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Duality for General TU-games Redefined

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Abstract

We criticize some conceptual weaknesses in the recent literature on coalitional TU-games and propose, based on our critics, a new definition of dual TU-games that coincides with the one in the literature on the class of super-additive games. We justify our new definition in four alternative ways: 1. Via an adequate definition of efficient payoff vectors. 2. Via a modification of the Bondareva-Shapley duality. 3. Via an explicit consideration of "coalition building". 4. Via associating general TU-games to coalition-production economies. Rather than imputations, we base our analysis on a modification of aspirations.

Keywords: TU-games, duality, core, c–Core, cohesive games, complete game efficiency

JEL Classification: C71

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1 Introduction

In our paper, we will present and justify a new concept of duality for general TU-games resulting from a careful reconsideration of some basic concepts of coalitional TU-games. Our work has been motivated by a deep dissatisfaction with the way conceptual problems for coalitional game theory are neglected, disdained or just ignored in a large part of the actual literature. TU-games are often just treated as elements of a vector space without simultaneous monitoring of the effects of mathematical operations on the underlying social or economic scenarios that are modelled in an aggregate stylized way by the respective TU-games. Questions as to which mathematical operations on games are coherent with the feasible actions and specific structures of the underlying social or economic scenario are asked only rarely in the literature. Exceptions are the classic works of von Neumann and Morgenstern (1944), McKinsey (1952), Luce and Raiffa (1957) and the later books by Rapoport (1970), Osborne and Rubinstein (1994) and, in particular, Myerson (1991).

To indicate what the most general understanding of the intention of coalitional cooperative games is, we quote from a very recent book whose authors have been also contributed significantly to game theory. In Chapter 16 on coalitional games with transferable utility of Maschler et al. (2013), we find on page 659:

"Coalitional games model situations in which players may cooperate to achieve their goals. It is assumed that every set of players can form a coalition and engage in a binding agreement that yields them a certain amount of profit. The maximal amount that a coalition can generate through cooperation is called the worth of the coalition."

And on page 672 we find:

"The main questions that are the focus of coalitional game theory include: 1. What happens when the players play the game? What coalitions will form, and if a coalition S is formed, how does it divide the worth $v(S)$ among its members? 2. What would a judge or an arbitrator recommend that the players do? The answers to these two questions are quite different. The question regarding the coalitional structure that the players can be expected to form is a difficult one, and will not be addressed in this book. We will often assume that the grand coalition N is formed and ask how will the players divide among them the worth $v(N)$."

This "*assumption that the grand coalition will be formed*" is implicit in a huge part of the treatments of coalitional games in the literature. It is even explicitly stated as a formal assumption already in Osborne and Rubinstein (1994) who write on page 258 in their Section 4 on coalitional games:

"...we assume that the coalitional games with transferable payoff that we study have the property that the worth of the coalition N of all players is at least as large as the sum of the worths of the members of any partition of N. This assumption ensures that it is optimal

that the coalition N of all players forms, as is required by our interpretations of the solution concepts we study (though formal analysis is meaningful without this assumption)".

Such games are called **cohesive** in Osborne and Rubinstein (1994) and **complete** in Sun et al. (2008). We shall use both terms in this article. While for cohesive games the standard notion of efficiency that takes $v(N)$ as the benchmark is justified, it is inadequate and even defective for non-cohesive games. This fact is the reason for our skeptical view on the use of the predominant definition of a dual game for **all** TU-games rather than only for **super-additive** ones. Cohesiveness will play **the** crucial role in our paper.

The definition of a TU-game as a pair (N, v) where v is a mapping from 2^N, the power set of the set of players N, to the real numbers that associates to every subset S of N its worth $v(S)$ is so general that it not only allows the "game theoretic" representation of a huge class of various, very different scenarios but also admits many games that hardly allow a meaningful direct economic or social interpretation. It is therefore very likely and in fact often the case that certain specific classes of games require specific treatments and solution concepts and, in particular, an **explicit mentioning** of the restriction to the considered specific class. We feel that most important inadequate definitions for **general** TU-games in the recent literature are those of **feasible** and of **efficient** payoff vectors. Based on the grand coalition's worth $v(N)$ as a benchmark, they lead via preimputations, imputations and additional axioms to solutions of games. While these notions of feasibility and efficiency appear adequate on cohesive games they represent only a "second best" feasibility and efficiency in the case of general games. Three different definitions of efficiency that we will have to distinguish are explicitly defined and discussed in Bejan and Gomez (2012). The careful comparison of their impacts on the interpretation of those coalitional functions they are applied to will be crucial. It is in fact instrumental for our definition of a dual game that is applicable and meaningful for all TU-games and will turn out to coincide with the usual one on the class of super-additive games.

Two other concepts whose exact meaning and use in the literature needs a careful investigation are those of a **coalition** and of a **coalitional function**. Finally, and most importantly for our expressed goal, we will have to deal with duality **in** TU-games and, in particular with duality **of** TU-games. This is the content of Section 4. In Section 2, we shall introduce concepts, notation and most of the definitions. In Section 3, the concepts of **coalition** and **coalitional function** are considered in detail. We discuss the varied positions concerning their meanings in the literature since their introduction by von Neumann and Morgenstern (1944), where the coalitional functions had been introduced as characteristic function forms for non-cooperative games in usual (or strategic) form.

Section 5 builds the central part of our article. There, our duality concept is defined and justified. This section has six subsections. In Subsection 5.1, we discuss the feasibility and efficiency of payoff vectors in TU-games. Subsection 5.2 is devoted to the interpretation of duality of TU-games in the literature. Subsection 5.3 provides a justification of our use of partitions rather than balanced families. Subsection 5.4 is dealing with the relation of our duality to a modification of Peleg's (1986) reduced game property. In Subsection 5.5, we seperate the use of Bondavera-Shapley balancing weights from the question of core existence. We modify the Bondavera-Shapley duality in such a way that the joint optimal value of the underlying dual programs becomes smaller and equals the worth of the grand coalition in the cohesive hull v^c of the game v. Finally, Subsection 5.6 discusses the close relation of the duality of TU-games with a duality of coalition production economies introduced in Sun et al. (2008), extended and simplified in Inoue (2012).

In Section 6 we collect our results on duality which indicate differences or similarities with the recent duality literature. Section 7 concludes with remarks on future research.

2 Notation, Definitions and Terminology

2.1 General Mathematical Tools

The symbols $\subseteq [\subset], \setminus$ represents the [strict] inclusion and subtraction for sets, respectively.

$\mathbb{N}$ is the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\mathbb{R}$ denotes the field and one-dimensional vector space of real numbers. The canonical [weak] ordering on $\mathbb{R}$ is denoted by $>$ $(\geq)$.

$\mathbb{R}^N$ is the real vector space of functions $x : N \to \mathbb{R}$. The natural weak ordering $\geq$ on $\mathbb{R}^N$ is defined pointwise. For $x, y \in \mathbb{R}^N$, we define $x > y$ by $[x \geq y$ and $x \neq y]$ and $x \gg y$ by $[x_i > y_i$ for all $i \in N]$.

For any $S \subseteq N$, the indicator function $\mathbb{1}_S : N \to \mathbb{R}$ is defined by $\mathbb{1}_S(i) = 1$ for all $i \in S$, otherwise 0.

The set of all subsets of N is the power set of N, denoted by 2^N. We define $\mathcal{N} := 2^N \setminus \{\emptyset\}$.

$(\mathbb{R}^N)^*$ and $(\mathbb{R}^{\mathcal{N}})^*$ are the **dual vector space** of $\mathbb{R}^N$ and $\mathbb{R}^{\mathcal{N}}$, respectively. They are the spaces of real valued linear forms on $\mathbb{R}^N$ and $\mathbb{R}^{\mathcal{N}}$, respectively.

We define bilinear maps on $\mathbb{R}^N \times (\mathbb{R}^N)^*$ and $\mathbb{R}^{\mathcal{N}} \times (\mathbb{R}^{\mathcal{N}})^*$ by

$$\langle .,. \rangle : \mathbb{R}^N \times (\mathbb{R}^N)^* \to \mathbb{R} : (x, x^*) \longmapsto \langle x, x^* \rangle := \sum_{i \in N} x_i x_i^*,$$

$$\Big\langle .,. \Big\rangle : \mathbb{R}^{\mathcal{N}} \times (\mathbb{R}^{\mathcal{N}})^* \to \mathbb{R} : (v, \lambda) \longmapsto \Big\langle v, \lambda \Big\rangle := \sum_{S \in \mathcal{N}} v(S)\lambda(S).$$

For any $S \in \mathcal{N}$, we have $\langle x, \mathbb{1}_S \rangle = \sum_{i \in N} x_i \mathbb{1}_S(i) = \sum_{i \in S} x_i =: \mathbf{x}(S)$. We use this bold face version $\mathbf{x}(S)$ in deviation of the confusing use of $x(S)$ in the TU-literature that conflicts with the standard mathematical definition $x(S) := \{x(i) \mid i \in S\}$. It allows us to visibly distinguish between a vector $x \in \mathbb{R}^N$ and the additive game $\mathbf{x} \in \mathbb{R}^{\mathcal{N}}$ generated by x.

In complete anology to our notational conventions in the context of $\mathbb{R}^N$, we use the notation in the context of $\mathbb{R}^{\mathcal{N}}$.

For any $v \in \mathbb{R}^{\mathcal{N}}$, $\lambda \in (\mathbb{R}^{\mathcal{N}})^*$ and $ß \subseteq \mathcal{N}$, we define $v_ß := (v(T))_{T \in ß}$, $\lambda_ß := (\lambda(T))_{T \in ß}$, and $\left\langle .,. \right\rangle^{ß} : \mathbb{R}^{ß} \times (\mathbb{R}^{ß})^* \to \mathbb{R} : (v_ß, \lambda_ß) \longmapsto \sum_{S \in ß} v_ß(S)\lambda_ß(S)$.

For $ß = \mathcal{N}$, we have $\left\langle .,. \right\rangle^{\mathcal{N}} =: \left\langle .,. \right\rangle$ and $\lambda_{\mathcal{N}} = \lambda$. Therefore, we have for any $ß \subseteq \mathcal{N}$,

$$\langle (v_ß, 0), (\lambda_ß, 0) \rangle = \left\langle v_ß, \lambda_ß \right\rangle^{ß} = \sum_{S \in ß} v_ß(S)\lambda_ß(S) = \sum_{S \in \mathcal{N}} v(S)\lambda(S)\mathbb{1}_ß(S).$$

Consider some set $ß \subseteq \mathcal{N}$. A **partition of unity** on N **subordinate to** $ß$ is a mapping $\tilde{\lambda} : \mathcal{N} \times N \to \mathbb{R}_+$ such that

1. For all $j \in N$, Supp $\tilde{\lambda}(\bullet, j) := \{T \in ß \mid \tilde{\lambda}(T, j) \neq 0\} \in ß$
2. For all $j \in N$, $\sum_{S \in ß} \tilde{\lambda}(S, j) = 1$.

The collection $ß \subseteq \mathcal{N}$ in the definition of the partition of unity on N is **balanced** if it satisfies for all $S \in ß$ and $i \in N$, $\tilde{\lambda}(S, i) = \lambda_ß(S)\mathbb{1}_S(i) = \lambda(S)\mathbb{1}_S(i)$. The real numbers in $\{\lambda(S)\}_{S \in ß}$ are called **balancing weights** for the subsets $S \in ß$. For each balanced collection $ß$, the system $\lambda_ß \in \mathbb{R}^{ß}_+$ of balancing weights is an element of $([0,1]^{ß})^* \subset (\mathbb{R}^{ß}_+)^* \equiv (\mathbb{R}^{ß})^*_+ \subset (\mathbb{R}^{ß})^*$. We denote the set of balanced collections by $ß \subseteq \mathcal{N}$ by $\mathcal{B}$.

A **partition** of $S \subseteq N$ is a set $\{T_1, T_2, \ldots T_m\}$ of pairwise disjoint subsets $T_i \subseteq S$ covering S. The set of partitions of $S \subseteq N$ is denoted $\Pi(S)$. A **bi-partition** of $S \subseteq N$ is a set $\{T, S \setminus T\}$ with $T \subset S$. The set of bi-partitions of $S \subseteq N$ is denoted by $\Pi_2(S)$, and maybe identified with $2^S \setminus \{\emptyset\}$.

2.2 TU-games

A cooperative **coalitional game with transferable utility**, for short TU-game, is an ordered pair (N, v) where $N = \{1, 2, \ldots n\}$ represents the set of players and $v \in \mathbb{R}^{2^N}$ with $v(\emptyset) = 0$. The **coalition function** v is completely described by its restriction to $\mathbb{R}^{\mathcal{N}}$, which by slight abuse of notation we also denote by v. We define $V := \mathbb{R}^{\mathcal{N}}$ as the vector space of TU-games, and denote its dual space by $V^* := (\mathbb{R}^{\mathcal{N}})^*$.

Frequently used properties of TU-games are

- **monotonicity**: $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$,

- **super-additivity**: $v(S) + v(T) \leq v(S \cup T)$ for all disjoint $S, T \subseteq N$,
- **convexity**: $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$,
- **sub-addivity**: v is sub-additive if and only if $-v$ is super-additive,
- **concavity**: v is concave if and only if $-v$ is convex.

The **super-additive** (resp. **convex**) **hull** of a TU-game (N, v) is the smallest super-additive (resp. convex) TU-game (N, w) such that for any $S \subseteq N$, $w(S) \geq v(S)$. The super-additive and convex hulls of (N, v) are denoted by $(N, \tilde{v})$ and $(N, \hat{v})$, respectively.

A TU-game (N, v) is **cohesive** (or **complete**) if $v(N) = \tilde{v}(N) = \max_{\pi \in \Pi(N)} \sum_{T \in \pi} v(T)$. In a cohesive game, the presumption that the grand coalition N will form is justified by the efficiency requirement (see Osborne and Rubinstein, 1994; Arnold and Schwalbe, 2002; Sun et al., 2008). The **cohesive hull** or **completion** of a TU-game (N, v) is the TU-game (N, v^c) such that $v^c(S) = v(S)$ for all $S \subset N$ and $v^c(N) = \tilde{v}(N)$.

A TU-game (N, v) is **balanced** if for each **balanced collection** ß of subsets of N and associated system $\lambda_ß = \{\lambda(S)\}_{S \in ß}$ of balancing weights holds: for all $i \in N$, $v(N) \geq \left\langle v, \lambda_ß \right\rangle^{ß} = \sum_{S \in ß} v(S)\lambda(S)$. The **balanced hull** of (N, v) is the TU-game (N, v^b) with $v^b(S) = v(S)$ for all $S \subset N$ and $v^b(N) := \max_{ß \in \mathcal{B}} \sum_{T \in ß} \lambda(T)v(T)$.

A **subgame** (S, v_S) of (N, v) is defined by $v_S := v|_{2^S \setminus \{\emptyset\}}$. A TU-game (N, v) is **totally balanced** if all subgames (S, v_S) with $S \subseteq N$ are balanced. The **totally balanced hull** or **cover** of (N, v) is $(N, \bar{v})$, which is the smallest totally balanced game with $\bar{v}(S) \geq v(S)$ for all $S \subseteq N$, *i. e.*, with $\bar{v}(S) = v_S^b(S)$ for all $S \subseteq N$.

We will restrict in this paper the standard definition of duality for TU-games to the class of super-additive games. Later in this work, we shall give a definition of duality (called **c-duality** in Aslan and Duman (2018)) that holds for all TU-games and coincides with the *duality defined next on super-additive games.

A TU-game (N, v^*) is the **dual game** of the super-additive game (N, v) if $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.

In contrast to almost all of the recent literature dealing with TU-duality, the Wikipedia article on "Cooperative game theory" (2019) defines duality in the described way above, but restricts this to "profit games" and "value games" as opposed to "cost games". The interpretation of v^* offered there is that (N, v^*) represents the "opportunity costs for S of not joining the grand coalition N". The dual game (N, v^*) is then denoted as the "cost game". The article interprets "value games" and "cost games" as two "in some sense equivalent" alternative ways of modelling the same problem.

2.3 Remarks on Our Terminology

There are three notions in colloquial English as well as in technical terms that describe in various contexts "cooperative" interaction of people in groups or sets in more or less obliging specified legal or codified ways. These notions are **coalition**, **syndicate** and **cartel**. All of them have been used in different varied meanings in economics. There is certainly some freedom in using these concepts as technical concepts with different specifications of their interpretations. We think that it is unavoidable to make use of this freedom in order to get rid of obvious inconsistencies in the use of the term "coalition" in coalitional game theory. There is an extensive literature focused on the problem and processes of coalition building in game theory. But this has not yet succeeded in scrutinizing the prevalent presumption in large parts of the literature of coalitional games that the grand coalition will be formed.

A terminology where coalitions defined just as subsets are used as "forming" of coalitions is meaningless. As we will explicate later in detail, we need to distinguish different sorts of player subsets and different steps of negotiations of players that take place outside and "before" the commitment of players to participate in a TU-game (N, v). Following Harsanyi (1959) and Rapoport (1970), we shall replace the term **coalition** as a synonym for a subset of players by **syndicate**. Syndicates are potential coalitions. They may negotiate about the redistribution of total payoffs in various coalitions. But they may also negotiate about binding cooperation between coalitions once they have been received their respective worths. Coalitions have formed when they have formally committed to participate in the game and to cooperation of their members in accordance with the rules embodied in v. This interpretation of a coalition is supported by Hart and Kurz (1983) who write on page 1047: *"By a "coalition" is meant a group of "players" (e.g. economic or political agents) which decide to act together, as one unit, relative to the rest of the players."*

Cartels are also syndicates with a commitment structure. They may consist of players who are members of different coalitions built by enterring the game and committing themselves to obey the rules of v. Members of cartels are bound to contribute to the cartels' certain prespecified shares of their total payoffs received as members of possibly different coalitions in the game.

Each player $i \in N$ will be a member of exactly one of coalition $S \subseteq N$. She is a member of each syndicate $T \subseteq N$ and maybe a member of several cartels $C \subseteq N$. A cartel may form a coalition or a union of disjoint coalitions but can be relevant for the game only under these aspects but not in its function as a cartel. We shall not deal with the pre-plaey negoatiations of syndicates on which cartels would build. While syndicates are just subsets of the set of all players, coalitions and cartels are subsets supplemented by certain commitment structures. The detailed specifications of these structures are not described.

3 Coalitions and Coalition Functions

This section is an indispensable preparation for Section 4 where we will try to distil the essence of the concept of duality for TU-games before in Section 5 we motivate, introduce and also justify our duality concept.

3.1 Coalitions

Although the term coalition is **the** central notion in defining coalitional games most of the literature is wasting its conceptual power by just identifying it with a subset of N, the set of players, in a TU-game (N, v). Like in the opaque but coherent definition of an extensive form game, one could associate with some pure mathematical objects names that reflect their interpretation in the social or economic model that one intends to build also for the model denoted **coalitional game with transferable utility**. Obviously, members of a set S of natural numbers are not per se players. They are **representing** players when the interpretation of what this means in the model is clear. An apt example is mathematics itself, where sets are denoted spaces when they carry a specific mathematical structure. Often this results in the description by some tuple collecting the set under question and the structures it is endowed with. Once introduced in that way often only the symbol for the set is written when the space is considered.

In cooperative game theory, the notion of a **coalition** is not treated in an analogous way. The reason for this becomes apparent when a set of players has **formed a coalition**. It is not described or even represented by exact mathematical terms what exactly the coalition's and its players' contractually defined potential actions, rights and duties are. That becomes evident when we use the **identification of a subset** S **of** N **with the coalition** S and ask one of the two fundamental questions of cooperative game theory: **When will players in a coalition** S **(subset of** N**) build the coalition** S**?** So coalitions in a game (N, v) must, as already argued in Subsection 2.3, be more than just subsets of N.

The first game theorists have been obviously aware of this fact. Already Neumann (1928) writes in his Section 4: "**the case** $n = 3$ discusses the problems of deriving a value in analogy to two person games and hints to the possibility of any subset of two players **to build a coalition** in order to achieve higher expected payoffs." Other early quotations confirm this view:

McKinsey (1952, p. 304): "The theory of games is largely concerned with the questions of what combinations of players ("**coalitions**") **will be formed** and what payments the players can be expected to make to each other as inducements to join the various **coalitions**."

Luce and Raiffa (1957, p. 156): "Thus any theory of collusion, i.e., of **coalition formation**, has a distinct ad hoc flavor."

Aumann and Peleg (1960, p. 174): "… **a coalition** B is **effective for a payoff vector** x if the members of B, by joining forces, can play so that each player i in B receives at least …"

This intuitive definition is open to a number of interpretations. A rather conservative one had been adopted by von Neumann and Morgenstern by assuming that the most the members of B can count on is what they can get if the **players** of $N \setminus B$ **form a coalition** whose purpose it is to minimize the payoffs of B.

Shapley and Shubik (1969, p. 11): "… a game is an ordered pair (N, v), where N is a finite set [the players] and v is a function from the subsets of N [**coalitions**] to the reals satisfying $v(\emptyset) = 0$, called the characteristic function."

Rapoport (1970, p. 170-180)(referring to Harsanyi (1959)): "… bargaining positions of the players in the **process of coalition formation**"; p.171-172: "**These subsets** [of N] **are called syndicates**. They are **distinguished from coalitions** in that **each player is considered as a member of every syndicate viewed as a set of players which includes him**. Actually, the **syndicates** can be viewed as **the potential coalitions**."; p. 286: "… two fundamental questions: 1) Which coalitions are likely to form? 2) How will the members of a **coalition** apportion their joint payoff?"; p.287: "Game theory is virtually silent on the question of **which coalitions are likely to form**."

Myerson (1991, p. 418): "Any nonempty subset of the set of players may be called a **coalition**."

Osborne and Rubinstein (1994, p. 257): "… subset S of N(**a coalition**)"

Peleg and Sudhölter (2003, p. 10): "Let S be a sub-**coalition** of N. If S forms … then its members get the amount $v(S)$ of money…" and "if a **coalition** S forms it may divide $v(S)$ among its members in any feasible way."

Peters (2008, p. 151): "It is important to note that the term '**coalition**' is used for any subset of the set of players. **So a coalition is not necessarily formed**."

Maschler et al. (2013, p. 660): "Cooperative game theory concentrates on questions such as which sets of players' **coalitions** will agree to conclude binding agreements?"

The search for a definition of a **coalition** in articles dealing with the duality of TU-games led us to the following:

Funaki (1998): "The value of a **coalition** in a given game…"

Kikuta (2007): "A **coalition** is a subset of N."

Oishi and Nakayama (2009): "$v(S)$ is the worth that **coalition** S can obtain by itself."

Oishi et al. (2016): "$v(S)$ represents what **coalition** S can achieve on its own."

We can conclude from the totality of quotations above that during the 75 years since the publication of von Neumann and Morgenstern (1944), the definition of the technical term coalition has not only undergone some change but is up to now not consistently treated as one well-defined technical term. The possible actions of coalitions or their members may well depend on the interpretation of the term coalition. In Section 3.2, we will be confronted with a similar terminological vagueness when dealing with the classical **characteristic function** and their later, broader defined version called now mostly **coalition function**.

With the exception of Kikuta (2007), the articles from 1998 and later concerned with duality of games, do not even define the concept of a coalition explicitly but rather leave it to the readers to conclude from their notation and terminology that they identify them with subsets of players. This is a distressing fact because a plausible distinction of what players in **syndicates** (or potential coalitions) could do as compared with what formed **coalitions**, committed to specified kinds of cooperation, could do to the rest of the players is in our views indispensable for a soundly based definition of dual TU-games. For instance, in a non-super-additive game (N, v) there must be a part of legal constraints embodied in v that prevents some (formed) **coalitions** S from splitting into **sub-coalitions** and thereby increasing their joint total payoffs. The **players** of these sub-coalitions would have been able, as not yet committed **syndicates**, to **form sub-coalitions** of S and thereby to achieve **under the "rules"** embodied in v aggregate payoffs that due to the lack of super-additivity are out of reach for them as the formed **coalition** S in the game (N, v).

All works after Aumann and Peleg (1960) quoted above, with the exception of Rapoport (1970) define coalitions explicitly as subsets of players. The formulation in Peleg and Sudhölter (2003) is the one in which the target of our criticism can be best recognized, namely the uncommented double use of the term **coalition** for two different objects: *formed* and *not yet formed* coalitions. Peters (2008), though also identifying coalitions with subsets of games, states explicitly that he is using this notation for formed **and** not formed coalitions. That alerts the reader but does not make it coherent! The verbal distinction between coalitions and formed coalitions that can be found occasionally in the game theory literature is almost nowhere fortified by a corresponding formal definition.

The only satisfactory way of dealing with this (termino)logical problem is in our opinion that by Rapoport (1970), based on Harsanyi (1959). **Coalition** and **syndicate** are in colloquial language often close to synonyms, and Harsanyi might well have used them as technical terms in the converse order of meanings. His merit is the **explicit distinction** of mere subsets of players, **syndicates** in his terminology, from subsets of players loaded with certain contractual commitments of its members, denoted **coalitions**. The excellent book of Rapoport (1970) is the only one in which this is discussed in a detailed and transparent manner. We shall adopt his terminology in the rest of this article.

Our discussion of **syndicates** (potential coalitions) versus (formed) **coalitions** underlying contractual restrictions had not included the following two kinds of situations:

1. The game TU-game (N, v) is augmented by an exogenously given partition of the

player set N, called a **coalition structure**.

2. The **forming of coalitions** by players in the underlying syndicates is broadened by allowing players to commit themselves to allocate their cooperation efforts to several coalitions with possibly different rules for the kinds of cooperation.

The **first** class had been analyzed under the name "games in partition function form" by Thrall and Lucas (1963) and is treated in some details in Rapoport (1970, chapter 9). Some solution concepts based on coalitional structures had been introduced in the 1960s. Actual work on this, as sketched in Peleg and Sudhölter (2003) under the name "games with coalition structures", has its roots and its name from the seminal article by Aumann and Dreze (1974).

The **second** class, the "balanced games", defined independently by Bondareva (1963) and Shapley (1967) had been proven by Shapley and Shubik (1969) to coincide with "TU market games", representing or being generated by the specific class of cardinal concave pure exchange economies.

In both classes an adequate analysis has to abstain from the assumption of **cohesiveness** - local super-additivity at N in the words of Aumann and Peleg (1960) - that ensures that any feasible payoff allocation negotiated in coalitions can be afforded by the grand coalition N. A consequence is the renouncement from the assumption of super-additivity that is necessarily satisfied for any **classical** (i.e. von Neumann-Morgenstern) **characteristic function**. But just the super-additive games build in our opinion the only class on which the standard definition of a *dual game can be reasonably justified.

3.2 Coalition Functions

The concept of a **coalition function** v as the essential component of a TU-game (N, v) was defined in Section 2. Its history and use in game theory and its interpretation are of fundamental importance for the theory of coalitional TU-games and, in particular, for the discussion of a suitable concept of a dual game. The dual game of **any** TU-game (N, v) is defined in the literature as the TU-game (N, v^*) satisfying the equality $v^*(S) = v(N) - v(N \setminus S)$ for every subset S of the player set N. Any attempt to justify or interpret this definition relies on the meaning of the worth of a coalition in a TU-game (N, v). In case of the dual game v^* this leads to a potential conflict between two interpretations of v^* that need to be compatible; first, the interpretation of v^* as a coalition function and the worth generated by it, and secondly, the interpretation of it as a dual game.

We start with a review of various explanations and interpretations of coalitional and characteristic functions in the literature. The first appearance of the coalition function in game theory had been in the treatment of non-cooperative n-person zero-sum games in von

Neumann and Morgenstern (1944). Application of the mini-max theorem for two-person zero-sum games to each bi-partition $\{S, N \setminus S =: T\}$ of the player set N in the strategic game G, with the coalitions S and T as "players" choosing coordinated joint strategies of their members led to a **value** for each of these two-person games, represented by $v(S)$ and $v(T) = -v(S)$. The function associating to every S the **worth** $v(S)$ is characteristic for the underlying game G and had therefore been called "the characteristic function of the game G". This notion had then been extended later in the book to general n-person games. Such a **von Neumann - Morgenstern characteristic function** is necessarily super-additive, and for every super-additive TU-game (N, v) there exist an n-person game G whose von Neumann-Morgenstern characteristic function coincides with v (see McKinsey (1952, Theorem 17.1)). The next detailed, more transparent treatments of the "von Neumann-Morgenstern theory" or "classical theory" as it is called by Aumann and Peleg (1960) can be found in McKinsey (1952) and Luce and Raiffa (1957). Both define a characteristic function without linking it to a given game G just by super-additivity and a worth 0 for the empty set. Also, Aumann and Peleg (1960) in their foundation of NTU-games use the term "characteristic function" for their coalitional correspondence which they assume to be super-additive. Later Rapoport (1970) contains a treatment of the classical theory using the classic terminology. But to the best of our knowledge, he was the first to derive a different (not classical) coalitional function from a three-person version of the Prisoners' Dilemma game that fails to be super-additive but is still in a specific way characteristic for that game and hence also called **characteristic function**. This non-super-additive characteristic function is, though not quoted there, a forerunner of Myerson's (1991) systematic treatment of three different "characteristic functions" for a game only one of them, the classical von Neumann-Morgenstern characteristic function, is necessarily super-additive. Here super-additivity appears as an additional assumption that may be imposed on a characteristic function. According to Myerson, "A characteristic function can also be called a game in coalitional form or a coalitional game".

In the next very influential book on game theory, that also covers cooperative games, Osborne and Rubinstein (1994) adopt this terminology and define TU-games as coalitional games with transferable payoffs without using, however, the term characteristic function or even giving a name to the function v at all. From the most popular books covering TU-games published after the year 2000, the term characteristic function without the assumption of super-additivity is used in Peters (2008). The same definition can be found already in Peleg and Sudhölter (2003) and later in Maschler et al. (2013). But here the term characteristic function is replaced by coalition function and coalitional function, respectively, terms that have been adopted in the recent literature on TU-games.

How important the difference between **super-additive** characteristic functions and general coalition functions is will become clear once we think about what syndicates can do in their negotiations and what coalitions can hope to receive in terms of joint payoffs. That will be one of the topics of Section 5 where we will motivate, introduce and also defend our duality concept.

The definition of duality at the beginning of this section relates two generally different coalitional functions v and v^*. These are not assumed to be super-additive, not even to be cohesive. As two super-additive TU-games that are dual to each other are necessarily identical, one can hardly defend a duality concept the meaning of which is dubious for non-super-additive games.

Next we shall collect and compare some interpretations of TU-games and of dual TU-games from the literature and check their mutual compatibility. That will lead us to a distinction between von Neumann-Morgenstern (or classical) characteristic functions, characteristic functions (per se) that are derived from a strategic game but fail to be super-additive, and coalition functions that associate a worth to every syndicate S in a player set N paid off to it once it has formed by mutual agreement on modes of legal cooperation as a coalition. So every coalitional function that is derivable from a strategic game is a characteristic function. It is classical if and only if it is super-additive.

4 Duality in the Theory of TU-games

We start this section with a short historical sketch of duality in the theory of TU-games. The first definition of a **dual game** we are aware of had been given in Rapoport (1970) for the very restricted domain of simple games. These are TU-games (N, v) with $v(S)$ either 0 or 1 for all subsets S of N. A coalition S is winning or losing if $v(S) = 1$ or 0, respectively. For any simple game (N, v), the **dual game** (N, v^*) is defined by $v^*(S) = 1$ if and only if $v(N \setminus S) = 0$ that implies that for all S, the number $v^*(S)+v(N \setminus S)$ equals to $v(N) = 1$. This duality, for simple games is the one induced by the actual standard definition. Although the presumed property that supersets of winning syndicates are winning according to Rapoport reflects that "the super-addivity property of characteristic function" is "preserved", it is formally not the case for many simple games. However, the "worth" that can be reached in these games is always derived from the same "public outcome" established by a winning coalition. Transfers of payoffs or adding them up are not meaningful operations here, even if technically possible.

Aumann and Maschler (1985) defined **bankruptcy problems** and their dual problems as well as solutions and dual solutions, and represented them by TU-games and their solutions. This approach is presented as one example of dual games and solutions in Oishi and Nakayama (2009). Funaki (1998) refers to a duality concept for TU-games due to Tadenuma (1990) and writes:

"Tadenuma (1990) also defines a dual game and examines dual axiomatizations of the

core and the anti-core. However, his duality is simply defined by a minus value of the original game. This game is well-defined mathematically, but it is not natural for a class of normal games with non-negative values. For example, our dualization operator is closed in a class of monotonic games, but not for his."

Interestingly, our criticism of the now standard duality defined for **all** TU-games is quite similar for the interesting classes of cohesive, super-additive, or balanced games which fail to be closed under dualization.

Quite a different approach to duality of TU-games has been proposed by Martinez-Legaz (1996). He considered an informationally equivalent representation of any TU-game by a non-increasing polyhedral convex function, the **indirect function**, that is related to the **characteristic function** via Fenchel-Moreau generalized conjugation theory. But here the dual of a TU-game is not a TU-game associated with this game, but rather a dual (non-TU-game) representation of the original TU-game.

Another approach to duality could be based on the fact that the set of n-person TU-games is a finite dimensional vector space. Hence a **dual game** could just be defined as an element of its dual vector space; i.e., as a linear functional on the space of TU-games. The balancing weights defined in Section 2 are examples of dual games in this sense. With this consideration, we are already close to the **duality** that is underlying the characterization of games with a non-empty core as balanced games in the Bondareva-Shapley Theorem.

We are not aware of any attempts in the literature to check for any potential relations between those different duality concepts.

The standard definition of duality at the beginning of Section 3.2 relates two generally different coalitional functions v and v^*. These are not assumed to be super-additive, not even to be cohesive. Two super-additive TU-games that are dual to each other are necessarily identical. Therefore, one can hardly defend a duality concept, the meaning of which for non-super-additive games, is dubious.

We are going to collect and compare now some interpretations of TU-games and of dual TU-games from the literature.

Aumann (1967) (for NTU-games with set valued characteristic functions; applicable here as TU-games are special NTU-games): "The characteristic function associates with each $S \subset N$, a subset $v(S)$ of E^S. Intuitively, $v(S)$ represents the set of payoffs that S can assure itself. Note that the characteristic function $v(S)$ does not necessarily have to be interpreted as the set of payoff vectors that S can assure itself; if preferred, it may be interpreted in any other way, such as what a coalition 'thinks it can get'. It is also possible that a game is given a priori in characteristic function form."

Rapoport (1970): "This way of defining the characteristic function is equivalent to assuming that **every coalition can expect that the counter-coalition will do 'its**

worst'; i.e., will act in such a way as to minimize the joint payoff of the first coalition. Thus the value [**worth**] of the game to each coalition is supposed to be **the very least it can get if it acts in concert (regardless of what common interest dictates to the counter-coalition)**.[That means that any number larger than $v(S)$ could be prevented by suitable coordinated actions of players in $N \setminus S$.]"

Aumann and Maschler (1985): "$v(S)$ **represents the total amount of payoff that the coalition S can get by itself, without the help of other players**; it is called the **worth** of S."

Myerson (1991): "Here $v(S)$ is called the **worth** of coalition S, and **it represents the total amount of transferable utility that the members of S could earn without any help from the players outside of S**."

Peleg and Sudhölter (2003): "If S forms in G then its members get the amount $v(S)$ of money. The number $v(S)$ is called the worth of S. $v^*(S)$ describes the amount which can be given to S, if the complement $N \setminus S$ receives what it can reach by cooperation. The complement $N \setminus S$ cannot prevent S from [receiving] the amount $v^*(S)$."

Maschler et al. (2013): "The maximal amount that a coalition can generate through cooperation is called the worth of the coalition."

Funaki (1998): "The value of a coalition in the dual of a given game is considered as an optimistic valuation of the game situation if the original game is considered as a pessimistic valuation like maximin standard."

Oishi and Nakayama (2009): "The dual v^* of a game v assigns to each coalition S the dual value $v^*(S)$, which is the amount that $N \setminus S$ cannot prevent S from obtaining."

Oishi et al. (2016): "For all $S \in 2^N$, $v(S)$ represents what coalition S can achieve on its own. The worth $v^*(S)$ represents the amount that the other agents $N \setminus S$ cannot prevent S from obtaining in v."

What can we distill from these various explanations of the meaning of TU-games and their duals? In order to answer these questions, we have to take regard of the type of coalition functions the respective authors had in mind. We start with TU-games.

The interpretation of v offered by Oishi et al. (2016), that is essentially also that of Funaki (1998), seems to coincide with that of Rapoport (1970), Aumann and Maschler (1985), Myerson (1991), Maschler et al. (2013), who are all referring to the von Neumann-Morgenstern characteristic function. But the meaning of "can achieve of its own" for a coalition function, that is not a classical characteristic function, is very different! In the classical context, the possible actions to achieve something are actions of the coalitions, as "players", defined as joint coordinated strategy choices of their members agreed upon in an underlying strategic n-person game. In the context of non-characteristic general coalition functions of Oishi et al. (2016), it has to be described in the non-formal socio-economic scenario underlying the specification of v and confirmed in the binding coalition building contract between the members of a syndicate, what the possible actions of the coalition, or of the players in the syndicate, are by which monetary payoffs can be achieved. Also, Aumann (1967) **shares the interpretation expressed by** "can achieve of its own", but in the NTU context of extended characteristic correspondences. But he admits the possibility of other interpretations and even coalitional correspondences not derived from strategic games.

Peleg and Sudhölter (2003), also in the general coalition function context, agree with the explanation of Oishi et al. (2016), without, however, explicitly giving room to any specific possible actions, beyond signing the coalition forming contract by the members of a syndicate. It is implicitly taken for granted in this standard interpretation of a coalition function that a breach of contract is out of the scope, so that any coalition S can be sure to get its worth paid out in money if its members do not violate the rules of the coalition contract. As cohesiveness is not a part of the definition of the non-classical TU-games, the case $v(S)+v(N\setminus S) > v(N)$ for some coalition S cannot be excluded. The grand coalition would be unable then to afford the total payment. So who is providing the amounts $v(S)$ and $v(N \setminus S)$? The logical consequence is either to restrict the class of TU-games to super-additive (or at least cohesive) games or to alter the interpretation of the worth, as indicated by Aumann (1967).

We will now focus on dual games. As by definition the set of dual TU-games and of TU-games are identical we get first for each dual game the same interpretations as offered in the previous paragraph. We ignore this fact for a while and consider the interpretations of duality offered in the above quoted literature. Aumann and Maschler (1985) define solutions and dual solutions for bankruptcy problems. Later they derive TU-games from bankruptcy problems but do not explicitly mention a duality concept for games. Kikuta (2007) related a bankruptcy game with another TU-game derived from a "dual" bankruptcy problem used in Aumann and Maschler (1985) in their definition of a dual solution for bankruptcy problems. He then introduced twisted pairs of TU-games (N, v) and (N, w) defined by satisfying the requirement that the game $(N, v+w)$ is additive and shows that the two TU-games derived from the two bankruptcy games in Aumann and Maschler (1985) are twisted.

Proposition 2 in Oishi and Nakayama (2009) on anti-duality of two bankruptcy games is just a reformulation of this observation in Kikuta (2007) expressed via their new notion of **anti-dual games**. Oishi et al. (2016) introduce also **anti-dual solutions for TU-games** on the basis of anti-dual games. This concept of **anti-duality** had been introduced already by Kikuta (2007) under the name **negative duality**. (There is another concept of **twisted duality** of TU-games introduced by Kikuta (2007) we will not deal with in our paper.)

The presently standard duality concept defined in Section 3.2 appears according to our knowledge first in Funaki (1998) who interprets the dual v^* as an "optimistic valuation of the game" in contrast to the pessimistic valuation represented by v. The problem with Funaki's interpretations, that remind of the different characteristic functions analyzed in Myerson

(1991) based on different strategic behavior of players in an underlying non-cooperative game, is the following: it remains completely unclear what the **game situation** is that v and v^* valuate differently when (N, v) and (N, v^*) are themselves the games to be compared. The identical interpretation of v^* in Peleg and Sudhölter (2003), Oishi and Nakayama (2009) and Oishi et al. (2016) is clear and understandable. But is it compatible with the interpretation of v? Consider the formulation of Peleg and Sudhölter (2003):

"$v^*(S)$ describes the amount which can be given to S, if the complement $N \setminus S$ receives what it can reach by cooperation. The complement $N \setminus S$ cannot prevent S from [receiving] the amount $v^*(S)$."

Assuming the legal right of coalition S to receive its worth $v(S)$, it is obvious that S cannot "be given" an amount smaller than $v(S)$. As $v^*(S)$ "can be given" to S provided $N \setminus S$ receives its guaranteed payoff $v(N \setminus S)$, we conclude that $v^*(S)$ cannot be smaller than $v(S)$. Therefore $v^*(S) = v(N) - v(N \setminus S) \geq v(S)$, hence $v(N) \geq v(S) + v(N \setminus S)$. Therefore, the concurrent validity of the interpretations of TU-games and dual TU-games in the above quoted literature is not self-contradicting **only** if the game (N, v) is cohesive. Applying the same interpretations also to subgames is therefore only consistent for super-additive games. In the case of super-additive, hence cohesive games the interpretation of v^* found in the above quotations is equivalent to the following one:

$x^* := v^*(S)$ **is the unique aggregate payoff to coalition** S that makes the payoff allocation $(x^*, v(N \setminus S))$ for the bi-partition $\{S, N \setminus S\}$ of N **attainable** and **Pareto efficient**. This interpretation can be literally adopted for all TU-games, as instead of the grand coalition the most "productive" partition is the benchmark for efficiency. We shall base our alternative definition of dual games on this observation.

5 A new definition of duality for TU-games

The maybe most troublesome deficiency of the current theory of coalitional games is the combination of an extension of the classical theory of characteristic functions in which the assumption of super-additivity as a part of the definition of the game is waived, but the feasibility and efficiency of payoff vectors are still determined with respect to the grand coalition. Already Rapoport (1970, p. 287) remarked: "Game theory is virtually silent on the question of which coalitions are likely to form." The importance of altering this situation has been recognized by many game theorists, yet, without much change. So we find that despite countless heavy criticisms of this fact, starting with Böhm (1974), Neuefeind (1974) and Guesnerie and Oddou (1979), no strong lasting effect can be observed in the literature. This state of coalitional game theory is exemplified by two statements in the most recent book on game theory authored by highly reputed game theorists! In Maschler et al. (2013, p. 672), we find:

"The main questions that are the focus of coalitional game theory include: What happens when the players play the game? What coalitions will form, and if a coalition S is formed, how does it divide the worth $v(S)$ among its members?"

And some lines later:

"The answers to these two questions are quite different. The question regarding the coalitional structure that the players can be expected to form is a difficult one, and will not be addressed in this book. We will often assume that the grand coalition N is formed and ask how will the players divide among them the worth $v(N)$."

Peleg and Sudhölter (2003) do consider games with coalition structure. But that is exogenously given and the efficiency of that structure is not an issue. Also, duality in this modified framework is not discussed or only defined.

5.1 Feasibility and efficiency

We shall start our approach to duality by changing the notions of feasibility and efficiency in such a way that they are applicable in **all** TU-games and coincide with the versions now dominating the literature once the considered games are super-additive. So instead of taking the worth of the grand coalition as the upper bound for affordable aggregate payoffs to all players in N, we take the upper bound over all coalition structures (*i. e.*, partitions of N) of the sums of worths of their member syndicates. This number is $v^c(N) = \tilde{v}(N)$. The effect of this change on the duality discussion is that once a coalition $N \setminus S$ has received her worth $v(N \setminus S)$, the payoff $\tilde{v}(N) - v(N \setminus S)$ to S is still affordable and not smaller than $v(S)$, its legally guaranteed worth. So it could be "given to S". In effect, we choose the middle one in size of the three versions of sets of feasible payoff vectors for a game (N, v) discussed in Bejan and Gomez (2012), and denoted by them as $X^*_\Pi(N, v) = \{x \in \mathbb{R}^N \mid \mathbf{x}(N) \leq v(\pi)$ for some $\pi \in \Pi(N)\}$. We have already argued at length why we do not constrain ourselves to $X^*(N, v) = \{x \in \mathbb{R}^N \mid \mathbf{x}(N) \leq v(N)\}$. We will explain later why we do not follow Bejan and Gomez (2012) in considering $X^*_{\mathcal{B}}(N, v) = \{x \in \mathbb{R}^N \mid \mathbf{x}(N) \leq v(\lambda_ß)$ for some $ß \in \mathcal{B}\}$.

5.2 What $N \setminus S$ cannot prevent S from receiving

In this section we shall discuss how the interpretation in the literature of the meaning of v^* in our new context of feasibility and efficiency can be formally represented in a coherent way.

Apparently, the result depends on whether we interpret S and $N \setminus S$ as just syndicates, *i. e.*, subsets of N, or as coalitions that have already been formed. As these considerations should be part of the pre-play negotiations about coalition building, we may assume that a syndicate is likely to build a coalition only if forming that coalition is not Pareto dominated by any partition of it into sub-coalitions. So all players in all syndicates may assume that whatever coalition S they finally will be a member of, the players in the syndicate $N \setminus S$ have organized themselves in sub-syndicates of $N \setminus S$ forming coalitions. For these coalitions T, we must have $v(T) = \tilde{v}(T)$. These considerations are similarly motivated as those that had led to the use of aspirations in the literature. See our discussion in Subsection 5.5.

If it is common knowledge among the players in N that the players in a syndicate S know that there remains at most $\tilde{v}(N) - \tilde{v}(N \setminus S)$ to be allocated to them, they will organize themselves in such way in a partition of S that the sum of the worths of the resulting subsets becomes maximal, *i. e.*, $(v|_S)^c(S) = \tilde{v}(S)$. But this is affordable for the players in S. All these considerations determine for each syndicate S that has formed a coalition or a partition of coalitions committed to cooperate that the worst their members may expect from the complement syndicate $N \setminus S$ is that it absorbs $\tilde{v}(N \setminus S)$. So while the players in S in its final structure of sub-coalitions have a legal right to get the aggregate payoff $\tilde{v}(S)$ they may expect $\tilde{v}(N) - \tilde{v}(N \setminus S)$ as the largest amount that the players in $N \setminus S$ cannot prevent them from receiving. Clearly, that cannot be smaller than $\tilde{v}(S)$.

These considerations justify our following definition of a dual game.

Definition 1. *Let (N, v) be a TU-game. Then its dual game (N, v^d) is defined by $v^d(S) = \tilde{v}(N) - \tilde{v}(N \setminus S)$ for all subsets S of N.*

Lemma 1. *For every super-additive TU-game (N, v), one has $v^d = v^*$.*

Remark 1. *Our definition of duality reflects the idea that the players of any syndicate that is forming a coalition in (N, v) do so only if $v(S) = \tilde{v}(S)$.*

5.3 Why partitions rather than balanced families?

In the definition of a TU-game as a pair (N, v) the coalitional function associates to any subset (= syndicate) S of N a worth $v(S)$ that S has a legal right to receive once it has committed to forming a coalition in which its members cooperate according to the rules embodied in v. That the assumption of super-additivity, that is not a defining property of a game, implies a restriction of the class of TU-games has been the reason for our introduction of our new duality concept. The classical question of coalitional game theory has been:

"What coalition will form, and if a coalition S is formed, how does it divide the worth $v(S)$ among its members?"

The forming of coalitions necessarily results in a coalition structure, *i. e.*, in a partition of the player set N and for each coalition the worth $v(S)$ to be divided among the members of S. Partial simultaneous memberships in several coalitions is, though a coherent concept, not part of the classical problem. In order to formalize it one needs to **extend** the notion of a game from (N, v) to (N, w) with $w : 2^N \times \mathbb{R}_+ \to \mathbb{R}$ such that $w(S, t) := tv(S)$ for any $(S, t) \in 2^N \times \mathbb{R}_+$. Depending on the socio-economic scenario represented by (N, v) this may or may not be a coherent way of defining an "extended" TU game. The resulting analogue of the classical question quoted above would then be: What families of coalitions with part-time memberships will be formed? How are the part-times for different coalitions and members related? How is the worth for a part-time coalition defined and divided among the members?

While in the context of **partitions** a coalition's worth is just a real number as membership is indivisible, in the context of **balanced families** a coalition's worth is a function mapping degrees of membership to shares of the coalition's worth in its classical meaning. Answers to the above questions are indeed given for balanced or even totally balanced games. We shall discuss that context also later, when we discuss a derivation of our duality from the Bondareva-Shapley duality of linear programs [cf. for instance Myerson (1991, p. 432-433)].

5.4 The relation between our duality and reduced games

Among the axioms on which solution concepts for TU-games are based, in the literature the notion of a reduced game for some syndicate S is contingent on an aggregate payoff $\mathbf{x}(N \setminus S)$ for the players in $N \setminus S$. The most prominent version of a reduced game is the one defined in Davis and Maschler (1965) that, however, is based on the implicit assumption that the grand coalition is formed. In order to get rid of this restriction, Bejan and Gomez (2012) adopted in their analysis a modified definition of a reduced game which according to them had been used, among others, by Moldovanu and Winter (1994) and Hokari and Kıbrıs (2003). Peleg and Sudhölter (2003, Remark 2.3.12) describe the worth of any subset T of a coalition S, given general agreement among all players in the game about $\mathbf{x}(N \setminus S)$ as the aggregate payoff for the syndicate $N \setminus S$, by $v_{S,x}(T)$. This worth is interpreted as the (maximal) total payoff that the coalition T expects to get. The TU-game $v_{S,x}$ is the **reduced game** with respect to S and x. According to them, $(S, v_{S,x})$ is "not a game in the ordinary sense", whatever that means. It serves only to determine the distribution of $v_{S,x}(S)$ to the members of S. Apart from the special treatment of the grand coalition in the definition of the Davis-Maschler-reduced game it coincides with that of the modified version. The latter one is defined by $v_{S,x}(T) := \max\{v(T \cup Q) - \mathbf{x}(Q)\}$ over all subsets Q of $N \setminus S$.

We had already argued above that in the simultaneous pre-play negotiations by all players, performed in all syndicates they are members of, rational players will form coalitions efficiently. That means for a syndicate S whose members have committed to cooperate in the game they will generate an optimal partition of S whose members will form the coalitions cooperating in the game accordingly. The best that S can afford by its members is, therefore, $\tilde{v}(S)$. The most what they may expect is what is left from $\tilde{v}(N)$ if the members of $N \setminus S$ behave in the same way as those of S did, namely $\tilde{v}(N) - \tilde{v}(N \setminus S) =: v^d(S)$. If our game v happens to be cohesive and the $\mathbf{x}(N \setminus S)$ above equals to $\tilde{v}(N \setminus S)$ then we get $v^d(S) = \tilde{v}(N) - \tilde{v}(N \setminus S) = v(N) - \tilde{v}(N \setminus S) = v_{S,x}(S)$. So our definition of a dual game is consistent with associating with any coalition (as distinguished from syndicate) the worth $v_{S,x}(S)$, *i. e.*, the maximal aggregate payoff that the syndicate $N \setminus S$ cannot prevent the coalition S from receiving. Only if $\mathbf{x}(N) = \tilde{v}(N)$ we have $\tilde{v}(S) = \mathbf{x}(S)$ and $\tilde{v}(N \setminus S) = \mathbf{x}(N \setminus S)$, hence $v^d(S) + \tilde{v}(N \setminus S) = \tilde{v}(N) = \tilde{v}(S) + v^d(N \setminus S)$.

5.5 Duality of TU-games via Bondareva-Shapley duality

One of the most important concepts (not only) in cooperative game theory is the **core** of a game. Defined for TU and NTU-games, it is of fundamental importance for economic theory. Introduced by Edgeworth (1881) under the name of **contract curve** in his analysis of perfect competition in pure exchange economies, it had been identified by Shubik (1959) as a forerunner of the core of TU-games defined by Gillies (1953) and Shapley (1953) (cf. Zhao, 2018) and used to analyze market games. The famous Core-Walras equivalence had been established in various settings in seminal articles by Debreu and Scarf (1963), Aumann (1964) and Anderson (1978). A version for TU-games can be derived from Theorem 1 in Shapley and Shubik (1975).

In the context of general TU-games the set of feasible payoff vectors is usually defined as $X^*(N, v) := \{x \in \mathbb{R}^N \mid \mathbf{x}(N) \leq v(N)\}$ and the set of Pareto efficient payoff vectors, also called pre-imputations, is $X(N, v) := \{x \in \mathbb{R}^N \mid \mathbf{x}(N) = v(N)\}$. The set $I(N, v) := \{x \in X(N, v) \mid x(i) \geq v(i) \text{ for all } i \in N\}$ is called set of imputations of (N, v). As explained earlier the employed notions of feasibility and Pareto efficiency are perfectly adequate for cohesive games, they are incompatible for non-cohesive games where there exist subsets S and $N \setminus S$ of N for which a payoff vector y with $\mathbf{y}(N) := v(S) + v(N \setminus S) = \tilde{v}(S) + \tilde{v}(N \setminus S) > v(N)$ is a Pareto improvement upon x with $\mathbf{x}(N) = v(N)$. In this game rational players will not join the grand coalition with a maximal aggregate payoff $v(N)$.

An alternative to imputations that avoids this inconsistency offer **aspirations**. According to Bennett (1983) "they had been first proposed by Cross (1967), were reinvented and carefully studied in Turbay's dissertation (1977)", and again "reinvented and applied to coalitional economies in Wooders' working paper (1978). Albers (1979) presented several variants of core and kernel solution concepts (von Neumann-Morgenstern solution, core, bargaining set, kernel and nucleolus) to the space of aspirations."

A careful assessment of aspirations versus imputations involving balanced but not necessarily super-additive games is provided by Bejan and Gomez (2012). They also give an axiomatization of the "aspiration core" a solution concept that satisfies non-emptiness on the class of all TU-games and coincides with the core whenever it is non-empty, *i. e.*, on the class of balanced games. This equivalence of balancedness of a TU-game and the non-emptiness of its core is the content of the famous Bondareva-Shapley Theorem that had independently been proven in Bondareva (1963) and Shapley (1967).

Before we consider these dual problems, we need some more definitions which we partially adopt from Bejan and Gomez (2012). We will contrast aspirations with pre-imputations and compare them with our new concept of **allotments**. These three different classes of payoff vectors result from different definitions of feasibility and efficiency.

The classical definition of a feasible set of payoff vectors for a TU-game (N, v) is $X^*(N, v) = \{x \in \mathbb{R}^N \mid \mathbf{x}(N) \leq v(N)\}$. It is meaningful under the restriction to superadditive or at least cohesive games. But in the context of **all** TU-games it is a serious hardly justifiable constraint. The classical efficiency concept based on this version of feasibility is embodied in the following version of a Pareto optimal set of feasible payoffs (cf Peleg and Sudhölter, 2003), the so-called set of pre-imputations: $X(N, v) = \{x \in X^*(N, v) \mid \mathbf{x}(N) = v(N)\} = \text{argmax}\{\mathbf{x}(N) \mid x \in X^*(N, v)\}$. Having the whole class of TU-games in mind this is a **constrainted efficiency** or, in economics terms, **second best efficiency**.

The core (Gillies, 1959) is defined as $C(N, v) = \{x \in X^*(N, v) \mid \mathbf{x}(S) \geq v(S)$ for all $S \subseteq N\}$.

The contrast program to approaching solutions of games in terms of pre-imputations or, if they are individually rational, imputations had been initiated mainly by Bennett (1983). She had in mind the players' visions and expectations in the preplay coalitional negotiations. First, she defined for any game (N, v) an **anticipation** as a payoff vector x such that for each player i, there exists a syndicate S with i as member, such that $\mathbf{x}(S) \leq v(S)$. Then, based on this notion, she defined *aspirations* as special anticipations and proved the following more transparent characterization to be equivalent to her definition:

The payoff vector $x \in \mathbb{R}^N$ is an aspiration if and only if it satisfies the followings:

(1) $\mathbf{x}(S) \geq v(S)$ for every syndicate S in N,

(2) For each i in N, there exists a syndicate S containing i such that $\mathbf{x}(S) = v(S)$.

The aspiration x is a *balanced* aspiration if $\mathbf{x}'(S) \geq \mathbf{x}(S)$ for every aspiration x' and any $S \in \mathcal{N}$.

This corresponds to the minimization problem (9.2) in Myerson (1991, p. 432) with the exception that Myerson did not use the restriction 2 of Bennett. As (2) is not implied by (1), the payoff vectors used by Myerson are not aspirations. Nevertheless, both problems have identical solutions, namely the **balanced aspirations** of (N, v). The set of balanced aspirations of a game (N, v) is called the *aspiration core* of (N, v) and denoted $Asp(N, v)$.

The term balanced is derived from the solution of the dual maximization problem (9.3) of Myersons' minimization problem (9.2) formulated as:

$$\max_{\mu \in \mathbb{R}_+^{\mathcal{N}}} \left\langle \mu, v \right\rangle \text{ subject to } \left\langle \mu, \mathbb{1}_{\bullet}(i) \right\rangle = 1 \text{ for all } i \in N,$$

where μ's are the vectors of balancing weights. The solutions of both problems x^* and μ^* result in the joint optimal value $x^*(N) = v^b(N)$.

Next we look at the feasible and efficient sets underlying the aspiration core.

The set of feasible payoff vectors of (N, v) is $X^*_{\mathcal{B}}(N, v) = \{x \in \mathbb{R}^N \mid \mathbf{x}(N) \leq v(\lambda_{ß})$ for some $ß \subseteq \mathcal{N}\}$. The set of efficient payoff vectors for every (N, v) is defined as $X_{\mathcal{B}}(N, v) = \text{argmax}\{\mathbf{x}(N) \mid x \in X^*_{\mathcal{B}}(N, v)\}$. $AC(N, v) = \{x \in X^*_{\mathcal{B}}(N, v) \mid \mathbf{x}(S) \geq v(S)$ for all $S \subseteq N\}$ is the aspiration core of the game (N, v).

In a non-balanced game v, we have $v(N) < v^b(N)$ and the partial memberships in coalitions by which $v^b(N)$ can be established is out of the scope in the game (N, v). No partition of N can legally claim that amount. Therefore, we do not share the view that the aspiration core should be considered as a solution concept for non-balanced games. As mentioned already earlier, only in "share-extended" TU-games (N, w) one should consider $AC(N, w)$ as a solution set. That is obviously applicable also in share-extended TU-games where the v-part of w is not balanced.

We had rejected two of the three feasibility and efficiency concepts in Bejan and Gomez (2012). The classical one involved a drastic restriction to the class of cohesive games which by definition exclude Pareto superior imputations. The other one, favored by Bejan and Gomez (2012), extended the classical problem of coalition building in a way similar to extending domains of sets by including *fuzzy sets*. For many socio-economic scenarios represented aptly by non-balanced games, the feasible set determined via balanced families of syndicates is simply to large and degrades the adequate efficiency unjustly to a second-best level.

As we know, partitions of N are special balanced families and support their associated balancing weights. Given a partition $\pi \in \Pi(N)$ (as a balanced family of coalitions), $\left\langle \mu, \mathbb{1}_{\bullet}(i) \right\rangle$ degenerates for every player i in N to $\left\langle \mathbb{1}_{\pi}(\bullet), \mathbb{1}_{\bullet}(i) \right\rangle$. Therefore, the problem (9.3) will be modified to the following maximization problem (B) with

(B) $\max_{\pi \in \Pi(N)} \left\langle \mathbb{1}_{\pi}(\bullet), v \right\rangle$ subject to for all $i \in N$, $\left\langle \mathbb{1}_{\pi}(\bullet), \mathbb{1}_{\bullet}(i) \right\rangle = 1$.

The dual problem (A) of problem (B) is

(A) $\min_{x \in \mathbb{R}^N} \mathbf{x}(N)$ subject to for all $S \subseteq N$, $\mathbf{x}(S) \geq \max_{\pi \in \Pi(S)} \sum_{T \in \pi} v(T)$.

Any $x \in \mathbb{R}^N$ satisfying the feasibility condition defined by the constraint in (A) is called an **allotment**. Any allotment x satisfies for all $S \in \mathcal{N}$, $\mathbf{x}(N) \geq \tilde{v}(S) + \tilde{v}(N \setminus S)$.

The joint value for the optima of the dual problems (A) and (B) is smaller than for those in (9.2) and (9.3) of Myerson (1991). The domain in (A) was extended by relaxing the constraints while that of (B) shrank to the partitions of N. The optima for (A) and (B) are, respectively, $\hat{x}$ and $\hat{\pi}$ with $\hat{x}(N) = \left\langle \mathbb{1}_{\hat{\pi}}(\bullet), v \right\rangle = v^c(N) = \tilde{v}(N)$.

If we presume that any syndicate S as a prospective coalition considers during preplay negotiations to proceed in the subgame guided by optimization problems of types (A) and (B) they will end up with $(v|_S)^c(S) = \tilde{v}(S)$. That means that in the considerations of S about what a coalition $N \setminus S$ could not prevent them from receiving they should assume that $\tilde{v}(N \setminus S)$ would be given in total to the coordinated sub-coalitions which the syndicate would form by organizing itself optimally. So the only version of Bondareva-Shapley duality that is adequate for all TU-games leads again to our duality.

Despite the efficieny of self-organization of the syndicates S and $N \setminus S$ resulting in $\tilde{v}(S)$ and $\tilde{v}(N \setminus S)$, the overall self-organization of the players in N may not be efficient. We may have $\tilde{v}(N) > \tilde{v}(S) + \tilde{v}(N \setminus S)$. So the simultaneous payoffs $v^d(S)$ and $v^d(N \setminus S)$ together may still be an **utopia allocation**. Notice, however, that it is possible to rewrite $\sum_{T \in \tilde{\pi}} v(T)$ as $\tilde{v}(S^*) + \tilde{v}(N \setminus S^*)$ for some $S^* \in \mathcal{N}$. For such S^*, we have $\tilde{v}(N) - \tilde{v}(S^*) = \tilde{v}(N \setminus S^*) = v^d(N \setminus S^*)$ and $\tilde{v}(N) - \tilde{v}(N \setminus S^*) = \tilde{v}(S^*) = v^d(S^*)$. This split of N in to S^* and $N \setminus S^*$ respresents an efficient organization of all players in N in a bi-partition in which the utopia worths $v^d(S^*)$, $v^d(N \setminus S^*)$ become feasible in (N, v).

The game (N, v^c) is dual to (N, v) if for every syndicate S in (N, v) one has $v^c(N \setminus S) = \tilde{v}(N) - \tilde{v}(S)$. It is easy to see that this duality coincides with the *duality on the class of super-additive games. But what happens on balanced games? Assume that (N, v) is balanced. In that case $v(N) = v^c(N) = \tilde{v}(N) = v^b(N)$. That shows that also for balanced and totally balanced games, our duality definition is adequate.

We know that for any game (N, v), one has $v(N) \leq v^c(N) = \tilde{v}(N) \leq v^b(N)$. We consider now which further modification (A') and (B') of our problems (A) and (B) will result in $v(N)$. If we define (B') by restricting the set of partitions of N in problem (B) to the singleton partition, there remains $\{N\}$. This is the only feasible partition satisfying $\left\langle \mathbb{1}_{\pi}(\bullet), \mathbb{1}_{\bullet}(i) \right\rangle = 1$ for all $i \in N$ and therefore optimal in (B') and yields the optimal value $v(N)$.

Replacing in (A) the term for all "$S \subseteq N$" to just "$S = N$" the problem degenerates to

(A') $\min_{x \in \mathbb{R}^N} \mathbf{x}(N)$ subject to $\mathbf{x}(N) \geq \sum_{T \in \{N\}} v(T) = v(N)$.

This is equivalent to $\min_{x \in \mathbb{R}^N} \mathbf{x}(N)$ subject to $\mathbf{x}(N) \geq v(N)$ where the optimal value is clearly $v(N)$.

The problems (A') and (B') debunk how restrictive it is to define feasibility and efficiency with regard to $v(N)$. In our terminology that is the case where N is the only coalition, where players of all syndicates can at best use the various worths they would have received if they would have formed a coalition as arguments in the internal negotiation of payoffs within the grand coalition. Only if (N, v) happened to be cohesive, the grand coalition can be assumed to form.

5.6 Duality of TU-games and Market Representations

There is a literature on markets and NTU-games, that is, however, not of our concern in this article. Representative articles are Scarf (1967), Billera (1974), Billera and Bixby (1974), Qin (1993) and Inoue (2012).

The usual meaning of "market games" is based on work of Shapley and Shubik, who following their earlier works Shapley (1953) and Shubik (1959), had in Shapley and Shubik (1969) introduced **market games** as a specific class of TU-games and proven that these are exactly the **totally balanced games**. In their following work on this topic, Shapley and Shubik (1975) had analyzed the relation between payoff vectors in the core of the TU-game and their representation of competitive equilibria in the markets inducing this game.

Later the class of this TU-games that result from economies of various sort had been extended. Garratt and Qin (1996, 1997, 2000) introduced market models that can be represented by super-additive games [see also Bejan and Gomez (2012)]. Sun et al. (2008) presented a coalition production economy that is induced by an arbitrary TU-game (N, v). They proved the equivalence between the c-Core of (N, v) and the utility allocation of competitive equilibria of the induced economy. Inoue (2012) simplified this model by reducing the number of output commodities to just one ("*money*") allowing two versions, one with divisible and one with indivisible labor input of agents, permitting respective excluding "multiple jobbing" of agents, i.e., working part time in different coalitions. Multiple jobbing is based on the possibility of the players to organize themselves in *balanced families* rather than only in partitions of N. Therefore, only the "single jobbing" model of Inoue (2012) is adequate for representing TU-games in our *classical* framework.

While explanation of coalition building via competitive equilibria of representing economies or the core of the games is only possible if the games are at least c-balanced (cf. Sun et al., 2008), it is still possible via efficient production and the coalitions being active in this production for arbitrary TU-games and their induced "single jobbing" coalition production economies. We will focus now on this model. For technicalities and detailed interpretation, we refer to the relatively short article Inoue (2012).

Like in Shapley and Shubik (1969) market games, each player of the game has one indivisible unit of an idiosyncratic endowment good, "his labor time". Each coalition $S \in \mathcal{N}$

has a technology $Y_v(S)$ by which a coalition S can produce $tv(S)$, $t \geq 0$ if each member of S provides t of its input good labor. Indivisibility implies for each player his t to be either 0 or 1. So the coalition is built if its production set is activated by all members of S providing their full endowments. If the "right" coalitions build, they form a partition π of N that in total produces $\sum\limits_{T\in\pi} v(T) = v^c(N) = \tilde{v}(N)$.

This is even the case if the considered game is a (totally balanced) market game. As Shapley and Shubik (1975) remark, the competitive (core) payoff vectors of (N, v) are exactly those which maximize the players' total utility (payoff). That is, $\bar{v}(N) = v^c(N) = \tilde{v}(N)$.

Notice that several coalition structures $\pi \in \Pi(N)$ may be able to produce the aggregate amount $v^c(N)$.

As Inoue's representation of TU-games with the characteristic function v by "single jobbing" coalition production economies $\mathcal{E}_v$ holds for all TU-games, it also applies to the dual v^d. We know already that for any active coalition in $\mathcal{E}_{v^d}$ the efficient production is $v^d(S)$. Remember, what $v^d(S)$ is meaning with respect to the underlying game (N, v):

$$v^d(S) = \tilde{v}(N) - \tilde{v}(N \setminus S) \geq \tilde{v}(S) \text{ for all } S \in \mathcal{N}.$$

Therefore, as $v^d(S) - \tilde{v}(S)$ describes the residual amount that S might receive if, given $N \setminus S$ receives its maximal legally guaranteed payoff $\tilde{v}(N \setminus S)$, the allocation $(v^d(S), \tilde{v}(N \setminus S))$ would be efficient.

$v^d(S)$ and $v^d(N \setminus S)$ describe the **residual efficiency gap** that need to be paid to S and $N \setminus S$, respectively, if $N \setminus S$ or S are already determined to get $\tilde{v}(N \setminus S)$ or $\tilde{v}(S)$, respectively.

An allocation $(v^d(S), v^d(N \setminus S))$ is in general not feasible in the economy $\mathcal{E}_v$ or the game (N, v). But an optimal aggregate payoff in (N, v) can be produced by a suitable partition $\pi^* \in \Pi(N)$. So we get $\sum\limits_{T\in\pi^*} v(T) = v^c(N) = \tilde{v}(N)$.

Any bi-partition $\{S, N \setminus S\}$ generated via unions of syndicates in π^* satisfies $\tilde{v}(S) + \tilde{v}(N \setminus S) = \tilde{v}(N)$. For such "efficient" bi-partition, we get again $\tilde{v}(N \setminus S) = \tilde{v}(N) - \tilde{v}(S) = v^d(N \setminus S)$ and $\tilde{v}(S) = \tilde{v}(N) - \tilde{v}(N \setminus S) = v^d(S)$. This implies immediately that $\tilde{v}(S) = v^d(S)$ and $\tilde{v}(N \setminus S) = v^d(N \setminus S)$, which means that neither S nor $N \setminus S$ is suffering from forgone opportunities, since both of the residual efficiency gaps are zero.

On the other hand, we know that for arbitrary $S \in \mathcal{N}$, we have $v^d(S) + v^d(N \setminus S) = \tilde{v}(N) - \tilde{v}(S) + \tilde{v}(N) - \tilde{v}(N \setminus S) = 2\tilde{v}(N) - (\tilde{v}(S) + \tilde{v}(N \setminus S)) \geq \tilde{v}(N)$.

What we have received is that $\min\limits_{S\in\mathcal{N}} (v^d(S) + v^d(N \setminus S)) = \max\limits_{S\in\mathcal{N}} (\tilde{v}(S) + \tilde{v}(N \setminus S)) = v^c(N) = \tilde{v}(N)$. So if each $i \in N$ is active in some coalition in the economy $\mathcal{E}_{v^d}$ the minimum that can be produced in the aggregate makes it just feasible and efficient in the economy

$\mathcal{E}_v$. So the dual economy $\mathcal{E}_{v^d}$ of $\mathcal{E}_v$ just produces via inefficiency in the production $\mathcal{E}_v$. One could say that $\mathcal{E}_{v^d}$ "produces" opportunity costs $v^d(S) - \tilde{v}(S)$ while $\mathcal{E}_v$ produces revenues $\tilde{v}(S)$. If revenues are maximal due to the efficient production and organisation of coalitions in a partition, opportunity cost generated in $\mathcal{E}_{v^d}$ are $v^d(S) - \tilde{v}(S) = 0$.

6 Some Results on Duality

Lemma 1. *If (N, v) is super-additive, $v^d(S) = v(N) - v(N \setminus S) = v^*(S)$ for all $S \subseteq N$.*

Proof. The super-additivity of (N, v) implies that $\tilde{v}(S) = v(S)$ for all $S \subseteq N$. Thus, $v^d(S) = \tilde{v}(N) - \tilde{v}(N \setminus S) = v(N) - v(N \setminus S) = v^*(S)$ for all $S \subseteq N$.

Proposition 1. *If (N, v) is a convex game, then the dual game (N, v^d) is concave.*

Proof. Let (N, v) be a convex game. As convexity implies super-additivity, we conclude from Lemma 1 that $v^d(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.

Let S, T be two arbitrary subsets of N.

$v^d(S) + v^d(T) = v(N) - v(N \setminus S) + v(N) - v(N \setminus T) = 2v(N) - (v(N \setminus S) + v(N \setminus T))$

$\geq 2v(N) - (v(N \setminus (S \cup T)) + v(N \setminus (S \cap T))) = v^d(S \cup T) + v^d(S \cap T)$ where the inequality is obtained by the convexity of (N, v).

Definition 2. *The negative-dual (or anti-dual) game of a game (N, v) is the game (N, v^{nd}) with $v^{nd} = -v^d$.*

Proposition 2. *If (N, v) is a convex game, then so is its negative-dual game (N, v^{nd}).*

Proof. (N, v) is super-additive as it is convex. So, it is enough to show that $(N, -v^*)$ is convex since $v^{nd} = -v^d = -v^*$.

We know from Oishi and Nakayama (2009) that the set of convex games is invariant under the negative-dual (anti-dual) operator. Thus, $(N, -v^*)$ is convex, and so is (N, v^{nd}).

Proposition 3. *If (N, v) is balanced, then (N, v^{nd}) is balanced.*

Proof. Let (N, v) be balanced. Then, $Core(N, v) \neq \emptyset$. Take an arbitrary x from $Core(N, v)$.

We know $\mathbf{x}(S) \geq v(S)$ for all $S \subset N$, and also $\mathbf{x}(N) = v(N) = \tilde{v}(N)$ by the fact that balancedness of a TU-game implies its cohesiveness.

Consider $-x$. Clearly, $-\mathbf{x}(N) = -\tilde{v}(N) = v^{nd}(N)$.

Take any coalition $T \subset N$.

$v^{nd}(T) = -v^d(T) = -\tilde{v}(N) + \tilde{v}(N \setminus T) = -v(N) + \max\limits_{P \in \mathbb{P}(N \setminus T)} \sum_{C \in P} v(C)$

$\leq -v(N) + \max\limits_{P \in \mathbb{P}(N \setminus T)} \sum_{C \in P} \mathbf{x}(C)$

$= -v(N) + \mathbf{x}(N \setminus T) = -\mathbf{x}(N) + \mathbf{x}(N \setminus T) = -\mathbf{x}(T)$,

where the inequality is clear by the fact that $\mathbf{x}(S) \geq v(S)$ for all $S \subset N$.

Thus, $-x \in Core(N, v^{nd})$.

Remark 2. *By Proposition 3, we have the following results:*

- *The set of balanced games is invariant under the negative-dual operator,*
- *$Core(N, v^{nd})$ is non-empty if (N, v) is a balanced game.*

Proposition 4. *If (N, v) is a balanced game, then $Core(N, v) = -Core(N, v^{nd})$.*

Proof. By Proposition 3, we have $Core(N, v) \subseteq -Core(N, v^{nd})$.

Let $x \in Core(N, v^{nd})$. Then, $\mathbf{x}(N) = v^{nd}(N) = -v^d(N) = -\tilde{v}(N) = -v(N)$ and $\mathbf{x}(S) \geq v^{nd}(S)$ for all $S \subset N$.

Consider $-x$. Since x is in $Core(N, v^{nd})$, we get $-\mathbf{x}(N \setminus S) \leq -v^{nd}(N \setminus S) = \tilde{v}(N) - \tilde{v}(S) = v(N) - \tilde{v}(S) \leq v(N) - v(S) = -\mathbf{x}(N) - v(S)$. Thus, $-\mathbf{x}(N \setminus S) \leq -\mathbf{x}(N) - v(S)$, *i. e.*, $v(S) \leq -\mathbf{x}(S)$.

Since $-\mathbf{x}(S) \geq v(S)$ and $-\mathbf{x}(N) = v(N)$, we have $-x \in Core(N, v)$.

By Proposition 4, one can conclude that on the domain of balanced games, the core is "self-anti-dual" as Oishi et al. (2016) call it.

Definition 3. *A TU-game (N, v) is called c-balanced game if its completion (N, v^c) is a balanced game.*

Proposition 5. *If (N, v) is a c-balanced game, so is (N, v^{nd}).*

Proof. By the hypothesis, we know (N, v^c) is a balanced game and so is its negative-dual. Then, by the cohesiveness of (N, v^c), on any coalition we have $(v^c)^{nd} = -(v^c)^d = -v^d = v^{nd}$. Hence, (N, v^{nd}) is a balanced game. Since a balanced game (N, v) is also c-balanced, (N, v^{nd}) is a c-balanced game.

Proposition 6. *The core is not self-anti-dual on the set of c-balanced games.*

Proof. Let (N, v) be c-balanced, but not balanced. So $Core(N, v) = \emptyset$ and (N, v^c) is balanced.

By Proposition 4, we also have $Core(N, v^c) = -Core(N, (v^c)^{nd})$. Additionally, $(v^c)^{nd} = v^{nd}$ as shown in the previous proposition. Thus, $Core(N, v^{nd}) = Core(N, (v^c)^{nd}) = -Core(N, v^c) \neq \emptyset$.

Since $-Core(N, v^{nd}) \neq Core(N, v)$, the core is not self-anti-dual on c-balanced games.

Combining the results above, we get the fact that unlike the core, the c-core is self-anti-dual on the set of c-balanced games: (N, v^{nd}) is c-balanced whenever (N, v) is c-balanced. By the help of Proposition 4, $c\text{–}Core(N, v) = Core(N, v^c) = -Core(N, (v^c)^{nd}) = -Core(N, v^{nd})$. Clearly, (N, v^{nd}) is balanced (so cohesive) as its core is non-empty, which implies $Core(N, v^{nd}) = c\text{–}Core(N, v^{nd})$. Thus, we get $c\text{–}Core(N, v) = -\ c\text{–}Core(N, v^{nd})$.

7 Concluding remarks: Summary and Outlook

7.1 Summary

We have introduced a new concept of duality for general TU-games that is not based on the second best efficiency defined via the grand coalition as the reference point, but rather on Pareto efficiency with the most productive coalition structures as a benchmark. These two concepts coincide on super-additive games, where the grand coalition builds its most productive (singleton) partition.

The predominant interpretation of the duality, that is meaningful for super-additive games, leads in general to logical inconsistency but can be maintained with our duality concept. We have provided various different strands of arguments for the use of our new concept and provided in Section 6 results paralleling or deviating from results found in the recent duality literature, predominantly by Oishi and Nakayama (2009) and Oishi et al. (2016).

The key concept of our definition of duality, the cohesive hull (or completion) of a TU-game admitted to cover also balanced and totally balanced games although in a strict sense these are linear homogeneous extensions of coalitional TU-games.

Still our much more general concept does not satisfactorily cover all games in all its interpretations, like the restricted class of super-additive games that suggests an interpretation of the worth as a gain that players want to maximize. Like the usual *duality, also our duality implicitly interprets worth as gain, profit, or saving rather than a loss, cost or debt. From a puristic point of view our duality, like the *duality, only an "upper duality" defined, and in contrast to *duality meaningfully interpretable for all coalitional TU-games. It is neither the sign (− versus +) nor the notation (c versus v) that characterizes TU-games as cost games. These can, however, according to Peleg and Sudhölter (2003, p. 14) be "associated with ordinary games, called savings games" and are "not games from the point of view of applications, because the cost function is not interpreted as an ordinary coalition function". As any game is strategically equivalent to positive games and to negative games any cost game can be represented as positive and any "ordinary" game as negative. It is just the interpretation of a game, i.e., of its payoffs as utilities or revenues as opposed to dis-utilities or costs that determines, what efficient payoff vectors are. A sound analysis of options for syndicates of players in pre-play negotiations presumes their agreement on the interpretation of efficiency, hence the choice between maximizing and minimizing the aggregate worth.

7.2 Outlook

A common understanding among players in a game of what efficiency in that game means is necessary for potential cooperation. The focus on players agreement on efficiency suggests naturally the idea of defining and analyzing also a "lower duality". Based on efficiency rather than on maximization of gains or minimization of costs, one may get such universally applicable duality notion for all TU-games, embodying upper and lower duality.

In this context we would like to hint to the remarkable article by Derks et al. (2014). Although they are not directly concerned with duality and the grand coalition still serves as a benchmark, their analysis may be seminal for further work on duality as their solution concepts cover the core as well as the anti-core. In their abstract, they write:

"*We consider several related set extensions of the core and the anticore of games with transferable utility. An efficient allocation is undominated if it cannot be improved, in a specific way, by sidepayments changing the allocation or the game. The set of all such allocations is called the undominated set, and we show that it consists of finitely many polytopes with a core-like structure. One of these polytopes is L_1-center, consisting of all efficient allocations that minimize the sum of the absolute values of excesses. The excess Pareto optimal set contains the allocations that are Pareto optimal in the set obtained by ordering the sums of the absolute values of the excesses of coaltions and the absolute values of the excesses of their complements. The L_1-center is contained in the excess Pareto-optimal set, which in turn is contained in the undominated set. For three-person games all these sets coincide. These three sets also coincide with the core for balanced games and with the anticore for antibalanced games. We study properties of these sets and provide characterizations in terms of balanced collections of coalitions.*"

It may be worthwhile to try to build on these ideas but with Pareto efficiency rather than second best efficiency, with $\tilde{v}(N)$ for $v(N)$ and with partitions of N for balanced families.

References

Albers, W. (1979). Core-and kernel-variants based on imputations and demand profiles. **Game Theory and Related Topics** (pp. 3-16). North-Holland, Amsterdam.

Anderson, R. M. (1978). An elementary core equivalence theorem. **Econometrica: Journal of the Econometric Society**, 1483-1487

Arnold, T., & Schwalbe, U. (2002). Dynamic coalition formation and the core. **Journal of Economic Behavior and Organization**, 49(3), 363-380.

Aslan, F., & Duman, P. (2018). c-Duality and Anti-c-Duality. Manuscript, Istanbul Bilgi University.

Aumann, R. J. (1964). Markets with a continuum of traders. **Econometrica**, 32.

Aumann, R. J. (1967). A survey of cooperative games without side payments. **Essays in Mathematical Economics**, 3-27.

Aumann, R.J., & Dreze, J.H. (1974). Cooperative games with coalition structures. **International Journal of Game Theory**, 3(4), 217-237.

Aumann, R.J., & Maschler, M. (1985). Game Theoretic Analysis of a Bankruptcy Problem from the Talmud. **Journal of Economic Theory**, 36(2), 195-213.

Aumann, R. J., & Peleg, B. (1960). Von Neumann-Morgenstern solutions to cooperative games without side payments. **Bulletin of the American Mathematical Society**, 66(3), 173-179.

Bejan, C., & Gomez, J.C. (2012). Axiomatizing Core Extensions. **International Journal of Game Theory**, 41(4), 885-898.

Bennett, E. (1983). The aspiration approach to predicting coalition formation and payoff distribution in sidepayment games. **International Journal of Game Theory**, 12(1), 1-28.

Billera, L. J. (1974). On games without side payments arising from a general class of markets. **Journal of Mathematical Economics**, 1(2), 129-139.

Billera, L. J., & Bixby, R. E. (1974). Market representations of n-person games. **Bulletin of the American Mathematical Society**, 80(3), 522-526.

Böhm, V. (1974). The core of an economy with production. **The Review of Economic Studies**, 41(3), 429-436.

Bondareva, O. N. (1963). Some applications of linear programming methods to the theory of cooperative games. **Problemy Kibernetiki**, 10, 119-139.

Cross, J. G. (1967). Some theoretic characteristics of economic and political coalitions. **Journal of Conflict Resolution**, 11(2), 184-195.

Davis, M., & Maschler, M. (1965). The kernel of a cooperative game. **Naval Research Logistics Quarterly**, 12(3), 223-259.

Debreu, G., & Scarf, H. (1963). A limit theorem on the core of an Economy. **International Economic Review**, 4(3), 235-246

Derks, J., Peters, H., & Sudhölter, P. (2014). On extensions of the core and the anticore of transferable utility games. **International Journal of Game Theory**, 43(1), 37-63.

Edgeworth, F. Y. (1881). **Mathematical Psychics**. Paul Kegan, London.

Funaki, Y. (1998). Dual Axiomatizations of Solutions of Cooperative Games, mimeo.

Garratt, R., & Qin, C. Z. (1996). Cores and competitive equilibria with indivisibilities and lotteries. **Journal of Economic Theory**, 68(2), 531-543.

Garratt, R., & Qin, C. Z. (1997). On a market for coalitions with indivisible agents and lotteries. **Journal of Economic Theory**, 77(1), 81-101.

Garratt, R., & Qin, C. Z. (2000). On market games when agents cannot be in two places at once. **Games and Economic Behavior**, 31(2), 165-173.

Gillies, D. (1953). Some theorems on n-person games. Ph. D. Dissertation, Princeton University, Department of Mathematics.

Gillies, D. B. (1959). Solutions to general non-zero-sum games. **Contributions to the Theory of Games**, 4, 47-85.

Guesnerie, R., & Oddou, C. (1979). On Economic Games Which Are Not Necessarily Superadditive: Solution Concepts and Application to a Local Public Good Problem With a Few Agents. **Economics Letters**, 3(4), 301-306.

Harsanyi, J. C. (1959). A bargaining model for the cooperative n-person game. **Contributions to the Theory of Games**, 4, 325-355.

Hart, S., & Kurz, M. (1983). Endogenous Formation of Coalitions. **Econometrica**, 51(4), 1047-1064.

Hokari, T., & Kıbrıs, Ö. (2003). Consistency, converse consistency, and aspirations in TU-games. **Mathematical social sciences**, 45(3), 313-331.

Inoue, T. (2012). Representation of transferable utility games by coalition production economies. **Journal of Mathematical Economics**, 48(3), 143-147.

Kikuta, K. (2007). Twisted Dual Games and Their Properties. **International Game Theory Review**, 9(2), 285-306.

Luce, R. D., & Raiffa, H. (1957). **Games and decisions: Introduction and critical survey**. Courier Corporation.

Martinez-Legaz, J. E. (1996). Dual representation of cooperative games based on Fenchel-Moreau conjugation. **Optimization**, 36(4), 291-319.

Maschler, M., Solan, E., & Zamir, S. (2013). **Game Theory**. Cambridge University Press.

McKinsey, J.C.C. (1952). **Introduction to the theory of games**. New York: McGraw-Hill

Moldovanu, B., & Winter, E. (1994). Core implementation and increasing returns to scale for cooperation. **Journal of Mathematical Economics**, 23(6), 533-548.

Myerson, R. B. (1991) **Game Theory: Analysis of Conflict**. The President and Fellows of Harvard College, USA.

Neuefeind, W. (1974). A stochastic bargaining process for n-person games. **Journal of Mathematical Economics**, 1(2), 175-191.

von Neumann, J. (1928). Zur theorie der gesellschaftsspiele. **Mathematische Annalen**, 100(1), 295-320.

von Neumann, J., & Morgenstern, O. (1944). **Theory of games and economic behavior**. Princeton: Princeton University Press.

Oishi, T., & Nakayama, M. (2009). Anti-dual of Economic Coalitional TU Games. **Japanese Economic Review**, 60(4), 560-566.

Oishi, T., Nakayama, M., Hokari, T., & Funaki, Y. (2016). Duality and Anti-duality in TU Games Applied to Solutions, Axioms, and Axiomatizations. **Journal of Mathematical Economics**, 63, 44-53.

Osborne, M. J., & Rubinstein, A. (1994). **A course in game theory**. MIT press.

Peleg, B., & Sudhölter, P. (2003). **Introduction to the theory of cooperative Games**. Kluwer Academic Publishers.

Peleg, B. (1986). On the reduced game property and its converse. **International Journal of Game Theory**, 15(3), 187-200.

Peters, H. (2008). **Game theory: A multi-leveled approach**(pp. 121-131). Springer.

Qin, C. Z. (1993). A conjecture of Shapley and Shubik on competitive outcomes in the cores of NTU market games. **International Journal of Game Theory**, 22(4), 335-344.

Rapoport, A. (1970). N**-person game theory: concepts and applications**. Ann Arbor: University of Michigan Press.

Scarf, H. E. (1967). The core of an N person game. **Econometrica**, 35(1), 50-69.

Shapley, L. S. (1953). A value for n-person games. **Contributions to the Theory of Games**, 2(28), 307-317.

Shapley, L. S. (1967). On balanced sets and cores. **Naval Research Logistics Quarterly**, 14(4), 453-460.

Shapley, L. S., & Shubik, M. (1969). On market games. **Journal of Economic Theory**, 1(1), 9-25.

Shapley, L. S., & Shubik, M. (1975). Competitive outcomes in the cores of market games. **International Journal of Game Theory**, 4(4), 229-237.

Shubik, M. (1959). Edgeworth market games. **Contributions to the Theory of Games**, 4, 267-278.

Sun, N., Trockel, W., & Yang Z. (2008). Competitive Outcomes and Endogenous Coalition Formation in an n-person Game. **Journal of Mathematical Economics**, 44(7), 853-860.

Tadenuma, K. (1990). Dual axiomatizations of the core and the anti-Core (pp. 90-41). Hitotsubashi University RUEE Working Paper.

Thrall, R. M., & Lucas, W. F. (1963). N-person games in partition function form. **Naval Research Logistics Quarterly**, 10(1), 281-298.

Turbay, G. J. (1977). On value theories for n-person cooperative games. Doctoral dissertation, Rice University.

Wikipedia. (2019). Cooperative Game Theory. Resource Document. https://en.wikipedia.org/wiki/Cooperative-game-theory

Wooders, M. (1978). Quasi-cores and quasi-equilibria in coalition economics with transferable utility. In Working Paper 184. Economics Department, State Univ. of New York at Stony Brook Stony Brook, NY.

Zhao, J. (2018). A reexamination of the coase theorem. **The Journal of Mechanism and Institution Design**, 3(1), 111-132.