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Abstract

The paper examines the role of fiscal and monetary policy on the dynamics of monetary expansion in a macroeconomy. Its microeconomic structure defined by producers with neoclassical production functions, heterogeneous OLG consumers, and a stationary fiscal and monetary policy induces a consistent dynamic closed macroeconomic model of the AS-AD type. Existence and uniqueness of a temporary competitive monetary equilibrium are shown in a two-market economy (determining prices, wages, output, and employment) under a standard set of neoclassical conditions on production, consumer preferences, fiscal and monetary parameters. Comparative statics on prices, wages, and allocations for all levels of the state variables: money balances, debt, and expectations are shown.

The dynamic development of temporary equilibria is defined by orbits of a dynamical system generated by three mappings of the one-step (recursive) time change, one for each state variable. The paper defines and describes explicitly the forecasting rules for prices as functions (so-called perfect predictors) which induce perfect foresight along orbits of the economy. It establishes sufficient conditions for their existence and uniqueness and provides a constructive characterization of perfect predictors for the AS-AD economy.

Given existence of a globally perfect predictor, perfect foresight holds along all orbits of the economy. The results show that constant intertemporal allocations are uniquely generated by orbits of balanced expansion of both money balances and public debt. Generically, depending on parameters, there exist two or no balanced paths while stationary equilibria with zero inflation exist only on a small (non-open) set of parameters.

For a benchmark case (defined by isoelastic utility and production functions) perfect foresight dynamics exist globally and are monotonic (no cycles). There exist at most two balanced paths one of which is always unstable. Their existence and stability are influenced in a decisive way by fiscal and monetary parameters determining steady state inflation rates, allocations, as well as bounds for sustainable debt-to-GDP ratios.

JEL codes: E21, E23, E24, E31, E40, E62, E63, C62, D84, H60, H62, H63

Keywords: monetary/fiscal policy, deficit, monetary growth, stability, perfect foresight

1 Introduction

Contributions to current macroeconomic theory are often directed toward the analysis of specific partial economic issues without a description of a fully specified closed flow monetary model within which real allocations (output, employment, consumption, and investment) are determined *simultaneously* with the level of the corresponding nominal prices, wages, and interest rates. The class of the so-called real models (of growth or intertemporal general equilibrium theory as for example by Azariadis, 1993; Mankiw, 1994; Sargent, 1979, 1987; Ljungqvist & Sargent, 2000) or (the main chapters in Romer, 2005, 2012) do not contain money or other nominal assets. As a consequence, questions of the determination of the level of prices and wages are left aside. The role and influence of fiscal or monetary policy on nominal entities such as the rate of inflation, on level of money balances or debt cannot be discussed properly. These models also seem unnecessarily restrictive in their dynamic description for a time series oriented analysis of intertemporal allocations with perfect foresight or rational expectations.

Conversely, models of the strictly *monetary approach* (i.e. of the Keynesian type) (as in Abel & Bernanke, 2005; Blanchard, 2003; Blanchard & Fischer, 1989) present mostly comparative statics features of temporary equilibrium configurations for linearized versions (reduced form models) leaving allocations or market clearing trades unspecified. In other words, the relationship between the values of transactions (nominal GDP, wage income, or profit income) and the underlying real trades (output and employment) are not described consistently, lacking an analysis of the role of the monetary variables and their impact/interaction in the determination of nominal (price and wage levels) and the real variables (output, employment, and consumption). The macroeconomic analysis takes on a partial equilibrium rather than a general equilibrium characteristic. To match/compare results of a monetary model with those from the pure allocative theory sometimes lacks consistency of cross-market effects or incomes consistency and does not explain interactions between the real and the monetary features of the economy. The extensive discussions of the role and significance of deficits and debt (in the final chapters of Abel & Bernanke, 2005; Blanchard, 2003; Blanchard & Fischer, 1989; Romer, 2012) reveal an awareness of the structural interdependence of monetary and real phenomena and a need for a consistent analysis.

The paper indicates that the chosen closed monetary model in an AS-AD format can be used successfully as a consistent macroeconomic representation of an Arrow-Debreu-Hicks-Patinkin-Keynes approach to analyze consistently the interaction of monetary and real phenomena in *temporary equilibrium* with monetary assets as well as the *intertemporal* issues associated with allocative market clearance, the evolution of prices and their expectations, of money balances, and of public debt. This is achieved by extending the equilibrium principles for allocations and prices of an Arrow-Debreu model to an intertemporal extension with money, prices, and expectations (combining ideas from Keynes, 1936; Hicks, 1937, 1939, 1950; Patinkin, 1965; Grandmont, 1983) and invoking the notion of *Temporary Monetary Equilibrium* at given expectations and asset holdings. Using a forward recursive approach to the formation of expectations guarantees a sequential framework allowing in the end the analysis of a dynamical system representing the macroeconomy to investigate the main issues of *intertemporal monetary macroeconomics:* describing time series under rational expectations as *orbits* of a dynamical system generating inflation, deficits, and debt; to examine their stability, the existence and convergence/divergence to paths of balanced monetary expansion.

2 An AS–AD Framework with Money and Debt

2.1 Markets and Agents

The paper extends the prototype model given in Böhm (2017) with two competitive markets, one for labor which is traded at a wage rate $w > 0$ and one for a homogeneous output traded at a commodity price denoted $p > 0$ to include public debt.

A government consumes $g \geq 0$ of the produced good and imposes a proportional income tax at the rate $0 \leq \tau \leq 1$ on wage and profit income. Fiscal policy consists in a choice of the parameters (g, τ). A central bank offers an interest rate payment on public debt which is issued by the government earning a nominal interest $r > -1$. Monetary policy consists in the choice of an interest rate r.

There are two types of consumers with an overlapping generations structure: workers and share holders (non-workers) each living and consuming for two periods receiving income only when young. Workers save in the form of money $M \geq 0$ while share holders save in the form of public debt B earning the nominal interest $r > -1$.

Production is carried out by a finite number of firms employing labor as input to produce the output using a smooth neoclassical production function. The technology does not allow storage or inventories. Firms are owned by the share-holder consumers who receive all profits.

Figure 1 portrays the interaction of trades and transfers between the sectors and across markets in the economy. Trades between sectors on either of the two markets are real flows or deliveries indicated by dashed lines with arrows indicating the direction of delivery. The payments for delivery, i.e. the associated monetary flows are indicated by solid lines with arrows typically oriented in opposite direction of the real trade. Feasibility of trade or market balance implies that for each of the two markets the sum of ingoing real flows equals the sum of outgoing flows. If all real flows are evaluated by the same prices and wages ingoing and outgoing monetary flows balance on each market as well.

Payments between the consumption and the public sector like tax revenue and interest payments are direct monetary transfers which have no real flow counterpart related to a market. Income consistency for consumers in a closed economy induces the usual budget implications making net intertemporal savings of the consolidated private sector equal to the consolidated deficit of the public sector.

All agents take prices, wages, and tax rates as given and the two markets are assumed to operate in a competitive fashion. Transactions are carried out at competitive prices and wages which are determined simultaneously by clearing demand and supply on the two markets. The AS-AD format allows the convenient two-step analysis of market clearing conditions of the supply side (the labor market) and the demand side (the commodity market) if the underlying microstructure of the model satisfies the separability/homotheticity conditions used for consumer workers.

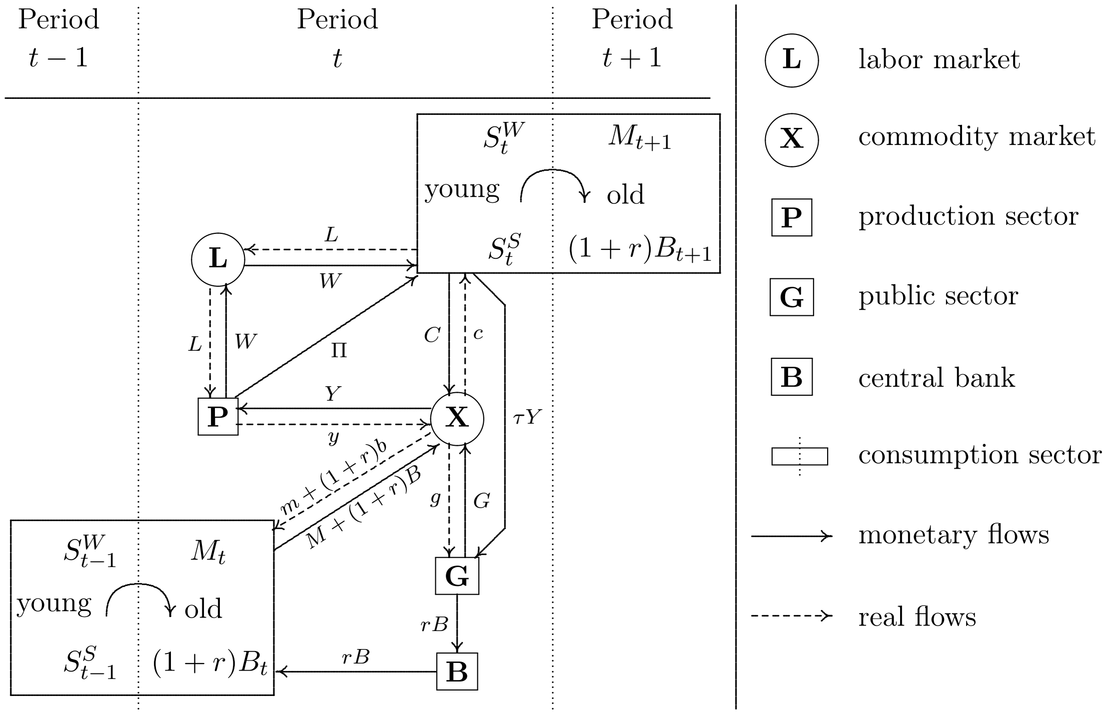


Figure 1: Patterns of real and monetary flows in a two-market economy with OLG consumers, fiscal policy, and public debt

2.2 The Aggregate Supply Function

Competitive Labor Market Clearing

Without loss of generality assume that there exist one producer and one young worker-consumer. The technology is given by a production function $y = F(L)$, $F : \mathbb{R}_+ \to \mathbb{R}_+$, which is C^2, strictly monotonically increasing, and strictly concave.

The worker's preferences are given by a utility function $U(c, c^e) - v(\ell)$ where U is strictly increasing, strictly concave, and homothetic while the disutility of work v is strictly increasing and strictly convex. Then, the indirect intertemporal utility of a unit of real labor income is a strictly decreasing function $V(\theta^e)$ where $\theta^e = p^e/p$ is the expected gross rate of inflation ($p =$ current price and $p^e =$ expected future price). Therefore, the competitive labor supply L by the young worker as a function of (p, w, p^e) must satisfy equality of the net real wage $(1-\tau)\alpha$ ($\alpha \equiv w/p$) with his marginal willingness to work, i.e.

$$\alpha = \frac{v'(L)}{(1-\tau)V(\theta^e)} \tag{2.1}$$

must hold which defines the inverse labor supply function, see Appendix A.

The worker and the share holder are assumed to pay a proportional income tax with common rate $0 \leq \tau \leq 1$ on their income when young. Let $L(\theta^e)$ denote the employment level which clears the labor market under competitive conditions

$$L(\theta^e) := \left(\frac{v'}{F'}\right)^{-1} ((1-\tau)V(\theta^e)) \quad \Longleftrightarrow \quad F'(L) = \frac{v'(L)}{(1-\tau)V(\theta^e)}. \tag{2.2}$$

This induces the equilibrium real wage function

$$\widetilde{W}(\theta^e) := F'(L(\theta^e)) \equiv \frac{v'(L(\theta^e))}{(1-\tau)V(\theta^e)} \tag{2.3}$$

and the aggregate supply function as

$$AS(\theta^e) := F(L(\theta^e)). \tag{2.4}$$

Therefore, in nominal terms, the market clearing wage level is homogeneous of degree one in (p, p^e) while the equilibrium employment level is homogeneous of degree zero in (p, p^e). If F and v satisfy the Inada conditions (concave and convex respectively) then, $\widetilde{W}$ is strictly increasing, and AS is surjective and strictly decreasing. Both are isolelastic if F, U, and v are isoelastic.

2.3 Income Consistent Aggregate Demand with Money and Debt

Intertemporal two-period preferences of both consumers between current and future consumption are assumed to be homothetic. Therefore, the propensity to consume out of income is given by $0 \leq c_w(\theta^e) \leq 1$ with indirect utility $V(\theta^e)$ for a worker, while $\theta^e/(1+r)$ is the relevant intertemporal rate of substitution for expected consumption for share holders with propensity $c_s(\theta^e/(1+r))$. The propensities c_i, $i = w, s$, may be decreasing or increasing functions while the indirect utility V is strictly decreasing in expected inflation.

Income consistency of nominal GDP $= Y$ with aggregate expenditure in any period implies that

$$Y = M + (1+r)B + pg + c_w(\theta^e)(1-\tau)\text{Wages} + c_s(\theta^e/(1+r))(1-\tau)(Y - \text{Wages}). \tag{2.5}$$

Labor market clearing implies a competitive wage rate and an aggregate income share of wage income to GDP (i.e. of Wages/GDP) equal to the elasticity of labor in production $E_F(L)$. Therefore, under competitive income conditions equation (2.5) implies a level of aggregate real demand

$$\begin{aligned} y^D &:= \frac{1}{p} \cdot \frac{M + (1+r)B + pg}{1 - c_w(\theta^e)(1-\tau)E_F(\theta^e) - c_s(\theta^e/(1+r))(1-\tau)(1-E_F(\theta^e))} \\ &= \frac{1}{\tilde{c}(\theta^e, r)} \left(\frac{W}{p} + g \right) =: AD\left(\frac{W}{p}, \theta^e, r \right) \end{aligned} \tag{2.6}$$

where $W := M + (1+r)B$ is *effective nominal private wealth*[1], $E_F(\theta^e) := E_F(L(\theta^e))$ is the labor share, and

$$\begin{aligned} \tilde{c}(\theta^e, r) &:= 1 - c_w(\theta^e)(1-\tau)E_F(\theta^e) - c_s(\theta^e/(1+r))(1-\tau)(1-E_F(\theta^e)) \\ &= 1 - (1-\tau)\left[c_s(\theta^e/(1+r)) - \left(c_w(\theta^e) - c_s(\theta^e/(1+r))\right)E_F(\theta^e)\right] \end{aligned} \tag{2.7}$$

is the demand multiplier. The form (2.6) of income-consistent aggregate demand reveals the interplay of the functional income distribution $E_F(\theta^e)$ with preferences, taxation, expectations, and the interest rate.

The economically interesting configurations which imply a constant multiplier balance off the effects from the functional income distribution against the heterogeneity of consumption propensities. The functional income distribution has an influence on the multiplier if and only if propensities to consume are heterogeneous, i.e. if $c_w(\theta^e) \neq c_s(\theta^e(1+r))$. If the production function is isoelastic E_F is constant. Then, the heterogeneity of propensities to consume influence

[1] W should not be confused with the real wage function $\widetilde{W}$ or with the term 'Wages' used in equation (2.5).

the multiplier. Expected inflation and the interest rate through the propensity to consume of the share holder exert an effect on the multiplier. If consumer preferences are isoelastic the multiplier is independent of expected inflation and of the interest rate. If in addition, consumer preferences are identical, the income distribution does not matter and the multiplier is constant. The situation with isoelastic consumers and producers will be considered as a benchmark. In this case the multiplier $\tilde{c}$ is constant and the rate of interest has a positive linear influence on aggregate demand through aggregate effective wealth only.

2.4 Temporary Equilibrium with Money and Public Debt

Definition 2.1. *Given* (M, B, p^e) *a temporary equilibrium is a price* $p > 0$ *which solves*

$$AD\left(\frac{W}{p}, \frac{p^e}{p}, r\right) = AS\left(\frac{p^e}{p}\right), \quad \textit{with} \quad W = M + (1+r)B. \tag{2.8}$$

Under a large set of reasonable conditions there exists a unique positive solution of (2.8) written as

$$p = \mathcal{P}(M, B, p^e). \tag{2.9}$$

The continuous mapping $\mathcal{P} : \mathbb{R}^3_+ \to \mathbb{R}_+$ is referred to as the *price law* which is homogeneous of degree one in (M, B, p^e). For the following analysis the policy parameters (g, τ, r) are assumed to be given and are suppressed as arguments of the price law when not needed. Under the conditions assumed for existence and continuity the price law has no specific effects a priori with respect to the interest rate or expected inflation. These will be discussed and introduced for the benchmark case.

The homogeneity of the price law implies that its unit-level contour

$$\begin{aligned}
\mathcal{E}_{\mathcal{P}} &:= \left\{(M, B, p^e) \in \mathbb{R}^3_+ \ \mid\ 1 = \mathcal{P}(M, B, p^e)\right\} \\
&= \left\{\left(\frac{M}{p}, \frac{B}{p}, \frac{p^e}{p}\right) \in \mathbb{R}^3_+ \,\middle|\, AS(p^e/p) = \frac{1}{\tilde{c}(p^e/p, r)}\left(\frac{W}{p} + g\right), W = M + (1+r)B\right\} \qquad (2.10) \\
&= \left\{(m, b, \theta^e) \in \mathbb{R}^3_+ \ \mid\ \tilde{c}(\theta^e, r)AS(\theta^e) - g = m + (1+r)b\right\}
\end{aligned}$$

is identical to the so called equilibrium set of the economy. This contains all the information necessary to describe the implications for the real allocations in the economy under a constant fiscal and monetary policy (g, τ, r). In other words, all equilibrium triples of real money balances, real debt, and expected inflation rates must lie in this set. Moreover, all time series of the economy under any form of expectations formation or learning belong to this set under stationary fiscal and monetary policy. Thus, $\mathcal{E}_{\mathcal{P}}$ is the unique time-invariant set which should/could be the object of an empirical analysis. Its analytical form in the third equation of (2.6) stipulates essentially that in any period real effective wealth is a function of the expected rate of inflation, a structural relationship which could be tested empirically or should be observed in numerical experiments. For a given interest rate r

$$\mathcal{E}_{\mathcal{P}} := \left\{(M, B, p^e) \in \mathbb{R}^3_+ \ \mid\ 1 = \mathcal{P}(M, B, p^e)\right\} \tag{2.11}$$

is a curved simplex in $\mathbb{R}^3_+$. Its projection onto the first two components (M, B) is a straight line. Its boundaries depend on the boundedness of labor supply, on the Inada conditions, and on the level of government demand.

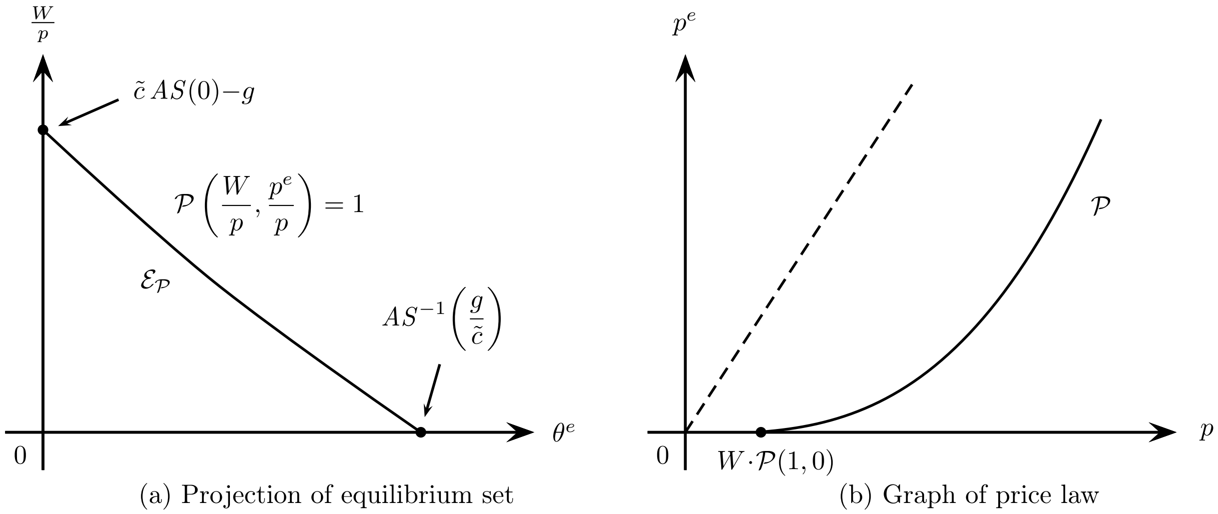


Figure 2: Geometry of the price law with bounded production for a given interest rate r

All other properties of equilibrium allocations in any time period can be shown to be defined as mappings induced by the price law. The market clearing nominal wage rate w is given by

$$w = \mathcal{P}\left(M, B, p^e\right) \widetilde{W}\left(\frac{p^e}{\mathcal{P}\left(M, B, p^e\right)}\right). \tag{2.12}$$

A pair of equilibrium mappings defines the level of employment using (2.2)

$$\mathcal{L}\left(M, B, p^e\right) := L\left(\frac{p^e}{\mathcal{P}\left(M, B, p^e\right)}\right) \tag{2.13}$$

and aggregate output

$$\begin{aligned}
\mathcal{Y}\left(M, B, p^e\right) &:= F\left(L\left(\frac{p^e}{\mathcal{P}\left(M, B, p^e\right)}\right)\right) \equiv F\left(\mathcal{L}\left(M, B, p^e\right)\right) \\
&\equiv AS\left(\frac{p^e}{\mathcal{P}\left(M, B, p^e\right)}\right) \\
&= AD\left(\frac{W}{\mathcal{P}\left(M, B, p^e\right)}, \frac{p^e}{\mathcal{P}\left(M, B, p^e\right)}\right).
\end{aligned} \tag{2.14}$$

Both functions $\mathcal{L}, \mathcal{Y}$ are homogeneous of degree zero in (M, B, p^e).

The Benchmark Case

The price law has strong structural properties in the benchmark case. Let $0 < \tilde{c}(\theta^e) \equiv \tilde{c} < 1$ be constant and let the aggregate supply function be strictly monotonically decreasing and convex satisfying

$$\lim_{\theta^e \to \infty} AS(\theta^e) = 0. \tag{2.15}$$

Then, the equilibrium condition (2.8) has the form

$$AS(\theta^e) = \frac{M/p + (1+r)B/p + g}{\tilde{c}} \tag{2.16}$$

implying the *explicit* functional equilibrium relation

$$\frac{W}{p} = \frac{M}{p} + (1+r)\frac{B}{p} = \tilde{c}AS(\theta^e) - g \tag{2.17}$$

which is an explicit linear form of the unit contour of the associated price law $\mathcal{P}(M, B, p^e)$, i.e. of $\mathcal{E}_\mathcal{P}$ in the first two components. Notice that the interest rate has no influence on aggregate supply AS and that it induces a partial effect on aggregate demand/aggregate wealth only. Since real wealth is linear in (M, B) the unit contour has linear level sets for fixed θ^e in that subspace defining a one-dimensional manifold in $\mathbb{R}^2_+$. In other words, there exists a uniquely defined decreasing function determining the level of effective real wealth in temporary equilibrium for each level of the expected inflation rate. Its representation coincides with the equilibrium set $\mathcal{E}_\mathcal{P}$ in $\mathbb{R}^3_+$ as shown in Figure 2(a). Real wealth is strictly negatively correlated with expected inflation.

3 Dynamics of Money, Debt, Expectations, and Prices

As shown in the previous section, the dynamic *evolution of temporary equilibria* of such a monetary economy occurs on the graph of the price law, which, geometrically speaking is the unique time invariant cone in $\mathbb{R}^4_+$ for constant values of the policy parameters. To describe the macroeconomic time series on that cone as an orbit of a dynamical system the associated time one maps for assets and for expectations have to be specified. The time one maps of the assets originate from the intertemporal budget equations for the respective sectors of the economy. They are time invariant (stationary) and straightforward. The description of the expectation formation will be restricted to so-called *stationary* forecasting rules, i.e. to those which are described by time invariant functions applying an identical mapping in each period to an updated list of observations. This includes large but not all classes of statistical learning. It implies that the evolution of the monetary economy is described by an autonomous dynamical system.

Let (g, τ, r) be given. Time series along balanced paths of the economy are associated with rays (half lines) in the space $(M, B, p^e) \in \mathbb{R}^3_+$, the domain of the price law (2.9). To determine their properties and to show existence and stability is the subject of the sections below. For the benchmark case the result is closely related to the results for the two-dimensional case of a monetary economy without government debt.

3.1 Dynamics of Money and Debt

Assume for simplicity $c_w = c_s \equiv c$. For the given state (M, B, p^e), let $\theta^e = p^e/\mathcal{P}(M, B, p^e)$ denote the expected inflation factor in equilibrium and define next period's money holdings by the worker and debt holdings by the shareholder as

$$M_1 = \mathcal{M}(M, B, p^e) := (1-c)(1-\tau)E_F(\theta^e)\mathcal{P}(M, B, p^e)\mathcal{Y}(M, B, p^e) \tag{3.1}$$

$$B_1 = \mathcal{B}(M, B, p^e) := (1-c)(1-\tau)(1-E_F(\theta^e))\mathcal{P}(M, B, p^e)\mathcal{Y}(M, B, p^e). \tag{3.2}$$

Both are functions of aggregate nominal income through aggregate demand and thus of aggregate effective wealth $W = M + (1+r)B$. Therefore, one finds for the development of aggregate effective wealth

$$\begin{aligned} W_1 &= M_1 + (1+r)B_1 \\ &= (1-c)(1-\tau)\Big(E_F(\theta^e) + (1+r)(1-E_F(\theta^e))\Big)\mathcal{P}(W,p^e)\mathcal{Y}(W,p^e) \\ &= \frac{(1-c)(1-\tau)}{\tilde{c}}\Big(E_F(\theta^e) + (1+r)(1-E_F(\theta^e))\Big)(W + \mathcal{P}(W,p^e)g) \\ &=: \mathcal{W}(W,p^e) \end{aligned} \tag{3.3}$$

which is independent of the distribution of wealth between money and debt. However, for $r \neq 0$, it is not independent of the income distribution. The ratio of money to debt

$$\frac{M_1}{B_1} = \frac{E_F(\theta^e)}{1 - E_F(\theta^e)} \tag{3.4}$$

depends on the income distribution and is constant over time with isolelastic production functions. Thus, for the benchmark case with constant income distribution, it is sufficient for the dynamic analysis to consider the time-one map $\mathcal{W}$ in (3.3) of aggregate effective wealth which is one-dimensional and homogeneous of degree one in (W, p^e).

3.2 Dynamics with Perfect Foresight

A function $\psi : I_T \to \mathbb{R}_+$ defined on a finite dimensional information set I_T to the positive real numbers is called a forecasting rule or a *predictor*, i.e. ψ is a mapping determining the forecast $p^e_{t,t+1}$ as a function of a finite list of observed past data. For example,

$$p^e_{t,t+1} = \psi(p_{t-T}, \dots, p_{t-1}; M_{t-T}, \dots, M_{t-1}; p^e_{t-T}, \dots, p^e_{t-1}) \equiv \psi(q)$$

where $q := (p_{t-T}, \dots, p_{t-1}; M_{t-T}, \dots, M_{t-1}; p^e_{t-T}, \dots, p^e_{t-1}) \in I_T$ is the list of prices, money balances, and expectations with memory T. Then, the three mappings $(\mathcal{M}, \mathcal{B}, \psi) : \mathbb{R}^3_+ \times I_T \to \mathbb{R}^3_+$ determine the one-step change of money, debt, and expectations $(M, B, q) \mapsto (\mathcal{M}(M, B, \psi(q)), \mathcal{B}(M, B, \psi(q)), \psi(q)) = (M_1, B_1, p^e_1)$. In other words, the induced delay system on $\mathbb{R}^3_+ \times I_T$ describes the dynamics of money, debt, and expectations as an autonomous dynamical system on $\mathbb{R}^2_+ \times I_T$ for any such forecasting rule.

In order to discuss dynamics with perfect foresight it is useful to consider the extended state space for $(M, B, p^e, p) \in \mathbb{R}^4_+$ in spite of the fact that the dynamics needs to be defined only in $\mathbb{R}^3_+$. Let $(M, B, p^e, p) = (M, B, p^e, \mathcal{P}(M, B, p^e)) \in \mathbb{R}^4_+$ be the monetary current state with associated real values $(m, b, \theta^e) := \frac{1}{\mathcal{P}(M,B,p^e)}(M, B, p^e) \in \mathcal{E}_{\mathcal{P}}$.

Definition 3.1. *An orbit* $\left\{(M_t, B_t, p^e_{t,t+1}, p_t)\right\}_{t=0}^{\infty}$ *of an economy is said to have the perfect foresight property if*

$$p_t = p^e_{t-1,t} \qquad \textit{for all } t. \tag{3.5}$$

Therefore, a predictor which induces the perfect foresight property along an orbit of a system when used will be called a *perfect predictor*.

Definition 3.2. *A predictor* ψ^* *(with associated information set* I*) is said to be* perfect *if it induces an orbit* $\left\{(M_t, B_t, p^e_{t,t+1}, p_t)\right\}_{t=0}^{\infty}$ *with the perfect foresight property, i.e. if for all* t

$$\begin{aligned} M_t &= \mathcal{M}(M_{t-1}, B_{t-1}, p^e_{t-1,t}) \\ B_t &= \mathcal{B}(M_{t-1}, B_{t-1}, p^e_{t-1,t}) \\ p^e_{t,t+1} &= \psi^*(I_{t-1}) \\ p_t &= \mathcal{P}(\mathcal{M}(M_{t-1}, p^e_{t-1,t}), \mathcal{B}(M_{t-1}, B_{t-1}, p^e_{t-1,t}), \psi^*(I)) = p^e_{t-1,t}. \end{aligned} \tag{3.6}$$

It is called globally perfect *if all orbits have the perfect foresight property.*

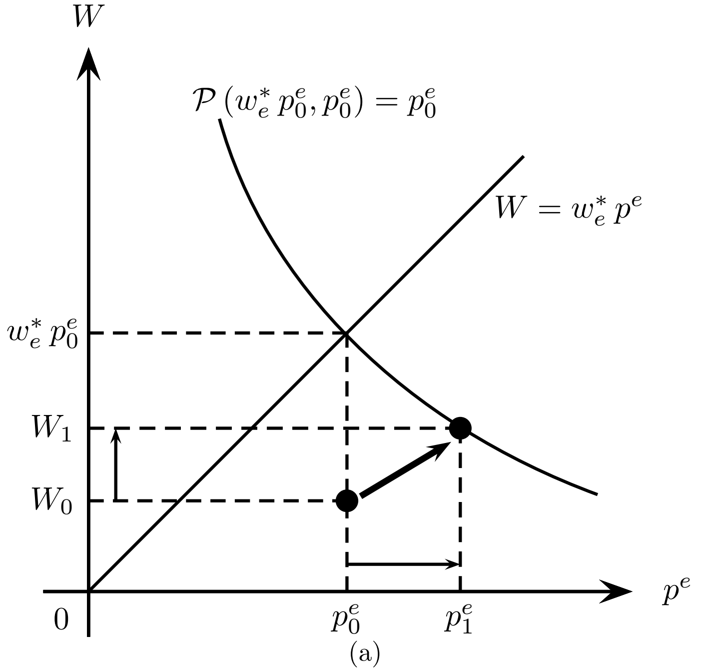


Figure 3: One-step perfect prediction in benchmark case: $p^e_0 = \mathcal{P}(W_1, p^e_1)$; $W := (1+r)B + M$

Due to the *expectational lead* in the price law predictors which induce orbits with perfect foresight must have a specific three-dimensional information set. The fourth condition of perfect foresight of (3.6) in Definition 3.2 suggests that it suffices to consider predictors which use the *previous* prediction and *current* asset levels $(M, B, p^e_{-1}) \in \mathbb{R}^3_+$ as information set. Let $\psi : \mathbb{R}^3_+ \to \mathbb{R}_+$, $(M, B, p^e_{-1}) \mapsto \psi(M, B, p^e_{-1}) = p^e$ denote a predictor of the macroeconomic model. One obtains the following representation lemma for a globally perfect predictor.

Lemma 3.1. *A* ***predictor*** $\psi : \mathbb{R}^3_+ \to \mathbb{R}_+$, $(M, B, p^e_{-1}) \mapsto \psi(M, B, p^e_{-1}) = p^e$ *is globally perfect if it satisfies*

$$\mathcal{P}(M, B, \psi(M, B, p^e_{-1})) = p^e_{-1} \quad \textit{for all} \quad (M, B, p^e_{-1}) \in \mathbb{R}^3_+. \tag{3.7}$$

The lemma suggests a geometric procedure for the construction of a perfect predictor[2]. Let $e(M, B, p^e, p^e_{-1}) := \mathcal{P}(M, B, p^e) - p^e_{-1}$ denote the error function for an arbitrary forecast p^e and define its zero-contour (or the general inverse of $\mathcal{P}$ with respect to p^e) as

$$\mathcal{P}^e(M, B, p^e_{-1}) := \begin{cases} p^e & \mathcal{P}(M, B, p^e) = p^e_{-1} \\ 0 & \text{otherwise} \end{cases}. \tag{3.8}$$

[2]For more details see Böhm & Wenzelburger (1999, 2004).

This is essentially the inverse of the price law $\mathcal{P}$ with respect to p^e, which, in general, is a set-valued mapping. Therefore, a perfect predictor ψ^*, if it exists, must be a selection of the inverse of the price law with respect to p^e, i.e. $\psi^* \in \mathcal{P}^e$. Figure 3 shows a geometric construction of the one-step perfect prediction $p_1^e = \mathcal{P}^e(M_1, B_1, p_0^e)$ for any given state (M_1, B_1, p_0^e) and (M_0, B_0).

In the benchmark case, the error function depends on aggregate wealth only and not on its distribution. Therefore, the problem to be solved is analogous to the situation with money as the only asset (see Lemma 4.1.1 and Proposition 4.1.1 in Böhm, 2017).

Proposition 3.1. *A unique globally perfect predictor ψ^* exists if and only if the price law is globally invertible for each level of aggregate wealth.*

Corollary 3.1. *Let the assumptions of the benchmark case hold and assume that the aggregate supply function is globally invertible. The unique globally perfect predictor is given by*

$$p^e = \psi^*(W, p) := pAS^{-1}(D(W)) = pAS^{-1}\left(\frac{M/p + (1+r)B/p + g}{\tilde{c}}\right) \tag{3.9}$$

As a consequence, one obtains the *three-dimensional* dynamical system under perfect foresight

$$\begin{aligned} M_t &= \mathcal{M}(M_{t-1}, B_{t-1}, p^e_{t-1,t}) \\ B_t &= \mathcal{B}(M_{t-1}, B_{t-1}, p^e_{t-1,t}) \\ p^e_{t,t+1} &= \psi^*(M_t + (1+r)B_t, p^e_{t-1,t}). \end{aligned} \tag{3.10}$$

which satisfies

$$p_t = \mathcal{P}(\mathcal{M}(M_{t-1}, p^e_{t-1,t}), \mathcal{B}(M_{t-1}, B_{t-1}, p^e_{t-1,t}), \psi^*(M_t + (1+r)B_t, p^e_{t-1,t})) = p^e_{t-1,t} \tag{3.11}$$

for all t by construction. Moreover, under the bench mark assumption, one obtains the equivalent dynamics under perfect foresight in the *two-dimensional* nominal (W, p)-space

$$\begin{aligned} W_1 &= \frac{(1-c)(1-\tau)}{\tilde{c}}\Big(E_F(\theta^e) + (1+r)(1 - E_F(\theta^e))\Big)(W + pg) \\ p_1 &= pAS^{-1}\left(\frac{W/p + g}{1 - c(1-\tau)}\right) \end{aligned} \tag{3.12}$$

and the *one-dimensional* perfect foresight dynamics in real wealth as

$$w_1 = \mathcal{F}(w) := \tilde{C}\,\frac{\dfrac{w+g}{\tilde{c}}}{AS^{-1}\left(\dfrac{w+g}{\tilde{c}}\right)} \tag{3.13}$$

with

$$\tilde{C} := (1-c)(1-\tau)\Big(E_F(\theta^e) + (1+r)(1 - E_F(\theta^e))\Big). \tag{3.14}$$

The multiplier $\tilde{C}$ satisfies $0 < \tilde{C} < 1$ for $-1 \le r \le 0$, but it may be larger than one for a large enough positive interest rate.

4 Existence and Stability of Balanced Paths

4.1 Dynamics of Real Wealth under Perfect Foresight

The previous analysis shows that, under the specific assumptions of the benchmark case, the dynamics of the three-dimensional monetary system (3.10) has an underlying dynamics of real wealth and expected inflation on the equilibrium set $\mathcal{E}_{\mathcal{P}}$ which is one-dimensional and displayed in Figure 5.

The two systems (3.10) and (3.12) are homogeneous of degree one. For such systems, fixed points are either zero or they consist of continua. For example, for any positive fixed point $(\bar{W}, \bar{p}) \gneqq 0$ of (3.12) there exists a continuum of fixed points consisting of the whole halfline $\lambda(\bar{W}, \bar{p})$, $\lambda > 0$. Moreover, because of the specific form of the second equation, stationarity, i.e. $p = p_1$ is rare, ('non-generic') in the space of policy parameters (g, τ, r). Thus, stationary equilibria with constant values of prices, money balances, debt, and expectations are rare.

Economically, the important orbits in nominal space are the so-called balanced paths, i.e. those along which nominal variables expand or contract at a constant rate while real allocations are constant. Such balanced paths correspond to nonzero fixed points of the system (3.13). Their properties are given in the next theorem .

Theorem 4.1. *Let the benchmark case hold and the interest rate be fixed. Assume that the aggregate supply function AS defined in* (2.4) *is differentiable, strictly downward sloping, strictly convex, and globally invertible.*

(a) There exists a unique globally defined predictor $\psi^* \equiv \mathcal{P}^{-1}$.

(b) The time-one map of effective real wealth under perfect foresight $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+$, $w_1 = \mathcal{F}(w) \equiv \mathcal{F}(w + g)$ *defined in* (3.13) *is a strictly convex monotonically increasing function with* $\mathcal{F}(0 + g) > 0$ *if* $g > 0$. *All perfect foresight dynamics are monotonic.*

(c) For $g = 0$*:*

  a. $\mathcal{F}(\cdot + 0)$ *satisfies the convex Inada conditions,*

  b. $\mathcal{F}$ *has two steady states* $\bar{w}_0 = 0 < \bar{w}_3$ *with* $\theta_3 := AS^{-1}(w_3/\tilde{c}) < 1$.

(d) There exists a unique $g^\star > 0$ *with* $0 < \mathcal{F}(w^\star + g^\star) = w^\star$ *and associated inflation factor* $0 < \theta^\star := AS^{-1}((w^\star + g^\star)/\tilde{c})$ *such that* $\mathcal{F}$ *has no fixed points for* $g > g^\star$.

(e) For each $0 < g < g^\star$*:*

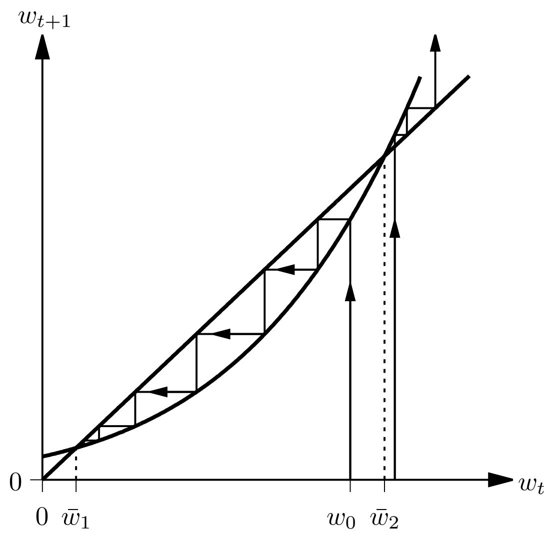
  a. there exist exactly two positive fixed points $0 < \bar{w}_1 < w^\star < \bar{w}_2 < \bar{w}_3$ *such that*

  b. $\bar{w}_1$ *is asymptotically stable with basin of attraction* $[0, \bar{w}_2)$ *while* $\bar{w}_2$ *is unstable,*

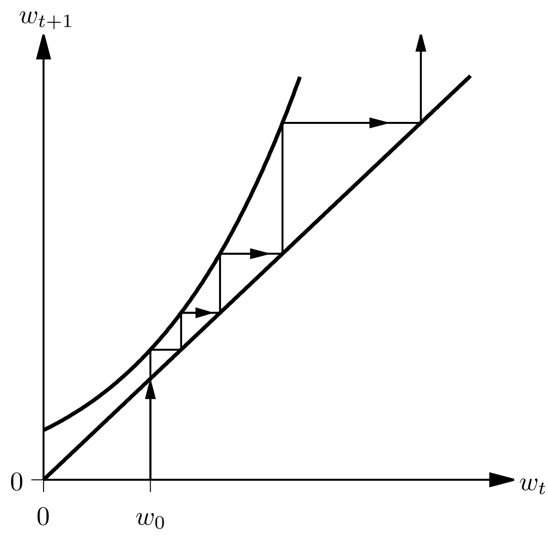
  c. the associated balanced inflation factors satisfy $\theta_1 > \theta^\star > \theta_2 > \theta_3$.

(f) There exists a positive interest rate $r_3 > 0$, *such that for all* $-1 < r < r_3$ *and* $g > 0$ *small, the associated steady state satisfies* $0 < \bar{w}(g) < \bar{w}_3$ *with* $\theta_3 < \theta(g) < 1$.

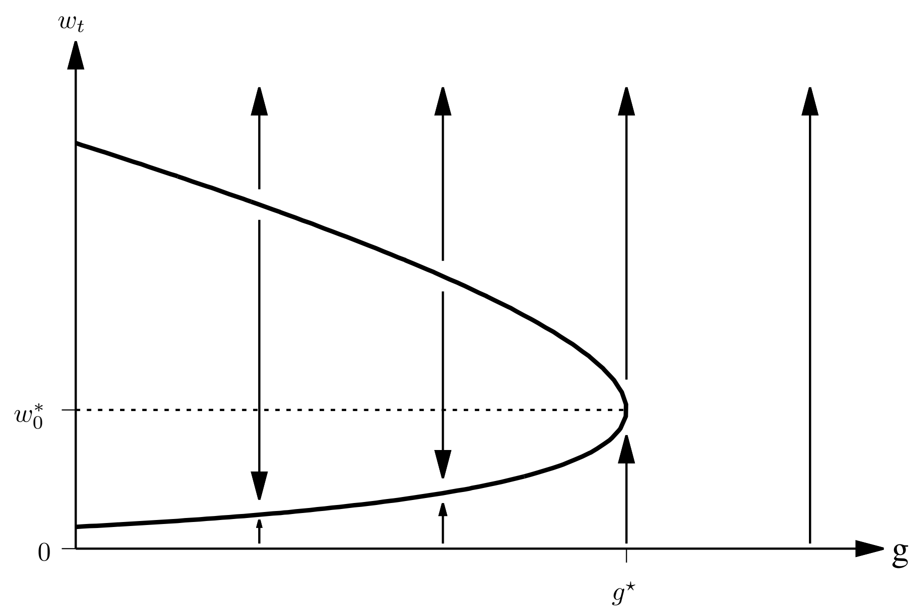
Proof. In the benchmark case, perfect foresight dynamics are driven exclusively by the properties of the aggregate supply function AS. The monotonicity of AS implies that $\mathcal{F}$ is strictly monotonically increasing which excludes cycles. The standard assumptions guarantee global invertibility and monotonicity of the aggregate supply function. With $\tilde{c}$ and $\tilde{C}$ being constant $\mathcal{F}$ is strictly monotonically increasing and convex with additive left shift in g for every constant



(a) Two balanced paths: $0 < g < g^\star$



(b) No positive balanced path: $g > g^\star$



(c) Fixed points $\bar{w}_1(g) < \bar{w}_2(g)$ and stability

Figure 4: Existence and stability of balanced paths: the role of government demand

interest rate $r > 0$. Thus, there are never more than two steady states, see Appendix A.1 for details. □

Figures 4 (a)-(c) illustrate the results of the theorem. Figure 5 provides a description of the dynamics of real wealth induced on the equilibrium set.

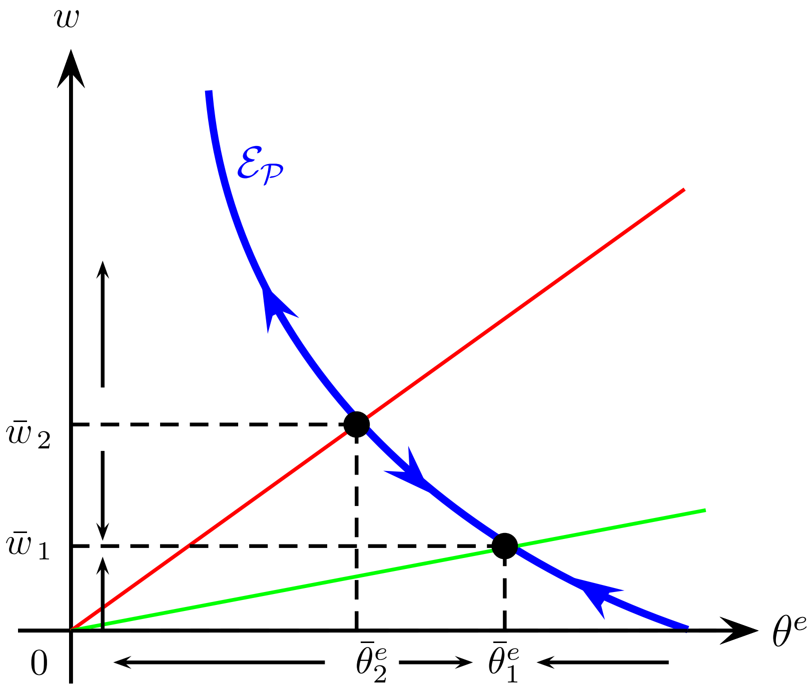


Figure 5: Real dynamics on equilibrium set with two balanced paths: $0 < g < g^\star$

4.2 Properties of Balanced Paths

The existence of the critical level $g^\star$ is one of the major structural features of the model. Government demand $g > g^\star$ implies divergence with perfect foresight in real and monetary terms along unbalanced orbits, i.e. large government consumption makes *balanced expansion* in nominal units impossible with $\lim \theta_t = 0$ and output and real wealth becoming unbounded.

- In general, the inflation factor $\theta^\star$ associated with the critical level of government demand $g^\star$ can be larger or smaller than one. The value depends *jointly* on the parameters of production and of demand. Therefore, $\theta_1 \neq 1, \theta_2 \neq 1$, so that stationary balanced paths are non-generic (and non-hyperbolic). Therefore, stable fixed points of (3.13) will have inflation ($\theta > 1$) as well as deflation ($\theta < 1$) at the given interest rate.
- With a fixed interest rate no cyclical movements are possible in the benchmark case. Cycles occur, as in the monetary case, when perfect predictors are no longer monotonic or unique, i.e. when aggregate supply is no longer globally invertible, when there are expectations effects in demand, or from income distribution under consumption heterogeneity.

One unexpected general property of monetary economies of the AS-AD type is the fact that they typically do not have stationary equilibria, i.e. equilibrium allocations with constant prices (zero inflation), constant money balances, and constant government debt do not exist in general. To verify this feature one observes that constant nominal wealth (equation 1 of (3.12)) defines a unique level of real wealth equal to

$$w = \frac{\tilde{C}}{\tilde{c} - \tilde{C}} g \tag{4.1}$$

while constant prices (equation 2 of (3.12)) requires a level of real wealth equal to

$$w = \tilde{c} AS(1) - g. \tag{4.2}$$

These show that constant real wealth requires restrictions from the demand side while constant prices stipulate conditions determined primarily from the supply side. Equality cannot hold simultaneously in general since their right hand sides are determined by lists of different independent parameters. Thus, stationary equilibria do not exist on open sets in the space of parameters of the economy, i.e. stationary solutions of the system (3.12) are rare, they do not exist generically.

From the homogeneity of the mapping (3.12) it follows that if a stationary equilibrium $(W, p) \gg 0$ with perfect foresight exists, nominal levels of prices, money balances, and of government debt are not uniquely defined, since for all $\lambda > 0$, $(\lambda W, \lambda p)$ are stationary equilibria as well. In other words, in such cases there exists a continuum of steady states, each of which is nonhyperbolic with at least one unit eigenvalue. Therefore, for generic values of the parameters of the economy, balanced paths have either positive or negative inflation, i.e. they are expanding or contracting in nominal terms.

The conditions of balanced expansion also reveal that in the benchmark case the debt-to-GDP ratio is bounded above. It measures the length of time (number of periods) required to repay the debt in full using a repeated payment equal to all of GDP in each period, like an annuity. Thus, its dimension is time. Let

$$w = m + (1+r)b = \tilde{C}\frac{(w+g)/\tilde{c}}{AS^{-1}((w+g)/\tilde{c})} \tag{4.3}$$

denote the steady state level of aggregate real wealth along a balanced path and $\theta = AS^{-1}((w+g)/\tilde{c})$ the associated inflation factor. Let $r \geq 0$. Then, one has the following inequalities

$$\frac{b}{y} < \frac{(1+r)b}{y} < \frac{m+(1+r)b}{y} = \frac{w}{y} = \frac{\tilde{C}}{\theta} = \frac{(1-c)(1-\tau)(1+r(1-E_F)))}{\theta} < \frac{1+r}{\theta}. \tag{4.4}$$

The ratio $\tilde{C}/\theta$ is the effective binding upper bound which is determined by a list of parameters and the endogenous level of balanced inflation. In other words, balanced monetary expansion with positive inflation cannot have a debt-to-GDP ratio larger than one. With a positive interest rate and positive inflation, the number of time periods is never more than $1+r$. Along all stable orbits the debt-to-GDP ratios will be bounded by $(1+r)/\theta$ which will be less than $(1+r)$ if $\theta_2 \geq 1$.

Surely, it may be larger than one under deflation. More subtle and varying estimates are obtained when the interest rate is negative. Under different time objectives, other criteria of sustainability (like debt-to-revenue, debt-service-to-revenue) could be considered from a policy point of view which are readily computable with their respective upper bounds for the benchmark case.

For most policy triples (g, τ, r), the government deficit is not zero along a balanced path associated with a steady state w. In other words, in steady states with perfect foresight the budget deficit grows or shrinks at the same rate as all other nominal variables. The deficit is zero if and only if

$$rb + g = \tau AS(\theta) \qquad \text{where} \qquad \theta = \frac{\tilde{C}}{w}\frac{w+g}{\tilde{c}}. \tag{4.5}$$

The condition shows that there are tradeoffs between the three policy parameters (g, τ, r) to influence the government deficit through output, debt, and inflation.

The main comparative dynamics properties with respect to government demand g are essentially given by the statements of the theorem. Those with respect to the tax rate τ can be derived using standard techniques. Their effects imply in general nonlinear possibly nonmonotonic changes since τ enters in a multiplicative way in the multiplier and in the aggregate supply function.

In the benchmark case the interest rate enters in a linear way only into the dynamic multiplier $\tilde{C}$ creating a clear positive wealth effect on demand and savings. Therefore, one obtains clear comparative dynamics properties.

- An *increase* of the interest rate induces a pointwise upward shift of the mapping $\mathcal{F}$ since $\partial\mathcal{F}/\partial r > 0$ *increasing* the lower steady state value of real wealth $\bar{w}_1$, of output and employment, and *decreasing* steady state inflation θ_1, and decreasing $\bar{w}_2$.
- A higher interest rate induces a lower $g^\star$, the critical level of government demand through the slope $\mathcal{F}'(w^\star + g^\star)$ (see Appendix A).
- An interest rate equal to zero (making the model equivalent to the money only situation) has no apparent qualitatively distinguishing properties. It is unclear whether high positive interest rates imply additional instabilities which do not occur for $r = 0$.
- Negative interest rates, in particular $r \to -1$ does not destroy the convexity or monotonicity of the mapping (3.13) in the benchmark case.

When consumption propensities are no longer constant the effects of changes of the interest rates are no longer clear cut. A negative effect on balanced levels of output and employment may in fact appear.

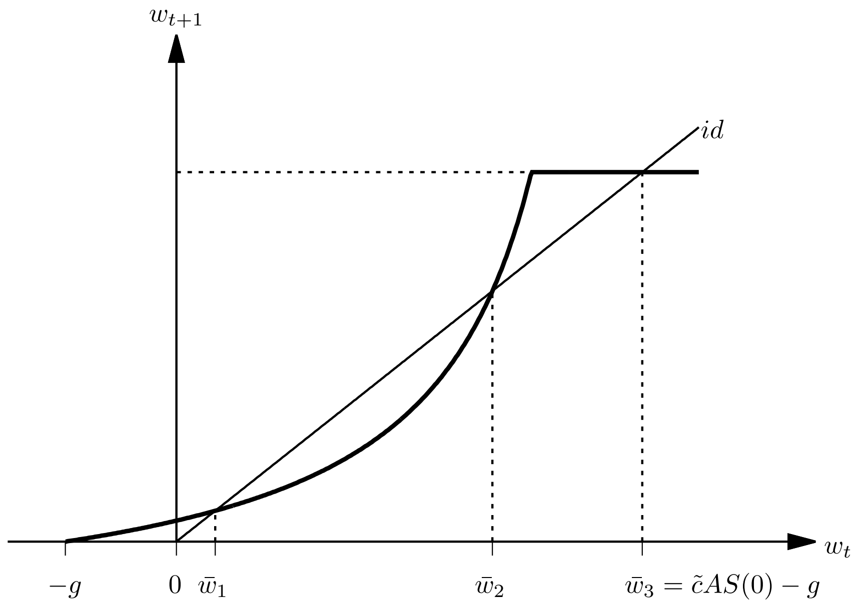


Figure 6: Time one map (3.13) with perfect foresight and bounded production: $g < g^\star$

With bounded production/labor supply the aggregate supply function (and, therefore, the price law) is no longer globally invertible and the equilibrium set must be bounded. Choosing a perfect predictor as in (3.8) implies a bounded 'maximal' time one map with perfect foresight dynamics, as shown in Figure 6. In this case, three balanced paths appear for which the principle features of the two interior steady states remain unchanged as stated in the theorem.

4.3 Dynamics in State Space and Stability of Balanced Paths

It is well-known that for two dimensional systems defined by maps which are homogeneous of degree one convergence in intensity form is only a necessary condition for convergence in the state space (see Deardorff, 1970; Böhm, Pampel & Wenzelburger, 2005; Pampel, 2009)[3]. In other words, convergence of an orbit $\gamma(w_0)$ in real wealth for the mapping (3.13) to the level $0 < \bar{w} \neq w_0$ does not imply the convergence of an associated orbit $\gamma(p_0, W_0) = \{(p_t, W_t)\}_{t=0}^{\infty}$, $w_0 = W_0/p_0$ of the system (3.12) to the half line defined by $\bar{w}$.

To examine the convergence define the distance between any state (p, W) from the half line by $\Delta((p, W), \bar{w}) := W - \bar{w}p$. Hence, for an orbit $\gamma(p_0, W_0)$, at any $t \geq 0$, the distance of the state (p_t, W_t) from the half line is given by

$$\Delta_t := W_t - \bar{w}p_t = (w_t - \bar{w})p_t. \tag{4.6}$$

Applying the same formula for $t + 1$ one obtains

$$\Delta_{t+1} = (w_{t+1} - \bar{w})p_{t+1} = AS^{-1}\left(D(w_t)\right) \frac{\mathcal{F}(w_t) - \bar{w}}{w_t - \bar{w}} \Delta_t \tag{4.7}$$

which is a function of (w_t, Δ_t). Therefore, along any orbit in the state space the sequence of distances is well defined. Moreover, $\lim_{t\to\infty} \Delta_t = 0$ implies that the orbit converges pointwise to the half line when $\lim_{t\to\infty} w_t = \bar{w}$. In this case we will say that the balanced path is asymptotically stable.

Figure 7 displays qualitatively the two possible scenarios: convergence (green) vs. divergence (red) in the state space $(p, W) \in \mathbb{R}^2_+$ as defined above analytically. It is assumed that real wealth decrease, i.e. $w_t > w_{t+1}$, where $(W_t, p_t) \mapsto (W_{t+1}, p_{t+1})$ portrays the diverging case with an associated increase of $\Delta_t < \Delta_{t+1}$ while $(W_t, p_t) \mapsto (W_{t+1}, p_{t+1})$ shows the converging case with a decrease of $\Delta_t > \Delta_{t+1}$. The reason for the convergence/divergence in the state space $\mathbb{R}^2_+$ for the *same* convergence scenario in real wealth arises from the fact that the inflation factor in the convergent case is substantially smaller than the one in the divergence case. Thus, convergence or divergence depends on the rates of expansion, as can also be seen directly from the analytical result presented in Theorem 4.2.

Since the mapping $\mathcal{F}$ is monotonically increasing, the sequence $\{\Delta_t\}$ is either positive or negative for all t depending on whether the initial $\Delta_0 := W_0 - \bar{w}p_0$ is positive or negative. Combining (4.7) with (3.13) one obtains a two-dimensional dynamical system

$$\begin{aligned} w_{t+1} &= \mathcal{F}(w_t) \\ \Delta_{t+1} &= AS^{-1}(D(w_t)) \frac{\mathcal{F}(w_t) - \bar{w}}{w_t - \bar{w}} \Delta_t \end{aligned} \tag{4.8}$$

whose fixed points are $(\bar{w}, 0)$ with $\bar{w} = \mathcal{F}(\bar{w})$. As eigenvalues one obtains

$$\nu_1 = \mathcal{F}'(\bar{w}) \quad \text{and} \quad \nu_2 = AS^{-1}(D(\bar{w}))\mathcal{F}'(\bar{w}), \tag{4.9}$$

which are both positive. Therefore, the fixed point $(\bar{w}_i, 0)$ is asymptotically stable if and only if

$$\max\left(\mathcal{F}'(\bar{w}_i)\,,\, AS^{-1}(D(\bar{w}_i))\mathcal{F}'(\bar{w}_i)\right) < 1, \qquad i = 1, 2. \tag{4.10}$$

[3]Common usage and folklore in macroeconomics often assert incorrect conclusions about the convergence in state space from the stability in intensive form. Most contributions in growth theory ignore the issue altogether.

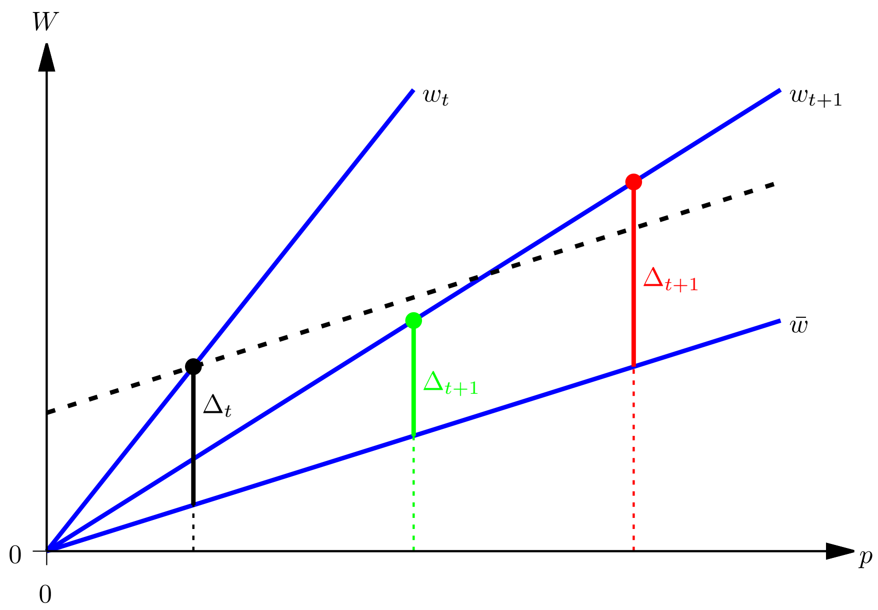


Figure 7: Stability of balanced paths: convergence vs. divergence in state space

Therefore, for the two possible fixed points $0 < \bar{w}_1 < \bar{w}_2$ of $\mathcal{F}$, the higher one $(\bar{w}_2, 0)$ is unstable since $\mathcal{F}'(\bar{w}_2) > 1$. It is either a saddle or a source of (4.8). The lower one $(\bar{w}_1, 0)$ is either a saddle or a sink depending on the parameters of the aggregate demand function and the aggregate supply function. Orbits are monotonic, cycles do not appear.

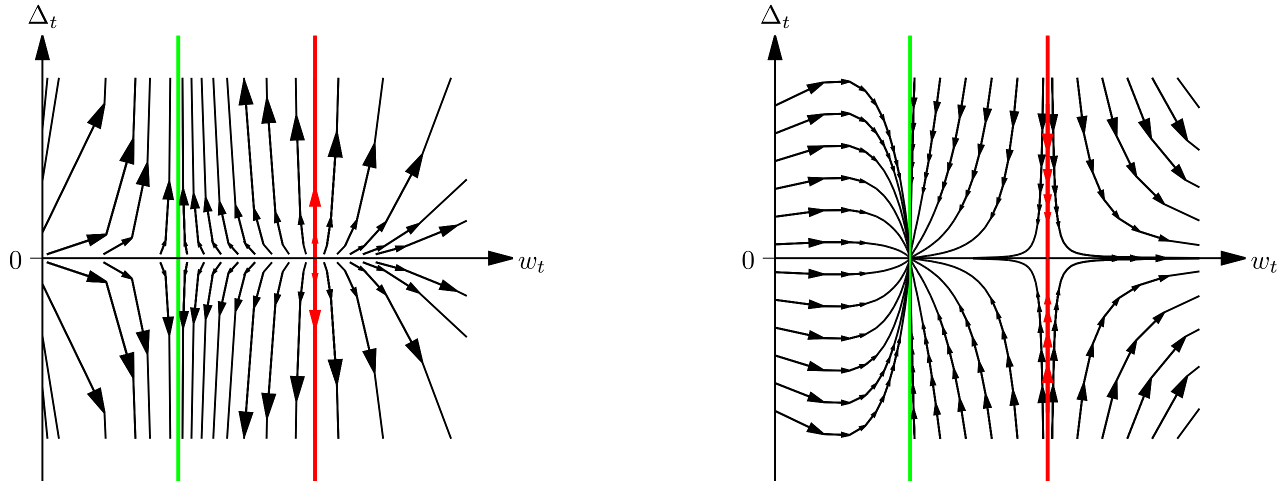
Using the features of the asymptotic convergence properties of the augmented mapping (4.8) one can show that the distance from the balanced path either converges to zero or becomes unbounded with $\lim_{t\to\infty} |\Delta_t| = \infty$, as stated in the following theorem.

Theorem 4.2.
Let $\mathcal{F}$ be differentiable and let $\bar{w} > 0$ be an asymptotically stable fixed point of (3.13) *and $w_0 \in \mathcal{B}(\bar{w})$ the basin of attraction of $\bar{w}$. Let $\gamma(p_0, W_0)$ be an orbit of* (3.12) *with $w_0 := W_0/p_0 \neq \bar{w}$, $\Delta_0 := W_0 - \bar{w}p_0 \neq 0$, and $\gamma(w_0, \Delta_0)$ be the associated orbit of* (4.8)*. Then, the following result holds:*

$$\text{If} \quad \mathcal{F}'(\bar{w})\, AS^{-1}(D(\bar{w})) > 1, \quad \text{then} \quad \lim_{t\to\infty} |\Delta_t| = \infty. \tag{4.11}$$

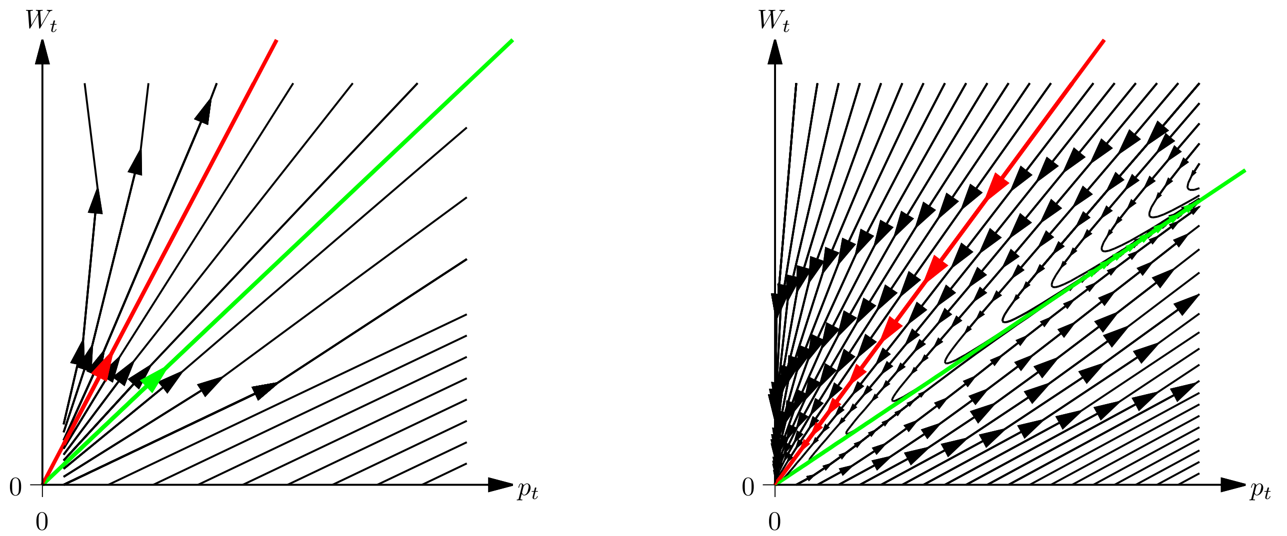
$$\text{If} \quad \mathcal{F}'(\bar{w})\, AS^{-1}(D(\bar{w})) < 1, \quad \text{then} \quad \lim_{t\to\infty} |\Delta_t| = 0. \tag{4.12}$$

The convergence/divergence condition stipulates that the product $\mathcal{F}'(\bar{w}) \cdot AS^{-1}(D(\bar{w}))$ of two numbers should be less than one, the first being the rate of contraction $\mathcal{F}'(\bar{w})$ of the real system (3.13) and the second one $AS^{-1}(D(\bar{w})) = \bar{\theta}$ being the rate of inflation along the balanced path (see Appendix A.2 for a proof). Thus, stability of an inflationary balanced path can be assured if the rate of real contraction $\mathcal{F}'(\bar{w})$ is sufficiently small to balance off a large inflation factor $\bar{\theta} > 1$. It is clear that the numerical values of the two multipliers are interrelated, both being determined by features of the supply side *and* the demand side. In a parametric version of the model their tradeoffs can be directly calculated as the unit contour of the product of $\mathcal{F}'(\bar{w}) \cdot AS^{-1}(D(\bar{w}))$ in the respective parameter space (see Böhm, 2017, for the isoelastic case). Figure 8 displays the dynamic behavior for the two possible cases of either two unstable or one stable and one unstable fixed point of the system (4.8). Figure 9 shows the implications in either case for convergence and divergence in the state space.



(a) Two unstable balanced paths (b) One stable – one unstable path

Figure 8: Stability of balanced paths in (Δ, w)-space



(a) Two unstable balanced paths (b) One stable – one unstable path

Figure 9: Stability of balanced paths in state space

5 Summary, Extensions, and Conclusions

The Benchmark Case

For each list of policy parameters (g, τ, r), in the benchmark case the perfect foresight dynamics is well defined and it exhibits no cycles. It may be monotonically diverging making balanced expansion impossible or asymptotically unstable. Specifically:

- The convexity and invertibility of the aggregate supply function AS, i.e. features of the supply side of the economy are responsible for the existence of dynamics under perfect foresight. They induce the global properties of the time one map of aggregate real wealth. Features of the demand side or the interest rate are not interfering.
- The monotonicity and the convexity of the time-one map induce the existence of two or no positive balanced paths for almost all pairs of parameters.
- With two balanced paths, the one with the lower level of aggregate effective wealth and lower output and employment may be stable under perfect foresight dynamics.
- For each (τ, r) there exists a critical level of government demand $g^\star(\tau, r)$ such that balanced positive expansion is impossible if $g > g^\star(\tau, r)$. The parametric tradeoffs between the parameters determining the critical level show the possibilities for the tradeoffs between fiscal policies (g, τ) and monetary policy given by an interest rate $r > -1$ to guarantee existence and stability of balanced expansion.
- There exists a threshold of wealth $\bar{w}_2(g, \tau, r)$ below which stable perfect monotonic dynamics under perfect foresight occur whose orbits induce GDP-to-Debt ratios which are bounded above by the interest factor $1 + r$. For positive interest rates these ratios are less than one period along balanced paths under positive inflation. More tight boundaries relative to the rate of inflation can be determined using more detailed properties of the parameters in the benchmark case.

Beyond the Benchmark

There exist three structural sources for qualitative changes of the monotonic deterministic dynamics under perfect foresight.

- If the technology is no longer isoelastic there appear effects on the dynamics from the functional income distribution. A non-invertibility of the aggregate supply function AS may induce zero or negative expectations effects for the price law, implying local possibilities for non-unique perfect prediction and cycles.
- Under more general assumptions from the demand side, for example more consumer diversity and/or expectations effects, the demand multiplier may become strongly dependent on expectations. In this case, the existence of unique globally perfect predictors is at stake and non-perfect adaptive predictors or a selection among multiple locally perfect ones will have to be used. In either case, the monotonicity or stability of orbits have to be reexamined.
- The assumption of group specific asset demand (workers save in money while share holders save in debt) was chosen to keep the dimension of the dynamical system of effective real wealth to one. Its generalization seems most desirable from an economic point of view

leaving the mathematical condition (the invertibility of the price law) for the existence of globally perfect predictors unchanged. Nevertheless, the dynamical system under perfect foresight would become two-dimensional making the analysis more complex requiring more demanding economic conditions with fewer intuitive results.

A Proofs

A.1 Proof of Theorem 4.1

(a) Proposition 3.1 and Corollary 3.1 imply that the predictor is given by

$$\psi^\star(W,p) := pAS^{-1}\left(\frac{W/p+g}{\tilde{c}}\right) \tag{A.1}$$

for $p^e_{-1} \equiv p$.

(b) The time-one map of real wealth under perfect foresight is given as in (3.13) by

$$w_1 = \mathcal{F}(w) := \tilde{C}\,\frac{\dfrac{w+g}{\tilde{c}}}{AS^{-1}\left(\dfrac{w+g}{\tilde{c}}\right)} \tag{A.2}$$

with

$$\tilde{C} = (1-c)(1-\tau)\Big(E_F(\theta^e) + (1+r)(1-E_F(\theta^e))\Big). \tag{A.3}$$

The multiplier $\tilde{C}$ satisfies $0 < \tilde{C} < 1$ for $-1 \leq r \leq 0$, but it may be larger than one for a large enough positive interest rate. Since the aggregate supply function AS is differentiable, strictly downward sloping, strictly convex, and globally invertible, its inverse AS^{-1} has the same properties. Therefore, $\mathcal{F}$ is strictly increasing and strictly convex which excludes cycles. The parameter $g \geq$ induces an additive left-shift such that $\lim_{w\to -g}\mathcal{F}(w) = 0$ for each $g \geq 0$.

(c) Let $y := w + g \geq 0$ and define $F : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$F(y) := \tilde{C}\frac{y/\tilde{c}}{AS^{-1}\,(y/\tilde{c})}. \tag{A.4}$$

The function $F(y)/y$ is strictly increasing and

$$\lim_{y\to 0}\frac{F(y)}{y} = 0 \qquad \text{and} \qquad \lim_{y\to\infty}\frac{\mathcal{F}(y)}{y} = \infty. \tag{A.5}$$

Thus, there exists a unique positive fixed point $0 < \bar{w}_3 = F(\bar{w}_3)$ and $0 < y^* < \bar{w}_3$ satisfying

$$0 = F(0) < y^* < \bar{w}_3 = F(\bar{w}_3) \qquad \text{with} \qquad F'(y^*) = 1 < F'(\bar{w}_3). \tag{A.6}$$

(d) Define $0 < g^* = y^* - F(y^*)$ and $0 < w^* = F(y^*)$, i.e. $w^* + g^* = y^*$. Therefore, w^* is a fixed point of $\mathcal{F}$

$$w^* = \mathcal{F}(w^* + g^*) \tag{A.7}$$

at g^*. It is the unique global minimizer of the strictly convex function $\mathcal{F}(w + g^*) - w$ since $\mathcal{F}'(w^* + g^*) = F'(y^*) = 1$. As a consequence, for every $\epsilon > 0$,

$$\mathcal{F}(w + g^* + \epsilon) > w \qquad \text{for all} \qquad w \geq 0 \tag{A.8}$$

holds. Thus, for $g > g^*$, one finds $\mathcal{F}(w + g) > w$ for all $w \geq 0$.

(e) For $0 < g < g^*$, the strict convexity of F implies that there exist two positive fixed points $\bar{w}_1 < \bar{w}_2$

$$0 < \bar{w}_1 = \mathcal{F}(\bar{w}_1) < \bar{w}_2 = \mathcal{F}(\bar{w}_2) < \bar{w}_3, \tag{A.9}$$

so that

$$\begin{aligned} \theta_1 = AS^{-1}\left(\frac{\bar{w}_1 + g}{\tilde{c}}\right) &> \theta^* = AS^{-1}\left(\frac{w^* + g^*}{\tilde{c}}\right) \\ > \theta_2 = AS^{-1}\left(\frac{\bar{w}_2 + g}{\tilde{c}}\right) &> \theta_3 = AS^{-1}\left(\frac{\bar{w}_3}{\tilde{c}}\right). \end{aligned} \tag{A.10}$$

(f) Consider the fixed point $\bar{w}_3$ for $g = 0$ with $\theta_3 = AS^{-1}(\bar{w}_3/\tilde{c})$, i.e.

$$\bar{w}_3 = \tilde{C}\frac{\bar{w}_3/\tilde{c}}{AS^{-1}(\bar{w}_3/\tilde{c})}. \tag{A.11}$$

Then,

$$\begin{aligned} \theta_3 = AS^{-1}(\bar{w}_3/\tilde{c}) = \frac{\tilde{C}}{\tilde{c}} &= \frac{(1-c)(1-\tau)}{1-c(1-\tau)}\Big(E_F(\theta_3) + (1+r)(1 - E_F(\theta_3))\Big) \\ &= \frac{(1-c)(1-\tau)}{1-c(1-\tau)}\Big(1 + r(1 - E_F(\theta_3))\Big) \\ &< 1 + r(1 - E_F(\theta_3)). \end{aligned} \tag{A.12}$$

There exists $r_3 > 0$ so that $\theta_3 < 1$ for all $-1 < r < r_3$. From continuity it follows that for such r there exists $g > 0$ small such that the larger fixed point $\bar{w}_2 = \mathcal{F}(\bar{w}_2 + g)$ has an associated inflation factor $\theta_3 < \theta_2 < 1$.

(g) The comparative statics results with respect to the interest r for $0 < g < g^*(\tau, r)$ are

$$\frac{\partial y^*}{\partial r} < 0 \quad \text{and} \quad \frac{\partial g^*}{\partial r} < 0 \quad \text{since} \quad \frac{\partial \tilde{C}}{\partial r} > 0 \tag{A.13}$$

implying for the displacements of the two balanced paths

$$\frac{\partial \bar{w}_2}{\partial r} < 0 < \frac{\partial \bar{w}_1}{\partial r}. \tag{A.14}$$

□

A.2 Proof of Theorem 4.2

Proof. Let $w_0 > 0$ and $w_0 \neq \bar{w}$. Then

$$\Delta_{t+1} = W_{t+1} - \bar{w} p_{t+1} = AS^{-1} D(w_t) \frac{\mathcal{F}(w_t) - \bar{w}}{w_t - \bar{w}} \Delta_t \qquad \forall t \in \mathbb{N}$$

implies

$$\frac{\Delta_{t+1}}{\Delta_t} = AS^{-1} D(w_t) \frac{\mathcal{F}(w_t) - \bar{w}}{w_t - \bar{w}}.$$

Since w_t converges to $\bar{w}$ and $w_0 \in \mathcal{B}(\bar{w})$,

$$\lim_{t \to \infty} \frac{\Delta_{t+1}}{\Delta_t} = AS^{-1} (D(\bar{w})) \, \mathcal{F}'(\bar{w}).$$

This implies $|\frac{\Delta_{t+1}}{\Delta_t} - AS^{-1}(D(\bar{w})) \, \mathcal{F}'(\bar{w})| < \epsilon$ for t larger than some t_0. Therefore,

$$[AS^{-1}(D(\bar{w})) \, \mathcal{F}'(\bar{w}) - \epsilon] |\Delta_t| < |\Delta_{t+1}| < [AS^{-1}(D(\bar{w})) \, \mathcal{F}'(\bar{w}) + \epsilon] |\Delta_t|, \quad t \geq t_0,$$

and by induction

$$[AS^{-1}(D(\bar{w})) \, \mathcal{F}'(\bar{w}) - \epsilon]^\tau |\Delta_{\tau + t_0}| < |\Delta_{t + t_0}| < [AS^{-1}(D(\bar{w})) \, \mathcal{F}'(\bar{w}) + \epsilon]^\tau |\Delta_{t_0}|, \quad \tau > 0.$$

Therefore, for ϵ sufficiently small,

$$\mathcal{F}'(\bar{w}) \, AS^{-1}(D(\bar{w})) < 1 \quad \Rightarrow \quad \mathcal{F}'(\bar{w}) \, AS^{-1}(D(\bar{w})) + \epsilon < 1$$

so that $\lim_{t \to \infty} \Delta_t = 0$. Conversely,

$$\mathcal{F}'(\bar{w}) \, AS^{-1}(D(\bar{w})) > 1 \quad \Rightarrow \quad \mathcal{F}'(\bar{w}) \, AS^{-1}(D(\bar{w})) - \epsilon > 1$$

so that $\lim_{t \to \infty} |\Delta_t| = \infty$. □

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