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Stable Balanced Expansion in Homogeneous Dynamic Models

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Abstract

This paper examines the stability of balanced paths of expansion or contraction in closed macroeconomic models as typical cases of homogeneous dynamical systems. Examples of known two-dimensional deterministic and stochastic models are discussed.

The appendix presents the mathematical tools and concepts to prove the stability of expanding/contracting paths in homogeneous systems. These are described by so-called Perron-Frobenius solutions. Since convergence of orbits of homogeneous systems in intensive form is only a necessary condition for convergence in state space additional requirements are derived for the general n-dimensional case. For deterministic dynamic economies, as in most models of economic growth, of international trade, or monetary macro, conditions of existence and stability are obtained applying the features of the non-linear generalization of the Perron-Frobenius Theorem. In the stochastic case, the conditions for the stability of balanced paths are derived using a recent extension of the Perron-Frobenius Theorem provided by Evstigneev & Pirogov (2010) and Babaei, Evstigneev & Pirogov (2018).

JEL codes: C02, C62, E13, E24, E30, E31, O41, O42

Keywords: balanced growth, stability, stochastic balanced expansion, random fixed points, Perron-Frobenius solution

Contents

1 Introduction

Since the early contribution by Deardorff (1970) economists have been aware of the fact that convergence to a balanced growth path is not guaranteed by considering only the stability of the growth model in its intensive form. In spite of this fact the issue has been largely ignored in the growth literature. In fact, standard text books (such as Barro & Sala-I-Martin, 1995; Romer, 1996; Aghion & Howitt, 1998; De La Croix & Michel, 2002) seem to presume that convergence in per-capita terms or in growth rates implies convergence in state space as well.

Figure 1 displays the occurrence of convergence or divergence to the balanced path in a standard two-dimensional growth model of the Solow type indicated by a decrease or an increase of the distance of an orbit from the stable balanced path in intensity form. The additional conditions required for the convergence of orbits to balanced growth paths in two-dimensional growth models were given in Böhm, Pampel & Wenzelburger (2005) for models in discrete time and in Pampel (2009) for continuous time. The conditions of stability of the intensive form with

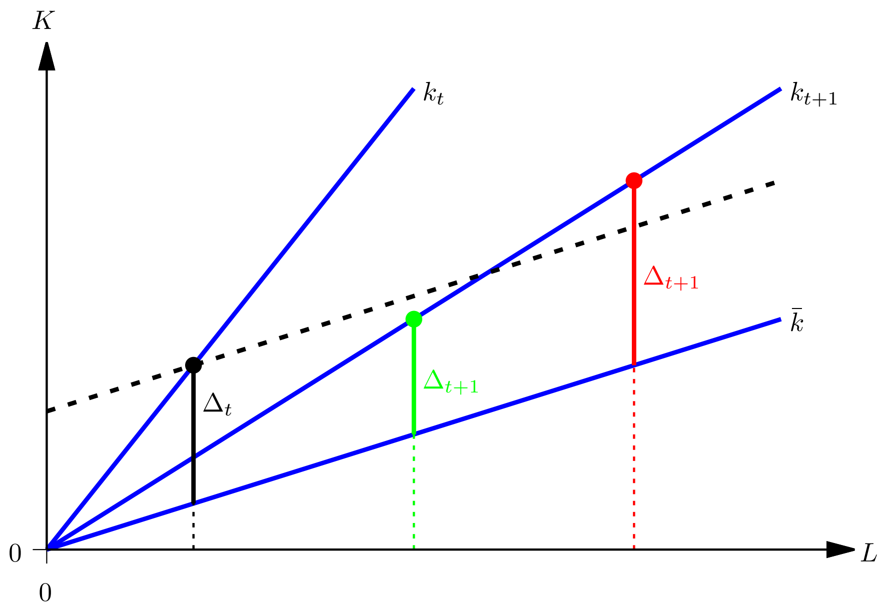


Figure 1: Convergence to/divergence from balanced path in $(L, K)-$space for $k_t \to \bar{k}$.

respect to labor is essentially equivalent to assuming that labor supply is constant contradicting the basic assumption of an economy with a growing population. Surely, the assumption is meaningless in a model with respect to one variable which grows endogenously. In models of growth of dimension higher than two (for example a trade model with two countries, a model with two financial assets) the choice of an intensive form with respect to one particular variable is arbitrary and needs to be taken with care to exhibit convergence of the associated 'real model'.

Orbits of homogeneous dynamical systems describing balanced contraction or expansion in state space are best described by the non-linear extension of so-called Perron-Frobenius solutions. They are not fixed points of an equivalent dynamical system on a compactified state space, often referred to as the intensive form of the system. Since convergence of orbits of homogeneous systems in intensive form is only a necessary condition for convergence in state space additional mathematical tools and concepts to prove the stability of expanding/contracting paths in homogeneous systems have to be developed.

The results for the general n-dimensional situation for continuous homogeneous time-one maps in discrete time are given in Appendix A and B. For deterministic dynamic economies, as in most models of economic growth, of international trade, or monetary macro, conditions of existence and stability are obtained applying the features of the non-linear generalization of the Perron-Frobenius Theorem. In the stochastic case, the conditions for the stability of balanced paths are derived using a recent extension of the Perron-Frobenius Theorem provided by Evstigneev & Pirogov (2010) and Babaei, Evstigneev & Pirogov (2018). The main two sections of this note present examples and discuss extensions and applications of the mathematical results to different economic homogeneous dynamic models.

2 Examples of Homogeneous Dynamical Systems

2.1 The Solow Growth Model

Let $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ be the concave homogeneous production function inducing the Solow growth model $(K, L) \mapsto (\mathcal{G}(K, L), \mathcal{L}(K, L))$ (Solow, 1956, 1988, 1999) given by

$$\begin{aligned} K' &= \mathcal{K}(K, L) := (1-\delta)K + sA\,F(K, L) \\ L' &= \mathcal{L}(K, L) := (1+n)L \end{aligned} \tag{2.1}$$

with parameters (n, δ, A, s) which induces the common one-dimensional mapping in intensity form

$$k' = G(k) := \frac{1}{1+n}\left((1-\delta)k + sA\,f(k)\right), \qquad k := K/L \quad f(k) := F(K/L, 1). \tag{2.2}$$

A balanced path of the Solow model is defined by a triple $\lambda > 0$, $(\bar{K}, \bar{L}) \gneqq 0$, $|(\bar{K}, \bar{L})| = 1$ (a so-called Perron-Frobenius solution of the homogeneous system $(\mathcal{K}, \mathcal{L})$) which solves

$$\lambda \begin{pmatrix} \bar{K} \\ \bar{L} \end{pmatrix} = \begin{pmatrix} (1-\delta)\bar{K} + sAF(\bar{K}, \bar{L}) \\ (1+n)\bar{L} \end{pmatrix}. \tag{2.3}$$

Balanced orbits of the Solow model are of the form $\gamma(\alpha(\bar{K}, \bar{L})) = \left\{\lambda^t \alpha(\bar{K}, \bar{L})\right\}_{t \geq 0}$, $\alpha > 0$. They are all contained in the set $L(\bar{k}) := \left\{(K, L) \in \mathbb{R}^2_+ \mid K = \bar{k}L, \bar{k} = \bar{K}/\bar{L} \gneqq 0\right\}$ which is the halfline through $(\bar{K}, \bar{L}) \gneqq 0$, the balanced ray.

If $f(k)$ satisfies the Inada conditions (Inada, 1963), then, for every (n, δ, A, s), there exists a unique Perron-Frobenius solution $\lambda > 0$, $(\bar{K}, \bar{L}) \gg 0$ satisfying

$$\begin{aligned} \frac{f(\bar{k})}{\bar{k}} &:= \frac{n+\delta}{s\,A}, \qquad\qquad \bar{k} := \bar{K}/\bar{L} \\ \lambda &:= 1+n, \\ M &:= \lim_{k \to \bar{k}} \frac{G(k) - G(\bar{k})}{k - \bar{k}} =: G'(\bar{k}). \end{aligned} \tag{2.4}$$

(1) The steady state $\bar{k} = G(\bar{k})$ of the model in intensity form is asymptotically stable if and only if $E_f(\bar{k}) < 1$ since

$$\begin{aligned} G'(\bar{k}) &= \frac{1}{1+n}\left(1 - \delta + Asf'(\bar{k})\right) = \frac{1}{1+n}\left(1 - \delta + (n+\delta)\frac{\bar{k}f'(\bar{k})}{f(\bar{k})})\right) \\ &= 1 + \frac{\delta+n}{1+n}\left(E_f(\bar{k}) - 1\right) < 1 \qquad \iff \qquad E_f(\bar{k}) < 1 \end{aligned} \tag{2.5}$$

which is guaranteed by the concavity of F (respectively of f).

(2) The condition (2.5) is only necessary to guarantee convergence of the orbit in state space to the balanced ray $L(\bar{k})$ (as observed by Deardorff, 1970). According to Lemma A.1 of the appendix (also Böhm, Pampel & Wenzelburger, 2005; Pampel, 2009) orbits $\gamma(K_0, L_0)$ of the Solow model converge to the halfline through $(\bar{K}, \bar{L})$ if and only if $\lambda M = \lambda G'(\bar{k}) < 1$ which is satisfied if and only if

$$E_f(\bar{k}) < \frac{\delta}{n+\delta}. \tag{2.6}$$

In other words, convergence to the balanced ray (halfline) is guaranteed only for levels of the elasticity E_f smaller than the relative rate of depreciation $\delta/(n+\delta)$. Surely, if $n \leq 0$ stability holds for all pairs $0 \leq (E_f(\bar{k}), \delta) \leq 1$. Conversely, if $n > 0$, all orbits are diverging from the balanced path if $\delta = 0$, which was the result established by Deardorff (1970). In general, for $0 \ll (\delta, n, E_f(\bar{k}))$, (2.6) describes the trade-off between (δ, n) and $E_f(\bar{k})$ to maintain stability, i.e. near the boundary of the stability region a decrease in capital depreciation has to be offset by a decrease in elasticity to maintain stability[1]. The same stability issue arises in models of optimal growth requiring similar conditions for convergence.

Figure 2 displays the ranges of $(E_f(\bar{k}), \delta) \in [0,1]^2$ (shaded region) for which *unstable* positive balanced growth occurs for given $n > 0$. In such economies orbits diverge from the balanced path whenever initial conditions are not equal to $\bar{k}$. This occurs in particular for the respective cases with Cobb-Douglas production functions with constant elasticity $E_f(\bar{k})$. Moreover, for

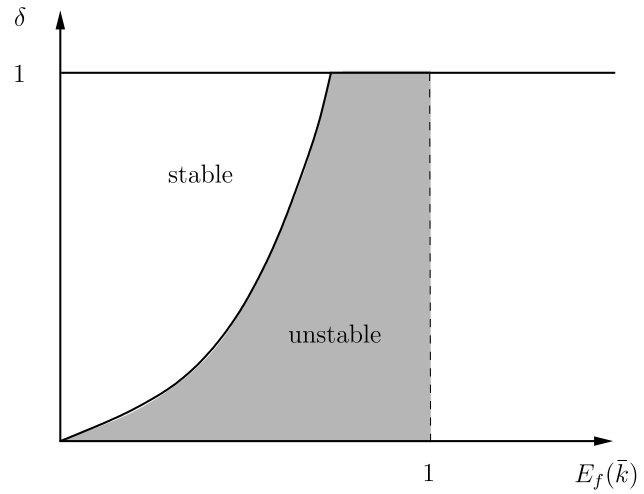


Figure 2: Regions of $(E_f(\bar{k}), \delta) \in [0,1]^2$ with unstable balanced growth; $n > 0$

more general monotonic homogeneous systems $(\mathcal{K}, \mathcal{L})$ multiple balanced paths may occur, all of which may be unstable for large open sets of parameters.

2.2 Economic Growth with an Aging Workforce and Vintage Capital

Consider a workforce with an overlapping generations structure where the productivity of each generation diminishes with age. Assume that total lifetime of each generation is finite, identical, and equal to some length $N > 2$. Let $L = (L_1, L_2, \ldots, L_N)$ denote the typical vector of the

[1]Equivalent conditions are derived for growth models in continuous time by Pampel (2009).

number of workers in an arbitrary period grouped by age, where L_i, $i = 1, \dots, L_N$ denotes the number of workers with remaining lifetime i.

Assume that the evolution of the workforce follows a linear regeneration process of population dynamics defined by a matrix $L' = \boldsymbol{N} L$ such that

$$L' = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1+n_1 & 1+n_2 & 1+n_3 & & \cdots & 1+n_N \end{pmatrix} L \tag{2.7}$$

where $\boldsymbol{n} := (1{+}n_1, 1{+}n_2, \cdots, 1{+}n_n) \geq 0$ are the fertility rates or growth factors from surviving generations, i.e. the contributions of each generation to the next youngest cohort. The matrix $\boldsymbol{N}$ contains an $N-1$ dimensional unit matrix $\boldsymbol{I}$ in the upper right hand corner while the first column is often assumed to consist of zeroes only. A more elaborate model could include differential death rates $0 \leq \boldsymbol{d} := (0, d_2, d_3, \cdots, d_n) \leq 1$ of generations altering the population matrix to

$$L' = \begin{pmatrix} 0 & 1-d_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1-d_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1-d_4 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-d_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-d_N \\ 1+n_1 & 1+n_2 & 1+n_3 & & \cdots & 1+n_N \end{pmatrix} L. \tag{2.8}$$

Assume that capital has a fixed finite life time $M > 2$ and that it is non-malleable once produced in time. Let $K = (K_1, K_2, \cdots, K_M)$ denote the vector of the capital equipment in the economy, where K_i is the number of machines with remaining operating life time i, $i = 1, \dots, M$.

Output across time is homogeneous and produced using a homogeneous production function $F : \mathbb{R}^M_+ \times \mathbb{R}^N_+ \to \mathbb{R}_+$, $(K, L) \mapsto F(K, L)$. Then, the formation of new capital under a Solow type hypothesis implies that

$$K'_M = sF(K, L). \tag{2.9}$$

The development of the vintage composition follows a linear decay process. Let the list of rates of decay of each vintage machine be given as $\boldsymbol{\delta} := (0, \delta_2, \delta_3, \dots, \delta_M)$ with $0 \leq \delta_i \leq 1$, $i = 2, \dots, M$. Then, the one-step mapping for the change of the vintage capital becomes $K' = \boldsymbol{M} K$ where

$$K' = \begin{pmatrix} 0 & 1-\delta_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1-\delta_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1-\delta_4 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-\delta_{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\delta_M \\ 0 & 0 & & & \cdots & k_M(K, L) \end{pmatrix} K. \tag{2.10}$$

Under the Solow hypothesis the entry $k_M(K,L) = sF(K,L)/K_M$ denotes the growth factor of new capital with respect to the previous/latest capital generated from aggregate savings. Thus, one obtains a homogeneous time-one map $(\mathcal{K},\mathcal{L}) : \mathbb{R}^M_+ \times \mathbb{R}^N_+ \to \mathbb{R}^M_+ \times \mathbb{R}^N_+$ of a dynamical system defined by

$$\begin{aligned} K' &= \mathcal{K}(K,L) := \boldsymbol{M}(K,L)K \\ L' &= \mathcal{L}(K,L) := \boldsymbol{N}L \end{aligned} \tag{2.11}$$

which is linear except for the last component in the vintage capital formation. It describes the joint evolution of capital accumulation and the demographic development of the work force of a real (non-monetary) economy under a constant aggregate savings propensity $0 < s < 1$ according to a Solow-type savings assumption. The demographic structure of workers as well as the vintage composition of capital is modeled in a linear parametrized form with arbitrary finite lengths of lifetimes of workers and vintage capital which induces a multidimensional Solow model. For $N = M = 1$ and malleability (additivity) of new and old capital the model reduces to the standard Solow model given by (2.1).

The results for existence, sustainability, and stability from the two-dimensional case can be generalized almost in a one-to-one fashion to the multidimensional model after defining an appropriate intensive form with the positive unit sphere as state space.

(1) A balanced path of the extended Solow model is defined by a triple $(\lambda,\bar{K},\bar{L}) \gg 0$, $|(\bar{K},\bar{L})| = 1$ which solves

$$\lambda \begin{pmatrix} \bar{K} \\ \bar{L} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}(\bar{K},\bar{L}) \\ \boldsymbol{N}\bar{L} \end{pmatrix}. \tag{2.12}$$

Thus, balanced orbits of $(\mathcal{K},\mathcal{L})$ are of the form $\gamma(\alpha(\bar{K},\bar{L})) = \left\{\lambda^t\alpha(\bar{K},\bar{L})\right\}_{t\geq 0}$, $\alpha > 0$, which are all contained in the half line $\{(K,L) \,|\, (K,L) = \alpha(\bar{K},\bar{L}), \alpha > 0\}$.

Since $\boldsymbol{N}$ is a matrix with constant coefficients independent of capital accumulation, $\lambda\bar{L} = \boldsymbol{N}\bar{L}$ must hold. Thus, λ is an eigenvalue and $\bar{L}$ is an eigenvector of the population matrix $\boldsymbol{N}$ representing the stationary distribution of the workforce along balanced orbits. Both are determined endogenously by $(\boldsymbol{d},\boldsymbol{n})$. A priory λ can be greater or less than one and $\bar{L}$ exhibits typically a non-uniform age distribution of workers across generations. The specific structure of the population matrix $\boldsymbol{N}$ of (2.8) implies a simple test for the size of the growth factor λ which turns out to be a leading real eigenvalue of $\boldsymbol{N}$ with multiplicity one[2].

Proposition 2.1. *Let the population matrix* $\boldsymbol{N}$ *be given by* (2.8) *and define* $\bar{p} := \sum_{\ell=1}^{N}\left((1+n_\ell)\cdot\prod_{k=\ell+1}^{N}(1-d_k)\right)$. *Then:*

$$\begin{aligned} &\textit{if} \quad 0 < \bar{p} < 1, \quad \textit{then} \quad \bar{p} < \bar{\lambda} < \bar{p}^{\frac{1}{N}} < 1, \\ &\textit{if} \quad\quad 1 < \bar{p}, \quad\quad \textit{then} \quad 1 < \bar{p}^{\frac{1}{N}} < \bar{\lambda} < \bar{p}. \end{aligned} \tag{2.13}$$

(2) Let $S \subset \mathbb{R}^{M+N}_+$ denote the nonnegative subset of the unit sphere and define the 'intensive form' of this Solow model by the mapping $g : S \to S$, $k \mapsto g(k)$ where $k : (K,L)/|(K,L)|$ and

$$g(k) := \frac{(\mathcal{K}(K,L),\mathcal{L}(K,L))}{|(\mathcal{K}(K,L),\mathcal{L}(K,L))|}. \tag{2.14}$$

[2] I am indebted to T. Pampel for pointing out this result.

By construction $\bar{k} := (\bar{K}, \bar{L}) = g(\bar{k}) \in S$ is a fixed point of g and $(\lambda, \bar{K}, \bar{L})$ defines a balanced path.

Let the production function F be strictly monotonically increasing and strictly concave for all x, y, $y \neq \alpha x$ (off rays), and satisfy a generalized weak Inada condition. Then, there exists a unique interior fixed point of $g(\bar{k}) = g(\bar{K}, \bar{L}) = (\bar{K}, \bar{L}) = \bar{k} \in S$ for all $(s, \boldsymbol{d}, \boldsymbol{n}, \boldsymbol{\delta})$. This condition is satisfied in particular when F is isoelastic, i.e. of the Cobb-Douglas type.

(3) If $\bar{k}$ is asymptotically stable under g, for every $k, k' \in \mathcal{B}(\bar{k}) \subset S$, the basin of attraction of $\bar{k}$, one has $|g^m(k) - g^m(k')| < |k - k'|$ for some $m \geq n$, and $\lim k_n = \lim g^n(k) = \lim g^n(k') = \bar{k}$. This implies a contractivity factor $M(\bar{k})$ as

$$M(\bar{k}) := \lim_{k_n \to \bar{k}} \frac{|g(k_n) - \bar{k}|}{|k_n - \bar{k}|} < 1, \tag{2.15}$$

which depends jointly on the curvature features of the production function and on $(\boldsymbol{d}, \boldsymbol{n}, \boldsymbol{\delta})$, see Lemma A.1.

(4) Lemma A.1 of the appendix states that orbits $\gamma(K_0, L_0)$ in state space converge to the half line $\{(K, L) \in \mathbb{R}_+^{M+N} \,|\, (K, L) = \alpha(\bar{K}, \bar{L}), \alpha > 0\}$ if $\lambda\, M(\bar{k}) < 1$. They diverge if $\lambda\, M(\bar{k}) > 1$. Therefore, as in the two-dimensional case without demographic or vintage structures, the convergence to balanced expansion depends on an interplay between production elasticities embedded in the technology F and the decay or regeneration/renewal parameters of capital and the work force.

There are obvious further applications of these methods to extensions and generalizations within the class of Solow type growth models to investigate the stability of balanced growth:

- all models with endogenous determination of the savings behavior, as in optimal growth, determination by income groups, or with public debt (Diamond, 1965), whose state spaces are at least three-dimensional;
- all models with differential savings behavior of heterogeneous agents, (as suggested by Kaldor, 1957; Pasinetti, 1962; Samuelson & Modigliani, 1966) or models of international trade (Oniki & Uzawa, 1965), whose convergence properties could be compared with those given by Mountford (1998, 1999);
- models with expanded commodity spaces induced by heterogeneous inputs, natural resources, or public goods;
- in Two-Sector Growth Models (Drandakis, 1963; Inada, 1963; Uzawa, 1961, 1963) comparing convergence properties with those given by Galor (1992);
- general multisector growth models of the von Neumann type: von Neumann (1937) (see Solow & Samuelson, 1953; Gale, 1956; Kemeny, Morgenstern & Thompson, 1956; Evstigneev & Schenk-Hoppé, 2008).

2.3 Examples of Monetary Models

All consistent and complete intertemporal macroeconomic models which describe time series of monetary data (satisfying the principles of national income accounting) belong to the class of homogeneous systems. If their sequences are induced by a forward recursive time-one map these will be homogeneous. Therefore, the dynamic features described and discussed by the mathematical results apply and the stability issues between the state space and an appropriate intensive form of the economic model arise. The completed and consistent versions of

- the AS-AD Model with money (Chapters 4.1-4.2 Böhm, 2017)
- the AS-AD Model with money and sovereign debt (Böhm, 2018)

provide two explicit applications of the results of the appendix. They are micro-based completions of the Keynesian IS-LM Model.

3 Examples of Random Homogeneous Dynamical Systems

3.1 The Two-Dimensional Stochastic Solow Growth Model

- The stochastic dynamic version of the Solow model in intensity form arises when one or several of the parameters (n, δ, A, s) are subjected to an exogenous random perturbation (see Schenk-Hoppé & Schmalfuß, 2001).
- Theorem B.2 of the appendix provides conditions to examine the stability of random balanced paths in state space for the intensive form used by Schenk-Hoppé & Schmalfuß (2001) (see also Böhm, Pampel & Wenzelburger, 2005; Pampel, 2009).
- A multidimensional stochastic version of the extended Solow model with an aging workforce and vintage capital of Section 2.2 arises when the two matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ have random coefficients $(\boldsymbol{d}, \boldsymbol{n}, \boldsymbol{\delta})$.
- Böhm & Hillebrand (2007) provides an application of an intensive form model examining the efficiency of Pay-As-You-Go pension systems in a stochastic economy with multiperiod overlapping generations of consumers where compulsory public retirement savings coexists with private savings and assets.

3.2 The Two-Dimensional Stochastic AS-AD Model with Money

The introduction of a multiplicative random production shock in the deterministic model with perfect foresight (see Chapter 4.1-4.2, Böhm, 2017) turns the parametrized deterministic system into a two-dimensional homogeneous random dynamical system (as described in Chapter 4.3 – 4.5, Böhm, 2017) under rational expectations[3].

Let the random effect be given by a bounded positive multiplicative (Hicks neutral) production shock defined by a random variable $Z : \Omega \to [Z_{\min}, Z_{\max}]$, $0 < Z_{\min} < Z_{\max} < \infty$ for a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with time-shift $\vartheta : \Omega \to \Omega$, see Appendix B. This implies two homogeneous stochastic difference equations under rational expectations given by

$$\begin{aligned} M_{t+1} &= M_t \left(\frac{\tilde{c} - \tau^\star}{\tilde{c}} \right) \big(1 + g\mathcal{P}(1, \psi^*(1, p^e_t/M_t), Z(\vartheta^t\omega))\big) \\ p^e_{t+1} &= M_t\, \psi^*(1, p^e_t/M_t). \end{aligned} \tag{3.1}$$

Here, $\mathcal{P}$ is the random price law which is homogeneous of degree one for each level of Z. The variable $p^e_t \equiv p^e_{t-1,t}$ denotes the prediction of the mean future price in period t made in $t-1$, and ψ^* is the predictor (a forecasting rule) making the mean prediction unbiased along the orbit.

[3]This stochastic AS-AD model with money is one version of a closed demand consistent temporary equilibrium model with a stochastic aggregate supply function of the Lucas-type.

This leads to the one-dimensional random difference equation in intensive form in $q^e := p^e/M$

$$q^e_{t+1} = S(Z_t, q^e_t) := \left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right) \frac{\psi^*(1, q^e_t)}{1 + g\mathcal{P}(1, \psi^*(1, q^e_t), Z_t)}. \tag{3.2}$$

Stationary solutions (often wrongly referred to as *stochastic steady states*) of the intensive form model are given by so-called *random fixed points* which are the stochastic analogue of the concept of a deterministic fixed point (steady states) of stochastic difference equations, see Definition B.2.

Definition 3.1. *A random fixed point of the intensive form model* (3.2) *is a random variable* $q^* : \Omega \to \mathbb{R}_+$ *solving*

$$q^\star(\vartheta\omega) = S(Z(\omega), q^\star(\omega)) := \left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right) \frac{\psi^*(1, q^\star(\omega))}{1 + g\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}, \quad \mathbb{P}\text{-}a.s. \tag{3.3}$$

$$= \psi^*(1, q^\star(\omega)) \frac{\left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right)}{1 + g\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}, \tag{3.4}$$

where $\vartheta : \Omega \to \Omega$ *is the time-shift operator generating the noise process (see Appendix B).*

As in stochastic growth models random balanced expansion of the two-dimensional AS-AD-system (3.1) with rational expectations are strongly related to the stationary solution generated by the random fixed points of its intensive form (3.2) in the following way. For any $\omega \in \Omega$ and $(M_0, p^e_0) \gg 0$, the two dimensional random difference equation system (3.1) generates an orbit of nominal money balances and expectations $\{(M_t, p^e_t)\}_{t=0}^\infty$ where $M_t = M(t, \omega, (M_0, p^e_0))$ and $p^e_t = p^e(t, \omega, (M_0, p^e_0))$ are defined as successive iterates of the stochastic difference equations *respecting* the time-shift of the noise process (see (B.1) - (B.3) in Appendix B).

Definition 3.2 (Balanced random orbits)**.** *Given* $\omega \in \Omega$*, an orbit*

$$\{(M_t, p^e_t)\} = \{(M(t, \omega, (M_0, p^e_0)), p^e(t, \omega, (M_0, p^e_0))\}$$

of (3.1)*, is called* ***balanced****, if there exists a random fixed point* $q^\star : \Omega \to \mathbb{R}_+$ *of* (3.2)*, i.e. a random variable solving*

$$q^\star(\vartheta\omega) = S(Z(\omega), q^\star(\omega)) := \left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right) \frac{\psi^*(1, q^\star(\omega))}{1 + g\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}, \quad \mathbb{P}\text{-}a.s. \tag{3.5}$$

satisfying for all $t \geq 0$ *:*

$$\frac{p^e_t}{M_t} = q^e_t = q^\star(\vartheta^t\omega) \tag{3.6}$$

$$M_{t+1} = M_t \left(\frac{\tilde{c} - \tau^\star}{\tilde{c}}\right) \left(1 + g\mathcal{P}(1, \psi^*(1, q^\star(\vartheta^t\omega)), Z(\vartheta^t\omega))\right) \tag{3.7}$$

$$p^e_{t+1} = M_t\, \psi^*(1, q^\star(\vartheta^t\omega)) = p^e_t \frac{\psi^*(1, q^\star(\vartheta^t\omega))}{q^\star(\vartheta^t\omega)}. \tag{3.8}$$

Equations (3.7) and (3.8) show that along random balanced orbits money holdings and rational price predictions are generated by two linear random difference equations. Their growth rates $M_{t+1}/M_t \equiv \mu : \Omega \to \mathbb{R}_+$ and $p^e_{t+1}/p^e_t \equiv \pi : \Omega \to \mathbb{R}_+$ are correlated random variables induced by the random fixed point (3.5) satisfying

$$\pi(\omega)q^*(\omega) = \psi^*(1, q^\star(\omega)) = \frac{1 + g\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}{\left(\frac{\tilde{c}}{\tilde{c} - \tau^\star}\right)} q^*(\vartheta\omega) = \mu(\omega)q^*(\vartheta\omega), \ \mathbb{P}\text{-a.s.}. \tag{3.9}$$

Define the distance of an orbit of (3.1) to the balanced path associated with $q^\star$ as

$$\Delta_t = \Delta(t,\omega,(M_0,p_0^e)) := p^e(t,\omega,(M_0,p_0^e)) - q^\star(\vartheta^t\omega)\cdot M(t,\omega,(M_0,p_0^e)). \tag{3.10}$$

Definition 3.3. *A balanced orbit is called* ***asymptotically stable*** *if, for all (M_0,p_0^e) in a neighborhood $\mathcal{U}(\bar M_0,\bar p_0^e,\omega)$, $\bar p_0^e = \bar M_0\, q^*(\omega)$,*

$$\lim_{t\to\infty} |q^e(t,\omega,p_0^e/M_0) - q^\star(\vartheta^t\omega)| = 0 \quad \textit{and} \quad \lim_{t\to\infty} |\Delta(t,\omega,(M_0,p_0^e))| = 0, \quad \mathbb{P}\text{-}a.s.. \tag{3.11}$$

The first condition imposes that the random fixed point in intensive form is asymptotically stable while the second one requires that the distance converges pointwise to zero (see Appendix B). Figure 3 describes the convergence issue in the state space of stochastic paths. The two rays $q^\star(\omega)$ and $q^\star(\vartheta\omega)$ describe the one-step movement of the stochastic fixed point moving within the cone of the two blue dotted lines. The one-step move of the orbit in intensive form converging to the random fixed point is indicated by the pair q_t^e and q_{t+1}^e. For (M_t,p_t^e) with distance $\Delta_t \equiv \Delta(t,\omega)$ the two possible cases of convergence Δ_{t+1} or of divergence with $\Delta_{t+1} > \Delta_t > \Delta_{t+1}$ are indicated for the same $\omega\in\Omega$. The diagram visualizes the possibility of convergence or divergence depending on the rate of monetary expansion which is shown to be larger for Δ_{t+1} than for Δ_{t+1} in the one-step description for an arbitrary point (M_t,p_t^e) along the orbit. Theorem B.2 states that convergence (divergence) occurs when the expansionary

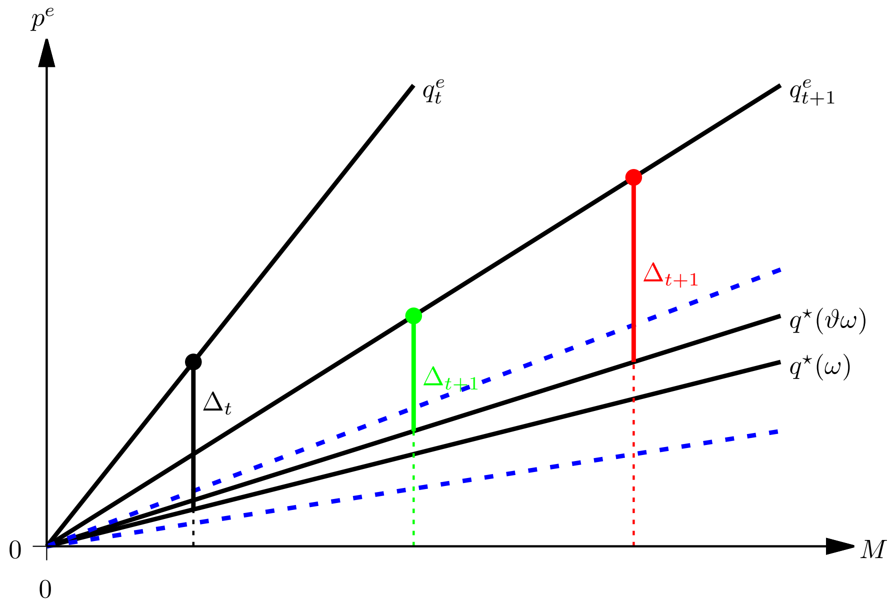


Figure 3: Convergence/divergence to balanced path in (M,p^e)-space

forces are appropriately dominated (or not dominated) *on average* by the rate of contraction of the random fixed point, i.e. keeping the random intensity sufficiently bounded relative to the rate of expansion along the random fixed point.

Generically, the benchmark case of the parametric deterministic model exhibits two balanced paths, one stable and one unstable for each level of the production shock Z (Chapter 4.1-4.2, Böhm, 2017) if government demand is positive and not too large. For the stochastic case this implies that there exist two random fixed points $q_2^\star : \Omega \to I_2$ and $q_3^\star : \Omega \to I_3$ in intensity form, one unstable and one asymptotically stable, with values in disjoint intervals $I_2 \cap I_3 = \emptyset$, $I_2 \subset \mathbb{R}_+$, $I_3 \subset \mathbb{R}_+$.

As in the deterministic situation convergence of the orbits of the stochastic intensive system (3.2) to $q_3^\star : \Omega \to I_3$ is only a necessary condition for convergence to a balanced random path in the space of money balances and expectations, so that asymptotic convergence to balanced random orbits requires additional conditions. These are provided in the following theorem.

Theorem 3.1.
Let S be differentiable and increasing with respect to q^e and let $q^\star$ be an asymptotically stable random fixed point of

$$q^e_{t+1} = S(\vartheta^t\omega, q^e_t) := \left(\frac{\tilde{c}}{\tilde{c}-\tau^\star}\right) \frac{\psi^*(q^e_t)}{1+g\mathcal{P}(1,\psi^*(q^e_t), Z(\vartheta^t\omega))}.$$

Then, for almost all $\omega \in \Omega$ and any $q^e_0 \in I_3$, $q^e_0 \neq q^\star(\omega)$ with $\lim_{t\to\infty} |q^e(t,\omega,q^e_0) - q^\star(\vartheta^t\omega)| = 0$ the distance $\Delta_t := p^e(t,\omega,(M_0,p^e_0)) - q^\star(\vartheta^t\omega)\, M(t,\omega,(M_0,p^e_0))$ satisfies $\mathbb{P}$-a.s.:

$\lim_{t\to\infty} |\Delta_t| = 0$ *if*

$$\mathbb{E}\log(S'(\omega,q^\star(\omega)) + \mathbb{E}\log\left(\frac{\tilde{c}-\tau^\star}{\tilde{c}}\right)(1+g\mathcal{P}(1,\psi^*(q^\star(\omega)), Z(\omega))) < 0 \tag{3.12}$$

$\lim_{t\to\infty} |\Delta_t| = \infty$ *if*

$$\mathbb{E}\log(S'(\omega,q^\star(\omega)) + \mathbb{E}\log\left(\frac{\tilde{c}-\tau^\star}{\tilde{c}}\right)(1+g\mathcal{P}(1,\psi^*(q^\star(\omega)), Z(\omega))) > 0. \tag{3.13}$$

The distance function Δ defines a second random difference equation $\Delta : \Omega \times I_3 \times \mathbb{R} \to \mathbb{R}$, making the pair (S,Δ) a two dimensional random dynamical system $(S,\Delta) : \Omega \times I_3 \times \mathbb{R} \to I_3 \times \mathbb{R}$ with fixed point $(q^\star, 0) : \Omega \to I_3 \times \mathbb{R}$.

Proof. From the definition $\Delta_t = M_t(q^e_t - q^\star(\vartheta^t\omega))$ and (3.2) one has

$$\begin{aligned}
\Delta_{t+1} &= M_{t+1}(q^e_{t+1} - q^\star(\vartheta^{t+1}\omega)) = M_{t+1}(S(\vartheta^t\omega, q^e_t) - S(\vartheta^t\omega, q^\star(\vartheta^t\omega))) \\
&= \frac{M_{t+1}}{M_t}\frac{S(\vartheta^t\omega, q^e_t) - S(\vartheta^t\omega, q^\star(\vartheta^t\omega))}{q^e_t - q^\star(\vartheta^t\omega)}\Delta_t \\
&= \frac{\tilde{c}-\tau^*}{\tilde{c}}\left(1+g\mathcal{P}(1,\psi^*(q^e_t), Z(\vartheta^t\omega))\right)\frac{S(\vartheta^t\omega, q^e_t) - S(\vartheta^t\omega, q^\star(\vartheta^t\omega))}{q^e_t - q^\star(\vartheta^t\omega)}\Delta_t
\end{aligned}$$

implying

$$\frac{\Delta_{t+1}}{\Delta_t} = \frac{\tilde{c}-\tau^*}{\tilde{c}}\left(1+g\mathcal{P}(1,\psi^*(q^e_t), Z(\vartheta^t\omega))\right)\frac{S(\vartheta^t\omega, q^e_t) - S(\vartheta^t\omega, q^\star(\vartheta^t\omega))}{q^e_t - q^\star(\vartheta^t\omega)}. \tag{3.14}$$

Since $\lim_{t\to\infty} |q^e_t - q^\star(\vartheta^t\omega)| = 0, \mathbb{P}$-a.s., there exists an $\varepsilon > 0$ sufficiently small and $t_0 = t_0(\varepsilon,\omega) > 0$ sufficiently large such that

$$\left|\mathcal{P}(1,\psi^*(q^e_t), Z(\vartheta^t\omega)) - \mathcal{P}(1,\psi^*(q^\star(\vartheta^t\omega), Z(\vartheta^t\omega))\right| < \varepsilon \tag{3.15}$$

$$\left|\frac{S(\vartheta^t\omega, q^e_t) - S(\vartheta^t\omega, q^\star(\vartheta^t\omega))}{q^e_t - q^\star(\vartheta^t\omega)} - S'(\vartheta^t\omega, q^\star(\vartheta^t\omega))\right| < \varepsilon. \tag{3.16}$$

By induction we have $\underline{\Delta}_t \leq |\Delta_t| \leq \overline{\Delta}_t$ for all $t \geq t_0$, for the two linear random dynamical systems

$$\overline{\Delta}_{t+1} = \left(\frac{\tilde{c}-\tau^*}{\tilde{c}}\right)\left[\left(1+g\mathcal{P}(1,\psi^*(q^\star(\vartheta^t\omega)), Z(\vartheta^t\omega))\right) S'(\vartheta^t\omega, q^\star(\vartheta^t\omega)) + \varepsilon\right]\overline{\Delta}_t \tag{3.17}$$

$$\underline{\Delta}_{t+1} = \left(\frac{\tilde{c}-\tau^*}{\tilde{c}}\right)\left[\left(1+g\mathcal{P}(1,\psi^*(q^\star(\vartheta^t\omega)), Z(\vartheta^t\omega))\right) S'(\vartheta^t\omega, q^\star(\vartheta^t\omega) - \varepsilon)\right]\underline{\Delta}_t \tag{3.18}$$

with $\overline{\Delta}_{t_0} = |\Delta_{t_0}|$ and $\underline{\Delta}_{t_0} = |\Delta_{t_0}|$.

Therefore, assumption (3.12) implies that the upper bound (3.17) converges to zero $\mathbb{P}$-a.s. while the condition (3.13) implies that the lower bound grows to infinity eventually. Thus, in this case the distance of an orbit to the balanced path diverges under assumption (3.13). □

To display properties of time series in the stable case, consider the numerical exercise for an isoelastic version of the economy (for details see Chapter 4 of Böhm, 2017) with a two point production shock $Z \sim \{Z_{\min}, Z_{\max}\}$ with equal probability. This implies an isoelastic random

Table 1: Standard parametrization a

$Z_{\min}$	$Z_{\max}$	B	C	c	τ	g	g^*	g^{**}
1.0	1.01	0.6	0.6	0.5	0.7	0.8240	0.8285	0.8328

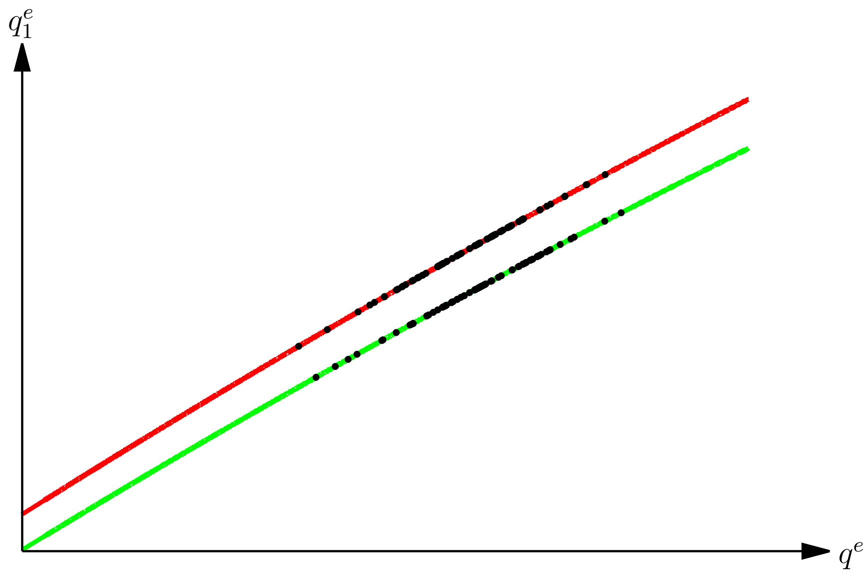
aggregate supply function and a deterministic aggregate demand function with a constant multiplier. For the first set of numerical experiments the values of the respective parameters in consumption, production, and for the government are chosen as in Table 1.

Figure 4 displays the convergence features when the balanced orbit associated with $q_3^\star$ is asymptotically stable, a situation which occurs for the parameters given in Table 2. The main differ-

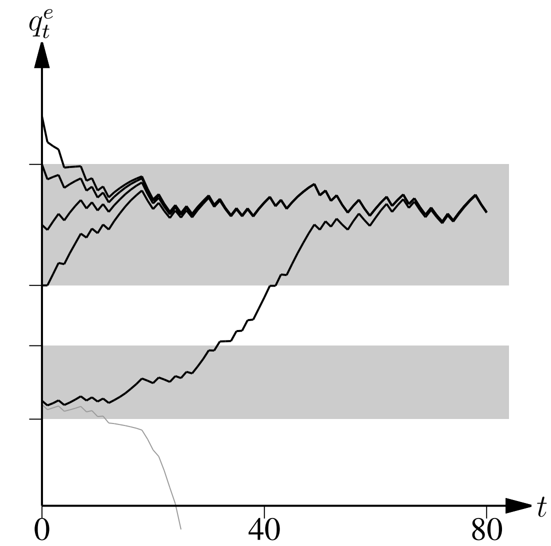
Table 2: Standard parametrization b

$Z_{\min}$	$Z_{\max}$	B	C	δ	c_s	τ_w	τ_π	g	g^*	g^{**}
1.0	1.01	0.6	0.6	1.0	0.5	0.75	0.75	0.8392		0.8449

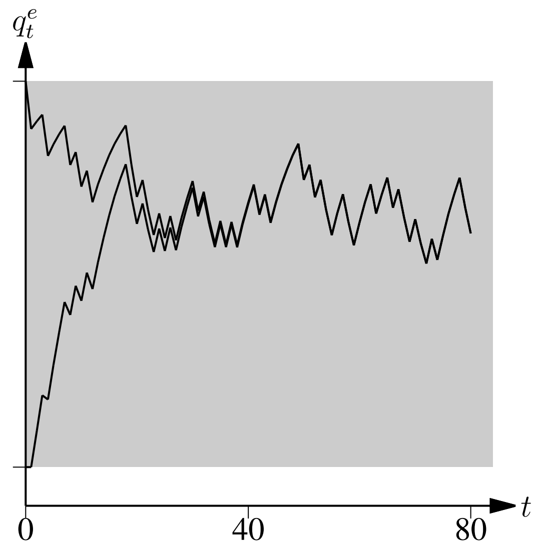
ences to the values in Table 1 consist in a slightly lower government demand and in higher tax rates. This causes lower deficits and thus lower rates of inflation at any one time. For these values – with a small production shock – the time one map S becomes almost linear on I_3, subfigure **a**. Panel **b** and **c** show the evolution for six different initial conditions in the space of real expectations (five converging and one diverging) while **d** and **e** display the convergence in (q^e, Δ)-space for the same ω. Notice the difference in scale between the subfigures **b** and **c**.



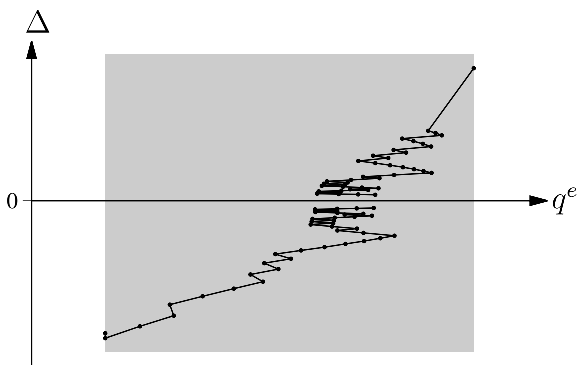
(a) S becomes almost linear on I_3 for values in Table 2



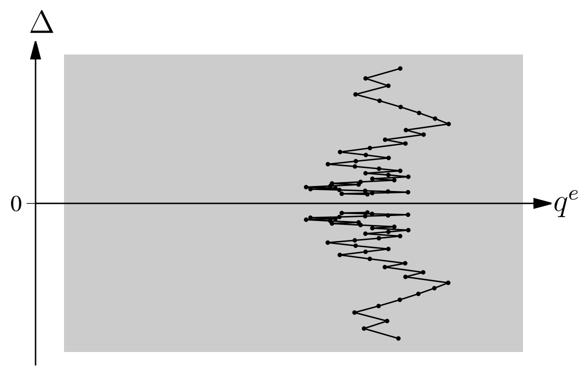
(b) Convergence from I_2 and I_3 to $q^\star \in I_3$



(c) Convergence in I_3, $t \in [0, 60]$



(d) Convergence in I_3: $t \in [0, 30]$



(e) Symmetry of orbits for $t > 30$

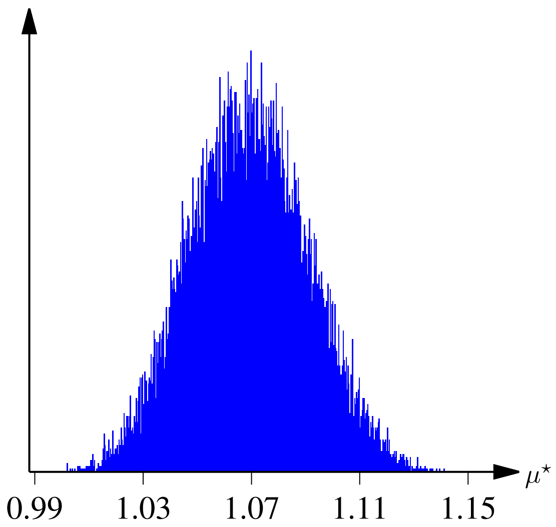
Figure 4: Convergence in (q^e, Δ)-space

3.2.1 Convergence and Growth Rates of Monetary Expansion

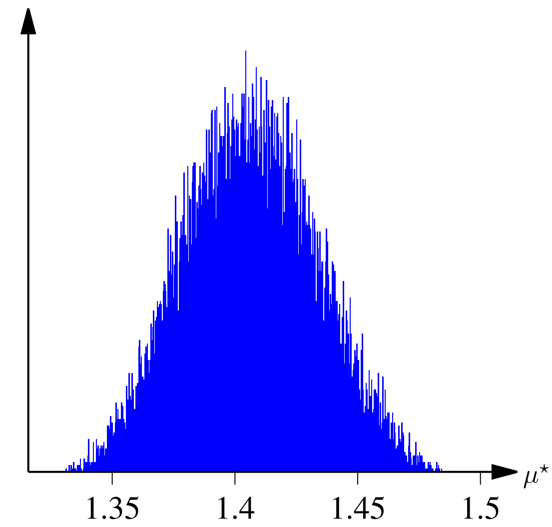
In order to understand the conditions (3.12) and (3.13) for stability/instability it is useful to consider the central equation (3.14)

$$\Delta_{t+1} = \frac{\tilde{c} - \tau^*}{\tilde{c}} \left(1 + g\mathcal{P}(1, \psi^*(q_t^e), Z(\vartheta^t\omega))\right) \cdot \frac{S(\vartheta^t\omega, q_t^e) - S(\vartheta^t\omega, q^\star(\vartheta^t\omega))}{q_t^e - q^\star(\vartheta^t\omega)} \cdot \Delta_t$$

again which defines a linear random dynamical system in Δ whose coefficient consists of a product of two random variables. Since $\lim_{t\to\infty} |q_t^e(t, \omega, q_0^e) - q^\star(\vartheta^t\omega)| = 0, \mathbb{P} - a.s.$ the second term converges to the derivative of S while the first converges to the growth rate of money $\mu_t := M_t/M_{t-1}$ along $q_3^\star$ (see (3.1)). Thus, convergence of the distance Δ to zero occurs if the



(a) Stable balanced growth: $\mathbb{E}\mu^\star = 1.0690$



(b) Unstable balanced growth: $\mathbb{E}\mu^\star = 1.4063$

Figure 5: Stationary growth rates of money: $T = 2 \cdot 10^4$

growth factor $\mu^\star(\vartheta^t\omega) \cdot S'(q_3^\star(\vartheta^t\omega))$ of the linear system is mean contracting (see Arnold & Crauel, 1992), i.e. if and only if $\mathbb{E}\left(\mu^\star(\vartheta^t\omega) \cdot S'(q_3^\star(\vartheta^t\omega))\right) < 1$, which corresponds to condition (3.12). Since $S'(q_3^\star(\vartheta^t\omega)) < 1, \mathbb{P} - a.s.$, given the assumption for S on I_3, the stability requirement stipulates that the rate of monetary expansion may well be larger than one along the whole orbit of $q_3^\star$, but it should make the product with S' less than one on average. Figure 5 displays the histograms of the rates of monetary expansion in the two cases with their respective means. While money grows at a rate of about 7 percent in the stable case, subfigure **a**, it is about 40 percent in **b** indicating clearly the reason for the instability of the balanced orbit in the case of the parameters of Table 1.

A Balanced Expansion of Homogeneous Systems

Let $(F_i)_{i=1}^n$ denote a list of continuous mappings $F_i : \mathbb{R}_+^n \to \mathbb{R}_+$ and define $(F_i)_{i=1}^n \equiv F : \mathbb{R}_+^n \to \mathbb{R}_+^n$, $x \mapsto F(x)$.

F is *homogeneous* if all functions $F_i, i = 1, \ldots, n$ are homogeneous of degree one, i.e. if for all $\lambda \geq 0$ and all $x \in \mathbb{R}_+^n$: $F(\lambda x) = \lambda F(x)$.

F is *concave* if all functions $F_i, i = 1, \dots, n$, are concave.

F is *monotonically increasing* if all functions $F_i, i = 1, \dots, n$, are monotonically increasing with respect to $\gneqq$, or $\gg$, i.e. if F preserves the respective order.

Define the mapping in so-called *intensive form* associated with the homogeneous map F from (the positive part of) the unit sphere S into itself by $f : S \to S$, $y \mapsto f(y) := F(y)/|F(y)|$.

Definition A.1. *An orbit $\gamma(x) = \{F^t(x)\}_{t \geq 0}$ is called balanced if for all $t \geq 0$: $F^t(x) = \lambda^t x$, for some $\lambda > 0$. Let $(\lambda, \bar{x})$ denote a solution of*

$$\lambda\bar{x} = F(\bar{x}) \qquad \textit{with} \qquad |\bar{x}| = 1 \tag{A.1}$$

λ is called a growth factor (or an eigenvalue) of F. Then, $\gamma(\alpha\bar{x}) = \{F^t(\alpha\bar{x}\}_{t=0}^{\infty} = \{\lambda^t(\alpha\bar{x})\}_{t=0}^{\infty}$ is a balanced orbit for all $\alpha > 0$.

For differentiable homogeneous functions F the identity $DF(x)x = F(x)$ implies that $\lambda\bar{x} = DF(\bar{x})\bar{x}$ which justifies to say that λ is an eigenvalue and $\bar{x}$ is an eigenvector for F.

The set $\left\{x^n \in \mathbb{R}^n_+ \,|\, x^n = \lambda^n\alpha\bar{x}, n = 0, 1, \dots\right\}$ is also referred to as a balanced path.
The union of all balanced paths $\cup_{\alpha\geq 0}\left\{x^n \in \mathbb{R}^n_+ \,|\, x^n = \lambda^n\alpha\bar{x}\right\}$ is a subset of the ray or halfline $L(\bar{x}) := \left\{x \in \mathbb{R}^n_+ \,|\, x = \alpha\bar{x}, \alpha \geq 0\right\}$ associated with $\bar{x}$, referred to as the balanced ray or halfline.

Define the distance of $x \in \mathbb{R}^n_+$ from $L(\bar{x})$ as

$$\Delta := d(x, L(\bar{x})) = \min_{\alpha\geq 0} |x - \alpha\bar{x}| = |x - \langle x, \bar{x}\rangle\bar{x}| \tag{A.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product.

Definition A.2. *An orbit $\gamma(x)$ is said to converge to a balanced path (to $L(\bar{x})$) if*

$$\Delta_t := d(F^t(x), L(\bar{x})) = |F^t(x) - \langle F^t(x), \bar{x}\rangle\bar{x}| \tag{A.3}$$

converges to zero for $t \to \infty$.

A.1 A Contraction Lemma for Deterministic Systems

Lemma A.1.
Consider a continuous and homogeneous time-one map $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and its associated mapping in intensive form $f : S \to S$ where $S := \left\{x \in \mathbb{R}^n_+ \,|\, |x| = 1\right\}$.

Let $(\lambda, \bar{x})$ denote a Perron-Frobenius solution for F, i.e. $\lambda\bar{x} = F(\bar{x})$ with $|\bar{x}| = 1$ and $\lambda > 0$. Assume that $\bar{x} \in S$ is an asymptotically stable fixed point of f with contractivity $0 < M < 1$, i.e. $|f^m(y) - f^m(x)| \leq M|y - x|$ for $m \geq m_0$.

Let $\gamma(x_0)$ be an orbit of F and define $y_0 := x_0/|x_0| \neq \bar{x}$, and $\Delta_0 := |x_0 - \langle x_0, \bar{x}\rangle\bar{x}| \neq 0$. Let $0 < \bar{x} \in S$ be an asymptotically stable fixed point of f and $y_0 \in \mathcal{B}(\bar{x}) \subset S$, its basin of attraction. Then, for all $x_0/|x_0| \in \mathcal{B}(\bar{x})$:

$$\textit{If} \qquad \lambda\, M > 1, \qquad \textit{then} \qquad \lim_{t\to\infty} |\Delta_t| = \infty. \tag{A.4}$$

$$\textit{If} \qquad \lambda\, M < 1, \qquad \textit{then} \qquad \lim_{t\to\infty} |\Delta_t| = 0. \tag{A.5}$$

Proof. Since

$$
\begin{aligned}
\Delta_1 &= |F(x) - \langle F(x), \bar{x}\rangle \bar{x}| \\
&= \frac{|F(x)|}{|x|} \cdot \frac{|F(x)/|F(x)| - \langle F(x)/|F(x)|, \bar{x}\rangle \bar{x}|}{|x/|x| - \langle x/|x|, \bar{x}\rangle \bar{x}|} \Delta \\
&= |F(x/|x|)| \cdot \frac{|F(x)/|F(x)| - \langle F(x)/|F(x)|, \bar{x}\rangle \bar{x}|}{|x/|x| - \langle x/|x|, \bar{x}\rangle \bar{x}|} \Delta \\
&= |F(y)| \cdot \frac{|f(y) - \langle f(y), \bar{x}\rangle \bar{x}|}{|y - \langle y, \bar{x}\rangle \bar{x}|} \Delta =: \mathcal{D}(y, \Delta), \qquad y \in S
\end{aligned}
\tag{A.6}
$$

the last equation defines a time-one map $\mathcal{D}$ for Δ as a function of (y, Δ). In other words, the mapping F induces a time-one map $(f, \mathcal{D}) : S \times \mathbb{R}_+ \to S \times \mathbb{R}_+$ of an auxiliary system whose fixed points are $(\bar{x}, 0)$. Thus, asymptotic convergence of $\{y_t, \Delta_t\}$ to $(\bar{x}, 0)$ holds if and only if the orbit $\gamma(x_0)$ converges to $L(\bar{x})$ in state space of the original dynamical system. Therefore,

$$
\lim |F(y_n)| = |F(\bar{x})| = |\lambda \bar{x}| = \lambda > 0 \tag{A.7}
$$

and

$$
\begin{aligned}
\lim \frac{|f(y_n) - \langle f(y_n), \bar{x}\rangle \bar{x}|}{|y_n - \langle y_n, \bar{x}\rangle \bar{x}|} &= \lim \frac{|f(y_n) - \langle \bar{x}, \bar{x}\rangle \bar{x}|}{|y_n - \langle \bar{x}, \bar{x}\rangle \bar{x}|} = \lim \frac{|f(y_n) - |\bar{x}|^2 \bar{x}|}{|y_n - |\bar{x}|^2 \bar{x}|} \\
&= \lim \frac{|f(y_n) - \bar{x}|}{|y_n - \bar{x}|} = M
\end{aligned}
\tag{A.8}
$$

imply

$$
\lim_{n \to \infty} \frac{\Delta_{n+1}}{\Delta_n} = \lambda\, M. \tag{A.9}
$$

This means $|\frac{\Delta_{t+1}}{\Delta_t} - \lambda\, M| < \epsilon$ for t larger than some t_0. Thus,

$$
[\lambda\, M - \epsilon] |\Delta_t| < |\Delta_{t+1}| < [\lambda\, M + \epsilon] |\Delta_t|, \quad t \geq t_0,
$$

and by induction

$$
[\lambda\, M - \epsilon]^\tau |\Delta_{\tau + t_0}| < |\Delta_{t + t_0}| < [\lambda\, M + \epsilon]^\tau |\Delta_{t_0}|, \quad \tau > 0.
$$

Therefore, for ϵ sufficiently small,

$$
\lambda\, M < 1 \quad \Rightarrow \quad \lambda\, M + \epsilon < 1
$$

so that $\lim_{t \to \infty} \Delta_t = 0$. Conversely,

$$
\lambda\, M > 1 \quad \Rightarrow \quad \lambda\, M - \epsilon > 1
$$

so that $\lim_{t \to \infty} |\Delta_t| = \infty$ (see also Theorem A.2.1 in Böhm, 2017). □

The two-dimensional cases treated in Böhm, Pampel & Wenzelburger (2005); Pampel (2009); Böhm (2017) use a different distance function than the one in the lemma.

B Balanced Expansion of Random Homogeneous Systems

Following Evstigneev & Pirogov (2007, 2010), let $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ denote an ergodic dynamical system with probability space $(\Omega, \mathcal{F}, \mathbb{P})$, its automorphism $\vartheta : \Omega \to \Omega$, (i.e. a one-to-one mapping such that ϑ and ϑ^{-1} are measurable and preserve the measure $\mathbb{P}$), and $F(\omega) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ an associated random family of continuous, homogeneous time-one maps which are $\mathcal{F}$-measurable in ω. $F(\omega)$ induces the random difference equation

$$x_t = F(\vartheta^{t-1}\omega)x_{t-1} \qquad \text{for all}\, t. \tag{B.1}$$

Random orbits $\gamma(\omega, x_0) := \{C(t,\omega)x_0\}_t^{\infty}$ of $F(\omega)$ are generated by the mapping

$$x_t = C(t,\omega)x_0 := \begin{cases} F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\omega)x_0 & t > 0 \\ id_X & t = 0 \end{cases} \tag{B.2}$$

which satisfies

$$C(t+s,\omega) = C(t, \vartheta^s\omega) \circ C(s,\omega) \qquad \text{for all}\ t, s. \tag{B.3}$$

The mapping $C(t,\omega)$ is a cocyle over the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ (see Arnold, 1998). In the following, the notational convention $C(t,\omega)x$ and $F(\omega)x$ will be used for the result of the application of the corresponding function to the point x (as in Evstigneev & Pirogov, 2010).

Definition B.1. *If $F(\omega) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is homogeneous of degree one, its associated* mapping in intensive form *$f(\omega) \equiv F(\omega)/|F(\omega)| : S \to S$, $y \mapsto f(\omega)y$ is defined as*

$$f(\omega)y \equiv (F(\omega)/|F(\omega)|)y := \frac{1}{|F(\omega)y|} \cdot F(\omega)y = F(\omega)\frac{y}{|F(\omega)y|} \tag{B.4}$$

where $S := \{y \in \mathbb{R}^n_+ \mid |y| = 1\}$.

Definition B.2. *A random fixed point of $f(\omega)$ is a random variable $\xi : \Omega \to S$ such that*

$$\xi(\vartheta\omega) = f(\omega)\xi(\omega) \quad \mathbb{P}\text{-}a.s. \tag{B.5}$$

Definition B.3. *Let $F(\omega)$ be homogeneous and $\xi : \Omega \to S$ be a random fixed point of $f(\omega)$. An orbit $\gamma(\omega, \xi(\omega)) = \{(C(t,\omega)\xi(\omega))\}$ of F is called balanced if there exists a random variable $\lambda : \Omega \to \mathbb{R}_{++}$ such that*

$$C(t,\omega)\xi(\omega) = F(\vartheta^{t-1}\omega) \circ \cdots \circ\ F(\omega)\xi(\omega) = \left(\prod_{\tau=1}^{t} \lambda(\vartheta^{\tau-1}\omega)\right) \cdot c(t,\omega)\xi(\omega) \tag{B.6}$$

where $c(t,\omega) := f(\vartheta^{t-1}\omega) \circ \cdots \circ f(\omega)$ is the cocycle associated with the mapping $f(\omega)$, see (B.2)*.*

Definition B.4 (Perron-Frobenius)**.** *A pair of random variables $(\lambda(\omega), \xi(\omega))$, $\lambda : \Omega \to \mathbb{R}_+$, $\xi : \Omega \to \mathbb{R}^N_+$ is called a Perron-Frobenius solution of $F(\omega)$ if $\mathbb{P}$-a.s. :*

$$\lambda(\omega)\xi(\vartheta\omega) = F(\omega)\xi(\omega) \quad \textit{with} \quad |\xi(\omega)| = 1 \quad \textit{and} \quad \lambda(\omega) > 0 \tag{B.7}$$

(Evstigneev & Pirogov, 2010).

Theorem B.1 (Evstigneev & Pirogov (2010)).
Let $F(\omega)$ be homogeneous and strictly monotone[4]*. There exists a unique Perron-Frobenius solution* $(\xi, \lambda) : \Omega \to \mathbb{R}^n_+ \times \mathbb{R}_+$

$$\lambda(\omega)\xi(\vartheta\omega) = F(\omega)\xi(\omega) \quad \textit{with} \quad |\xi(\omega)| = 1 \quad \textit{and} \quad \lambda(\omega) > 0 \tag{B.8}$$

Lemma B.1.
Every Perron-Frobenius solution (λ, ξ) induces a balanced orbit $\{C(t,\omega)\xi(\omega)\}_0^\infty$ of $F(\omega)$.

Proof. The homogeneity of F implies that $F(\omega)\alpha x = \alpha \cdot F(\omega)x$ for all $\alpha > 0$ and $x \gneqq 0$. Therefore,

$$\begin{aligned}
C(t,\omega)\xi(\omega) &= F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\omega)\xi(\omega) \\
&= F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta\omega) \circ \lambda(\omega) \cdot f(\omega)\xi(\omega) \\
&= \lambda(\omega) \cdot F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta\omega) \circ f(\omega)\xi(\omega) \\
&= \lambda(\omega) \cdot F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta\omega) \circ f(\omega)\xi(\omega) \\
&= \lambda(\omega) \cdot F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta\omega)\xi(\vartheta\omega) \\
&= \lambda(\omega) \cdot F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta^2\omega) \circ F(\vartheta\omega)\xi(\vartheta\omega) \\
&= \lambda(\omega) \cdot F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta^2\omega) \circ \lambda(\vartheta\omega) \cdot f(\vartheta\omega)\xi(\vartheta\omega) \\
&= \lambda(\omega) \cdot \lambda(\vartheta\omega) \cdot F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta^2\omega) \circ f(\vartheta\omega)\xi(\vartheta\omega) \\
&= \lambda(\omega) \cdot \lambda(\vartheta\omega) \cdot F(\vartheta^{t-1}\omega) \circ \cdots \circ F(\vartheta^2\omega) \circ f(\vartheta\omega) \circ f(\omega)\xi(\omega)
\end{aligned} \tag{B.9}$$

and by induction

$$\begin{aligned}
&= \lambda(\omega) \cdot \lambda(\vartheta\omega) \cdots \lambda(\vartheta^{t-1}\omega) \cdot f(\vartheta^{t-1}\omega) \circ \cdots \circ f(\omega)\xi(\omega) \\
&= \left(\prod_{\tau=1}^{t} \lambda(\vartheta^{\tau-1}\omega)\right) \cdot c(t,\omega)\xi(\omega) =: \Lambda(\vartheta^t\omega) \cdot c(t,\omega)\xi(\omega)
\end{aligned}$$

□

Lemma B.2 (Stochastic Case).
Assume that $\xi^\star : \Omega \to S$ is a random fixed point of $f(\omega)$. There exists $\lambda : \Omega \to \mathbb{R}_+$, $\lambda(\omega) > 0$ such that $\mathbb{P}$-a.s.

$$F(\omega)\xi^\star(\omega) = \lambda(\omega) \cdot \xi^\star(\vartheta\omega) = \lambda(\omega) \cdot f(\omega)\xi^\star(\omega) = \lambda(\omega) \cdot \frac{F(\omega)}{|F(\omega)|}\xi^\star(\omega) \tag{B.10}$$

Proof. Define the random variable

$$\lambda^\star(\omega) := \frac{|F(\omega)\xi^\star(\omega)|}{|(F(\omega)/|F(\omega)|)\xi^\star(\omega)|} \tag{B.11}$$

Then,

$$\begin{aligned}
\lambda^\star(\omega) \cdot \xi^\star(\vartheta\omega) &= \frac{|F(\omega)\xi^\star(\omega)|}{|(F(\omega)/|F(\omega)|)\xi^\star(\omega)|} \cdot f(\omega)\xi^\star(\omega) \\
&= \frac{|F(\omega)\xi^\star(\omega)|}{|(F(\omega)/|F(\omega)|)\xi^\star(\omega)|} \cdot \frac{F(\omega)\xi^\star(\omega)}{|F(\omega)\xi^\star(\omega)|} \\
&= \frac{|F(\omega)\xi^\star(\omega)|}{|f(\omega)\xi^\star(\omega)|} \cdot \frac{F(\omega)\xi^\star(\omega)}{|F(\omega)\xi^\star(\omega)|} = F(\omega)\xi^\star(\omega)
\end{aligned} \tag{B.12}$$

[4] F is called strictly monotone if $x \gneqq y$ implies $F(\omega)x \gg F(\omega)y$, for all ω, see Evstigneev & Pirogov (2010), Theorem 1.

Thus, the pair $(\lambda^\star(\omega), \xi^\star(\omega))$ is a Perron-Frobenius solution inducing random balanced orbits given by

$$\begin{aligned} \gamma(\omega, \xi^\star(\omega)) &= \{C(t,\omega)\xi^\star(\omega)\}_{t=0}^{\infty} \\ C(t,\omega)\xi^\star(\omega) &= \left(\prod_{\tau=1}^{t} \lambda^\star(\vartheta^{\tau-1}\omega)\right) \cdot c(t,\omega)\xi^\star(\omega). \end{aligned} \tag{B.13}$$

. □

B.1 A Contraction Theorem for Random Systems

Definition B.5. *A random fixed point $\xi : \Omega \to S$ is called asymptotically stable if*

$$\lim_{t\to\infty} |c(t,\omega)x_0 - \xi(\vartheta^t\omega)| = 0 \quad \mathbb{P}\text{-}a.s. \tag{B.14}$$

for all $x_0 \in \mathcal{B}(\xi(\omega))$, the basin of attraction of ξ, where $c(t,\omega)$ is the cocycle associated with $f(\omega)$, see equation (B.2).

Definition B.6. *Define the distance of an orbit $\{C(t,\omega)X_0\}$ of F with $|X_0 = 1|$ to the balanced one $\{C(t,\omega)\xi(\omega)\}$ associated with the random fixed point $\xi : \Omega \to S$ as*

$$\Delta_t = \Delta(t,\omega)X_0 := |C(t,\omega)X_0 - C(t,\omega)\xi(\omega))| = |C(t,\omega)X_0 - \Lambda(\vartheta^t\omega)\xi(\vartheta^t\omega))|. \tag{B.15}$$

An orbit $\{C(t,\omega)X_0\}$ is said to ***converge*** *to a balanced orbit if for all $x_0 \in \mathcal{B}(\xi(\omega)) \subset S$ and for all $X_0 = x_0 \neq \xi(\omega)$:*

$$\lim_{t\to\infty} |c(t,\omega)x_0 - \xi(\vartheta^t\omega)| = 0 \quad \text{and} \quad \lim_{t\to\infty} |\Delta(t,\omega)X_0| = 0, \quad \mathbb{P}\text{-}a.s.. \tag{B.16}$$

Theorem B.2.
Let $\xi^\star : \Omega \to \mathbb{R}^n_+$ be an asymptotically stable random fixed point of $f(\omega)$ inducing the rate of contraction

$$M(\omega, \xi^\star(\omega)) := \lim_{x_0 \to \xi^\star(\omega)} \left| \frac{|f(\omega)x_0 - f(\omega)\xi^\star(\omega)|}{|x_0 - \xi^\star(\omega)|} \right| < 1, \quad \mathbb{P}\text{-}a.s. \tag{B.17}$$

of f at $\xi^\star(\omega)$.
Then, for almost all $\omega \in \Omega$ and any $x_0 \in \mathcal{B}(\omega)$, $x_0 \neq \xi^\star(\omega)$ with $\lim_{t\to\infty} |c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)| = 0$ the distance $\Delta_t := |C(t,\omega)X_0 - \Lambda(t,\omega) \cdot \xi^\star(\vartheta^t\omega))|$ satisfies $\mathbb{P}$-a.s.:

$$\lim_{t\to\infty} |\Delta_t| = 0 \qquad \text{if} \qquad \mathbb{E}\log(\lambda(\omega, \xi^\star(\omega)) + \mathbb{E}\log M(\omega, \xi^\star(\omega)) < 0 \tag{B.18}$$

$$\lim_{t\to\infty} |\Delta_t| = \infty \qquad \text{if} \qquad \mathbb{E}\log(\lambda(\omega, \xi^\star(\omega)) + \mathbb{E}\log M(\omega, \xi^\star(\omega)) > 0 \tag{B.19}$$

Proof. Let $\gamma(\omega, \xi^\star(\omega)) = \{C(t,\omega)\xi^\star(\omega)\}_{t=0}^{\infty}$ denote the balanced orbit associated with the random fixed point $\xi^\star$ of f given by the two associated cocycles and $C(t,\omega)$ resp. $c(t,\omega)$

$$C(t,\omega)\xi^\star(\omega) = \left(\prod_{\tau=1}^{t} \lambda^\star(\vartheta^{\tau-1}\omega)\right) \cdot c(t,\omega)\xi^\star(\omega) \equiv \Lambda(\vartheta^t\omega) \cdot c(t,\omega)\xi^\star(\omega). \tag{B.20}$$

From the definition (B.15) one has

$$\begin{aligned}\Delta_t = \Delta(t,\omega)X_0 =& |C(t,\omega)X_0 - \Lambda(\vartheta^t\omega)\cdot\xi^\star(\vartheta^t\omega))| \\ =& \Lambda(\vartheta^t\omega)\cdot|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)| \\ & \text{and} \\ \Delta_{t+1} = \Delta(t+1,\omega)X_0 =& |C(t+1,\omega)X_0 - \Lambda(\vartheta^{t+1}\omega)\cdot\xi^\star(\vartheta^{t+1}\omega))| \\ =& \lambda(\vartheta^t\omega)\cdot\Lambda(\vartheta^t\omega)\cdot|c(t+1,\omega)x_0 - \xi^\star(\vartheta^{t+1}\omega)| \\ =& \lambda(\vartheta^t\omega)\cdot\Lambda(\vartheta^t\omega)\cdot|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|\end{aligned} \tag{B.21}$$

implying

$$\begin{aligned}\frac{\Delta_{t+1}}{\Delta_t} =& \frac{\lambda(\vartheta^t\omega)\cdot\Lambda(\vartheta^t\omega)\cdot|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|}{\Lambda(\vartheta^t\omega)\cdot|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)|} \\ =& \frac{\lambda(\vartheta^t\omega)\cdot|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|}{|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)|}\end{aligned} \tag{B.22}$$

Since $\lim_{t\to\infty}|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)| = 0, \mathbb{P}$-a.s., there exists an $\varepsilon > 0$ sufficiently small and $t_0 = t_0(\varepsilon,\omega) > 0$ sufficiently large such that

$$\left|\frac{|f(\vartheta^{t+1}\omega)\circ c(t,\omega)x_0 - f(\vartheta^{t+1}\omega)\xi^\star(\vartheta^t\omega)|}{|c(t,\omega)x_0 - \xi^\star(\vartheta^t\omega)|} - M(\vartheta^t\omega,\xi^\star(\omega))\right| < \varepsilon \tag{B.23}$$

for $t \geq t_0$. By induction, for all $t \geq t_0$, there are upper and lower bounds satisfying $\underline{\Delta}_t \leq |\Delta_t| \leq \overline{\Delta}_t$ for the two linear random dynamical systems

$$\overline{\Delta}_{t+1} = \left[\lambda(\vartheta^t\omega)\cdot\ M(\vartheta^t\omega,\xi^\star(\omega)) + \varepsilon\right]\overline{\Delta}_t \tag{B.24}$$

$$\underline{\Delta}_{t+1} = \left[\lambda(\vartheta^t\omega)\cdot\ M(\vartheta^t\omega,\xi^\star(\omega)) - \varepsilon\right]\underline{\Delta}_t \tag{B.25}$$

with $\overline{\Delta}_{t_0} = |\Delta_{t_0}|$ and $\underline{\Delta}_{t_0} = |\Delta_{t_0}|$. Therefore,

$$\mathbb{E}\log(\lambda(\omega,\xi^\star(\omega)) + \mathbb{E}\log M(\omega,\xi^\star(\omega)) < 0$$

implies that the upper bound (B.24) converges to zero $\mathbb{P}$-a.s. so that $\lim_{t\to\infty}|\Delta_t| = 0$.

Conversely,

$$\mathbb{E}\log(\lambda(\omega,\xi^\star(\omega)) + \mathbb{E}\log M(\omega,\xi^\star(\omega)) > 0$$

implies that the lower bound grows to infinity and $\lim_{t\to\infty}|\Delta_t| = \infty$. □

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