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The Role of Inequality and Housing Subsidies

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Residential Segregation: The Role of Inequality and Housing Subsidies

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Abstract

Residential segregation is a key public policy issue that is driven by economic factors on the one side, and individual attitudes towards ethnic diversity on the other side. We assume a modeling framework that consists of a population of two ethnic groups, a rental market for each neighborhood, and household's utility which depends on consumption and housing. Accounting for income disparities and heterogeneous preferences for living in ethnically diverse neighborhoods, we examine the residential segregation patterns that occur when households make their neighborhood choice by taking economic and diversity related aspects into account. The investigation reveals that ethnic income disparities and heterogeneous preferences are antagonistic forces such that a certain level of income stratification is the price for residential integration. In light of these findings, we discuss to which extent and under which conditions housing subsidy policies can favor residential integration.

Keywords: Residential segregation; Ethnic income disparities; Housing subsidies; Evolutionary dynamics. **JEL codes:** C73, C63, R23, J15, I30.

1 Introduction

Many metropolitan centers in Western Europe and the USA exhibit substantial levels of residential segregation by ethnicity. Extreme segregation can have a negative impact on the economic system and is therefore often considered as a serious threat to social stability. The most important public economic issues related to residential segregation include school quality and accessibility of education (see, e.g, Nechyba, 2003), underachievement and disadvantages in the labor market (see, e.g, Wilson, 1996), difficulties in public health assistance (see, e.g., Baughman, 2004) and inequality in the supply of public goods (see, e.g., Brender, 2005). Residential segregation is thereby not only detrimental to the residents of segregated districts but also to the welfare of the entire metropolitan population as it affects the local short- and long-term growth negatively (e.g. Li et al., 2013). Beside these traditional public economic aspects, a number of related public policy issues have emerged in the last few years, such as a lack of shared language, cultural values and norms. This makes social cohesion and coordination more difficult and the whole idea of a peaceful society with its constitutional and civic liberties at risk (see, e.g., Scheffer, 2000; Pancs and Vriend, 2007).

The problem of residential segregation by ethnicity has attracted scholars' attention and already several decades ago Thomas Schelling pointed out that the driving forces behind persistent segregation are many and include both individual preferences for ethnic diversity and economic aspects (see Schelling, 1969, 1971). Since then, a burgeoning area of research has focused on studying the effect of ethnic factors such as limited levels of tolerance towards other ethnic groups in order to explain the phenomenon of residential segregation. These

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contributions include spacial proximity models (e.g. Zhang, 2004, 2011) and game-theoretical models (e.g. Bischi and Merlone, 2011), and they reveal that even strong preferences for integration at the individual level produce residential segregation patterns at the macro level. This remarkable result is confirmed in many other contributions, among which we mention Pancs and Vriend (2007), where residential segregation is proved to occur even when households have a strict preference for integration, Fagiolo et al. (2007), where segregation is observed even in proximity models in which households interact under general social network structures, and Radi and Gardini (2018), where residential segregation is observed under heterogeneous preferences for ethnic integration.

In general, the dynamical Schelling-type models of segregation do not account for economic aspects that may also drive the decentralized dynamics of neighborhood choices – although the link between ethnic and economic segregation is empirically well documented (e.g. Borjas, 1998; Massey et al., 2009). Concerning economic drivers of residential segregation, there is a large body of literature on the sorting of neighborhoods when individuals differ with respect to their incomes. Empirical evidences suggest that segregation by income has become a common urban pattern in Western societies in the last decades where this increase can be explained by growing inequality (see, e.g., Watson, 2009). The neighborhood sorting is thereby typically on the basis of neighborhood characteristics such as local taxation and the provision of public goods (see, e.g., Watson, 2006; Epple and Platt, 1998). A strand of this literature argues that persistent ethnic income disparities, by themselves, generate segregation by ethnicity as many minority group members, because of their socioeconomic status, cannot afford to live in affluent neighborhoods. At the same time, individuals convert socioeconomic gains into higher quality housing, often by leaving neighborhoods populated by ethnic minorities (see Charles, 2003; Iceland and Wilkes, 2006, for discussions of the so-called spatial assimilation theory). According to this intuition, however, the level of residential segregation should have decreased with narrowing ethnic income disparities as observed in the last years. But residential segregation by ethnicity remains a striking feature of the urbane landscape such that this intuition does not find confirmation in reality (see, e.g., Reardon et al., 2015).

In light of these theoretical contributions and empirical evidences, we present in this paper a dynamic framework that introduces both, ethnic and economic aspects in the residential choices of households. We use a dynamic evolutionary model to demonstrate how ethnic and economic factors may interact and thereby shape residential outcomes. Our main finding is that, if income and preferences for integration are correlated with ethnicity in a way that members of the on average less affluent ethnic group have higher preferences for diverse neighborhoods, then ethnic and economic drivers of segregation are antagonistic forces that either let the more tolerant or the more affluent ethnic group prevail in an initially integrated neighborhood in the long run. But in cases when the Schelling-type dynamics and the economic dynamics driven by ethnic income stratification are well balanced such that they offset each other, then the interaction of the drivers of segregation can give rise to ethnic integration. In such a situation, narrowing ethnic income gaps that weaken the economic driver of segregation can lead to more segregation, which is consistent with the persistence in segregation despite narrowing income disparities observed in recent years. Our second main finding is that income gaps that generate almost perfectly mixed neighborhoods represent tipping points at which small perturbations of the income gap can tip the residential outcome from perfect integration into full segregation and *vica versa*. Our two main findings bear stark policy implications as we show that specific housing subsidies applied to support low-income households have similar effects as narrowing ethnic income disparities. Hence, under certain circumstances, the introduction of housing subsidies can either transform a fully segregated neighborhood into an almost perfectly integrated one, or they decrease the degree of integration and can even lead to full segregation.

Our work is closely related to two streams of the theoretical literature on neighborhood sorting. On the one hand, there is the dynamic literature on residential segregation descended from Schelling (1969, 1971) as outlined above. On the other hand, there is a literature that considers static equilibrium models of neighborhood choice that take account of economic aspects by incorporating market adjustments of rents. Some contributions in this literature highlight the importance of prejudice or discriminatory motives for the emergence of segregated neighborhoods (see, e.g., Anas, 2002; Yinger, 1976; Kern, 1981). However, empirical evidences for the US suggest that preferences for fully segregated neighborhoods have substantially decreased in the last decades, and nowadays the desired neighborhood composition is characterized by some degree of integration with a bias towards members of the own group (see, e.g., Clark, 2009; Charles, 2003; Schuman et al., 1997). Furthermore, after the passage of the Fair Housing Act in 1968 and the Home Mortgage Disclosure Act in 1975 in the US, discriminatory housing practices are unlawful, at least in theory. Hence, the question arises whether prejudice and housing market discrimination can still explain why segregation has not decreased with narrowing ethnic income disparities. Sethi and Somanathan (2004) respond to this critique and propose a static equilibrium framework in which households sort neighborhoods on the basis of their own preferences for the ethnic composition of the district, the price of rents, and the average income of residents. They find that a narrowing ethnic income gap is consistent with extreme and even rising levels of segregation in cities. Our paper confirms this result.

We contribute to the theoretical literature on residential segregation by linking the static equilibrium approach that takes into account economic and ethnic aspects of the neighborhood choice to the dynamic Schelling-type literature. This allows to identify tipping points in the interaction of ethnic and economic divers of segregation. At the same time, the dynamic approach allows us to discuss the stability other than the existence of an equilibrium of integration, thereby highlighting a trade off between the equilibrium-level of integration and the stability of the equilibrium itself. Focusing on the convergence towards the equilibrium, the dynamic approach underlines the fragility of an equilibrium of integration due to the co-existence of an always-stable equilibrium of segregation.

Moreover, we address the dynamic impact of public housing policies on residential segregation, an issue that has not received much attention in the theoretical literature. In fact, there is a tremendous variety of housing policies across different countries but a cornerstone of most of those programs is to provide support for low-income households in the rental market. Some of these housing policies, however, cause residential segregation by default. In the US, this is the case for *supply-side housing assistance programs* which subsidize the construction and operation of housing projects for less affluent households. Given the still persistent correlation between income and ethnicity, this policy might imply to subsidize the emergence of segregated neighborhoods by construction. To overcome this issue, different housing subsidy schemes have been designed in form of the so-called *rental assistance programs* which aim at subsidizing the cost of rentals for low-income households in the private rental market (see, e.g., Anderson et al., 2003; McCarty et al., 2014). The aim of these policies is to assure affordable houses for low-income families without impacting their residential location choices. However, implementing those assistance programs can lead to distortions at the housing market and it is not clear to which extent these distortions affect the residential pattern.

The modeling framework developed to address these issues consists of two populations that make repeated residential decisions and differ with respect to their ethnic traits, preferences for ethnic diversity and mean income. The first population is on average wealthier while the second one is more attracted by ethnic diversity in a residential context. The residential choice of the two household groups depends on the attractiveness of ethnically integrated residential districts for both groups, which is expressed by the payoff achieved by households who are living in the district in relation to the payoff of some outside option. The neighborhood entry and exit decision of households is made on the basis of information about the relative payoff of the neighborhood which has been collected via direct communication within the group. On the one hand, these payoffs take into account the economic utility from spending the income for consumption and housing, where the local housing price is endogenously determined by the average affluence of the neighborhood; on the other hand, they incorporate the level of ethnic diversity where all households have a higher preference for integrated over segregated neighborhoods, which is consistent with empirical observations.

Methodologically, we use a two-stage approach. In a first step, we develop a deterministic evolutionary model of residential segregation represented by a one-dimensional map in discrete time. The underlying replicator-like dynamics is based on the assumption that heterogeneity only occurs across but not within sub populations. This population-based formulation allows us to characterize the behavior of the model analytically and to compute the equilibria numerically. In a second step, we follow an agent-based approach to simulate the same environment but including inequality within sub populations. The comparison of the agent-based approach and the deterministic evolutionary game approach using a common theoretical framework as a basis enables us to explore the long-term residential pattern in a highly stylized environment from an analytical perspective and then to check the robustness of these findings by simulating the agent-based model under more realistic assumptions by, e.g., incorporating a more realistic income distribution. For further discussions on the relationship between evolutionary games and the agent-based approach, see, e.g., Adami et al. (2016); Dawid (2007); Carpenter (2002).

The road map of the paper is as follows. Section 2 introduces the game-theoretical model in a highly stylized environment and discusses the assumptions. Section 3 shows that residential integration occurs when there is a mix of income inequality and heterogeneous individual preferences for integration. Moreover, it focuses on the policy measure introduced to subsidize housing of low-income households and the related implications in terms of residential segregation. Section 4 generalizes the framework by adopting an agent-based approach that includes income stratification within ethnic groups. Simulations of the agent-based model illustrate that the implications obtained from the stylized game-theoretical model are robust. Section 5 concludes. All the proofs are in the Appendix.

2 The Model

Households live in a metropolitan area and can be classified into two groups $i \in \{1, 2\}$ based on ethnicity. Besides these different ethnic backgrounds, we assume that the two groups can be distinguished by income and their propensity towards ethnic diversity in households' residential environment. In particular, the members of the first population are the wealthier ones while the members of the second population are the more tolerant ones toward no-like neighbors. A household sorts at each time $t \in \mathbb{N}$ between a set of segregated neighborhoods, where there are only like-type habitants, and a set of integrated neighborhoods in which the ethnic composition is an endogenous variable that evolves over time. Thus, the model takes the form of a multiple-neighborhood model, where each household has always the possibility to choose to live in segregated neighborhoods with co-ethnic neighbors as a sort of *outside option*. This modeling choice is motivated by the current high levels of residential segregation that makes the outside option realistic and always offers households the possibility to choose the totally segregated solution.

More precisely, we consider a city with an undefined number of neighborhoods and we assume a urban density equilibrium in the allocation of habitants such that each neighborhood has a population normalized to one. Let $n_{1,t}$ denote the number of households of the first group that live in an integrated district at time t and let $n_{2,t}$ denote the number of households of the second group that live in an integrated district at time t. Then, the ethnic composition of an integrated neighborhood is given by $x_t = n_{1,t}/\left(n_{1,t} + n_{2,t}\right)$. The fitness of a household of type i that lives in an integrated neighborhood is $\Pi_i\left(x_t\right)$ while the fitness of the same household that lives in a segregated neighborhood is δ_i.

The evolution of residential patterns is driven by the flow of information generated by a word-of-mouth communication, see, e.g., Bischi et al. (2003). Specifically, at each time $t \in \mathbb{N}$, a household living in an integrated neighborhood samples a like-type household from a segregated neighborhood and they exchange information about their current relative payoffs. A household changes his/her residential location anytime his/her fitness is lower than the one of the sampled household. The relative fitness is affected by idiosyncratic preferences which are measured by a random variable ε_t. It follows that $\mathbb{P}\left(\Pi_i\left(x_t\right) < \delta_i + \varepsilon_t\right)$ is the probability that a household living in an integrated neighborhood changes his/her residential location choice and $\mathbb{P}\left(\Pi_i\left(x_t\right) > \delta_i + \varepsilon_t\right)$ is the probability that a household from a segregated neighborhood moves to an integrated location. Let Θ be the cumulative distribution function of the random variable ε_t, then the number of households of type i that live in an integrated neighborhood evolves according to the following difference equation:

$$n_{i,t+1} = 2\Theta\left(\Pi_i\left(x_t\right) - \delta_i\right) n_{i,t}. \tag{1}$$

In the following we assume that ε_t is a continuous random variable which follows a logistic distribution with zero mean, i.e.

$$\Theta\left(\Pi_i\left(x_t\right)-\delta_i\right)=\frac{1}{1+\exp\left(\beta\left(\delta_i-\Pi_i\left(x_t\right)\right)\right)} \tag{2}$$

where $1/\beta$ is the scale parameter proportional to the standard deviation. Hence, the dynamic Equation (1) indicates that the number of members of population i that live in an integrated district increases whenever living in an integrated neighborhood offers to an agent of type i a higher payoff than living in a segregated neighborhood, while it remains constant when the two payoffs are equal and it decreases otherwise.

Living in a neighborhood implies renting a house, where the price of a housing unit is denoted by p_s. Moreover, ω_i is the budget of an agent of type i, where we assume $\omega_1 \geq \omega_2$. Then at each time $t \in \mathbb{N}$ his/her housing demand, which we denote by s_i, solves the following optimization problem:

$$\begin{aligned}\left(s_i, z_i\right) \quad = \quad & \arg\max_{s,z} U\left(s,z\right)\\ & \text{s. t.}\\ & \quad p_z z + p_s s = \omega_i\\ & \quad s \geq 0;\ z \geq 0\end{aligned} \tag{3}$$

where U is households' utility function, z_i is the demand of the all of the consumption goods at a constant price p_z and the equality constraint represents the budget equation. Assuming the following Cobb-Douglas utility function:

$$U\left(s,z\right)=s^{\alpha}z^{1-\alpha} \tag{4}$$

with $\alpha \in (0,1)$ and solving optimization problem (3), we obtain the Walrasian demand functions for houses and consumption goods, respectively,

$$s_i\left(p_s\right)=\frac{\alpha\omega_i}{p_s} \quad \text{and} \quad z_i\left(p_z\right)=\frac{\left(1-\alpha\right)\omega_i}{p_z} \tag{5}$$

from which we observe that the parameter α measures the fraction of nominal income spent on housing. Note that Cobb-Douglas preferences for consumption and housing are common assumptions in recent macroeconomic models studying housing, see, e.g., Davis and Heathcote (2005), Iacoviello (2005), Kiyotaki et al. (2011) and Davis and Ortalo-Magné (2011). In fact, as revealed by empirical analysis, these assumptions allow to capture the two main striking features of the housing market, which are a constant expenditure share on housing and a certain degree of complementary in utility between consumption goods and houses. In the following, for the sake of computational simplicity, we assume $\alpha = 0.5$. Moreover, we assume that the price of the consumption goods is exogenously given and is normalized to one. On the other hand, the housing price is endogenously determined according to the classical market clearing mechanism and solves the following price equation for each neighborhood:

$$s_1\left(p_s\right)x_t+s_2\left(p_s\right)\left(1-x_t\right)=c \tag{6}$$

where c is the supply of houses. Without loss of generality, it is normalized to 1. It follows that

$$p_s\left(x_t\right)=\frac{\omega_1 x_t+\omega_2\left(1-x_t\right)}{2}. \tag{7}$$

It should be noted that Equation (7) implies that the market price of a house in a district depends on the ethnic composition of the district itself. In particular, the larger is the fraction of households of type 1, the higher is the price that residents pay for a house. This is because agents of type 1 are the more affluent households and therefore their presence implies a larger housing demand that drives up the market price. Given the market clearing price, the demands for houses are

$$s_1\left(p_s\left(x_t\right)\right)=\frac{\omega_1}{\omega_1 x_t+\omega_2\left(1-x_t\right)} \quad \text{and} \quad s_2\left(p_s\left(x_t\right)\right)=\frac{\omega_2}{\omega_1 x_t+\omega_2\left(1-x_t\right)}. \tag{8}$$

From the optimal choices of household i with respect to housing and consumption, one can derive the indirect utility function which is

$$V_i(x_t) = \frac{\omega_i}{\sqrt{2(\omega_1 x_t + \omega_2(1 - x_t))}}. \tag{9}$$

Besides the economic utility, a household cares about the ethnic composition of the neighborhood where preferences for integration are represented by an unimodal function $f_i(x)$. Similarly to Zhang (2004) and Pancs and Vriend (2007), we assume a maximum satisfaction level when the neighborhood is perfectly integrated, i.e. $x = 0.5$. Furthermore, each household has a bias in favor of agents of the same kind, hence $f_1(1),\ f_2(0) > 0$. This bias implies that, although people do not prefer segregated neighborhoods, they feel better if they belong to the majority group rather than the minority group. The particular case of a segregated neighborhood only inhabited by unlike agents is associated with a zero preference, i.e. $f_1(0) = f_2(1) = 0$. Formally, we assume

$$f_1(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ (2-d) - 2(1-d)x & \frac{1}{2} < x \leq 1 \end{cases} \quad \text{and} \quad f_2(x) = bf_1(1-x) \tag{10}$$

where $b \geq 1$ and $d \in [0, 1)$. Here, parameter b measures the relative importance of integration between group 1 and group 2, in particular the condition $b > 1$ indicates that, consistently with empirical evidences, see, e.g., Clark (1991) and Schuman et al. (1997), one group appreciates equally ethnically-mixed neighborhoods more than the other group. At the same time, parameter d measures the bias toward members of one's own group, see e.g. Charles (2003).

The overall fitness of a household generated by living in a certain neighborhood is then given by a combination of economic utility and his/her satisfaction from the ethnic composition in the neighborhood. Let the parameter η measure the relative importance of individual preferences for integration over economic utility, it follows that the fitness measure of an agent of type i is given by

$$\Pi_i(x_t) = V_i(x_t) + \eta f_i(x_t). \tag{11}$$

Notably, the fitness depends only on the ethnic composition of the neighborhood which, consistently with the dynamics of $n_{1,t}$ and $n_{2,t}$ in (1), evolves according to the following population-based evolutionary process:

$$x_{t+1} = \frac{x_t}{x_t + (1 - x_t)\frac{1+\exp(\beta(\delta_1 - \Pi_1(x_t)))}{1+\exp(\beta(\delta_2 - \Pi_2(x_t)))}} \tag{12}$$

where δ_1 is equal to $\Pi_1(1)$ and δ_2 is equal to $\Pi_2(0)$ and the parameter space Φ is given by:

Assumption 1 $\Phi = \{(b, d, \eta, \omega_1, \omega_2, \beta) | \, b \geq 1,\ d \in (0, 1),\ \eta \geq 0,\ \omega_1 \geq \omega_2 \geq 0$ *and* $\beta > 0\}$

Considering the parameter space defined as in Assumption 1, Model (12) is characterized by few general features that are summarized in the following Lemma.

Lemma 1 *Consider the parameter space* Φ *defined in Assumption 1 and define*

$$\Delta\Pi(x) = \delta_1 - \Pi_1(x) - \delta_2 + \Pi_2(x). \tag{13}$$

Then the model in (12) *is such that:*

1. $x_{t+1} > x_t\ \forall x_t \in (0, 1)$ *s.t.* $\Delta\Pi(x_t) < 0$ *and* $x_{t+1} < x_t\ \forall x_t \in (0, 1)$ *s.t.* $\Delta\Pi(x_t) > 0$*;*
2. *The set* $[0, 1]$ *is invariant (*$x_{t+1} \in [0, 1]$ *for whatever* $t \geq 0$ *and* $x_t \in [0, 1]$*);*
3. $x_0^S = 0$ *and* $x_1^S = 1$ *are equilibria representing segregation;*
4. $x^* \in (0, 1)$ *is an equilibrium if and only if* $\Delta\Pi(x^*) = 0$.

It should be noted that Equation (12) is a replicator-type dynamics that is, however, different from the replicator dynamics in the classical sense as put forward in, e.g., Hofbauer and Sigmund (2003). In fact, Equation (12) is the result of decision processes of two separated groups of individuals each of which has two strategies available. Instead, the classical replicator dynamics simulates either the Darwinian selection of species or the selection of strategies by a single population. Despite these differences, both mechanisms have in common that selection is driven by the relative performance of the strategies. This similarity implies that the results that follow for Model (12) would still hold true even if we adopted a replicator dynamics such as the exponential one employed, e.g., in Radi and Gardini (2018) to study segregation patterns in a Schelling-type model. Differently from the modeling setup employed in Radi and Gardini (2018), the dynamic Equation (12) allows a straightforward generalization to the agent-based setup as we will see later.

3 The Dynamic Patterns of Integration and Segregation

3.1 Economic and Ethnic Segregation

The model outlined in the previous section represents a deterministic evolutionary game in discrete time. In this section, we analyze the long-term behavior of this evolutionary game with respect to possible steady state levels of ethnic diversity x. Model (12) indicates that the ethnic composition of a neighborhood evolves over time according to a difference equation on the subspace $[0, 1]$, which is an invariant region. Segregation occurs either when the neighborhood is populated by members of population 1 only, or when the neighborhood is populated by members of population 2 only. These two residential layouts are steady states of the model and are indicated by $x_1^S = 1$ and $x_0^S = 0$, respectively. Consistently with the previous findings, see, e.g., Schelling (1971), Pancs and Vriend (2007) and Bischi and Merlone (2011), integration is extremely fragile and these *equilibria of segregation* are the unique asymptotically stable long run location patterns when there is no income inequality but heterogeneous distributions of tolerances/preferences toward unlike neighbors.

Theorem 1 (Ethnic segregation) *Consider Model* (12) *with* $\eta > 0$ *and* $b > 1$ *and without ethnic income inequality, i.e.* $\omega_1 = \omega_2$*. Then, segregation is the only stable location pattern.*

The residential segregation driven by heterogeneous distribution of tolerances/preferences is denoted *ethnic segregation*. The other form, or dimension, of residential segregation accounted by the model is segregation by income, also known as *economic segregation*, which occurs when wealthy people live in affluent neighborhoods and poor people live in lower-income neighborhoods, see, e.g., Anderson et al. (2003) and Watson (2009). Considering this second dimension of residential segregation only, the model forecasts metropolitan areas divided in wealthy neighborhoods and low-income neighborhoods.

Theorem 2 (Economic segregation) *Consider Model* (12) *with* $\eta = 0$ *and* $\omega_1 > \omega_2$*. Then, an initially integrated neighborhood will be populated by the wealthy population only in the long run.*

Summarizing, excluding income inequalities, the coexistence of different ethnic groups in the same neighborhood is not possible because in the long run the group prevails that records the highest level of satisfaction with the current composition of the neighborhood. Similarly, excluding ethnic preferences, the coexistence of wealthy and poor people is threatened by the inflow of the more affluent people.

3.2 The Combined Effect of Economic and Ethnic Segregation

In Theorem 1 and 2 we pointed out that there exist two different forces potentially driving the long-term dynamics of the model towards residential segregation. On the one hand, there is ethnic segregation that is mainly driven by relative differences in terms of preferences for integration between the two groups; on the other hand, there is economic segregation which is driven by ethnic income differentials. Although this sounds like a definitive condemnation to segregation, the question can be asked whether there is also room for residential integration stemming from the negative correlation between income and preference for integration at the ethnic level. In order to answer this question, we first describe a numerical analysis of subsets of the parameter space. Based on these numerical exercises, in what follows, we characterize the global dynamics of the system graphically by means of bifurcation diagrams, in which the attractors towards which the dynamical system tends to evolve in the long run and the corresponding basins of attraction are depicted as functions of a bifurcation parameter (on the horizontal axis) and initial levels of integration $x_0 \in [0, 1.0]$ (on the vertical axis).

In Figure 1 we show those bifurcation diagrams for each of the four permutations of $b \in \{1.5, 2.0\}$ and $\eta \in \{0.25, 1.0\}$, where b represents the relative importance of integration between group 1 and 2 and η indicates the general weight both types of agents put on integration compared to economic utility. Here we use the ethnic income gap ω_2 with $0 < \omega_2 \leq \omega_1 = 1.0$ as bifurcation parameter, where this parameter can be interpreted as the strength towards economic segregation.

Generally speaking, one can observe that, depending on the initial conditions and parameter choices, either ethnic segregation or integration can be a stable long-term outcome where both, the drivers of ethnic and economic segregation, can have a substantial impact on the long-term dynamics of the model. To be more precise, let us first consider the case in which both types of households put only low weight on their individual preferences for integration, i.e. $\eta = 0.25$ (upper panels of Figure 1). Apparently, there are only mild qualitative differences between $b = 1.5$ and $b = 2.0$ whereas a variation of the income level ω_2 changes the long-term behavior substantially. In particular, a sufficiently high level of ethnic income inequality drives the initial sub population of the less affluent households of type 2 out of the previously integrated neighborhoods. Thus, in scenarios in which the gap between ω_2 and ω_1 is sufficiently large, economic segregation dominates thereby preventing integration in the long run. Intuitively, this occurs because wealthy households of type 1 have higher incentives to enter mixed-neighborhoods that offer lower housing prices compared to those neighborhoods that are entirely populated by their co-ethnic fellows. A higher share of high-income households in turn increases the housing prices of the ethnically diverse neighborhoods making them less attractive for household of type 2, which is consistent with a common gentrification pattern (see, e.g., Helms, 2003; Smith, 1996).

But what happens if the ethnic income gap narrows? Then, one can observe a substantial change in the long-term behavior of the model. For both considered values of b one can identify a threshold value of ω_2, denoted by ω_2^{FB1}, at which the transition from residential segregation to possible long run location patterns of integration can occur. This transition reflects a fold bifurcation at which two new equilibria appear, the one stable and the other unstable. Notably, the critical point for $b = 1.5$ occurs already at a lower level of ω_2 compared to $b = 2$ suggesting that a larger relative preference for integration of type 2 households enhances the force towards ethnic segregation. Nevertheless, in both cases, a further narrowing of ethnic income disparities leads to lower levels of integration such that, at a second threshold level of ω_2, denoted by ω_2^{TB}, a transition from possible patterns of integration back to full segregation can be observed. This transition reflects a transcritical bifurcation at which the unique locally asymptotically stable steady state of integration crosses the unique unstable equilibrium of segregation x_0^S and they exchange their stability. Thus, for sufficiently small ethnic income gaps, ethnic segregation rather than economic segregation is the dominating force leading to a situation in which initially integrated neighborhoods will be only populated by households of type 2 in the long run. Summarizing, the counteracting forces of ethnic and economic segregation are balanced and give rise to ethnic integration for values of ω_2 that lie between the two critical points ω_2^{FB1} and ω_2^{TB}. It should, however, be noted that segregation remains a stable long-run location pattern that originates when initially the ethnic group 2 is a sufficiently small minority in the mixed neighborhood.

We now turn to the case in which both households care more about integration, i.e. we consider $\eta = 1.0$ (lower panels of Figure 1). When households put more emphasis on the individual preferences for integration, the long-term residential patterns are qualitatively different even though the negative effect of a narrowing income gap on the level of integration, which could be observed for $\omega_2 \in \left(\omega_2^{FB1}, \omega_2^{TB}\right)$ in case of $\eta = 0.25$, persists and

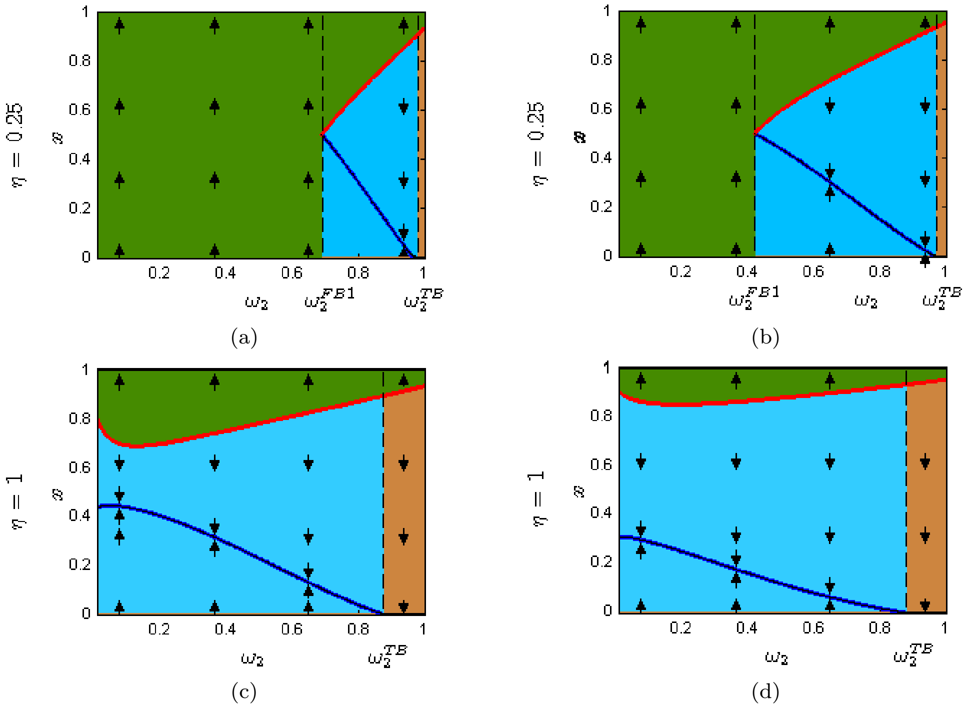


Figure 1: One-dimension bifurcation diagrams with respect to parameter ω_2 (income of a member of population 2). For each value of ω_2 in dark-green is depicted the basin of attraction of the equilibrium of segregation x_1^S, in cyan the basin of attraction of the internal equilibrium x_l^I, i.e. the equilibrium of integration, and in dark-yellow the basin of attraction of the second equilibrium of segregation, i.e. x_0^S. The red curve represents the unstable equilibrium of integration, denoted by x_u^I, while in the blue curve represents the stable internal equilibrium of integration, say x_l^I. The arrows indicate the direction of trajectories. Parameters: $d = 0.05$, $\omega_1 = 1$, $\beta = 1$. Panel (a), $b = 3/2$, $\eta = 0.25$. Panel (b), $b = 2$, $\eta = 0.25$. Panel (c), $b = 3/2$, $\eta = 1$. Panel (d), $b = 2$, $\eta = 1$.

can now even be seen over the full support of ω_2. In fact, we observe that high levels of income inequalities do not necessarily imply segregation since the more weight put on the individual preferences for integration makes the ethnically mixed neighborhood, despite the large income disparities, more attractive for ethnic group 2. With respect to the different values of b, again, the global dynamics are qualitatively robust where a higher b seems to strengthen the effect of ethnic segregation.

It should be noted that the dynamics of the model discussed so far is only valid for specific parameter settings. To address the complete behavior of the model given any possible parameter configuration we relay on analytical results. In particular, in order to classify all the possible qualitative dynamics of the Model (12) under Assumption 1, we start with the following Lemma (see Proof in Appendix) which describes the main fundamental features of Model (12).

Lemma 2 *Consider the values of the parameters as in Assumption 1 with the exception that $\eta > 0$, $b > 1$ and $\omega_1 > \omega_2$. Furthermore, let $\mathcal{B}(\cdot)$ denote the basin of attraction of an equilibrium, let $\Delta\Pi(x)$ be as in Equation (13) and define:*

$$\omega_2^{TB} = \frac{\omega_1^2}{\left(\sqrt{\omega_1} + \eta d\sqrt{2}\right)^2}. \tag{14}$$

Then, Model (12) has the following properties:

1. *Equilibrium x_0^S is locally asymptotically stable $\forall \omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$ and unstable $\forall \omega_2 \in \left(0, \omega_2^{TB}\right)$;*

2. *Equilibrium x_1^S is locally asymptotically stable;*

3. *There are at most two equilibria in $(0,1)$, which are $x_l^I \in (0, 0.5]$ and $x_u^I \geq x_l^I$;*

4. $\forall \omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$ x_u^I *is the unique equilibrium in* $(0,1)$ *and equilibria* x_0^S *and* x_1^S *are both locally asymptotically stable with* $\mathcal{B}\left(x_0^S\right) = \left[0, x_u^I\right)$ *and* $\mathcal{B}\left(x_1^S\right) = \left(x_u^I, 1\right]$*;*

5. $\forall \omega_2 \in \left(0, \omega_2^{TB}\right)$ x_1^S *is the unique asymptotically stable equilibrium with* $\mathcal{B}\left(x_1^S\right) = (0,1]$ *otherwise* x_l^I *and* x_u^I *exist and* x_u^I *is unstable.*

Here, it is worth noting that, independently of the ethnic income gap, a previously integrated neighborhood can flip into a segregated one exclusively populated by households of the more affluent group 1. This occurs if the population of type 1 in the neighborhood is already sufficiently large. On the contrary, a previously integrated neighborhood can flip into a segregated one populated by households of the less affluent group 2, but only if the ethnic income gap is sufficiently narrow. Moreover, there is at most one stable steady state of residential integration and in this one households of type 2 are never the minority. In any case, some ethnic income inequality has to be tolerated in order to obtain residential integration between ethnic groups.

Using the main properties of Map (12) enumerated in Lemma 2, it is possible to prove (see proof in Appendix) the following Theorem. It describes all the possible configurations of the dynamics of the model that emerge when varying the ethnic income gap.

Theorem 3 *Consider the values of the parameters as in Assumption 1 with the exception that* $\eta > 0$, $b > 1$ *and* $\omega_1 > \omega_2$. *Furthermore, define the bifurcation values* $\omega_2^{FB1} \in (0, \omega_1)$ *and* $\omega_2^{FB2} \in \left(0, \omega_2^{FB1}\right)$ *at which a fold bifurcation occurs through which the internal equilibria* x_l^I *and* x_u^I *either appear or disappear. Then, for* $b \geq 1/(1-d)$, *one of the following three scenarios occurs:*

- ***Scenario 1:*** *only* ω_2^{FB1} *exists and for* $\omega_2 \in \left(0, \omega_2^{FB1}\right)$, *there are no internal equilibria and we have* $\mathcal{B}\left(x_1^S\right) = (0,1]$. *For* $\omega_2 \in \left(\omega_2^{FB1}, \omega_2^{TB}\right)$, *there are two internal equilibria, say* x_u^I *which is unstable and* x_l^I *which is either locally asymptotically stable or unstable. Moreover, the equilibrium of segregation* x_0^S *is unstable and* x_1^S *is locally asymptotically stable. For* $\omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$, *a unique internal equilibrium exists, say* x_u^I, *and equilibria* x_1^S *and* x_0^S *are both locally asymptotically stable with* $\mathcal{B}\left(x_0^S\right) = \left[0, x_u^I\right)$ *and* $\mathcal{B}\left(x_1^S\right) = \left(x_u^I, 1\right]$.

- ***Scenario 2:*** ω_2^{FB1} *and* ω_2^{FB2} *do not exist and, for* $\omega_2 \in \left(0, \omega_2^{TB}\right)$ *there are two internal equilibria, say* x_u^I *which is unstable and* x_l^I *which is either locally asymptotically stable or unstable, equilibrium* x_0^S *is unstable and equilibrium* x_1^S *is locally asymptotically stable. For* $\omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$, *a unique internal equilibrium exists, and equilibria* x_1^S *and* x_0^S *are both locally asymptotically stable with* $\mathcal{B}\left(x_0^S\right) = \left[0, x_u^I\right)$ *and* $\mathcal{B}\left(x_1^S\right) = \left(x_u^I, 1\right]$.

- ***Scenario 3:*** ω_2^{FB2} *and* ω_2^{FB1} *exist and for* $\omega_2 \in \left(0, \omega_2^{FB2}\right)$, *there are two internal equilibria, say* x_u^I *which is unstable and* x_l^I *which is either locally asymptotically stable or unstable. The equilibrium of segregation* x_0^S *is unstable and* x_1^S *is locally asymptotically stable. For* $\omega_2 \in \left(\omega_2^{FB2}, \omega_2^{FB1}\right)$, *there are no internal equilibria and* $\mathcal{B}\left(x_1^S\right) = (0,1]$. *For* $\omega_2 \in \left(\omega_2^{FB1}, \omega_2^{TB}\right)$, *there are two internal equilibria, say* x_u^I *which is unstable and* x_l^I *which is either locally asymptotically stable or unstable,* x_0^S *is unstable and* x_1^S *is locally asymptotically stable. Finally, for* $\omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$ *a unique internal equilibrium exists, say* x_u^I, *and equilibria* x_0^S *and* x_1^S *are both locally asymptotically stable with* $\mathcal{B}\left(x_0^S\right) = \left[0, x_u^I\right)$ *and* $\mathcal{B}\left(x_1^S\right) = \left(x_u^I, 1\right]$.

For $b < 1/(1-d)$ *either one of the three dynamic scenarios above occurs or, alternatively, if* x_l^I *exists and is locally asymptotically stable, then* $\omega_2 \in \left(0, \omega_2^{FB1}\right)$.

Scenarios 1 and 2 are those already discussed in the numerical analysis above. In Scenario 1, which is illustrated in the upper panels of Figure 1, ethnic integration is not possible for both high and low ethnic income gaps. In Scenario 2, which is illustrated in the lower panels of the same figure, integration is not feasible only for low levels of income gaps. Scenario 3 represents an intermediate dynamic configuration between Scenarios 1 and 2; it features two windows of ethnic income gaps at which segregation is the only outcome that the model predicts. This scenario is depicted in Panel (a) of Figure 2. In general, Scenario 1 can be observed for relatively low values of η. If increasing the value of η while keeping the other parameters constant, the configuration of the dynamics changes first to Scenario 3, and with a further increase of η the structure of the dynamics eventually switches to that described by Scenario 2. Scenario 3 is, however, a particular case that can only be observed for narrow ranges of η.

The condition $b > 1/(1-d)$ specified in the theorem is sufficient to have one of these three scenarios and when this condition is not satisfied then additional scenarios may occur where either integration is not possible or only possible for extreme ethnic income stratifications. An example of these additional scenarios, namely one in which no integration emerges whatsoever, is depicted in Figure 2 (b). Intuitively, this condition implies that a large bias d towards members of the own group must be compensated by a higher relative propensity b towards integration of households of type 2 in order to facilitate integration for intermediate levels of ethnic income gaps.

So far we focused our analysis on the long-term composition of a possible integrated neighborhood, which indicates whether a certain level of diversity of a neighborhood is stable over time. This, however, does not necessarily ensure that in the long run the population that lives in an integrated neighborhood will not decrease as time elapses. In fact, according to Equation (1), the number of households that locate in an integrated neighborhood does not decrease over time if and only if $\Pi_i(x) \geq \delta_i$, for each $i \in \{1,2\}$. Accordingly, we introduce the concept of sustainability of a steady state with the following definition.

Definition 1 *A sustainable steady state is a fixed point x^+ of Model* (12) *such that* $\Pi_i(x^+) \geq \delta_i$, $\forall i \in \{1,2\}$.

Then, the following Lemma (see proof in Appendix) specifies two sufficient conditions under which any asymptotically stable steady state of integration is sustainable.

Lemma 3 *Each asymptotically stable fixed point $x^+ \in (0,1)$ such that either $x^+ > d/2$ or*

$$d < \frac{\left(\sqrt{2}-1\right)\omega_1 - \sqrt{\omega_1\omega_2}}{\eta\sqrt{2\left(\omega_1+\omega_2\right)}} \tag{15}$$

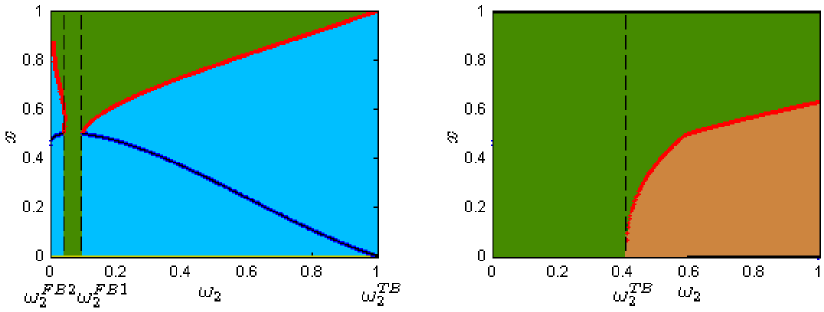
is a sustainable steady state.

This lemma implies that as long as the bias is relatively small the unique asymptotically fixed point of integration of Model (12) is always sustainable according to Definition 1.

It is worth pointing out that the bias d is important to determine whether residential segregation is persistent and sustainable in the long run. In particular, if d tends to be not too large then the condition $b > 1/(1-d)$ is satisfied for any b which is not too close to 1. On the other hand, if d is sufficiently small then steady states of integration tend to be sustainable. Let us further point out that all steady states of integration considered in our investigations are sustainable and a numerical test underlines that a steady state of integration is unsustainable only in the extreme case in which the ethnic income gap is very large (ω_2 is close to 0), the relative importance of integration between group 1 and group 2, say b, is close to 1, and the bias d tends to 1.

Summarizing, the global dynamics of the model reveals a trade-off between narrowing ethnic income gap and residential integration stemming from the counteracting forces of economic and ethnic segregation. Integration can only be achieved if the two forces are well balanced and this balance is only obtainable for specific, and sometimes rather narrow, intervals of ethnic income gaps. At the same time, the balance may be fragile as small changes in the ethnic income gap may cause sudden switches between segregation and integration. This is due to the fact that the impact of the ethnic income gap on segregation as its economic drivers is twofold. Once the income gap increases, there is an increasing level of integration but, at the same time, there is a weakening stability of the steady state of integration.[1] In particular, a 50-50 integrated neighborhood is extremely fragile and it will not resist to a small increase of the number of households of the more affluent ethnic group.

[1] Here we conceive the stability of a fixed point as the minimum Euclidean distance between this fixed point and the border of its basin of attraction.



(a) (b)

Figure 2: One-dimension bifurcation diagrams with respect to parameter ω_2 (income of a member of population 2) that show in Panel (a) an example of Scenario 3 and in Panel (b) a scenario in which there is not any long-run stable pattern of integration. For each value of ω_2 in dark-green is depicted the basin of attraction of the equilibrium of segregation x_1^S, in cyan the basin of attraction of the internal equilibrium x_l^I, i.e. the equilibrium of integration, and in dark-yellow the basin of attraction of the second equilibrium of segregation, i.e. x_0^S. The red curve represents the unstable equilibrium of integration, denoted by x_u^I, while in the blue curve represents the stable internal equilibrium of integration, say x_l^I. Panel (a), parameters: $\beta = 1$, $b = 2$, $d = 0.00$, $\omega_1 = 1$ and $\eta = 0.375$. Panel (b), parameters: $\beta = 1$, $b = 1.1$, $d = 0.2$, $\omega_1 = 1$ and $\eta = 1.5$.

In order to put these findings into perspective, let us relate them to the results of Sethi and Somanathan (2004). In their paper the authors find that, if there is a significant minority population in a city, segregation will be stable as long as ethnic income disparities are either sufficiently large or sufficiently small. This finding is consistent with the results we obtain from the analysis of our model for the case of Scenario 1. Indeed, we can identify two critical income thresholds (ω_2^{FB1} and ω_2^{TB}) between which integration is potentially stable in the long run. However, the result cannot be observed in Scenarios 2 and 3. In fact, in Scenario 2 and in Scenario 3, which are associated with a higher η, we observe that integration can occur also for extreme ethnic income disparities. Altogether, the current investigation expands the findings of Sethi and Somanathan (2004) in several directions: first, the observation that integration only occurs for intermediate levels of ethnic income inequalities is limited to those cases in which the weight η that households put on their preference for integrated neighborhoods is not too strong. Second, the premise that the low-income group must be the minority of the population can be relaxed as, given the parameter choice of η, integration only for intermediate levels of ethnic inequality can be obtained without any restriction on the population. And, finally, the emergence of ethnic and economic drivers of segregation allows us to determine the net flows into the mixed neighborhoods of the ethnic sub populations. As a result, we can specify two different types of steady states of segregation characterized by the exclusive presence of either households of type 1 or those of type 2 in the previously integrated neighborhood.

3.3 The Impact of Government Housing Subsidies on Residential Segregation

The so far conducted investigation reveals that the ethnic income gap has a substantial impact on the long-term residential outcome and changes in the income gap can alter these outcomes dramatically. From a policy perspective, this raises another relevant question, namely whether social programs to support low-income households can have similar impacts on ethnic diversity in metropolitan districts. In fact, over the last decades, there have been many examples of public housing subsidy programs and a considerable number of these measures has been focused on low-income households (see, e.g., Anderson et al., 2003; McCarty et al., 2014). Those measures have been implemented in several forms. In the U.S., for example, the most common practice has been the *supply-side housing assistance programs* which subsidize the construction and operation of housing for low-income households. Given the correlation between income and ethnicity, this implies the threat of subsidizing the emergence of segregated neighborhoods by construction. The second form of housing subsidy has been *rental assistance programs* which aim at subsidizing rents for low-income households in the private rental market. Besides *tenant-based rental assistance programs* that hand out vouchers directly to low-income households, there are the so-called *project-based rental assistance programs* defining neighborhoods in which houses are rented to the more affluent households at the market price and to low-income families at subsidized prices.

In the following we focus on *project-based rental assistance programs* and we want to analyze the possible effects of this policy measure on residential integration. In order to do so, we assume that the government housing subsidies are reserved for low-income people that live in an initially integrated neighborhood. In terms of our model, this implies that the budget equation of members of the less affluent group 2 becomes:

$$p_z z_2 + p_s (1 - \phi) s_2 = \omega_2 \tag{16}$$

with ϕ representing a subsidy rate, whereas the rest of the model setup remains unchanged.[2]

What is the effect of introducing such a subsidy scheme? First of all, a subsidy ϕ has a positive effect on the housing demand of the low-income households. Quantitatively, the subsidized demand is thereby consistent with a quantity demanded by a non-subsidized household with a reference income $\bar{\omega} > \omega_2$. The reference income $\bar{\omega}$ and the subsidy rate ϕ are thereby linked as follows:

$$\bar{\omega} = \frac{\omega_2}{1 - \phi} \quad \text{or, equivalently,} \quad \phi = \frac{\bar{\omega} - \omega_2}{\bar{\omega}}. \tag{17}$$

Since there is a fixed housing supply normalized to 1, the subsidy shifts the relative housing demand in an integrated neighborhood from affluent tenants to low-income households. This reallocation of housing demand emerges through the price channel as the higher demand of the subsidized low-income households drives up the market price of houses in integrated neighborhoods which, in turn, lowers the demand of high-income households. However, a too high subsidization can lead to a situation in which the beneficiaries of the housing subsidies are better off in terms of housing units compared to non-subsidized tenants, which might bear the risk of social conflicts. In order to avoid such a situation, the subsidy rate has to satisfy the following condition:

Definition 2 (Fairness condition) $\omega_1 > \bar{\omega}$ *or, equivalently,* $\phi \in \left[0, \frac{\omega_1 - \omega_2}{\omega_1}\right]$.

At the same time, the changing price driven by the subsidized demand of low-income households in integrated neighborhoods has a negative impact on the utility of the wealthy households such that these neighborhoods become, in relative terms, less attractive compared to their outside option. As a result, the introduction of housing subsidies might eventually increase the risk of segregation.

Based on a numerical analysis, we illustrate in Figure 3 the implications of the subsidy policy where we restrict our attention to those parameter constellations already considered in Figure 1, which are consistent with Scenario 1 and 2 of Theorem 3. Here, we show for a fixed income gap $\omega_2 = 0.5$ the effect of variations of the reference income $\bar{\omega}$ on the residential outcome in the long run. As mentioned above, such a variation is equivalent to an increase of the subsidy rate ϕ. In Panel (a), which depicts the long-term residential patterns for $\eta = 0.25$ and $b = 1.5$, one can see that in case of no subsidies (i.e. $\bar{\omega} = 0.5$) the long-term outcome is characterized by full segregation with only households of type 1. Increasing the subsidy, or equivalently, increasing $\bar{\omega}$, we observe that the residential steady state can switch from full segregation to integration, where the transition is characterized by a classical fold bifurcation. A further increase of the subsidy, however, decreases the achievable levels of integration and, if the subsidy rate is too high, the steady state can even fall back to full segregation marking a transcritical bifurcation. In this case, the previously integrated neighborhoods are exclusively populated by households of type 2.

In the other cases illustrated in Figure 3 (b) to (d), which depict the other possible combinations of b and η considered in Figure 1, the long-term residential patterns at $\omega_2 =$

[2] It should be noted that the subsidy is linked to the income and not to the ethnicity of a household. Since we assume homogeneous incomes within each group and $\omega_1 > \omega_2$, only households of type 2 are eligible to receive these subsidies. In Section 4 we will relax the assumption of within-group homogeneity of incomes.

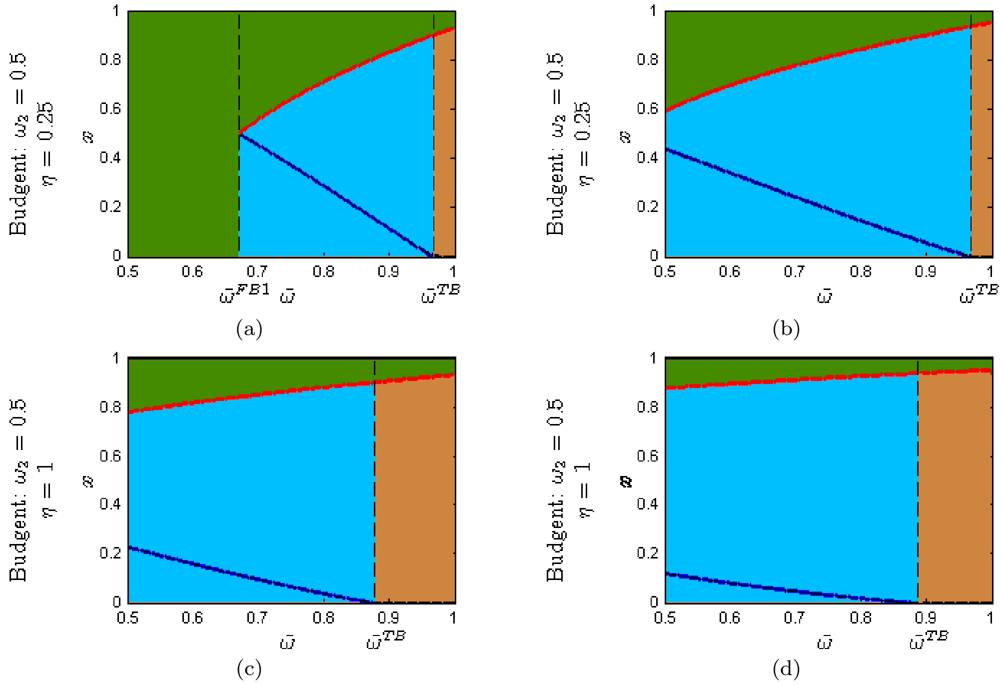


Figure 3: One-dimension bifurcation diagrams with respect to parameter $\bar{\omega}$ representing the reference income for the subsidies. For each value of $\bar{\omega}$ in dark-green is depicted the basin of attraction of the equilibrium of segregation x_1^S, in cyan the basin of attraction of the internal equilibrium x_1^I, i.e. the equilibrium of integration and in dark-yellow the basin of attraction of the second equilibrium of segregation, i.e. x_0^S. Parameters: $d = 0.05$, $\omega_1 = 1$, $\beta = 1$.

0.5 without subsidies are characterized by the possibility of integration. An increase of the reference income determining the subsidy rate leads in these cases to lower levels of integration and eventually to the emergence of full segregation, again only with households of type 2. Altogether, the described outcomes of the numerical analysis suggest that the long-term effects of subsidizing type-2 households with an income $\omega_2 = 0.5$ are qualitatively identical to those we can observe for narrowing the ethnic income gap starting at $\omega_2 = 0.5$.

As a conclusion, this policy illustration points out that applying housing subsidy schemes can have substantial effects on ethnic diversity in the residential patterns which are qualitatively very similar to the effects one obtains when reducing the ethnic income disparities. However, housing subsidies only affect the housing demand while narrowing the income gap has also an impact on the demand for consumption goods. Moreover, the results show that there can be a trade-off between integration and the standard of living of low-income people in the sense that a policy measure that aims at improving the housing situation of low-income households may eventually lead to lower levels of integration. If, on the other hand, due to a large ethnic income gap, the residential distribution is characterized by segregation in the first place, then the government housing subsidy can favor integration. Moreover, under certain conditions, even very small changes in the policy can imply strong effects on the long-term residential outcomes, especially if the system rests in a steady state close to a politically desirable level of 50-50 integration. Hence, there is also another trade-off, namely between high levels of integration and the fragility of the steady state. More precisely, increasing the subsidies reduces the level of integration but, at the same time, it lowers the fragility of the steady state. In any case, it should be clear that policy makers must be aware of potential trade-offs and should take them into account when they implement economic measures such as housing subsidies.

4 Robustness Check: The Effect of Within-group Inequality

4.1 An Agent-based Model of Residential Segregation

In the analysis of residential outcomes and possible implications of housing subsidies considered so far we made a rather restrictive assumption, namely that there is only income inequality across but not within ethnic groups. This assumption was essentially imposed to analyze the residential dynamics as an analytically tractable population-based evolutionary process in which the evolution of x_t can be described by the one-dimensional Map (12). But what happens if this assumption is relaxed? Stay our findings intact when introducing heterogeneity with respect to income among households of the same type?

Answering these questions is not straightforward. The reason is that it is hardly feasible to tackle within-group heterogeneity in the simple evolutionary game framework described in Section 2. In order to address the question of the robustness nonetheless, we now formulate a variant of the model in which we relax some key assumptions that made it possible to describe the model by a one-dimensional map in the first place without, however, changing the basic structure of the model. In particular, we relax the notation of an inherent equilibrium allocation of a continuum of households across an undefined number of locations. Instead, we analyze the residential dynamics in a unique neighborhood in which the local population is modeled as a discrete set of households. The individual income $\omega_j > 0$ of a household j of a type i is thereby drawn from a type-specific distribution represented by log-normal distributions with $\omega_j \sim \mathcal{L}(\tilde{\omega}_i, \sigma^2_\omega)$. Here, $\tilde{\omega}_i$ is a type-specific distribution mean, with $\tilde{\omega}_1 \geq \tilde{\omega}_2$, and $\sigma^2_\omega > 0$ is an exogenous model parameter governing the degree of within-group inequality.

Households are explicitly modeled as autonomous decision making entities implying a shift from a population-based evolutionary game to an agent-based setup in which the dynamic behavior of the model is the aggregate outcome of individual decisions of heterogeneous and interacting agents in an adaptive and evolutionary environment. Consistently with the original model, the residential choice of an individual household is driven by a comparison of expected payoffs associated with the choice of living either in the integrated neighborhood or in an outside option characterized by full segregation. The fitness measure $\tilde{\Pi}_{j,t}$ of household j is thereby influenced by the economic utility from consumption and housing as well as the appreciation of ethnic diversity in the neighborhood.

Again, information that is relevant for reconsidering the residential choice is transmitted between households by word of mouth and flows only in a constellation in which a household located in the integrated neighborhood (labeled neighborhood 1) meets another one living in his/her outside option (labeled neighborhood 2 for household of type 1 and neighborhood 3 for households of type 2). We assume that a household of the integrated neighborhood meets one co-ethnic resident from the outside option per period whose income is drawn from the log-normal distribution $\mathcal{L}(\tilde{\omega}_i, \sigma^2_\omega)$.

Denote by H_t the set of households living in neighborhood 1 at time t, and suppose (j, k) with $j \in H_t$ and $k \notin H_t$ is a pair of households of the same kind that has been matched successfully in period t. Then, there is a bilateral flow of information and both parties reconsider their location choice by taking the obtained information into account to assess the expected fitness of living either in the neighborhood or in the outside option. The actual residential decision is modeled as a logit discrete choice problem where for household j the probability to move from neighborhood 1 to his/her outside option is

$$\mathbb{P}(j \text{ moves out}) = 1 - \frac{1}{1 + \exp(\beta(\delta_{j,t} - \tilde{\Pi}_{j,t})}, \tag{18}$$

whereas for household k, the probability to move into the neighborhood is

$$\mathbb{P}(k \text{ moves in}) = \frac{1}{1 + \exp(\beta(\delta_{k,t} - \tilde{\Pi}_{k,t}))}. \tag{19}$$

The number of households of type i in the integrated neighborhood at the beginning of the next period $N_{i,t+1}$ is then the sum of all households of that type opting for neighborhood 1.

Consequently, the level of integration in period $t+1$, which is our variable of interest, is

$$X_{t+1} = \frac{N_{1,t+1}}{N_{1,t+1} + N_{2,t+1}}. \tag{20}$$

The demand side of the local housing market is composed of the set of households H_t that decided to live in the neighborhood. The individual demand for housing units s is thereby driven by the market price $p_{s,t}$ and the individual income ω_j. The demand s_j as well as the demand for outside consumption z_j is derived from solving the same utility maximization problem under budget constraint as described in Section 2. Given the local housing price $p_{s,t}$ and the normalized price for the consumption good $p_z = 1.0$ and assuming $\alpha = 0.5$, the corresponding Walrasian demand functions are

$$s_j(p_{s,t}) = \frac{\omega_j}{2 \cdot p_{s,t}} \text{ and } z_j = \frac{\omega_j}{2}. \tag{21}$$

The stock of houses in the neighborhood is assumed to be heterogeneous with respect to housing standard. However, to keep the analysis simple, the heterogeneous stock is represented by an aggregate stock with average standard $\tilde{q}$. The housing market yields a market clearing price $p_{s,t}$ that equals the average demand for housing units with the average housing standard in the neighborhood. Put formally, we have

$$\frac{1}{n_t} \sum_{j \in H_t} s_j(p_{s,t}, \omega_j) = \tilde{q}, \tag{22}$$

where $n_t = |H_t|$ represents the population size of neighborhood 1. Normalizing $\tilde{q} = 1$, the market clearing price is then

$$p_{s,t} = \frac{1}{2n_t} \sum_{j \in H_t} \omega_j, \tag{23}$$

which is consistent with the price one obtains in the population-based model described in Section 2 (compare Equation (7)).

Now we introduce a modification to the model setup that aims at limiting the demographic dynamics that emerge by relaxing the assumption of a population density equilibrium. In fact, since we consider a single neighborhood and at the same time it cannot be ruled out that $\delta_{j,t} < \tilde{\Pi}_{j,t}$ holds for a large majority of households, there is the possibility of a high population growth that might lead to an overpopulated neighborhood. However, as we are only interested in the long-term dynamics of the ethnic composition in the neighborhood, we introduce an additional component to the payoff function that reduces the payoff of living in an overpopulated neighborhood and therefore mitigates the demographic dynamics.[3] More precisely, the negative demographic effect is captured by a penalizing term g in the payoff function that depends on a measure of the population density $\rho_t = \frac{n_t}{m}$, where m can be interpreted as the area of the neighborhood. Function g has the properties

$$g(\rho_t) = \rho_t^{\psi} \text{ with } \psi > 0. \tag{24}$$

Below we will show that this demographic component in the payoff function does not affect the residential dynamics with respect to the ethnic composition.

Altogether, the expected payoff of household j of type i that is associated with living in the neighborhood populated by the set of households H_t is given by

$$\tilde{\Pi}_j = \tilde{V}_j(H_t) + \eta f_i(X_t) - g(\rho_t), \tag{25}$$

where f_i captures the type-specific preferences for ethnic diversity and is identical with Equation (10) in Section 2. $\tilde{V}_j$ denotes the indirect utility function of household j that depends on the average income of the neighborhood's population H_t, i.e.,

$$\tilde{V}_j(H_t) = \sqrt{\frac{n_t}{2 \sum_{k \in H_t} \omega_k}} \cdot \omega_j. \tag{26}$$

[3]In fact, a negative relationship between payoff and population density is consistent with empirical evidences as studies show that low-density settlement is the preferred choice for residential living, see, e.g., Gordon and Richardson (1997).

The outside option represents the *rest of the world* in which a household can always choose to live in full segregation. The in- and outflow into neighborhood 1 does not affect the demographics in the outside option, therefore the population density and consequently g is constant. Without loss of generality, we set $g = 0$ in the two outside options. The payoff of the outside option is then

$$\delta_{j,t} = \begin{cases} \sqrt{\frac{1}{2\tilde{\omega}_1}} \cdot \omega_j + f_1(1) & \text{if } j \text{ is of type } i = 1, \\ \sqrt{\frac{1}{2\tilde{\omega}_2}} \cdot \omega_j + f_2(0) & \text{else.} \end{cases} \tag{27}$$

The evolution of X_t can be described as a stochastic evolutionary process that is driven by the co-evolution of the number of households of the two types living in the neighborhood. The following proposition demonstrates that, despite the differences in the setup, the stochastic framework has the same dynamic core as the deterministic model of Section 2 such that, as long as the stochastic model is set up without within-group inequality (i.e. $\sigma_\omega^2 = 0$), both models generate the same long-term residential dynamics. This allows to use the stochastic model to assess the robustness of the residential dynamics with respect to the introduction of within-group inequality in the first place (proof see appendix).

Proposition 1 *Suppose $\sigma_\omega^2 = 0$. If the population size n_t in the neighborhood is sufficiently large, then the long-term behavior of the deterministic population-based dynamics of Map (12) can be approximated by the stochastic agent-based process of X_t.*

To illustrate this issue we now show simulation results of the agent-based model with a setup with $\sigma_\omega = 0$. In particular, we show the long-term dynamics for two values $\eta = 0.25$ and $\eta = 1.0$ in order to capture the two scenarios 1 and 2 defined in Theorem 3. The remaining parameter values are chosen as summarized in Table 1.

In the following discussion, the simulation results are presented in an analog way as we discussed the numerical results of the evolutionary game. To this end, we carried out different sets of simulations for which we varied the initial ethnic mix X_0 together with the average income of the low-income ethnic group $\tilde{\omega}_2$. In order to capture the stochastic nature of the emerging agent-based dynamics, we carried out 20 batch runs for each considered pair $(X_0, \tilde{\omega}_2)$.[4] The simulation output is then used to construct one-dimensional bifurcation diagrams for the long-term values of $X \in [0, 1]$ indicating the degree of integration in the neighborhood. These diagrams illustrate the stochastic equilibria of the simulations as well as their basins of attraction in the considered one-dimensional parameter sub space. Concerning the stochastic equilibria, we will refer to X^0 and X^1 as the stochastic analogous of the equilibria of segregation x_0^S and x_1^S of the evolutionary model in the previous section, and to X^I as the stochastic analogous of the equilibrium of integration x_l^I.[5]

Note that we use here a stochastic concept of basins of attraction. Serdukova et al. (2016), for example, describe stochastic basins of attraction as the set of initial conditions whose solutions have a small probability to escape the neighborhood of an attractor and have a high probability to return to the vicinity of the same attractor. This concept is used in the context of metastability, which describes the property of a system when long periods of stasis are punctuated by sudden changes of the regime (see, e.g., Young, 2006). In general, these tipping phenomena can occur if the random noise of the stochastic process accumulates such that its trajectory escapes the basin of attraction (in a deterministic sense) of the current equilibrium and enters the basin of another one. In the present stochastic framework, metastability is related to the aspects of fragility discussed in Section 3. In order to address the issue of fragility in the context of the stochastic process, we mark any initial condition $(X_0, \tilde{\omega}_2)$ as fragile if there is at least one of the 20 batch runs ending up in a different equilibrium than the others. Any initial condition that is not marked as fragile, in contrast, represents a stochastic basin of attraction of a stochastic equilibrium.

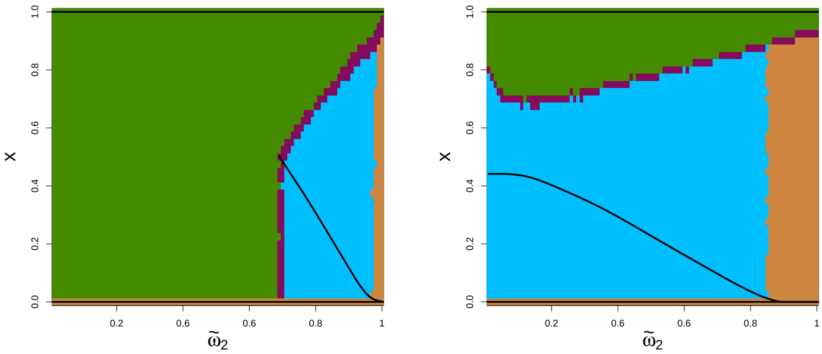
---

[4] We consider discrete sets with $X_0 \in \{0.0, 0.025, 0.05, ..., 0.975, 1.0\}$ and $\tilde{\omega}_2 \in \{0.01, 0.02, ..., 0.99, 1.0\}$.

[5] In the following, we will define the stochastic equilibria of segregation by $X^1 = \{X|X > 0.99\}$ and $X^0 = \{X|X < 0.01\}$, where X is the average X_t the time interval $500 \leq t \leq 600$. The stochastic equilibria of integration X^I are defined accordingly. The curve representing X^I in the figures are estimated by applying Generalized Additive Models with penalized splines (see, e.g., Wood, 2011) based on all observation X_t in $500 \leq t \leq 600$ from all 20 batch runs that are in the interval [0.01,0.99].

Parameter	Description	Value
n_0	Initial population size in the neighborhood	10.000
β	Intensity of choice	1.0
d	Bias towards members of the own type	0.05
b	Parameter ethnic tolerance type 2	1.5
m	Area of the neighborhood	20.000
ψ	Effect of population density	4.0
$\tilde{\omega}_1$	Average income group 1	1.0

Table 1: Parameters of the agent-based model.



(a) (b)

Figure 4: One-dimension bifurcation diagrams of the agent-based model with $\sigma_\omega = 0.0$ for $\eta = 0.25$ (panel a) and $\eta = 1.0$ (panel b). For each value of $\tilde{\omega}_2$ in dark-green is depicted the basin of attraction of the stochastic equilibrium of segregation X^1, in cyan the basin of attraction of the stochastic internal equilibrium X^I, i.e. the equilibrium of integration and in dark-yellow the basin of attraction of the second equilibrium of segregation, i.e. X^0. The dark-red region marks the regions from which the system either converges to X^I or X^1.

Let us now consider how the simulation model and the evolutionary game behave in direct comparison. Figure 4 shows the global dynamics of the agent-based model with respect to a variation of parameter $\tilde{\omega}_2 \in [0.01, 1.0]$ for the two values of $\eta = 0.25$ (left) and $\eta = 1.0$ (right). As suggested by Proposition 1, comparing Figure 4 with Figure 1 reveals that the global dynamics generated by the simulation model are indeed very similar to those of the evolutionary game. Moreover, there are only small areas at the border of the basins of attraction of X^I and X^1 in which the system shows signs of fragility.

4.2 The Robustness of the Evolutionary Dynamics

The previous discussion illustrates that the simulation output of the agent-based model with no inequality within ethnic groups is very close to the numerical benchmark of the deterministic evolutionary dynamics. But how does the behavior change if we introduce within-group heterogeneity? The upper panels of Figure 5 illustrate for $\eta = 0.25$ and 1.0 the global dynamics of the simulation model using the same setup as above but with $\sigma_\omega = 0.5$ instead of $\sigma_\omega = 0.0$. Thus, we introduce a substantial income inequality into both sub populations. By comparing these panels with the benchmark without intra-group heterogeneity in Figure 4, one can clearly see that the global dynamics are qualitatively very similar to those of the simulations without inequality. Therefore, one can conclude that the findings with respect to the interaction of ethnic and economic segregation are robust if extending the analysis by incorporating within-group inequality. The lower panels show for fixed levels of $\tilde{\omega}_2$ the effect of a change in the level of within-group inequality. These graphics confirm the assertion that for a wide range of within-group inequality there are hardly any effects on the long-term level of residential integration.

Now we turn to the robustness of the results of the policy analysis. In Section 3.3, we considered the long-term residential pattern of the evolutionary game under a governmental

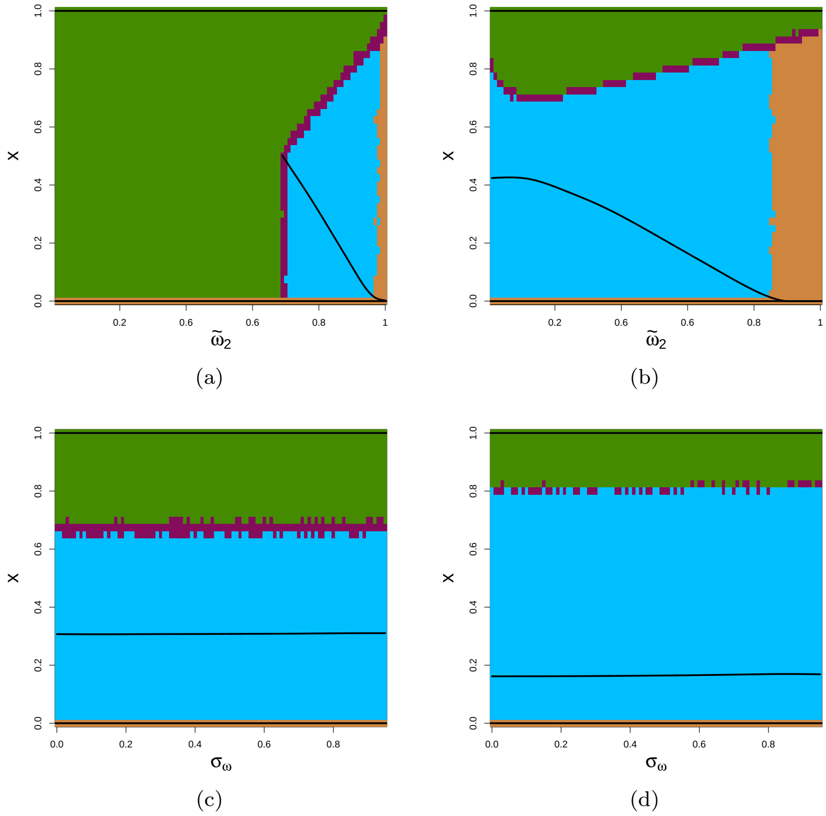


Figure 5: One-dimension bifurcation diagrams of the agent-based model for $\eta = 0.25$ (left panels) and $\eta = 1.0$ (right panels) with $\sigma_\omega = 0.5$ and $\tilde{\omega}_2$ as bifurcation parameter (upper panels) and $\tilde{\omega}_2 = 0.8$ (panel c) or, respectively, $\tilde{\omega}_2 = 0.6$ and σ_ω as bifurcation parameter (lower panels). In dark-green is depicted the basin of attraction of the stochastic equilibrium of segregation X^1, in cyan the basin of attraction of the stochastic internal equilibrium X^I, i.e. the equilibrium of integration and in dark-yellow the basin of attraction of the second equilibrium of segregation, i.e. X^0. The dark-red region marks the regions from which the system either converges to X^I or X^1.

subsidy regime that is supposed to lower housing prices for low-income households in order to guarantee a minimum housing standard. However, since we considered the absence of within-group heterogeneity, the beneficiaries of the policy always belonged to the group $i = 2$.

When introducing within-group inequality, the income distributions might overlap such that the least affluent households of group 1 earn less than the most affluent households of the other group. In such a case, the beneficiaries of the subsidy must not necessarily belong to group 2. In the following, we assume that again low-income households receive a subsidy that allows them to live in a house that corresponds to a minimum standard that is consistent with the demand of a unsubsidized household with a threshold income $\bar{\omega}$. But before we discuss the policy design in more detail, we have to adjust the fairness condition introduced in Section 3.3, which makes sure that beneficiaries of the housing subsidies are not better off with respect to housing units compared to non subsidized tenants:

Definition 3 (Fairness condition II) *Suppose household k and l live in the target neighborhood 1, i.e. $k, l \in H_t$ and suppose that $\omega_k > \omega_l$. Then, we need to have the following condition:*

$$\frac{\omega_k}{2p_{1,t}\,(1-\phi_k)} \geq \frac{\omega_l}{2p_{1,t}\,(1-\phi_l)}. \tag{28}$$

In the following, we assume that household j, if his/her income ω_j is below the threshold income $\bar{\omega}$, receives a subsidy on the housing price that depends on the individual gap between

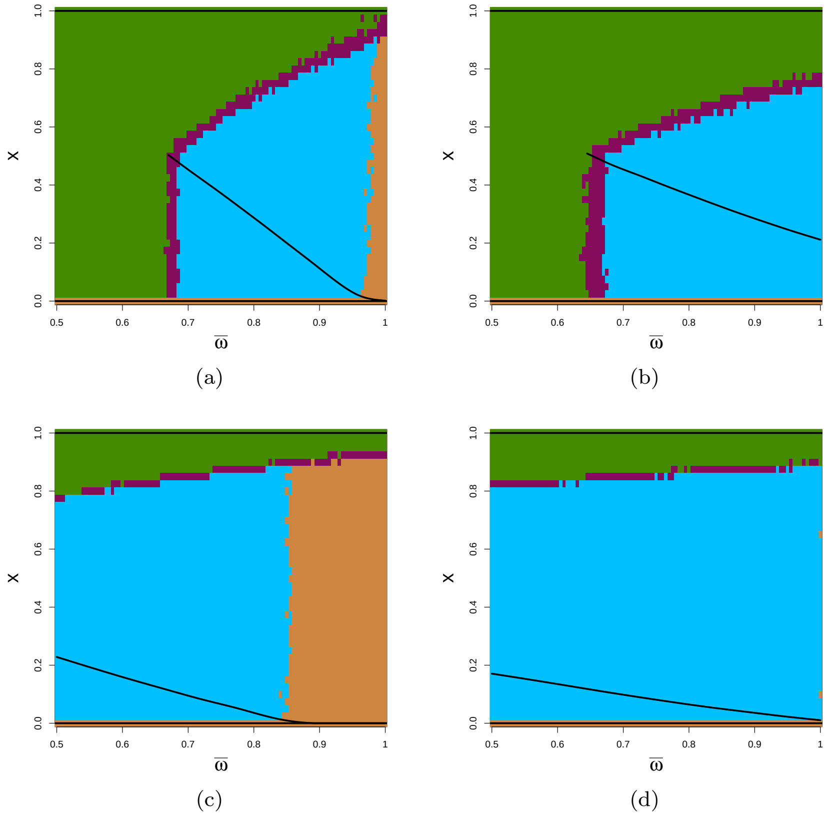


Figure 6: One-dimension bifurcation diagrams of the agent-based model with $\sigma_\omega = 0.0$ and $\eta = 0.25$ (panel a), $\sigma_\omega = 0.5$ and $\eta = 0.25$ (panel b), $\sigma_\omega = 0.0$ and $\eta = 1.0$ (panel c) and $\sigma_\omega = 0.5$ and $\eta = 1.0$ (panel d). For each value of $\bar{\omega}$ in dark-green is depicted the basin of attraction of the stochastic equilibrium of segregation X^1, in cyan the basin of attraction of the stochastic internal equilibrium X^I, i.e. the equilibrium of integration and in dark-yellow the basin of attraction of the second equilibrium of segregation, i.e. X^0. The dark-red region marks the regions from which the system either converges to X^I or X^1.

the own income and the threshold. Put formally, we have

$$\phi_j = \max\left(0, \frac{\bar{\omega} - \omega_j}{\bar{\omega}}\right), \tag{29}$$

where it is easy to see that (29) satisfies the fairness condition.

In Figure 6, we again depict diagrams for $\eta = 0.25$ (upper panels) and $\eta = 1.0$ (lower panels) for the cases $\sigma_\omega = 0.0$ and 0.5. Here, the threshold income $\bar{\omega}$ is varied between 0.5 and 1.0, where the lower bound corresponds to $\tilde{\omega}_2$ and the upper bound to $\tilde{\omega}_1$. Thus, we can compare how the presence of within-group heterogeneity affects the implications of the subsidy policy. Note that again, when comparing the left panels with an homogeneous within-group income with Figure 3, then it becomes apparent that the dynamics of the agent-based model are a close approximation of the deterministic evolutionary dynamics.

Apparently, the issue of robustness becomes more ambiguous for the policy scenario. Qualitatively, the introduction of within-group inequality leads to a similar impact of the housing subsidy as in the scenario without inequality. In particular, if the individual weight put on ethnic aspects of the neighborhood choice are minor ($\eta = 0.25$), then an increasing threshold income $\bar{\omega}$ can give rise to a rapid transition from complete segregation with only households of type 1 to full integration. A further increase of $\bar{\omega}$ weakens the effect of economic segregation thereby leading to an increasing dominance of type 2 households, which can, if $\bar{\omega}$ is sufficiently large, result in a situation in which the neighborhood is exclusively populated by households of type 2. For $\eta = 1.0$, a weakening of economic segregation in response to an increasing $\bar{\omega}$ can also be observed but at a substantially weaker level.

Comparing the left and right panels of Figure 6 illustrates, however, that in case of within-group income heterogeneity the weakening effect of an increasing minimum housing standard on the economic drivers of segregation is less pronounced. As a consequence, especially for higher levels of $\bar{\omega}$, more integration can be achieved when the two sub populations have heterogeneous incomes. Moreover, the presence of within-group inequality increases the fragility of the long-term equilibrium, which can be seen from the enlarged region representing the punctuated equilibrium behavior at the basins of attraction of X^I close to 0.5.

The reason for these differences can be found in the fact that within-group income inequality changes the ethnic composition of the potential beneficiaries of the subsidy such that the higher the heterogeneity the more diverse is the pool of potential recipients of the policy. More precisely, for a given level of $\bar{\omega}$ with $\tilde{\omega}_2 < \bar{\omega} < \tilde{\omega}_1$, in the case without heterogeneity (i.e. $\sigma_\omega = 0$) the recipients of the subsidy exclusively belong to group 2. A rising σ_ω, however, might increase the fraction of members of group 1 whose income is below the threshold income $\bar{\omega}$ and is hence eligible for the subsidy. At the same time, it reduces the number of potential beneficiaries in group 2. Since the subsidies increase the attractiveness of neighborhood 1 for recipients of the subsidy, more within-group heterogeneity leads to a higher share of households of type 1 in that neighborhood.

At the same time, subsidizing low income households in the potentially integrated neighborhood leads to higher housing prices in this neighborhood, which is the result of a higher housing demand of the beneficiaries of the policy. Due to the convexity of the indirect utility function with respect to housing prices, the differential in terms of economic utility from living in neighborhood 1 versus households' outside options, i.e. neighborhood 2 for type 1 and neighborhood 3 for type 2 households, is on average smaller for households of type 1, given that the emerging housing prices can be ordered by $p^3_{s,t} < p^1_{s,t} < p^2_{s,t}$, where $p^k_{s,t}$ is the market price for housing units in neighborhood k. As a result, there is an additional shift in the relative attractiveness of neighborhood 1 in favor of type 1 households although there is also a counteracting effect in favor of type-2 households that is caused by a higher X together with the assumption that type 2 households are the more tolerant households with respect to ethnically mixed neighborhoods ($b > 1$). Altogether, by introducing within-group inequality, the integrated neighborhood 1 becomes more attractive for type 1 households, which weakens the effect of ethnic segregation and, under specific conditions, allows to achieve higher levels of integration. Nevertheless, the results show also for the policy analysis a high level of robustness such that the qualitative conclusions obtained from the analysis of the evolutionary game still hold.

To sum up, we consider the following conjecture which summarizes the implications of introducing within-group income inequality.

Conjecture 1 *The introduction of within-group inequality does not affect the residential dynamics as long as there are no housing subsidies for low-income households. Otherwise, a high level of income inequality within ethnic groups leads to a weakening of the force towards economic segregation such that aspects of ethnicity play a more prominent role in determining the long-term residential pattern.*

5 Discussion and Conclusions

Residential segregation by income and ethnicity represent a persistent phenomena in many western societies with a strong impact on social cohesion. At the same time, governments spend considerable funds for housing policies aimed at supporting low-income households, and it is not a priori clear how these policies affect the residential patterns. The aim of this paper was to provide a theoretical framework for explaining the emergence of residential integration and segregation and to develop a model that can serve as a tool for analyzing the effects of housing policies on residential outcomes. To this end, we introduced a dynamic evolutionary model where households select the residential location based on individual preferences for the ethnic composition of the neighborhoods and endogenous market prices of housing. Households belong to two different ethnic groups. The first is made of people that put less emphasis on ethnic diversity but are more affluent than the member of the second one. The sorting dynamics is described by an evolutionary-like game.

The model reveals that residential integration is possible only when segregation by ethnicity is counterbalanced by stratification by income. Thus, a certain level of ethnic income disparities is the price the society has to pay for residential integration. In that sense, ethnic factors and economic factors combined together facilitate integration whereas narrowing ethnic income disparities result in *increasing* residential segregation. Nevertheless, this is not a generic observation of the model as these results do not hold true for the entire parameter space. In fact, for low heterogeneities in the individual preferences for diversity between the two ethnic groups, economic segregation prevails in the long run for whatever level of ethnic income gap. On the contrary, in case of significant differences in the propensity towards integration, the ethnic income disparities play a crucial role in determining whether segregation or integration prevails in the long run.

This underlines that a policy measure supposed to strengthen residential integration must be able to balance the economic and ethnic drivers of segregation. Since it is difficult to reshape individuals' preferences at least in the short run, the policy has to target the economic forces towards segregation stemming from ethnic income disparities. In this respect, we analyzed the impact of a housing subsidy scheme that mimics the practice of supporting low-income households in the rental market. We found that the effect of the government housing subsidies for low-income households on residential integration is ambiguous. In particular, if the ethnic income gap is sufficiently large such that economic segregation prevails, then government housing subsidies favor integration. On the other hand, if ethnic segregation prevails, the introduction of government housing subsidies fosters segregation.

Furthermore, the dynamic framework of our model allows to discuss asymptotic and structural stability of the patterns of integration. In this respect, our investigation reveals that the unique asymptotically stable equilibrium of integration is such that the less tolerant ethnic group is always the minority. Moreover, the higher the level of integration in a residential location equilibrium, the more fragile is the global stability of this equilibrium. Emblematic is the equilibrium of perfect integration, where half of the households of a neighborhood belongs to the wealthy ethnic group and the other half to the other ethnic group. This particular configuration is only marginally stable representing a tipping point at which a slight change in the ethnic mix of the neighborhood might trigger residential patterns of segregation. Moreover, this equilibrium is also structurally unstable, in the sense that a variation in the parameter space might also drive the system from integration to full segregation. This especially holds for changes in the ethnic income gap and the housing subsidy rates revealing possible policy trade-offs between high level of integration and the stability of the residential equilibrium patterns.

The model used to carry out this analysis is a highly stylized population-based evolutionary model. The simplicity of this model allows to have a easy-to-use and easy-to-understand tool for policy analysis. Although simple models are always preferable with respect to complicated ones, the risk is to have a limited view of the phenomenon and to miss some important aspects of the problem that may have a fundamental impact in the final output of the real system. In this respect, stylized models must go through robustness tests to be deemed trustworthy. In order to provide a robustness analysis for our stylized model, we developed a related agent-based model where we additionally included income inequality within ethnic groups. A comparison of the outcomes of the two models provides further confirmations that residential integration can be achieved through a combination of ethnic income disparities and individual preferences. Furthermore, the robustness test suggests that the population-based stylized model without intra-ethnic income inequality performs sufficiently well as a tool for policy analysis.

What policy implications can be derived from this study? Our policy analysis shows that a housing policy that subsidizes low-income households can have diametrically opposed effects on the level of ethnic integration and the direction in which the policy works depends on the starting condition. If the ethnic income differential is sufficiently large such that the economic aspects dominate driving the residential outcome towards full segregation, then a housing subsidy for low-income households can give rise to integration by weakening the economic drivers of segregation. In this scenario, the housing policy can achieve two goals, a higher housing standard for low-income households and more integration. If, however, the ethnic and economic drivers offset each other such that residential integration prevails, then the policy's weakening effect on the economic drivers of segregation reduces the level of integration. This implies a possible conflict of interest between improving the housing standard of low-income households and achieving ethnic diversity. On the other hand, in this scenario, the subsidy policy would increase the stability of the state of integration, which can be considered as a positive aspect from a policy perspective. Thus, there is another trade-off which a policy maker has to take into account, namely a trade-off between stability and high levels of ethnic diversity. Altogether, policy makers should be aware that the residential implications of housing subsidies supporting low-income households are highly ambiguous and particularly sensitive to the state of the residential environment at which the policy is going to be implemented. In that sense, those housing subsidies must be considered with caution when introduced as a policy to foster residential integration.

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Appendix: Proofs

Proof of Lemma 1. Assuming $x_t \in (0,1)$, let us note that $x_{t+1} \gtrless x_t$ when the denominator of Map (12) is smaller/larger than one. Since the denominator of Map (12) is smaller/larger than one when $\Delta\Pi(x_t) \lesseqgtr 0$, property 1 follows. Let us further note that $\forall x_t \in [0,1]$ the denominator of Map (12) is positive and greater or equal than the numerator which is non negative, then $x_{t+1} \in [0,1]$ and property 2 follows. Finally, imposing the equilibrium condition for Map (12), properties 3 and 4 follows. ■

Proof of Theorem 1. By Lemma 1 x_0^S and x_1^S are always equilibria of Model (12). Moreover, since $\omega_1 = \omega_2$ by assumption, $\Delta\Pi(x)$ defined in (13) is such that $\Delta\Pi(x) = \eta(d - 2(1 - b + db)x) > 0 \; \forall x \in (0, 0.5)$, which implies (see properties 1 and 4 in Lemma 1) $x_{t+1} < x_t \; \forall x_t \in (0, 0.5)$ and the absence of equilibria in $(0, 0.5)$. Moreover, $\Delta\Pi(x) = \eta(2d - 2 + 2b)(1 - x) - bd \; \forall x \in (0.5, 1)$, which implies (see properties 1 and 4 in Lemma 1) the presence of the following unique equilibrium (it always exists) of Model (12) in $(0.5, 1)$

$$x^* = 1 - \frac{bd}{2(b - 1 + d)}, \tag{30}$$

$x_{t+1} < x_t \; \forall x_t \in (0.5, x^*)$ and $x_{t+1} > x_t \; \forall x_t \in (x^*, 1)$. Hence, the statement of the Theorem follows by straightforward considerations. ■

Proof of Theorem 2. By Lemma 1 x_0^S and x_1^S are always equilibria of Model (12). Moreover, $\eta = 0$ and $\omega_1 > \omega_2$ by assumption. Then, $\Delta\Pi(x)$ defined in (13) is given by

$$\sqrt{\frac{\omega_1}{2}}\left(1 - \sqrt{\frac{\omega_1}{\omega_1 x + \omega_2(1-x)}}\right) - \sqrt{\frac{\omega_2}{2}}\left(1 - \sqrt{\frac{\omega_2}{\omega_1 x + \omega_2(1-x)}}\right) \tag{31}$$

and $\Delta\Pi(x) < 0$ $\forall x \in [0,1]$, which implies (see property 1 in Lemma 1) $x_{t+1} - x_t > 0$ $\forall x_t \in (0,1)$. Hence, the statement of the Theorem follows by straightforward considerations. ■

Proof of Lemma 2. Let us prove one by one all the five properties of the Lemma:

1. Since

$$\Delta\Pi(0) = \sqrt{\frac{\omega_1}{2}} + \eta d - \frac{\omega_1}{\sqrt{2\omega_2}} \tag{32}$$

we have that $\Delta\Pi(0) \gtrless 0$ for $\omega_2 \gtrless \omega_2^{TB}$. By continuity of function $\Delta\Pi(x)$ the same holds true in a non empty neighborhood of the equilibrium $x_0^S = 0$. Then, property 1 follows from Lemma 1.

2. Since

$$\Delta\Pi(1) = \frac{\omega_2}{\sqrt{2\omega_1}} - \sqrt{\frac{\omega_2}{2}} - \eta bd < 0 \tag{33}$$

for all constellations of the values of the parameters in Φ. By continuity of function $\Delta\Pi(x)$ the same holds true in a non empty neighborhood of the equilibrium $x_1^S = 1$. Then, property 2 follows from Lemma 1.

3. Let us start showing that there are at most two equilibria in $(0,1)$. To this aim, let us first prove that at most one equilibrium in $(0.5,1)$ exist and at most two equilibria in $(0,0.5]$ exist. As specified in Lemma 1, $x^* \in (0,1)$ is an equilibrium of Map (12) if and only if $\Delta\Pi(x^*) = 0$. For $x < 0.5$ we have $\Delta\Pi(x) = 0$ when

$$\frac{\omega_2 - \omega_1}{\sqrt{2(\omega_1 x + \omega_2(1-x))}} + 2\eta b(1-d)x - 2\eta x + \sqrt{\frac{\omega_1}{2}} + \eta d - \sqrt{\frac{\omega_2}{2}} = 0 \tag{34}$$

Operating the change of variable

$$z = \sqrt{\omega_1 x + \omega_2(1-x)} \geq \sqrt{\omega_2} \tag{35}$$

the number of real solutions of Equation (34) corresponds to the number of real and greater than $\sqrt{\omega_2}$ (≥ 0) solutions of the following equation

$$z^3 + p_1 z + q_1 = 0 \tag{36}$$

where

$$p_1 = \frac{\left(\sqrt{\frac{\omega_1}{2}} + \eta d - \sqrt{\frac{\omega_2}{2}}\right)(\omega_1 - \omega_2)}{2\eta(b(1-d)-1)} - \omega_2 \quad \text{and} \quad q_1 = \frac{-(\omega_2 - \omega_1)^2}{\eta(b(1-d)-1)\sqrt{8}} \tag{37}$$

Since either $q_1 < 0$ (when $b(1-d) - 1 > 0$) or $q_1 > 0$ and $p_1 < 0$ (when $b(1-d) - 1 < 0$), from Cardano-Descartes rule of signs we know that Equation (36) has at most one real positive solution when $b(1-d) - 1 > 0$ and at most two positive real roots otherwise. Then, at most a unique equilibrium of Model (12) exists in $(0,0.5)$ when $b(1-d) - 1 > 0$ and at most two otherwise. For $x \in (0.5,1)$ the equilibrium condition $\Delta\Pi(x) = 0$ is given by

$$\frac{\omega_2 - \omega_1}{\sqrt{2(\omega_1 x + \omega_2(1-x))}} - 2\eta(1-d) + 2\eta b + 2\eta((1-d) - b)x + \sqrt{\frac{\omega_1}{2}} - \sqrt{\frac{\omega_2}{2}} - \eta bd = 0 \tag{38}$$

To prove that there could be at most one equilibrium (real solution of Equation (38)) in $(0.5,1)$, let us employ the change of variable (35). Then, defining

$$q_2 = \frac{(\omega_1 + \omega_2)^2}{\eta(b-1+d)\sqrt{8}}; \quad p_2 = \frac{\left(\sqrt{\frac{\omega_1}{2}} - \sqrt{\frac{\omega_2}{2}} + \eta(2(b-1)(1-d) + bd)\right)(\omega_1 - \omega_2)}{2\eta(1-d-b)} - \omega_2$$

and $h(z) = z^3 + p_2 z + q_2$, we obtain

$$h(z) = 0 \tag{39}$$

The maximum number of solutions of Equation (39) in the interval $\left(\sqrt{\frac{\omega_1+\omega_2}{2}}, \sqrt{\omega_1}\right)$ corresponds to the maximum number of solutions of equation $\Delta\Pi(x) = 0$ in the interval $(0.5, 1)$. Since $q_2 > 0$ and $p_2 < 0$, by Cardano-Descartes rule of signs Equation (39) has at most two positive solutions, therefore at most two solutions in the interval $\left(\sqrt{\frac{\omega_1+\omega_2}{2}}, \sqrt{\omega_1}\right)$. Since function $h(z)$ is a third-degree polynomial such that $h(z) \to +\infty$ as $z \to +\infty$ which has at most two positive zeros and let $\overline{z}$ be a positive real value such that $\frac{\partial h(\overline{z})}{\partial z} = 0$, two solutions of Equation (39) in the interval $\left(\sqrt{\frac{\omega_1+\omega_2}{2}}, \sqrt{\omega_1}\right)$ imply the following necessary conditions

$$\begin{cases} \overline{z} = \sqrt{\frac{-p_2}{3}} \in \left(\sqrt{\frac{\omega_1+\omega_2}{2}}, \sqrt{\omega_1}\right) \\ h\left(\sqrt{\omega_1}\right) > 0 \\ h\left(\sqrt{\frac{\omega_1+\omega_2}{2}}\right) > 0 \\ h(\overline{z}) < 0 \end{cases} \Leftrightarrow \begin{cases} -3\frac{\omega_1+\omega_2}{2} > p_2 > -3\omega_1 \\ p_2 > -\frac{q_2}{\sqrt{\omega_1}} - \omega_1 \\ p_2 > -\frac{q_2}{\sqrt{\frac{\omega_1+\omega_2}{2}}} - \frac{\omega_1+\omega_2}{2} \\ q_2 < \sqrt{\frac{-4}{27}(p_2)^3} \end{cases} \tag{40}$$

Using the first one of these necessary conditions (specifically $-p_2 < 3\omega_1$), the last one implies $q_2 < 2\omega_1\sqrt{\omega_1}$. Then, the third one requires $p_2 > -3\frac{\omega_1+\omega_2}{2}$, which in turn contradicts the first one. Hence, there is at most one root of Equation (39) in the interval $\left(\sqrt{\frac{\omega_1+\omega_2}{2}}, \sqrt{\omega_1}\right)$ and equation $\Delta\Pi(x) = 0$ has at most a unique solution in the interval $(0.5, 1)$. Then, at most a unique equilibrium of Model (12) exists in $(0.5, 1)$ and its existence implies ($\Delta\Pi$ is a continuous function and $\Delta\Pi(1) < 0$ always) $\Delta\Pi(0.5) > 0$. Finally, to prove that the inner equilibria are at most two, let us recap that two equilibria in $(0, 0.5]$ require $b(1-d) - 1 < 0$, then the function on the left-hand side of Equation (34) (which is equal to $\Delta\Pi(x)$ $\forall x \in (0, 0.5]$) tends to $-\infty$ as x tends to $+\infty$ and it does not have roots for $x \geq 0.5$. Therefore, the continuity of the function implies $\Delta\Pi(0.5) < 0$. As already verified, the latter inequality implies no equilibria in $(0.5, 1)$. Hence, at most two equilibria exist in $(0, 1)$. Let us denote the greater of the these two equilibria by x_u^l and by x_l^l the smaller one. Then, property 3 follows by noting that x_l^l has to belong to the interval $(0, 0.5]$ since two equilibria in $(0.5, 1)$ cannot exist.

4. Let us consider $\omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$, which implies (first two properties of the current Lemma) that $\Delta\Pi(1) < 0$ and $\Delta\Pi(0) > 0$. Then, by continuity of function $\Delta\Pi$ the equation $\Delta\Pi(x) = 0$ has either one or more than two solutions in $(0, 1)$. This implies that Model (12) has to have either one or more than two equilibria in $(0, 1)$. According to property 3 of the current Lemma more than two equilibria cannot exist in $(0, 1)$. Hence, $\forall\omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$ a unique equilibrium x_l^I exists in $(0, 1)$. Moreover, since $\Delta\Pi(0) > 0$, by continuity of function $\Delta\Pi$ we have that $\Delta\Pi(x) > 0$ $\forall x \in \left(0, x_l^I\right)$ and $\Delta\Pi(x) < 0$ $\forall x \in \left(x_l^I, 1\right)$. Then, from property 1 in Lemma 1 it follows that $\mathcal{B}\left(x_0^S\right) = \left[0, x_l^I\right)$ and $\mathcal{B}\left(x_1^S\right) = \left(x_l^I, 1\right]$, which proves property 4.

5. Let us consider $\omega_2 \in \left(0, \omega_2^{TB}\right)$, which implies (first two properties of the current Lemma) $\Delta\Pi(1) < 0$ and $\Delta\Pi(0) < 0$. Then, by continuity of function $\Delta\Pi$ the equation $\Delta\Pi(x) = 0$ has either zero solutions or at least two solutions in $(0, 1)$. Therefore, $\Delta\Pi(x) < 0$ $\forall x \in (0, 1)$ if there are not internal equilibria, otherwise by property 3 two equilibria, x_l^I and x_u^I, exist in $(0, 1)$ and $\Delta\Pi(x) < 0$ $\forall x \in \left(x_u^I, 1\right)$. Hence, property 1 in Lemma 1 implies that x_u^I is unstable. This proves property 5.

■

Proof of Theorem 3. By properties 3 in Lemma 1 x_0^S and x_1^S are equilibria of Model (12) which has, by property 3 in Lemma 2, at most two more (internal) equilibria in $(0, 1)$. Then, only three possible configurations of rest points for Model (12) are possible. The first one characterized by two internal equilibria. The second one characterized by a unique internal equilibrium. The third one characterized by no internal equilibria. By continuity of Map (12), the transition from one configuration to another occurs through bifurcations at which equilibria merge. From bifurcation theory, see e.g. Devaney (1989), fold and transcritical are the only two bifurcations through which equilibria can appear/disappear from an invariant set of a one-dimensional continuous map as (12). A *transcritical bifurcation* occurs at $\omega_2 = \omega_2^{TB}$, since x_0^S is an equilibrium before and after it, the associated eigenvalue is equal to one and by properties 4 and 5 of Lemma 2 an equilibrium has either to enter or to exit the invariant set $[0,1]$ when ω_2 increases and crosses ω_2^{TB}. Let us label by ω_2^{TBa} the bifurcation value ω_2^{TB} when a transcritical bifurcation occurs such that an internal equilibrium appears and exists in $[0,1]$ for ω_2 in a left-side neighborhood of ω_2^{TB} and ω_2^{TBb} otherwise. Since equilibrium x_1^S is always locally asymptotically stable as stated in Lemma 1, equilibrium x_u^I is always unstable and x_0^S changes stability only at ω_2^{TB}, see Lemma 2, by straightforward considerations there are no further transcritical bifurcations. Concerning *fold bifurcations*, they imply a couple of fixed points that collide and annihilate each other and thus the existence of an $\bar{x}$ such that $\bar{x} = x_u^I = x_l^I$ and $\Delta\Pi(\bar{x}) = 0$. Since $x_l^I \in (0, 0.5]$ as stated in property 3 of Lemma 2, it follows that $\bar{x} \in (0, 0.5]$. Let us assume $\left(\overline{x}_1, \omega_2^{FB1}\right)$ and $\left(\overline{x}_2, \omega_2^{FB2}\right)$ to be the values of the state variable and of parameter ω_2 at which a fold bifurcation occur and without loss of generality let us assume $\omega_2^{FB1} \geq \omega_2^{FB2}$. The appearance/disappearance of two internal fixed points in case of fold bifurcations implies $\omega_2^{FB1}, \omega_2^{FB2} \in \left(0, \omega_2^{TB}\right)$, see properties 4 and 5 in Lemma 2. Since $\Delta\Pi(0) < 0$ and $\Delta\Pi(1) < 0$ when $\omega_2 \in \left(0, \omega_2^{TB}\right)$, as implicit in property 5 of Lemma 2, the existence of two equilibria for $\omega_2 \in \left(\omega_2^{FB2}, \omega_2^{FB1}\right)$ implies an $\tilde{x} \in (0,1)$ such that $\Delta\Pi(\tilde{x}) > 0$ for some $\omega_2 \in \left(\omega_2^{FB2}, \omega_2^{FB1}\right)$ and $\Delta\Pi(\tilde{x}) < 0$ for some $\omega_2 \in \left(0, \omega_2^{FB2}\right) \cup \left(\omega_2^{FB1}, \omega_2^{TB}\right)$, which is impossible since $\Delta\Pi(\tilde{x})$ is a strictly convex function w.r.t ω_2. Then, two equilibria cannot exist in $\left(\omega_2^{FB2}, \omega_2^{FB1}\right)$, therefore neither fold bifurcations, and (from the strict convexity of $\Delta\Pi(\overline{x}_1)$ and $\Delta\Pi(\overline{x}_2)$ w.r.t ω_2) $\Delta\Pi(\overline{x}_1) > 0$ for all $\omega_2 \in \left(0, \omega_2^{FB2}\right)$ and $\Delta\Pi(\overline{x}_2) > 0$ for all $\omega_2 \in \left(\omega_2^{FB2}, \omega_2^{TB}\right)$, therefore at least two internal equilibria exist for $\omega_2 \in \left(0, \omega_2^{FB1}\right) \cup \left(\omega_2^{FB2}, \omega_2^{TB}\right)$. Since more than two equilibria for $\omega_2 \in \left(0, \omega_2^{FB2}\right) \cup \left(\omega_2^{FB1}, \omega_2^{TB}\right)$ cannot exist, see property 3 in Lemma 2, further fold bifurcations cannot occur for $\omega_2 \in \left(0, \omega_2^{FB2}\right) \cup \left(\omega_2^{FB1}, \omega_2^{TB}\right)$ either. Hence, varying ω_2 in $(0, \omega_1)$ at most two fold bifurcations can occur, one at $\omega_2^{FB2} < \omega_2^{FB1}$ and the other one at $\omega_2^{FB1} < \omega_2^{TB}$. Without loss of generality assume that ω_2^{FB2} implies ω_2^{FB1}. Then, all the possible sequences of bifurcations are:

1. $\omega_2^{FB2} < \omega_2^{FB1} < \omega_2^{TBa}$;
2. $\omega_2^{FB2} < \omega_2^{FB1} < \omega_2^{TBb}$;
3. $\omega_2^{FB1} < \omega_2^{TBa}$ (ω_2^{FB2} does not exist);
4. $\omega_2^{FB1} < \omega_2^{TBb}$ (ω_2^{FB2} does not exist);
5. ω_2^{TBa} (ω_2^{FB2} and ω_2^{FB1} do not exist);
6. ω_2^{TBb} (ω_2^{FB2} and ω_2^{FB1} do not exist);

Let us start noting that, independently of which sequence of bifurcation occurs, the dynamics of Model (12) for $\omega_2 \in \left(\omega_2^{TB}, \omega_1\right)$ (where ω_2^{TB} is either ω_2^{TBa} or ω_2^{TBb}) does not change and it is described by property 4 in Lemma 2. Then, by Lemmas 1 and 2, the first, the third and the fifth sequences of bifurcations imply scenarios 3, 1 and 2, respectively, and the numerical simulations in Figure 2, Panel (a) and in Figure 1, Panel (a) and Panel (b), prove their existence. The second sequence of bifurcations is impossible since it would imply the existence of two internal equilibria for $\omega_2 \in \left(\omega_2^{FB2}, \omega_2^{FB1}\right)$ which we already proved to be impossible. As a last step, it remains to discuss the bifurcation sequences 4 and 6. Concerning these two cases, let us note that condition $b(1-d) > 1$ implies $\Delta\Pi(0.5) > 0$ when $\Delta\Pi(0) = 0$ at $\omega_2 = \omega_2^{TBb}$. Thus, since $\Delta\Pi(1) < 0$ always and $\Delta\Pi$ is a continuous function, an equilibrium lays in the subspace $(0.5, 1)$ when $\omega_2 = \omega_2^{TBb}$. Moreover, the continuity of Map (12) implies that it exists an equilibrium in $(0.5, 1)$ even for ω_2 slightly larger than ω_2^{TBb}. Then, by property 4 of Lemma 2 an equilibrium in $(0, 0.5)$ that converges to 0 when ω_2 converges to ω_2^{TBb} from the right does not exist. This implies that the sequences of bifurcations 4 and 6 are not possible. However, they can occur for $b(1-d) < 1$ and, by Lemmas 1 and 2, they imply that either no locally asymptotically stable equilibrium exists in $(0,1)$ for any value of $\omega_2 \in (0,\omega_1)$ (it is the case of sequence 6) or it exists only for $\omega_2 \in \left(0, \omega_2^{FB1}\right)$ (it is the case of sequence 4).

Since $\omega_2^{TB} = \omega_2^{TBa} = \omega_2^{TBb}$, the statement of the Lemma follows. ∎

Proof of Lemma 3. By Lemma 2 (the last item), we know that x_l^I is the only steady state of residential integration that can be locally asymptotically stable. Moreover, since x_l^I is a steady state and it lies in $(0, 0.5)$, we have

$$\Pi_1\left(x_l^I\right) - \delta_1 = \Pi_2\left(x_l^I\right) - \delta_2 \tag{41}$$

Then, according to definition 1, x_l^I is sustainable if and only if

$$\Pi_1\left(x_l^I\right) - \delta_1 > 0 \tag{42}$$

Since, $x_l^I > d/2$ implies (42), and condition (15) implies

$$\Pi_1\left(x_l^I\right) - \delta_1 > \frac{\omega_1}{\sqrt{\omega_2+\omega_1}} - \sqrt{\frac{\omega_1}{2}} - \eta d > 0 \tag{43}$$

the statement of the lemma follows. ∎

Proof of Proposition 1. Since each individual takes a stochastic residential decision given the current state $\mathbf{N}_t = (N_{1,t}, N_{2,t})$, the evolution of $\mathbf{N}_t$ and therefore of X_t is a Markov process on the state space $\mathscr{L} \subset \mathbb{N}^2$. We now show that if the population size in the neighborhood $n_t = N_{1,t} + N_{2,t}$ is sufficiently large, then we can describe the joint evolution of $N_{1,t}$ and $N_{2,t}$ and therefore of X_t by considering the deterministic dynamics generated by the conditional expectation formation (see Dawid, 2007). Let us denote by $\{\mathbf{n}_t\}_{t=0}^{\infty}$ with $\mathbf{n}_t = (n_{1,t}, n_{2,t})$ the dynamical system generated by $\mathbf{n}_{t+1} = \mathbb{E}\left[\mathbf{N}_{t+1}|\mathbf{N}_t = \mathbf{n}_t\right]$. Given that the probability of an individual household to choose the neighborhood over the outside option is $\mathbb{P}(\tilde{\Pi}_i'(\mathbf{n}_t) + \varepsilon_t > \delta_i')$, where $\tilde{\Pi}_i' = \tilde{\Pi}_j$ and $\delta_i' = \delta_j$ for all j of type i, and the noise terms ε_t are assumed to be independent across periods and follow a logistic distribution with zero mean represented by a cumulative distribution function Θ, we obtain two dynamic equations describing the evolution of the number of households of type i residing in the neighborhood over time, given by

$$n_{i,t+1} = 2\Theta\left(\tilde{\Pi}_i'\left(\mathbf{n}_t\right) - \delta_i'\right) n_{i,t} \quad \text{where} \quad \Theta\left(\tilde{\Pi}_i'\left(\mathbf{n}_t\right) - \delta_i'\right) = \frac{1}{1+\exp\left(\beta\left(\delta_i' - \tilde{\Pi}_i'\left(\mathbf{n}_t\right)\right)\right)}, \tag{44}$$

with $i = 1, 2$. Then, one obtains a difference equation describing the deterministic residential integration dynamics $\tilde{x}_t$ with

$$\tilde{x}_{t+1} = \frac{\tilde{x}_t}{\tilde{x}_t + (1-\tilde{x}_t)\frac{1+\exp\left(\beta\left(\delta_1' - \tilde{\Pi}_1'(\mathbf{n}_t)\right)\right)}{1+\exp\left(\beta\left(\delta_2' - \tilde{\Pi}_2'(\mathbf{n}_t)\right)\right)}}. \tag{45}$$

Given that the probability of an individual household to choose the neighborhood is $\mathbb{P}(\tilde{\Pi}_i + \varepsilon_t > \delta_i)$ and the error term ε_t is stochastically independent given state $\mathbf{N}_t = \mathbf{Y}(X_t, n_t)$ and follows a logistic distribution with mean zero, from the standard law of large numbers one can show that the following condition holds

$$\lim_{n_t \to \infty} \sup_{\mathbf{Y}(X_t, n_t) \in \mathscr{L}} \left[Var\left[N_{i,t+1}|\mathbf{N}_t = \mathbf{Y}(X_t, n_t)\right]\right] = 0 \tag{46}$$

for $i = \{1, 2\}$ and $X_t \in [0, 1]$. Then, following Proposition 3 and 4 in Dawid (2007), for any given T and any given $\gamma > 0$ and $\xi > 0$ there is a minimal population size $\bar{n}$ such that for any $n_t > \bar{n}$:

$$\mathbb{P}\left(\max_{t=1,\dots,T} |N_{i,t} - n_{i,t}| > \gamma\right) < \xi \;\forall\; \mathbf{N}_0 = \mathbf{n}_0 \in \mathscr{L} \text{ and } i \in \{1, 2\}. \tag{47}$$

Since the evolution of $N_{i,t}$ can be approximated by $n_{i,t}$ for sufficiently large populations, also the trajectory of $\tilde{x}_t$ can be used as an approximation of the trajectory of X_t.

Now let us note that $\delta_1 = \delta_1'$ and $\delta_2 = \delta_2'$. Hence, the Map (45) can be obtained from (12) substituting $\Pi_1(x)$ with $\tilde{\Pi}_1'(x)$ and $\Pi_2(x)$ with $\tilde{\Pi}_2'(x)$. Then, since the results about the dynamics of Map (12) in Theorems 1, 2, and 3 and Lemmas 2 and 3 depend on $\Delta\Pi = \delta_1 - \delta_2 + (\Pi_1(x) - \Pi_2(x))$ only and $\Pi_1(x) - \Pi_2(x) = \tilde{\Pi}_1(x) - \tilde{\Pi}_2(x)$, the long term dynamics of Map (12) and Map (45) are identical and the statement of the proposition follows. ■

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