rangle_\alpha := \int_E f(z)\overline{g(z)}\,\mathrm{d}A_\alpha(z)\ , \quad f, g \in A^2_\alpha. \tag{4.24}$$

One might immediately ask, whether it is possible to provide an orthogonal basis for this space. The result is quite simple and surprising and presented in the following Theorem 4.5. It constitutes one of our main results.

For any non-negative integer n and real parameter $\alpha > -1$ let us define the polynomials

$$p_n^{(\alpha)}(z) \quad := \quad \frac{1}{\sqrt{h_n}} C_n^{(1+\alpha)}\left(\frac{z}{c}\right)\ , \quad n = 0, 1, \ldots \tag{4.25}$$

where $C_n^{(1+\alpha)}(x)$ are the standard Gegenbauer polynomials on the real line having real coefficients, now taken with a complex argument. We recall that the ellipse E in (4.6) defining the inner product (4.24) is parametrised by the real numbers $a > b > 0$. The constant

$$c = \sqrt{a^2 - b^2} > 0 \tag{4.26}$$

provides the location of the right focus of the ellipse E, and we define by

$$h_n \quad := \quad h_n(a,b) = \frac{1+\alpha}{1+\alpha+n} C_n^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right) > 0\ . \tag{4.27}$$

Theorem 4.5. *The set of polynomials* $\{p_n^{(\alpha)}\}_{n\in\mathbb{N}}$ *defined in* (4.25) *forms a orthonormal basis for* A^2_α *for any* $\alpha > -1$.

We will prove the orthogonality of the sequence of Gegenbauer polynomials in the following lemma, whereas the completeness of this sequence is deferred to the very end of this section.

Lemma 4.6. *Let E be the elliptic domain (4.6), $dA_\alpha(z)$ the density (4.7) over E and $\alpha > -1$. Then, the sequence of Gegenbauer polynomials $\{C_n^{(1+\alpha)}(z)\}_{n\in\mathbb{N}}$ satisfies the following orthogonality relation*

$$\int_E C_m^{(1+\alpha)}\left(\frac{z}{c}\right) C_n^{(1+\alpha)}\left(\frac{\overline{z}}{c}\right) \mathrm{d}A_\alpha(z) \quad = \quad \frac{1+\alpha}{1+\alpha+n} C_n^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right)\delta_{nm} \ , \quad (4.28)$$

where $a > b > 0$ and $c = \sqrt{a^2-b^2}$.

Proof. It is sufficient to show that for all $m \in \mathbb{N}$

$$\int_E C_m^{(1+\alpha)}\left(\frac{z}{c}\right)\left(\frac{\overline{z}}{c}\right)^j \mathrm{d}A_\alpha(z) = 0 \ , \quad \text{for } j = 0, 1, ..., m-1 \ . \quad (4.29)$$

Due to the fact that the ellipse is a central symmetric domain and the integration measure (4.7) is invariant under the reflection $z \to -z$, as well as the reflection properties (2.49) of Gegenbauer polynomials $C_n^{(1+\alpha)}(-z) = (-1)^n C_n^{(1+\alpha)}(z)$, without restriction we assume that either $m = 2n$ and $j = 2l$ are both even, or $m = 2n+1$ and $j = 2l+1$ are both odd, and $l < n$. In the following we will only present the even-even case. The odd-odd case follows from the same line of arguments and it can be found in [2, App. A]

We rewrite the integral (4.29) in terms of elliptic coordinates

$$\mathrm{Re}(z) = ar\cos(\theta) \ , \ \mathrm{Im}(z) = br\sin(\theta) \ , \quad \text{with } \ r \in [0,1), \ \theta \in [0, 2\pi] \ . \quad (4.30)$$

The pullback of the measure $dA_\alpha(z)$ in E to $[0,1) \times [0, 2\pi]$ reads

$$dA_\alpha(z) = \frac{1+\alpha}{\pi}(1-r^2)^\alpha \, r dr d\theta \, , \quad (4.31)$$

and we obtain for the complex arguments

$$\frac{z(r,\theta)}{c} \quad = \quad \frac{r}{2}(Re^{i\theta} + R^{-1}e^{-i\theta}) \ , \quad \text{with } R := \frac{a+b}{c} = \sqrt{\frac{a+b}{a-b}} \ . \quad (4.32)$$

This leads to the following expression

$$\int_E C_{2n}^{(1+\alpha)}\left(\frac{z}{c}\right)\left(\frac{\overline{z}}{c}\right)^{2l} \mathrm{d}A_\alpha(z) =$$

$$= \frac{1+\alpha}{\pi}\int_0^1 \mathrm{d}r r \int_0^{2\pi} \mathrm{d}\theta \, C_{2n}^{(1+\alpha)}\left(\frac{z(r,\theta)}{c}\right)\left(\frac{\overline{z(r,\theta)}}{c}\right)^{2l} (1-r^2)^\alpha. \quad (4.33)$$

The even Gegenbauer polynomials can be written in terms of Gauß' hypergeometric function (2.59) in the following way

$$
\begin{aligned}
C_{2n}^{(1+\alpha)}\left(\frac{z(r,\theta)}{c}\right) &= \frac{(-1)^n(1+\alpha)_n}{n!}F\left(-n, n+\alpha+1; \frac{1}{2}; \frac{z(r,\theta)^2}{c^2}\right) \\
&= \frac{(-1)^n}{2\Gamma(1+\alpha)n!}\sum_{p=0}^{n}\sum_{k=0}^{2p}(-1)^p\binom{n}{p}\binom{2p}{k}\frac{\Gamma(1+\alpha+n+p)\Gamma(p)}{\Gamma(2p)}r^{2p}R^{2(k-p)}e^{2i\theta(k-p)} \\
&= \frac{(-1)^n}{2\Gamma(1+\alpha)n!}\sum_{p=0}^{n}\sum_{k=0}^{2p}(-1)^p\binom{n}{p}\binom{2p}{k}\frac{\Gamma(1+\alpha+n+p)\Gamma(p)}{\Gamma(2p)}r^{2p}R^{2(p-k)}e^{2i\theta(p-k)}.
\end{aligned}
\tag{4.34}
$$

Here, we introduced two representations to be both used below, using the binomial theorem for (4.32) in two equivalent ways. In order to prepare the integration in (4.33), we spell out the complex conjugated variable to the power $2l$:

$$
\begin{aligned}
\left(\frac{\overline{z(r,\theta)}}{c}\right)^{2l} &= \left(\frac{r}{2}\right)^{2l}\left(Re^{-i\theta}+R^{-1}e^{i\theta}\right)^{2l} \\
&= \left(\frac{r}{2}\right)^{2l}\left[\sum_{k=1}^{l}\binom{2l}{k+l}R^{-2k}e^{2i\theta k}+\binom{2l}{l}+\sum_{k=1}^{l}\binom{2l}{k+l}R^{2k}e^{-2i\theta k}\right].
\end{aligned}
\tag{4.35}
$$

From the radial integral in (4.33) we obtain, including all prefactors,

$$
\frac{1+\alpha}{\pi}\int_0^1 \mathrm{d}r\, r^{2p+1}\frac{r^{2l}}{2^{2l}}(1-r^2)^{\alpha} = \frac{1+\alpha}{2^{2l+1}\pi}B(1+\alpha, 1+p+l). \tag{4.36}
$$

For the remaining angular integration we thus have

$$\int_E C_{2n}^{(1+\alpha)}\left(\frac{z}{c}\right)\left(\frac{\overline{z}}{c}\right)^{2l} \mathrm{d}A_\alpha(z) =$$

$$\begin{aligned}
&= \frac{(1+\alpha)(-1)^n}{2^{2l+1}\pi}\sum_{k'=1}^{l}\sum_{p=0}^{n}\sum_{k=0}^{2p}\binom{2l}{k'+l}\frac{(-1)^p\Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!k!(2p-k)!\Gamma(2+\alpha+l+p)} \\
&\qquad \times R^{2(p-k-k')}\int_0^{2\pi}\mathrm{d}\theta\, e^{2i\theta(p-k+k')} \\
&\quad + \frac{(1+\alpha)(-1)^n}{2^{2l+1}\pi}\sum_{p=0}^{n}\sum_{k=0}^{2p}\binom{2l}{l}\frac{(-1)^p\Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!k!(2p-k)!\Gamma(2+\alpha+l+p)} \\
&\qquad \times R^{2(p-k)}\int_0^{2\pi}\mathrm{d}\theta\, e^{2i\theta(p-k)} \\
&\quad + \frac{(1+\alpha)(-1)^n}{2^{2l+1}\pi}\sum_{k'=1}^{l}\sum_{p=0}^{n}\sum_{k=0}^{2p}\binom{2l}{k'+l}\frac{(-1)^p\Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!k!(2p-k)!\Gamma(2+\alpha+l+p)} \\
&\qquad \times R^{2(k-p+k')}\int_0^{2\pi}\mathrm{d}\theta\, e^{2i\theta(k-p-k')}.
\end{aligned} \tag{4.37}$$

In the first step we have already simplified the binomial factors and Gamma-functions from (4.34). Notice that in the first two terms, obtained from integrating over the first two contributions on the right-hand side of (4.35), we have used the second identity in (4.34), whereas for the last sum from (4.35) we have used the first form of identity in (4.34). We now evaluate each of the multiple sums in (4.37) individually. In the last triple sum we have $k = p + k'$ due to the angular integration, because of $k \leq 2p$, this force $k' \leq p$ we obtain for it

$$\frac{(1+\alpha)(-1)^n}{2^{2l}}\sum_{k'=1}^{l}\binom{2l}{k'+l}\sum_{p=k'}^{n}\frac{(-1)^p\Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!(k'+p)!(p-k')!\Gamma(2+\alpha+l+p)}R^{4k'}. \tag{4.38}$$

Clearly, this is a polynomials in R of degree $4l$. Let

$$\begin{aligned}
a_k(n,l) &= \sum_{p=k}^{n}\frac{(-1)^p\Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!(k+p)!(p-k)!\Gamma(2+\alpha+l+p)} \\
&= \frac{(-1)^k}{(n-k)!}\sum_{p=0}^{n-k}(-1)^p\binom{n-k}{p}\frac{\Gamma(1+\alpha+n+k+p)\Gamma(1+l+k+p)}{(2k+p)!\Gamma(2+\alpha+l+k+p)}.
\end{aligned} \tag{4.39}$$

From the second term in (4.37), the double sum, the angular integration contribute only when $p = k$, i.e. we obtain the term a_0 with R rise to the power 0. For the first triple sum, we have again $k = p + k'$ thus $k' \leq p$, so after collecting all the terms we have

$$
\begin{aligned}
&\int_E C_{2n}^{(1+\alpha)}\left(\frac{z}{c}\right)\left(\frac{\overline{z}}{c}\right)^{2l} \mathrm{d}A_\alpha(z) \\
&= \frac{(1+\alpha)}{(-1)^n 2^{2l}}\left(\sum_{k=1}^{l}\binom{2l}{k+l}a_k R^{4k} + \binom{2l}{l}a_0 + \sum_{k=1}^{l}\binom{2l}{k+l}a_k R^{-4k}\right) \\
&= \frac{(1+\alpha)}{(-1)^n 2^{2l}}\left(\sum_{k=0}^{l-1}\binom{2l}{k}a_{l-k} R^{4(l-k)} + \binom{2l}{l}a_0 + \sum_{k=l+1}^{2l}\binom{2l}{k}a_{k-l} R^{-4(k-l)}\right). \quad (4.40)
\end{aligned}
$$

So, if we can show that all coefficients $a_k(n,l)$ vanish for $k = 0, 1, \ldots, l$ when $l < n$, we are done. This can be seen as follows. From the definition (4.39) we have,

$$
\begin{aligned}
a_k &= \frac{(-1)^k}{(n-k)!}\sum_{p=0}^{n-k}(-1)^p\binom{n-k}{p}\frac{\Gamma(1+\alpha+n+k+p)\Gamma(1+l+k+p)}{(2k+p)!\Gamma(2+\alpha+l+k+p)} \\
&= \frac{(-1)^k}{(n-k)!\Gamma(1+\alpha)}\int_0^1 \mathrm{d}x\, x^{l+k}(1-x)^\alpha \sum_{p=0}^{n-k}(-1)^p\binom{n-k}{p}\frac{\Gamma(1+\alpha+n+k+p)}{\Gamma(1+2k+p)}x^p \\
&= \frac{(-1)^k(1+\alpha)_{n+k}}{(n-k)!(2k)!}\int_0^1 \mathrm{d}x\, x^{l+k}(1-x)^\alpha F(-n+k, 1+\alpha+n+k; 1+2k; x).
\end{aligned}
\quad (4.41)
$$

By Lemma 2.6, with $\gamma = 2k+1$ and $\rho = l+k+1$, the last integral vanishes for $l < n$, and for $l = n$, we have

$$
a_k(n,n) = \frac{(-1)^n\Gamma(1+\alpha+n+k)\Gamma(1+\alpha+n-k)}{\Gamma(1+\alpha)\Gamma(2n+\alpha+2)}\ . \quad (4.42)
$$

Note that $a_{n-k}(n,n) = a_{k-n}(n,n)$ and $a_0(n,n) = a_{n-n}(n,n)$, taking into consideration this symmetry, from (4.40) we get the result for the integral at $n = l$ as

$$
\int_E C_{2n}^{(1+\alpha)}\left(\frac{z}{c}\right)\left(\frac{\overline{z}}{c}\right)^{2l} \mathrm{d}A_\alpha(z) = \delta_{n,l}\frac{(2n)!2^{-2n}}{(2+\alpha)_{2n}}\sum_{k=0}^{2n}\frac{(1+\alpha)_k(1+\alpha)_{2n-k}}{k!(2n-k)!}R^{4(n-k)}.
$$

The leading coefficient of $C_{2l}^{\alpha}(z)$ can be obtained from (2.59)

$$
C_{2l}^{(1+\alpha)}(z/c) = \frac{(1+\alpha)_{2l}2^{2l}}{(2l)!}(z/c)^{2l} + O((z/c)^{2l-2})\ . \quad (4.43)
$$

Because the lower powers give zero, combined with (4.43) we have

$$\int_E C_{2n}^{(1+\alpha)}\left(\frac{z}{c}\right)C_{2l}^{(1+\alpha)}\left(\frac{\overline{z}}{c}\right)\mathrm{d}A_\alpha(z) \;=\; \frac{(1+\alpha)\delta_{2n,2l}}{1+\alpha+2n}\sum_{k=0}^{2n}\frac{(1+\alpha)_k(1+\alpha)_{2n-k}}{k!(2n-k)!}R^{4(n-k)}.$$

The remaining sum can be related to a single Gegenbauer polynomial as follows. Because this sum is invariant under $k \to 2n-k$, we can write it as

$$\begin{aligned}
&= \frac{1}{2}\sum_{k=0}^{2n}\frac{(1+\alpha)_k(1+\alpha)_{2n-k}}{k!(2n-k)!}(R^{4(n-k)}+R^{-4(n-k)}) \\
&= \sum_{k=0}^{2n}\frac{(1+\alpha)_k(1+\alpha)_{2n-k}}{k!(2n-k)!}\cosh[(2n-2k)\ln(R^2)] \\
&= C_{2n}^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right)\ .
\end{aligned} \tag{4.44}$$

In the last step we have used the (analytically continued) relation [69, 18.5.11]

$$C_m^{(1+\alpha)}(\cos\theta) = \sum_{l=0}^{m}\frac{(1+\alpha)_l(1+\alpha)_{m-l}}{l!(m-l)!}\cos((m-2l)\theta)\ , \tag{4.45}$$

together with

$$\cosh[\ln(R^2)] = \frac{1}{2}(R^2+R^{-2}) = \frac{a^2+b^2}{a^2-b^2}\ . \tag{4.46}$$

This completes the proof for even indices. The proof for the odd polynomials follows exactly in the same way and as was mentioned above it can be found in [2]. □

Remark 4.7. We can establish contact with the usual orthogonality relation for the Gegenbauer polynomials on the real interval $[-1,1]$. The change of variables for the imaginary part $y = \frac{b}{a}\hat{y}$ maps the ellipse to a disc of radius a. Together with $\mathrm{d}A_\alpha(z) = (1+\alpha)(1-(x/a)^2-(y/b)^2)^\alpha \mathrm{d}x\mathrm{d}y/(ab\pi)$, this allows us to take the limit $b \to 0$ on (4.28)

$$\begin{aligned}
&\lim_{b\to 0}\int_E C_m^{(1+\alpha)}\left(\frac{z}{c}\right)C_n^{(1+\alpha)}\left(\frac{\overline{z}}{c}\right)\mathrm{d}A_\alpha(z) = \\
&= \int_{-a}^{a} C_m^{(1+\alpha)}\left(\frac{x}{a}\right)C_n^{(1+\alpha)}\left(\frac{x}{a}\right)\left(1-\frac{x^2}{a^2}\right)^\alpha \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(1-\frac{\hat{y}^2}{a^2-x^2}\right)^\alpha \frac{(1+\alpha)\mathrm{d}\hat{y}\mathrm{d}x}{a^2\pi} \\
&= \int_{-1}^{1} C_m^{(1+\alpha)}(x)C_n^{(1+\alpha)}(x)\left(1-x^2\right)^{\alpha+\frac{1}{2}} F\left(\frac{1}{2},-\alpha;\frac{3}{2};1\right)\frac{2(1+\alpha)}{\pi}\mathrm{d}x \\
&= \frac{1+\alpha}{1+\alpha+n}C_n^{(1+\alpha)}(1)\delta_{n,m}\ .
\end{aligned} \tag{4.47}$$

The value of Gegenbauer polynomials at $x=1$ follows from (2.47)

$$C_n^{(1+\alpha)}(1) = \frac{\Gamma(2+2\alpha+n)}{\Gamma(2+2\alpha)\Gamma(n+1)} \ , \tag{4.48}$$

and for Gauß' hypergeometric function [47, 9.122] at unity

$$F\left(\frac{1}{2}, -\alpha; \frac{3}{2}; 1\right) = \frac{\sqrt{\pi}\Gamma(1+\alpha)}{2\Gamma(\alpha+3/2)} \ , \tag{4.49}$$

this yield to the standard orthogonality relation

$$\int_{-1}^{1} C_n^{(\alpha+1)}(x)\, C_m^{(\alpha+1)}(x)(1-x^2)^{\alpha+\frac{1}{2}}\mathrm{d}x \ = \ \frac{2^{1-2(1+\alpha)}\pi\Gamma(2+2\alpha+n)}{(1+\alpha+n)\Gamma^2(1+\alpha)n!}\delta_{n,m} \ . \tag{4.50}$$

We can now finish the proof of Theorem 4.5 by showing the completeness of the system of orthogonal polynomials.

Proof. Let $f \in A^2_\alpha$ with $\langle f, p_n\rangle_\alpha = 0$ for all $n = 0, 1, 2, ...$ Then

$$0 = \lim_{b\to 0}\langle f, p_n\rangle_\alpha = \int_{-1}^{1} \mathrm{d}x\, f(ax)\, C_n^{(1+\alpha)}(x)(1-x^2)^{\alpha+\frac{1}{2}} \ . \tag{4.51}$$

Hence $f(ax) = 0$ for all $x \in (-1, 1)$, see [38] for the completeness of the Jacobi polynomials on the real line. Since f is regular in E, it follows that $f \equiv 0$, i.e. $\{p_n^\alpha\}$ defined above form an orthonormal basis for A^2_α. □

As remarks, we will establish a series of immediate consequences of Theorem 4.5.

Remark 4.8. In the case $\alpha = 0$ we recover the orthogonality relation for Chebyshev polynomials of the second kind, due to $U_n(x) = C_n^{(1)}(x)$, which goes back to [45]. We will come back to this statement in Section 5.

Remark 4.9. In the limit $c \to 0$, when the ellipse E becomes a disc, we obtain for integer values of α the weight function that results from the complex eigenvalues of the ensemble of truncated unitary random matrices studied in [14], with monomials as orthogonal polynomials. This can be seen as follows: We have from eq. (4.34) that the monic Gegenbauer polynomials occurring in (4.28) read:

$$\tilde{p}_n^{(\alpha)}(z) := \frac{n!c^n}{2^n(1+\alpha)_n}C_n^{(1+\alpha)}(z/c) \ . \tag{4.52}$$

Multiplying (4.28) with the corresponding factors, we can take the limit $b \to a$, implying $c \to 0$ in this orthogonality relation, to obtain

$$\int_{x^2+y^2<a^2} z^m\bar{z}^n(1+\alpha)\left(1-\frac{|z|^2}{a^2}\right)^\alpha \frac{\mathrm{d}^2z}{\pi a^2} \ = \ \frac{n!(1+\alpha)}{(1+\alpha)_n(1+\alpha+n)}a^{2n}\delta_{n,m} \ , \tag{4.53}$$

where $z = x+iy$. After rescaling $z \to az$, and dividing (4.53) by $(1+\alpha)$, we arrive at the weight function and monic polynomials for the complex eigenvalues in the ensemble of truncated unitary random matrices [14] on the unit disc. It is defined starting from the circular unitary ensemble of Haar distributed unitary random matrices of size $N \times N$, and truncating these to the upper left block of size $M \times M$ with $N > M$, by removing

$N-M$ rows and columns. The weight function reads $w(z) = (1-|z|^2)^{N-M-1}$, that is we have to identify $\alpha = N-M-1 \geq 0$. In this case there is no singularity on the boundary of the circle, and we may extend our integration from inside the disc to include the boundary, cf. (4.24). In this ensemble this is important as for large M and small truncation $N-M$ a substantial fraction of eigenvalues of the truncated unitary matrix may remain on the unit circle. We refer to [14] for a further discussion of the limiting behaviour.

In analogy to the relation between the Ginibre ensemble and its elliptic version, our Gegenbauer polynomials can thus be viewed as the orthogonal polynomials of an elliptic deformation of the truncated unitary ensemble [14], with an appropriate random matrix realisation yet to be constructed.

Remark 4.10. We can make contact with the holomorphic Hermite polynomials (Thm. 3.5) in the plane. Setting $z \to z/\sqrt{1+\alpha}$ and taking α to infinity in (4.28), we have from (2.67)

$$\lim_{\alpha\to\infty} (1+\alpha)^{-\frac{n}{2}} C_n^{(1+\alpha)}\left((1+\alpha)^{-\frac{1}{2}}x\right) = H_n(x)/n! \ , \tag{4.54}$$

leading to

$$\int_{\mathbb{C}} H_m(z/c)\, H_n(\overline{z}/c)\, e^{-Q(z)} dA(z) \quad = \quad \pi n!\ ab \left(2\frac{a^2+b^2}{a^2-b^2}\right)^n \delta_{n,m}\ , \tag{4.55}$$

with $Q(z)$ defined in (4.5). This reproduces the known orthogonality relation for Hermite polynomials in the complex plane, obtained by van Eijndhoven and Meyers [57, Eq.(0.5)] for $a = \sqrt{\frac{1}{1-A}}$ and $b = \sqrt{\frac{A}{1-A}}$, with $0 < A < 1$, see also [21].

Remark 4.11. Gegenbauer polynomials may be written in terms of symmetric Jacobi polynomials (2.56), the orthogonality relations given by lemma 4.6 imply

$$\int_E P_n^{(\alpha+\frac{1}{2},\alpha+\frac{1}{2})}(z/c)\, P_m^{(\alpha+\frac{1}{2},\alpha+\frac{1}{2})}(\overline{w}/c) dA_\alpha(w) = \frac{\delta_{n,m}(1+\alpha)}{1+\alpha+n} \frac{(\alpha+\frac{3}{2})_n}{(2(1+\alpha))_n} P_n^{(\alpha+\frac{1}{2},\alpha+\frac{1}{2})}\left(\frac{a^2+b^2}{a^2-b^2}\right) \tag{4.56}$$

where $a > b > 0$ and $c = \sqrt{a^2-b^2}$.

Remark 4.12. Gegenbauer polynomials satisfy a non-Hermitian orthogonality condition. With the above notation, let us define the bilinear form $B_\alpha(f,g)$ by

$$B_\alpha(f,g) := \int_E f(z)g(z) dA_\alpha(z), \quad f, g \in L^2(E, dA_\alpha). \tag{4.57}$$

Lemma 4.13. *The sequence of Gegenbauer polynomials* $\{C_n^{(1+\alpha)}(z/c)\}_{n\in\mathbb{N}}$ *satisfy the following non-Hermitian orthogonality relations*

$$B_\alpha\left[C_n^{(1+\alpha)}, C_m^{(1+\alpha)}\right] = \frac{1+\alpha}{1+\alpha+n} C_n^{(1+\alpha)}(1)\, \delta_{nm}. \tag{4.58}$$

Proof. Unsing the same notations as in lemma 4.6, we recall that even Gegenbauer polynomials can be written in the following two equivalent forms

$$C_{2n}^{(1+\alpha)}\left(\frac{z(r,\theta)}{c}\right)=\frac{(-1)^n\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(1+\alpha)}F\left(-n,n+\alpha+1;\frac{1}{2};\frac{z(r,\theta)^2}{c^2}\right)$$

$$=\frac{(-1)^n}{2\Gamma(1+\alpha)n!}\sum_{p=0}^{n}\sum_{k=0}^{2p}(-1)^p\binom{n}{p}\binom{2p}{k}\frac{\Gamma(1+\alpha+n+p)\Gamma(p)}{\Gamma(2p)}r^{2p}R^{2(k-p)}e^{2i\theta(k-p)} \tag{4.59}$$

$$=\frac{(-1)^n}{2\Gamma(1+\alpha)n!}\sum_{p=0}^{n}\sum_{k=0}^{2p}(-1)^p\binom{n}{p}\binom{2p}{k}\frac{\Gamma(1+\alpha+n+p)\Gamma(p)}{\Gamma(2p)}r^{2p}R^{2(p-k)}e^{2i\theta(p-k)}, \tag{4.60}$$

and that under the change of variable (4.30) $z(r,\theta)$ it can be expanded using Newton's binomial, but this time without taking complex conjugation

$$\begin{aligned}\left(\frac{z(r,\theta)}{c}\right)^{2l} &= \left(\frac{r}{2}\right)^{2l}\left(Re^{i\theta}+R^{-1}e^{-i\theta}\right)^{2l}\\ &= \left(\frac{r}{2}\right)^{2l}\left[\sum_{k=1}^{l}\binom{2l}{k+l}R^{-2k}e^{-2i\theta k}+\binom{2l}{l}+\sum_{k=1}^{l}\binom{2l}{k+l}R^{2k}e^{2i\theta k}\right].\end{aligned} \tag{4.61}$$

If we combine (4.59) with the first term in the right hand side of (4.61) and (4.60) with the last term in (4.61) we will obtain the same angular contribution as in (4.40), but now with R rised to the power 0. Replacing $R=1$ in (4.46) we complete the proof. □

4.2. **Legendre polynomials.** From Lemma 4.6 at $\alpha=-1/2$ we obtain as a special case the orthogonality of the Legendre polynomials $P_n(x)=C_n^{(1/2)}(x)$:

Corollary 4.14. *The Legendre polynomials P_n are orthogonal with respect to the weight function $\mathrm{d}A_\alpha$ defined in (4.7) at $\alpha=-1/2$:*

$$\int_E P_m\left(\frac{z}{c}\right)P_n\left(\frac{\overline{z}}{c}\right)\mathrm{d}A_{-\frac{1}{2}}(z)=\frac{1}{1+2n}P_n\left(\frac{a^2+b^2}{a^2-b^2}\right)\delta_{n,m}\ . \tag{4.62}$$

We have not been able to find this result in the literature either.

5. The weighted Bergman space $\mathbf{A}^{\mathbf{p}}_{\alpha,\pm\frac{1}{2}}$

The quadratic transformations (2.57) and (2.58), which relates the Gegenbauer polynomials with a subfamily of non-symmetric Jacobi polynomials, suggest that the underlying map together with Theorem 4.5 will lead to new orthogonality relations for the non-symmetric Jacobi polynomials at $(\alpha+1/2, \pm 1/2)$. Let us start, first, with the effect of such quadratic transformation on the ellipse E (4.6).

Proposition 5.1. *Let $a > b > 0$, and for*

$$\tilde{a} = \frac{a^2+b^2}{c}, \quad \tilde{b} = \frac{2ab}{c}, \quad c = \sqrt{a^2-b^2}$$

consider the ellipse $E_{\tilde{a},\tilde{b}}$, with parameters $\tilde{a}, \tilde{b}$. Then, the pullback of the measure $dA_\alpha(z)$ in E to $E_{\tilde{a},\tilde{b}}$ is

$$dA_\alpha(z) \;=\; \frac{(1+\alpha)}{2\pi\tilde{b}}\frac{(1-J(w))^\alpha}{|\tilde{c}+w|}dA(w) \tag{5.1}$$

where,

$$J(w) = \frac{\tilde{a}}{\tilde{b}^2}|\tilde{c}+w| - \frac{\tilde{c}}{\tilde{b}^2}\operatorname{Re}(\tilde{c}+w). \tag{5.2}$$

Proof. We introduce first an auxiliary ellipse $E^{cut}_{\tilde{a},\tilde{b}} = E_{\tilde{a},\tilde{b}} \setminus \{\operatorname{Re}(w)+c<0\}$ and the domain $E^+_{a,b} = \{z \in E : \operatorname{Re}(z) > 0\}$. We procede in showing that the function

$$\begin{aligned} \varphi\colon E^{cut}_{\tilde{a},\tilde{b}} &\longrightarrow E^+_{a,b} \\ w &\longmapsto c\sqrt{\frac{w+c}{2c}} \end{aligned} \tag{5.3}$$

maps conformally the domain $E^{cut}_{\tilde{a},\tilde{b}}$ onto $E^+_{a,b}$. We recall the function $Q(z) = A|z|^2 - B\operatorname{Re}(z^2)$ parametrizes the ellipse E with the parameters A, B given in (4.4), these two quantities satisfy

$$A^2 - B^2 = \frac{2B}{c^2} \quad\text{and}\quad \frac{Bc^2}{2} + 1 = \frac{(a^2+b^2)^2}{4a^2b^2}\;. \tag{5.4}$$

Thus, we can write for $z = \varphi(w), w = u+iv$

$$\begin{aligned} 1 > A|z|^2 - B\operatorname{Re}(z^2) \;&=\; \frac{cA}{2}|w+c| - \frac{cB}{2}\operatorname{Re}(w+c) \\ &=\; \frac{cA}{2}\sqrt{(u+c)^2+v^2} - \frac{cB}{2}(u+c), \end{aligned} \tag{5.5}$$

which is the defining equation for the new domain. The claim that it is again given by an ellipse, with new parameters $\tilde{a}$ and $\tilde{b}$, can be seen as follows. From (5.5) we have

$$\begin{aligned}
& 0 < \frac{cA}{2}\sqrt{(u+c)^2+v^2} < 1 + \frac{cB}{2}(u+c) \\
\Leftrightarrow \quad & \frac{c^2}{4}(A^2-B^2)u^2 + uc\left(\frac{c^2(A^2-B^2)}{2} - B\right) + \frac{c^2A^2}{4}v^2 < 1 - \frac{c^4(A^2-B^2)}{4} + Bc^2 \\
\Leftrightarrow \quad & \frac{c^2}{(a^2+b^2)^2}u^2 + \frac{c^2}{4a^2b^2}v^2 < 1 \ ,
\end{aligned} \tag{5.6}$$

which is obtained after squaring the inequality, using (5.4) and multiplying with $4a^2b^2/(a^2+b^2)^2$. Clearly φ is analytic in $E_{\tilde{a},\tilde{b}}$, therefore the mapping function φ has the required properties. It remains to show (5.1), to this end, we use the fact that the measure dA_α and the domain E are invariant under reflection, so without restriction, it is enough to consider twice the measure dA_α with support in $E^+_{a,b}$. We note that $\tilde{c}^2 = \tilde{a}^2 - \tilde{b}^2 = a^2 - b^2 = c^2$ follows, and

$$\begin{aligned}
1 - Q(\varphi(w)) &= 1 - \frac{cA}{2}|w+c| + \frac{cB}{2}\,\mathrm{Re}(w+c). \\
&= 1 - J(w) \ ,
\end{aligned} \tag{5.7}$$

In the last equation we have used $cA/2 = \tilde{a}/\tilde{b}^2$ and $cB/2 = \tilde{c}/\tilde{b}^2$. The contribution of the Jacobian $|\varphi'(w)|^2$ gives the extra factor in the denominator of (5.1).

The restriction on the square root cut, which would lead to a *slit domain*, can be removed. To showcase this, we multiply the equation defining $E_{a,b}$

$$\begin{aligned}
& \frac{u^2}{a^2} + \frac{v^2}{b^2} < 1 \\
\Leftrightarrow \quad & a^2\left((u+c)^2 + v^2\right) < (b^2 + c(u+c))^2 \ ,
\end{aligned} \tag{5.8}$$

by a^2b^2, to arrive at the second line. While it is clear that the left hand side is always non negative, we can take the square root here without crossing zero, due to the following fact. It holds that $b^2 + c(u+c) = cu + a^2$ inside the square on the right hand side is always positive for $u \in (-a, +a)$. □

As an important consequence of Proposition 5.1, the function

$$J(z) = \frac{a}{b^2}|c+z| - \frac{c}{b^2}\,\mathrm{Re}\,(c+z), \quad a > b > 0 \tag{5.9}$$

provides an explicit parametrisation of the interior of an ellipse E:

$$E = \{z \in \mathbb{C} : J(z) < 1\} \ . \tag{5.10}$$

For $0 < p < \infty$ and $-1 < \alpha < \infty$, we define $A^p_{\alpha,-\frac{1}{2}} := A^p_{\alpha,-\frac{1}{2}}(E, dB_{\alpha,-\frac{1}{2}}) \subseteq L^p(E, dB_{\alpha,-\frac{1}{2}})$, the two parameters $(\alpha, -1/2)$ weighted Bergman space of the ellipse E, where

$$\mathrm{d}B_{\alpha,-\frac{1}{2}}(z) \ := \ \frac{(1+\alpha)}{2\pi b}\frac{(1-J(z))^{\alpha}}{|c+z|}dA(z). \tag{5.11}$$

For $1 \leq p < \infty$ the standard L^p-norm is definded by

$$||f||_{p,\alpha,-\frac{1}{2}} = \left(\int_E |f(z)|^p \, \mathrm{d}B_{\alpha,-\frac{1}{2}}(z)\right)^{1/p}, \tag{5.12}$$

and for $0 < p < 1$ the corresponding metric is given by

$$d(f,g) \ = \ \int_E |f(z)-g(z)|^p \, \mathrm{d}B_{\alpha,-\frac{1}{2}}(z) \ . \tag{5.13}$$

Moreover, for $0 < p < \infty$ and $-1 < \alpha < \infty$, we define $A^p_{\alpha,\frac{1}{2}} := A^p_{\alpha,\frac{1}{2}}(E, dB_{\alpha,\frac{1}{2}}) \subseteq L^p(E, dB_{\alpha,\frac{1}{2}})$, the two parameters $(\alpha, 1/2)$ weighted Bergman space of the ellipse E, where

$$\mathrm{d}B_{\alpha,\frac{1}{2}}(z) \ := \ \frac{(1+\alpha)(2+\alpha)}{2\pi ab}(1-J(z))^{\alpha}dA(z). \tag{5.14}$$

For $1 \leq p < \infty$ the standard L^p-norm is definded by

$$||f||_{p,\alpha,\frac{1}{2}} = \left(\int_E |f(z)|^p \, \mathrm{d}B_{\alpha,\frac{1}{2}}(z)\right)^{1/p}, \tag{5.15}$$

and for $0 < p < 1$ the corresponding metric is given by

$$d(f,g) \ = \ \int_E |f(z)-g(z)|^p \, \mathrm{d}B_{\alpha,\frac{1}{2}}(z) \ . \tag{5.16}$$

First, for $p \geq 1$ we will show (but now with a short argument) that the Bergman space $A^p_{\alpha,-\frac{1}{2}}$ shares the same properties as A^2_α, i.e the Proposition 4.1 and Corollary 4.2 hold true for $A^p_{\alpha,-\frac{1}{2}}$. This means that Bergman space $A^p_{\alpha,-\frac{1}{2}}$ is a Banach space. Later, the same proof can by carried out for $A^p_{\alpha,\frac{1}{2}}$, with $0 < p < \infty$ and $\alpha > -1$.

Proposition 5.2. *Let $1 \leq p < \infty$ and $-1 < \alpha < \infty$, and K be a compact subset of E, with positive minimum distance to ∂E. Then, there is a positive constant C such that*

$$\sup_K |f(z)|^p \leq C\|f\|^p_{p,\alpha,-\frac{1}{2}} \ ,$$

for all $f \in A^p_{\alpha,-\frac{1}{2}}$.

Proof. We procede as in Proposition 4.1, from (5.8) it follows that $J(z) = 1$ if and only if $z \in \partial E$. It is easy to see that there are no local extrema for $J(z)$ inside the smaller ellipse E_ρ (4.11) , therefore $0 < \max_{z\in E_\rho} j(z) = j(z_*) < 1$ for some $z_* \in E_\rho$.

Thus, given $f \in A^p_{\alpha.-\frac{1}{2}}$, $B(t,\varepsilon) \subseteq E$ with positive minimum distance to the boundary ∂E, with $0 < r < \mathrm{dist}(B(t,\varepsilon), \partial E)$, we can find a positive constant $C > 0$ such that

$$\begin{aligned} |f(z)| &\leq \frac{4}{\pi r^2} \int_{B(z,r/2)} |f(w)|\, dA(w) \\ &\leq C_1 \int_{B(z,r/2)} |f(w)|\, (1 - J(w))^\alpha dA(w) \\ &\leq C_2 \left[\int_E |f(w)|^p \, dB_{\alpha,-\frac{1}{2}}(w) \right]^{\frac{1}{p}} \left[\int_E |w + c|^{\frac{q}{p}} (1 - J)^\alpha dA \right]^{\frac{1}{q}} \\ &= C \|f\|_{p,\alpha,-\frac{1}{2}} \qquad \text{for } z \in B(t,\varepsilon)\ . \end{aligned} \tag{5.17}$$

In the second step we use the fact that $B(z, r/2) \subseteq E_r$ which allows to introduce the weight $(1 - J)^\alpha$ as in proposition 4.1, following by Hölder's inequality with $p, q \geq 1$, $1/p + 1/q = 1$, the second integral in the right hand side of (5.17) is constant and can be absorbed in C. □

Corollary 5.3. *For every $1 \leq p < \infty$, $-1 < \alpha < \infty$, the weighted Bergman space $A^p_{\alpha,-\frac{1}{2}}$ is closed in $L^p(E, dB_{\alpha,-\frac{1}{2}})$.*

Proposition 5.4. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$, and K be a compact subset of E, with positive minimum distance to ∂E. Then, there is a positive constant C such that*

$$\sup_K |f(z)|^p \leq C \|f\|^p_{p,\alpha,\frac{1}{2}}\ ,$$

for all $f \in A^p_{\alpha,\frac{1}{2}}$.

Corollary 5.5. *For every $0 < p < \infty$, $-1 < \alpha < \infty$, the weighted Bergman space $A^p_{\alpha,\frac{1}{2}}$ is closed in $L^p(E, dB_{\alpha,\frac{1}{2}})$.*

5.1. **Holomorphic Jacobi polynomials $P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}$.** For $p \geq 1$ it follows from Corollary 5.3 that the Bergman space is a Banach space, in particular for $p = 2$ a Hilbert space with the inner product defined as

$$\langle f, g \rangle_{\alpha,-\frac{1}{2}} := \int_E f(z) \overline{g(z)}\, dB_{\alpha,-\frac{1}{2}}(z)\ , \quad f, g \in A^2_{\alpha,-\frac{1}{2}}. \tag{5.18}$$

For any non-negative integer n and real parameter $\alpha > -1$ let us define the polynomials

$$J_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}(z) \ := \ \frac{1}{\sqrt{h_n}} P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})} \left(\frac{z}{c} \right)\ , \tag{5.19}$$

where $P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}(x)$ are the standard Jacobi polynomials (2.46), now taken with a complex argument. We recall that the ellipse E in (5.10) defining the inner product (5.18) is parametrised by the real numbers $a > b > 0$. The constant

$$c := \sqrt{a^2 - b^2} > 0 \tag{5.20}$$

provides the location of the right focus of the ellipse E, and we define by

$$h_n \;:=\; h_n(a,b) = \frac{1+\alpha}{1+\alpha+2n}\frac{(1/2)_n}{(1+\alpha)_n}P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}\left(\frac{a^2+b^2}{a^2-b^2}\right)\;, \tag{5.21}$$

Theorem 5.6. *The sequence of polynomials* $\{J_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}(z)\}_{n\in\mathbb{N}}$ *defined in* (5.29) *forms a orthonormal basis for* $A^2_{\alpha,-\frac{1}{2}}$ *for any* $\alpha > -1$.

We show that Jacobi polynomials $P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}(z)$ satisfy the orthogonality condition with norm (5.21), the completeness of the sequence of Jacobi polynomials follows by the same line of argument in Theorem 4.5.

Lemma 5.7. *Let E be the elliptic domain* (5.10)*, $dB_{\alpha,-\frac{1}{2}}(z)$ the density* (5.11) *over E and $\alpha > -1$. Then sequence of Jacobi polynomials $\{P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}(z)\}_{n\in\mathbb{N}}$ satisfy the following orthogonality relations*

$$\int_E P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}(w/c)P_m^{(\alpha+\frac{1}{2},-\frac{1}{2})}(\overline{w}/c)dB_{\alpha,-\frac{1}{2}}(w) = \frac{\delta_{n,m}(1+\alpha)}{1+\alpha+2n}\frac{(1/2)_n}{(1+\alpha)_n}P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}\left(\frac{a^2+b^2}{a^2-b^2}\right). \tag{5.22}$$

where $a > b > 0$ and $c = \sqrt{a^2-b^2}$.

Proof. Using the quadratic transformation (2.57), we have for all even Gegenbauer polynomials

$$C_{2n}^{(\alpha+1)}(z/c) = \frac{(\alpha+1)_n}{\left(\frac{1}{2}\right)_n}P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}\left(2\left(\frac{z}{c}\right)^2-1\right)\;, \tag{5.23}$$

By Proposition 5.1, with $z = \varphi(w)$, we have

$$\begin{aligned}
\langle C_{2n}^{\alpha+1}, C_{2m}^{\alpha+1}\rangle_\alpha &= \int_E C_{2n}^{(\alpha+1)}(z/c)C_{2l}^{(\alpha+1)}(\bar z/c)dA_\alpha(z)\\
&= 2\int_{E^+_{a,b}} C_{2n}^{(\alpha+1)}(z/c)C_{2l}^{(\alpha+1)}(\bar z/c)dA_\alpha(z)\\
&= \frac{(\alpha+1)_n^2}{\left(\frac{1}{2}\right)_n^2}\int_{E_{\tilde a,\tilde b}} P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}(w/\tilde c)P_m^{(\alpha+\frac{1}{2},-\frac{1}{2})}(\overline{w}/\tilde c)\,dB_{\alpha,-\frac{1}{2}}(w),
\end{aligned} \tag{5.24}$$

with

$$J(w) = \frac{\tilde a}{\tilde b^2}|\tilde c + w| - \frac{\tilde c}{\tilde b^2}\,\mathrm{Re}\,(\tilde c + w). \tag{5.25}$$

Now, we use the norm of even Gegenbauer polynomials, Lemma 4.6, is given by

$$\langle C_{2n}^{\alpha+1}, C_{2m}^{\alpha+1}\rangle_\alpha = \frac{1+\alpha}{1+\alpha+2n}C_{2n}^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right)\delta_{nm} \tag{5.26}$$

the quantity $(a^2+b^2)/(a^2-b^2) = \tilde{a}/\tilde{c}$, and using again the quadratic transformation (5.23), we have

$$C_{2n}^{(\alpha+1)}(\tilde{a}/\tilde{c}) = \frac{(\alpha+1)_n}{\left(\frac{1}{2}\right)_n} P_n^{(\alpha+\frac{1}{2},-\frac{1}{2})}\left(\frac{\tilde{a}^2+\tilde{b}^2}{\tilde{a}^2-\tilde{b}^2}\right) , \tag{5.27}$$

Multiplying (5.24) with $(1/2)_n^2/(\alpha+1)_n^2$ and dropping the tilde on all quantities we arrive at the statement in (5.22). □

5.2. **Holomorphic Jacobi polynomials $P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}$.** For $p \geq 1$ it follows from Corollary 5.5 the Bergman space is a Banach space, in particular for $p=2$ a Hilbert space with the inner product defined as

$$\langle f,g\rangle_{\alpha,\frac{1}{2}} := \int_E f(z)\overline{g(z)}\, dB_{\alpha,\frac{1}{2}}(z) , \quad f,g \in A^2_{\alpha,\frac{1}{2}}. \tag{5.28}$$

For any non-negative integer n and real parameter $\alpha > -1$ let us define the polynomials

$$J_n^{(\alpha+\frac{1}{2},\frac{1}{2})}(z) \;:=\; \frac{1}{\sqrt{h_n}} P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}\left(\frac{z}{c}\right) , \tag{5.29}$$

where $P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}(z)$ are the standard Jacobi polynomials (2.46), now taken with a complex argument. We recall that the ellipse E in (5.10) defining the inner product (5.28) is parametrised by the real numbers $a > b > 0$. The constant

$$c := \sqrt{a^2-b^2} > 0, \tag{5.30}$$

provides the location of the right focus of the ellipse E, and we define by

$$h_n \;:=\; h_n(a,b) = \frac{2+\alpha}{2+\alpha+2n}\frac{(3/2)_n}{(2+\alpha)_n} P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}\left(\frac{a^2+b^2}{a^2-b^2}\right) . \tag{5.31}$$

Theorem 5.8. *The sequence of polynomials $\{J_n^{(\alpha+\frac{1}{2},\frac{1}{2})}(z)\}_{n\in\mathbb{N}}$ defined in* (5.29) *forms a orthonormal basis for $A^2_{\alpha,\frac{1}{2}}$ for any $\alpha > -1$.*

The proof of this Theorem is analogous to the theorem 5.6, we will only establish the result of orthogonality with a short comment after this

Lemma 5.9. *Let E be the elliptic domain* (5.10)*, $dB_{\alpha,\frac{1}{2}}(z)$ the density* (5.14) *over E and $\alpha > -1$. Then sequence of Jacobi polynomials $\{P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}(z)\}_{n\in\mathbb{N}}$ satisfy the following orthogonality relations*

$$\int_E P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}(w/c) P_m^{(\alpha+\frac{1}{2},\frac{1}{2})}(\overline{w}/c) dB_{\alpha,\frac{1}{2}}(w) = \frac{\delta_{n,m}(2+\alpha)}{2+\alpha+2n}\frac{(3/2)_n}{(2+\alpha)_n} P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}\left(\frac{a^2+b^2}{a^2-b^2}\right), \tag{5.32}$$

where $a > b > 0$ and $c = \sqrt{a^2-b^2}$.

Proof. We use the quadratic transformation (2.58)

$$C_{2n+1}^{(\alpha+1)}(z/c) \;=\; \frac{(\alpha+1)_{n+1}}{\left(\frac{1}{2}\right)_{n+1}}\frac{z}{c}P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}(2(z/c)^2-1)\;. \tag{5.33}$$

together with Proposition 5.1. Apart from the additional constant factors, we obtain from (5.33) an additional factor

$$\left|\frac{z}{c}\right|^2 = \frac{|w+c|}{2c} \tag{5.34}$$

which cancels the pole from the Jacobian $|\varphi'(z)|^2$. □

5.3. **Orthogonality of the Chebyshev polynomials.** In this section we will prove the orthogonality of the Chebyshev polynomials of first to fourth kind as a direct consequence of Theorems 5.6 and 5.8. The following statement is due to [46], where the notation for the polynomials of third and fourth kind is interchanged compared to ours, $V_n \leftrightarrow W_n$.

Corollary 5.10. *The Chebyshev polynomials satisfy the following orthogonality relations on the ellipse defined in* (4.6)*, with $r = a+b$ and $c^2 = a^2 - b^2$:*

$$\int_E T_n(z/c)T_m(\overline{z}/c)\,\frac{\mathrm{d}^2 z}{|z^2-c^2|} \;=\; \begin{cases} \dfrac{\pi}{4n}((r/c)^{2n}-(c/r)^{2n})\delta_{n,m} & \text{for } n>0, m\geq 0,\\[2ex] 2\pi\ln(r/c) & \text{for } n=m=0\;, \end{cases} \tag{5.35}$$

$$\int_E U_n(z/c)U_m(\overline{z}/c)\,\mathrm{d}^2 z \;=\; \frac{\pi c^2}{4(1+n)}((r/c)^{2n+2}-(c/r)^{2n+2})\delta_{n,m}\;, \tag{5.36}$$

$$\int_E V_n(z/c)V_m(\overline{z}/c)\,\frac{\mathrm{d}^2 z}{|c+z|} \;=\; \frac{\pi c}{1+2n}((r/c)^{2n+1}-(c/r)^{2n+1})\delta_{n,m}\;, \tag{5.37}$$

$$\int_E W_n(z/c)W_m(\overline{z}/c)\,\frac{\mathrm{d}^2 z}{|c-z|} \;=\; \frac{\pi c}{1+2n}((r/c)^{2n+1}-(c/r)^{2n+1})\delta_{n,m}\;. \tag{5.38}$$

Note that for better comparison with [46][2] our statements are with respect to the flat measure $\mathrm{d}^2 z = d\,\mathrm{Re}(z)d\,\mathrm{Im}(z)$. Because the proof for Chebyshev polynomials U_n, V_n and W_n is similar, we show only one case and leave the others for the reader.

Proof. The Chebyshev Polynomials of the third kind V_n are related to Jacobi polynomials as in (2.54):

$$V_n(z/c) = \frac{1+2n}{P_n^{(1/2,-1/2)}(1)}P_n^{(1/2,-1/2)}(z/c)\;. \tag{5.39}$$

[2]In contrast to the orthogonality of the Chebyshev polynomials on the contour given by the boundary of the ellipse ∂E stated in [46] too, the weight function we find here differs from the classical weight on the real line, continued to the ellipse.

Setting $\alpha = 0$ in (5.22) we obtain

$$\int_E V_n(w/c)V_m(\overline{w}/c)\,\frac{\mathrm{d}^2 w}{|c+w|} = \frac{2\pi b(1)_n}{(1+2n)(1/2)_n} P_n^{(\frac{1}{2},\frac{1}{2})}\left(2\left(\frac{a}{c}\right)^2 - 1\right)\delta_{n,m} = \frac{2\pi b}{1+2n} C_{2n}^{(1)}\left(\frac{a}{c}\right)\delta_{n,m} \tag{5.40}$$

Here, we have included the pre-factors in (5.39). Using (4.45) for an even index, we have

$$C_{2n}^{(1)}(\cos(i\ln(r/c))) = \sum_{k=0}^{2n}\cos((2n-2k)i\ln(r/c)) = \sum_{k=0}^{2n}\left(\frac{r}{c}\right)^{2n-k}\left(\frac{c}{r}\right)^{k} = \frac{c}{2b}((r/c)^{2n+1} - (c/r)^{2n+1})\ . \tag{5.41}$$

Recalling $r = a + b$ and $c^2 = a^2 - b^2$, we have

$$\frac{1}{2}\left(\frac{r}{c} + \frac{c}{r}\right) = \frac{a}{c} \quad \text{and} \quad \frac{1}{2}\left(\frac{r}{c} - \frac{c}{r}\right) = \frac{b}{c}\ , \tag{5.42}$$

which upon replacing $C_{2n}^{(1)}(a/c)$ in (5.40) leads to the statement (5.37).

We turn to the orthogonality for the Chebyshev polynomials of the first kind T_n. The relation [69, 18.7.18]

$$T_{2n+1}(x) = xW_n(2x^2 - 1) \tag{5.43}$$

allows us to find the corresponding weight function and orthogonality of the odd polynomials, starting from (5.38):

$$\int_{E_{\tilde{a},\tilde{b}}} W_n(z'/c)W_m(\overline{z'}/c)\,\frac{\mathrm{d}^2 z'}{|z'-c|} = 8c\int_{E^+_{a,b}} \frac{z}{c}W_n(2(z/c)^2 - 1)\frac{\overline{z}}{c}W_m(2(\overline{z}/c)^2 - 1)\frac{\mathrm{d}^2 z}{|z^2 - c^2|} = 4c\int_E T_{2n+1}(z/c)T_{2m+1}(\overline{z}/c)\,\frac{\mathrm{d}^2 z}{|z^2 - c^2|}\ . \tag{5.44}$$

Here, we use the inverse transformation in Proposition 5.1. Thus the polynomials $\{T_n\}$ are orthogonal w.r.t. $\frac{1}{|z^2-c^2|}\mathrm{d}^2 z$. The following well known relation [46] holds for the Joukowsky map $z/c = \frac{1}{2}(w/c + c/w)$:

$$T_n(z/c) = \frac{1}{2}((w/c)^n + (c/w)^n) \quad \text{for} \quad n \geq 0, \tag{5.45}$$

which maps the ellipse E to the annulus $A := \{w \in \mathbb{C} : c < |w| < r\}$. Thus we obtain for $n > 0, m \geq 0$

$$\begin{aligned}
&\int_E T_n(z/c) T_m(\overline{z}/c) \, \frac{\mathrm{d}^2 z}{|z^2 - c^2|} \\
= & \int_A T_n(z(w)/c) T_m(\overline{z(w)}/c) \frac{\mathrm{d}^2 w}{|w|^2} \\
= & \frac{1}{4} \int_c^r \frac{\mathrm{d}s}{s} \int_0^{2\pi} \mathrm{d}\theta \left((s/c)^n e^{in\theta} - (c/s)^n e^{-in\theta} \right) \left((s/c)^m e^{-im\theta} - (c/s)^m e^{im\theta} \right) \\
= & \frac{\pi}{4n} ((r/c)^{2n} - (c/r)^{2n}) \delta_{n,m} \\
= & \frac{\pi b}{2nc} C_{2n-1}^{(1)} \left(\frac{a}{c} \right) \delta_{n,m} \ ,
\end{aligned} \tag{5.46}$$

by changing to polar coordinates $w = s\, e^{i\theta}$. Performing the elementary integrations we need to assume that $n > 0$ and $m \geq 0$. Then, the first part of (5.35) follows while in the last step we have used (4.45) in order to compare it to the previous orthogonality relations. Following the same computation for $n = m = 0$, we have with $T_0(x) = 1$,

$$\int_E \frac{1}{|z^2 - c^2|} \mathrm{d}^2 z \ = \ 2\pi \ln(r/c) \ , \tag{5.47}$$

which ends the proof of Corollary 5.10.

□

5.4. **Bergman polynomials and finite-term recurrence.** All the orthogonal polynomials on an ellipse we encountered in the previous section satisfy a three-step recursion relation, as they result from classical polynomials on the real line. It was shown by Khavinson and Stylianopoulos in [44], if the planar orthogonal polynomials on a domain E with regular enough boundary satisfy a finite recurrence relation, then the size of the recursion is three and the domain is an ellipse. This result was demonstrated for the unweighted case, also known as Carleman's polynomials, as summarised in Theorem 5.12 below. Using the Gegenbauer polynomials from Lemma 4.6 that are orthogonal on a weighted ellipse, we will construct an example of a family of orthogonal polynomials with respect to a non-flat weight function, that has no finite-term recursion on such ellipse. Therefore *the elliptic domain is not special and weight functions which leads to finite-term recursion are exceptional.* In this sense, it remains an open question to characterise the positive Borel measures supported on an ellipse, such that the associated planar orthogonal polynomials do satisfy a three-terms recurrence relation.

Let E be a bounded domain in the complex plane, let $\mathrm{d}\mu(z) = w(z)\mathrm{d}A(z)$ be a measure on E, where $\mathrm{d}A$ is the planar Lebesgue measure, and w a non-negative weight function on E. By $p_n(z) = \gamma_n z^n + \ldots$, $\gamma_n > 0$, we denoted the associated orthonormal Bergman polynomials. The multiplication operator acting on polynomials can always be represented by expanding $z\,p_n(z)$ as a series of the Bergman polynomials being a basis:

$$z\,p_n(z) \;=\; \sum_{l=0}^{n+1} c_{l,n}\,p_l(z)\;,n=0,1,2,\dots \tag{5.48}$$

The Fourier coefficients $c_{l,n}$ are then given by

$$c_{l,n} \;=\; \int_E z\,p_n(z)\overline{p_l(z)}w(z)\,\mathrm{d}A(z)\;. \tag{5.49}$$

These coefficients $c_{l,n}$ constitute the entries of an infinite upper Hessenberg matrix

$$M=\begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} & \cdots \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} & \cdots \\ 0 & c_{2,1} & c_{2,2} & c_{2,3} & \cdots \\ 0 & 0 & c_{3,2} & c_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

This matrix provides a representation of the Bergman Shift operator, which is linear and defined by $(T_z\,f)(z)=zf(z)$ with respect to the basis $\{p_n\}_{n\in\mathbb{N}}$.

Definition 5.11 (see [44])**.** We say that the upper Hessenberg matrix is *banded* or, equivalently, that the orthogonal polynomials p_n satisfy a *finite* $(d+1)$*-term recurrence* if there exists a positive integer d such that

$$c_{l,n}=0,\ \text{for}\ 0\le l<n+1-d\;. \tag{5.50}$$

In [44] Khavinson and Stylianopoulos proved the following

Theorem 5.12. *If the Bergman polynomials orthogonal with respect to the flat measure, on a bounded simply-connected domain D with regular enough boundary, satisfy a $(d+1)$-term recurrence relation with $2\le d$, then D is an ellipse and $d=2$.*

All polynomials in the previous sections satisfy three term recurrence relations, as they come form the real line, then a natural question is if the above Theorem 5.12 extends to the weighted case, or at least if the weighted ellipse is special. The answer is negative and is given in the following proposition

Proposition 5.13. *Let $v\in\mathbb{C}$, $a>b>0$, E be the ellipse with parameters a,b, dA_α the density over E defined in (4.7) and*

$$d\mu_\alpha(z)=:|v-z|^2dA_\alpha(z). \tag{5.51}$$

Define the polynomial $P_n^{(1)}(z)$ by

$$P_N^{(1)}(z)=\frac{\kappa_{N+1}(z,\bar v)p_{N+1}(v)-\kappa_{N+1}(v,\bar v)p_{N+1}(z)}{(v-z)\sqrt{\kappa_{N+1}(v,\bar v)\kappa_{N+2}(v,\bar v)}}\;, \tag{5.52}$$

where $\kappa_N(z,w)$ is the polynomial kernel constituted by the orthonormal Gegenbauer polynomials $p_n(z)=C_n^{\alpha+1}(z/c)/\sqrt{h_n}$.

Then, the sequence of polynomials $\left\{P_n^{(1)}(z)\right\}_{n\in\mathbb{N}}$

(1) form a orthonormal system in $L^2(E, d\mu_\alpha)$.

(2) the sequence of polynomials $\{P_n^{(1)}\}$ *do not satisfy any finite term recurrence relations in the sense of* (5.50).

Proof. We use Theorem 3.3 for $K = 2$ and $L = 1$. Following the Heine formula (3.9), the polynomials $\{P_n^{(1)}\}_{n\in\mathbb{N}}$ orthogonal w.r.t. $|v-z|^2 dA_\alpha(z)$ can be expressed in terms of the polynomials p_n orthogonal with respect to $dA_\alpha(z)$ (the Gegenbauer polynomials in this case). They are reading in monic normalisation

$$\begin{aligned} \tilde{P}_N^{(1)}(z) &= \left\langle \prod_{i=1}^{N}(z-z_i) \right\rangle_{N,d\mu_\alpha} = \frac{\left\langle \prod_{i=1}^{N}(z-z_i)|v-z_i|^2 \right\rangle_{N,dA_\alpha}}{\left\langle \prod_{i=1}^{N}|v-z_i|^2 \right\rangle_{N,dA_\alpha}} \\ &= h_{N+1}^{\frac{1}{2}} \frac{\kappa_{N+1}(z,\bar{v})p_{N+1}(v) - \kappa_{N+1}(v,\bar{v})p_{N+1}(z)}{(v-z)\kappa_{N+1}(v,\bar{v})} \ . \end{aligned} \tag{5.53}$$

Their respective squared norms $\tilde{h}_N^{(1)}$ are not difficult to compute, using the orthonormality of the underlying polynomials $\tilde{p}_n$ (3.10):

$$\begin{aligned} \tilde{h}_N^{(1)} &= \int \tilde{P}_N^{(1)}(z)\overline{\tilde{P}_N^{(1)}(z)}\ |v-z|^2 \mathrm{d}A_\alpha(z) \\ &= \frac{h_{N+1}}{\kappa_{N+1}(v,\bar{v})}\left(\kappa_{N+1}(v,\bar{v})|\tilde{P}_N^{(1)}(v)|^2 + \kappa_{N+1}(v,\bar{v})^2\right) \\ &= \frac{h_{N+1}\kappa_{N+2}(v,\bar{v})}{\kappa_{N+1}(v,\bar{v})} \ . \end{aligned} \tag{5.54}$$

This leads to the orthonormal polynomials (5.52) and complete the first part of the proposition.

For the second part, we will to show that the Fourier coefficients $c_{l,n}$ of

$$zP_n^{(1)}(z) = \sum_{l=0}^{n+1} c_{l,n}\, P_l^{(1)}(z) \tag{5.55}$$

are (in general) non-zero for $l \leq n-2$.

Here, we may use that the orthonormalised Gegenbauer polynomials (4.18) in the complex plane also satisfy a *three-term recurrence* relation (2.61), reading

$$z\, p_n^{(\alpha)}(z) = a_{n+1}p_{n+1}^{(\alpha)}(z) + b_n p_{n-1}^{(\alpha)}(z) \ , \tag{5.56}$$

with

$$a_{n+1} = \frac{c(n+1)}{2(n+\alpha+1)}\sqrt{\frac{h_{n+1}}{h_n}} \ , \quad b_n = \frac{c(n+2\alpha+1)}{2(n+\alpha+1)}\sqrt{\frac{h_{n-1}}{h_n}} \ . \tag{5.57}$$

Here, we use the definition from (4.27) for the squared norms h_n of the Gegenbauer polynomials. Notice that in contrast to the recursion for orthonormal Gegenbauer polynomials on the real line, the recurrence (5.56) is not symmetric, $a_n \neq b_n$. This is due to the difference in norm for $[-1,1]$ and E. From now on we will use the following notation for $\kappa_{i+1}(v,\bar{v}) := \kappa_{i+1}$. A simple calculation implies that the coefficients

$$c_{l,n} \;=\; \int_E z\, P_n^{(1)}(z)\overline{P_l^{(1)}(z)}\;|v-z|^2 \mathrm{d}A_\alpha(z) \tag{5.58}$$

are given by

$$\begin{aligned} c_{l,n} =& \frac{1}{\sqrt{\kappa_{n+1}\kappa_{n+2}\kappa_{l+1}\kappa_{l+2}}} \Bigg[\left(\sum_{k=1}^{l} a_k p_k^{(\alpha)}(v) p_{k-1}^{(\alpha)}(\bar{v}) - \sum_{k=0}^{min\{l,n-1\}} b_{k+1} p_k^{(\alpha)}(v) p_{k+1}^{(\alpha)}(\bar{v}) \right) \\ & \times p_{l+1}^{(\alpha)}(\bar{v}) p_{n+1}^{(\alpha)}(v) - \Big(a_{l+1} p_l^{(\alpha)}(\bar{v}) \Theta(n-l) - b_{l+2} p_{l+2}^{(\alpha)}(\bar{v}) \Theta(n-l-2) \Big) \kappa_{l+1} p_{n+1}^{(\alpha)}(v) \\ & + \kappa_{n+1}\kappa_{n+2} a_{n+2} \delta_{n+1,l} - \kappa_{n+1} b_{n+1} p_n^{(\alpha)}(v) p_{l+1}^{(\alpha)}(\bar{v}) \Theta(l-n) + \kappa_{n+1}\kappa_n a_{n+1} \delta_{n-1,l} \Bigg], \end{aligned} \tag{5.59}$$

where we have used the recursion (5.56) and introduced the step function

$$\Theta(x) := \begin{cases} 1 & \text{for } x \geq 0 \;, \\ 0 & \text{for } x < 0 \;. \end{cases} \tag{5.60}$$

If we only restrict ourselves to those indices $l \leq n-2$ which spoil the three-step recurrence, the remaining terms are simplified considerably and we obtain

$$\begin{aligned} c_{l,n} \;=\; & \frac{p_{n+1}^{(\alpha)}(v)}{\sqrt{\kappa_{n+1}\kappa_{n+2}\kappa_{l+1}\kappa_{l+2}}} \Bigg[\Big(v\kappa_l + b_l p_{l-1}^{(\alpha)}(v) p_l^{(\alpha)}(\bar{v}) + b_{l+1} p_l^{(\alpha)}(v) p_{l+1}^{(\alpha)}(\bar{v}) \Big)\, p_{l+1}^{(\alpha)}(\bar{v}) \\ & - \Big(a_{l+1} p_l^{(\alpha)}(\bar{v}) + b_{l+2} p_{l+2}^{(\alpha)}(\bar{v}) \Big)\, \kappa_{l+1} \Bigg] . \end{aligned} \tag{5.61}$$

Let us first check that we recover the three-term recurrence in the real limit $b \to 0$, where we have to show that indeed $c_{l,n} = 0$ in this limit for $l \leq n-2$. When $b = 0$ and the corresponding normalisation constants are understood as $h_n = h_n(a,0)$, the recursion coefficients (5.57) become symmetric, $a_n = b_n$, as it is known for Gegenbauer polynomials on $[-1,1]$ [69], cf. Remark 4.7. We thus obtain for the bracket in (5.61) at $b = 0$

$$\begin{aligned} & p_{l+1}^{(\alpha)}(\bar{v}) \Big(v\kappa_l(v,\bar{v}) + b_l p_{l-1}^{(\alpha)}(v) p_l^{(\alpha)}(\bar{v}) + b_{l+1} p_l^{(\alpha)}(v) p_{l+1}^{(\alpha)}(\bar{v}) - \bar{v}\kappa_{l+1}(v,\bar{v}) \Big) \\ & = p_{l+1}^{(\alpha)}(\bar{v}) \left(\sum_{i=0}^{l-1} b_{i+1} p_{i+1}^{(\alpha)}(v) p_i^{(\alpha)}(\bar{v}) + \sum_{i=0}^{l+1} b_i p_{i-1}^{(\alpha)}(v) p_i^{(\alpha)}(\bar{v}) \right. \\ & \qquad \left. - \sum_{i=0}^{l} p_i^{(\alpha)}(v) \Big(b_{i+1} p_{i+1}^{(\alpha)}(\bar{v}) + b_i p_{i-1}^{(\alpha)}(\bar{v}) \Big) \right) \\ & = 0 \;. \end{aligned} \tag{5.62}$$

Here, we have used the notation $p_{-1}^{(\alpha)} = 0$, and after relabelling the sums they cancel.

To see that the expression (5.61) is non vanishing in general for $b > 0$, we consider the leading coefficient of (5.61) as a polynomial in $\bar{v}$, which is of degree $2l + 2$. We thus have to focus on

$$\left(b_{l+1} p_{l+1}^{(\alpha)}(\bar{v})^2 - b_{l+2} p_l^{(\alpha)}(\bar{v}) p_{l+2}^{(\alpha)}(\bar{v})\right) p_l^{(\alpha)}(v) \ . \tag{5.63}$$

Because the polynomials of degree l and $l+1$ do not have common zeros, it is sufficient to consider the leading coefficients inside the bracket, which read

$$\begin{aligned} &\frac{c(l+3+2\alpha)}{2(l+2+\alpha)} \sqrt{\frac{h_l}{h_{l+1}}} \frac{1}{h_{l+1}} \left(\frac{2^{l+1}\Gamma(2+l+\alpha)}{\Gamma(1+\alpha)(l+1)! c^{l+1}} \right)^2 \\ &- \frac{c(l+3+2\alpha)}{2(l+3+\alpha)} \frac{\sqrt{h_{l+1}}}{h_{l+2}} \frac{1}{\sqrt{h_l}} \frac{2^{2l+2}\Gamma(1+\alpha+l)\Gamma(3+\alpha+l)}{\Gamma(1+\alpha)^2 l! (l+2)! c^{2l+2}} \ , \end{aligned} \tag{5.64}$$

upon using (5.57). Inserting (4.27) and recalling (4.48)

$$C_l^{(1+\alpha)}(1) = \frac{\Gamma(2+2\alpha+l)}{\Gamma(2+2\alpha) l!} \ , \tag{5.65}$$

it can be shown that (5.64) vanishes only if the following equality holds:

$$\frac{\left(C_{l+1}^{(1+\alpha)}(x)\right)^2}{\left(C_{l+1}^{(1+\alpha)}(1)\right)^2} - \frac{C_l^{(1+\alpha)}(x)\, C_{l+2}^{(1+\alpha)}(x)}{C_l^{(1+\alpha)}(1)\, C_{l+2}^{(1+\alpha)}(1)} = 0 \ , \tag{5.66}$$

where

$$x = \frac{a^2 + b^2}{a^2 - b^2} \ . \tag{5.67}$$

The expression on the left hand side of (5.66), usually denoted by $\Delta_n(x)$, is know as Turán determinant. By [70, Theorem 1] $\Delta_n(x) = 0$ if and only if $x = \pm 1$. Thus $c_{l,n} \equiv 0$ for $0 \le l \le n-2$ in the limit $b \to 0$, that is when $x \to 1$, which brings us back to the real line with a three-step recursion. For $x > 1$ all Fourier coefficients are non-vanishing, $c_{l,n} \neq 0$ for $0 \le l \le n-2$, in our example of polynomials (5.52) and thus there exists no *finite-term recurrence* relation.

□

5.5. **Conjecture on general Jacobi polynomials $P_n^{\alpha,\beta}$.** Let $a > b > 0$, $r = a + b$, $c^2 = a^2 - b^2$, by E we will denote the ellipse with parameters a, b. Jacobi polynomials are given by the generating function (5.68), see [69, 18.12.3], valid for $|t| < 1$, in particular for $|t| < c/r$

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\beta)_n} \frac{1+\alpha+\beta+2n}{1+\alpha+\beta} P_n^{(\alpha,\beta)}(x) t^n \\ = \ &\frac{1-t}{(1+t)^{\alpha+\beta+2}} F\left(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}, \beta+1, \frac{2t(x+1)}{(1+t)^2}\right), \ |t| < \frac{c}{r} < 1. \end{aligned} \tag{5.68}$$

Then, after substitution $\alpha \to \alpha + 1/2$, $t \to c/w$ and $x \to z/c$, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(3/2+\alpha+\beta)_n}{(1+\beta)_n} \frac{3/2+\alpha+\beta+2n}{3/2+\alpha+\beta} P_n^{(\alpha+\frac{1}{2},\beta)}\left(\frac{z}{c}\right) \frac{c^n}{w^n} \\ = & \frac{1-c/w}{(1+c/w)^{\alpha+\beta+5/2}} F\left(\frac{\alpha+\beta+5/2}{2}, \frac{\alpha+\beta+7/2}{2}, \beta+1, \frac{2(z+c)}{w(1+c/w)^2}\right), |w|>r. \end{aligned} \tag{5.69}$$

So, we can choose $R > 0$ big enough such that the Laurent series (5.69) converges uniformly for $|w| > R \geq r$ and for all $z \in E$. According to the previous introduced notation, we defined

$$dB_{\alpha,\beta}(z) = \begin{cases} dA_\alpha(z) & \text{for } \beta = \alpha + \frac{1}{2}\,, \\ dB_{\alpha,\pm\frac{1}{2}}(z) & \text{for } \beta = \pm\frac{1}{2}. \end{cases} \tag{5.70}$$

Let

$$f_{\alpha,\beta}(z,w) := \sum_{n=0}^{\infty} \frac{(3/2+\alpha+\beta)_n}{(1+\beta)_n} \frac{3/2+\alpha+\beta+2n}{3/2+\alpha+\beta} P_n^{(\alpha+\frac{1}{2},\beta)}\left(\frac{z}{c}\right) \frac{c^n}{w^n}. \tag{5.71}$$

Our orthogonality relations (4.56), (5.22) and (5.32) give

$$\begin{aligned} & \int_E f_{\alpha,\beta}(z,w)\overline{f_{\alpha,\beta}(z,s)}dB_{\alpha,\beta}(z) \\ = & \sum_{n=0}^{\infty} \frac{(3/2+\alpha+\beta)_n}{(1+\beta)_n} \frac{3/2+\alpha+\beta+2n}{3/2+\alpha+\beta} P_n^{(\alpha+\frac{1}{2},\beta)}\left(\frac{a^2+b^2}{a^2-b^2}\right) \frac{c^{2n}}{(w\overline{s})^n} \\ = & \frac{1-c^2/w\overline{s}}{(1+c^2/w\overline{s})^{\alpha+\beta+5/2}} F\left(\frac{\alpha+\beta+5/2}{2}, \frac{\alpha+\beta+7/2}{2}, \beta+1, \frac{2(a^2+b^2+c^2)}{w\overline{s}(1+c^2/w\overline{s})^2}\right). \end{aligned} \tag{5.72}$$

Taking into account the second line in (5.72)

We conjecture: for a certain probability density $dB_{\alpha,\beta}$ over the ellipse E, being (5.70) *its particular cases, the Jabobi polynomials with $\alpha, \beta > -1$ satisfy the orthogonality relations*

$$\langle P_n^{\alpha+\frac{1}{2}}, P_m^{\beta+\frac{1}{2}} \rangle_{\alpha,\beta} = \frac{(3/2+\beta)_n}{(2+\alpha+\beta)_n} \frac{2+\alpha+\beta}{2+\alpha+\beta+2n} P_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}\left(\frac{a^2+b^2}{a^2-b^2}\right) \delta_{n,m}. \tag{5.73}$$

6. Static 2D Coulomb gases on an ellipse

This chapter introduces two particular two-dimensional, static one-component Coulomb gases that we will solve. We will first deal with the case induced by the weight function for Gegenbauer polynomials in the complex plane and at the very end of this section we will treat the case induced by the weight function for non-symmetric Jacobi polynomials.

We recall, with $a > b > 0$ the ellipse $E \subset \mathbb{C}$ is parametrized by

$$Q(z) = A|z|^2 - B\,\mathrm{Re}(z^2), \quad A = \frac{a^2+b^2}{2a^2b^2},\ B = \frac{a^2-b^2}{2a^2b^2}. \tag{6.1}$$

From now on, we specialize the ellipse to one parameter $0 < \tau < 1$ by choosing

$$a = \sqrt{\frac{1+\tau}{2\tau}}, b = \sqrt{\frac{1-\tau}{2\tau}}, \quad Q(z) = \frac{2\tau}{1-\tau^2}|z|^2 - \frac{2\tau^2}{1-\tau^2}\,\mathrm{Re}(z^2) \tag{6.2}$$

$$E = \left\{ z \,\middle|\, \frac{2\tau}{1-\tau^2}|z|^2 - \frac{2\tau^2}{1-\tau^2}\,\mathrm{Re}(z^2) < 1 \right\}, \quad 0 < \tau < 1. \tag{6.3}$$

We consider a two-dimensional, static one-component Coulomb gas with a Hamiltonian

$$H = \sum_{j=1}^{N} V(z_j) - \sum_{j<l}^{N} \log|z_j - z_l| \ . \tag{6.4}$$

For the particles interacting logarithmically in the plane we impose a *hard wall* constrain, where we completely confine the system $\{z_i\}_{i=1}^N$ to the ellipse by setting $V = \infty$ outside of E. The one-particle potential in the Hamiltonian (6.4) is given by

$$V(z) = -\frac{\alpha}{2}\log\left(1 - Q(z)\right)\cdot \mathbb{1}_E(z) + \infty \cdot \mathbb{1}_{\mathbb{C}\setminus E}(z)\ , \quad \alpha > -1\ . \tag{6.5}$$

Note that, this potential mimics a charged mirror at the boundary of the ellipse which is either attractive ($\alpha < 0$) or repulsive ($\alpha > 0$). In this section, and unless we state the contrary, $\alpha > -1$ will be arbitrary but fixed, $\alpha = O(1)$.

The resulting Boltzman density function for the particles to be at equilibrium at an inverse temperature $1/(k_BT) = \beta = 2$ is known to be

$$P(z_1, z_2, \cdots, z_N) = \frac{1}{Z_N}e^{-\beta H} = \frac{1}{Z_N}\prod_{j=1}^{N} w(z_j)\prod_{j<l}^{N}|z_j - z_l|^2\ . \tag{6.6}$$

Here, the one-particle weight function is

$$w(z) = (1 - Q(z))^{\alpha}\cdot \mathbb{1}_E(z) = e^{-\beta V(z)}, \tag{6.7}$$

The partition function that normalises the distribution (6.6) is defined as

$$Z_N = \prod_{j=1}^{N}\int_E d^2z_j\ w(z_j)\prod_{i<l}^{N}|z_i - z_l|^2,\ \ d^2z_j = d\,\mathrm{Re}(z_j)d\,\mathrm{Im}(z_j)\ . \tag{6.8}$$

The point process in (6.6) is determinantal,

$$\rho(z_1, z_2, \cdots, z_k) = \det\left[K_N(z_j, z_l)\right]_{j,l=1,2,\cdots,k}, \tag{6.9}$$

with correlation kernel given by the sum over the orthonormalised Gegenbauer polynomials $C_n^{\alpha+1}(z)$ over the ellipse E,

$$K_N(z_j, z_l) = \frac{2\tau}{\pi\sqrt{1-\tau^2}}\sqrt{w(z_j)w(\bar{z}_l)}\sum_{n=0}^{N-1}\frac{n+\alpha+1}{C_n^{(\alpha+1)}(1/\tau)}C_n^{(\alpha+1)}(z_j)C_n^{(\alpha+1)}(\bar{z}_l). \tag{6.10}$$

Note that under the choice of the ellipse given by one parameter τ, the position of the foci is now at $z = \pm 1$, which simplifies the argument of the polynomials that constitute the kernel and that we have incorporated the area of the wighted ellipse $\pi ab/(1+\alpha)$ in the norm of these polynomials. Futhermore, note that the point process (6.9) may be written as

$$\rho(z_1, z_2, \cdots, z_k) = \det_{1\le j,l\le k}\left[\kappa_N(z_j, z_l)\right]\prod_{i=1}^{k}\frac{2\tau(1+\alpha)}{\pi\sqrt{1-\tau^2}}(1-Q(z_i))^{\alpha}, \tag{6.11}$$

here κ_N is the polynomial kernel

$$\kappa_N(z_j, z_l) = \frac{1}{\alpha+1}\sum_{n=0}^{N-1}\frac{n+\alpha+1}{C_n^{(\alpha+1)}(1/\tau)}C_n^{(\alpha+1)}(z_j)C_n^{(\alpha+1)}(\bar{z}_l). \tag{6.12}$$

Sometimes this equivalent representation is more convenient for understanding scale processes and not counting it twice.

Let us point out several limits of the point process (6.9) (or equivalently of the distribution (6.6)) that relates our point process to known ones in RMT. For that aim is enough to consider correlation kernel (6.10)

First, we consider the rotationally invariant limit. Here, we have to rescale the positions as

$$z_j \mapsto z_j/\sqrt{2\tau}, \tag{6.13}$$

and then take the limit $\tau \to 0$. The Jacobian cancels the term 2τ in (6.10). In this limit only the leading coefficients of the polynomials contribute to the sum, it can be seen by introducing the term $(2\tau)^n/(2\tau)^n$. The ellipse E in (6.3) becomes the unit disc and the weight function becomes

$$w_{truncated}(z) = \left(1-|z|^2\right)^{\alpha}, \quad \alpha > -1, \tag{6.14}$$

which is radially symmetric. For an integer α the limiting point process from (6.9) then agrees with the point process of the complex eigenvalues of the ensemble of truncated unitary random matrices introduced in [14]. It is obtained from a unitary matrix $U \in U(N)$ distributed according to the Haar measure, truncated to the upper left block of U of size $M \times M$, with $N > M$ and the resulting parameter

$$\alpha = N - M - 1. \tag{6.15}$$

The complex eigenvalue correlation functions of such a truncated unitary matrix were computed in [14], using monomials z^n as orthogonal polynomials with respect to the weight (6.14).

In the second limit, we want to make contact with the eigenvalues of Hermitian RMT and the corresponding Dyson gas of particles confined to (a subset of) the real line, while still interacting logarithmically, that is with Coulomb interaction in two dimensions. This is easy to see by writing the ellipse in terms of real coordinates $z = x + iy$, we have

$$E = \left\{ z = x + iy \,\middle|\, \frac{2\tau}{1+\tau}x^2 + \frac{2\tau}{1-\tau}y^2 \leq 1 \right\} , \quad 0 < \tau < 1. \tag{6.16}$$

Here, we have to rescale the imaginary part $\mathrm{Im}(z_j) = y_j$ of the positions as

$$y_j \mapsto \sqrt{\frac{1-\tau}{1+\tau}}\, y_j \ , \tag{6.17}$$

Taking the limit $\tau \to 1$, the Jacobian cancels the pole $1/\sqrt{1-\tau}$ in (6.10). The ellipse E become the unit disc. In this limit the argument of the correlation kernel (6.10) is projected to the real parts $\mathrm{Re}(z_j) = x_j \in [-1, 1]$ $(j = 1, 2, \cdots, N)$.

Because the initial measure is in two dimensions, in (6.6) we still have to integrate out the imaginary parts $\mathrm{Im}(z_j) = y_j$, leading to an additional contribution to the weight function, see Remark 4.7 for details. We arrive at the following limiting weight function

$$w_{Jacobi}(x) = \left(1 - x^2\right)^{\alpha + \frac{1}{2}} , \tag{6.18}$$

It agrees with a special case of the weight where the eigenvalues result from the Jacobi ensemble of Hermitian random matrices [71, 72]. The eigenvalue correlation functions are computed with the help of the Jacobi polynomials, in our case with symmetric indices, when the Jacobi polynomials reduce to the Gegenbauer polynomials.

Finally a map to the elliptic Ginibre ensemble exists, this is achieved by making the scaling transformation

$$z_j \mapsto z_j/\sqrt{2\tau\alpha} \ , \tag{6.19}$$

under this transformations the new parameters of the ellipse E are $a = \sqrt{\alpha(1+\tau)}$ and $b = \sqrt{\alpha(1-\tau)}$. Then taking the limit $\alpha \to \infty$, E becoming the entire complex, thus removing the hard wall constraint in (6.5) for the potential V and we find the limiting weight function (6.7) which is a Gaussian,

$$w_{Ginibre}(z) \ = \ \exp\left(-G(z)\right), \text{ with } G(z) = \frac{1}{1-\tau^2}|z|^2 - \frac{\tau}{1-\tau^2}\,\mathrm{Re}(z^2). \tag{6.20}$$

The limit of Gegenbauer polynomials with rescaled argument, as required by (6.19),

$$\lim_{\alpha\to\infty} \alpha^{-n/2} C_n^{(\alpha)}(z/\sqrt{\alpha}) = \frac{1}{n!} H_n(z) \ , \tag{6.21}$$

for the denominator of the correlation kernel (6.10) we also need the corresponding limit *without* rescaling the arguments. It follows from the generating function for Gegenbauer polynomials [69, 18.12.4]

$$\sum_{n=0}^{\infty} C_n^{(\alpha)}(x) r^n = (1 - 2rx + r^2)^{-\alpha} \ . \tag{6.22}$$

After rescaling $r \to r/\alpha$ and taking $\alpha \to \infty$,

$$\lim_{\alpha\to\infty} \sum_{n=0}^{\infty} \frac{1}{\alpha^n} C_n^{(\alpha)}(x) r^n \ = \ e^{2rx} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} r^n \ , \tag{6.23}$$

we obtain the relation

$$\lim_{\alpha\to\infty} a^{-n} C_n^{(\alpha)}(x) = \frac{1}{n!}(2x)^n \ . \tag{6.24}$$

Putting these together and rescaling as in (6.19), we obtain

$$\begin{aligned} K_N^{Ginibre}(z_1, z_2) \ &= \ \lim_{\alpha\to\infty} \frac{1}{2\tau\alpha} K_N\left(\frac{z_1}{\sqrt{2\tau\alpha}}, \frac{z_2}{\sqrt{2\tau\alpha}}\right) \\ &= \ \exp\left[-G(z_1)/2 - G(z_2)/2\right] \\ &\quad \times \frac{1}{\pi\sqrt{1-\tau^2}} \sum_{n=0}^{N-1} \left(\frac{\tau}{2}\right)^n \frac{1}{n!} H_n\left(\frac{z_1}{\sqrt{2\tau}}\right) H_n\left(\frac{\bar{z}_2}{\sqrt{2\tau}}\right) . \end{aligned} \tag{6.25}$$

So, in the limit $\alpha \to \infty$, the resulting point process with kernel (6.10) agrees with that of the complex eigenvalues of the elliptic Ginibre ensemble of complex random matrices [73], including the rotationally invariant Ginibre ensemble at $\tau = 0$. The elliptic Ginibre ensemble was analysed as a Coulomb gas in [21], deriving and using the orthogonality property of the Hermite polynomials with respect to the weight (6.20). All complex eigenvalue correlation functions of the elliptic Ginibre ensemble were derived later in [74].

6.1. **Local correlations at week non-Hermiticity.** In this section we introduce the local scale limit that allows us to present our new results (next section) and will mainly be concerned with local correlation functions in the limit regime in non-Hermitian RMT introduced by Fyodorov, Khoruzhenko and Sommers in [75] known as weak non-Hermiticity limit. Although *our methods in this section will not be rigorous*, as we have already pointed out in the previous part, our ensemble at finite N is mapped to already known in RMT, this serves as a test floor to compare our results for large N.

The analysis of the correlation function (6.10) in the weakly non-Hermitian situation will lead to a new two-parameter family of limiting point processes. The weak non-Hermiticity parameter s, which we will specify below, projects our statistics on the real line and we obtain the well-known Sine and Bessel kernel. Furthermore, this will help us to indirectly find the leading asymptotic term for the correlation kernel (6.10) at the origin, the so-called limiting kernel at *strong* non-Hermiticity, that we conjecture to be universal after a suitable scaling limit.

The weak non-Hermiticity limit, both in the bulk and at the edge of the spectrum, is defined by taking the limit $\tau \to 1$ such that

$$\frac{1}{\tau} = 1 + \frac{s^2}{2N^2}, \quad 0 < s < \infty, \tag{6.26}$$

with $N \to \infty$, and the weak non-Hermiticity parameter s is kept fixed[3]. For later use we collect the following expressions

$$\tau = \frac{1}{1 + \dfrac{s^2}{2N^2}}, \quad \frac{\tau}{1-\tau} = \frac{2N^2}{s^2}, \quad \frac{\tau}{1+\tau} = \frac{2N^2}{4N^2 + s^2}. \tag{6.27}$$

Weak non-Hermiticity means a double scaling limit $N \to \infty$ and $\tau \to 1$, the Hermitian limit, taken such that the global density collapses to the real line, the interval $[-1, 1]$ in our case, whereas local correlation functions still extend into the complex plane.

In our ensemble, with edge we mean the vicinity of the endpoints ± 1, and with bulk we mean the vicinity of interior points of the open interval $(-1, 1)$, away from the edges.

Given that the Gegenbauer polynomials can be expressed in terms of the Jacobi polynomials, (2.56), (2.57) and (2.58) , it turns out that in both the bulk and edge limits the following asymptotic form of the general Jacobi polynomials $P^{(\alpha,\gamma)}(z)$ will be useful, [69, 18.11.5]:

$$P_n^{(\alpha,\gamma)}\left(1 - \frac{Z}{2n^2}\right) \sim n^{\alpha}\left(\frac{\sqrt{Z}}{2}\right)^{-\alpha} J_{\alpha}\left(\sqrt{Z}\right), \quad n \to \infty, \tag{6.28}$$

with fixed real α and γ, and $Z = X + iY$ (X and Y are real) kept fixed. Note that the asymptotic form (6.28) zooming into the vicinity of $+1$ is independent of γ.

We consider the bulk scaling limit in the vicinity of the origin, by rescaling the complex variables inside the kernel (6.10) as

$$z_j = x_j + iy_j = \frac{\hat{z}_j}{N}, \quad j = 1, 2, \tag{6.29}$$

where $\hat{z}_j = \hat{x}_j + i\hat{y}_j$ ($\hat{x}_j$ and $\hat{y}_j$ are real) are kept fixed when $N \to \infty$.

Because the Gegenbauer polynomials have parity $C_n^{(\alpha+1)}(-x) = (-1)^n C_n^{(\alpha+1)}(x)$, without loss of generality, we consider the weak non-Hermiticity limit at the edge of the spectrum around the focus at $+1$, choosing the scaling

$$z_j = 1 - \frac{Z_j}{2N^2}, \quad j = 1, 2, \tag{6.30}$$

together with the weak non-Hermiticity limit (6.26). Here, the complex numbers $Z_j = X_j + iY_j$ are fixed (X_j and Y_j are real).

In the kernel (6.10) the sum will turn into an integral. Because we split the sum into its even and odd parts, let us present the details of this step. For f_n some continuous

[3]Note that in [74] this parameter is typically found to be proportional to $\sim (1-\tau)N$.

and integrable function depending on n we have

$$\sum_{n=0}^{N-1}(n+a+1)f_n = \sum_{\ell=0}^{\lfloor\frac{N-1}{2}\rfloor}(2\ell+a+1)f_{2\ell}+\sum_{\ell=0}^{\lfloor\frac{N-2}{2}\rfloor}(2\ell+a+2)f_{2\ell+1}$$
$$\sim \frac{N^2}{2}\int_0^1 dc\, c\left(f\left(\frac{2\ell}{N}=c\right)+f\left(\frac{2\ell+1}{N}=c\right)\right), \qquad (6.31)$$

in the limit $N\to\infty$, where $\ell=\lfloor n/2\rfloor$. We also introduced the integration variable

$$c=\frac{n}{N}=\frac{2\ell}{N} \text{ or } \frac{2\ell+1}{N}\in[0,1]\ , \qquad (6.32)$$

and use that

$$\frac{2}{N}\sum_{\ell=0}^{\mathcal{L}}\to\int_0^1 dc, \quad \text{for } \mathcal{L}=\left\lfloor\frac{N-1}{2}\right\rfloor \text{ or } \left\lfloor\frac{N-2}{2}\right\rfloor. \qquad (6.33)$$

6.2. **Weak non-Hermiticity in the bulk.** With a short calculation for the scaling limit (given by (6.26) and (6.29)) of the pre-factors of the kernel in the first line of (6.10), that originates from the weight function, we obtain

$$\lim_{N\to\infty}\left(1-\frac{2\tau}{1+\tau}x_j^2-\frac{2\tau}{1-\tau}y_j^2\right)^{\alpha/2}=\left(1-4\frac{\hat{y}_j^2}{s^2}\right)^{\alpha/2}, \qquad (6.34)$$

for $j=1,2$. Here, only the imaginary part of the scaling variable $\hat{z}_j=\hat{x}_j+i\hat{y}_j$ appears. From this limit we can read off the domain of the scaling variables $\hat{z}_j$ ($j=1,2$) in the bulk limit:

$$D_{\text{Bulk}}=\left\{\hat{z}\,\middle|\,\frac{s^2}{4}\geq\hat{y}^2 \text{ and } -\infty<\hat{x}<\infty\right\}, \qquad (6.35)$$

with $\hat{z}=\hat{x}+i\hat{y}$ ($\hat{x}$ and $\hat{y}$ are real).

For the asymptotic form of the Gegenbauer polynomials inside the sum of (6.10), we can apply the asymptotic form of the Jacobi polynomials (6.28). We begin with the even Gegenbauer polynomials. Using (2.49) and (2.57), we have

$$C_{2\ell}^{(\alpha+1)}(x) = \frac{(\alpha+1)_\ell}{(1/2)_\ell}P_\ell^{(\alpha+\frac{1}{2},\,-\frac{1}{2})}(2x^2-1)$$
$$= \frac{\Gamma(\ell+\alpha+1)\ \Gamma(1/2)}{\Gamma(\alpha+1)\ \Gamma\left(\ell+\frac{1}{2}\right)}(-1)^\ell P_\ell^{(-\frac{1}{2},\,\alpha+\frac{1}{2})}(1-2x^2), \qquad (6.36)$$

From (6.28) we thus obtain

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^\alpha}(-1)^\ell C_{2\ell}^{(\alpha+1)}\left(\frac{\hat{z}}{N}\right) &= \frac{\sqrt{\pi}c^\alpha}{2^\alpha\Gamma(\alpha+1)}\lim_{N\to\infty}\ell^{\frac{1}{2}}P_\ell^{(-\frac{1}{2},\ \alpha+\frac{1}{2})}\left(1-2\frac{\hat{z}^2}{N^2}\right)\\ &= \frac{\sqrt{\pi}c^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(\frac{c\hat{z}}{2}\right)^{\frac{1}{2}}J_{-\frac{1}{2}}(c\hat{z})\\ &= \frac{c^\alpha}{2^\alpha\Gamma(\alpha+1)}\cos(c\hat{z}). \end{aligned}\tag{6.37}$$

Here, $c = 2\ell/N$ is fixed in the limit $N \to \infty$, and in the last step we have used [47, 8.464.2]

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos(z).\tag{6.38}$$

The very same steps can be taken for the asymptotic form of the odd Gegenbauer polynomials. Using (2.49) and (2.58), we start from the map

$$\begin{aligned}C_{2\ell+1}^{(\alpha+1)}(x) &= \frac{(\alpha+1)_{\ell+1}}{(1/2)_{\ell+1}}xP_\ell^{(\alpha+\frac{1}{2},\ \frac{1}{2})}(2x^2-1)\\ &= \frac{\Gamma(\ell+\alpha+2)\ \Gamma(1/2)}{\Gamma(\alpha+1)\ \Gamma\left(\ell+\frac{3}{2}\right)}(-1)^\ell xP_\ell^{(\frac{1}{2},\ \alpha+\frac{1}{2})}(1-2x^2).\end{aligned}\tag{6.39}$$

Once again (6.28) leads to

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^\alpha}(-1)^\ell C_{2\ell+1}^{(\alpha+1)}\left(\frac{\hat{z}}{N}\right) &= \frac{\sqrt{\pi}c^\alpha}{2^\alpha\Gamma(\alpha+1)}\lim_{N\to\infty}\frac{\hat{z}}{N}\ell^{\frac{1}{2}}P_\ell^{(\frac{1}{2},\ \alpha+\frac{1}{2})}\left(1-2\frac{\hat{z}^2}{N^2}\right)\\ &= \frac{\sqrt{\pi}c^\alpha}{2^a\Gamma(\alpha+1)}\left(\frac{c\hat{z}}{2}\right)^{\frac{1}{2}}J_{\frac{1}{2}}(c\hat{z})\\ &= \frac{c^\alpha}{2^\alpha\Gamma(\alpha+1)}\sin(c\hat{z}).\end{aligned}\tag{6.40}$$

Here, $c = (2\ell+1)/N$ is fixed in the limit $N \to \infty$, and in the last step we have used [47, 8.464.1]

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z).\tag{6.41}$$

For the Gegenbauer polynomials from the normalisation in the denominator inside the sum of (6.10), the argument is $1/\tau$. Using (6.26), we see that we can directly use (6.28) together with the standard map (2.56), valid for both even and odd polynomials alike. By analytic continuation of the asymptotic (6.28) to imaginary argument, $Z \to iZ$, we obtain for the normalising Gegenbauer polynomial of the scaling variable (6.26)

$$\lim_{n\to\infty}\frac{1}{N^{2\alpha+1}}C_n^{(\alpha+1)}\left(1+\frac{s^2}{2N^2}\right) = \frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(2\alpha+2)}\left(\frac{2}{cs}\right)^{\alpha+\frac{1}{2}}I_{\alpha+\frac{1}{2}}(cs),\tag{6.42}$$

with $c = n/N$ fixed. Here, $I_\alpha(z)$ is the modified Bessel function of the first kind.

Putting all the above together we obtain the following result for the bulk scaling limit of the kernel (6.10) around the origin:

$$\begin{aligned} K_{\text{Bulk}}(\hat{z}_1, \hat{z}_2) &= \lim_{N\to\infty} \frac{1}{N^2} K_N\left(\frac{\hat{z}_1}{N}, \frac{\hat{z}_2}{N}\right) \\ &= \left(1 - \frac{4\hat{y}_1^2}{s^2}\right)^{\frac{\alpha}{2}} \left(1 - \frac{4\hat{y}_2^2}{s^2}\right)^{\frac{\alpha}{2}} \frac{1}{\pi s} \frac{s^{\alpha+\frac{1}{2}}\Gamma(2\alpha+2)}{2^{3\alpha+\frac{1}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma(\alpha+1)^2} \\ &\quad \times \int_0^1 dc\, \frac{c^{\alpha+\frac{1}{2}}\left(\cos(c\hat{z}_1)\cos(c\bar{\hat{z}}_2) + \sin(c\hat{z}_1)\sin(c\bar{\hat{z}}_2)\right)}{I_{\alpha+\frac{1}{2}}(cs)} \\ &= \frac{2}{s\pi^{\frac{3}{2}}\Gamma(\alpha+1)} \left(1 - \frac{4\hat{y}_1^2}{s^2}\right)^{\frac{\alpha}{2}} \left(1 - \frac{4\hat{y}_2^2}{s^2}\right)^{\frac{\alpha}{2}} \int_0^1 dc \frac{(cs/2)^{\alpha+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(cs)} \cos(c(\hat{z}_1 - \bar{\hat{z}}_2)). \end{aligned} \tag{6.43}$$

In the second step we have used an addition theorem for the trigonometric functions and the duplication formula for the Gamma function (2.5).

The corresponding microscopic level density only depends on the imaginary part, see (6.43), and reads

$$\varrho(\hat{y}) = K_{\text{Bulk}}(\hat{x} + i\hat{y}, \hat{x} + i\hat{y}). \tag{6.44}$$

In what follows we will take three limits of the bulk kernel (6.43), in order to compare to other known results in RMT. We begin with the Hermitian limit as a consistency check.

(1) The Hermitian limit $s \to 0$:

In this limit the local bulk kernel is mapped back to the real axis. This can be seen from the support (6.35) of length s in the $\hat{y}$-direction. After rescaling by $s/2$ (in the $\hat{y}$-direction), the support of the imaginary part of z_i is restricted to $|\operatorname{Im}(z_i)| < 1$, while the argument of the correlation kernel is projected on the real line in the limit $s \to 0$.

For the denominator of the integrand we have the small argument asymptotic relation of the modified Bessel-function, see e.g. in [47, 8.445]

$$I_{\alpha+\frac{1}{2}}(cs) \sim \frac{(cs/2)^{\alpha+\frac{1}{2}}}{\Gamma\left(\alpha+\frac{3}{2}\right)}, \quad s \to 0. \tag{6.45}$$

Before taking the limit $s \to 0$ we have to recall that the point process with kernel (6.43) is equivalent to one of the form (6.11) and that the underlying measure is the planar Lebesgue measure, that is, in the scaling limit the extra factor $\Gamma(\alpha + \frac{3}{2})/\sqrt{\pi}\Gamma(\alpha + 1)$ does not contribute to the statistics and is trivially canceled with the integral

$$\int_{-1}^{1} (1 - y^2)^\alpha \, dy \tag{6.46}$$

So, in the limit $s \to 0$, we obtain

$$\begin{aligned}
\lim_{s\to 0}\frac{s}{2}\ K_{\text{Bulk}}(\hat{x}_1+i\frac{s}{2}\hat{y}_2,\hat{x}_2+i\frac{s}{2}\hat{y}_2) &= \frac{1}{\pi}\int_0^1 dc\cos(c(\hat{x}_1-\hat{x}_2)) \\
&= \frac{1}{\pi}\frac{\sin(\hat{x}_1-\hat{x}_2)}{\hat{x}_1-\hat{x}_2}\ . \qquad (6.47)
\end{aligned}$$

It results into the well-known universal sine-kernel. It is known to hold for the Jacobi ensemble in the bulk of the spectrum [71], as well as for other ensembles within the same symmetry class.

(2) The strong non-Hermiticity limit $s \to \infty$:

This limit is expected to reproduce the limiting kernel at strong non-Hermiticity, when rescaling $\tilde{z}_j = \tilde{x}_j + i\tilde{y}_j = \hat{z}_j/s$ for $j = 1, 2$ ($\tilde{x}_j$ and $\tilde{y}_j$ are real). The same mechanism was applied in the elliptic Ginibre ensemble in [74]. The corresponding domain (6.35) gets mapped to

$$D_{\text{Bulk, strong}} = \left\{ \tilde{z} \,\middle|\, \frac{1}{4} \geq \tilde{y}^2 \ \text{ and } \ -\infty < \tilde{x} < \infty \right\}, \qquad (6.48)$$

with $\tilde{z} = \tilde{x} + i\tilde{y}$ ($\tilde{x}$ and $\tilde{y}$ are real). It is an infinite strip of unit width parallel to the $\tilde{x}$-axis. We obtain the following expression for the limit of the integral in (6.43):

$$\begin{aligned}
\mathcal{J}_\alpha &= \lim_{s\to\infty} s\int_0^1 dc\frac{(cs/2)^{\alpha+\frac{1}{2}}}{I_{a+\frac{1}{2}}(cs)}\cos(c(\hat{z}_1-\bar{\hat{z}}_2)) \\
&= \lim_{s\to\infty}\int_0^s dt\frac{(t/2)^{\alpha+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(t)}\cos(t(\tilde{z}_1-\bar{\tilde{z}}_2)) \\
&= \int_0^\infty dt\frac{(t/2)^{a+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(t)}\cos(t(\tilde{z}_1-\bar{\tilde{z}}_2)). \qquad (6.49)
\end{aligned}$$

Here, we have changed the integration variable to $t = cs$. The final answer for the limiting kernel at strong non-Hermiticity on the domain (6.48) thus reads

$$\begin{aligned}
K_{\text{Strong,bulk}}(\tilde{z}_1,\tilde{z}_2) &= \lim_{s\to\infty} s^2 K_{\text{Bulk}}(s\tilde{z}_1,s\tilde{z}_2) \\
&= \frac{2}{\pi^{\frac{3}{2}}\Gamma(\alpha+1)}\left(1-4\tilde{y}_1^2\right)^{\frac{a}{2}}\left(1-4\tilde{y}_2^2\right)^{\frac{\alpha}{2}}\int_0^\infty dt\frac{(t/2)^{\alpha+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(t)}\cos(t(\tilde{z}_1-\bar{\tilde{z}}_2)).
\end{aligned} \qquad (6.50)$$

Although we have derived the kernel (6.50) indirectly via the weak non-Hermiticity limit at the origin, we conjecture it to be universal, after an appropriate shift of the weight away from the origin plus rescalings. Because the appropriate Mehler or Poisson formula for the kernel (6.10) is lacking, when extending the sum to infinity, we have been unable to directly take the strong non-Hermiticity limit. Notice that a different Poisson kernel exists for the general Jacobi polynomials, cf. [76], but is relies on the fact that the sum is a geometric-type sum.

(3) From the strong non-Hermiticity to the Ginibre Kernel in the limit $\alpha \to \infty$:

Let us explain how to recover the Ginibre kernel in the limit $\alpha \to \infty$. A series expansion [69, 10.25.2]

$$I_{\alpha+\frac{1}{2}}(t) = \left(\frac{t}{2}\right)^{\alpha+\frac{1}{2}} \sum_{\ell=0}^{\infty} \frac{(t^2/4)^\ell}{\ell!\ \Gamma\left(\ell+\alpha+\frac{3}{2}\right)} \tag{6.51}$$

is known for the modified Bessel function. Introducing a new variable $\hat{t} = t/\sqrt{\alpha}$ and using the asymptotic relation

$$\frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma\left(\ell+\alpha+\frac{3}{2}\right)} \sim \alpha^{-\ell}, \quad \alpha \to \infty, \tag{6.52}$$

for a fixed non-negative integer ℓ, we obtain

$$I_{\alpha+\frac{1}{2}}(\sqrt{\alpha}\hat{t}) \sim \left(\frac{\sqrt{\alpha}\hat{t}}{2}\right)^{\alpha+\frac{1}{2}} \frac{e^{\hat{t}^2/4}}{\Gamma\left(\alpha+\frac{3}{2}\right)}, \quad \alpha \to \infty, \tag{6.53}$$

from (6.51). Here $\hat{t}$ is fixed. We put this asymptotic form into (6.49) and find

$$\begin{aligned} \mathcal{J}_\alpha &= \sqrt{\alpha} \int_0^\infty d\hat{t} \frac{\left(\sqrt{\alpha}\hat{t}/2\right)^{\alpha+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(\sqrt{\alpha}\hat{t})} \cos(\sqrt{\alpha}\hat{t}(\tilde{z}_1 - \bar{\tilde{z}}_2)) \\ &\sim \sqrt{\alpha}\ \Gamma\left(\alpha+\frac{3}{2}\right) \int_0^\infty d\hat{t}\ e^{-\hat{t}^2/4} \cos\left(\hat{t}(u_1-\bar{u}_2)\right) \\ &= \sqrt{\pi\alpha}\ \Gamma\left(\alpha+\frac{3}{2}\right) e^{-(u_1-\bar{u}_2)^2}, \end{aligned} \tag{6.54}$$

where $u_j = \sqrt{a}\tilde{z}_j$ $(j = 1, 2)$. Then it follows that

$$\begin{aligned} \widetilde{K}_{Ginibre}(u_1, u_2) &= \lim_{a\to\infty} K_{\text{Strong, bulk}}\left(u_1/\sqrt{a}, u_2/\sqrt{a}\right)/a \\ &= \frac{2}{\pi} \exp\left[-|u_1|^2 - |u_2|^2 + 2u_1\bar{u}_2 - i\,\mathrm{Im}(u_1^2 - u_2^2)\right]. \end{aligned} \tag{6.55}$$

This kernel is equivalent to the Ginibre kernel $K_{Ginibre}(u_1, u_2)$, presented in the introduction (1.19). Note that two kernels are equivalent if they agree up to multiplication by $f(u_1)/f(u_2)$ as they yield the same correlation functions in (6.9), with $f(u_1) = e^{-i\,\mathrm{Im}\,u_1^2}$ here.

6.3. **Weak non-Hermiticity at the edge.** In this subsection we consider the weak non-Hermiticity limit at the edge of the spectrum. We magnify the region around the focus at $+1$, choosing the following scaling limit

$$z_j = 1 - \frac{Z_j}{2N^2}, \quad j = 1, 2, \tag{6.56}$$

together with the weak non-Hermiticity limit (6.26). Here the complex numbers $Z_j = X_j + iY_j$ are fixed (X_j and Y_j are real). In this limit the pre-factors of the kernel (6.10) from the weight turn into

$$\left(1 - \frac{2\tau}{1+\tau}x_j^2 - \frac{2\tau}{1-\tau}y_j^2\right)^{\alpha/2} \sim N^{-\alpha}\left(\frac{s^2}{4} + X_j - \left(\frac{Y_j}{s}\right)^2\right)^{\alpha/2}, \tag{6.57}$$

in the limit $N \to \infty$ as (6.26) and (6.56). Once again we keep the parameter α fixed in this limit. Eq. (6.57) implies that the limiting domain of the scaled particle positions (X_j, Y_j) becomes the parabolic domain

$$D_{\mathrm{Edge}} = \left\{(X,Y)\,\middle|\, X \geq \left(\frac{Y}{s}\right)^2 - \frac{s^2}{4}\right\}, \tag{6.58}$$

which is a magnified part around the right focus of the ellipse, that is the right endpoint of $[-1, 1]$.

The pre-factor of the sum in the second line of (6.10) is easily evaluated by using (6.27), to give

$$\frac{2\tau}{\pi\sqrt{1-\tau^2}} = \frac{2}{\pi}\sqrt{\frac{\tau}{1-\tau}\frac{\tau}{1+\tau}} \sim \frac{2N}{s\pi}. \tag{6.59}$$

Due to the relation (2.56) of the Gegenbauer polynomials to the symmetric Jacobi polynomials, and their asymptotic form (6.28) in the vicinity of unity, we find the following asymptotic relation,

$$\begin{aligned} C_n^{(\alpha+1)}(z_j) &= C_n^{(\alpha+1)}\left(1 - \frac{Z_j}{2N^2}\right) \\ &\sim N^{2\alpha+1}\frac{\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(2\alpha+2)}\left(\frac{\sqrt{Z_j}}{2c}\right)^{-\alpha-\frac{1}{2}} J_{\alpha+\frac{1}{2}}\left(c\sqrt{Z_j}\right). \end{aligned} \tag{6.60}$$

Because the limit of the squared norms does not depend on the point we magnify, we may use again the asymptotic (6.42) from the previous subsection.

Inserting (6.57), (6.59), (6.60) and (6.42) together in (6.10), and replacing the sum by an integral, yields the following asymptotic formula for the limiting kernel at the edge

$$\begin{aligned} K_{\mathrm{Edge}}(Z_1, Z_2) &= \lim_{N\to\infty}\frac{1}{4N^4}K_N(z_1, z_2) \\ &= \frac{1}{4\sqrt{\pi}\Gamma(\alpha+1)}\left(\frac{s}{2}\right)^{a-\frac{1}{2}}\left(\frac{s^2}{4} + X_1 - \left(\frac{Y_1}{s}\right)^2\right)^{\frac{\alpha}{2}}\left(\frac{s^2}{4} + X_2 - \left(\frac{Y_2}{s}\right)^2\right)^{\frac{\alpha}{2}} \\ &\quad\times\left(\sqrt{Z_1\bar{Z}_2}\right)^{-\alpha-\frac{1}{2}}\int_0^1 dc\,\frac{c^{\alpha+\frac{3}{2}}}{I_{\alpha+\frac{1}{2}}(cs)}J_{\alpha+\frac{1}{2}}\left(c\sqrt{Z_1}\right)J_{\alpha+\frac{1}{2}}\left(c\sqrt{\bar{Z}_2}\right), \end{aligned} \tag{6.61}$$

with a fixed $\alpha > -1$. This limiting kernel is a deformation of the Bessel-kernel into the complex plane, holding inside the domain (6.58) where the two pre-factors from the weight have non-negative arguments. From symmetry the same limiting kernel is obtained at the left edge of the ellipse. Not only the pre-factors from the weight but also the pre-factor in the integrand inversely proportional to the modified I-Bessel function differs from the pre-factor of the deformed Bessel-kernel of the chiral ensemble [12], given by an exponential. There, $\alpha+\frac{1}{2}=\nu$, and for integer values it corresponds to the number of zero-modes therein. This difference remains valid for any fixed values $\alpha>-1$, and shows the influence of the boundary. It pertains also for large arguments, as we will see below. We expect that the limiting edge-kernel (6.61) is also universal.

Again we define a microscopic density which depends this time on both the real and imaginary parts, due to the loss of translation invariance, i.e.,

$$\hat{\varrho}(X,Y)=K_{\mathrm{Edge}}(X+iY,X+iY). \tag{6.62}$$

Its dependence on an increasing non-Hermiticity s and an increasing charge α. The spectrum lies in a constant competition between s, which tries to spread and squeeze it into the boundary, and α, which creates a repulsion from exactly the same boundary.

Below we will take two limits of the kernel (6.61) to compare with known asymptotic kernels in RMT, the Hermitian and strong non-Hermiticity limit. In addition we take a third limit of large argument, that brings us back to the result in the bulk from the previous subsection.

(4) The Hermitian limit $s\to 0$:

In this limit, after rescaling by s in the Y-direction (6.58) , the support of the imaginary part of Z_i is restricted to $|\,\mathrm{Im}(Z_i)|\leq\sqrt{\mathrm{Re}(Z_i)}$ with $\mathrm{Re}(Z_i)\geq 0$, while the argument of the correlation kernel (6.61) is projected on the real line in the limit $s\to 0$.

For the s dependent factor inside the integral in (6.61) we may use again the asymptotic (6.45). We recall that the point process with kernel (6.61) is equivalent to one of the form (6.11) and that the underline measure is the planar Lebesgue measure. In the limit $s\to 0$, for the (non) constant pre-factor in (6.61), including the contribution of (6.45), we have

$$\frac{\Gamma(\alpha+\frac{3}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)}\frac{1}{x^{\alpha+\frac{1}{2}}}\int_{-\sqrt{x}}^{\sqrt{x}}(x-y^2)^{\alpha}\,dy=\frac{1}{2}\ , \tag{6.63}$$

this leads to the following result:

$$\begin{aligned}
&\lim_{s\to 0} s\,K_{\mathrm{Edge}}(X_1+isY_1,X_2+isY_2)\\
=\ &\frac{1}{2}\int_0^1 dc\ c\ J_{\alpha+\frac{1}{2}}\left(c\sqrt{X_1}\right)J_{\alpha+\frac{1}{2}}\left(c\sqrt{X_2}\right)\\
=\ &\frac{1}{4}\int_0^1 dc\ J_{\alpha+\frac{1}{2}}\left(\sqrt{cX_1}\right)J_{\alpha+\frac{1}{2}}\left(\sqrt{cX_2}\right)\\
=\ &\frac{J_{\alpha+\frac{1}{2}}\left(\sqrt{X_1}\right)\sqrt{X_2}J'_{\alpha+\frac{1}{2}}\left(\sqrt{X_2}\right)-\sqrt{X_1}J'_{\alpha+\frac{1}{2}}\left(\sqrt{X_1}\right)J_{\alpha+\frac{1}{2}}\left(\sqrt{X_2}\right)}{2(X_1-X_2)}\,.
\end{aligned} \tag{6.64}$$

with a fixed $\alpha > -1$. This reproduces a well-known universal result in RMT, the so-called Bessel-kernel. Note the last equality may be obtained from a Christoffel-Darboux type of argument, see [77].

(5) The strong non-Hermiticity limit $s \to \infty$:

Let us next consider the opposite limit $s \to \infty$, to obtain the limiting kernel at strong non-Hermiticity. For that purpose, we introduce new scaling variables

$$\tilde{X}_j = \frac{2}{s} X_j + \frac{s}{2}, \quad \tilde{Y}_j = \frac{2}{s} Y_j, \tag{6.65}$$

where we keep $\tilde{X}_j$ and $\tilde{Y}_j$ fixed when taking the limit $s \to \infty$. In terms of these new variables the determining equation for the domain (6.58) becomes $\frac{s}{2}\tilde{X}_j \geq \frac{\tilde{Y}_j^2}{4}$. Thus in the limit the scaled particle positions $(\tilde{X}_j, \tilde{Y}_j)$ are confined to the half plane, that is $0 \leq \tilde{X}_j < \infty$ and $-\infty < \tilde{Y}_j < \infty$. Now we use the asymptotic formula [38] for $u \to \infty$,

$$J_b\left(uz\right) \sim \left(\frac{2}{\pi u z}\right)^{1/2} \cos\left(uz - \frac{\pi}{2} b - \frac{\pi}{4}\right), \tag{6.66}$$

for a fixed real index b and a fixed complex z, to obtain

$$\left(\sqrt{Z_j}\right)^{-\alpha-\frac{1}{2}} J_{\alpha+\frac{1}{2}}\left(c\sqrt{Z_j}\right) \sim \left(\frac{s}{2}\right)^{-a-\frac{1}{2}} \left(\pi c s\right)^{-1/2} \exp\left[\frac{cs}{2}\left(1 - \frac{1}{s}\left(\tilde{X}_j + i\tilde{Y}_j\right)\right)\right], \tag{6.67}$$

for $s \to \infty$. Together with the large-s asymptotic for the modified Bessel functions, cf. [47, 8.451.5],

$$I_{\alpha+\frac{1}{2}}\left(cs\right) \sim \left(2\pi c s\right)^{-1/2} e^{cs}, \quad s \to \infty\ , \tag{6.68}$$

valid for any fixed α, it then follows for the scaling (6.65) that

$$\begin{aligned} K_{\text{Strong,edge}}(\tilde{Z}_1, \tilde{Z}_2) &= \lim_{s\to\infty} \frac{s^2}{4} K_{\text{Edge}}(Z_1, Z_2) \\ &= \frac{\left(\tilde{X}_1 \tilde{X}_2\right)^{\alpha/2}}{4\pi\Gamma(\alpha+1)} \int_0^1 dc\ c^{\alpha+1} \exp\left[-\frac{c}{2}(\tilde{X}_1 + \tilde{X}_2) - i\frac{c}{2}(\tilde{Y}_1 - \tilde{Y}_2)\right], \end{aligned} \tag{6.69}$$

with a fixed $\alpha > -1$. This limiting kernel is not new and agrees with the kernel found for truncated unitary matrices [14, eq. (21)] in what the authors call weakly non-unitary limit. But their terminology is different from ours, in [14] the large matrix size is M, while from $\alpha = N - M - 1$, with $L = N - M$ it is fixed.

In what follows and as a consistency check with our results we will use the fact that in the large limit argument the correlations at the edge get mapped back to the correlations in the bulk, see e.g. [78].

(6) From the edge to the bulk limit:

Let us introduce scaled complex variables $\hat{z}_j = \hat{x}_j + i\hat{y}_j$ for the arguments of the edge kernel (6.61) as

$$Z_j = \kappa h - 2\sqrt{h}\hat{z}_j, \tag{6.70}$$

where $\kappa > 0$ and $\hat{z}_j$ remain fixed, and we will take the limit of h positive to become large, $h \to \infty$. In these variables the defining equation for the domain (6.58) with $Z = X + iY = \kappa h - 2\sqrt{h}\hat{z}$ becomes

$$\kappa h - 2\sqrt{h}\hat{x} \geq \frac{4h\hat{y}^2}{s^2} - \frac{s^2}{4}\ , \tag{6.71}$$

leading to the domain

$$D_{\text{Bulk}} = \left\{ \hat{z} \,\middle|\, \frac{s^2}{4}\kappa \geq \hat{y}^2 \quad \text{and} \quad -\infty < \hat{x} < \infty \right\}, \tag{6.72}$$

where $\hat{z} = \hat{x} + i\hat{y}$.

For the scaling (6.70) we can see that

$$\sqrt{Z_j} \sim \sqrt{\kappa h} - \frac{\hat{z}_j}{\sqrt{\kappa}}, \quad h \to \infty. \tag{6.73}$$

Then, we can use (6.66) to find that

$$J_{\alpha+\frac{1}{2}}\left(c\sqrt{Z_j}\right) \sim \left(\frac{2}{c\pi\sqrt{\kappa h}}\right)^{1/2} \cos\left(c\sqrt{\kappa h} - \frac{c\hat{z}_j}{\sqrt{\kappa}} - \frac{\pi}{2}\alpha - \frac{\pi}{2}\right), \quad h \to \infty. \tag{6.74}$$

Putting the above asymptotic results for the scaling (6.70) together in (6.61), we obtain

$$\begin{aligned} K_{\text{Bulk}}(\hat{z}_1, \hat{z}_2) &= \lim_{h\to\infty} 4hK_{\text{Edge}}(Z_1, Z_2) \\ &= \frac{2}{s\pi^{\frac{3}{2}}\Gamma(\alpha+1)\kappa^{a+1}} \\ &\quad \times \left(\kappa - \frac{4\hat{y}_1^2}{s^2}\right)^{\frac{\alpha}{2}} \left(\kappa - \frac{4\hat{y}_2^2}{s^2}\right)^{\frac{\alpha}{2}} \int_0^1 dc \frac{(cs/2)^{\alpha+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(cs)} \cos\left(\frac{c}{\sqrt{\kappa}}(\hat{z}_1 - \bar{\hat{z}}_2)\right), \end{aligned} \tag{6.75}$$

which is similar to the asymptotic kernel (6.43) computed at the origin, in agreement with our conjecture.That is, a similar asymptotic form to the kernel (6.43) is valid in the entire bulk.

6.4. **2D Coulomb gas induced by non-symmetric Jacobi polynomials.** First we consider the case induced by Jacobi polynomials with parameters $(\alpha + 1/2, 1/2)$ and as a remark will treat the case $(\alpha + 1/2, -1/2)$. With the above notations, the one-particle weight function $w(z)$ like in (6.6) defining a 2D Coulomb gas now takes the form

$$w_+(z) = (1 - \mu(z))^\alpha, \tag{6.76}$$

and

$$\mu(z) = \frac{2\tau}{1-\tau}\left(\sqrt{\frac{1+\tau}{2\tau}}\sqrt{(1+x)^2+y^2}-1-x\right), \tag{6.77}$$

with $z = x + iy$. This weight function is different from (6.7), except in the case $\alpha = 0$, when the indices of the Jacobi polynomials again become symmetric.

Under the one parameter $0 < \tau < 1$ ellipse assumption, the kernel again is given by the sum over the orthonormalised polynomials Lemma 5.9 and take the form

$$\begin{aligned} K_N(z_1, z_2) &= \frac{1}{4}(1-\mu(z_1))^{\alpha/2}(1-\mu(\bar{z}_2))^{\alpha/2}\sqrt{\frac{2\tau}{1-\tau}}\frac{1}{\Gamma(\alpha+1)^2} \\ &\times \sum_{n=0}^{N-1} \frac{(2n+\alpha+2)\Gamma(n+\alpha+2)^2}{\Gamma(n+\frac{3}{2})^2 C_{2n+1}^{(\alpha+1)}\left(\sqrt{(1+\tau)/(2\tau)}\right)} P_n^{\left(\alpha+\frac{1}{2},\frac{1}{2}\right)}(z_1) P_n^{\left(\alpha+\frac{1}{2},\frac{1}{2}\right)}(\bar{z}_2). \end{aligned} \tag{6.78}$$

In the following, we will evaluate the asymptotic forms of this kernel in the weak non-Hermiticity limit at the edges, that is around the foci of the ellipse $z = +1$ and $z = -1$. Like in limit (6) we have seen that the bulk limit can be recovered from the edge limit, we will first derive the latter. However, due to the indices of the Jacobi polynomials now being non-symmetric, we expect the limits at the endpoints ± 1 to be different, because of the lack of parity symmetry, cf. (2.49).

(7) Edge limit at the focus $z = +1$, non-symmetric case $(\alpha + 1/2, 1/2)$:

In order to magnify this region, we recall the weak non-Hermiticity limit (6.26)

$$\frac{1}{\tau} = 1 + \frac{s^2}{2N^2}, \tag{6.79}$$

and the rescaling (6.56) around the right focus $+1$:

$$z_j = 1 - \frac{Z_j}{2N^2}, \quad j = 1, 2. \tag{6.80}$$

We will take the double scaling limit $N \to \infty$ and $\tau \to 1$ such that the positive number s and complex numbers $Z_j = X_j + iY_j$ are kept fixed. In this scaling limit the function inside the weight (6.76) gets mapped to

$$1 - \mu\left(1 - \frac{Z}{2N^2}\right) \sim \frac{1}{4N^2}\left(\frac{s^2}{4} + X - \frac{Y^2}{s^2}\right), \tag{6.81}$$

from which we can read off the domain of our scaling variables, being in the parabolic domain (6.58). Here $Z = X + iY$ is kept fixed. In analogy to (6.42) we have

$$C_{2n+1}^{(\alpha+1)}\left(\sqrt{\frac{1+\tau}{2\tau}}\right) \sim N^{2\alpha+1}\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(2\alpha+2)}\left(\frac{s}{8c}\right)^{-\alpha-\frac{1}{2}} I_{\alpha+\frac{1}{2}}(cs), \tag{6.82}$$

with the ratio $c = n/N$ being kept fixed. Using (6.28), we can find the asymptotic for the polynomials

$$P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}\left(1-\frac{Z}{2N^2}\right) \sim N^{\alpha+\frac{1}{2}}\left(\frac{\sqrt{Z}}{2}\right)^{-\alpha-\frac{1}{2}} J_{\alpha+\frac{1}{2}}\left(c\sqrt{Z}\right). \tag{6.83}$$

Putting these asymptotic formulas together with the identities (6.27) into (6.78), and replacing the sum by an integral, we obtain exactly the same asymptotic formula (6.61) for $K_{\mathrm{Edge}}(Z_1, Z_2) = \lim_{N\to\infty} K_N(z_1, z_2)/(4N^4)$. This fact indicates the universality of this kernel.

The Hermitian and strongly non-Hermitian limit as well as the bulk limit then follow as in Subsection 6.3.

(8) Edge limit at the focus $z = -1$, non-symmetric case $(\alpha + 1/2, 1/2)$:

Next, we use the scaling in the weak non-Hermiticity limit (6.79) and magnify the region around the left focus $z = -1$ in the same way as in (6.80):

$$z_j = -1 + \frac{Z_j}{2N^2}, \quad j = 1, 2, \tag{6.84}$$

with $s > 0$ and $Z_j = X_j + iY_j$ fixed in the limit $N \to \infty$. It is straightforward to derive the asymptotic form of the weight function

$$1 - \mu\left(-1 + \frac{Z}{2N^2}\right) \sim 1 - \frac{2}{s^2}\left(\sqrt{X^2+Y^2} - X\right). \tag{6.85}$$

Here $Z = X + iY$ is kept fixed. For this factor to be non-negative it can be seen that the points (X_j, Y_j) have to lie inside the parabolic domain (6.58). For the asymptotic form of the Jacobi polynomials with non-symmetric indices we have

$$P_n^{(\alpha+\frac{1}{2},\frac{1}{2})}\left(-1+\frac{Z}{2N^2}\right) = (-1)^n P_n^{(\frac{1}{2},\alpha+\frac{1}{2})}\left(1-\frac{Z}{2N^2}\right) \sim (-1)^n N^{\frac{1}{2}}\left(\frac{\sqrt{Z}}{2}\right)^{-\frac{1}{2}} J_{\frac{1}{2}}\left(c\sqrt{Z}\right), \tag{6.86}$$

in the limit $N \to \infty$, after using (2.49) and (6.28). These asymptotic formulas together with (6.82) are put into the kernel (6.78) and yield

$$\begin{aligned}
K_{\mathrm{Edge}}(Z_1, Z_2) &= \lim_{N\to\infty} \frac{1}{4N^4} K_N(z_1, z_2) \\
&= \frac{(s/2)^{\alpha-\frac{1}{2}}}{4\sqrt{\pi}\Gamma(\alpha+1)} \left(1 - \frac{2}{s^2}\left(|Z_1| - X_1\right)\right)^{\alpha/2} \left(1 - \frac{2}{s^2}\left(|Z_2| - X_2\right)\right)^{\alpha/2} \\
&\quad \times \left(\sqrt{Z_1 \bar{Z}_2}\right)^{-\frac{1}{2}} \int_0^1 dc\, \frac{c^{\alpha+\frac{3}{2}}}{I_{\alpha+\frac{1}{2}}(cs)} J_{\frac{1}{2}}\left(c\sqrt{Z_1}\right) J_{\frac{1}{2}}\left(c\sqrt{\bar{Z}_2}\right). \\
&= \frac{(s/2)^{\alpha-\frac{1}{2}}}{2\pi^{3/2}\Gamma(a+1)} \left(1 - \frac{2}{s^2}\left(|Z_1| - X_1\right)\right)^{\alpha/2} \left(1 - \frac{2}{s^2}\left(|Z_2| - X_2\right)\right)^{\alpha/2} \\
&\quad \times \frac{1}{\sqrt{Z_1 \bar{Z}_2}} \int_0^1 dc\, \frac{c^{\alpha+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(cs)} \sin\left(c\sqrt{Z_1}\right) \sin\left(c\sqrt{\bar{Z}_2}\right). \qquad (6.87)
\end{aligned}$$

In the last step the J-Bessel functions are expressed in terms of sine, using (6.41). For $\alpha \neq 0$ this edge kernel is clearly different from the one obtained for the Gegenbauer polynomials in (6.61) in Subsection 6.3. While the local asymptotic form of the Jacobi polynomials around this focal point yields $J_{\frac{1}{2}}$ (represented by means of the sine function), the influence of the edge is obviously still present through the dependence of the other factors on α.

In the Hermitian limit $s \to 0$, the coordinates (X_j, Y_j) are confined to the domain satisfying $X_j \geq 0$ and $|Y_j| \leq \sqrt{X_j}$, as we saw already in the Point (4) in Subsection 6.3. Using (6.45) and (6.63), we find the asymptotic formula

$$\lim_{s\to 0} s\, K_{\mathrm{Edge}}(Z_1, Z_2) = \frac{1}{2}\int_0^1 dc\; c\; J_{\frac{1}{2}}\left(c\sqrt{X_1}\right) J_{\frac{1}{2}}\left(c\sqrt{X_2}\right). \qquad (6.88)$$

It agrees with the Bessel-kernel of the Jacobi ensemble (6.64) at $\alpha = 0$.

In the strong non-Hermiticity limit $s \to \infty$ we use the scaling variables $\tilde{X}_j$ and $\tilde{Y}_j$ defined in (6.65), together with the asymptotic relation

$$1 - \frac{2}{s^2}\left(|Z_j| - X_j\right) \sim \frac{2}{s}\tilde{X}_j, \quad s \to \infty, \qquad (6.89)$$

and (6.66). The resulting limit $\lim_{s\to\infty}(s^2/4)K_{\mathrm{Edge}}(Z_1, Z_2)$ exactly reproduces the formula (6.69).

The bulk limit $h \to \infty$, with the scaling variables $\hat{z}_j = \hat{x}_j + i\hat{y}_j$ defined as in (6.70) by $Z_j = \kappa h - 2\sqrt{h}\hat{z}_j$ ($\kappa > 0$), can be evaluated by means of the relation

$$1 - \frac{2}{s^2}\left(|Z_j| - X_j\right) \sim 1 - \frac{4}{\kappa s^2}\hat{y}_j^2, \quad h \to \infty\ , \qquad (6.90)$$

and (6.66). As a result we obtain exactly the same formula (6.75) for the asymptotic kernel $K_{\mathrm{Bulk}}(\hat{z}_1, \hat{z}_2) = \lim_{h\to\infty} 4hK_{\mathrm{Edge}}(Z_1, Z_2)$. From this, we again conjecture that the bulk scaling limit has a similar form, when we zoom into any point $x_0 \in (-1, 1)$. Thus all three limits of the kernel (6.87) lead back to the classes we have already found in Section 6.2 and 6.3.

Remark 6.1. The case induced by non-symmetric Jacobi polynomials $(\alpha+1/2,-1/2)$ and orthogonality given in Lemma 5.9, can be treated along the same line as the previous part, here we will only indicate some important steps

For this case of 2D Coulomb gas, the wight $w(z)$ in (6.6) takes the form

$$w_-(z)=\frac{(1-\mu(z))^\alpha}{|1+z|}, \tag{6.91}$$

with $\mu(z)$ defined in (6.77). Notice that also for $\alpha=0$ the polynomials and weight are different from those in Section 6.2 and 6.3.

The kernel function $K_N(z_1,z_2)$ in (6.9) take the form

$$\begin{aligned} K_N(z_1,z_2) &= \frac{(1-\mu(z_1))^{\alpha/2}(1-\mu(\bar z_2))^{\alpha/2}}{2|1+z_1|^{1/2}|1+z_2|^{1/2}}\sqrt{\frac{2\tau}{1-\tau}}\frac{1}{\Gamma(\alpha+1)^2} \\ &\times\sum_{n=0}^{N-1}\frac{(2n+\alpha+1)\Gamma(n+\alpha+1)^2}{\Gamma(n+\frac12)^2\, C_{2n}^{(\alpha+1)}\left(\sqrt{(1+\tau)/(2\tau)}\right)}P_n^{(\alpha+\frac12,-\frac12)}(z_1)P_n^{(\alpha+\frac12,-\frac12)}(\bar z_2). \end{aligned} \tag{6.92}$$

As in the previous part we can first determine the weak non-Hermiticity limit at the edges.

(9) Edge limit at the focus $z=+1$, non-symmetric case $(\alpha+1/2,-1/2)$:

In the vicinity of the focus $+1$, we can again utilize the scalings (6.79) and (6.80), finding the same domain (6.58) as before. From (6.28), we find

$$P_n^{(\alpha+\frac12,-\frac12)}\left(1-\frac{Z}{2N^2}\right)\sim N^{\alpha+\frac12}\left(\frac{\sqrt{Z}}{2}\right)^{-\alpha-\frac12}J_{\alpha+\frac12}\left(c\sqrt{Z}\right), \tag{6.93}$$

in the limit $N\to\infty$. It agrees with (6.86) because of its independence of the second index of the Jacobi polynomials.

We put this together with (6.82) - which does not change to leading order under the shift $2n+1\mapsto 2n$ - and (6.81) into (6.92), and again find exactly the same asymptotic formula (6.61) for $K_{\text{Edge}}(Z_1,Z_2)=\lim_{N\to\infty}K_N(z_1,z_2)/(4N^4)$. After the analysis of the previous subsection this universality is not unexpected. The corresponding limits to Hermiticity, strong non-Hermiticity and the bulk thus follow alike.

(10) Edge limit at the focus $z=-1$, non-symmetric case $(\alpha+1/2,-1/2)$:

Finally we use the scalings (6.79) and (6.84) to study the asymptotic behaviour of the kernel in the vicinity of $z=-1$. As in the previous subsection the coordinates (X_j,Y_j) are in the domain (6.58). For the asymptotic behaviour we now find

$$P_n^{(\alpha+\frac12,-\frac12)}\left(-1+\frac{Z}{2N^2}\right)\sim(-1)^nN^{-1/2}\left(\frac{\sqrt{Z}}{2}\right)^{1/2}J_{-1/2}\left(c\sqrt{Z}\right), \tag{6.94}$$

in the limit $N \to \infty$, due to (2.49) and (6.28). This formula (6.82) being also true for shifted index $2n+1 \mapsto 2n$, and (6.85) are put into the kernel (6.92). The result is

$$\begin{aligned}
K_{\mathrm{Edge}}(Z_1, Z_2) &= \lim_{N\to\infty} \frac{1}{4N^4} K_N(z_1, z_2) \\
&= \frac{(s/2)^{\alpha-\frac{1}{2}}}{4\sqrt{\pi}\Gamma(a+1)} \left(1 - \frac{2}{s^2}\left(|Z_1| - X_1\right)\right)^{\alpha/2} \left(1 - \frac{2}{s^2}\left(|Z_2| - X_2\right)\right)^{\alpha/2} \\
&\quad \times \left(\frac{\sqrt{Z_1 \bar{Z}_2}}{|Z_1 Z_2|}\right)^{\frac{1}{2}} \int_0^1 dc\, \frac{c^{\alpha+\frac{3}{2}}}{I_{\alpha+\frac{1}{2}}(cs)} J_{-\frac{1}{2}}\left(c\sqrt{Z_1}\right) J_{-\frac{1}{2}}\left(c\sqrt{\bar{Z}_2}\right) \\
&= \frac{(s/2)^{\alpha-\frac{1}{2}}}{2\pi^{3/2}\Gamma(a+1)} \left(1 - \frac{2}{s^2}\left(|Z_1| - X_1\right)\right)^{\alpha/2} \left(1 - \frac{2}{s^2}\left(|Z_2| - X_2\right)\right)^{\alpha/2} \\
&\quad \times |Z_1 Z_2|^{-1/2} \int_0^1 dc\, \frac{c^{\alpha+\frac{1}{2}}}{I_{\alpha+\frac{1}{2}}(cs)} \cos\left(c\sqrt{Z_1}\right) \cos\left(c\sqrt{\bar{Z}_2}\right). \qquad (6.95)
\end{aligned}$$

In the last step we used (6.38), expressing the J-Bessel functions through cosine. Once again this edge kernel is different from that in (6.61) in Subsection 6.3, with the influence of the edge clearly visible through the dependence on α.

In the Hermitian limit $s \to 0$. As before, in the Point (4) in Subsection 6.3 (6.45) leads to

$$\lim_{s\to 0} s\, K_{\mathrm{Edge}}(Z_1, Z_2) = \frac{1}{2} \int_0^1 dc\; c\; J_{-\frac{1}{2}}\left(c\sqrt{X_1}\right) J_{-\frac{1}{2}}\left(c\sqrt{X_2}\right), \qquad (6.96)$$

which agrees with (6.64) continued to $\alpha = -1$,

In the strong non-Hermiticity limit $s \to \infty$ we use the scalings (6.65) and the asymptotic relations (6.89) and (6.66). It follows that $\lim_{s\to\infty}(s^2/4)K_{\mathrm{Edge}}(Z_1, Z_2)$ is identical to the result in (6.69).

The bulk limit $h \to \infty$ with the scaling (6.70) can be treated along the same line as in the previous subsection, by using (6.90) and (6.66). We find exactly the same formula (6.75) for $K_{\mathrm{Bulk}}(\hat{z}_1, \hat{z}_2) = \lim_{h\to\infty} 4hK_{\mathrm{Edge}}(Z_1, Z_2)$. We again conjecture that a similar bulk asymptotic form holds for this model. Also for these polynomials all three limits lead back to known results.

7. Summary and Outlook

In this thesis, our journey has begun by searching for orthogonality relations for the classical Jacobi polynomials on an elliptic domain in the complex plane. This has been motivated to a large extent by the fact, that complex eigenvalues in RMT behave like particles in a 2D Coulomb Gas, and that if the underlying orthogonal polynomials are known, these polynomials ensure the integrability of the model and provide an exactly soluble 2D Coulomb gas, like in the elliptic Ginibre ensemble and the chiral Ginibre ensemble. These ensembles have, as associated planar polynomials, the classical Hermite and Laguerre polynomials, respectively. In this matter, our main findings are that the Gegenbauer polynomials provide an orthonormal basis for a weighted Bergman space of the ellipse. The same holds true for a subfamily of non-symetric Jacoby polynomials, being orthonormal bases for its corresponding weighted Bergman space. Also, inspired by the random matrix model GUE with an external source, we have provided an extension for Hermite polynomials as planar multiple orthogonal polynomials, we believe that this orthogonality relations can illuminate the path to find a random matrix model in the plane, under the influence of an external field. A matrix model with an external source in the plane has not been found yet due to the lack of the corresponding group integral.

Once the polynomials were found, this has allowed us to introduce new families of exactly soluble 2D Coulomb gases. In the analysis of local fluctuations of the correlation kernel, we have found that in the large argument limit they share the same correlation functions. This phenomenon is known as universality. In this regard, we have found two important correlation kernels, which we have called *Strong-edge* and *Strong-bulk* kernels. The former is not new, however it had only appeared in the context where the confinement domain is the unit disk. This leads us to conjecture that as long as the boundary of the confinement domain is a *smooth, simple and closed curve*, the correlation kernel *Strong-edge* will always appear in the eigenvalues statistics at the edge. That is, *Strong-edge* plays the role as the complementary error function does in free boundary ensembles in RMT. We would like to comment that we have not been able to find such statement in the literature. We showed that the *Strong-bulk* correlation kernel was directly linked –in some limit– to the Ginibre kernel, in this matter, we also believe that the *Strong-bulk* kernel is universal. Here it is important to mention that our methods in the asymptotic analysis of the correlation functions were not rigorous, this is a pending matter, the rigorous mathematical analysis would support our conjectures.

Maybe at the end of this thesis we are left with more questions than answers, for instance, is it possible to find the underlying matrix space and its density function in such way that the families of 2D Coulomb gases that we have defined in this thesis are the corresponding joint probability distributions for their eigenvalues.

Based on our conjecture for the norms of the Jacobi polynomials with general parameters α and β, it is possible to generate a rigorous method to recover the integration measure in such way that these polynomials became a family of orthogonal polynomials on the ellipse. Perhaps the reader is familiar with the proof of the Selberg Integral on the line. There, the weight function is the Beta distribution, this function is symmetric in its parameters and this symmetry plays an important role in the proof. Moreover, the Beta distribution is the shift of the Jacobi weight, so we believe that the problem about general Jacobi polynomials on the ellipse deserves to be considered, its solution could lead to a general Selberg-type integral in the plane.

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