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Well-Posedness for a Class of Degenerate Itô Stochastic Differential Equations with Fully Discontinuous Coefficients

Haesung Lee [1] and Gerald Trutnau [2,*]

[1] Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany; fthslt@math.uni-bielefeld.de
[2] Department of Mathematical Sciences and Research Institute, Mathematics of Seoul National University, 1 Gwanak-Ro, Gwanak-Gu, Seoul 08826, Korea
* Correspondence: trutnau@snu.ac.kr

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Abstract: We show uniqueness in law for a general class of stochastic differential equations in $\mathbb{R}^d$, $d \geq 2$, with possibly degenerate and/or fully discontinuous locally bounded coefficients among all weak solutions that spend zero time at the points of degeneracy of the dispersion matrix. Points of degeneracy have a d-dimensional Lebesgue–Borel measure zero. Weak existence is obtained for a more general, but not necessarily locally bounded drift coefficient.

Keywords: degenerate stochastic differential equation; uniqueness in law; martingale problem; weak existence; strong Feller semigroup.

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1. Introduction

The question whether a solution to a stochastic differential equation (hereafter SDE) on $\mathbb{R}^d$ exists that is pathwise unique and strong occurs widely in the mathematical literature; for instance, see the introduction of [1] for a recent detailed, but possibly incomplete development. Sometimes, strong solutions that are roughly described as weak solutions for a given Brownian motion are required, for instance, in signal processing, where a noisy signal is implicitly given. Sometimes, it may be impossible to obtain a strong solution, only weak solutions are important to consider, or only the strong Markov property of the solution is needed for some reason. Then, uniqueness in law, i.e., the question whether, given an initial distribution, the distribution of any weak solution no matter on which probability space it is considered is the same, plays an important role. It might also be that pathwise uniqueness and strong solution results are just too restrictive, so that one is naturally led to consider weak solutions and their uniqueness. Here, we consider weak uniqueness of an SDE with respect to all initial conditions $x \in \mathbb{R}^d$ as defined, for instance, in [2] (Chapter 5); see also Definition 2 below.

To explain our motivation for this work, fix symmetric matrix $C = (c_{ij})_{1 \leq i,j \leq d}$ of bounded measurable functions c_{ij}, such that, for some $\lambda \geq 1$,

$$\lambda^{-1}\|\xi\|^2 \leq \langle C(x)\xi, \xi\rangle \leq \lambda\|\xi\|^2, \quad \text{for all } x, \xi \in \mathbb{R}^d,$$

and vector $\mathbf{H} = (h_1, ..., h_d)$ of locally bounded measurable functions. Let

$$\mathcal{L}f = \sum_{i,j=1}^{d} \frac{c_{ij}}{2} \partial_{ij} f + \sum_{i=1}^{d} h_i \partial_i f \tag{1}$$

be the corresponding linear operator and

$$X_t = x + \int_0^t \sqrt{C}(X_s)\, dW_s + \int_0^t \mathbf{H}(X_s)\, ds, \quad t \geq 0,\ x \in \mathbb{R}^d, \tag{2}$$

be the corresponding Itô-SDE. If the c_{ij} are continuous and the h_i bounded, then Equation (2) is well-posed, i.e., there exists a solution and it is unique in law (see [3]). If the h_i are bounded, then Equation (2) is well-posed for $d = 2$ (see [3] Exercise 7.3.4); however, if $d \geq 3$, there exists an example of a measurable discontinuous C for which uniqueness in law does not hold [4]. Hence, even in the nondegenerate case, well-posedness for discontinuous coefficients is nontrivial, and one is naturally led to search for general subclasses in which well-posedness holds. Some of these are given when C is not far from being continuous, i.e., continuous up to a small set (e.g., a discrete set or a set of α-Hausdorff measure zero with sufficiently small α; else, see, for instance, introductions of [4,5] for references). Another special subclass is given when C is a piecewise constant on a decomposition of $\mathbb{R}^d$ into a finite union of polyhedrons [6], and the h_i are locally bounded with at most linear growth at infinity. The work in [6] is one of our sources of motivation for this article. Though we do not perfectly cover the conditions in [6], we complement them in many ways. In particular, we consider arbitrary decompositions of $\mathbb{R}^d$ into bounded disjoint measurable sets (choose, for instance, $\sqrt{\frac{1}{\psi}} = \sum_{i=1}^{\infty} \alpha_i 1_{A_i}$, with $\mathbb{R}^d = \dot\cup_{i=1}^{\infty} A_i$, $(\alpha_i)_{i\in\mathbb{N}} \subset (0,\infty)$ in Equation (4) below). A further example for a discontinuous C, where well-posedness holds, can be found in [7]. There, discontinuity is along the common boundary of the upper- and lower-half spaces. In [5], among others, the problem of uniqueness in law for Equation (2) is related to the Dirichlet problem for $\mathcal{L}$ as in Equation (1), locally on smooth domains. This method was also used in [4] using Krylov's previous work. In particular, a shorter proof of the well-posedness results of Bass and Pardoux [6] and Gao [7] is presented in [5] (Theorems 2.16 and 3.11). However, the most remarkable is the derivation of well-posedness for a special subclass of processes with degenerate discontinuous C. Though discontinuity is only along a hyperplane of codimension one, and coefficients are quite regular outside the hyperplane, it seems to be one of the first examples of a discontinuous degenerate C where well-posedness still holds ([5] (Example 1.1)). This intriguing example was another source of our motivation. As was the case for results in [6], we could not perfectly cover [5] (Example 1.1), but we again complement it in many ways. As a main observation besides the above considerations, it seems that no general subclass has been presented so far where C is degenerate (or also nondegenerate if $d \geq 3$) and fully discontinuous, but well-posedness holds nonetheless. This is another main goal of this paper, and our method strongly differs from techniques used in [5,6] and in the past literature. Our techniques involve semigroup theory, elliptic and parabolic regularity theory, the theory of generalized Dirichlet forms (i.e., the construction of a Hunt process from a sub-Markovian C_0-semigroup of contractions on some L^1-space with a weight), and an adaptation of an idea of Stroock and Varadhan to show uniqueness for the martingale problem using a Krylov-type estimate. Krylov-type estimates have been widely used to simultaneously obtain a weak solution and its uniqueness, in particular, pathwise uniqueness. The advantage of our method is that the weak existence of a solution and uniqueness in law are shown separately of each other using different techniques. We used local Krylov-type estimates (Theorem 9) to show uniqueness in law. Once uniqueness in law holds, we could improve the original Krylov estimate, at least for the time-homogeneous case (see Remark 4). In particular, our method typically implies weak-existence results that are more general than uniqueness results (see Theorem 8 here and in [1,8]).

Now, let us describe our results. Let $d \geq 2$, and $A = (a_{ij})_{1\leq i,j\leq d}$ be a symmetric matrix of functions $a_{ij} \in H^{1,2d+2}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, such that, for every open ball $B \subset \mathbb{R}^d$, there exist constants $\lambda_B, \Lambda_B > 0$ with

$$\lambda_B \|\xi\|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda_B \|\xi\|^2, \quad \text{for all } \xi \in \mathbb{R}^d,\ x \in B.$$

Let $\psi \in L^q_{loc}(\mathbb{R}^d)$, with $q > 2d+2$, $\psi > 0$ a.e., such that $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$. Here, we assumed that expression $\frac{1}{\psi}$ stood for an arbitrary but fixed Borel measurable function satisfying $\psi \cdot \frac{1}{\psi} = 1$ a.e., and $\frac{1}{\psi}(x) \in [0,\infty)$ for any $x \in \mathbb{R}^d$. Let $\mathbf{G} = (g_1, \ldots, g_d) \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ be a vector of Borel measurable functions. Let $(\sigma_{ij})_{1\leq i\leq d, 1\leq j\leq m}$, $m \in \mathbb{N}$ arbitrary but fixed, and be any matrix consisting of continuous functions, such that $A = \sigma\sigma^T$. Suppose there exists a constant $M > 0$, such that

$$-\frac{\langle (\frac{1}{\psi}A)(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\mathrm{trace}\big((\frac{1}{\psi}A)(x)\big) + \langle \mathbf{G}(x), x\rangle \leq M\left(\|x\|^2+1\right)\left(\ln(\|x\|^2+1)+1\right) \tag{3}$$

for a.e. $x \in \mathbb{R}^d$. The main result of our paper (Theorem 13) was that weak existence and uniqueness in law, i.e., well-posedness, then holds for stochastic differential equation

$$X_t = x + \int_0^t \Big(\sqrt{\frac{1}{\psi}} \cdot \sigma\Big)(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{4}$$

among all weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, X_t = (X^1_t, \ldots, X^d_t), W = (W^1, \ldots, W^m), \mathbb{P}_x)$, $x \in \mathbb{R}^d$, such that

$$\int_0^\infty 1_{\left\{\sqrt{\frac{1}{\psi}}=0\right\}}(X_s)ds = 0 \qquad \mathbb{P}_x\text{-a.s. } \forall x \in \mathbb{R}^d. \tag{5}$$

Here, the solution and integrals involving the solution in Equation (4) may a priori depend on Borel versions chosen for $\sqrt{\frac{1}{\psi}}$ and $\mathbf{G}$. but Condition (5) is exactly the condition that makes these objects independent of the chosen Borel versions (cf. Lemma 2). $\sqrt{\frac{1}{\psi}}$ may, of course, be fully discontinuous, but if it takes all its values in $(0,\infty)$; then, Equation (5) is automatically satisfied. However, since $\psi \in L^q_{loc}(\mathbb{R}^d)$, it must be a.e. finite, so that zeros Z of $\sqrt{\frac{1}{\psi}}$ have Lebesgue–Borel measure zero. Nonetheless, our main result comprehends the existence of a whole class of degenerate (on Z) diffusions with fully discontinuous coefficients for which well-posedness holds. This seems to be new in the literature. For another condition that implies Equation (5), we refer to Lemma 2. For an explicit example for well-posedness, which reminds the Engelbert/Schmidt condition for uniqueness in law in dimension one (see [9]), we refer to Example 2.

We derived weak existence of a solution to Equation (4) up to its explosion time under quite more general conditions on the coefficients, see Theorem 8. In this case, for nonexplosion, one only needs that Equation (3) holds outside an arbitrarily large open ball (see Remark 3ii). Moreover, Equation (5) is always satisfied for the weak solution that we construct (see Remark 3), and our weak solution originated from a Hunt process, not only from a strong Markov process.

The techniques that we used for weak existence are as follows. First, any solution to Equation (4) determines the same (up to a.e. uniqueness of the coefficients) second-order partial differential operator L on $C_0^\infty(\mathbb{R}^d)$,

$$Lf = \sum_{i,j=1}^d \frac{\frac{1}{\psi}a_{ij}}{2}\partial_{ij}f + \sum_{i=1}^d g_i \partial_i f, \quad f \in C_0^\infty(\mathbb{R}^d).$$

In Theorem 4, we found a measure $\mu := \rho\psi\, dx$ with some nice regularity of ρ, which is an infinitesimally invariant measure for $(L, C_0^\infty(\mathbb{R}^d))$, i.e.,

$$\int_{\mathbb{R}^d} Lf\, d\mu = \int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^{d} \frac{\frac{1}{\psi} a_{ij}}{2} \partial_{ij} f + \sum_{i=1}^{d} g_i \partial_i f \Big)\, d\mu = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d). \tag{6}$$

Then, using the existence of a density to the infinitesimally invariant measure, we adapted the method from Stannat [10] to our case and constructed a sub-Markovian C_0-semigroup of contractions $(T_t)_{t\geq 0}$ on each $L^s(\mathbb{R}^d, \mu)$, $s \geq 1$ of which the generator extended $(L, C_0^\infty(\mathbb{R}^d))$, i.e., we found a suitable functional analytic frame (see Theorem 3 that further induced a generalized Dirichlet form; see (19)) to describe a potential infinitesimal generator of a weak solution to Equation (4). This is done in Section 4, where we also derive, with the help of the results about general regularity properties from Section 3, the regularity properties of $(T_t)_{t\geq 0}$ and its resolvent (see Section 4.3). Then, crucially using the existence of a Hunt process for a.e. starting point related to $(T_t)_{t\geq 0}$ in Proposition 3 (which follows similarly to [11] (Theorem 6)) this leads to a transition function of a Hunt process that not only weakly solves (4), but also has a transition function with such nice regularity that many presumably optimal classical conditions for properties of a solution to Equation (4) carry over to our situation. We mention, for instance, nonexplosion Condition (3) and moment inequalities (see Remark 2). However, irreducibility and classical ergodic properties, as in [1], could also be studied in this framework by further investigating the influence of $\frac{1}{\psi}$ on properties of the transition function. Similarly to the results of [1], the only point where Krylov-type estimates were used in our method was when it came up to uniqueness. Here, because of the possible degeneracy of $\sqrt{\frac{1}{\psi}}$, we needed Condition (5) to derive a Krylov-type estimate that held for any weak solution to Condition (4) (see Theorem 9 which straightforwardly followed from the original Krylov estimate [12] (2. Theorem (2), p. 52)). Again, our constructed transition function had such a nice regularity that a time-dependent drift-eliminating Itô-formula held for function $g(x,t) := P_{T-t} f(x)$, $f \in C_0^\infty(\mathbb{R}^d)$. In fact, it held for any weak solution to Condition (4), so that for all these, the one-dimensional and, hence, all finite-dimensional marginals coincided (cf. Theorem 12). This latter technique goes back to an idea of Stroock/Varadhan ([3]), and we used the treatise of this technique as presented in [2] (Chapter 5).

2. Article Structure and Notations

The main parts of this article are Sections 4 and 5. Section 4 contains the analytic results, and Section 5 contains the probabilistic results. Section 3 also contains auxiliary analytical results that are important on their own. Section 3 could be skipped in a first reading, so the reader may directly start with Section 4. The proofs for all statements of this article and further auxiliary statements were collected in Appendix A.

Throughout, we used the same notations as in [1,8], and $d \geq 2$. Additionally, for an open-set U in $\mathbb{R}^d$ and a measure μ on $\mathbb{R}^d$, let $L^q(U, \mathbb{R}^d, \mu) := \{\mathbf{F} = (f_1, \ldots, f_d) : U \to \mathbb{R}^d \mid f_i \in L^q(U, \mu), 1 \leq i \leq d\}$, equipped with the norm, $\|\mathbf{F}\|_{L^q(U,\mathbb{R}^d,\mu)} := \| \|\mathbf{F}\| \|_{L^q(U,\mu)}$, $\mathbf{F} \in L^q(U, \mathbb{R}^d, \mu)$. If $\mu = dx$, we write $L^q(U)$, $L^q(U, \mathbb{R}^d)$ for $L^q(U, dx)$, $L^q(U, \mathbb{R}^d, dx)$ respectively, and even $\|\mathbf{F}\|_{L^q(U)}$ for $\|\mathbf{F}\|_{L^q(U,\mathbb{R}^d)}$. Denote by $C^k(\overline{U})$, $k \in \mathbb{N} \cup \{0\}$, the usual space of k-times continuously differentiable functions in U, such that the partial derivatives of an order less or equal to k extend continuously to $\overline{U}$ (as defined, for instance, in [13]). In particular, $C(\overline{U}) := C^0(\overline{U})$ is the space of continuous functions on $\overline{U}$ with supnorm $\|\cdot\|_{C(\overline{U})}$ and $C^\infty(\overline{U}) := \bigcap_{k\in\mathbb{N}} C^k(\overline{U})$. If I is an open interval in $\mathbb{R}$ and $p, q \in [1, \infty]$, we denoted by $L^{p,q}(U \times I)$ the space of all Borel measurable functions f on $U \times I$ for which

$$\|f\|_{L^{p,q}(U\times I)} := \| \|f(\cdot,\cdot)\|_{L^p(U)} \|_{L^q(I)} < \infty,$$

and let $\text{supp}(f) := \text{supp}(|f|dxdt)$. For a locally integrable function g on $U \times I$ and $i \in \{1, \dots, d\}$, we denoted by $\partial_i g$ the i-th weak spatial derivative on $U \times I$, by $\nabla g := (\partial_1 g, \dots, \partial_d g)$ the weak spatial gradient of g, by $\nabla^2 g := (\partial_{ij} g)_{1 \le i,j \le d}$ the weak spatial Hessian matrix, and by $\partial_t g$ the weak time derivative on $U \times I$, provided these existed. For $p, q \in [1, \infty]$, let $W^{2,1}_{p,q}(U \times I)$ be the set of all locally integrable functions $g : U \times I \to \mathbb{R}$ such that $\partial_t g$, $\partial_i g$, $\partial_i \partial_j g \in L^{p,q}(U \times I)$ for all $1 \le i, j \le d$. Let $W^{2,1}_p(U \times I) := W^{2,1}_{p,p}(U \times I)$.

3. New Regularity Results

In this section, we develop some new regularity estimates (Theorems 1 and 2). Theorem 1 was used to obtain the semigroup regularity in Theorem 6, and Theorem 2 was used to obtain the resolvent regularity in Theorem 5.

3.1. Regularity Estimate for Linear Parabolic Equations with Weight in Time Derivative Term

Throughout this subsection, we assume the following condition:

(I) $U \times (0, T)$ is a bounded open set in $\mathbb{R}^d \times \mathbb{R}$, $T > 0$, $A = (a_{ij})_{1 \le i,j \le d}$ is a (possibly nonsymmetric) matrix of functions on U that is uniformly strictly elliptic and bounded, i.e., there exist constants $\lambda > 0$, $M > 0$, such that, for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, $x \in U$, it holds

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \lambda \|\xi\|^2, \qquad \max_{1 \le i,j \le d} |a_{ij}(x)| \le M,$$

$\mathbf{B} \in L^p(U, \mathbb{R}^d)$ with $p > d$, $\psi \in L^q(U)$, $q \in [2 \vee \frac{p}{2}, p)$, and there exists $c_0 > 0$, such that $c_0 \le \psi$ on U, and finally

$$u \in H^{1,2}(U \times (0, T)) \cap L^\infty(U \times (0, T)).$$

Assuming Condition **(I)**, we considered a divergence form linear parabolic equation with a singular weight in the time derivative term as follows

$$\iint_{U \times (0,T)} (u \partial_t \varphi) \psi \, dxdt = \iint_{U \times (0,T)} \langle A \nabla u, \nabla \varphi \rangle + \langle \mathbf{B}, \nabla u \rangle \varphi \, dxdt, \tag{7}$$

which is supposed to hold for all $\varphi \in C_0^\infty(U \times (0, T))$.

Let $(\bar{x}, \bar{t})$ be an arbitrary but fixed point in $U \times (0, T)$, and $R_{\bar{x}}(r)$ be the open cube in $\mathbb{R}^d$ of edge length $r > 0$ centered at $\bar{x}$. Define $Q(r) := R_{\bar{x}}(r) \times (\bar{t} - r^2, \bar{t})$.

Theorem 1. *Suppose that $Q(3r) \subset U \times (0, T)$. Under the assumption* **(I)** *and* (7)*, we have*

$$\|u\|_{L^\infty(Q(r))} \le C \|u\|_{L^{\frac{2p}{p-2}, 2}(Q(2r))}, \tag{8}$$

where $C > 0$ is a constant depending only on r, λ, M and $\|\mathbf{B}\|_{L^p(R_{\bar{x}}(3r))}$.

3.2. Elliptic Hölder Regularity and Estimate

The following theorem is an adaptation of [14] (Théorème 7.2) using [15] (Theorem 1.7.4). It might already exist in the literature, but we could not find any reference for it, and we therefore provide a proof (in Appendix A).

Theorem 2. *Let U be a bounded open ball in $\mathbb{R}^d$. Let $A = (a_{ij})_{1\le i,j\le d}$ be as in* **(I)***. Assume* $\mathbf{B} \in L^p(U,\mathbb{R}^d)$, $c \in L^q(U)$, $f \in L^{\tilde{q}}(U)$ *for some $p > d$, $q, \tilde{q} > \frac{d}{2}$. If $u \in H^{1,2}(U)$ satisfies*

$$\int_U \langle A\nabla u, \nabla \varphi\rangle + (\langle \mathbf{B}, \nabla u\rangle + cu)\,\varphi\, dx = \int_U f\varphi\, dx, \quad \text{for all } \varphi \in C_0^\infty(U), \tag{9}$$

then for any open ball U_1 in $\mathbb{R}^d$ with $\overline{U}_1 \subset U$, we have $u \in C^{0,\gamma}(\overline{U}_1)$ and

$$\|u\|_{C^{0,\gamma}(\overline{U}_1)} \le C\left(\|u\|_{L^1(U)} + \|f\|_{L^{\tilde{q}}(U)}\right),$$

where $\gamma \in (0,1)$ and $C > 0$ are constants which are independent of u and f.

4. L^1-Generator and Its Strong Feller Semigroup

In this section, we precisely describe the potential infinitesimal generator, its semigroup and resolvent, of a weak solution to Condition (4) in a suitable functional analytic frame, originally due to Stannat (Theorem 3 and (19)). Subsequently, using the regularity results from Section 3, we derived regularity properties for the resolvent and semigroup (Theorems 5 and 6). One key tool for this method is the existence of an infinitesimally invariant measure with nice density (Theorem 4).

4.1. Framework

Let $\rho \in H^{1,2}_{loc}(\mathbb{R}^d) \cap L^\infty_{loc}(\mathbb{R}^d)$, $\psi \in L^1_{loc}(\mathbb{R}^d)$ be a.e. strictly positive functions satisfying $\frac{1}{\rho}, \frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$. Here, we assumed that expressions $\frac{1}{\rho}, \frac{1}{\psi}$, denoted any Borel measurable functions satisfying $\rho \cdot \frac{1}{\rho} = 1$ and $\psi \cdot \frac{1}{\psi} = 1$ a.e., respectively (later, especially in Section 5 it is important which measurable Borel version $\frac{1}{\psi}$ we choose, but for the moment it does not matter). Set $\mu := \rho\psi\, dx$. If U is any open subset of $\mathbb{R}^d$; then, bilinear form $\int_U \langle \nabla u, \nabla v\rangle dx$, $u, v \in C_0^\infty(U)$ is closable in $L^2(U,\mu)$ by [16] (Subsection II.2a)). Define $\widehat{H}^{1,2}_0(U,\mu)$ as the closure of $C_0^\infty(U)$ in $L^2(U,\mu)$ with respect to norm $\left(\int_U \|\nabla u\|^2 dx + \int_U u^2 d\mu\right)^{1/2}$. Thus $u \in \widehat{H}^{1,2}_0(U,\mu)$, if and only if there exists $(u_n)_{n\ge 1} \subset C_0^\infty(U)$ such that

$$\lim_{n\to\infty} u_n = u \ \text{ in } L^2(U,\mu), \qquad \lim_{n,m\to\infty} \int_U \|\nabla(u_n - u_m)\|^2 dx = 0; \tag{10}$$

moreover, $\widehat{H}^{1,2}_0(U,\mu)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{\widehat{H}^{1,2}_0(U,\mu)} = \lim_{n\to\infty} \int_U \langle \nabla u_n, \nabla v_n\rangle dx + \int_U uv\, d\mu,$$

where $(u_n)_{n\ge 1}, (v_n)_{n\ge 1} \subset C_0^\infty(U)$ are arbitrary sequences that satisfy Equation (10).

If $u \in \widehat{H}^{1,2}_0(V,\mu)$ for some bounded open subset V of $\mathbb{R}^d$, then $u \in H^{1,2}_0(V) \cap L^2(V,\mu)$ and there exists $(u_n)_{n\ge 1} \subset C_0^\infty(V)$, such that

$$\lim_{n\to\infty} u_n = u \quad \text{in } H^{1,2}_0(V) \ \text{ and } \ \text{in } L^2(V,\mu).$$

Consider a symmetric matrix of functions $A = (a_{ij})_{1\le i,j\le d}$ satisfying

$$a_{ij} = a_{ji} \in H^{1,2}_{loc}(\mathbb{R}^d), \quad 1 \le i,j \le d,$$

and assume A is locally uniformly strictly elliptic, i.e., for every open ball B, there exist constants $\lambda_B, \Lambda_B > 0$, such that

$$\lambda_B \|\xi\|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda_B \|\xi\|^2, \quad \text{for all } \xi \in \mathbb{R}^d,\ x \in B. \tag{11}$$

Define $\widehat{A} := \frac{1}{\psi} A$. By [16] (Subsection II.2b)), the symmetric bilinear form

$$\mathcal{E}^0(f,g) := \frac{1}{2} \int_{\mathbb{R}^d} \langle \widehat{A} \nabla f, \nabla g \rangle d\mu, \quad f, g \in C_0^\infty(\mathbb{R}^d),$$

is closable in $L^2(\mathbb{R}^d, \mu)$, and its closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric Dirichlet form in $L^2(\mathbb{R}^d, \mu)$ (see [16] ((II. 2.18))). Denote the corresponding generator of $(\mathcal{E}^0, D(\mathcal{E}^0))$ by $(L^0, D(L^0))$. Let $f \in C_0^\infty(\mathbb{R}^d)$. Using integration by parts, for any $g \in C_0^\infty(\mathbb{R}^d)$,

$$\mathcal{E}^0(f,g) \quad = \quad -\int_{\mathbb{R}^d} \Big(\frac{1}{2} \text{trace}(\widehat{A} \nabla^2 f) + \langle \underbrace{\frac{1}{2\psi} \nabla A + \frac{A \nabla \rho}{2\rho\psi}}_{=:\beta^{\rho,A,\psi}}, \nabla f \rangle \Big) g \, d\mu.$$

Thus, $f \in D(L^0)$. This implies $C_0^\infty(\mathbb{R}^d) \subset D(L^0)$ and

$$L^0 f = \frac{1}{2} \text{trace}(\widehat{A} \nabla^2 f) + \langle \beta^{\rho,A,\psi}, \nabla f \rangle \in L^2(\mathbb{R}^d, \mu). \tag{12}$$

Let $(T_t^0)_{t>0}$ be the sub-Markovian C_0-semigroup of contractions on $L^2(\mathbb{R}^d, \mu)$ associated with $(L^0, D(L^0))$. It is well-known that $T_t^0|_{L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)}$ can be uniquely extended to a sub-Markovian C_0-semigroup of contractions $(\overline{T_t^0})_{t>0}$ on $L^1(\mathbb{R}^d, \mu)$.

Now, let $\mathbf{B} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$ be weakly divergence-free with respect to μ, i.e.,

$$\int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u \rangle d\mu = 0, \quad \text{for all } u \in C_0^\infty(\mathbb{R}^d). \tag{13}$$

Assume

$$\rho\psi \mathbf{B} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d). \tag{14}$$

By routine arguments, Equation (13) extends to all $u \in \widehat{H}_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$, and

$$\int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u \rangle v d\mu = -\int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla v \rangle u d\mu, \quad \text{for all } u, v \in \widehat{H}_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}.$$

Define $Lu := L^0 u + \langle \mathbf{B}, \nabla u \rangle,\ u \in D(L^0)_{0,b}$. Then, $(L, D(L^0)_{0,b})$ is an extension of

$$\frac{1}{2} \text{trace}(\widehat{A} \nabla^2 u) + \langle \beta^{\rho,A,\psi} + \mathbf{B}, \nabla u \rangle, \quad u \in C_0^\infty(\mathbb{R}^d).$$

For any bounded open subset V of $\mathbb{R}^d$,

$$\mathcal{E}^{0,V}(f,g) := \frac{1}{2} \int_V \langle \widehat{A} \nabla f, \nabla g \rangle d\mu, \quad f, g \in C_0^\infty(V).$$

is also closable on $L^2(V, \mu)$ by [16] (Subsection II.2b)). Denote by $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$ the closure of $(\mathcal{E}^{0,V}, C_0^\infty(V))$ in $L^2(V, \mu)$. Using (11) and $0 < \inf_V \rho \leq \sup_V \rho < \infty$, it is clear that $D(\mathcal{E}^{0,V}) = \widehat{H}_0^{1,2}(V, \mu)$

since the norms $\|\cdot\|_{D(\mathcal{E}^{0,V})}$ and $\|\cdot\|_{\widehat{H}_0^{1,2}(V,\mu)}$ are equivalent. Denote by $(L^{0,V}, D(L^{0,V}))$ the generator of $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$.

4.2. L^1-Generator

In this section, we use all notations and assumptions from Section 4.1.

The technique of [10] (Chapter 1) to obtain a closed extension of a densely defined diffusion operator and, subsequently, a generalized Dirichlet form carried nearly one by one over to our situation; only a small structural difference occurred. Since we considered a degenerate diffusion matrix in the definition of the underlying symmetric Dirichlet form via a function ψ that also acts on the μ-divergence free antisymmetric part of drift (see Equation (13)), we considered local convergence in space $\widehat{H}_0^{1,2}(V,\mu)$ and imposed Assumption (14) on the antisymmetric part, while [10] (Chapter 1) dealt with local convergence in space $H_0^{1,2}(V,\mu)$. As a first step, the following proposition was derived in a nearly identical manner to [10] (Proposition 1.1). We therefore omitted the proof.

Proposition 1. *Let V be a bounded open subset of $\mathbb{R}^d$.*

(i) *Operator $(L^V, D(L^{0,V})_b)$ on $L^1(V,\mu)$ defined by*

$$L^V u := L^{0,V}u + \langle \mathbf{B}, \nabla u\rangle, \quad u \in D(L^{0,V})_b$$

is dissipative, and hence closable on $L^1(V,\mu)$. Closure $(\overline{L}^V, D(\overline{L}^V))$ generates a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t^V)_{t>0}$ on $L^1(V,\mu)$.

(ii) *$D(\overline{L}^V)_b \subset \widehat{H}_0^{1,2}(V,\mu)$ and*

$$\mathcal{E}^{0,V}(u,v) - \int_V \langle \mathbf{B}, \nabla u\rangle\, v d\mu = \int_V \overline{L}^V u \cdot v d\mu, \quad \textit{for all } u \in D(\overline{L}^V)_b,\ v \in \widehat{H}_0^{1,2}(V,\mu)_b. \tag{15}$$

Now, let V be a bounded open subset of $\mathbb{R}^d$. Denote by $(\overline{G}_\alpha^V)_{\alpha>0}$ the resolvent associated with $(\overline{L}^V, D(\overline{L}^V))$ on $L^1(V,\mu)$. Then, $(\overline{G}_\alpha^V)_{\alpha>0}$ could be extended to $L^1(\mathbb{R}^d,\mu)$ by

$$\overline{G}_\alpha^V f := \begin{cases} \overline{G}_\alpha^V(f 1_V) & \text{on } V \\ 0 & \text{on } \mathbb{R}^d \setminus V, \end{cases} \qquad f \in L^1(\mathbb{R}^d,\mu), \tag{16}$$

Let $g \in L^1(\mathbb{R}^d,\mu)_b$. Then $\overline{G}_\alpha^V(g 1_V) \in D(\overline{L}^V)_b \subset \widehat{H}_0^{1,2}(V,\mu)$, hence $\overline{G}_\alpha^V g \in \widehat{H}_0^{1,2}(V,\mu)$.

If $u \in D(\mathcal{E}^{0,V})$, then by definition it holds $u \in D(\mathcal{E}^0)$ and $\mathcal{E}^{0,V}(u,u) = \mathcal{E}^0(u,u)$. Therefore, we obtained

$$\mathcal{E}^0(\overline{G}_\alpha^{V_n} g, \overline{G}_\alpha^{V_n} g) = \mathcal{E}^{0,V_n}(\overline{G}_\alpha^{V_n}(g 1_{V_n}), \overline{G}_\alpha^{V_n}(g 1_{V_n})). \tag{17}$$

By means of Proposition 1, the following Theorem 3 was also derived in a nearly identical manner to [10] (Theorem 1.5).

Theorem 3. *There exists a closed extension $(\overline{L}, D(\overline{L}))$ of $Lu := L^0 u + \langle \mathbf{B}, \nabla u\rangle$, $u \in D(L^0)_{0,b}$ on $L^1(\mathbb{R}^d,\mu)$ satisfying the following properties:*

(a) *$(\overline{L}, D(\overline{L}))$ generates a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t)_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$.*

(b) *Let $(U_n)_{n\geq 1}$ be a family of bounded open subsets of $\mathbb{R}^d$ satisfying $\overline{U}_n \subset U_{n+1}$ and $\mathbb{R}^d = \bigcup_{n\geq 1} U_n$. Then $\lim_{n\to\infty} \overline{G}_\alpha^{U_n} f = (\alpha - \overline{L})^{-1} f$ in $L^1(\mathbb{R}^d,\mu)$, for all $f \in L^1(\mathbb{R}^d,\mu)$ and $\alpha > 0$.*

(c) $D(\overline{L})_b \subset D(\mathcal{E}^0)$ *and for all* $u \in D(\overline{L})_b$, $v \in \widehat{H}_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}$ *it holds*

$$\mathcal{E}^0(u,u) \leq -\int_{\mathbb{R}^d} \overline{L}u \cdot u d\mu, \quad \text{and} \quad \mathcal{E}^0(u,v) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u \rangle v d\mu = -\int_{\mathbb{R}^d} \overline{L}u \cdot v d\mu.$$

4.3. Existence of Infinitesimally Invariant Measure and Strong Feller Properties

Here, we state some conditions that were used as our assumptions.

(A1) $p > d$ is fixed, and $A = (a_{ij})_{1 \leq i,j \leq d}$ is a symmetric matrix of functions that are locally uniformly strictly elliptic on $\mathbb{R}^d$, such that $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ for all $1 \leq i,j \leq d$. $\psi \in L^1_{loc}(\mathbb{R}^d)$ is a positive function, such that $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$ and $\mathbf{G}$ is a Borel measurable vector field on $\mathbb{R}^d$ satisfying $\psi\mathbf{G} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$.

(A2) $\psi \in L^q_{loc}(\mathbb{R}^d)$ with $q \in (\frac{d}{2}, \infty]$. Fix $s \in (\frac{d}{2}, \infty)$ such that $\frac{1}{q} + \frac{1}{s} < \frac{2}{d}$.

(A3) $q \in [\frac{p}{2} \vee 2, \infty]$.

Theorem 4. *Under Assumption* **(A1)**, *there exists* $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ *satisfying* $\rho(x) > 0$ *for all* $x \in \mathbb{R}^d$ *such that*

$$\int_{\mathbb{R}^d} \langle \mathbf{G} - \beta^{\rho,A,\psi}, \nabla \varphi \rangle \rho\psi dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d), \tag{18}$$

or equivalently, (6) *holds. Moreover,* $\rho\psi\, \mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, *where* $\mathbf{B} := \mathbf{G} - \beta^{\rho,A,\psi}$.

From now on, we assume that Condition **(A1)** holds and fix A, ψ, ρ, $\mathbf{B}$ as in Theorem 4. Then, A, ψ, ρ, $\mathbf{B}$ satisfy all assumptions of Section 4.1. As in Section 4.1 $\mu := \rho\psi\, dx$, $\widehat{A} := \frac{1}{\psi} A$.

By Theorem 3, there existed a closed extension $(\overline{L}, D(\overline{L}))$ of

$$Lf = L^0 f + \langle \mathbf{B}, \nabla f \rangle, \quad f \in D(L^0)_{0,b},$$

on $L^1(\mathbb{R}^d,\mu)$ that generates a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t)_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$. Restricting $(\overline{T}_t)_{t>0}$ to $L^1(\mathbb{R}^d,\mu)_b$, it is well-known by Riesz–Thorin interpolation that $(\overline{T}_t)_{t>0}$ could be extended to a sub-Markovian C_0-semigroup of contractions $(T_t)_{t>0}$ on each $L^r(\mathbb{R}^d,\mu)$, $r \in [1,\infty)$. Denote by $(L_r, D(L_r))$ the corresponding closed generator with graph norm

$$\|f\|_{D(L_r)} := \|f\|_{L^r(\mathbb{R}^d,\mu)} + \|L_r f\|_{L^r(\mathbb{R}^d,\mu)},$$

and by $(G_\alpha)_{\alpha>0}$ the corresponding resolvent. $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ can also be uniquely defined on $L^\infty(\mathbb{R}^d,\mu)$, but are no longer strongly continuous there.

For $f \in C_0^\infty(\mathbb{R}^d)$, we have

$$Lf = L^0 f + \langle \mathbf{B}, \nabla f \rangle = \frac{1}{2}\text{trace}(\widehat{A}\nabla^2 f) + \langle \mathbf{G}, \nabla f \rangle.$$

Define

$$L^* f : \;=\; L^0 f - \langle \mathbf{B}, \nabla f \rangle = \frac{1}{2}\text{trace}(\widehat{A}\nabla^2 f) + \langle \mathbf{G}^*, \nabla f \rangle,$$

with

$$\mathbf{G}^* := (g_1^*, \ldots, g_d^*) = 2\beta^{\rho,A,\psi} - \mathbf{G} = \beta^{\rho,A,\psi} - \mathbf{B} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu).$$

Denote by $(L_r^*, D(L_r^*))$ operators corresponding to L^* for the cogenerator on $L^r(\mathbb{R}^d,\mu)$, $r \in [1,\infty)$, $(T_t^*)_{t>0}$ for the cosemigroup, $(G_\alpha^*)_{\alpha>0}$ for the coresolvent. As in ([10], Section 3), we obtained a corresponding bilinear form with domain $D(L_2) \times L^2(\mathbb{R}^d,\mu) \cup L^2(\mathbb{R}^d,\mu) \times D(L_2^*)$ by

$$\mathcal{E}(f,g) := \begin{cases} -\int_{\mathbb{R}^d} L_2 f \cdot g\, d\mu & \text{for } f \in D(L_2),\ g \in L^2(\mathbb{R}^d,\mu), \\ -\int_{\mathbb{R}^d} f \cdot L_2^* g\, d\mu & \text{for } f \in L^2(\mathbb{R}^d,\mu),\ g \in D(L_2^*). \end{cases} \tag{19}$$

$\mathcal{E}$ is called the *generalized Dirichlet form associated with* $(L_2, D(L_2))$.

Theorem 5. *Assume Conditions* **(A1)** *and* **(A2)**, *and let* $f \in \cup_{r\in[s,\infty]} L^r(\mathbb{R}^d,\mu)$. *Then,* $G_\alpha f$ *has a locally Hölder continuous* μ*-version* $R_\alpha f$ *on* $\mathbb{R}^d$. *Furthermore for any open balls* B, B' *satisfying* $\overline{B} \subset B'$, *we have the following estimate:*

$$\|R_\alpha f\|_{C^{0,\gamma}(\overline{B})} \le c_2 \left(\|f\|_{L^s(B',\mu)} + \|G_\alpha f\|_{L^1(B',\mu)} \right), \tag{20}$$

where $c_2 > 0$, $\gamma \in (0,1)$ *are constants that are independent of* f.

Let $f \in D(L_r)$ for some $r \in [s,\infty)$. Then $f = G_1(1-L_r)f$; hence, by Theorem 5, f has a locally Hölder continuous μ-version on $\mathbb{R}^d$ and

$$\|f\|_{C^{0,\gamma}(\overline{B})} \quad \le \quad c_3 \|f\|_{D(L_r)},$$

where $c_3 > 0$, $\gamma \in (0,1)$ are constants independent of f. In particular, $T_t f \in D(L_r)$ and $T_t f$ hence has a continuous μ-version, say $P_t f$, with

$$\|P_t f\|_{C^{0,\gamma}(\overline{B})} \le c_3 \|P_t f\|_{D(L_r)}. \tag{21}$$

c_3 is independent of $t \ge 0$ and f. The following lemma is quite important later to show the joint continuity of $P_\cdot g(\cdot)$ for $g \in \cup_{\nu\in[\frac{2p}{p-2},\infty]} L^\nu(\mathbb{R}^d,\mu)$. Due to Equation (21), it can be proven as in [1] (Lemma 4.13).

Lemma 1. *Assume Conditions* **(A1)**, **(A2)**. *For any* $f \in \bigcup_{r\in[s,\infty)} D(L_r)$, *map*

$$(x,t) \mapsto P_t f(x)$$

is continuous on $\mathbb{R}^d \times [0,\infty)$.

Theorem 6. *Assume Conditions* **(A1)**, **(A2)**, *and* **(A3)**, *and let* $f \in \bigcup_{\nu\in[\frac{2p}{p-2},\infty]} L^\nu(\mathbb{R}^d,\mu)$, $t > 0$. *Then,* $T_t f$ *has a continuous* μ*-version* $P_t f$ *on* $\mathbb{R}^d$, *and* $P_\cdot f(\cdot)$ *is continuous on* $\mathbb{R}^d \times (0,\infty)$. *For any bounded open set* U, V *in* $\mathbb{R}^d$ *with* $\overline{U} \subset V$ *and* $0 < \tau_3 < \tau_1 < \tau_2 < \tau_4$, *i.e.,* $[\tau_1,\tau_2] \subset (\tau_3,\tau_4)$, *we have the following estimate for all* $f \in \cup_{\nu\in[\frac{2p}{p-2},\infty]} L^\nu(\mathbb{R}^d,\mu)$:

$$\|P_\cdot f(\cdot)\|_{C(\overline{U}\times[\tau_1,\tau_2])} \le C_1 \|P_\cdot f(\cdot)\|_{L^{\frac{2p}{p-2},2}(V\times(\tau_3,\tau_4))}, \tag{22}$$

where C_1 *is a constant that depends on* $\overline{U} \times [\tau_1,\tau_2]$, $V \times (\tau_3,\tau_4)$, *but is independent of* f.

By Theorems 5 and 6, exactly as in [1] (Remark 3.7), we obtained resolvent kernels and resolvent kernel densities $R_\alpha(x,dy)$, $r_\alpha(x,y)$, corresponding to resolvent $(R_\alpha)_{\alpha>0}$, as well as transition kernels and transition-kernel densities $P_t(x,dy)$, $p_t(x,y)$, corresponding to transition function $(P_t)_{t\ge 0}$.

Proposition 2. *Assume Conditions* **(A1)**, **(A2)**, *and* **(A3)**, *and let* $t, \alpha > 0$. *Then, it holds that*

(i) $G_\alpha g$ *has a locally Hölder continuous* μ*-version*

$$R_\alpha g = \int_{\mathbb{R}^d} g(y) R_\alpha(\cdot, dy) = \int_{\mathbb{R}^d} g(y) r_\alpha(\cdot, y)\mu(dy), \ \forall g \in \bigcup_{r\in[s,\infty]} L^r(\mathbb{R}^d, \mu). \tag{23}$$

In particular, Equation (23) *extends by linearity to all* $g \in L^s(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$, *i.e.,* $(R_\alpha)_{\alpha>0}$ *is* $L^{[s,\infty]}(\mathbb{R}^d, \mu)$*-strong Feller.*

(ii) $T_t f$ *has a continuous* μ*-version*

$$P_t f = \int_{\mathbb{R}^d} f(y) P_t(\cdot, dy) = \int_{\mathbb{R}^d} f(y) p_t(\cdot, y)\mu(dy), \ \forall f \in \bigcup_{\nu\in[\frac{2p}{p-2},\infty]} L^\nu(\mathbb{R}^d, \mu). \tag{24}$$

In particular,Equation (24) *extends by linearity to all* $f \in L^{\frac{2p}{p-2}}(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$, *i.e.,* $(P_t)_{t>0}$ *is* $L^{[\frac{2p}{p-2},\infty]}(\mathbb{R}^d, \mu)$*-strong Feller.*

Finally, for any $\alpha > 0, x \in \mathbb{R}^d$, $g \in L^s(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$

$$R_\alpha g(x) = \int_0^\infty e^{-\alpha t} P_t g(x)\, dt.$$

5. Well-Posedness

With the help of the regularity results, Theorems 5 and 6 of Section 4, and the mere existence of a Hunt process for a.e. starting point (Proposition 3), we constructed a weak solution to Equation (4) (Theorems 7 and 8). Then, using a local Krylov-type estimate and Itô-formula (Theorems 9 and 10), uniqueness in law was derived for weak solutions to Equation (4) that spend zero time at the points of degeneracy of the dispersion matrix (Theorems 12 and 13). The method to derive uniqueness in law is an adaptation of the Stroock and Varadhan method ([3]) via the martingale problem.

5.1. Weak Existence

The following assumption in particular is necessary to obtain a Hunt process with transition function $(P_t)_{t\geq 0}$ (and consequently a weak solution to the corresponding SDE for every starting point). It is first used in Theorem 7 below.

(A4) $\mathbf{G} \in L^s_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$, where s is as in **(A2)**.

Condition **(A4)** is not necessary to get a Hunt process (and consequently a weak solution to the corresponding SDE for merely quasi-every starting point) as in the following proposition.

Proposition 3. *Assume Conditions* **(A1)**, **(A2)**, *and* **(A3)**. *Then, there exists a Hunt process*

$$\tilde{\mathbb{M}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t\geq 0}, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$$

with life time $\tilde{\zeta} := \inf\{t \geq 0 \,|\, \tilde{X}_t = \Delta\}$ *and cemetery* Δ, *such that* $\mathcal{E}$ *is (strictly properly) associated with* $\tilde{\mathbb{M}}$ *and for strictly* $\mathcal{E}$*-q.e.* $x \in \mathbb{R}^d$,

$$\tilde{\mathbb{P}}_x\big(\{\omega \in \tilde{\Omega} \mid \tilde{X}_\cdot(\omega) \in C\big([0,\infty), \mathbb{R}^d_\Delta\big),\ \tilde{X}_t(\omega) = \Delta,\ \forall t \geq \zeta(\omega)\}\big) = 1.$$

Remark 1. *(i)* *Assume Conditions* **(A1)**, **(A2)**, **(A3)**, *and* $\mathbf{G} \in L^{\frac{sq}{q-1}}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. *Then, for any bounded open subset V of $\mathbb{R}^d$, it holds that*

$$\int_V \|\mathbf{G}\|^s d\mu \leq \|\mathbf{G}\|^s_{L^{\frac{sq}{q-1}}(V)} \|\rho\psi\|_{L^q(V)};$$

hence, Condition **(A4)** *is satisfied.*

(ii) *Two simple examples where Conditions* **(A1)**, **(A2)**, **(A3)**, *and* **(A4)** *are satisfied are given as follows: for the first example, let A, ψ satisfy the assumptions of* **(A1)**, $\psi \in L^p_{loc}(\mathbb{R}^d)$, $s = \frac{dp}{2p-d} + \varepsilon$, *and* $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$; *for the second, let A, ψ satisfy the assumptions of* **(A1)**, $\psi \in L^{2p}_{loc}(\mathbb{R}^d)$, $s = \frac{2pd}{4p-d} + \varepsilon$ *and* $\mathbf{G} \in L^{2p}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. *In both cases, $\varepsilon > 0$ can be chosen to be arbitrarily small.*

Analogously to [1] (Theorem 3.12), we obtained

Theorem 7. *Under Assumptions* **(A1)**, **(A2)**, **(A3)**, **(A4)**, *there exists a Hunt process*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$$

with state space $\mathbb{R}^d$ and life time

$$\zeta = \inf\{t \geq 0 \,|\, X_t = \Delta\} = \inf\{t \geq 0 \,|\, X_t \notin \mathbb{R}^d\},$$

having transition function $(P_t)_{t\geq 0}$ as the transition semigroup, such that $\mathbb{M}$ has continuous sample paths in the one-point compactification $\mathbb{R}^d_\Delta$ of $\mathbb{R}^d$ with cemetery Δ as point at infinity, i.e., for any $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\big(\{\omega \in \Omega \mid X_\cdot(\omega) \in C([0,\infty), \mathbb{R}^d_\Delta),\ X_t(\omega) = \Delta,\ \forall t \geq \zeta(\omega)\}\big) = 1.$$

Remark 2. *The analogous results to [1] (Lemma 3.14, Lemma 3.15, Proposition 3.16, Proposition 3.17, Theorem 3.19) hold in the situation of this paper. One of the main differences is that $q = \frac{dp}{d+p} > \frac{d}{2}$ of [1] is replaced by $s > \frac{d}{2}$ of* **(A2)**. *A Krylov-type estimate for $\mathbb{M}$ of Theorem 7 especially holds as stated in Equation (25) right below. Let $g \in L^r(\mathbb{R}^d, \mu)$ for some $r \in [s, \infty]$ be given. Then, for any ball B, there exists a constant $C_{B,r}$, depending in particular on B and r, such that for all $t \geq 0$,*

$$\sup_{x\in\overline{B}} \mathbb{E}_x\left[\int_0^t |g|(X_s)\,ds\right] < e^t C_{B,r} \|g\|_{L^r(\mathbb{R}^d,\mu)}. \tag{25}$$

The derivation of Equation (25) is based on Theorem 5, of which the proof uses the elliptic Hölder estimate of Theorem 2. This differs from the proof of the Krylov-type estimates in [1,8] that are based on an elliptic $H^{1,p}$-estimate. Finally, one can get the analogous conservativeness and moment inequalities to [1] (Theorem 4.2, Theorem 4.4(i)) in this paper.

The following theorem can be proved exactly as in [1] (Theorem 3.19).

Theorem 8. *Assume Conditions* **(A1)**, **(A2)**, **(A3)**, *and* **(A4)** *are satisfied. Consider Hunt process $\mathbb{M}$ from Theorem 7 with co-ordinates $X_t = (X^1_t, \ldots, X^d_t)$. Let $(\sigma_{ij})_{1\leq i\leq d, 1\leq j\leq m}$, $m \in \mathbb{N}$ arbitrary but fixed, be any locally uniformly strictly elliptic matrix consisting of continuous functions for all $1 \leq i \leq d$, $1 \leq j \leq m$, such that $A = \sigma\sigma^T$, i.e.,*

$$a_{ij}(x) = \sum_{k=1}^m \sigma_{ik}(x)\sigma_{jk}(x), \ \ \forall x \in \mathbb{R}^d,\ 1 \leq i, j \leq d.$$

Set

$$\widehat{\sigma} = \sqrt{\frac{1}{\psi}} \cdot \sigma \text{ , i.e., } \widehat{\sigma}_{ij} = \sqrt{\frac{1}{\psi}} \cdot \sigma_{ij},\ 1 \le i \le d,\ 1 \le j \le m.$$

(Recall that expression $\frac{1}{\psi}$ denotes an arbitrary Borel measurable function satisfying $\psi \cdot \frac{1}{\psi} = 1$ a.e.).

Then, on a standard extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, which we denote for notational convenience again by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, there exists a standard m-dimensional Brownian motion $W = (W^1, \dots, W^m)$ starting from zero, such that $\mathbb{P}_x$-a.s. for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $i = 1, \dots, d$

$$X_t^i = x_i + \sum_{j=1}^m \int_0^t \widehat{\sigma}_{ij}(X_s)\, dW_s^j + \int_0^t g_i(X_s)\, ds, \quad 0 \le t < \zeta, \tag{26}$$

in short

$$X_t = x + \int_0^t \widehat{\sigma}(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad 0 \le t < \zeta.$$

If Equation (3) holds a.e. outside an arbitrarily large compact set, then $\mathbb{P}_x(\zeta = \infty) = 1$ for all $x \in \mathbb{R}^d$ (cf. [1] (Theorem 4.2)).

Example 1. *Given $p > d$, let $A = (a_{ij})_{1\le i,j\le d}$ be a symmetric matrix of functions on $\mathbb{R}^d$ that is locally uniformly strictly elliptic and $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ for all $1 \le i, j \le d$. Given $m \in \mathbb{N}$, let $\sigma = (\sigma_{ij})_{1\le i\le d, 1\le j\le m}$ be a matrix of functions satisfying $\sigma_{ij} \in C(\mathbb{R}^d)$ for all $1 \le i \le d, 1 \le j \le m$, such that $A = \sigma\sigma^T$. Let $\phi \in L^\infty_{loc}(\mathbb{R}^d)$ be such that for any open ball B, there exist strictly positive constants c_B, C_B such that*

$$c_B \le \phi(x) \le C_B \quad \text{for every } x \in B.$$

Let $\frac{1}{\psi}(x) := \frac{\|x\|^\alpha}{\phi(x)}$, $x \in \mathbb{R}^d$, for some $\alpha > 0$ and consider following conditions.

(a) $\alpha p < d$, $\mathbf{G} \in L^\infty(B_\varepsilon(0), \mathbb{R}^d) \cap L^p_{loc}(\mathbb{R}^d \setminus \overline{B}_\varepsilon(0), \mathbb{R}^d)$ *for some* $\varepsilon > 0$,

(b) $2\alpha p < d$, $\mathbf{G} \in L^{2p}(B_\varepsilon(0), \mathbb{R}^d) \cap L^p_{loc}(\mathbb{R}^d \setminus \overline{B}_\varepsilon(0), \mathbb{R}^d)$ *for some* $\varepsilon > 0$,

(c) $\alpha \cdot (\frac{p}{2} \vee 2) < d$, $\mathbf{G} \equiv 0$ *on* $B_\varepsilon(0)$ *and* $\mathbf{G} \in L^s_{loc}(\mathbb{R}^d \setminus \overline{B}_\varepsilon(0), \mathbb{R}^d)$ *for some* $\varepsilon > 0$, *where* $s > d$ *so that* $(\frac{p}{2} \vee 2)^{-1} + \frac{1}{s} < \frac{2}{d}$.

Any of Conditions (a), (b), or (c) imply Assumptions **(A1)**, **(A2)**, **(A3)**, *and* **(A4)**. *Indeed, for an arbitrary $\varepsilon > 0$ take $q = p$, $s = \frac{pd}{2p-d} + \varepsilon$ in the case of Condition (a), $q = 2p$, $s = \frac{2pd}{4p-d} + \varepsilon$ in the case of Condition (b), and $q = \frac{p}{2} \vee 2$, $s > d$ defined by Condition (c) in the case of Condition (c). Assuming Condition (a), (b), or (c), Hunt process $\mathbb{M}$ as in Theorem 8 solves weakly $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^d$,*

$$X_t = x + \int_0^t \|X_s\|^{\alpha/2} \cdot \frac{\sigma}{\sqrt{\phi}}(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad 0 \le t < \zeta \tag{27}$$

and is nonexplosive if Equation (3) holds a.e. outside an arbitrarily large compact set.

5.2. Uniqueness in Law

Consider

(A4)′: **(A1)** holds with $p = 2d + 2$, **(A2)** holds with some $q \in (2d + 2, \infty]$, $s \in (\frac{d}{2}, \infty)$ is fixed, such that $\frac{1}{q} + \frac{1}{s} < \frac{2}{d}$, and $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$.

Definition 1. *Suppose Assumptions* **(A1)**, **(A2)**, **(A3)**, *and* **(A4)** *hold (for instance, if* **(A4)$'$** *holds). Let expression $\frac{1}{\psi}$ denote an arbitrary but fixed Borel measurable function satisfying $\psi \cdot \frac{1}{\psi} = 1$ a.e. and $\frac{1}{\psi}(x) \in [0,\infty)$ for any $x \in \mathbb{R}^d$. Let*

$$\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{W}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d})$$

be such that for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$

(i) $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}}_x)$ *is a filtered probability space, satisfying the usual conditions,*

(ii) $(\widetilde{X}_t = (\widetilde{X}_t^1, \dots, \widetilde{X}_t^d))_{t\geq 0}$ *is an $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$-adapted continuous $\mathbb{R}^d$-valued stochastic process,*

(iii) $(\widetilde{W}_t = (\widetilde{W}_t^1, \dots, \widetilde{W}_t^m))_{t\geq 0}$ *is a standard m-dimensional $((\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}}_x)$-Brownian motion starting from zero,*

(iv) for the (real-valued) Borel measurable functions $\widehat{\sigma}_{ij}, g_i, \frac{1}{\psi}$, $\widehat{\sigma}_{ij} = \sqrt{\frac{1}{\psi}}\sigma_{ij}$, with σ is as in Theorem 8, it holds

$$\widetilde{\mathbb{P}}_x\Big(\int_0^t (\widehat{\sigma}_{ij}^2(\widetilde{X}_s) + |g_i(\widetilde{X}_s)|)ds < \infty\Big) = 1, \quad 1 \leq i \leq d,\ 1 \leq j \leq m,\ t \in [0,\infty),$$

and for any $1 \leq i \leq d$,

$$\widetilde{X}_t^i = x_i + \sum_{j=1}^m \int_0^t \widehat{\sigma}_{ij}(\widetilde{X}_s)\, d\widetilde{W}_s^j + \int_0^t g_i(\widetilde{X}_s)\, ds, \quad 0 \leq t < \infty, \quad \widetilde{\mathbb{P}}_x\text{-a.s.},$$

in short

$$\widetilde{X}_t = x + \int_0^t \widehat{\sigma}(\widetilde{X}_s)d\widetilde{W}_s + \int_0^t \mathbf{G}(\widetilde{X}_s)ds, \quad 0 \leq t < \infty, \quad \widetilde{\mathbb{P}}_x\text{-a.s.} \tag{28}$$

Then, $\widetilde{\mathbb{M}}$ is called a weak solution to Equation (28). In this case, $(t,\widetilde{\omega}) \mapsto \widehat{\sigma}(\widetilde{X}_t(\widetilde{\omega}))$ and $(t,\widetilde{\omega}) \mapsto \mathbf{G}(\widetilde{X}_t(\widetilde{\omega}))$ are progressively measurable with respect to $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$, and

$$\widetilde{D}_R := \inf\{t \geq 0 \mid \widetilde{X}_t \in \mathbb{R}^d \setminus B_R\} \nearrow \infty \quad \widetilde{\mathbb{P}}_x\text{-a.s. for any } x \in \mathbb{R}^d.$$

Remark 3. *(i) In Definition 1, the (real-valued) Borel measurable functions $\widehat{\sigma}_{ij}, g_i, \frac{1}{\psi}$ are fixed. In particular, the solution and the integrals involving the solution in Equation (28) may depend on the versions that we choose. When we fix the Borel measurable version $\frac{1}{\psi}$ with $\frac{1}{\psi}(x) \in [0,\infty)$ for all $x \in \mathbb{R}^d$, as in Definition 1, we always consider corresponding extended Borel measurable function ψ defined by*

$$\psi(x) := \frac{1}{\frac{1}{\psi}(x)}, \quad \text{if } \frac{1}{\psi}(x) \in (0,\infty), \qquad \psi(x) := \infty, \quad \text{if } \frac{1}{\psi}(x) = 0.pt$$

Thus, the choice of the special version for ψ depends on the previously chosen Borel measurable version $\frac{1}{\psi}$.

(ii) If $\mathbb{M}$ of Theorem 8 is nonexplosive (has infinite lifetime for any starting point), then it is a weak solution to Equation (28). Thus, a weak solution to Equation (28) exists just under Assumptions **(A1)**, **(A2)**, **(A3)**, *and* **(A4)**, *and a suitable growth condition (cf. Remark 2) on the coefficients. For this special weak solution, we know that integrals involving the solution do not depend on the chosen Borel versions. This follows similarly to [1] (Lemma 3.14(i)).*

Theorem 9 (Local Krylov-type estimate). *Assume* **(A4)$'$**, *and let $\widetilde{\mathbb{M}}$ be a weak solution to Equation (28). Let*

$$Z^{\widetilde{\mathbb{M}}}(\widetilde{\omega}) := \{t \geq 0 \mid \sqrt{\frac{1}{\psi}}(\widetilde{X}_t(\widetilde{\omega})) = 0\}$$

and

$$\Lambda(Z^{\widetilde{\mathbb{M}}}) := \{\widetilde{\omega} \in \widetilde{\Omega} \mid dt(Z^{\widetilde{\mathbb{M}}}(\widetilde{\omega})) = 0\}.$$

Assume that

$$\widetilde{\mathbb{P}}_x(\Lambda(Z^{\widetilde{\mathbb{M}}})) = 1 \quad \textit{for all } x \in \mathbb{R}^d. \tag{29}$$

Let $x \in \mathbb{R}^d$, $T > 0$, $R > 0$ and $f \in L^{2d+2,d+1}(B_R \times (0,T))$. Then, there exists a constant $C > 0$ that is independent of f such that

$$\widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} f(\widetilde{X}_s, s)ds\right] \le C\|f\|_{L^{2d+2,d+1}(B_R\times(0,T))},$$

where $\widetilde{\mathbb{E}}_x$ is the expectation w.r.t. $\widetilde{\mathbb{P}}_x$.

Using Theorem 9 and Equation (25), the proof of the following lemma is straightforward.

Lemma 2. *Let $\widetilde{\mathbb{M}}$ be a weak solution to Equation (28). Then, either of the following conditions implies Equation (29):*

(i) $\frac{1}{\psi}(x) \in (0,\infty)$ *for all $x \in \mathbb{R}^d$.*

(ii) *For each $n \in \mathbb{N}, T > 0$ and $x \in \mathbb{R}^d$ it holds*

$$\widetilde{\mathbb{E}}_x\left[\int_0^T 1_{B_n}\psi(\widetilde{X}_s)ds\right] < \infty,$$

where ψ denotes the extended Borel measurable version as explained in Remark 3(i). Moreover, Equation (5) is equivalent to Equation (29).

In particular, if the weak solution that is constructed in Theorem 8 is nonexplosive, then Condition (ii) always holds for this solution and (29) implies in general that integrals of the form $\int_0^t f(\widetilde{X}_s, s)ds$ are, whenever they are well-defined, independent of the particular Borel version that is chosen for f.

Theorem 10 (Local Itô-formula). *Assume* **(A4)′** *and let $\widetilde{\mathbb{M}}$ be a weak solution to (28) such that (29) holds. Let $R_0 > 0$, $T > 0$. Let $u \in W^{2,1}_{2d+2}(B_{R_0} \times (0,T)) \cap C(\overline{B}_{R_0} \times [0,T])$ be such that $\|\nabla u\| \in L^{4d+4}(B_{R_0} \times (0,T))$. Let $R > 0$ with $R < R_0$. Then $\widetilde{\mathbb{P}}_x$-a.s. for any $x \in \mathbb{R}^d$,*

$$u(\widetilde{X}_{T\wedge\widetilde{D}_R}, T\wedge\widetilde{D}_R) - u(x,0) = \int_0^{T\wedge\widetilde{D}_R} \nabla u(\widetilde{X}_s, s)\widehat{\sigma}(\widetilde{X}_s)d\widetilde{W}_s + \int_0^{T\wedge\widetilde{D}_R}(\partial_t u + Lu)(\widetilde{X}_s, s)ds,$$

where $Lu := \frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 u) + \langle \mathbf{G}, \nabla u\rangle$.

Theorem 11. *Assume* **(A4)′** *and let $f \in C_0^\infty(\mathbb{R}^d)$. Then there exists*

$$u_f \in C_b\left(\mathbb{R}^d \times [0,\infty)\right) \cap \Big(\bigcap_{r>0} W^{2,1}_{2d+2,\infty}(B_r \times (0,\infty))\Big)$$

satisfying $u_f(x,0) = f(x)$ for all $x \in \mathbb{R}^d$ such that

$$\partial_t u_f \in L^\infty(\mathbb{R}^d \times (0,\infty)),\ \partial_i u_f \in \bigcap_{r>0} L^\infty(B_r \times (0,\infty)) \ \textit{ for all } 1 \le i \le d,$$

and

$$\partial_t u_f = \frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 u_f) + \langle \mathbf{G}, \nabla u_f \rangle \quad \text{a.e. on } \mathbb{R}^d \times (0,\infty).$$

Definition 2. *We say that uniqueness in law holds for Equation (28) if, for any two weak solutions,*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (W_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$$

and

$$\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{W}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d})$$

of (28) it holds $\mathbb{P}_x \circ X^{-1} = \widetilde{\mathbb{P}}_x \circ \widetilde{X}^{-1}$ *for all* $x \in \mathbb{R}^d$. *We say that the stochastic differential Equation (28) is well-posed if there exists a weak solution to it, and uniqueness in law holds.*

Theorem 12. *Assume Condition* **(A4)′**. *Consider two arbitrarily given weak solutions to Equation (28),* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (W_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ *and* $\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{W}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d})$. *Suppose*

$$\mathbb{P}_x(\Lambda(Z^{\mathbb{M}})) = \widetilde{\mathbb{P}}_x(\Lambda(Z^{\widetilde{\mathbb{M}}})) = 1, \quad \text{for all } x \in \mathbb{R}^d. \tag{30}$$

Then, $\mathbb{P}_x \circ X^{-1} = \widetilde{\mathbb{P}}_x \circ \widetilde{X}^{-1}$ *for all* $x \in \mathbb{R}^d$. *In particular, under Assumption* **(A4)′**, *any weak solution to Equation (28) is a strong Markov process.*

Combining Theorem 12, Remark 2, and Theorem 8, we obtain the following result.

Theorem 13. *Assume Condition* **(A4)′**, *and suppose that* $\mathbb{M}$ *of Theorem 8 is nonexplosive. (This is, for instance, the case if (3) holds; see Theorem 8.) Then, Hunt process* $\mathbb{M}$ *forms a unique solution (in law) to Equation (28) that satisfies* $\mathbb{P}_x(\Lambda(Z^{\mathbb{M}})) = 1$, *for all* $x \in \mathbb{R}^d$. *Moreover, under the same conditions as in [1] (Theorem 4.4), but replacing* A, σ *there with* $\frac{1}{\psi}A$, $\sqrt{\frac{1}{\psi}}\sigma$, *respectively, the moment inequalities of the mentioned theorem also hold for our* $\mathbb{M}$ *here.*

Remark 4. *Once uniqueness in law holds for Equation (28), any weak solution to Equation (28) satisfies the improved (time-homogeneous) Krylov-type estimate (25). We illustrate this with respect to each other extreme cases. For the first case, suppose that Assumption* **(A4)′** *holds with* $q = 2d + 2 + \varepsilon$ *for some small* $\varepsilon > 0$. *Then, we may choose* $s = \frac{2}{3}d$ *and* $s_0 := \frac{sq}{q-1} = \frac{2}{3}d \cdot \frac{2d+2+\varepsilon}{2d+1+\varepsilon}$ *satisfies* $s_0 < \frac{4}{5}d$, *actually* $s_0 = \frac{4}{5}d - \delta$ *for small* $\delta > 0$; *for any bounded open set* V, *any ball* $B \subset \mathbb{R}^d$, *and* $g \in L^{s_0}(\mathbb{R}^d)_0$ *with* $supp(g) \subset V$, *we have by Equation (25) for any* $x \in \overline{B}$

$$\begin{aligned}\mathbb{E}_x\left[\int_0^T g(X_s)ds\right] &\leq C_{B,s,t}\,\|g\|_{L^s(\mathbb{R}^d,\mu)} \\ &\leq C_{B,s,t}\,\|\rho\psi\|^{1/s}_{L^q(V)}\left(\int_V |g|^{\frac{sq}{q-1}}dx\right)^{\frac{q-1}{sq}} = C_{B,s,t}\,\|\rho\psi\|^{1/s}_{L^q(V)}\|g\|_{L^{s_0}(V)}.\end{aligned}$$

On the other hand, if Assumption **(A4)′** *holds with* $q = \infty$, *and* $\frac{1}{\psi}$ *is supposed to be locally pointwise bounded below and above by strictly positive constants, we may choose* $s = \frac{d}{2} + \varepsilon$ *for arbitrarily small* $\varepsilon > 0$, *and we obtain for* $g \in L^s(\mathbb{R}^d)_0$ *with* $supp(g) \subset V$, V, B *and* x *as above,*

$$\mathbb{E}_x\left[\int_0^T g(X_s)ds\right] \leq C_{B,s,t}\,\|\rho\psi\|^{1/s}_{L^\infty(V)}\|g\|_{L^s(V)}.$$

Example 2. *Consider the situation in Example 1 except for Conditions (a), (b), (c). Let* $p := 2d + 2$*, and assume* $\mathbf{G} \in L^{\infty}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$*. Let* $\alpha \geq 0$ *be such that* $\alpha(2d + 2) < d$*. Take* $q \in (2d + 2, \frac{d}{\alpha})$*. Then* A, $\mathbf{G}$*, and* ψ *satisfy Assumption* **(A4)**$'$*. Therefore, Hunt process* $\mathbb{M}$ *of Theorem 8 solves weakly* $\mathbb{P}_x$*-a.s. for any* $x \in \mathbb{R}^d$,

$$X_t = x + \int_0^t \|X_s\|^{\alpha/2} \cdot \frac{\sigma}{\sqrt{\phi}}(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad 0 \leq t < \zeta. \tag{31}$$

Assume Equation (3). Then $\zeta = \infty$ *and by Theorem 13,* $\mathbb{M}$ *is the unique (in law) solution to Equation (31) that satisfies* $\mathbb{P}_x(\Lambda(Z^{\mathbb{M}})) = 1$*, for all* $x \in \mathbb{R}^d$*. If we choose the following Borel measurable version of* $\|x\|^{\alpha/2}$*, namely,*

$$f_\gamma(x) := \|x\|^{\alpha/2} 1_{\{x \neq 0\}}(x) + \gamma 1_{\{x=0\}}(x), \quad x \in \mathbb{R}^d$$

where γ *is an arbitrary but fixed strictly positive real number, then* $\mathbb{P}_x(\Lambda(Z^{\widetilde{\mathbb{M}}})) = 1$ *(here, of course,* $Z^{\widetilde{\mathbb{M}}}$ *is defined w.r.t.* $\sqrt{\frac{1}{\psi}} = \frac{f_\gamma}{\sqrt{\phi}}$*) is automatically satisfied by Lemma 2(i) for any weak solution* $\widetilde{\mathbb{M}}$ *to*

$$\widetilde{X}_t = x + \int_0^t \frac{f_\gamma \cdot \sigma}{\sqrt{\phi}}(\widetilde{X}_s)\, d\widetilde{W}_s + \int_0^t \mathbf{G}(\widetilde{X}_s)\, ds, \quad t \geq 0,\ x \in \mathbb{R}^d, \tag{32}$$

Thus, under Equation (3), the SDE (32) is well-posed for any $\gamma > 0$*, and* $\mathbb{M}$ *of Theorem 8 also solves (32).*

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Appendix A. Proofs and Auxiliary Statements

In this section, we collect all proofs of statements given in this article, and the statement of several auxiliary Lemmas.

The following lemma is a slight modification of [17] (Lemma 6) and involves a weight function ψ.

Lemma A1. *Let* U *be a bounded open subset of* $\mathbb{R}^d$ *and* $T > 0$*. Let* $w \in L^2(U \times (0,T))$ *be such that* $\text{supp}(w) \subset U \times (0, T]$ *and assume* $\partial_t w \in L^2(U \times (0,T))$, $\psi \in L^2(U)$*. Then, for a.e.* $\tau \in (0,T)$*, it holds*

$$\int_0^\tau \int_U \partial_t w \cdot \psi \, dxdt = \int_U w|_{t=\tau}\, \psi dx.$$

Proof of Lemma A1. Using an approximation $\lim_{n\to\infty} \psi_n = \psi$ in $L^2(U)$, with $\psi_n \in C_0^\infty(U)$, $n \geq 1$, and noting that $w\psi \in L^{1,2}(U \times (0,T))$ for any $\varphi \in C_0^\infty(U \times (0,T))$, we obtain

$$\iint_{U\times(0,T)} \partial_t \varphi \cdot w\psi \, dxdt \quad = \quad -\iint_{U\times(0,T)} \varphi \cdot (\partial_t w \cdot \psi) dxdt.$$

Thus, $\partial_t(w\psi) = \partial_t w \cdot \psi \in L^{1,2}(U \times (0,T))$. Now let $f(t) := \int_U w(x,t)\psi(x)dx$. Then $f(t)$ is defined for a.e. $t \in (0,T)$ and is in $L^1((0,T))$. Let $g \in C_0^\infty\big((0,T)\big)$ be given. Then

$$\int_0^T \partial_t g \cdot f\,dt \quad = \quad -\int_0^T g \cdot \left(\int_U \partial_t w \cdot \psi dx\right) dt.$$

Thus, $\partial_t f = \int_U \partial_t w \cdot \psi dx \in L^1\big((0,T)\big)$. Then, by [18] (Theorem 4.20), f has an absolutely continuous dx-version on $(0,T)$ and by the Fundamental Theorem of Calculus, for a.e $\tau_1, \tau \in (0,T)$ it holds

$$\int_{\tau_1}^{\tau} \int_U \partial_t w \cdot \psi dxdt = \int_{\tau_1}^{\tau} \partial_t f dt = \int_{\tau_1}^{\tau} f' dt = f(\tau) - f(\tau_1) = \int_U (w|_{t=\tau} - w|_{t=\tau_1})\,\psi dx.$$

Choosing τ_1 near 0, our assertion follows. □

Lemma A2. *Assume Conditions* **(I)** *and* (7). *Let* $\beta \geq 1$ *be a constant and* $\eta \in C^\infty(\overline{U} \times [0,T])$ *with* $supp(\eta) \subset U \times (0,T]$ *and* $\eta \geq 0$. *Then, for a.e.* $\tau \in (0,T)$

$$\begin{aligned} &\frac{1}{\beta+1}\int_U \eta^2 (u^+)^{\beta+1}\,|_{t=\tau}\,\psi dx + \frac{\lambda\beta}{2}\int_0^\tau \int_U \eta^2 (u^+)^{\beta-1}\|\nabla u^+\|^2 dxdt \\ &\leq \int_0^\tau \int_U \left(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\right)(u^+)^{\beta+1}dxdt + \frac{2}{\beta+1}\int_0^\tau \int_U \eta|\partial_t\eta|(u^+)^{\beta+1}\,\psi dxdt. \end{aligned} \tag{A1}$$

Proof of Lemma A2. Using integration by parts in the left hand term, Equation (7) is equivalent to

$$-\iint_{U\times(0,T)} (\partial_t u)\,\varphi\psi dxdt = \iint_{U\times(0,T)} \langle A\nabla u, \nabla\varphi\rangle + \langle \mathbf{B}, \nabla u\rangle \varphi dxdt, \tag{A2}$$

for all $\varphi \in C_0^\infty(U \times (0,T))$. Using the standard mollification on $\mathbb{R}^d \times \mathbb{R}$ to approximate functions in $\mathcal{A} := \{v \in L^\infty(U \times (0,T)) \mid \nabla v \in L^2(U \times (0,T)),\ \text{supp}(v) \subset U \times (0,T)\}$, (A2) extends to

$$-\iint_{U\times(0,T)} (\partial_t u)\,\varphi\psi dxdt = \iint_{U\times(0,T)} \langle A\nabla u, \nabla\varphi\rangle + \langle \mathbf{B}, \nabla u\rangle \varphi dxdt, \quad \forall \varphi \in \mathcal{A}. \tag{A3}$$

For $t \in \mathbb{R}$, define functions $G(t) := (t^+)^\beta$, $H(t) := \frac{1}{\beta+1}(t^+)^{\beta+1}$, where $t^+ := \max(0,t)$. Then, by [18] (Theorem 4.4), $G'(t) = \beta(t^+)^{\beta-1}1_{[0,\infty)}(t)$ and $H'(t) = G(t)$. Given $\tau \in (0,T)$, define $\widetilde{\varphi} := \eta^2 G(u) 1_{(0,\tau)}$. Then, by [18] (Theorem 4.4) (or [17] (Lemma 4)),

$$\nabla\widetilde{\varphi} = \begin{cases} \eta^2 G'(u)\nabla u + 2\eta\nabla\eta\, G(u), & 0 < t < \tau, \\ 0, & \tau \leq t < T. \end{cases}$$

Thus, $\widetilde{\varphi} \in \mathcal{A}$ and by Assumption (A3), we have

$$-\iint_{U\times(0,T)} (\partial_t u)\,\widetilde{\varphi}\psi dxdt = \iint_{U\times(0,T)} \langle A\nabla u, \nabla\widetilde{\varphi}\rangle + \langle \mathbf{B}, \nabla u\rangle \widetilde{\varphi} dxdt. \tag{A4}$$

By [18] (Theorem 4.4) (or [17] (Lemma 4)),

$$\partial_t(\eta^2 H(u)) = 2\eta\partial_t\eta\, H(u) + \eta^2 G(u)\partial_t u.$$

Thus, by Lemma A1

$$\iint_{U\times(0,T)} \widetilde{\varphi}\,(\partial_t u)\,\psi dxdt = \int_U \eta^2 H(u)\,|_{t=\tau}\,\psi dx - \int_0^\tau \int_U 2\eta\partial_t\eta\, H(u)\psi dxdt, \tag{A5}$$

for a.e. $\tau \in (0,T)$. By Assumptions (A4) and (A5), we get

$$\begin{aligned} &\int_U \eta^2 H(u)\,|_{t=\tau}\,\psi dxdt + \int_0^\tau \int_U \langle A\nabla u, \nabla\widetilde{\varphi}\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi} dxdt \\ = &\int_0^\tau \int_U 2\eta\,\partial_t\eta\, H(u)\,\psi dxdt, \end{aligned} \tag{A6}$$

for a.e. $\tau \in (0,T)$. On $\{\widetilde{\varphi} > 0\}$, it holds that $u > 0$, so that $\nabla u = \nabla u^+$. Thus, on $\{\widetilde{\varphi} > 0\}$,

$$\begin{aligned} &\langle A\nabla u, \nabla\widetilde{\varphi}\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi} \\ &\geq \eta^2 G'(u)\lambda\,\|\nabla u^+\|^2 - 2\eta G(u) dM \|\nabla\eta\|\|\nabla u^+\| - \eta^2 G(u)\|\mathbf{B}\|\|\nabla u^+\|. \end{aligned}$$

Moreover, on $\{\widetilde{\varphi} > 0\}$, it holds $(u^+)^{-\beta-1}\,G(u)^2 \leq G'(u)$. Hence, using Young's inequality, we obtain

$$2\eta G(u) dM\|\nabla\eta\|\|\nabla u^+\| \;\leq\; \frac{\lambda}{4}\eta^2 G'(u)\|\nabla u^+\|^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\,(u^+)^{\beta+1},$$

and

$$\eta^2 G(u)\|\mathbf{B}\|\|\nabla u^+\| \;\leq\; \frac{\lambda}{4}\eta^2 G'(u)\|\nabla u^+\|^2 + \frac{1}{\lambda}\|\mathbf{B}\|^2 (u^+)^{\beta+1}\eta^2.$$

Therefore, on $\{\widetilde{\varphi} > 0\}$, it holds that

$$\begin{aligned} &\frac{\lambda}{2}\eta^2 G'(u)\|\nabla u^+\|^2 \\ &\leq \langle A\nabla u, \nabla\widetilde{\varphi}\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi} + \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big)(u^+)^{\beta+1}. \end{aligned} \tag{A7}$$

Since $\{\widetilde{\varphi} = 0\} \cap \big(U \times (0,\tau)\big) = \{\eta = 0\} \cup \{u \leq 0\}$ and $\nabla u^+ = 0$ on $\{u \leq 0\}$, (A7) holds on $U \times (0,\tau)$. Combining Assumptions (A7) and (A6), we obtain Assumption (A1). □

Proof of Theorem 1. Let $\eta \in C^\infty\big(\overline{R}_{\bar{x}}(r) \times [\bar{t} - 9r^2, \bar{t}\,]\big)$ with $\operatorname{supp}(\eta) \subset R_{\bar{x}}(r) \times (\bar{t} - 9r^2, \bar{t}\,]$ and $\eta \geq 0$. Then, by Lemma A2, Assumption (A1) holds with $U \times (0,T)$ replaced by $Q(3r)$. Using appropriate scaling arguments (cf. [17] (proof of Theorem 2)), we may assume $r = \frac{1}{3}$. Set $v := (u^+)^\gamma$ with $\gamma := \frac{\beta+1}{2}$. Then $\|\nabla v\|^2 = \gamma^2 (u^+)^{\beta-1}\|\nabla u^+\|^2$. By Lemma A2, it holds for a.e. $\tau \in (\bar{t} - 1, \bar{t})$

$$\begin{aligned} &\frac{c_0}{2\gamma}\int_{R_{\bar{x}}(1)} \eta^2 v^2\,|_{t=\tau}\,dx + \frac{\lambda}{2\gamma^2}\int_{\bar{t}-1}^{\tau}\int_{R_{\bar{x}}(1)} \eta^2\|\nabla v\|^2 dxdt \\ &\quad\leq \iint_{Q(1)} \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big) v^2 dxdt + \iint_{Q(1)} \eta|\partial_t\eta| v^2\,\psi dx. \end{aligned}$$

Let l and l' be positive numbers satisfying $\frac{1}{3} < l' < l \le \frac{2}{3}$. Assume that $\eta \equiv 1$ in $Q(l')$, $\eta \equiv 0$ outside $Q(l)$, $0 \le \eta \le 1$, and $|\partial_t \eta|, \|\nabla \eta\| \le 2d(l-l')^{-1}$. Then,

$$\iint_{Q(1)} \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big) v^2 dxdt \\ \le \frac{4d^2}{\lambda}(l-l')^{-2}(\|\mathbf{B}\|^2_{L^p(R_{\bar{x}}(1))} + 4d^2M^2)\|v\|^2_{L^{\frac{2p}{p-2},2}(Q(l))},$$

and $\frac{2q}{q-1} \le \frac{2p}{p-2}$, it follows that

$$\int_{\bar{t}-1}^{\bar{t}} \int_{R(1)} \eta|\partial_t\eta| v^2\, \psi dx \quad \le \quad 2d(l-l')^{-2}\|\psi\|_{L^q(R_{\bar{x}}(1))}\|v\|^2_{L^{\frac{2p}{p-2},2}(Q(l))}.$$

Thus, we obtain

$$\lambda\|\eta\nabla v\|^2_{L^2(Q(1))} \le 2C_1(l-l')^{-2}\gamma^2\|v\|^2_{L^{\frac{2p}{p-2},2}(Q(l))}$$

and

$$\|\eta v\|^2_{L^{2,\infty}(Q(1))} \le 2c_0^{-1}C_1(l-l')^{-2}\gamma^2\|v\|^2_{L^{\frac{2p}{p-2},2}(Q(l))},$$

where $C_1 = \frac{4d^2}{\lambda}(\|\mathbf{B}\|^2_{L^p(R_{\bar{x}}(1))} + 4d^2M^2) + 2d\|\psi\|_{L^q(R_{\bar{x}}(1))}$.

Now set $\theta := 1 - \frac{d}{p}$ and $\sigma := 1 + \frac{\theta}{2}$ if $d = 2$, $\quad \sigma := 1 + \frac{2\theta}{d}$ if $d \ge 3$. Set $p_\sigma := \Big(\frac{\sigma p}{p-2}\Big)' = \frac{\sigma p}{\sigma p - p + 2}$, $q_\sigma := \sigma' = \frac{\sigma}{\sigma - 1}$. Then

$$\frac{d}{2p_\sigma} + \frac{1}{q_\sigma} < 1 \ \text{ if } d = 2, \qquad \frac{d}{2p_\sigma} + \frac{1}{q_\sigma} = 1 \ \text{ if } d \ge 3.$$

By [17] (Lemma 3),

$$\begin{aligned} \|v^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2},2}(Q(l'))} &\le \|(\eta v)^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2},2}(Q(1))} = \|\eta v\|^2_{L^{\frac{2\sigma p}{p-2},2\sigma}(Q(1))} = \|\eta v\|^2_{L^{2(p_\sigma)',2(q_\sigma)'}(Q(1))} \\ &\le K\Big(\|\eta v\|^2_{L^{2,\infty}(Q(1))} + 2\|\eta\nabla v\|^2_{L^2(Q(1))} + 8d^2(l-l')^{-2}\|v\|^2_{L^2(Q(l))}\Big) \\ &\le C_2(l-l')^{-2}\gamma^2\|v\|^2_{L^{\frac{2p}{p-2},2}(Q(l))}, \end{aligned} \tag{A8}$$

where $C_2 = K(4C_1\lambda^{-1} + 2C_1c_0^{-1} + 8d^2)$. Now, for $m \in \mathbb{N} \cup \{0\}$, set $l = l_m := 3^{-1}(1 + 2^{-m})$, $\ l' = l'_m := 3^{-1}(1 + 2^{-m-1})$, $\ \varphi_m := \|(u^+)^{\sigma^m}\|^{2/\sigma^m}_{L^{\frac{2p}{p-2},2}(Q(l_m))}$. Taking $\gamma = \sigma^m$ and $1/3 < l' = l'_m < l = l_m \le 2/3$ for $m \in \mathbb{N} \cup \{0\}$, we obtain using Assumption (A8)

$$\varphi_{m+1} \le (36C_2)^{\frac{1}{\sigma^m}}(2\sigma)^{\frac{2m}{\sigma^m}}\varphi_m. \tag{A9}$$

Iterating Assumption (A9), we get

$$\varphi_{m+1} \quad \le \quad (36C_2)^{\sum_{i=0}^m \frac{1}{\sigma^i}}(2\sigma)^{\sum_{i=0}^m \frac{2i}{\sigma^i}}\varphi_0 \ \le\ \underbrace{(36C_2)^{\frac{\sigma}{\sigma-1}}(2\sigma)^{\frac{2\sigma}{(\sigma-1)^2}}}_{=:C_3}\|u\|^2_{L^{\frac{2p}{p-2},2}(Q(2/3))}.$$

Letting $m \to \infty$, we get

$$\|u^+\|_{L^\infty(Q(1/3))} \le \sqrt{C_3}\|u\|_{L^{\frac{2p}{p-2},2}(Q(2/3))}.$$

Exactly in the same way, but with u replaced by $-u$, we obtain Equation (8) with $C = 2\sqrt{C_3}$. □

Lemma A3. *Let U be a bounded open ball in $\mathbb{R}^d$. Let $f \in L^{\widetilde{q}}(U)$ with $\frac{d}{2} < \widetilde{q} < d$. Then, there exists $\mathbf{F} = (f_1, \dots, f_d) \in H^{1,\widetilde{q}}(U, \mathbb{R}^d)$ such that $\mathrm{div}\mathbf{F} = f$ in U and*

$$\sum_{i=1}^{d} \|f_i\|_{H^{1,\widetilde{q}}(U)} \le C\|f\|_{L^{\widetilde{q}}(U)},$$

where $C > 0$ only depends on $\widetilde{q}$, U. In particular, applying the Sobolev inequality, we get

$$\sum_{i=1}^{d} \|f_i\|_{L^{\frac{d\widetilde{q}}{d-\widetilde{q}}}(U)} \le C'\|f\|_{L^{\widetilde{q}}(U)},$$

where $C' > 0$ only depends on $\widetilde{q}$, U.

Proof of Lemma A3. By [19] (Theorem 9.15 and Lemma 9.17), there exists $u \in H^{2,\widetilde{q}}(U) \cap H_0^{1,\widetilde{q}}(U)$ such that $\Delta u = f$ in U and

$$\|u\|_{H^{2,\widetilde{q}}(U)} \le C_1\|f\|_{L^{\widetilde{q}}(U)},$$

where $C_1 > 0$ is a constant only depending on $\widetilde{q}$, U. Let $\mathbf{F} := \nabla u$. Then $\mathbf{F} \in H^{1,\widetilde{q}}(U, \mathbb{R}^d)$ with $\mathrm{div}\mathbf{F} = f$ in U and it holds that

$$\begin{aligned}\sum_{i=1}^{d} \|f_i\|_{H^{1,\widetilde{q}}(U)} &\le \left(d + d^2\right)^{\frac{\widetilde{q}-1}{\widetilde{q}}} \left(\sum_{i=1}^{d} \|\partial_i u\|_{L^{\widetilde{q}}(U)}^{\widetilde{q}} + \sum_{i=1}^{d}\sum_{j=1}^{d} \|\partial_j\partial_i u\|_{L^{\widetilde{q}}(U)}^{\widetilde{q}}\right)^{\frac{1}{\widetilde{q}}} \\ &\le C_1 \left(d + d^2\right)^{\frac{\widetilde{q}-1}{\widetilde{q}}} \|f\|_{L^{\widetilde{q}}(U)}.\end{aligned}$$

□

Proof of Theorem 2. Without loss of generality, we may assume that $\frac{d}{2} < \widetilde{q} < d$. Let U_2 be an open ball in $\mathbb{R}^d$ satisfying $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$. By Lemma A3, we can find $\mathbf{F} = (f_1, \dots, f_d) \in H^{1,\widetilde{q}}(U_2, \mathbb{R}^d) \subset L^{\frac{d\widetilde{q}}{d-\widetilde{q}}}(U_2, \mathbb{R}^d)$ such that

$$\mathrm{div}\mathbf{F} = f \ \text{ in } U_2, \qquad \sum_{i=1}^{d} \|f_i\|_{L^{\frac{d\widetilde{q}}{d-\widetilde{q}}}(U_2)} \le C_1\|f\|_{L^{\widetilde{q}}(U_2)},$$

where $C_1 > 0$ is a constant only depending on $\widetilde{q}$ and U_2. Then, Equation (9) implies

$$\int_{U_2} \langle A\nabla u, \nabla\varphi\rangle + (\langle \mathbf{B}, \nabla u\rangle + cu)\,\varphi\, dx = \int_{U_2} \langle -\mathbf{F}, \nabla\varphi\rangle\, dx \ \text{ for all } \varphi \in C_0^\infty(U_2).$$

Given $x \in U_1$, $r > 0$ with $r < \mathrm{dist}(x, U_2)$, set $\omega_x(r) := \sup_{B_x(r)} u - \inf_{B_x(r)} u$. By [14] (Théorème 7.2) and Lemma A3,

$$\omega_x(r) \le K \left(\|u\|_{L^2(U_2)} + \sum_{i=1}^{d} \|f_i\|_{L^{\frac{d\tilde{q}}{d-\tilde{q}}}(U_2)} \right) r^\gamma \le K(1+C') \left(\|u\|_{L^2(U_2)} + \|f\|_{L^{\tilde{q}}(U_2)} \right) r^\gamma,$$

where $\gamma \in (0,1)$ and $K, C' > 0$ are constants that are independent of $x, r, u, \mathbf{F}, f$. Thus, we have

$$\int_{B_r(x)} |u(y) - u_{x,r}|^2 dy \le (K')^2 \left(\|u\|_{L^2(U_2)} + \|f\|_{L^{\tilde{q}}(U_2)} \right)^2 r^{d+2\gamma},$$

where $u_{x,r} := \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy$ and $(K')^2 := K^2 \cdot \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} (1 + C')^2$. Finally by [20] (Theorem 3.1), [15] (Theorem 1.7.4) (the VMO condition and symmetry of $A = (a_{ij})_{1\le i,j\le d}$ are not needed in [15] (Theorem 1.7.4), as we can see from its proof), we obtain

$$\begin{aligned}
\|u\|_{C^{0,\gamma}(\overline{U}_1)} &\le c\big(K' \left(\|u\|_{L^2(U_2)} + \|f\|_{L^{\tilde{q}}(U_2)} \right) + \|u\|_{L^2(U_2)}\big) \\
&\le (cK' \vee c) \left(\|u\|_{H^{1,2}(U_2)} + \|f\|_{L^{\tilde{q}}(U_2)} \right) \\
&\le (cK' \vee c) \left(C_1 \|u\|_{L^1(U)} + C_1 \|f\|_{L^{\tilde{q}}(U)} + \|f\|_{L^{\tilde{q}}(U_2)} \right) \\
&\le (C_1 + 1)\,(cK' \vee c)\,\left(\|u\|_{L^1(U)} + \|f\|_{L^{\tilde{q}}(U)} \right),
\end{aligned}$$

where $c > 0$, $C_1 > 0$ are constants that are independent of u and f. □

Proof of Theorem 4. By [8] (Theorem 3.6), there exists $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ satisfying $\rho(x) > 0$ for all $x \in \mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d} \langle \frac{1}{2} A \nabla \rho + (\frac{1}{2} \nabla A - \psi \mathbf{G}) \rho, \nabla \varphi \rangle dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d),$$

hence

$$\int_{\mathbb{R}^d} \langle \mathbf{G} - \frac{\nabla A}{2\psi} - \frac{A \nabla \rho}{2\rho\psi}, \nabla \varphi \rangle \rho \psi dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d);$$

moreover,

$$\rho\psi\,\mathbf{B} = \rho\psi\mathbf{G} - \frac{\rho}{2} \nabla A - \frac{A\nabla\rho}{2} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d).$$

The equivalence of Equation (18) and (6) follows since $Lf = L^0 f + \langle \mathbf{G} - \beta^{\rho,A,\psi}, \nabla f \rangle$, $f \in C_0^\infty(\mathbb{R}^d)$, where L^0 is as in Equation (12) and by elementary calculation $\int_{\mathbb{R}^d} L^0 f\, d\mu = 0$ for any $f \in C_0^\infty(\mathbb{R}^d)$. □

Proof of Theorem 5. Let $f \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$. Then, by Theorem 3, $G_\alpha f \in D(\overline{L})_b \subset D(\mathcal{E}^0)$ and

$$\mathcal{E}^0(G_\alpha f, \varphi) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla G_\alpha f \rangle \varphi d\mu \;=\; -\int_{\mathbb{R}^d} (\overline{L}\,\overline{G}_\alpha f)\, \varphi\, d\mu = \int_{\mathbb{R}^d} (f - \alpha G_\alpha f) \varphi\, d\mu, \tag{A10}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. Thus, Assumption (A10) implies for all $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}
&\int_{\mathbb{R}^d} \langle \frac{1}{2} \rho A \nabla G_\alpha f, \nabla \varphi \rangle dx - \int_{\mathbb{R}^d} \langle \rho\psi\mathbf{B}, \nabla G_\alpha f \rangle \varphi\, dx + \int_{\mathbb{R}^d} (\alpha\rho\psi G_\alpha f)\, \varphi dx \\
&= \int_{\mathbb{R}^d} (\rho\psi f)\, \varphi dx.
\end{aligned} \tag{A11}$$

ρ is locally bounded below and above on $\mathbb{R}^d$ and $\rho\psi\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $\alpha\rho\psi \in L^q_{loc}(\mathbb{R}^d)$. Let B and B' be open balls in $\mathbb{R}^d$ satisfying $\overline{B} \subset B'$. Since $\frac{1}{\psi} \in L^\infty(B')$, $G_\alpha f \in H^{1,2}(B')$. Thus, by Theorem 2, there exist a Hölder continuous μ-version $R_\alpha f$ of $G_\alpha f$ on $\mathbb{R}^d$ and constants $\gamma \in (0,1)$, $c_1 > 0$ that are independent of f, such that

$$\begin{aligned}\|R_\alpha f\|_{C^{0,\gamma}(\overline{B})} &\leq c_1\big(\|G_\alpha f\|_{L^1(B')} + \|\rho\psi f\|_{L^{(\frac{1}{q}+\frac{1}{s})^{-1}}(B')}\big)\\ &\leq c_2\big(\|G_\alpha f\|_{L^1(B',\mu)} + \|f\|_{L^s(B',\mu)}\big),\end{aligned} \tag{A12}$$

where $c_2 := c_1\big(\frac{1}{\inf_{B'}\rho\psi} \vee \frac{\|\rho\psi\|_{L^q(B')}}{(\inf_{B'}\rho\psi)^{1/s}}\big)$. Using the Hölder inequality and the contraction property, Assumption (A12) extends to $f \in \cup_{r\in[s,\infty)} L^r(\mathbb{R}^d, \mu)$. In order to extend Assumption (A12) to $f \in L^\infty(\mathbb{R}^d, \mu)$, let $f_n := 1_{B_n} \cdot f \in L^q(\mathbb{R}^d, \mu)_0$, $n \geq 1$. Then, $\|f - f_n\|_{L^s(B',\mu)} + \|G_\alpha(f - f_n)\|_{L^1(B',\mu)} \to 0$ as $n \to \infty$ by Lebesgue's Theorem. Hence, Assumption (A12) also extends to $f \in L^\infty(\mathbb{R}^d, \mu)$. □

The following well-known fact is stated without proof.

Lemma A4. *Let U be a bounded open subset of $\mathbb{R}^d$ and $T > 0$. Then, $C_0^2(U \times (0,T))$ is in the closure of $\{\sum_{i=1}^N f_i g_i \,|\, f_i \in C_0^\infty(U),\ g_i \in C_0^\infty((0,T)),\ i = 1,\dots,N,\ N \in \mathbb{N}\}$ w.r.t. $\|u\|_{C^2(\overline{U}\times[0,T])} := \|u\|_{C(\overline{U}\times[0,T])} + \sum_{i=1}^{d+1} \|\partial_i u\|_{C(\overline{U}\times[0,T])} + \sum_{i,j=1}^{d+1} \|\partial_i\partial_j u\|_{C(\overline{U}\times[0,T])}$.*

Proof of Theorem 6. First assume $f \in D(\overline{L})_b \cap D(L_s) \cap D(L_2)$. By means of Lemma 1, define $u \in C_b(\mathbb{R}^d \times [0,\infty))$ by $u(x,t) := P_t f(x)$. Note that for any bounded open set $O \subset \mathbb{R}^d$ and $T > 0$, it holds $u \in H^{1,2}(O \times (0,T))$ by Lemma A6 below. Let $\varphi_1 \in C_0^\infty(\mathbb{R}^d)$, $\varphi_2 \in C_0^\infty((0,T))$. Observe that $T_t f \in D(\overline{L})_b$, hence

$$\begin{aligned}&\iint_{\mathbb{R}^d\times(0,T)} \langle\frac{1}{2}\rho A\nabla u, \nabla(\varphi_1\varphi_2)\rangle - \langle\rho\psi\mathbf{B}, \nabla\,(T_t f)\rangle\varphi_1\varphi_2\,dxdt\\ &= \int_0^T \varphi_2(\mathcal{E}^0(T_t f, \varphi_1) - \int_{\mathbb{R}^d}\langle\mathbf{B}, \nabla T_t f\rangle\varphi_1\,d\mu)dt\\ &= \int_0^T -\varphi_2(\frac{d}{dt}\int_{\mathbb{R}^d}\varphi_1 T_t f\,\rho\psi dx)dt \;=\; \iint_{\mathbb{R}^d\times(0,T)} u\,\partial_t(\varphi_1\varphi_2)\rho\psi dxdt.\end{aligned} \tag{A13}$$

By Lemma A4, Assumption (A13) extends to

$$\iint_{\mathbb{R}^d\times(0,T)} \langle\frac{1}{2}\rho A\nabla u, \nabla\varphi\rangle - \langle\rho\psi\mathbf{B}, \nabla\,(T_t f)\rangle\varphi\,dxdt = \iint_{\mathbb{R}^d\times(0,T)} u\,\partial_t\varphi\cdot\rho\psi dxdt \tag{A14}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$. Let $\tau_2^* := \frac{\tau_2+\tau_4}{2}$ and take $r > 0$ so that

$$r < \frac{\sqrt{\tau_1 - \tau_3}}{2} \quad \text{and} \quad R_{\bar{x}}(2r) \subset V,\ \forall \bar{x} \in \overline{U}.$$

Then, for all $(\bar{x}, \bar{t}) \in \overline{U} \times [\tau_1, \tau_2^*]$, we have $R_{\bar{x}}(2r) \times (\bar{t} - (2r)^2, \bar{t}) \subset V \times (\tau_3, \tau_4)$. Using the compactness of $\overline{U} \times [\tau_1, \tau_2]$, there exist $(x_i, t_i) \in \overline{U} \times [\tau_1, \tau_2^*]$, $i = 1, \dots, N$, such that

$$\overline{U} \times [\tau_1, \tau_2] \subset \bigcup_{i=1}^N R_{x_i}(r) \times (t_i - r^2, t_i).$$

Using Theorem 1,

$$\begin{aligned}\|u\|_{C(\overline{U}\times[\tau_1,\tau_2])} &= \sup_{\overline{U}\times[\tau_1,\tau_2]} |u| \leq \max_{i=1,\ldots,N} \sup_{R_{x_i}(r)\times(t_i-r^2,t_i)} |u| \\ &\leq \max_{i=1,\ldots,N} c_i \|u\|_{L^{\frac{2p}{p-2},2}\left(R_{x_i}(2r)\times(t_i-(2r)^2,t_i)\right)} \\ &\leq \underbrace{\left(\max_{i=1,\ldots,N} c_i\right)}_{=:C_1} \|u\|_{L^{\frac{2p}{p-2},2}\left(V\times(\tau_3,\tau_4)\right)},\end{aligned}$$

where $c_i > 0$ $(1 \leq i \leq N)$ are constants that are independent of u. Thus, for $\nu \geq \frac{2p}{p-2}$,

$$\begin{aligned}\|P_\cdot f\|_{C(\overline{U}\times[\tau_1,\tau_2])} &\leq C_1 \|P_\cdot f\|_{L^{\frac{2p}{p-2},2}\left(V\times(\tau_3,\tau_4)\right)} \qquad \text{(A15)} \\ &\leq C_1 \left(\frac{1}{\inf_V \rho\psi}\right)^{\frac{p-2}{2p}} \left(\int_{\tau_3}^{\tau_4} \|T_t f\|^2_{L^{\frac{2p}{p-2}}(V,\mu)} dt\right)^{1/2} \\ &\leq \underbrace{C_1 \left(\frac{1}{\inf_V \rho\psi}\right)^{\frac{p-2}{2p}} \mu(V)^{\frac{1}{2}-\frac{1}{p}-\frac{1}{\nu}}}_{=:C_2} \left(\int_{\tau_3}^{\tau_4} \|T_t f\|^2_{L^\nu(V,\mu)} dt\right)^{1/2} \\ &\leq C_1 C_2 (\tau_4-\tau_3)^{1/2} \|f\|_{L^\nu(\mathbb{R}^d,\mu)}. \qquad \text{(A16)}\end{aligned}$$

Now, assume $f \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$. Then $nG_n f \in D(\overline{L})_b \cap D(L_s) \cap D(L_2)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} nG_n f = f$ in $L^\nu(\mathbb{R}^d,\mu)$. Thus, Assumption (A16) extends to all $f \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$. If $\nu \in [\frac{2p}{p-2},\infty)$, the above again extends to all $f \in L^\nu(\mathbb{R}^d,\mu)$ using the denseness of $L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$ in $L^\nu(\mathbb{R}^d,\mu)$. Finally, assume $f \in L^\infty(\mathbb{R}^d,\mu)$ and let $f_n := 1_{B_n} \cdot f$ for $n \geq 1$. Then, $\lim_{n\to\infty} f_n = f$ μ-a.e. on $\mathbb{R}^d$ and

$$T_t f = \lim_{n\to\infty} T_t f_n = \lim_{n\to\infty} P_t f_n, \ \mu\text{-a.e. on } \mathbb{R}^d. \qquad \text{(A17)}$$

Thus, using the sub-Markovian property and Lebesgue's Theorem in Assumption (A15), $(P_\cdot f_n(\cdot))_{n\geq 1}$ is a Cauchy sequence in $C(\overline{U}\times[\tau_1,\tau_2])$. Hence, we can again define

$$P_\cdot f := \lim_{n\to\infty} P_\cdot f_n(\cdot) \ \text{ in } \ C(\overline{U}\times[\tau_1,\tau_2]).$$

For each $t > 0$, $P_t f_n$ converges uniformly to $P_t f$ in U; hence, in view of Assumption (A17), $T_t f$ has continuous μ-version $P_t f$ and $P_\cdot f \in C(\overline{U}\times[\tau_1,\tau_2])$. Therefore, Assumption (A16) extends to all $f \in L^\infty(\mathbb{R}^d,\mu)$. Since U and $[\tau_1,\tau_2]$ were arbitrary, it holds for any $f \in \cup_{\nu\in[\frac{2p}{p-2},\infty]} L^\nu(\mathbb{R}^d,\mu)$, $P_\cdot f(\cdot)$ is continuous on $\mathbb{R}^d \times (0,\infty)$ and for each $t > 0$, $P_t f = T_t f$ μ-a.e. on $\mathbb{R}^d$. □

Proof of Proposition 3. The first shows the quasiregularity of the generalized Dirichlet form $(\mathcal{E}, D(L_2))$ associated with $(L_2, D(L_2))$, and the existence of a μ-tight special standard process associated with $(\mathcal{E}, D(L_2))$. This can be done exactly as in [10] (Theorem 3.5). One only has to take care that space $\mathcal{Y}$, as defined in the proof of [10] (Theorem 3.5), is replaced because of a seemingly uncorrected version of the paper by

$$\mathcal{Y} := \{u \in D(\overline{L})_b \,|\, \exists f,g \in L^1(\mathbb{R}^d,\mu)_b,\ f,g \geq 0, \text{ such that } u \leq G_1 f \text{ and } -u \leq G_1 g\}$$

to guarantee the convergence at the end of the proof. In particular, $D(\overline{L})_b$ is an algebra that can be proven in a similar way to [10] (Remark 1.7iii). Then, the assertion follows exactly as in [11] (Theorem 6), using for the proof instead $\mathcal{G}$ there the space $\mathcal{Y}$ defined above and defining $E_k \equiv \mathbb{R}^d$, $k \geq 1$. □

Proof of Theorem 9. Let $g \in L^{d+1}(B_R \times (0,T))$. (all functions defined on $B_R \times (0,T)$ are trivially extended on $\mathbb{R}^d \times (0,\infty) \setminus B_R \times (0,T)$.) Using [12] (2. Theorem (2), p. 52), there exists a constant $C_1 > 0$ that is independent of g, such that

$$
\begin{aligned}
& \widetilde{\mathbb{E}}_x\left[\int_{(0,T\wedge\widetilde{D}_R)\setminus Z^{\widetilde{\mathbb{M}}}} \left(2^{-\frac{d}{d+1}}\det(A)^{\frac{1}{d+1}}\cdot\left(\frac{1}{\psi}\right)^{\frac{d}{d+1}} g\right)(\widetilde{X}_s,s)ds\right] \\
= \;& \widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} \left(2^{-\frac{d}{d+1}}\det(A)^{\frac{1}{d+1}}\cdot\left(\frac{1}{\psi}\right)^{\frac{d}{d+1}} g\right)(\widetilde{X}_s,s)ds\right] \\
\leq \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}}\cdot\widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} e^{-\int_0^s\|\mathbf{G}(\widetilde{X}_u)\|du}\cdot\det\big(\widehat{A}/2\big)^{\frac{1}{d+1}} g(\widetilde{X}_s,s)ds\right] \\
\leq \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}}\cdot C_1\|g\|_{L^{d+1}(B_R\times(0,\infty))} \\
= \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}}\cdot C_1\|g\|_{L^{d+1}(B_R\times(0,T))}.
\end{aligned}
$$

Let $f \in L^{2d+2,d+1}(B_R \times (0,T))$. Let ψ denote the extended Borel measurable version as explained in Remark 3(i). Note that

$$
Z^{\widetilde{\mathbb{M}}}(\widetilde{\omega}) = \{s \geq 0 \,|\, \left(\frac{1}{\psi}\right)^{\frac{d}{d+1}}(\widetilde{X}_s(\widetilde{\omega}))\psi^{\frac{d}{d+1}}(\widetilde{X}_s(\widetilde{\omega})) \neq 1\}.
$$

Hence, by Equation (29),

$$
\widetilde{\mathbb{P}}_x\big(dt(\{s \geq 0 \,|\, \left(\frac{1}{\psi}\right)^{\frac{d}{d+1}}(\widetilde{X}_s)\psi^{\frac{d}{d+1}}(\widetilde{X}_s) \neq 1\}) = 0\big) = 1.
$$

Thus, replacing g with $2^{\frac{d}{d+1}} \cdot \det(A)^{-\frac{1}{d+1}}\psi^{\frac{d}{d+1}} f$, we get

$$
\begin{aligned}
& \widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} f(\widetilde{X}_s,s)ds\right] = \widetilde{\mathbb{E}}_x\left[\int_{(0,T\wedge\widetilde{D}_R)\setminus Z^{\widetilde{\mathbb{M}}}} f(\widetilde{X}_s,s)ds\right] \\
\leq \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}}\cdot C_1\|2^{\frac{d}{d+1}}\cdot\det(A)^{-\frac{1}{d+1}}\psi^{\frac{d}{d+1}} f\|_{L^{d+1}(B_R\times(0,T))} \\
\leq \;& \underbrace{2^{\frac{d}{d+1}} e^{T\|\mathbf{G}\|_{L^\infty(B_R)}}\cdot C_1\|\det(A)^{-\frac{1}{d+1}}\|_{L^\infty(B_R)}\|\psi\|_{L^{2d}(B_R)}^{\frac{2d}{2d+2}}}_{=:C}\|f\|_{L^{2d+2,d+1}(B_R\times(0,T))}.
\end{aligned}
$$

□

Proof of Theorem 10. Take $T_0 > 0$ satisfying $T_0 > T$. Extend u to $\overline{B}_{R_0} \times [-T_0, T_0]$ by

$$
u(x,t) = u(x,0) \;\text{ for } -T_0 \leq t < 0, \quad u(x,t) = u(x,T) \;\text{ for } T < t \leq T_0,\; x \in \overline{B}_{R_0}.
$$

Then, it holds that

$$
u \in W^{2,1}_{2d+2}(B_{R_0} \times (0,T)) \cap C(\overline{B}_{R_0} \times [-T,T]) \;\text{ and }\; \|\nabla u\| \in L^{4d+4}(B_{R_0} \times (-T_0,T_0)).
$$

For sufficiently large $n \in \mathbb{N}$, let ζ_n be a standard mollifier on $\mathbb{R}^{d+1}$ and $u_n := u * \zeta_n$. Then it holds $u_n \in C^\infty(\overline{B}_R \times [0,T])$, such that $\lim_{n\to\infty} \|u_n - u\|_{W^{2,1}_{2d+2}(B_R\times(0,T))} = 0$ and $\lim_{n\to\infty} \|\nabla u_n - \nabla u\|_{L^{4d+4}(B_R\times(0,T))} = 0$. By Itô's formula, for $x \in \mathbb{R}^d$, it holds for any $n \geq 1$

$$\begin{aligned} & u_n(\widetilde{X}_{T\wedge\widetilde{D}_R}, T\wedge\widetilde{D}_R) - u_n(x,0) \\ = & \int_0^{T\wedge\widetilde{D}_R} \nabla u_n(\widetilde{X}_s,s)\,\widehat{\sigma}(\widetilde{X}_s)d\widetilde{W}_s + \int_0^{T\wedge\widetilde{D}_R} (\partial_t u_n + Lu_n)(\widetilde{X}_s,s)ds, \quad \widetilde{\mathbb{P}}_x\text{-a.s.} \end{aligned} \tag{A18}$$

By Sobolev embedding, there exists a constant $C > 0$, independent of u_n and u, such that

$$\|u_n - u\|_{C(\overline{B}_R\times[0,T])} \leq C\|u_n - u\|_{W^{2,1}_{2d+2}(B_R\times(0,T))}.$$

Thus, $\lim_{n\to\infty} u_n(x,0) = u(x,0)$ and

$$u_n(X_{T\wedge\widetilde{D}_R}, T\wedge\widetilde{D}_R) \text{ converges } \mathbb{P}_x\text{-a.s. to } u(X_{T\wedge\widetilde{D}_R}, T\wedge\widetilde{D}_R) \text{ as } n\to\infty.$$

By Theorem 9,

$$\begin{aligned} & \widetilde{\mathbb{E}}_x\left[\left|\int_0^{T\wedge\widetilde{D}_R} (\partial_t u_n + Lu_n)(\widetilde{X}_s,s)ds - \int_0^{T\wedge\widetilde{D}_R} (\partial_t u + Lu)(\widetilde{X}_s,s)ds\right|\right] \\ & \leq \widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} |\partial_t u - \partial_t u_n|(\widetilde{X}_s,s)ds\right] + \widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} |Lu - Lu_n|(\widetilde{X}_s,s)ds\right] \\ & \leq C\|\partial_t u_n - \partial_t u\|_{L^{2d+2,d+1}(B_R\times(0,T))} + C\|Lu - Lu_n\|_{L^{2d+2,d+1}(B_R\times(0,T))} \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

where $C > 0$ is a constant that is independent of u and u_n. Using Jensen's inequality, Itô isometry, and Theorem 9, we obtain

$$\begin{aligned} & \widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} \left(\nabla u_n(\widetilde{X}_s,s) - \nabla u(\widetilde{X}_s,s)\right)\,\widehat{\sigma}(\widetilde{X}_s)dW_s\right] \\ & \leq \widetilde{\mathbb{E}}_x\left[\left|\int_0^{T\wedge\widetilde{D}_R} \left(\nabla u_n(\widetilde{X}_s,s) - \nabla u(\widetilde{X}_s,s)\right)\,\widehat{\sigma}(\widetilde{X}_s)dW_s\right|^2\right]^{1/2} \\ & = \widetilde{\mathbb{E}}_x\left[\int_0^{T\wedge\widetilde{D}_R} \left\|\left(\nabla u_n(\widetilde{X}_s,s) - \nabla u(\widetilde{X}_s,s)\right)\,\widehat{\sigma}(\widetilde{X}_s)\right\|^2 ds\right]^{1/2} \\ & \leq C\|(\nabla u_n - \nabla u)\widehat{\sigma}\|_{L^{4d+4,2d+2}(B_R\times(0,T))} \\ & \leq CC'\|\widehat{\sigma}\|_{L^\infty(B_R)}\|\nabla u_n - \nabla u\|_{L^{4d+4,2d+2}(B_R\times(0,T))} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Letting $n \to \infty$ in Assumption (A18), the assertion holds. □

Lemma A5. *Assume Assumption* **(A4)**$'$ *and let* $q_0 > 2d+2$ *be such that* $\frac{1}{q_0} + \frac{1}{q} = \frac{1}{2d+2}$. *If* $u \in D(L_{q_0})$; *then,* $u \in H^{2,2d+2}_{loc}(\mathbb{R}^d)$. *Moreover, for any open ball* B *in* $\mathbb{R}^d$, *there exists a constant* $C > 0$, *independent of* u, *such that*

$$\|u\|_{H^{2,2d+2}(B)} \leq C\|u\|_{D(L_{q_0})}.$$

Proof of Lemma A5. By Assumption **(A4)**$'$ and Theorem 4, $\rho \in H^{1,2d+2}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and $\rho\psi\mathbf{B} \in L^{2d+2}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Let $f \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$. Then by (A11), for all $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \langle \frac{1}{2}\rho A \nabla G_\alpha f, \nabla \varphi \rangle dx - \int_{\mathbb{R}^d} \langle \rho\psi\mathbf{B}, \nabla G_\alpha f\rangle \varphi \, dx + \int_{\mathbb{R}^d} (\alpha\rho\psi G_\alpha f)\, \varphi dx \\ = \int_{\mathbb{R}^d} (\rho\psi f)\, \varphi dx. \tag{A19}$$

Let $\widetilde{q} := \left(\frac{1}{2d+2} + \frac{1}{d}\right)^{-1}$. Then $\alpha\rho\psi \in L^{2d+2}_{loc}(\mathbb{R}^d) \subset L^{\widetilde{q}}_{loc}(\mathbb{R}^d)$, $\rho\psi f \in L^{2d+2}_{loc}(\mathbb{R}^d) \subset L^{\widetilde{q}}_{loc}(\mathbb{R}^d)$, hence by [15] (Theorem 1.8.3), $G_\alpha f \in H^{1,2d+2}_{loc}(\mathbb{R}^d)$. Moreover, using [15] (Theorem 1.7.4) and the resolvent contraction property, for any open balls V, V' in $\mathbb{R}^d$ with $\overline{V} \subset V'$, there exists a constant $\widetilde{C} > 0$, independent of f, such that

$$\|G_\alpha f\|_{H^{1,2d+2}(V)} \leq \widetilde{C}(\|G_\alpha f\|_{L^1(V')} + \|\rho\psi f\|_{L^{\widetilde{q}}(V')}) \\ \leq \widetilde{C}(\|G_\alpha f\|_{L^1(V')} + \|\rho\psi\|_{L^{2d+2}(V')}\|f\|_{L^d(V')}) \\ \leq \widetilde{C}\,\widetilde{C}_1 \|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}, \tag{A20}$$

where $\widetilde{C}_1 := \left(\frac{1}{\inf_{V'}\rho\psi}\right)^{\frac{1}{q_0}}(\alpha^{-1}dx(V')^{1-\frac{1}{q_0}} + \|\rho\psi\|_{L^{2d+2}(V')}dx(V')^{\frac{1}{d}-\frac{1}{q_0}})$. Using Morrey's inequality and Assumption (A20), there exists a constant $\widetilde{C}_2 > 0$ that is independent of f, such that

$$\|G_\alpha f\|_{L^\infty(V)} \leq \widetilde{C}_2\widetilde{C}\widetilde{C}_1\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}. \tag{A21}$$

Now, set

$$h_1 := \langle \rho\psi\mathbf{B}, \nabla G_\alpha f\rangle - \alpha\rho\psi G_\alpha f + \rho\psi f \in L^{d+1}_{loc}(\mathbb{R}^d).$$

Then, Assumption (A19) implies for all $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \langle \frac{1}{2}\rho A\nabla G_\alpha f, \nabla\varphi\rangle dx = \int_{\mathbb{R}^d} h_1 \varphi dx. \tag{A22}$$

Let U_1, U_2 be open balls in $\mathbb{R}^d$ satisfying $\overline{B} \subset U_1 \subset \overline{U}_1 \subset U_2$. Take $\zeta_1 \in C_0^\infty(U_2)$ such that $\zeta_1 \equiv 1$ on $\overline{U}_1$. Then, using integration by parts, and Assumption (A22), for all $\varphi \in C_0^\infty(U_2)$

$$\int_{U_2} \langle \frac{1}{2}\rho A\nabla(\zeta_1 G_\alpha f), \nabla\varphi\rangle dx = \int_{U_2} \langle \frac{1}{2}\rho A\nabla G_\alpha f, \zeta_1\nabla\varphi\rangle dx + \int_{U_2} \frac{1}{2}\langle A\nabla\zeta_1, \nabla\varphi\rangle \rho G_\alpha f dx \\ = \int_{U_2} \langle \frac{1}{2}\rho A\nabla G_\alpha f, \nabla(\zeta_1\varphi)\rangle dx - \int_{U_2} \underbrace{\langle \frac{1}{2}\rho A\nabla G_\alpha f, \nabla\zeta_1\rangle}_{=:h_2} \varphi dx \\ + \int_{U_2} \underbrace{-\frac{1}{2}\big(\langle G_\alpha f\nabla\rho + \rho\nabla G_\alpha f, A\nabla\zeta_1\rangle + \rho G_\alpha f\langle\nabla A, \nabla\zeta_1\rangle + \rho G_\alpha f \,\mathrm{trace}(A\nabla^2\zeta_1)\big)}_{=:h_3} \varphi dx \\ = \int_{U_2} (h_1\zeta_1 - h_2 + h_3)\varphi dx. \tag{A23}$$

Note that $h_2, h_3 \in L^{2d+2}_{loc}(\mathbb{R}^d)$. Let $h_4 := \langle \frac{1}{2}\nabla(\rho A), \nabla(\zeta_1 G_\alpha f)\rangle \in L^{d+1}_{loc}(\mathbb{R}^d)$. Using Assumption (A23),

$$\begin{aligned} &\int_{U_2} \langle \frac{1}{2}\rho A \nabla(\zeta_1 G_\alpha f), \nabla\varphi\rangle dx + \int_{U_2} \langle \frac{1}{2}\nabla(\rho A), \nabla(\zeta_1 G_\alpha f)\rangle \varphi dx \\ = &\int_{U_2} (h_1\zeta_1 - h_2 + h_3 + h_4)\varphi dx, \end{aligned} \tag{A24}$$

for all $\varphi \in C_0^\infty(U_2)$. We have $h := h_1\zeta_1 - h_2 + h_3 + h_4 \in L^{d+1}_{loc}(\mathbb{R}^d)$ and

$$\|h\|_{L^{d+1}(U_2)} \leq C_2(\|G_\alpha f\|_{H^{1,2d+2}(U_2)} + \|\rho\psi f\|_{L^{d+1}(U_2)}), \tag{A25}$$

where $C_2 > 0$ is a constant that is independent of f. By [19] (Theorem 9.15), there exists $w \in H^{2,d+1}(U_2) \cap H_0^{1,d+1}(U_2)$, such that

$$-\frac{1}{2}\text{trace}(\rho A \nabla^2 w) = h \quad \text{a.e. on } U_2. \tag{A26}$$

Furthermore, using [19] (Lemma 9.17), and Assumptions (A25) and (A20), there exists a constant $C_1 > 0$ that is independent of f, such that

$$\begin{aligned} \|w\|_{H^{2,d+1}(U_2)} \leq C_1\|h\|_{L^{d+1}(U_2)} &\leq C_1C_2\left(\|G_\alpha f\|_{H^{1,2d+2}(U_2)} + \|\rho\psi f\|_{L^{d+1}(U_2)}\right) \\ &\leq C_1C_2C_3\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}, \end{aligned}$$

where $C_3 := \widetilde{C}_1 + \|\rho\psi\|_{L^{2d+2}(U_2)} dx(U_2)^{\frac{1}{2d+2}-\frac{1}{q_0}} \left(\frac{1}{\inf_{U_2}\rho\psi}\right)^{\frac{1}{q_0}}$. Assumption (A26) implies

$$\int_{U_2} \langle \frac{1}{2}\rho A \nabla w, \nabla\varphi\rangle dx + \int_{U_2} \langle \frac{1}{2}\nabla(\rho A), \nabla w\rangle \varphi dx = \int_{U_2} h\varphi dx, \quad \forall \varphi \in C_0^\infty(U_2). \tag{A27}$$

Using the maximal principle of [21] (Theorem 1) and comparing Assumptions (A27) and (A24), we obtain $\zeta G_\alpha f = w$ on U_2, hence $G_\alpha f = w$ on U_1, so that $G_\alpha f \in H^{2,d+1}(U_1)$. Therefore, by Morrey's inequality, we obtain $\partial_i G_\alpha f \in L^\infty(U_1)$, $1 \leq i \leq d$, and

$$\|\partial_i G_\alpha f\|_{L^\infty(U_1)} \leq C_4\|G_\alpha f\|_{H^{2,d+1}(U_1)} \leq C_4\|w\|_{H^{2,d+1}(U_2)} \leq C_1C_2C_3C_4\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}, \tag{A28}$$

where $C_4 > 0$ is a constant that is independent of f. Thus, we obtain $h \in L^{2d+2}(U_1)$. Now, take $\zeta_2 \in C_0^\infty(U_1)$, such that $\zeta_2 \equiv 1$ on $\overline{B}$. Note that a.e. on U_1, it holds that

$$-\frac{1}{2}\text{trace}\big(\rho A \nabla^2(\zeta_2 G_\alpha f)\big) = -\frac{1}{2}\zeta_2 h - \frac{1}{2}G_\alpha f \cdot \text{trace}(\rho A \nabla^2 \zeta_2) - \langle \rho A \nabla\zeta_2, \nabla G_\alpha f\rangle =: \widetilde{h}.$$

Since $\|\nabla G_\alpha f\| \in L^\infty(U_1)$, $\widetilde{h} \in L^{2d+2}(U_1)$, by [19] (Theorem 9.15), we get $\zeta_2 G_\alpha f \in H^{2,2d+2}(U_1)$; hence, $G_\alpha f \in H^{2,2d+2}(B)$. Using [19] (Lemma 9.17), (A21), (A28), there exist constants $C_5, C_6 > 0$ that are independent of f, such that

$$\begin{aligned} \|G_\alpha f\|_{H^{2,2d+2}(B)} &\leq \|\zeta_2 G_\alpha f\|_{H^{2,2d+2}(U_1)} \leq C_5\|\widetilde{h}\|_{L^{2d+2}(U_1)} \\ &\leq C_5C_6(\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)} + \|\rho\psi f\|_{L^{2d+2}(U_1)}) \\ &\leq C_5C_6(\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)} + \|\rho\psi\|_{L^q(U_1)}(\inf_{U_1}\rho\psi)^{-1/q_0}\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}) \\ &\leq C\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}, \end{aligned} \tag{A29}$$

where $C := C_5C_6(1 \vee \|\rho\psi\|_{L^q(U_1)}(\inf_{U_1}\rho\psi)^{-1/q_0})$. Using the denseness of $C_0^\infty(\mathbb{R}^d)$ in $L^{q_0}(\mathbb{R}^d,\mu)$, (A29) extends to $f \in L^{q_0}(\mathbb{R}^d,\mu)$. Now let $u \in D(L_{q_0})$, Then $(1-L_{q_0})u \in L^{q_0}(\mathbb{R}^d,\mu)$, hence by (A29), it holds $u = G_1(1-L_{q_0})u \in H^{2,2d+2}_{loc}(\mathbb{R}^d)$ and

$$\|u\|_{H^{2,2d+2}(B)} = \|G_1(1-L_{q_0})u\|_{H^{2,2d+2}(B)} \le C\|(1-L_{q_0})u\|_{L^{q_0}(\mathbb{R}^d,\mu)} \le C\|u\|_{D(L_{q_0})}.$$

□

Lemma A6. *Assume Assumptions* **(A1)** *and* **(A2)**. *Let* $f \in D(\overline{L})_b \cap D(L_s) \cap D(L_2)$ *and define*

$$u_f := P_\cdot f \in C(\mathbb{R}^d \times [0,\infty))$$

as in Lemma 1. Then, for any open set U *in* $\mathbb{R}^d$ *and* $T>0$,

$$\partial_t u_f,\, \partial_i u_f \in L^{2,\infty}(U \times (0,T)) \text{ for all } 1 \le i \le d,$$

and for each $t \in (0,T)$, *it holds*

$$\partial_t u_f(\cdot,t) = T_t L_2 f \in L^2(U), \text{ and } \partial_i u_f(\cdot,t) = \partial_i P_t f \in L^2(U).$$

If we additionally assume Assumption **(A4)′** *and* $f \in D(L_{q_0})$, *where* q_0 *is as in Lemma A5, then* $\partial_i\partial_j u_f \in L^{2d+2,\infty}(U\times(0,T))$ *for all* $1 \le i,j \le d$, *and for each* $t \in (0,T)$, *it holds that*

$$\partial_i\partial_j u_f(\cdot,t) = \partial_i\partial_j P_t f \in L^{2d+2}(U).$$

Proof of Lemma A6. Assume Assumptions **(A1)** and **(A2)**. Let $f \in D(\overline{L})_b \cap D(L_s) \cap D(L_2)$ and $t>0$, $t_0 \ge 0$. Then, by Theorem 3(c),

$$P_{t_0} f = \overline{T}_{t_0} f \in D(\overline{L})_b \subset D(\mathcal{E}^0),$$

where $\overline{T}_0 := id$. Observe that, by Theorem 3(c), for any open ball B in $\mathbb{R}^d$ with $\overline{U} \subset B$,

$$\begin{aligned} \|\nabla P_t f - \nabla P_{t_0} f\|^2_{L^2(B)} &\le 2(\lambda_B \inf_B \rho)^{-1}\mathcal{E}^0(P_t f - P_{t_0} f, P_t f - P_{t_0} f) \\ &\le 4(\lambda_B \inf_B \rho)^{-1}\|f\|_{L^\infty(\mathbb{R}^d,\mu)}\|\overline{T}_t\overline{L}f - \overline{T}_{t_0}\overline{L}f\|_{L^1(\mathbb{R}^d,\mu)}. \end{aligned} \tag{A30}$$

Likewise,

$$\|\nabla P_t f\|^2_{L^2(B)} \le 2(\lambda_B \inf_B \rho)^{-1}\|f\|_{L^\infty(\mathbb{R}^d,\mu)}\|\overline{T}_t\overline{L}f\|_{L^1(\mathbb{R}^d,\mu)}.$$

For each $i = 1,\dots,d$, define a map

$$\partial_i P_\cdot f : [0,T] \to L^2(U), \quad t \mapsto \partial_i P_t f.$$

Then, by Assumption (A30) and the $L^1(\mathbb{R}^d,\mu)$-strong continuity of $(\overline{T}_t)_{t>0}$, map $\partial_i P_\cdot f$ is continuous with respect to the $\|\cdot\|_{L^2(B)}$-norm, hence by [22] (Theorem, p91), there exists a Borel measurable function u_f^i on $U \times (0,T)$ such that for each $t \in (0,T)$, it holds that

$$u_f^i(\cdot,t) = \partial_i P_t f \in L^2(U).$$

Thus, using Assumption (A30) and the $L^1(\mathbb{R}^d,\mu)$-contraction property of $(\overline{T}_t)_{t>0}$, it holds that $u_f^i \in L^{2,\infty}(U\times(0,T))$ and

$$\|u_f^i\|_{L^{2,\infty}(U\times(0,T))} \quad \leq \quad 2(\lambda_B \inf_B \rho)^{-1/2}\|f\|^{1/2}_{L^\infty(\mathbb{R}^d,\mu)}\|\overline{L}f\|^{1/2}_{L^1(\mathbb{R}^d,\mu)}.$$

Now, let $\varphi_1 \in C_0^\infty(U)$ and $\varphi_2 \in C_0^\infty((0,T))$. Then

$$\iint_{U\times(0,T)} u_f \cdot \partial_i(\varphi_1\varphi_2)dxdt \quad = \quad -\iint_U u_f^i \cdot \varphi_1\varphi_2 dxdt. \tag{A31}$$

Using the approximation as in Lemma A4, $\partial_i u_f = u_f^i \in L^{2,\infty}(U\times(0,T))$.

Now, define a map

$$T_\cdot L_2 f : [0,T] \to L^2(U), \quad t \mapsto T_t L_2 f,$$

where $T_0 := id$. Since

$$\|T_tL_2f - T_{t_0}L_2f\|_{L^2(U)} \leq (\inf_U \rho\psi)^{-1/2}\|T_tL_2f - T_{t_0}L_2f\|_{L^2(\mathbb{R}^d,\mu)},$$

using the $L^2(\mathbb{R}^d,\mu)$-strong continuity of $(T_t)_{t>0}$ and [22] (Theorem, p91), there exists a Borel measurable function u_f^0 on $U\times(0,T)$ such that for each $t \in (0,T)$ it holds that

$$u_f^0(\cdot,t) = T_tL_2f \in L^2(U).$$

Using the $L^2(\mathbb{R}^d,\mu)$-contraction property of $(T_t)_{t>0}$, it holds $u_f^0 \in L^{2,\infty}(U\times(0,T))$ and

$$\|u_f^0\|_{L^{2,\infty}(U\times(0,T))} \quad \leq \quad (\inf_U \rho\psi)^{-1/2}\|L_2f\|_{L^2(\mathbb{R}^d,\mu)}.$$

Observe that

$$\iint_{U\times(0,T)} u_f \cdot \partial_t(\varphi_1\varphi_2)dxdt \quad = \quad -\iint_U u_f^0 \cdot \varphi_1\varphi_2 dxdt.$$

Using the approximation of Lemma A4, we obtain $\partial_t u_f = u_f^0 \in L^{2,\infty}(U\times(0,T))$. Now assume **(A4′)**. Then, by Lemma A5, $P_{t_0}f \in D(L_{q_0}) \subset H^{2,2d+2}_{loc}(\mathbb{R}^d)$, and for each $1 \leq i,j \leq d$, it holds

$$\|\partial_i\partial_jP_tf - \partial_i\partial_jP_{t_0}f\|_{L^{2d+2}(U)} \leq \|T_tf - T_{t_0}f\|_{L^{q_0}(\mathbb{R}^d,\mu)} + \|T_tL_{q_0}f - T_{t_0}L_{q_0}f\|_{L^{q_0}(\mathbb{R}^d,\mu)}. \tag{A32}$$

Define a map

$$\partial_i\partial_jP_\cdot f : [0,T] \to L^2(U), \quad t \mapsto \partial_i\partial_jP_tf.$$

By the $L^{q_0}(\mathbb{R}^d,\mu)$-strong continuity of $(T_t)_{t>0}$ and (A32), map $\partial_i\partial_jP_\cdot f$ is continuous with respect to the $\|\cdot\|_{L^{2d+2}(U)}$-norm. Hence, by [22] (Theorem, p91), there exists a Borel measurable function u_f^{ij} on $U\times(0,T)$ such that, for each $t\in(0,T)$, it holds that

$$u_f^{ij}(\cdot,t) = \partial_i\partial_jP_tf.$$

Using Lemma A5 and the $L^{q_0}(\mathbb{R}^d,\mu)$-contraction property of $(T_t)_{t>0}$, $u_f^{ij} \in L^{2d+2,\infty}(U\times(0,T))$ and

$$\|u_f^{ij}\|_{L^{2d+2,\infty}(U\times(0,T))} \le \sup_{t\in(0,T)} \|P_t f\|_{H^{2,2d+2}(U)} \le C\|f\|_{D(L_{q_0})},$$

where $C > 0$ is a constant that is independent of f. Using the same line of arguments as in Assumption (A31), and the approximation as in Lemma A4,

$$\partial_i\partial_j u_f = u_f^{ij} \in L^{2d+2,\infty}(U\times(0,T)).$$

□

Proof of Theorem 11. Let $f \in C_0^\infty(\mathbb{R}^d)$. Then $f \in D(L_s)$. Define $u_f := P_\cdot f(\cdot)$. Then by Lemma 1, $u_f \in C_b(\mathbb{R}^d\times[0,\infty))$ and $u_f(x,0) = f(x)$ for all $x \in \mathbb{R}^d$. In particular, since $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d,\mathbb{R}^d)$, it holds $f \in D(L_{q_0})$, so that $P_t f \in D(L_{q_0})$ for any $t \ge 0$. By Lemma A6, for each $t > 0$, it holds $\partial_t u_f(\cdot,t) = T_t L_s f = T_t L f$ μ-a.e. on $\mathbb{R}^d$. Note that for each $t > 0$, using the sub-Markovian property,

$$\|\partial_t u_f(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} = \|T_t L f\|_{L^\infty(\mathbb{R}^d)} \le \|Lf\|_{L^\infty(\mathbb{R}^d,\mu)};$$

hence, $\partial_t u_f \in L^\infty(\mathbb{R}^d\times(0,\infty))$. By Lemma A6, for $1 \le i,j \le d$, $t>0$, $\partial_i u_f(\cdot,t) = \partial_i P_t f$, $\partial_i\partial_j u_f(\cdot,t) = \partial_i\partial_j P_t f$ μ-a.e. on $\mathbb{R}^d$. Using Lemma A5 and the $L^{q_0}(\mathbb{R}^d,\mu)$-contraction property of $(T_t)_{t>0}$, for any $R>0$ and for each $1 \le i,j \le d$, $t > 0$, it holds

$$\|\partial_i\partial_j u_f(\cdot,t)\|_{L^{2d+2}(B_R)} \le \|P_t f\|_{H^{2,2d+2}(B_R)} \le C\|f\|_{D(L_{q_0})},$$

where $C>0$ is as in Lemma A5 and independent of f and $t>0$. By Morrey's inequality, there exists a constant $C_{R,d} > 0$, independent of f and $t > 0$, such that for each $t > 0$, $1 \le i \le d$,

$$\|\partial_i u_f(\cdot,t)\|_{L^\infty(B_R)} \le \|\partial_i P_t f\|_{L^\infty(B_R)} \le C_{R,d}\|P_t f\|_{H^{2,2d+2}(B_R)} \le C_{R,d}C\|f\|_{D(L_{q_0})}.$$

Thus, $u_f \in W^{2,1}_{2d+2,\infty}(B_R\times(0,\infty))$ and $\partial_t u_f, \partial_i u_f \in L^\infty(B_R\times(0,\infty))$ for all $1 \le i \le d$. By Assumption (A14), we have for any $\varphi \in C_0^\infty(\mathbb{R}^d\times(0,\infty))$

$$\iint_{\mathbb{R}^d\times(0,\infty)} \langle \frac{1}{2}\rho A\nabla u_f, \nabla\varphi\rangle - \langle \rho\psi\mathbf{B}, \nabla u_f\rangle\varphi\, dxdt = \iint_{\mathbb{R}^d\times(0,\infty)} -\partial_t u_f\cdot\varphi\rho\psi dxdt,$$

and using integration by parts, we obtain

$$-\iint_{\mathbb{R}^d\times(0,\infty)} \big(\frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 u_f) + \langle \beta^{\rho,A,\psi} + \mathbf{B}, \nabla u_f\rangle\big)\varphi\, d\mu dt = \iint_{\mathbb{R}^d\times(0,\infty)} -\partial_t u_f\cdot\varphi\, d\mu dt$$

for any $\varphi \in C_0^\infty(\mathbb{R}^d\times(0,\infty))$. Therefore,

$$\partial_t u_f = \frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 u_f) + \langle \mathbf{G}, \nabla u_f\rangle \quad \text{a.e. on } \mathbb{R}^d\times(0,\infty).$$

□

Proof of Theorem 12. Let $x \in \mathbb{R}^d$ be arbitrary. Let $\mathbb{Q}_x = \mathbb{P}_x \circ X^{-1}$ and $\widetilde{\mathbb{Q}}_x = \widetilde{\mathbb{P}}_x \circ \widetilde{X}^{-1}$ respectively. Then $\mathbb{Q}_x$, $\widetilde{\mathbb{Q}}_x$ are two solutions of the time-homogeneous martingale problem with initial condition x and coefficients $(\widehat{\sigma}, \mathbf{G})$ as defined in [2] (Chapter 5, 4.15 Definition). Let $f \in C_0^\infty(\mathbb{R}^d)$. For $T > 0$, define $g(x,t) := u_f(x, T-t)$, $(x,t) \in \mathbb{R}^d \times [0,T]$, where u_f is defined as in Theorem 11. Then by Theorem 11,

$$g \in C_b\left(\mathbb{R}^d \times [0,T]\right) \cap \big(\bigcap_{r>0} W^{2,1}_{2d+2,\infty}(B_r \times (0,T))\big),$$
$$\partial_t g \in L^\infty(\mathbb{R}^d \times (0,T)), \quad \partial_i g \in \bigcap_{r>0} L^\infty(B_r \times (0,T)),\ 1 \le i \le d,$$

and it holds

$$\frac{\partial g}{\partial t} + Lg = 0 \quad \text{a.e. in } \mathbb{R}^d \times (0,T), \quad g(x,T) = f(x) \ \text{ for all } x \in \mathbb{R}^d.$$

Applying Theorem 9 to $\mathbb{M}$, for $x \in \mathbb{R}^d$, $R > 0$, it holds that

$$\mathbb{E}_x\left[\int_0^{T\wedge D_R} \left|\frac{\partial g}{\partial t} + Lg\right|(X_s, s)ds\right] = 0,$$

hence

$$\int_0^{T\wedge D_R} \big(\frac{\partial g}{\partial t} + Lg\big)(X_s, s)ds = 0, \quad \mathbb{P}_x\text{-a.s.},$$

and so by Theorem 10,

$$g(X_{T\wedge D_R}, T\wedge D_R) - g(x,0) = \int_0^{T\wedge D_R} \nabla g(X_s, s)\widehat{\sigma}(X_s)dW_s, \quad \mathbb{P}_x\text{-a.s.}$$

Therefore

$$\mathbb{E}_x\left[g(X_{T\wedge D_R}, T\wedge D_R)\right] = g(x,0).$$

Letting $R \to \infty$ and using Lebesgue's Theorem, we obtain

$$\mathbb{E}_x[f(X_T)] = \mathbb{E}_x[g(X_T, T)] = g(x,0).$$

Analogously for $\widetilde{\mathbb{M}}$, we obtain $\widetilde{\mathbb{E}}_x[f(\widetilde{X}_T)] = g(x,0)$. Thus,

$$\mathbb{E}_x[f(X_T)] = \widetilde{\mathbb{E}}_x[f(\widetilde{X}_T)].$$

Therefore, $\mathbb{Q}_x$ and $\widetilde{\mathbb{Q}}_x$ have the same one-dimensional marginal distributions, and we can conclude as in [2] (Chapter 5, proof of 4.27 Proposition) that $\mathbb{Q}_x = \widetilde{\mathbb{Q}}_x$.

For the last statement, see [2] (Chapter 5, 4.20 Theorem). □

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