Center for Mathematical Economics
Working Papers

640

Oktober 2020

Approximating Convex Bodies by Cephoids

Joachim Rosenmüller

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

Abstract

We consider a class of comprehensive compact convex polyhedra called *Cephoids*. A Cephoid is a Minkowski sum of finitely many standardized simplices ("deGua Simplices"). The Pareto surface of Cephoids consists of certain translates of simplices, algebraic sums of subsimplices etc. The peculiar shape of such a Pareto surface raises the question as to how far results for Cephoids can be carried over to general comprehensive compact convex bodies by approximation.

We prove that to any comprehensive compact convex body Γ, given a set of finitely many points on its surface, there is a Cephoid Π that coincides with Γ in exactly these preset points. As a consequence, Cephoids are dense within the set of comprehensive compact convex bodies with respect to the Hausdorff metric.

Cephoids appear in Operations Research (Optimization [10], [3]), in Mathematical Economics (Free Trade theory [7], [8]), and in Cooperative Game Theory (the Maschler–Perles solution [6]).

More generally in the context of Cooperative Game Theory, the notion of a Cephoid serves to construct "solutions" or "values" for bargaining problems and non–side payment games ([9]).

Therefore, the results of this paper open up an avenue for the extension of solution concepts from Cephoids to general compact convex bodies.

1 Notations and Definitions

A Cephoid is a specific compact convex comprehensive polyhedron located within the nonnegative orthant of $\mathbb{R}^n$. We review some concepts described extensively in [10], see also [3], [4].

Let $\boldsymbol{I} := \{1, \ldots, n\}$ denote the set of coordinates of $\mathbb{R}^n$, the positive orthant is $\mathbb{R}^n_+ := \{\boldsymbol{x} = (x_1, \ldots, x_n) \mid x_i \geq 0 \, , i \in \boldsymbol{I}\}$. Let $\boldsymbol{e}^i$ denote the i^{th} unit vector of $\mathbb{R}^n$ and $\boldsymbol{e} := (1, \ldots, 1) = \sum_{i=1}^n \boldsymbol{e}^i \in \mathbb{R}^n$ the “diagonal” vector. For min and max of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we write

$$\boldsymbol{x} \wedge \boldsymbol{y} = (\min\{x_i, y_i\})_{(i \in \boldsymbol{I})} \;, \quad \boldsymbol{x} \vee \boldsymbol{y} = (\max\{x_i, y_i\})_{(i \in \boldsymbol{I})} \; . \tag{1.1}$$

The notation $\boldsymbol{CovH}\, \boldsymbol{C}$ is used to denote the ***convex hull*** of a subset $\boldsymbol{C}$ of $\mathbb{R}^n_+$. Given a vector $\boldsymbol{a} = (a_1, \ldots, a_n) > \boldsymbol{0} \in \mathbb{R}^n_+$, we consider the n multiples $\boldsymbol{a}^i := a_i \boldsymbol{e}^i \;\; (i \in \boldsymbol{I})$ of the unit vectors. The the set

$$\Delta^{\boldsymbol{a}} := \boldsymbol{CovH} \left\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\} \tag{1.2}$$

is the ***Standard Simplex*** or for short, the ***Simplex*** resulting from $\boldsymbol{a}$ (we use capitals in this context). Figure 1.1 represents a Simplex in $\mathbb{R}^3_+$.

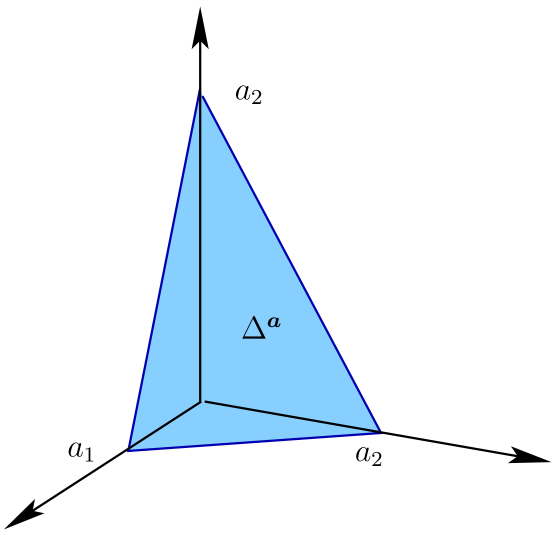

Figure 1.1: The Simplex in $\mathbb{R}^3_+$ generated by $\boldsymbol{a} = (a_1, a_2, a_3)$

Next, for $\boldsymbol{J} \subseteq \boldsymbol{I}$ we write $\mathbb{R}^n_{\boldsymbol{J}} := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\}$. Accordingly, we obtain the ***Standard Subsimplex*** or just ***Subsimplex***

$$\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Delta^{\boldsymbol{a}} \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\} = \Delta^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}} = \boldsymbol{CovH} \left\{\boldsymbol{a}^i \,\middle|\, i \in \boldsymbol{J}\right\} \; . \tag{1.3}$$

There is a second type of simplex we want to associate with a positive vector $\boldsymbol{a} \in \mathbb{R}^n_+$. This is the one spanned by the vectors $\boldsymbol{a}^i$ plus the vector $\boldsymbol{0} \in \mathbb{R}^n_+$, that is

$$\Pi^{\boldsymbol{a}} \; := \; \boldsymbol{CovH}\left\{\boldsymbol{0}, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\} \; . \tag{1.4}$$

In order to distinguish both types verbally we call $\Pi^{\boldsymbol{a}}$ the ***deGua Simplex*** associated to $\boldsymbol{a}$, paying homage to J.P. de Gua de Malves [1] who generalized the Pythagorean theorem for simplices of this type. Consistently we write, for any $\boldsymbol{J} \subseteq \boldsymbol{I}$ the corresponding ***deGua Subsimplex*** of $\Pi^{\boldsymbol{a}}$ as

$$\begin{aligned} \Pi^{\boldsymbol{a}}_{\boldsymbol{J}} \; &:= \; \{x \in \Pi^{\boldsymbol{a}} \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\} \\ &= \Pi^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}} \; = \; \boldsymbol{CovH}\left\{\{\boldsymbol{0}\}\{\boldsymbol{a}^i \,\middle|\, i \in \boldsymbol{J}\}\right\} \; . \end{aligned} \tag{1.5}$$

A set $A \subseteq \mathbb{R}^n_+$ is called ***comprehensive*** if, for any $\boldsymbol{x} \in A$ it contains all vectors $\boldsymbol{y} \in \mathbb{R}^n_+$ satisfying $\boldsymbol{y} \leq \boldsymbol{x}$ (inequalities between vectors to be interpreted coordinatewise). The ***comprehensive*** hull of a set $A \subseteq \mathbb{R}^n_+$ is given by

$$\boldsymbol{CmpH}\, A \; := \; \left\{\boldsymbol{y} \in \mathbb{R}^n_+ \,\middle|\, \exists \boldsymbol{x} \in A \; : \; \boldsymbol{y} \leq \boldsymbol{x}\right\} \; .$$

clearly we have also

$$\Pi^{\boldsymbol{a}} \; = \; \boldsymbol{CmpH}\, \Delta^{\boldsymbol{a}} \quad , \quad \Pi^{\boldsymbol{a}}_{\boldsymbol{J}} \; = \; \boldsymbol{CmpH}\, \Delta^{\boldsymbol{a}}_{\boldsymbol{J}} \; ,$$

and Figure 1.2 indicates the deGua Simplex $\Pi^{\boldsymbol{a}}$ generated by $\boldsymbol{a}$. All vectors below $\Delta^{\boldsymbol{a}}$ including the vector $\boldsymbol{0} \in \mathbb{R}^3_+$ are included.

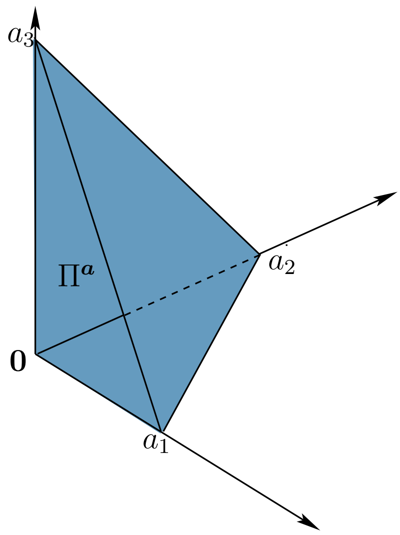

Figure 1.2: The deGua Simplex $\Pi^{\boldsymbol{a}}$; $\boldsymbol{a} = (a_1, a_2, a_3)$

In the terminology of Convex Analysis, $\Delta^{\boldsymbol{a}}$ is the ***maximal (outward) face*** of $\Pi^{\boldsymbol{a}}$. Here we prefer the MathEcon notation, calling $\Delta^{\boldsymbol{a}}$ the ***Pareto face*** of $\Pi^{\boldsymbol{a}}$.

A ***normal*** at/to some convex set $\boldsymbol{C}$ in some point $\bar{\boldsymbol{x}} \in \partial \boldsymbol{C}$ is a vector that generates a separating hyperplane. A vector that is a normal to some face $\boldsymbol{F}$ of a convex set $\boldsymbol{C}$ in *all* points of $\boldsymbol{F}$ is called normal to $\boldsymbol{F}$.

A deGua Simplex $\Pi^{\boldsymbol{a}}$ admits of a normal

$$\mathfrak{n}^{\boldsymbol{a}} \;:=\; \left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) \; .$$

to $\Delta^{\boldsymbol{a}}$. All other normals to $\Delta^{\boldsymbol{a}}$ are positive multiples of this one, i.e., the ***normal cone*** to $\Delta^{\boldsymbol{a}}$ is

$$\mathcal{N}^{\boldsymbol{a}} \;:=\; \{t\mathfrak{n}^{\boldsymbol{a}} \,|\, t > 0\} \; .$$

We refer to this situation saying that the normal of $\Delta^{\boldsymbol{a}}$ is "unique up to a multiple" or "essentially unique" etc.

The projection of $\mathfrak{n}^{\boldsymbol{a}}$ to $\mathbb{R}^n_{\boldsymbol{J}+}$ is denoted by $\mathfrak{n}^{\boldsymbol{a}}_{\boldsymbol{J}} \;:=\; \mathfrak{n}^{\boldsymbol{a}} \;{}_{|\,\mathbb{R}^n_{\boldsymbol{J}+}}$. The subface $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}}$ of the Pareto face admits of a normal cone $\mathcal{N}^{\boldsymbol{a}}_{\boldsymbol{J}}$ generated by the normals

$$\{\mathfrak{n}^{\boldsymbol{a}}_{\boldsymbol{J}'} \,|\, \boldsymbol{J} \subseteq \boldsymbol{J}' \subseteq \boldsymbol{I}\} \; .$$

Certain operations on convex sets are a standard in Convex Geometry. For two subsets $A, B \subseteq \mathbb{R}^n_+$ the ***algebraic*** or ***Minkowski*** sum is

$$A + B \;:=\; \{\boldsymbol{x} + \boldsymbol{y} \,|\, \boldsymbol{x} \in A,\; \boldsymbol{y} \in B\}$$

and for $\lambda \in \mathbb{R}_+$ the multiple of A is defined via

$$\lambda A \;:=\; \{\lambda \boldsymbol{x} \,|\, \boldsymbol{x} \in A\} \; .$$

If A and B are convex sets, then the sets $A+B$ and λA are also convex and if A and B are polytopes, so are $A+B$ and λA.

Now we are in the position to define the subject of this treatise. A ***Cephoid*** is a *Minkowski sum of deGua Simplices.* More precisely, using

$$\boldsymbol{K} \;:=\; \{1,\ldots,K\}$$

for some integer K, we have:

Definition 1.1. Let $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ denote a family of positive vectors and let

$$\Pi \;=\; \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \;=:\; \sum_{k\in\boldsymbol{K}} \Pi^{(k)} \tag{1.6}$$

be the Minkowski sum. Then Π is called a ***Cephoid***.

• ~~~~~~~ •

The surface of a polyhedron can be described by either a list of extremal points or by maximal faces. We focus on the *Pareto surface* of a Cephoid. For completeness, we provide the following

Definition 1.2. 1. A face $\boldsymbol{F}$ of a Cephoid Π is ***maximal*** if, for any face $\boldsymbol{F}^0$ of Π with $\boldsymbol{F} \subseteq \boldsymbol{F}^0$ it follows that $\boldsymbol{F} \;=\; \boldsymbol{F}^0$ is true.

2. The (***outward***) or ***Pareto surface*** of a compact convex set (specifically: of a Cephoid Π) is the set

$$(1.7) \qquad \partial\Pi \;:=\; \{\boldsymbol{x} \in \Pi \,|\; \nexists \boldsymbol{y} \in \Pi, \nexists i \in \boldsymbol{I} \;:\; \boldsymbol{y} \geq \boldsymbol{x},\; y_i > x_i\,\}\,.$$

3. The points of the Pareto surface are called ***Pareto efficient***.

4. Maximal faces in the Pareto surface are called ***Pareto faces***.

• ~~~~~~~ •

Clearly, $\Delta^{\boldsymbol{a}}$ is the only Pareto face of $\Pi^{\boldsymbol{a}}$; similarly for $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}}$ and $\Pi^{\boldsymbol{a}}_{\boldsymbol{J}}$.

The vector $\mathbf{0}$ is always an extremal point of a Cephoid in $\mathbb{R}^n$ but it is not Pareto efficient. All other extremal points of a Cephoid are Pareto efficient and referred to as ***vertices***.

Definition 1.3. Let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid and let $i \in \boldsymbol{I}$. Define

$$(1.8) \qquad \Pi^{(-i)} \;:=\; \Pi \cap \mathbb{R}_{\boldsymbol{I}\setminus\{i\}}\;.$$

Then $\Pi^{(-i)}$ constitutes a maximal face of Π but not a Pareto face. $\Pi^{(-i)}$ is called the ***i–face*** of Π.

• ~~~~~~~ •

Indeed, $\Pi^{(-i)}$ is clearly a maximal face but not located in the Pareto surface as not all points of $\Pi^{(-i)}$ are Pareto efficient (Definition 1.2). All maximal faces of a Cephoid Π are either Pareto faces or intersections of Π with some $\mathbb{R}_{\boldsymbol{I}\setminus\{i\}}$ as in (1.8). On the other hand, $\Pi^{(-i)} \subseteq \mathbb{R}^n_{\boldsymbol{I}\setminus\{i\}+}$ is a Cephoid in its own right, generated by the family of vectors

$$\left\{\boldsymbol{a}^{(k)}_{\boldsymbol{I}\setminus\{i\}}\right\}_{k \in \boldsymbol{K}}\;.$$

We also introduce a notation for the reduction of a Cephoid in members of the family as follows.

Definition 1.4. Let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid and let $k \in \boldsymbol{K}$. Define

$$(1.9) \qquad \Pi^{[-k]} = \sum_{k \in \boldsymbol{K}\setminus\{k\}} \Pi^{\boldsymbol{a}^{(k)}}$$

Then $\Pi^{[-k]}$ is called the ***k–missing*** Cephoid to Π. This is a Cephoid in $\mathbb{R}^n_+$.

• ~~~~~~~ •

The following well known theorem (see e.g. EWALD [2] or PALLASCHKE–URBAŃSKI [5]) is basic tool for testing Pareto efficiency of a sum of polyhedra.

Theorem 1.5. *Let A and B be compact convex sets and let $\boldsymbol{x} \in A$ and $\boldsymbol{y} \in B$ be Pareto efficient vectors of A and B respectively. Then $\boldsymbol{x} + \boldsymbol{y}$ is a Pareto efficient vector in $A + B$ if and only if the normal cone of A in $\boldsymbol{x}$ and the normal cone of B in $\boldsymbol{y}$ have a nonempty intersection. That is, if and only if A and B admit of a joint normal in $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.*

On the other hand, every extremal point $\boldsymbol{z}$ of $A + B$ is the sum $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ of two extremal points $\boldsymbol{x} \in A$ and $\boldsymbol{y} \in B$, such that the intersection of the normal cones of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ has a nonempty intersection.

• ∼∼∼ ◦ ∼∼∼ •

Similarly, we have for faces or extremal sets of two convex and compact sets the following

Theorem 1.6. *Let A and B be compact convex sets and let $\boldsymbol{F}^1 \in A$ and $\boldsymbol{F}^2 \in B$ be faces of A and B respectively. Then $\boldsymbol{F}^1 + \boldsymbol{F}^2$ is a face of $A + B$ if and only if the normal cone of $\boldsymbol{F}^1$ with respect to A and the normal cone of $\boldsymbol{F}^2$ with respect to B have a nonempty intersection. That is, if and only if A and B admit of a joint normal in $\boldsymbol{F}^1$ and $\boldsymbol{F}^2$ respectively.*

On the other hand, every face $\boldsymbol{F}$ of $A + B$ is the sum $\boldsymbol{F} = \boldsymbol{F}^1 + \boldsymbol{F}^2$ of two faces $\boldsymbol{F}^1$ of A and $\boldsymbol{F}^2$ of B, such that the intersection of normal cones of $\boldsymbol{F}, \boldsymbol{F}^1, \boldsymbol{F}^2$ have a nonempty intersection.

• ∼∼∼ ◦ ∼∼∼ •

2 Examples: Windmills

We recall the idea of the "canonical representation" of a Cephoid and offer some examples.

The "canonical representation" (see CHAPTER 2 of [10]) is a consistent bijection of the Pareto surface of a Cephoid Π onto a suitable multiple of Δ^{e}. This is done in a way such that the partially ordered ("PO") set of the Pareto faces is preserved. The details are found in [10].

Example 2.1. We present a Cephoid (or rather a family or type of Cephoids) called the Windmill. It appears within the classifications offered in CHAPTER 2 of [10].

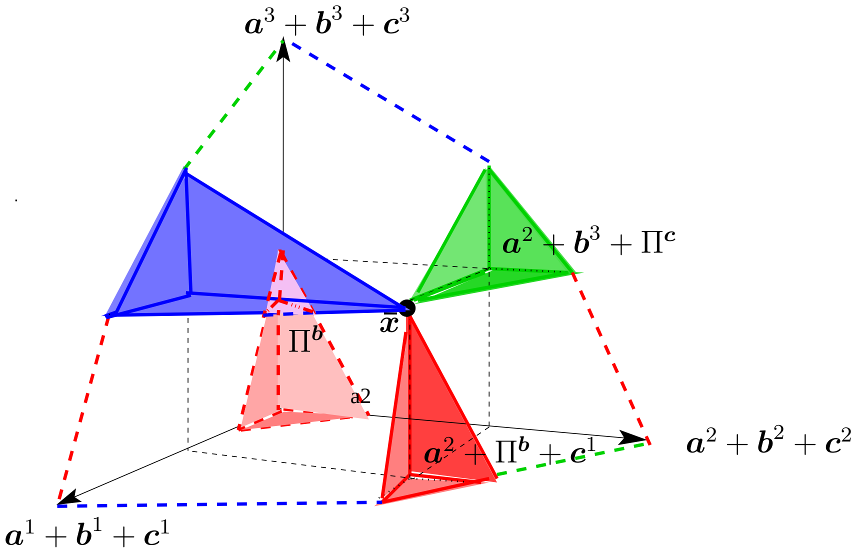

Figure 2.1: The Windmill

Figure 2.1 shows a Cephoid

$$\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}} + \Pi^{\boldsymbol{c}} \ ;$$

we sketch $\boldsymbol{a}$ in blue, $\boldsymbol{b}$ in red, and $\boldsymbol{c}$ in green. The deGua Simplex $\Pi^{\boldsymbol{b}}$ (red) is located in the origin. Its translate that appears on $\partial\Pi$ is

$$\boldsymbol{a}^2 + \Pi^{\boldsymbol{b}} + \boldsymbol{c}^1 \ .$$

Thus, the Pareto surface $\partial\Pi$ is indicated in Figure (2.1). This Cephoid has exactly one positive vertex which is

$$\bar{\boldsymbol{x}} \ = \ \boldsymbol{a}^2 + \boldsymbol{b}^3 + \boldsymbol{c}^1$$

The Canonical Representation of the Windmill is depicted in Figure 2.2. This figure reflects the PO set of the Windmill which consists of three deGua simplices and three rhombi; all ingredients being defined on the multiple $9\Delta^{e}$ of the unit Simplex.

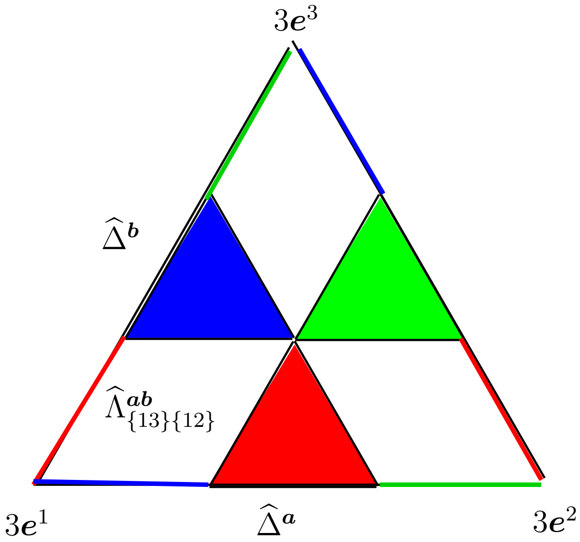

Figure 2.2: The Windmill: Canonical Representation

Here the size of the various Pareto faces is irrelevant, it is just the relative location that matters. The canonical representation reflects the structure of the PO set. It allows for a classification as there are only finitely many possible arrangements of the deGua Simplices and the rhombi.

∘ ∼∼∼∼∼∼∼ ∘

Example 2.2. The "inductive type" is a (family of) Cephoid(s) also described in the classification of [10].

The Cephoid *IND* is a sum of three deGua simplices, but the Pareto surface does not resemble a windmill (see Figure 2.3). Figure 2.4 shows the canonical representation of *IND*.

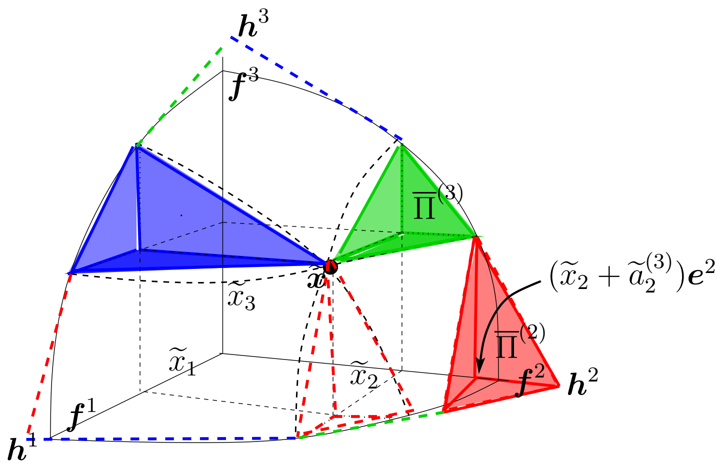

Figure 2.3: The Pareto set of *IND*

The central vertex (the unique positive one) of *IND* is denoted $\widetilde{\boldsymbol{x}}$. Obviously it writes

$$\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{a}}^{(1)2} + \widetilde{\boldsymbol{a}}^{(2)3} + \widetilde{\boldsymbol{a}}^{(3)1} \ .$$

The three translates of the deGua simplices involved are not ordered in a "cyclic" way. E.g., the blue deGua Simplex (i.e. $\Delta^{\widetilde{\boldsymbol{a}}^{(1)}}$) is translated to the Pareto surface as

$$\Delta^{\widetilde{\boldsymbol{a}}^{(1)}} + \widetilde{\boldsymbol{a}}^{(2)3} + \widetilde{\boldsymbol{a}}^{(3)1} \ = \ \Delta^{\widetilde{\boldsymbol{a}}^{(1)}}_{\{123\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(2)}}_{\{3\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(3)}}_{\{1\}} \ . \tag{2.1}$$

The green deGua Simplex (i.e. $\Delta^{\widetilde{\boldsymbol{a}}^{(3)}}$) is translated to

$$\widetilde{\boldsymbol{a}}^{(1)2} + \widetilde{\boldsymbol{a}}^{(2)3} + \Delta^{\widetilde{\boldsymbol{a}}^{(3)}} \ = \ \Delta^{\widetilde{\boldsymbol{a}}^{(1)}}_{\{2\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(2)}}_{\{3\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(3)}}_{\{123\}} \ . \tag{2.2}$$

This is the Pareto face of the translated version

$$\overline{\Pi}^{(3)} \ = \ \widetilde{\boldsymbol{a}}^{(1)2} + \widetilde{\boldsymbol{a}}^{(2)3} + \Pi^{\widetilde{\boldsymbol{a}}^{(3)}} \ . \tag{2.3}$$

However, the red deGua Simplex (i.e., $\Delta^{\widetilde{\boldsymbol{a}}^{(2)}}$) is translated to be

$$\Delta^{\widetilde{\boldsymbol{a}}^{(1)2}} + \Delta^{\widetilde{\boldsymbol{a}}^{(2)}} + \widetilde{\boldsymbol{a}}^{(3)2} \ . \tag{2.4}$$

Also, the two rhombi in the lower third of the sketch are

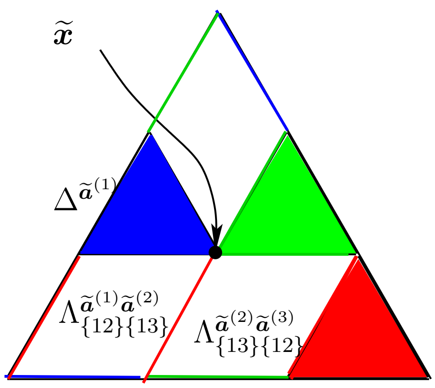

Figure 2.4: An Inductive Type – Canonical Representation

$$\Lambda^{\widetilde{\boldsymbol{a}}^{(1)}\widetilde{\boldsymbol{a}}^{(2)}}_{\{12\}\{13\}} = \Delta^{\widetilde{\boldsymbol{a}}^{(1)}}_{\{12\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(2)}}_{\{13\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(3)}}_{\{1\}} \text{ and } \Lambda^{\widetilde{\boldsymbol{a}}^{(2)}\widetilde{\boldsymbol{a}}^{(3)}}_{\{13\}\{12\}} = \Delta^{\widetilde{\boldsymbol{a}}^{(1)}}_{\{2\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(2)}}_{\{13\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(3)}}_{\{12\}} \ . \tag{2.5}$$

The point common to (2.1), (2.2),(2.5) is the unique positive vertex

$$\Delta^{\widetilde{\boldsymbol{a}}^{(1)}}_{\{2\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(2)}}_{\{3\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(3)}}_{\{1\}} \ = \ \widetilde{\boldsymbol{a}}^{(1)2} + \widetilde{\boldsymbol{a}}^{(2)3} + \widetilde{\boldsymbol{a}}^{(3)1} \ = \ \widetilde{\boldsymbol{x}} \ .$$

◦ ∼∼∼∼∼∼∼ ◦

3 The Pseudo Windmill

Let Γ be a smooth compact convex comprehensive set with boundary $\partial\Gamma$. We write $t_i := \max\{t \,|\, t\boldsymbol{e}^i \in \partial\Gamma\}$ $(i \in \boldsymbol{I})$; thus $\boldsymbol{f}^i = t_i\boldsymbol{e}^i$ is the maximal point located in Γ on the $i-$ axis. We denote these points in the tradition of Bargaining Theory:

Definition 3.1. The vectors $\boldsymbol{f}^i$ $(i \in \boldsymbol{I})$ are the ***bliss points*** of Γ.

• ~~~~~~~ •

Definition 3.2. Let $\widehat{\boldsymbol{x}} \in \Gamma$. Then

$$\Gamma_{\widehat{\boldsymbol{x}}} := (\Gamma - \widehat{\boldsymbol{x}}) \cap \mathbb{R}^3_+ \tag{3.1}$$

is called the ***calotte*** defined by $\widehat{\boldsymbol{x}}$. Define $\widehat{\boldsymbol{a}} = (\widehat{a}_1, \ldots, \widehat{a}_n)$ such that $\widehat{a}_i\boldsymbol{e}^i$ $(i \in \boldsymbol{I})$ are the bliss points of the calotte $\Gamma_{\widehat{\boldsymbol{x}}}$. Then the deGua Simplex $\Pi^{\widehat{\boldsymbol{a}}}$ is the ***vane*** defined by $\widehat{\boldsymbol{x}}$ (or by $\Gamma_{\widehat{\boldsymbol{x}}}$).

• ~~~~~~~ •

Example 3.3. Figure 3.1 shows the convex body Γ with surface $\partial\Gamma$ and calotte $\Gamma_{\widehat{\boldsymbol{x}}}$ in three dimensions. The sketch shows the translated version $\widehat{\boldsymbol{x}} + \Gamma_{\widehat{\boldsymbol{x}}}$ so that the surface $\partial\Gamma_{\widehat{\boldsymbol{x}}}$ is imbedded in the surface $\partial\Gamma$. The vane $\Pi^{\widehat{\boldsymbol{a}}}$ is also indicated (dotted red lines), also transferred via $\widehat{\boldsymbol{x}}$ such that the translated vertices

$$\widehat{\boldsymbol{x}} + \widehat{a}_i\boldsymbol{e}^i \tag{3.2}$$

appear as points on $\partial\Gamma$.

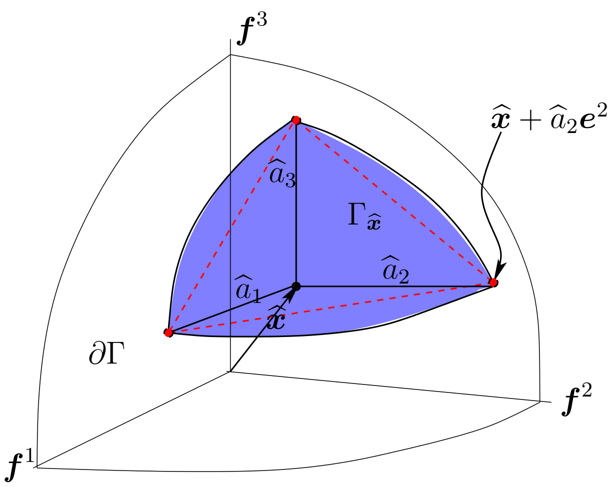

Figure 3.1: Transfering Calotte $\Gamma_{\widehat{\boldsymbol{x}}}$

◦ ~~~~~~~ ◦

Next, let $\widetilde{\boldsymbol{x}} \in \partial\Gamma$ be a point on the surface of Γ. Define

$$\widetilde{\boldsymbol{x}}^{(-i)} \;:=\; \widetilde{\boldsymbol{x}}\;\Big|_{\;\mathbb{R}^n_{\boldsymbol{I}\setminus\{i\}}}\;. \tag{3.3}$$

For $i \in \boldsymbol{I}$ consider the calotte

$$\Gamma_{\widetilde{\boldsymbol{x}}^{(-i)}} \tag{3.4}$$

with surface $\partial\Gamma_{\widetilde{\boldsymbol{x}}^{(-i)}}$. For $i \in \boldsymbol{I}$ the vane generated by this calotte has vertices

$$\widetilde{\boldsymbol{x}} - \widetilde{\boldsymbol{x}}^{(-i)} = \widetilde{x}_i \boldsymbol{e}^i \text{ as well as } \widetilde{a}_j \boldsymbol{e}^j \quad (j \in \boldsymbol{I} \setminus \{i\}) \tag{3.5}$$

Definition 3.4. 1. Let $\widetilde{\boldsymbol{x}} \in \partial\Gamma$. We denote $\partial\Gamma_{\widetilde{\boldsymbol{x}}^{(-i)}}$ to be the ***i–calotte*** of $\widetilde{\boldsymbol{x}}$.

2. The deGua Simplex $\widetilde{\Pi}^{(i)} = \Pi^{\widetilde{\boldsymbol{a}}^{(i)}}$ given by $\widetilde{\boldsymbol{a}} \;=\; \widetilde{\boldsymbol{a}}^{(i)}$ via

$$\widetilde{a}_i = \widetilde{x}_i \;, \quad \widetilde{a}_j \quad (j \in \boldsymbol{I} \setminus \{i\}) \tag{3.6}$$

is called the ***i–vane*** corresponding to $\widetilde{\boldsymbol{x}}$.

3. The Cephoid

$$\widetilde{\Pi} \;=\; \Pi^{(\widetilde{\boldsymbol{x}})} \;:=\; \sum_{i \in \boldsymbol{I}} \widetilde{\Pi}^{(i)} \tag{3.7}$$

is called the ***pseudo–windmill*** corresponding to $\widetilde{\boldsymbol{x}}$.

• ~~~~~~~ •

Remark 3.5. Writing the full notation, formula (3.6) reads for all $i \in \boldsymbol{I}$:

$$\widetilde{a}_i^{(i)} = \widetilde{x}_i \quad (i \in \boldsymbol{I}) \tag{3.8}$$

while $\widetilde{a}_j^{(i)}$ $(j \in \boldsymbol{I})$ is given by the blisspoints of the surface (3.4) and hence determined by the surface $\partial\Gamma_{\widetilde{\boldsymbol{x}}^{(i)}}$ (or $\partial\Gamma$ respectively). In particular we observe that

$$\widetilde{\boldsymbol{x}} \;=\; (\widetilde{x}_1, \ldots, \widetilde{x}_n) \;=\; (\widetilde{a}_1^{(1)}, \ldots, \widetilde{a}_n^{(n)}) \;=\; \sum_{i \in \boldsymbol{I}} \widetilde{\boldsymbol{a}}^{(i)i}\;. \tag{3.9}$$

holds true. Hence, $\widetilde{\boldsymbol{x}}$ is a sum of extremals of the deGua Simplices $\Pi^{\widetilde{\boldsymbol{a}}^{(i)}}$ $(i \in \boldsymbol{I})$. Consequently, $\widetilde{\boldsymbol{x}}$ is an extremal of the sum, i.e., the pseudo windmill $\widetilde{\Pi} \;=\; \Pi^{(\widetilde{\boldsymbol{x}})}$, *if and only if* there is a common normal to all the summands $\widetilde{\Pi}^{(i)}$ in the corresponding extremal $\widetilde{\boldsymbol{a}}^{(i)i}$. See Theorem 1.5, also for the present purpose see e.g., Theorem 1.5, CHAPTER I of [10]. This will be an essential detail within the following development.

∘ ~~~~~~~ ∘

Now consider the calotte $\Gamma_{\widetilde{\boldsymbol{x}}^{(-i)}}$ to be transferred back onto the surface $\partial\Gamma$ via $\widetilde{\boldsymbol{x}}^{(-i)}$. The bliss points are then transferred into the points

$$(3.10) \qquad \widetilde{\boldsymbol{x}} \text{ and } \widetilde{\boldsymbol{x}}^{(-i)} + \widetilde{a}_j \boldsymbol{e}^j \;=:\; \widetilde{\boldsymbol{x}}^{(-i+j)} \quad (j \in \boldsymbol{I}) \;.$$

The transferred deGua Simplices – the i–vanes – are

$$(3.11) \qquad \widetilde{\boldsymbol{x}}^{(-i)} + \widetilde{\Pi}^{(i)}$$

Superficially, transferring all $\widetilde{\Pi}^{(i)}$ via $\widetilde{\boldsymbol{x}}^{(-i)}$ onto the surface $\partial\Gamma$ suggests (in three dimensions) the shape of a "Windmill", cf. Example 2.1. We emphasize that this is a possible shape of the Pareto surface of the Cephoid

$$(3.12) \qquad \widetilde{\Pi} \;=\; \Pi^{(\widetilde{\boldsymbol{x}})} \;=\; \sum_{i \in \boldsymbol{I}} \widetilde{\Pi}^{(i)},$$

but not the only one. $\widetilde{\Pi}$ (in three dimensions) can have a different Pareto surface $\partial\widetilde{\Gamma}$; thus, Figure 3.2 (see below) may be misleading. This is why we call $\widetilde{\Pi}$ a "pseudo windmill". The following example sheds some light on this phenomenon.

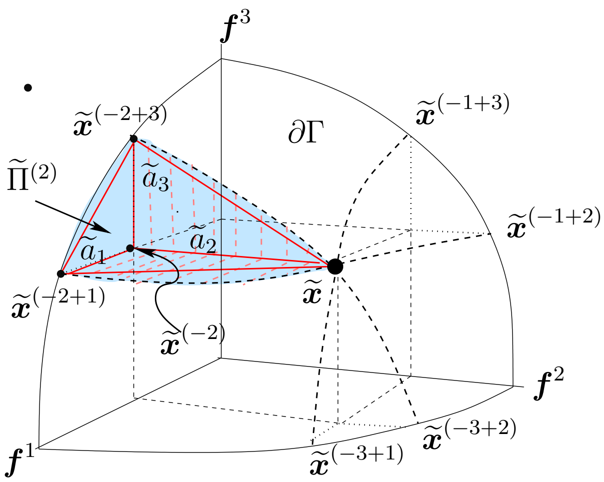

Figure 3.2: Constructing a i–Vane

Example 3.6. In three dimensions let $\widetilde{\boldsymbol{x}} \in \partial\Gamma$ and consider the three i–calottes generated together with the vanes $\widetilde{\Pi}^{(i)}$ $(i \in \boldsymbol{I})$. First, we focus on $\partial\Gamma_{\widetilde{\boldsymbol{x}}^{(-2)}}$, depicted in blue (Figure 3.2). The vane $\widetilde{\Pi}^{(2)} = \Pi^{\widetilde{\boldsymbol{a}}^{(2)}}$ is the deGua Simplex indicated by $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3)$. Here, $\widetilde{a}_2 = \widetilde{x}_2$ while $\widetilde{a}_1$ and $\widetilde{a}_2$ are determined by the calotte construction as boundary points of $\partial\Gamma$. $\widetilde{\boldsymbol{x}}^{(-2)}$ is the origin of $\Gamma_{\widetilde{\boldsymbol{x}}^{(-2)}}$.

In Figure 3.2, $\widetilde{\Pi}^{(2)}$ is transferred to the the surface $\partial\Gamma$ via $\widetilde{\boldsymbol{x}}^{(-2)}$, so one views

$$\widetilde{\boldsymbol{x}}^{(-2)} + \widetilde{\Pi}^{(2)} \;.$$

Note that $\widetilde{\boldsymbol{x}}$ is a vertex of $\widetilde{\boldsymbol{x}}^{(-2)} + \widetilde{\Pi}^{(2)}$. Moreover, as this Simplex is inscribed into $\partial\Gamma$, it follows that the normal $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ at $\partial\Gamma$ in $\widetilde{\boldsymbol{x}}$ is a normal to the deGua

Simplex $\widetilde{\Pi}^{(2)}$ in $\widetilde{\boldsymbol{a}}^2$. This holds true for the translated versions as well, that is, we have

$$\mathfrak{n}^{\widetilde{\boldsymbol{x}}} \quad \text{is a normal at} \quad \widetilde{\boldsymbol{x}}^{(-2)} + \widetilde{\Pi}^{(2)} \quad \text{in} \quad \widetilde{\boldsymbol{x}}\,. \tag{3.13}$$

Now this procedure is being performed for the calotte $\partial\Gamma_{\widetilde{\boldsymbol{x}}^{(-1)}}$ and $\partial\Gamma_{\widetilde{\boldsymbol{x}}^{(-3)}}$ as well resulting in two further deGua Simplices lifted to $\partial\Gamma$. The construction is depicted in Figure 3.3. It resembles the "Windmill" discussed in Example 2.1 of Section 2. Whether the construction ("lifting the vanes") does indeed result in an image of the sum $\widetilde{\Pi} \;=\; \Pi^{(\widetilde{\boldsymbol{x}})} = \sum_{i\in\{1,2,3\}} \widetilde{\Pi}^{(i)}$ depends on the various subsimplices making up the Pareto surface $\partial\widetilde{\Pi}$.

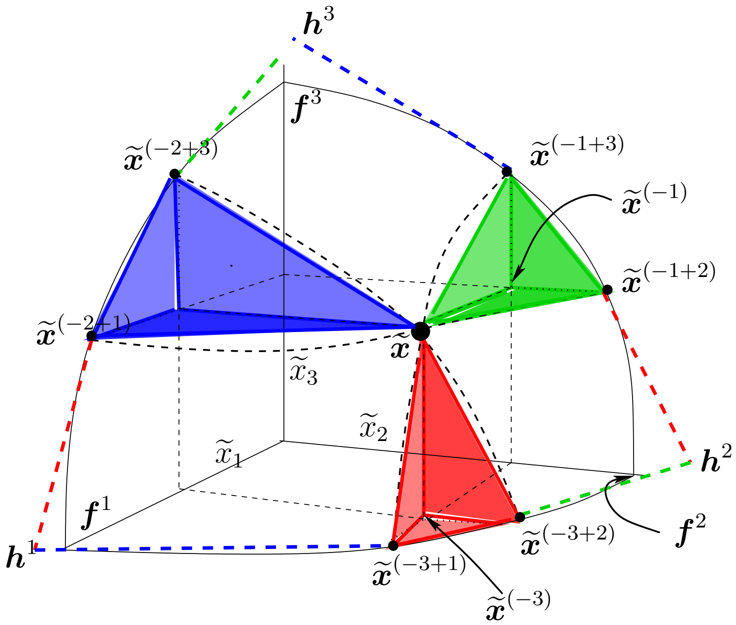

Figure 3.3: The Pseudo Windmill as a Windmill

Assuming for the moment that $\widetilde{\Pi}$ is a windmill (hence represented by Figure 3.3), we discuss the situation somewhat more in detail.

For example, the green Simplex $\widetilde{\boldsymbol{x}}^{(-1)} + \widetilde{\Pi}^{(1)}$ satisfies

$$(0, \widetilde{x}_2, \widetilde{x}_3) + \widetilde{\Pi}^{(1)} \;=\; (0, \widetilde{a}_2^{(2)}, \widetilde{a}_3^{(3)}) + \widetilde{\Pi}^{(1)} \;=\; \widetilde{\Pi}^{(1)} + \widetilde{\boldsymbol{a}}^{(2)2} + \widetilde{\boldsymbol{a}}^{(3)3}\,. \tag{3.14}$$

This means that it is a translate of $\widetilde{\Pi}^{(1)}$ via two extremals of $\widetilde{\Pi}^{(2)}$ and $\Pi^{(3)}$.

Analogously we have for the blue Simplex

$$\widetilde{\boldsymbol{x}}^{(-2)} + \widetilde{\Pi}^{(2)} \;=\; \widetilde{\boldsymbol{a}}^{(1)1} + \widetilde{\Pi}^{(2)} + \widetilde{\boldsymbol{a}}^{(3)3}\,. \tag{3.15}$$

and for the red one

$$\widetilde{\boldsymbol{x}}^{(-3)} + \widetilde{\Pi}^{(3)} \;=\; \widetilde{\boldsymbol{a}}^{(1)1} + \widetilde{\boldsymbol{a}}^{(2)2} + \widetilde{\Pi}^{(3)}\,. \tag{3.16}$$

All translates (3.14), (3.15), and (3.16) occur as summands of $\widetilde{\Pi}$. Obviously we have

$$\widetilde{\boldsymbol{x}} \;=\; (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \;=\; \widetilde{\boldsymbol{a}}^{(1)1} + \widetilde{\boldsymbol{a}}^{(2)2} + \widetilde{\boldsymbol{a}}^{(3)3}\,. \tag{3.17}$$

holds true. Thus, $\widetilde{\boldsymbol{x}}$ is a sum of three extremals of the deGua Simplices summing up to $\widetilde{\Pi}$. Moreover, according to what was said in (3.13), we observe that $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ is a normal at $\widetilde{\boldsymbol{x}}^{(-i)} + \widetilde{\Pi}^{(i)}$ $(i = 1, 2, 3)$ in $\widetilde{\boldsymbol{x}}$. This implies that actually

$$\begin{aligned} &(a)\ \widetilde{\boldsymbol{x}} \quad \text{is a vertex of } \widetilde{\Pi}, \\ &(b)\ \mathfrak{n}^{\widetilde{\boldsymbol{x}}} \quad \text{is a normal to } \widetilde{\Pi} \text{ in the vertex } \widetilde{\boldsymbol{x}}\ . \end{aligned} \tag{3.18}$$

∘ ∼∼∼∼∼∼∼ ∘

Accordingly, we prove the decisive argument to be true in general: $\widetilde{\boldsymbol{x}}$ is a vertex of the pseudo windmill $\Pi^{(\widetilde{\boldsymbol{x}})}$ and $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ is a common normal to this Cephoid and to Γ in the common point $\widetilde{\boldsymbol{x}}$. The actual shape of $\partial\widetilde{\Pi}$ (windmill or not) is not important.

Theorem 3.7 (***The Central Vertex Theorem***)**.**
Let $\widetilde{\Pi} = \Pi^{(\widetilde{\boldsymbol{x}})} = \sum_{i\in \boldsymbol{I}} \widetilde{\Pi}^{(i)}$ be the pseudo windmill generated by $\widetilde{\boldsymbol{x}}$. Then $\widetilde{\boldsymbol{x}}$ is a vertex of $\Pi^{(\widetilde{\boldsymbol{x}})}$ admitting the normal $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$. Thus, $\widetilde{\boldsymbol{x}}$ is the unique vertex of $\Pi^{(}\widetilde{\boldsymbol{x}})$ with positive coordinates.

• ∼∼∼ ∘ ∼∼∼ •

Proof: The last statement follows from non–degeneracy. We claim it without offering a proof as it is not relevant in the present context.

For $i \in \boldsymbol{I}$ let

$$\widetilde{\mathcal{K}}^i = \mathcal{K}^{\widetilde{\boldsymbol{a}}^{(i)i}}_{\widetilde{\Pi}^{(i)}} \tag{3.19}$$

denote the normal cone to $\widetilde{\Pi}^{(i)}$ in $\widetilde{\boldsymbol{a}}^{(i)i}$. The normal cone is the same for any translation of $\widetilde{\Pi}^{(i)}$. Therefore $\widetilde{\mathcal{K}}^i$ is as well the normal cone to

$$\sum_{j\in i\in \boldsymbol{I}\setminus\{i\}} \widetilde{\boldsymbol{a}}^{(j)j} + \widetilde{\Pi}^{(i)} \quad \text{in} \quad \sum_{j\in \boldsymbol{I}} \widetilde{\boldsymbol{a}}^{(j)j} = \widetilde{\boldsymbol{x}}\ ; \tag{3.20}$$

the last equation following from Remark 3.5, Formula (3.8).

The translate $\sum_{j\in i\in \boldsymbol{I}\setminus\{i\}} \widetilde{\boldsymbol{a}}^{(j)j} + \widetilde{\Pi}^{(i)}$ is inscribed into Γ and has exactly n points common with $\partial\Gamma$; these are

$$\begin{gathered} \sum_{j\in i\in \boldsymbol{I}\setminus\{i\}} \widetilde{\boldsymbol{a}}^{(j)j} + \widetilde{\boldsymbol{a}}^{(i)l} (l \in i \in \boldsymbol{I} \setminus \{i\}) \\ \text{and} \\ \sum_{j\in \boldsymbol{I}} \widetilde{\boldsymbol{a}}^{(j)j} = \widetilde{\boldsymbol{x}} \end{gathered} \tag{3.21}$$

Consequently, $\widetilde{\mathcal{K}}^i$ contains the normal $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ to $\partial\Gamma$ in $\widetilde{\boldsymbol{x}}$ for all $i \in \boldsymbol{I}$. As a result (e.g., Theorem 1.5, CHAPTER I of [10]),

$$\widetilde{\boldsymbol{x}} = \sum_{j\in \boldsymbol{I}} \widetilde{\boldsymbol{a}}^{(j)j} \tag{3.22}$$

is an extremal of $\widetilde{\Pi}$. In addition, as every sum of n simplices in $\mathbb{R}^n_+$ has exactly one vertex with positive coordinates, $\widetilde{\boldsymbol{x}}$ is precisely this vertex.

q.e.d.

The normal $\widetilde{\mathfrak{n}} = \mathfrak{n}^{(\widetilde{\boldsymbol{x}})}$ is common to all vertices $\widetilde{\boldsymbol{a}}^{(i)i}$ of the vanes $\widetilde{\Pi}^{(i)}$ $(i \in \boldsymbol{I})$. We note the relation to any other deGua simplex (a candidate for the vanes of other pseudo windmills) as follows.

Let $\widetilde{\boldsymbol{x}} \in \partial\Gamma$ and let $\Pi^{(0)} = \Pi^{\boldsymbol{a}^{(0)}}$ be a de Gua Simplex. Let $\widetilde{\mathfrak{n}} = \mathfrak{n}^{(\widetilde{\boldsymbol{x}})}$ denote the normal in $\widetilde{\boldsymbol{x}}$ at $\widetilde{\Pi}$ and let $\mathcal{K}^{(0)j}$ be the normal cone at $\Delta^{(0)}$ in the vertex $\boldsymbol{a}^{(0)j}$. Define

$$\boldsymbol{I}^{(0)}_{\{\widetilde{\boldsymbol{x}}\}} := \left\{ j \in \boldsymbol{I} \,\middle|\, \widetilde{\mathfrak{n}} \in \mathcal{K}^{(0)j} \right\} \ . \tag{3.23}$$

That is, we collect those vertices $\boldsymbol{a}^{(0)j}$ which allow for $\widetilde{\mathfrak{n}}$ as a normal in $\widetilde{\boldsymbol{x}}$ to $\widetilde{\Delta}$.

Then we have

Lemma 3.8. Let $\widetilde{\boldsymbol{x}} \in \partial\Gamma$. Let $\widetilde{\Pi} = \Pi^{(\widetilde{\boldsymbol{x}})}$ be the pseudo windmill generated by $\widetilde{\boldsymbol{x}}$ and let $\Pi^{(0)}$ be a de Gua Simplex. Then

$$\boldsymbol{I}^{(0)}_{\{\widetilde{\boldsymbol{x}}\}} \neq \emptyset \tag{3.24}$$

holds true.

• ~~~∘~~~ •

Proof:

This is verified by a standard argument, cf. Theorem 4.8, CHAPTER III of [10]. The normal cones $\mathcal{K}^{(0)j}$ $(j \in \boldsymbol{I})$ span the full $\mathbb{R}^n_+$.

q.e.d.

Based on these considerations we observe that we can identify further vectors that are common to a pseudo windmill and the surface $\partial\Gamma$.

Theorem 3.9. *Let $\widetilde{\boldsymbol{x}} \in \partial\Gamma$. Let $\widetilde{\Pi} = \Pi^{(\widetilde{\boldsymbol{x}})} = \sum_{i\in\boldsymbol{I}} \widetilde{\Pi}^{(i)}$ be the pseudo windmill generated by $\widetilde{\boldsymbol{x}}$. Then, for $i \in \boldsymbol{I}$ there exists $l \in \boldsymbol{I} \setminus \{i\}$ such that*

1. *$\widetilde{\boldsymbol{x}}^{(-i+l)}$ is a vertex of $\widetilde{\Pi}$.*
2. *$\widetilde{\boldsymbol{x}}^{(-i+l)} \in \partial\Gamma^{(-i)} = \partial\Gamma \cap \{\boldsymbol{x} \,|\, x_i = 0\}$.*
3. *$\widetilde{\mathfrak{n}}\,\big|_{\,\boldsymbol{I}\setminus\{i\}}$ is a normal to $\widetilde{\Pi}$ in $\widetilde{\boldsymbol{x}}^{(-i+l)}$.*

That is, $\partial\Gamma$ and $\widetilde{\Pi}$ have at least one additional point in common on each boundary $\partial\Gamma^{(-i)}$. Together with $\widetilde{\boldsymbol{x}}$ we, therefore, find indeed $n+1$ common points of $\partial\Gamma$ and $\widetilde{\Pi}$ that are extremal for both convex bodies involved. In particular, if $\widetilde{\Pi}$ is a windmill, then all vertices of the translated deGua simplices $\widetilde{\Pi}^{(i)}\ \ (i \in \boldsymbol{I})$ are located on $\partial\Gamma$.

Proof: For $i \in \boldsymbol{I}$ consider the projection $\widetilde{\boldsymbol{x}}^{(-i)} \ := \ \widetilde{\boldsymbol{x}}_{\ |\ \boldsymbol{I}-\{i\}}$. Then

$$\widetilde{\boldsymbol{x}}^{(-i)} = \sum_{l\in\boldsymbol{I}\setminus\{i\}} \boldsymbol{a}^{(l)l} \ . \tag{3.25}$$

The vector $\widetilde{\mathfrak{n}}^{(-i)} \ := \ \mathfrak{n}^{\widetilde{\boldsymbol{x}}}{}_{\ |\ \boldsymbol{I}\setminus\{i\}}$ is a normal in $\boldsymbol{a}^{(l)l}$ to $\Delta^{\boldsymbol{a}^{(l)}}$ for all $l \in \boldsymbol{I}\setminus\{i\}$, hence it is a normal in $\widetilde{\boldsymbol{x}}^{(-i)}$ to the Cephoid

$$\sum_{l\in\boldsymbol{I}\setminus\{i\}} \Delta^{\boldsymbol{a}^{(l)}}{}_{\ |\ \boldsymbol{I}-\{i\}} \tag{3.26}$$

The normal cones $\mathcal{K}^{(i)l}$ $(l \in \boldsymbol{I}\setminus\{i\})$ in $\boldsymbol{a}^{(i)l}$ at $\Delta^{\boldsymbol{a}^{(i)}}$ span the full $\mathbb{R}^n_{\boldsymbol{I}\setminus\{i\}}$, i.e.,

$$\boldsymbol{I}^{(-i)}_{\widetilde{\boldsymbol{x}}} \ := \ \{l \,\big|\, l \in \boldsymbol{I}\setminus\{i\}, \widetilde{\mathfrak{n}}^{(-i)} \in \mathcal{K}^{(i)l}\} \neq \emptyset \ . \tag{3.27}$$

Therefore, any vector $\boldsymbol{c}$ of the convex hull

$$\boldsymbol{C}^{(-i)}_{\widetilde{\boldsymbol{x}}} \ := \ \boldsymbol{CovH}\{\boldsymbol{a}^{(i)l}\ (l \in \boldsymbol{I}^{(-i)}_{\widetilde{\boldsymbol{x}}})\} \tag{3.28}$$

admits of a joint normal with all vectors $\boldsymbol{a}^{(l)l}$ $(l \in \boldsymbol{I}\setminus\{i\})$. Hence, in view of (3.25) we find that

$$\widetilde{\boldsymbol{x}}^{(-i)} + \boldsymbol{c} \ = \ \sum_{l\in\boldsymbol{I}\setminus\{i\}} \boldsymbol{a}^{(l)l} + \boldsymbol{c} \tag{3.29}$$

is Pareto efficient in

$$\sum_{l\in\boldsymbol{I}\setminus\{i\}} \Delta^{\boldsymbol{a}^{(l)}}{}_{\ |\ \boldsymbol{I}-\{i\}} + \Delta^{\boldsymbol{a}^{(i)}}{}_{\ |\ \{\boldsymbol{I}-\{i\}\}} \ = \ \widetilde{\Pi}_{\ |\ \{\boldsymbol{I}-\{i\}\}} \ = \ \widetilde{\Pi}^{(-i)} \ . \tag{3.30}$$

In particular, the extremals of $\boldsymbol{C}^{(-i)}_{\widetilde{\boldsymbol{x}}}$ yield the vectors

$$\widetilde{\boldsymbol{x}}^{(-i+l)} \ = \ \widetilde{\boldsymbol{x}}^{(-i)} + \boldsymbol{a}^{(i)l} \ \ (l \in \boldsymbol{I}^{(-i)}_{\widetilde{\boldsymbol{x}}}) \ . \tag{3.31}$$

All of these vectors are located on $\partial\Gamma^{(-i)}$ by construction. Clearly $\widetilde{\mathfrak{n}}^{(-i)}$ is normal to $\widetilde{\Pi}^{(-i)}$ and hence to $\widetilde{\Pi}$ in all of these vectors.

q.e.d.

Remark 3.10. Based on the above development we can now point out the possible shape of a pseudo windmill in 3 dimensions. Here, $\widetilde{\Pi}$ resembles a different type of a "sum of three" Cephoid in $\mathbb{R}^3_+$.

We return to Example 2.2, the "Inductive Type" *IND* (also described in [10]). This type of Cephoid in three dimensions is also a candidate for $\widetilde{\Pi}$, consistent with Theorem 3.7.

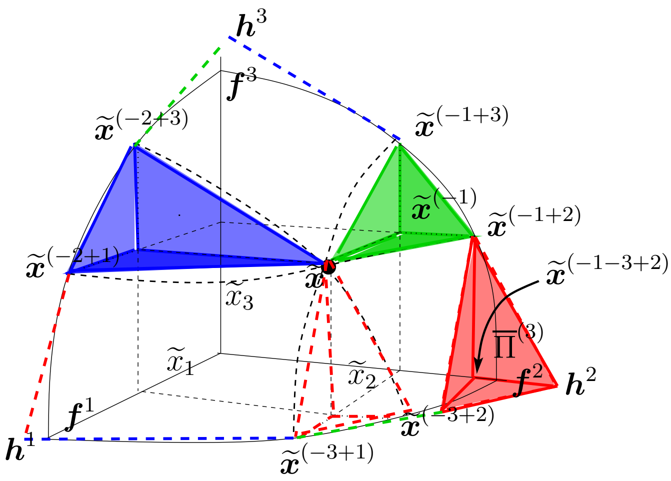

Figure 3.4: The Pseudo Windmill as *IND*

Compare Figure 3.4 which copied from Figure 2.3. The vector $\widetilde{\boldsymbol{x}}$ is the central vertex of *IND* but not all vectors $\widetilde{\boldsymbol{x}}^{(-i+l)}$ are vertices of $\widetilde{\Pi}$. Recall that the central vertex is

$$\widetilde{\boldsymbol{x}} \;= \Delta^{\widetilde{\boldsymbol{a}}^{(1)}}_{\{2\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(2)}}_{\{3\}} + \Delta^{\widetilde{\boldsymbol{a}}^{(3)}}_{\{1\}}$$

If we consider the translates located on the Pareto surface, then this vertex is common to the two rhombi

$$\Lambda^{\widetilde{\boldsymbol{a}}^{(1)}\widetilde{\boldsymbol{a}}^{(2)}}_{\{12\}\{13\}} \text{ and } \Lambda^{\widetilde{\boldsymbol{a}}^{(2)}\widetilde{\boldsymbol{a}}^{(3)}}_{\{13\}\{12\}}$$

and the two deGua Simplices

$$\Delta^{\widetilde{\boldsymbol{a}}^{(1)}}_{\{123\}} \text{ and } \Delta^{\widetilde{\boldsymbol{a}}^{(3)}}_{\{123\}} \;.$$

It is not a vertex of the translate of (the red deGua Simplex) $\Delta^{\widetilde{\boldsymbol{a}}^{(2)}}$.

Within the framework of Theorem 3.9 it is seen that $\boldsymbol{I}^{(-3)}_{\widetilde{\boldsymbol{x}}} \;= \{1\}$ and thus $\widetilde{\boldsymbol{x}}^{(-3+1)}$ is the common vertex of $\partial\Gamma$ and $\widetilde{\Pi}$ located on $\partial\Gamma^{(-3)}$.

◦ ∼∼∼∼∼∼∼ ◦

In what follows we adapt the construction of a pseudo windmill to a "local windmill", that is, a pseudo windmill defined w.r.t. a calotte $\Gamma_{\widehat{\boldsymbol{x}}}$ for some $\widehat{\boldsymbol{x}} \in \partial\Gamma$.

Definition 3.11. Let $\widehat{\boldsymbol{x}} \in \Gamma$, $\widetilde{\boldsymbol{x}} \in \partial\Gamma$ be such that $\widehat{\boldsymbol{x}} < \widetilde{\boldsymbol{x}}$. The ***local windmill*** $\Pi^{(\widetilde{\boldsymbol{x}})}_{\widehat{\boldsymbol{x}}}$ is the pseudo windmill generated by $\widetilde{\boldsymbol{x}} - \widehat{\boldsymbol{x}}$ via Definition 3.4 with respect to the calotte $\Gamma_{\widehat{\boldsymbol{x}}}$.

• ∼∼∼∼∼∼∼ •

Figure 3.5 is depicting the following procedure. First we transfer the calotte $\Gamma_{\widehat{\boldsymbol{x}}} = \Gamma \cap \{\boldsymbol{x} \mid \; \geq \widehat{\boldsymbol{x}}\}$ via $\widehat{\boldsymbol{x}}$ into the origin. Then we construct the pseudo

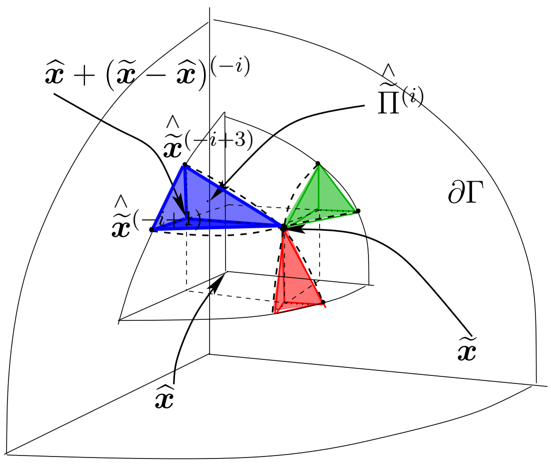

Figure 3.5: The Local Windmill $\Pi_{\widehat{\boldsymbol{x}}}^{(\widetilde{\boldsymbol{x}})}$

windmill w.r.t. $\widetilde{\boldsymbol{x}} - \widehat{\boldsymbol{x}} \in \partial\Gamma_{\widehat{\boldsymbol{x}}}$. Finally, this pseudo windmill is moved back so the calotte appears as a segment of $\partial\Gamma$. The transferred local windmill

$$\widehat{\boldsymbol{x}} + \Pi_{\widehat{\boldsymbol{x}}}^{(\widetilde{\boldsymbol{x}})} \tag{3.32}$$

appears inserted into Γ.

The local windmill is a sum of n deGua Simplices

$$\Pi_{\widehat{\boldsymbol{x}}}^{(\widetilde{\boldsymbol{x}})} \;=:\; \sum_{i \in \boldsymbol{I}} \Pi^{\overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(i)}} \;; \tag{3.33}$$

for short we write

$$\Pi^{\overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(i)}} \;=:\; \overset{\wedge}{\widetilde{\Pi}}^{(i)} \tag{3.34}$$

The vectors $\overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(i)}$ are obtained via Definition 3.4 (and formula (3.5)) *mutatis mutandis*. They are given by the blisspoints of $\Gamma_{\widehat{\boldsymbol{x}}}$ analogously to Remark 3.5. In particular it follows as in (3.8) that

$$\overset{\wedge}{\widetilde{a}}_i^{(i)} = \widetilde{\boldsymbol{x}}_i - \widehat{\boldsymbol{x}}_i \quad (i \in \boldsymbol{I}) \tag{3.35}$$

and hence, analogously to (3.9)

$$\widetilde{\boldsymbol{x}} \;=\; \widehat{\boldsymbol{x}} + (\overset{\wedge}{\widetilde{a}}_1^{(1)}, \ldots, \overset{\wedge}{\widetilde{a}}_n^{(n)}) \;=\; \widehat{\boldsymbol{x}} + \sum_{i \in \boldsymbol{I}} \overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(i)i} \,. \tag{3.36}$$

holds true. That is, up to shifting with $\widehat{\boldsymbol{x}}$, $\widetilde{\boldsymbol{x}}$ is a sum of extremals of the deGua Simplices $\Pi^{\overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(i)}}$ $(i \in \boldsymbol{I})$.

The transferred deGua Simplices are

$$\widehat{\boldsymbol{x}} + (\widetilde{\boldsymbol{x}} - \widehat{\boldsymbol{x}})^{(-i)} + \overset{\wedge}{\widetilde{\Pi}}^{(i)} \;=\; \widehat{\boldsymbol{x}} + \sum_{j \in \boldsymbol{I} \setminus -\{i\}} \overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(j)j} + \overset{\wedge}{\widetilde{\Pi}}^{(i)} \quad (i \in \boldsymbol{I}) \tag{3.37}$$

with the vertices

$$\widetilde{\boldsymbol{x}} \quad \text{as well as} \quad \widehat{\boldsymbol{x}} + (\widetilde{\boldsymbol{x}} - \widehat{\boldsymbol{x}})^{(-i)} + \overset{\wedge}{\widetilde{a}}_j \boldsymbol{e}^j \ := \ \overset{\wedge}{\widetilde{\boldsymbol{x}}}^{(-i+j)} \ \ (j \in \boldsymbol{I}) \tag{3.38}$$

in analogy to (3.11). Figure 3.5 shows the situation for three dimensions and the deGua Simplex $\overset{\wedge}{\widetilde{\Pi}}^{(i)}$ coloured in blue. The figure reflects the situation that the local windmill is actually a windmill which is not necessarily the case.

Again we emphasize the role of the normal $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ at $\partial\Gamma_{\widehat{\boldsymbol{x}}}$ in $\widetilde{\boldsymbol{x}}$ which is the same as the normal $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ at $\partial\Gamma$ in $\widetilde{\boldsymbol{x}}$. As the deGua Simplices $\overset{\wedge}{\widetilde{\Pi}}^{(i)}$ are inscribed into $\Gamma_{\widehat{\boldsymbol{x}}}$, we conclude

Corollary 3.12.

1. For all $i \in \boldsymbol{I}$ the normal $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ is a common normal to $\overset{\wedge}{\widetilde{\Pi}}^{(i)}$ in $\overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(i)i}$. This holds true also for the translates: $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ is a common normal to

$$\widehat{\boldsymbol{x}} + \sum_{j \in \boldsymbol{I} \setminus -\{i\}} \overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(j)j} + \overset{\wedge}{\widetilde{\Pi}}^{(i)} \quad \text{in} \quad \widehat{\boldsymbol{x}} + \sum_{j \in \boldsymbol{I}} \overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(j)j} \ = \ \widetilde{\boldsymbol{x}}\ . \tag{3.39}$$

2. Consequently,

$$\widehat{\boldsymbol{x}} + \sum_{j \in \boldsymbol{I}} \overset{\wedge}{\widetilde{\boldsymbol{a}}}^{(j)j} \ = \ \widetilde{\boldsymbol{x}}\ ; \tag{3.40}$$

is a common vertex to all translated $\overset{\wedge}{\widetilde{\Pi}}^{(i)}$ $(i \in \boldsymbol{I})$.

3. Thus, $\widetilde{\boldsymbol{x}}$ is a common to $\partial\Gamma$ and the translated local windmill $\widehat{\boldsymbol{x}} + \Pi^{(\widetilde{\boldsymbol{x}})}_{\widehat{\boldsymbol{x}}}$. Also, $\mathfrak{n}^{\widetilde{\boldsymbol{x}}}$ is a normal common to $\partial\Gamma$ and to $\Pi^{(\widetilde{\boldsymbol{x}})}_{\widehat{\boldsymbol{x}}} + \widehat{\boldsymbol{x}}$ in $\widetilde{\boldsymbol{x}}$.

Proof: Follows from Theorem 3.7 and (3.18).

q.e.d.

4 Approximating a Convex Body

Let Γ be a compact convex comprehensive set Γ with smooth boundary $\partial\Gamma$. Tentatively, we assume that the normal to $\partial\Gamma$ in every $\boldsymbol{x} \in \partial\Gamma$ is positive. Also, we assume that $\partial\Gamma$ has no "flat" areas, i.e., for any $\bar{\boldsymbol{x}} \in \partial\Gamma$ with normal $\mathfrak{n}^{\bar{\boldsymbol{x}}}$ it follows that $\{\boldsymbol{x} \mid \mathfrak{n}\boldsymbol{x} = \mathfrak{n}\bar{\boldsymbol{x}}\} \; = \; \{\bar{\boldsymbol{x}}\}$.

Given an exogenously preset number of of points located in $\partial\Gamma$ we construct a Cephoid $\Pi^\star$ such that both bodies coincide at these points.

As we have seen, a pseudo windmill can be constructed to *one* preset point $\widetilde{\boldsymbol{x}}$ in $\partial\Gamma$ such that this point is the sole positive Pareto efficient vertex of the pseudo windmill. Essentially this is possible since the deGua Simplices derived from the point in question (the vanes) have the normal to $\partial\Gamma$ in $\widetilde{\boldsymbol{x}}$ as a common normal. Hence it is a normal to $\widetilde{\boldsymbol{x}}$ seen as a vertex of the Pseudo windmill as the deGua Simplices are inscribed into Γ.

Naturally, the same holds true for any local windmill $\Pi^{\widetilde{\boldsymbol{x}}}_{\widehat{\boldsymbol{x}}}$ as long as $\widehat{\boldsymbol{x}} \leq \widetilde{\boldsymbol{x}}$ holds true.

The idea is as follows. Given a set of points on $\partial\Gamma$, we attempt to construct a "windmilled" Cephoid which is a sum of pseudo windmills. Each of these have a joint vertex coinciding with one of the prescribed points. The Pseudo Widmills are arranged in a way such that summing up all of them preserves this decisive property.

Example 4.1. A naive geometrical idea of how to approach the problem is presented by the following canonical representation of a "windmilled" Cephoid.

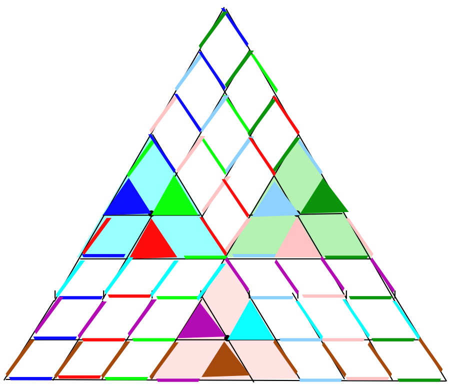

Figure 4.1: The windmilled Windmill

Figure 4.1 suggests a "windmilled Windmill": given 3 points in a surface each of them is supported by a local windmill.

∘ ∼∼∼∼∼∼∼ ∘

We introduce a notation for enumerating a set of vectors that are located on $\partial\Gamma$. For $Q \in \mathbb{N}$ we write $\boldsymbol{Q} \; := \; \{1, \ldots, Q\}$; a system of vectors located on

$\partial\Gamma$ is then enumerated by $\boldsymbol{Q}$; we write

$$(4.1) \qquad \widetilde{\boldsymbol{X}} = \left\{ \overset{\{q\}}{\widetilde{\boldsymbol{x}}} \in \mathbb{R}^n_+ \,\middle|\, (q \in \boldsymbol{Q}) \right\} \subseteq \partial\Gamma \; .$$

We also consider a corresponding system

$$(4.2) \qquad \widehat{\boldsymbol{X}} = \left\{ \overset{\{q\}}{\widehat{\boldsymbol{x}}} \in \mathbb{R}^n_+ \,\middle|\, \overset{\{q\}}{\widehat{\boldsymbol{x}}} \leq \overset{\{q\}}{\widetilde{\boldsymbol{x}}} \quad (q \in \boldsymbol{Q}) \right\} \subseteq \Gamma$$

of vectors dominated by the vectors of $\widetilde{\boldsymbol{X}}$. $\widehat{\boldsymbol{X}}$ is arbitrary under this condition but will be specified later. Both sets are also regarded as elements of $\mathbb{R}^{3\times\boldsymbol{Q}}$; we write

$$(4.3) \qquad \mathfrak{X} := \{\widehat{\boldsymbol{X}} \,|\, \widehat{\boldsymbol{X}} \text{ satisfies } \quad (4.2)\} \subseteq \mathbb{R}^{3\times\boldsymbol{Q}} \; .$$

Here is our main Theorem.

Theorem 4.2 (***Main Theorem of Approximation***)**.** *Let $\Gamma \subseteq \mathbb{R}^n_+$ be a compact comprehensive convex body with smooth surface $\partial\Gamma$ and let $Q \in \mathbb{N}$. Let $\widetilde{\boldsymbol{X}}$ be a set of positive vectors on $\partial\Gamma$. Then there exists a Cephoid $\Pi^\star$ such that*

1. *$\Pi^\star$ is a sum of nQ deGua Simplices,*
2. *$\Pi^\star$ is a sum of Q Windmills,*
3. *For $q \in \boldsymbol{Q}$ we have $\overset{\{q\}}{\widetilde{\boldsymbol{x}}} \in \partial\Pi^\star$, i.e., the preset points are Pareto efficient in $\Pi^\star$.*

• ~~~∘~~~ •

Proof:

1stSTEP :

Let $\widehat{\boldsymbol{X}} \in \mathfrak{X}$. For each $q \in \boldsymbol{Q}$ consider the calotte $\Gamma_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}$ generated according to Definition 3.2. Construct the the local windmill $\overset{\{q\}}{\Pi} := \Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}$ corresponding to $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ and $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ according to Definition 3.11. This Cephoid is a sum of n deGua Simplices (vanes), say

$$(4.4) \qquad \overset{\{q\}}{\Pi} := \Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}} = \sum_{i\in\boldsymbol{I}} \overset{\{q\}}{\Pi}{}^{(i)} \; .$$

The vanes

$$(4.5) \qquad \overset{\{q\}}{\Pi}{}^{(i)} = \Pi^{\overset{\{q\}}{\boldsymbol{a}}{}^{(i)}} \quad (i \in \boldsymbol{I})$$

are given by vectors

$$\left\{ \overset{\{q\}}{\boldsymbol{a}}{}^{(i)} \right\}_{i \in \boldsymbol{I}} \tag{4.6}$$

that result from the procedure described in 3.11 using the bliss points of the local windmill. All of this is done quite analogously to SECTION 3. Figure 4.2 shows, for some $q \in \boldsymbol{Q}$, a local windmill in three dimensions, shifted to the surface of Γ via $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$. The blue vane indicated is drawn for $i = 2$.

2$^{\text{nd}}$STEP :

Now we focus on Corollary 3.12. Accordingly we know for the translates via $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ that, for $q \in \boldsymbol{Q}$ and $i \in \boldsymbol{I}$,

$$\begin{array}{ccc} \mathfrak{n}^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}} \text{ is a} & \text{common normal to} & \\ \overset{\{q\}}{\widehat{\boldsymbol{x}}} + \sum\limits_{j \in \boldsymbol{I} \setminus \{i\}} \overset{\{q\}}{\boldsymbol{a}}{}^{(j)j} + \overset{\{q\}}{\Pi}{}^{(i)} & \text{in} & \sum\limits_{j \in \boldsymbol{I}} \overset{\{q\}}{\boldsymbol{a}}{}^{(j)j} + \overset{\{q\}}{\widehat{\boldsymbol{x}}} \;=\; \overset{\{q\}}{\widetilde{\boldsymbol{x}}}\ . \end{array} \tag{4.7}$$

Thus, for $q \in \boldsymbol{Q}$,

$$\overset{\{q\}}{\widetilde{\boldsymbol{x}}} \;=\; \sum_{j \in \boldsymbol{I}} \overset{\{q\}}{a}{}^{(j)j} + \overset{\{q\}}{\widehat{\boldsymbol{x}}} \tag{4.8}$$

is a common vertex to all translates of $\overset{\{q\}}{\Pi}{}^{(i)}$ $(i \in \boldsymbol{I})$. Also, $\mathfrak{n}^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}$ is a common normal to the translated Cephoid $\Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}} + \overset{\{q\}}{\widehat{\boldsymbol{x}}}$ in $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ which is a common point of this translated Cephoid and $\partial\Gamma$. Figure 4.2 repeats Figure 3.5 adjusted to

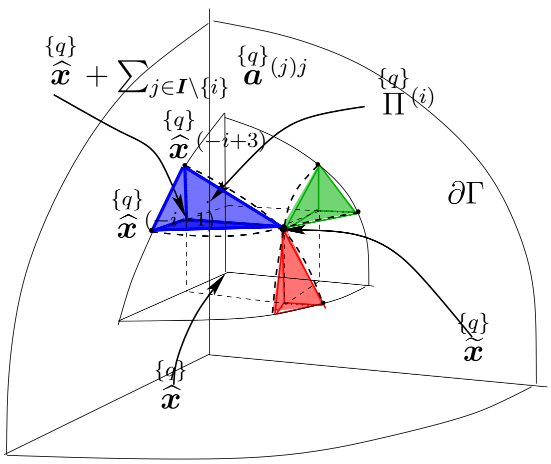

Figure 4.2: The Local Windmill $\overset{\{q\}}{\Pi}$

the current situation.

3$^{\mathrm{rd}}$STEP : The choice of

$$\widehat{\boldsymbol{X}} = \left\{ \overset{\{q\}}{\widehat{\boldsymbol{x}}} \,\middle|\, q \in \boldsymbol{Q} \right\}$$

so far has been free under the conditions listed in (4.2), i.e., under the conditions $\mathbf{0} \leq \overset{\{q\}}{\widehat{\boldsymbol{x}}} \leq \overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ $(q \in \boldsymbol{Q})$. If $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ varies, then the vectors $\overset{\{q\}}{\boldsymbol{a}}{}^{(i)}$ $(i \in \boldsymbol{I})$ are functions of $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$

$$\overset{\{q\}}{\widehat{\boldsymbol{x}}} \mapsto \left\{ \overset{\{q\}}{\boldsymbol{a}}{}^{(i)} \right\}_{i \in \boldsymbol{I}} \tag{4.9}$$

Correspondingly we imagine that $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ is mapped into the n deGua Simplices $\overset{\{q\}}{\Pi}{}^{(i)} = \Pi^{\overset{\{q\}}{\boldsymbol{a}}{}^{(i)}}$ $(i \in \boldsymbol{I})$ adding up to the pseudo windmill $\overset{\{q\}}{\Pi} = \Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}$ "at" $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$,

$$\overset{\{q\}}{\widehat{\boldsymbol{x}}} \mapsto \overset{\{q\}}{\Pi} = \Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}} \,. \tag{4.10}$$

More or less obviously, these functions are continuous and *antitone*: if we decrease the coordinates of $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$, the values of the data $\{\overset{\{q\}}{\boldsymbol{a}}{}^{(i)}\}_{i \in \boldsymbol{I}}$ in (4.6) will increase.

A closer inspection shows: for small $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ the calotte $\Gamma_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}$ approaches Γ, hence the local windmill $\overset{\{q\}}{\Pi} = \Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}$ approaches the ("global") pseudo windmill constructed via Definition 3.11. For $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ approaching $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ we observe that the calotte $\Gamma_{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}$ becomes arbitrarily small as $\overset{\{q\}}{\widetilde{\boldsymbol{x}}} \in \partial\Gamma$.

4$^{\mathrm{th}}$STEP : Now we apply Lemma 3.8 to the present situation. Accordingly, let $\overset{\{q\}}{\widetilde{\mathfrak{n}}}$ denote the normal at $\partial\Gamma$ in $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ and let $\overset{\{p\}}{\mathcal{K}}{}^{(l)j}$ be the normal cone at the vane $\overset{\{p\}}{\Delta}{}^{(l)}$ in $\overset{\{p\}}{\boldsymbol{a}}{}^{(l)j}$. Following Formula (3.23), define for $p, q \in \boldsymbol{Q}$ and $l \in \boldsymbol{I}$

$$\overset{\{p\}}{\boldsymbol{I}}{}^{(l)}_{\{q\}} := \left\{ j \in \boldsymbol{I} \,\middle|\, \overset{\{q\}}{\widetilde{\mathfrak{n}}} \in \overset{\{p\}}{\mathcal{K}}{}^{(l)j} \right\} . \tag{4.11}$$

That is, for some $l \in \boldsymbol{I}$, we collect those vertices $\overset{\{p\}}{\boldsymbol{a}}{}^{(l)j}$ $(j \in \boldsymbol{I})$ which allow for $\overset{\{q\}}{\widetilde{\mathfrak{n}}}$ as a normal at $\Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}}$ in $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ and hence can be added without disturbing the Pareto property. We know by Lemma 3.8 that

For $p, q \in \boldsymbol{Q}$ and $l \in \boldsymbol{I}$

$$\overset{\{p\}}{\boldsymbol{I}}{}^{(l)}_{\{q\}} \neq \emptyset \tag{4.12}$$

holds true.

5

5^{th}STEP :

Now we specify $\widehat{X}$ in a suitable way to prove our theorem. To this end define

$$\Pi^{\star} := \sum_{p \in \boldsymbol{Q}} \Pi^{\overset{\{p\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{p\}}{\widehat{\boldsymbol{x}}}} . \tag{4.13}$$

Rewriting (4.8) we obtain for $q \in \boldsymbol{Q}$

$$\begin{aligned} (\overset{\{p\}}{a}{}^{(1)}_{1}, \ldots, \overset{\{p\}}{a}{}^{(n)}_{n}) &= \sum_{j \in \boldsymbol{I}} \overset{\{p\}}{\boldsymbol{a}}{}^{(j)j} = \overset{\{q\}}{\widetilde{\boldsymbol{x}}} - \overset{\{q\}}{\widehat{\boldsymbol{x}}} . \\ &\text{that is,} \\ \overset{\{q\}}{\widetilde{\boldsymbol{x}}} &= \overset{\{q\}}{\widehat{\boldsymbol{x}}} + \sum_{j \in \boldsymbol{I}} \overset{\{q\}}{\boldsymbol{a}}{}^{(j)j} . \end{aligned} \tag{4.14}$$

Now, if we can represent $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ as a Pareto efficient vector of

$$\sum_{p \in \boldsymbol{Q} \setminus \{q\}} \Pi^{\overset{\{q\}}{\widetilde{\boldsymbol{x}}}}_{\overset{\{q\}}{\widehat{\boldsymbol{x}}}} , \tag{4.15}$$

then (4.14) and (4.15) imply that $\overset{\{q\}}{\widetilde{\boldsymbol{x}}} \in \Pi^{\star}$. Moreover, $\overset{\{q\}}{\widetilde{\boldsymbol{x}}} \in \partial\Pi^{\star}$ if and only if the representation of $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ by Pareto efficient vectors of (4.15) can be arranged so as to allow for common normals of all vectors involved.

To this end, for $p, q \in \boldsymbol{Q}, p \neq q$ and $l \in \boldsymbol{I}$ consider the convex hull

$$\overset{\{p\}}{\boldsymbol{C}}{}^{(l)}_{\{q\}} := \boldsymbol{CovH} \left\{ \overset{\{p\}}{\boldsymbol{a}}{}^{(l)j} \,\middle|\, j \in \overset{\{p\}}{\boldsymbol{I}}{}^{(l)}_{\{q\}} \right\} \neq \emptyset \tag{4.16}$$

and the resulting sum

$$\overset{\{q\}}{\boldsymbol{C}} := \sum_{p \in \boldsymbol{Q} \setminus \{q\}} \sum_{l \in \boldsymbol{I}} \overset{\{p\}}{\boldsymbol{C}}{}^{(l)}_{\{q\}} \neq \emptyset \quad (p \in \boldsymbol{Q} \setminus \{q\}) . \tag{4.17}$$

Next, consider the correspondence (set valued mapping)

$$\begin{aligned} &\mathfrak{X} \longmapsto \mathfrak{X} \\ &\overset{\{q\}}{\widehat{\boldsymbol{x}}} \longmapsto \boldsymbol{CovH} \left\{ \overset{\{q\}}{\boldsymbol{C}} \cup \{ \overset{\{q\}}{\widetilde{\boldsymbol{x}}} \} \right\} \cap \left\{ \boldsymbol{x} \leq \overset{\{q\}}{\widetilde{\boldsymbol{x}}} \right\} \quad (q \in \boldsymbol{Q}) \end{aligned} \tag{4.18}$$

The values of this correspondence are compact and convex; it is not hard to see that it is upper hemi continuous. According to Kakutani's fixed point theorem, this correspondence has a fixed point satisfying $\overset{\{q\}}{\widehat{\boldsymbol{x}}} \leq \overset{\{q\}}{\widetilde{\boldsymbol{x}}}\ \ (q \in \boldsymbol{Q})$. (Somewhat sloppily we do not use a special notation for the fixed point).

A fixed point cannot result in $\overset{\{q\}}{\widehat{\boldsymbol{x}}} = \mathbf{0}$ for some $q \in \boldsymbol{Q}$ in view of our remark on monotonicity in the $3^{rd}STEP$. By the same reasoning it cannot result in $\overset{\{q\}}{\widehat{\boldsymbol{x}}} = \overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ for some $q \in \boldsymbol{Q}$. Therefore we know that it satisfies

$$\overset{\{q\}}{\widehat{\boldsymbol{x}}} \in \overset{\{q\}}{\boldsymbol{C}},\ \ \overset{\{q\}}{\widehat{\boldsymbol{x}}} \ = \sum_{p \in \boldsymbol{Q}\setminus\{q\}} \sum_{l \in \boldsymbol{I}} \overset{\{p\}}{\boldsymbol{C}}{}^{(l)}_{\{q\}}\ \ (q \in \boldsymbol{Q}) \tag{4.19}$$

Accordingly, we can for any $q \in \boldsymbol{Q}$ choose vectors

$$\overset{\{p\}}{\boldsymbol{c}}{}^{(l)}_{\{q\}} \in \overset{\{p\}}{\boldsymbol{C}}{}^{(l)}_{\{q\}}\ \ (p \in \boldsymbol{Q}, l \in \boldsymbol{I}) \tag{4.20}$$

such that

$$\overset{\{q\}}{\widehat{\boldsymbol{x}}} \ = \sum_{p \in \boldsymbol{Q}\setminus\{q\}} \sum_{l \in \boldsymbol{I}} \overset{\{p\}}{\boldsymbol{c}}{}^{(l)}_{\{q\}} \tag{4.21}$$

holds true. In (4.21), each $\overset{\{p\}}{\boldsymbol{c}}{}^{(l)}_{\{q\}}$ is a convex combination of vertices

$$\overset{\{p\}}{\boldsymbol{a}}{}^{(l)j}\ \ \left(j \in \overset{\{p\}}{\boldsymbol{I}}{}^{(l)}_{\{q\}}\right)\ \text{i.e.}\ \overset{\{p\}}{\boldsymbol{a}}{}^{(l)j} \in \overset{\{p\}}{\Delta}{}^{(l)}_{\overset{\{p\}}{\boldsymbol{I}}{}^{(l)}_{\{q\}}}\ .$$

In view of the $4^{th}STEP$ and the defining equation (4.11), we conclude that $\overset{\{p\}}{\boldsymbol{c}}{}^{(l)}_{\{q\}}$ is a vector of the vane $\overset{\{p\}}{\Delta}{}^{(l)}$ that admits of $\overset{\{q\}}{\mathfrak{n}}$ as a normal to this vane (and its translates). Therefore, (4.21) demonstrates that $\overset{\{q\}}{\widehat{\boldsymbol{x}}}$ is a sum of vectors of all vanes of pseudo windmills other than q which can be added to the local windmill $\overset{\{q\}}{\Pi}$ without disturbing the Pareto property of the vertex $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$.

More precisely, combining (4.14) and (4.21) we obtain

$$\overset{\{q\}}{\widetilde{\boldsymbol{x}}} \ = \sum_{p \in \boldsymbol{Q}\setminus\{q\}} \sum_{l \in \boldsymbol{I}} \overset{\{p\}}{\boldsymbol{c}}{}^{(l)}_{\{q\}} + \sum_{j \in \boldsymbol{I}} \overset{\{q\}}{\boldsymbol{a}}{}^{(j)j} \in \Pi^{\star} \tag{4.22}$$

We know that $\overset{\{q\}}{\widetilde{\boldsymbol{x}}}$ is Pareto efficient in $\Pi^{\star}$ as the common normal condition is satisfied. This proves our Theorem.

q.e.d.

Remark 4.3. For the generation of a pseudo windmill a point $\widetilde{\boldsymbol{x}} \in \partial\Gamma$ is not necessarily positive. We find "degenerate pseudo windmills" if $\widetilde{\boldsymbol{x}}$ has zero coordinates. In particular, if $\widetilde{\boldsymbol{x}} = \boldsymbol{f}^{(i)}$ is a bliss point, then the pseudo windmill $\Pi^{(\widetilde{\boldsymbol{x}})}$ consists of just one vane; we have $\Pi^{(\widetilde{\boldsymbol{x}})} = \Pi$. Therefore the positivity assumption for vectors $\overset{\{q\}}{\widetilde{\boldsymbol{x}}} \in \widetilde{\boldsymbol{X}}$ can be dropped in Theorem 4.2. The same is true for $\widehat{\boldsymbol{X}}$.

2. Also, we do not have to assume a positive normal in all points of $\partial\Gamma$, neither do we have to assume that $\partial\Gamma$ has no flat areas.

∘ ∼∼∼∼∼∼ ∘

Combining we obtain

Corollary 4.4. The set of Cephoids is dense within the set of compact comprehensive convex bodies in $\mathbb{R}^n_+$ with respect to the Hausdorff metric.

• ∼∼∼∘∼∼∼ •

Based on these results we can now proceed with the establishment of values or bargaining solutions for general NTU games as presented e.g. in [9]. We may employ an approximating procedure similar to the one for the Maschler–Perles solution explained in [6].

References

[1] J. P. de Gua de Malves, *Article on : The n-dimensional Pythagorean Theorem*, in: Histoire de l'Academie des Sciences pour l'ann'ee 1783, cf. Guggenheimer, Mathematical Reviews 1136610, 1783.

[2] G. Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, vol. 168, Springer Verlag, New York, Berlin, Heidelberg, 1996.

[3] D. Pallaschke and J. Rosenmüller, *Cephoids: Minkowski sums of deGua simplexes*, Journal of Optimization **59(4)** (2010), 515–540.

[4] D. Pallaschke and J. Rosenmüller, *Cephoids: Duality and reference vectors*, Pacific Journal of Optimization **8** (2012), 157–184.

[5] D. Pallaschke and R. Urbański, *Pairs of compact convex sets — fractional arithmetic with convex sets*, Mathematics and its Applications, vol. 548, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[6] M. A. Perles and M. Maschler, *The superadditive solution for the Nash bargaining game*, International Journal of Game Theory **10** (1981), 163–193.

[7] D. Ricardo, *On the Principles of Political Economy and Taxation*, John Murray, Albemarle-Street, London, 1817.

[8] W.F. Richter and J. Rosenmüller, *Efficient specialization in Ricardian Production*, German Economic Review **13** (2012), 117–241.

[9] J. Rosenmüller, *A value for cephoidal NTU–games*, International Game Theory Review **11(3)** (2009), 247–272.

[10] ———, *Cephoids: Minkowski Sums of deGua Simplices*, Working Papers, vol. 629, Center for Mathematical Economics, Bielefeld, Germany, 2019.