On the Monopole Map in Three Dimensions

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Chapter 0

Introduction

0.1 Historical background and motivation

Compared to other dimensions, the world of smooth 4-dimensional manifolds remains largely mysterious. Methods that offer many important answers in others dimensions do not have an adaptation applicable to dimension 4. An important breakthrough came towards the end of the last century with the introduction of the ideas of gauge theory from physics into mathematics. This breakthrough started with Donaldson's theory [Don83], and continued with the inception of the Seiberg-Witten theory [Wit94].

The basic idea of the two theories is similar in principle. Both extract information about 4-manifolds by analysing the moduli space of solutions of certain differential equations on the manifold. The latter, however, has significant technical advantages over the former. As a result, the ensuing activity led to striking new insights in the world of smooth 4-dimensional manifolds ([Don96, §6], [Sco05, Ch. 10]). However, the Seiberg-Witten theory also has some limitations. One notable example is its inability to provide information about connected sums of 4-manifolds.

A stable cohomotopy invariant was proposed and constructed in [BF04] as a new, non-mainstream way of describing the Seiberg-Witten invariant. Rather than directly analysing the moduli space of monopoles (i.e. solutions of the Seiberg-Witten equations), the central object of interest in [BF04] (as well as in its immediate successor [Bau04b]) is the so called monopole map. The idea of this new approach is to use the monopole map in a certain stable homotopy setup and construct a class associated to the underlying $\mathrm{spin}^{\mathbb{C}}$ 4-manifold.

Since the Seiberg-Witten equations appear as the main ingredients of the monopole map, the resulting invariant is very closely related to the Seiberg-Witten invariant. It, however, yields some information on decomposable 4-manifolds undetected by the latter.

The Seiberg-Witten theory is originally a theory for 4-dimensional manifolds. Its success in providing information about 4-manifolds motivated efforts to use it in the research of 3-dimensional manifolds. Adaptations to the 3-dimensional world were provided through several different approaches (notably [KM07, Nic03]).

In this thesis, the possibility of applying the procedure from [BF04] to closed 3-dimensional manifolds is investigated. Although successful, direct application of the methods from [BF04] to the monopole map in three dimensions fails to yield interesting information about the underlying manifold.

For this reason, instead of the monopole map, a family of a certain type of perturbations parametrised by the complex plane is analysed. The main focus of this thesis is the study of the limit behaviour of this family. In particular, a certain technical condition (the so called boundedness property) is proved. This condition is needed in order to be able to extend the new monopole maps to the 1-point compactification of a certain Hilbert space and is therefore vital for exploiting the stable homotopic apparatus used in [BF04].

0.2 Thesis overview

Chapter 1 serves as a preparation for the main discussion in later chapters. Some general notational and computational conventions are covered in §1.1 and §1.2. Section 1.3 lists definitions of different scalar products and norms used in the concrete models for objects needed in the Seiberg-Witten theory which are presented in §1.4. These concrete models are defined using quaternions, which enable an elegant presentation of the objects in question and as a result simplify local calculations. In addition, some well-known related constructions are carried out explicitly in the given concrete setup.

Sections 1.5 and 1.6 treat certain subspace of differential forms as a Clifford module and the corresponding Dirac operator defined on it is briefly discussed.

The chapter's *raison d'être* is Section 1.7, where a detailed analysis of the quadratic term is carried out. First, the quadratic term is described in terms of the quaternionic model from previous sections and a relation between some scalar products is established. Next, an expression for the derivative of the quadratic term is provided in §1.7.2. This expression plays an important role in Chapter 2 in the proof of the boundedness property. Moreover, it is used in the final part of the section to obtain an estimate of Sobolev norms of the quadratic term needed in the proof of the boundedness property in Chapter 3.

In Chapter 2 the monopole map on a closed 3-dimensional manifold is defined along the lines of [BF04]. Its properties on a general closed 3-manifold are examined in detail, and its shortcomings are discussed.

A modification of the monopole map is proposed in Chapter 3. Some interesting new terms are added and after suitable renormalisation, a family of monopole maps is obtained in §3.1. Section 3.2 deals with the boundedness property and is divided into three parts. An appropriate modification of the boundedness property for this family is introduced in §3.2.1. Subsection 3.2.2 shows how to obtain a priori estimates, and in §3.2.3 the estimates needed in the boundedness property are attained with the help of a modified version of the bootstrapping argument. The result, which is the main result of the thesis, is summarised in §3.3. After proving the modified boundedness property, a further renormalisation of the map is carried out in §3.4.

Chapter 4 illustrates how the general discussion of the previous chapters applies to a concrete example: a 3-torus. The monopole map on a 3-torus is expliticly written down using concrete models and conventions from Chapter 1. This example was used to detect and understand subtle differences in the proofs of the boundedness property of different versions of the monopole map.

Some general and well-known facts needed in the discussion of the monopole map are included in the Appendix. Section A.1 summarises selected important results from the theory of elliptic operators and Sobolev spaces. Due to a large amount of freedom at choosing the conventions and the inconsistency of the choice in the literature, several relatively basic and standard calculations were performed in the sections on Hodge theory (§A.2) and Clifford and exterior algebras (§A.3). Section A.4 recalls the definition of the connection Laplacian, and contains a basic calculation.

Although their presence is not crucial, some auxiliary calculations are added with the intention of easing the readability. They appear in the form of endnotes in order to avoid lengthy digressions from the main text. Endnotes are indicated in the same way as footnotes, except the number is framed in order to distinguish the two. For example: an endnote[0], a footnote[0].

As a notational remark, throughout the thesis the letter Y is used to denote a closed 3-manifold, while M is reserved for a general closed n-manifold. Most of other notational conventions used in the text can be found in the list of notation.

The symbol ◄ is used to indicate the end of remarks and conventions. Theorems, Propositions, Lemmas etc. are all numbered by the same counter to make them easier to find.

0.3 A remark

Lastly, a remark regarding the proof of the boundedness property in Chapter 3. In the bootstrapping argument in §3.2.3, an estimate for the norm of the quadratic term and a similar estimate for Clifford product are used to carry the argument through. However, the general fact that Sobolev L^p_k-completion is a Banach algebra for $pk > n$ ([Pal68, Corollary 9.7]) is already enough to conclude the proof.

At the time the proof of the boundedness property was being compiled, the above-mentioned fact about Sobolev spaces managed to escape my attention, and so a weaker version of Sobolev's theorem (Theorem A.1.6) was used instead. Since using the weaker version only slightly prolongs the proof, the original version of the proof is left in the thesis. The shorter version would mean skipping §1.7.3 and the proof of (3.11), and using the above-mentioned fact to conclude the bootstrapping argument as explained in (appropriately located) Remark 3.2.4.

Chapter 1

Preliminaries

As mentioned in the introduction, this chapter will serve as a reminder of some results used later and also to fix notation and conventions used in the rest of the thesis. A significant part of the choice of conventions and models presented here is borrowed from [Bau12].

1.1 Notation and remarks

The symbol $\lesssim$ will denote inequalities up to a multiplication of the right-hand side by some positive constant. For example, if $l - \frac{n}{q} \leq k - \frac{n}{p}$ and $l \leq k$ holds, then instead of writing

$$\exists C = C_{p,k,l,q} > 0 \ \text{ s.t. } \ \|\,.\,\|_{L^q_l} \leq C\,\|\,.\,\|_{L^p_k},$$

we will write

$$\|\,.\,\|_{L^q_l} \lesssim \|\,.\,\|_{L^p_k}.$$

This notation will also be used to indicate that there is a bound on the set of objects (or rather on their norms). For example, in Chapters 2 and 3, an expression of the form $\|\psi\|_{L^p_k} \lesssim 1$ will occur frequently. This means, that there exists a fixed positive constant $R' > 0$ such that $\|\psi\|_{L^p_k} \leq R'$ holds *simultaneously for all* spinors ψ from some set we are interested in at the given moment.

It is worth stressing at this point that the expression $\|\psi\|_{L^p_k} \lesssim 1$ is *not* meant to indicate that $\|\psi\|_{L^p_k} < \infty$ (i.e. that ψ is an L^p_k-spinor); rather, it means that its L^p_k-norm is bounded by some fixed positive real number, as explained above.

The following simple inequality will be repeatedly used in the text without mentioning it

$$\|x+y\|^p \lesssim \|x\|^p + \|y\|^p. \tag{1.1}$$

Here, $\|\,.\,\|$ denotes an arbitrary norm (in the text usually one of the Sobolev norms, e.g. $\|\,.\,\|_{L^p}$) and $p \geq 0$. This can easily be shown as follows

$$\|x+y\|^p \leq \left(\|x\| + \|y\|\right)^p \leq \left(2\max\{\|x\|, \|y\|\}\right)^p \leq 2^p\left(\|x\|^p + \|y\|^p\right).$$

1.2 Some identifications and conventions

In this section we present a quaternionic model for spinors, Clifford bundle etc. The main motivation for presenting a concrete model for these objects will primarily be its use in the analysis of the quadratic term in §1.7. Also, it will be useful in describing relationships between scalar products of spinors as well as forms and endomorphisms (§1.3).

$\mathbb{R}^4$ will be identified with the quaternions $\mathbb{H}$ in the obvious way

$$\mathbb{R}^4 \ni h = (x_0, x_1, x_2, x_3) \equiv x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}. \tag{1.2}$$

Specifically,

$$\begin{aligned} \mathbb{R}^4 &\ni e_0 \equiv 1 \in \mathbb{H}, \\ \mathbb{R}^4 &\ni e_1 \equiv i \in \mathbb{H}, \\ \mathbb{R}^4 &\ni e_2 \equiv j \in \mathbb{H}, \\ \mathbb{R}^4 &\ni e_3 \equiv k \in \mathbb{H}, \end{aligned} \tag{1.3}$$

where e_0, e_1, e_2, e_3 denotes the canonical basis of $\mathbb{R}^4$.

Generally, when using $\mathbb{H}$ as a model for the spinor bundle, we will denote it by $\Delta_3 := \mathbb{H}$ and call its elements Dirac spinors or simply spinors[1]. The symbol $\Delta_3^{\mathbb{C}}$ will sometimes be used to stress the fact that we are interpreting Δ_3 as a complex space (see (1.17)).

Let $e_1, e_2, e_3 \in \mathbb{R}^3$ denote elements of the canonical ordered basis. $\mathbb{R}^3$ will be identified with the subspace of purely imaginary quaternions $\mathrm{Im}(\mathbb{H})$ via

$$e_1 \mapsto i \in \mathrm{Im}(\mathbb{H}), \qquad e_2 \mapsto j \in \mathrm{Im}(\mathbb{H}), \qquad e_3 \mapsto k \in \mathrm{Im}(\mathbb{H}). \tag{1.4}$$

[1] the notation and nomenclature are borrowed from [Fri97, p. 15]

The symbol $\Lambda^*(\mathbb{R}^n)$ will denote both the exterior algebra of $\mathbb{R}^n$ and the exterior algebra of its dual $(\mathbb{R}^n)^*$. Which interpretation will be used will depend on the context, and will generally not be mentioned explicitly. This imprecision is justified by the fact that $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ are canonically isomorphic via the standard inner product. In situations where elements of $\Lambda^*(\mathbb{R}^n)$ are indexed, a lower index will suggest that the elements are interpreted as vectors (e.g. $e_0, \ldots, e_{n-1} \in \mathbb{R}^n$), and an upper index will suggest that the elements are interpreted as covectors (e.g. $e^0, \ldots, e^{n-1} \in (\mathbb{R}^n)^*$).

The complexified exterior algebra of $\mathbb{R}^n$ and the complexified exterior algebra of its dual will be identified via the tensor product of the canonical isomorphism $\mathbb{R}^n \cong (\mathbb{R}^n)^*$ and the identity on $\mathbb{C}$. This means, we will not identify the two complexified algebras via dualising with the help of the complexified inner product on them[2].

In general, the Clifford algebra of $\mathbb{R}^n$ and the exterior algebra of $\mathbb{R}^n$ are canonically isomorphic (as vector spaces) via the assignment (cf. [LM89, p. 11])

$$C_n \ni e_I \longmapsto e^I \in \Lambda^*(\mathbb{R}^n), \tag{1.5}$$

where $I = (i_1, \ldots, i_k)$ is an ordered subset of $(0, \ldots, n-1)$, $e_I := e_{i_1} \underset{\mathrm{cl}}{\bullet} \ldots \underset{\mathrm{cl}}{\bullet} e_{i_k}$ and $e^I := e^{i_1} \wedge \ldots \wedge e^{i_k}$.

In accordance with the above-mentioned identification of the complexification of $\Lambda^*(\mathbb{R}^n)$ and its complexified dual, the complexified versions $C_n^{\mathbb{C}} = C_n \otimes_{\mathbb{R}} \mathbb{C}$ and $\Lambda^*_{\mathbb{C}}(\mathbb{R}^n) = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ will be identified via

$$C_n^{\mathbb{C}} \ni e_I \otimes \lambda \longmapsto e^I \otimes \lambda \in \Lambda^*_{\mathbb{C}}(\mathbb{R}^n), \tag{1.6}$$

i.e. via the vector space isomorphism (1.5) tensored with the identity on $\mathbb{C}$.

1.2.1. REMARK. The above isomorphisms will be understood without special mention when treating Clifford multiplication by a vector and its dual covector as the same operator. ◀

1.3 Scalar products and norms

The scalar product

$$\langle h, h' \rangle_{\mathbb{R}} := \mathrm{Re}(h \underset{\mathbb{H}}{\cdot} \bar{h}'), \tag{1.7}$$

[2] as an example, we will have $c(i \otimes e^1) = c(i \otimes e_1)$, and *not* $c(i \otimes e^1) = -c(i \otimes e_1)$, where c denotes the Clifford multiplication with the corresponding element

and the induced norm correspond under (1.2) to the standard Euclidean scalar product and the standard norm in $\mathbb{R}^4$. The standard Hermitian product reads

$$\langle h, h' \rangle_{\mathbb{C}} = \operatorname{Re}(h \cdot_{\mathbb{H}} \bar{h}') - i \operatorname{Re}(h \cdot_{\mathbb{H}} i \cdot_{\mathbb{H}} \bar{h}'). \tag{1.8}$$

On the space $\operatorname{End}_{\mathbb{H}}(\Delta_3) = \mathbb{H}$ of $\mathbb{H}$-linear endomorphisms of Δ_3 we define the following real inner product:

$$\langle a, b \rangle_{\operatorname{End}_{\mathbb{H}}(\Delta_3)} := \operatorname{Re}\operatorname{tr}(ab^*) = \operatorname{Re}(a\bar{b}). \tag{1.9}$$

Complexification of (1.9) reads

$$\langle a \otimes \lambda, b \otimes \tau \rangle = \operatorname{Re}(a\bar{b}) \cdot \lambda\bar{\tau}. \tag{1.10}$$

On the space $\operatorname{End}_{\mathbb{C}}(\Delta_3)$ of $\mathbb{C}$-linear endomorphisms of Δ_3 we define the following real inner product:

$$\begin{aligned} \langle a \otimes \lambda, b \otimes \tau \rangle_{\operatorname{End}_{\mathbb{C}}(\mathbb{H})} &:= \operatorname{Re}\operatorname{tr}\Big((a \otimes \lambda) \circ (b \otimes \tau)^*\Big) \\ &= \operatorname{Re}\operatorname{tr}\Big((a \otimes \lambda) \circ (\bar{b} \otimes \bar{\tau})\Big) \\ &= \operatorname{Re}\operatorname{tr}\Big((a\bar{b} \otimes \bar{\tau}\lambda)\Big) = \operatorname{Re}(a\bar{b}) \cdot \operatorname{Re}(\lambda\bar{\tau}). \end{aligned} \tag{1.11}$$

When considering $\mathbb{C}$-endomorphisms of Δ_3 as complex matrices of order 2, we will be using

$$\langle A, B \rangle_{\mathbb{C}(2)} := \frac{1}{2} \operatorname{Re}\operatorname{tr}(AB^*), \tag{1.12}$$

where $A, B \in \mathbb{C}(2)$, in line with the general definition

$$\langle A, B \rangle_{\mathbb{K}(n)} := \frac{1}{n} \operatorname{Re}\operatorname{tr}(AB^*), \qquad \mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}. \tag{1.13}$$

If matrices A and B correspond[3] to $a \otimes \lambda$ and $b \otimes \tau$ respectively, the values of the scalar products coincide. In that way, the value of the scalar product is independent of the interpretation.

The real scalar product (1.7) on $\mathbb{H}$ induces the standard Euclidean product on $\mathbb{R}^3$ through (1.4). This induces a scalar product on $\Lambda^*(\mathbb{R}^3)$ determined by the requirement

$$\left\langle e^I, e^J \right\rangle := \begin{cases} 1, & I = J, \\ 0, & I \neq J. \end{cases}, \tag{1.14}$$

[3] the correspondence will be specified in §1.4.1 (Conventions 1.4.6 and 1.4.10)

where I, J stand for strictly increasing ordered multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$, and e^I stands short for $e^{i_1} \wedge \dots \wedge e^{i_k}$.

In general, the standard inner product on $\mathbb{R}^n$ induces an inner product on $\Lambda^*(\mathbb{R}^n)$, and on $\Lambda^*_{\mathbb{C}}(\mathbb{R}^n) = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ a Hermitian product by

$$\langle \alpha \otimes \lambda, \beta \otimes \tau \rangle_{\mathbb{C}} \coloneqq \langle \alpha, \beta \rangle \cdot \lambda \bar{\tau},$$

and an inner product by

$$\langle \alpha \otimes \lambda, \beta \otimes \tau \rangle_{\mathbb{R}} \coloneqq \langle \alpha, \beta \rangle \cdot \mathrm{Re}(\lambda \bar{\tau}) = \mathrm{Re} \langle \alpha \otimes \lambda, \beta \otimes \tau \rangle_{\mathbb{C}}.$$

All these different inner and Hermitian products will be denoted by $\langle . , . \rangle_{\mathbb{R}}$ and $\langle . , . \rangle_{\mathbb{C}}$ respectively. Whenever possible, $\mathbb{R}$ will be omitted from $\langle . , . \rangle_{\mathbb{R}}$ to further simplify notation. Possible ambiguities will be left to the context to resolve. In places where the context is not clear enough, it will be explicitly mentioned which particular inner or Hermitian product is meant, or the appropriate suggestive notation will be used.

Finally, note that for a 1-covector $a \in \Lambda^1(\mathbb{R}^3)$ and a spinor $h \in \Delta_3$ we have

$$\langle i \mathbin{\dot{s}} a \bullet_{\mathrm{cl}} h, h \rangle_{\Delta_3} = \mathrm{Re}(a h \bar{i} \bar{h}) = \left\langle a, h i \bar{h} \right\rangle_{\mathrm{End}_{\mathbb{H}}(\Delta_3)} = \left\langle a, h i \bar{h} \right\rangle_{\Lambda^*(\mathbb{R}^3)}. \tag{1.15}$$

The last equality requires some explanation. Later in the chapter[4], we will specifiy a concrete isomorphism $\Lambda^{0,1}_{\mathbb{C}}(\mathbb{R}^3) \cong \mathrm{End}_{\mathbb{C}}(\Delta_3)$. With respect to the above inner products, this isomorphism becomes an isometry (regardless of how one choses to write elements of the latter space). Also, it is worth pointing out that under the mentioned isometry, $\mathrm{End}_{\mathbb{H}}(\Delta_3)$ corresponds to the space $\Lambda^{0,1}(\mathbb{R}^3)$.

1.4 Clifford algebras and spin groups

The general Dirac spinors[5] $\Delta_n = \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}}$ for $n = 3$ yield $\Delta_3 = \mathbb{C}^2 \equiv \mathbb{H}$. In particular, $\Delta_3 = \mathbb{H}$ is the unique irreducible real representation of $Spin(3)$, and[6] $\Delta_3 \equiv \mathbb{C}^2$ is the unique irreducible complex representation of $Spin^{\mathbb{C}}(3)$ ([LM89, §I.5]). For this reason, $\Delta_3 = \mathbb{H}$ will serve as a local model for the spinor bundle, on which we will have a Clifford module structure, as well as the structure of a complex vector space. Considering the chosen quaternionic model, there are two main options for realisation of the two above-mentioned module structures:

[4] see Convention 1.4.10

[5] [Fri97, p. 15]

[6] see (1.17)

(i) Clifford product as the left-hand side $\mathbb{H}$-multiplication, and scalar multiplication on the right-hand side;

(ii) Clifford product as the right-hand side $\mathbb{H}$-multiplication, and scalar multiplication on the left-hand side.

Since both module structures naturally belong on the left-hand side, there is a certain amount of unnaturality in both cases. Thus, it is probably fair to say that it is a matter of taste which one of them one choses. In this thesis we will use the first option.

1.4.1. **Convention.** Left-hand side multiplication by a quaternion will represent Clifford multiplication, and right-hand side multiplication by a conjugate will represent scalar multiplication. ◀

In order to avoid ambiguity with the Clifford product, a special symbol will sometimes be used to denote scalar multiplication. For a scalar λ (primarily from $\mathbb{C}$), and a spinor $h \in \Delta_3 = \mathbb{H}$ we will write

$$\lambda \cdot_{s} h := h \cdot_{\mathbb{H}} \bar{\lambda},$$

with $\cdot_{\mathbb{H}}$ denoting quaternionic multiplication. As a consequence of Convention 1.4.1, we have

1.4.2. **Convention.** The space of quaternions, when considered as a 2-dimensional complex vector space, will be identified with $\mathbb{C}^2$ via

$$\begin{aligned} x &= x_0 + ix_1 + jx_2 + kx_3 \equiv (x_0 + ix_1, x_2 - ix_3) \\ &= h_1 + jh_2 \equiv (h_1, h_2). \end{aligned} \tag{1.16}$$

In other words, instead of the more natural (i.e. orientation preserving) identification $\mathbb{H} = \mathbb{C} + \mathbb{C}j$, we have

$$\mathbb{H} = \mathbb{C} + j\,\mathbb{C} \equiv \mathbb{C}^2. \tag{1.17}$$

This, furthermore, carries over to $\mathbb{C}(2) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2) = \mathbb{R} \oplus \mathfrak{su}(2)$. Namely $\mathbb{C}(2)$ is now a complex vector space generated by the identity I and an antihermitian version $\{i\sigma_3, -i\sigma_2, -i\sigma_1\}$ of trace-free Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.18}$$

with the $\mathbb{C}$-module structure given by

$$\lambda \cdot_{S} A \equiv A\_\bar{\lambda} \equiv \left[x \mapsto Ax \cdot \bar{\lambda}\right].$$

As an example, the endomorphism $A = \left[\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right] \in \mathbb{C}(2)$ acts on a vector $\mathbb{H} \ni v_1 + jv_2 = v \equiv \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{C}^2$ as follows:

$$Av = A\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} iv_1 \\ iv_2 \end{smallmatrix}\right] \equiv v \cdot_{\mathbb{H}} i = -v \cdot_{\mathbb{H}} \bar{i} = -i \cdot_{S} v,$$

hence $A = -i \cdot_{S} I$. ◀

1.4.3. Remark. Following Convention 1.4.1, we have

$$C_3^{\mathbb{C}} = C_3 \otimes_{\mathbb{R}} \mathbb{C} \ni a \otimes u \equiv \left(h \mapsto a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{u}\right).$$

Every element of $C_3^{\mathbb{C}}$ can be written as a sum of elements of the form $a \otimes u$, with $a \in C_3$, $u \in \mathbb{C}$.

An element of $Spin^{\mathbb{C}}(3) = Spin(3) \times_{\mathbb{Z}_2} U(1)$ will be denoted by the suggestive symbol $[a, u]$, with $a \in Sp(1)$, $u \in U(1)$ to stress that the pair $(a, u) \in Sp(1) \times U(1)$ is uniquely determined up to sign. ◀

With the usual general definitions ([LM89, §I.4]), in dimension 3 we have the isomorphisms:

$$\begin{aligned} C_3 &\cong \mathbb{H} \oplus \mathbb{H}, \\ C_3^{\mathbb{C}} &\cong \mathbb{C}(2) \oplus \mathbb{C}(2), \\ Spin(3) &\cong SU(2) \cong Sp(1), \\ Spin^{\mathbb{C}}(3) &\cong U(2). \end{aligned}$$

We will actually identify C_3 with the subalgebra of diagonal matrices in $\mathbb{H}(2)$, and write $C_3 \subseteq \mathbb{H}(2)$. Similarly, $C_3^{\mathbb{C}}$ will be identified with the subalgebra of $\mathbb{C}(4)$ consisting of block diagonal matrices, with blocks being elements of $\mathbb{C}(2)$.

We proceed with making the above-mentioned identifications explicit. Convention (1.4) suggests the following assignment:

$$\begin{aligned} 1 &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{H}(2), & e_1e_2 &\mapsto \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \in \mathbb{H}(2), \\ e_1 &\mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \mathbb{H}(2), & e_1e_3 &\mapsto \begin{bmatrix} -j & 0 \\ 0 & -j \end{bmatrix} \in \mathbb{H}(2), \\ e_2 &\mapsto \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix} \in \mathbb{H}(2), & e_2e_3 &\mapsto \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \in \mathbb{H}(2), \\ e_3 &\mapsto \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix} \in \mathbb{H}(2), & e_1e_2e_3 &\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{H}(2). \end{aligned} \tag{1.19}$$

In short, we have:

$$\mathbb{R}^3 \overset{(1.4)}{\equiv} \operatorname{im}\mathbb{H} \ni v \mapsto \begin{bmatrix} v & 0 \\ 0 & -v \end{bmatrix} \in \mathbb{H}(2). \tag{1.20}$$

1.4.4. REMARK. The minus sign in the lower right entry for e_1, e_2 and e_3 serves to distinguish e_1 from e_2e_3 etc. ◄

Convention (1.17) implies the following correspondence

$$\begin{aligned} i &\mapsto i\sigma_3 \in \mathbb{C}(2), \\ j &\mapsto -i\sigma_2 \in \mathbb{C}(2), \\ k &\mapsto -i\sigma_1 \in \mathbb{C}(2), \end{aligned} \tag{1.21}$$

and hence we get the identifications for $C_3^{\mathbb{C}} \subseteq \mathbb{C}(4)$:

$$\begin{aligned}
1 &\mapsto \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \in \mathbb{C}(4), & e_1e_2 &\mapsto \begin{bmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{bmatrix} \in \mathbb{C}(4), \\
e_1 &\mapsto \begin{bmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{bmatrix} \in \mathbb{C}(4), & e_1e_3 &\mapsto \begin{bmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix} \in \mathbb{C}(4), \\
e_2 &\mapsto \begin{bmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{bmatrix} \in \mathbb{C}(4), & e_2e_3 &\mapsto \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{bmatrix} \in \mathbb{C}(4), \\
e_3 &\mapsto \begin{bmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{bmatrix} \in \mathbb{C}(4), & e_1e_2e_3 &\mapsto \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \in \mathbb{C}(4),
\end{aligned} \tag{1.22}$$

where, $\sigma_1, \sigma_2, \sigma_3$ denote the Pauli matrices, as before (1.18).

Using the universal property of Clifford algebras, it is easy to see that (1.20) determines an isomorphism between abstractly defined C_3 and ${}^{\mathbb{H}}\oplus{}_{\mathbb{H}} \subseteq \mathbb{H}(2)$. The same applies to $C_3^{\mathbb{C}}$ and ${}^{\mathbb{C}(2)}\oplus{}_{\mathbb{C}(2)} \subseteq \mathbb{C}(4)$ via (1.22).

Given the above isomorphisms, we make the following (re)definitions:

$$C_3 := {}^{\mathbb{H}}\oplus{}_{\mathbb{H}} = \left\{ \begin{bmatrix} h & 0 \\ 0 & h' \end{bmatrix} : h, h' \in \mathbb{H} \right\} \subseteq \mathbb{H}(2), \tag{1.23}$$

$$C_3^{\mathbb{C}} := {}^{\mathbb{C}(2)}\oplus{}_{\mathbb{C}(2)} = \left\{ \begin{bmatrix} h \otimes_{\mathbb{R}} u & 0 \\ 0 & h' \otimes_{\mathbb{R}} u' \end{bmatrix} : h, h' \in \mathbb{H},\ u, u' \in \mathbb{C} \right\} \subseteq \mathbb{C}(4). \tag{1.24}$$

Next we determine the subset of ${}^{\mathbb{H}}\oplus{}_{\mathbb{H}}$ which corresponds to the group $Spin(3)$. General definition of the spin group reads $Spin(n) = Pin(n) \cap C_n^0$, where

$$Pin(n) \subseteq C_n^{\times}$$

denotes the group generated by elements of the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$, and C_n^0 denotes the +1-eigenspace of the automorphism of C_n induced by the assignment $\mathbb{R}^n \ni v \mapsto -v \in \mathbb{R}^n$ ([LM89, §I.2]).

1.4.5. Lemma. *The group*

$$Pin(3) \cap C_3^0$$

corresponds to the group

$$G = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in Sp(1) \right\}.$$

under the identification (1.19).

Proof. First note that

$$\begin{aligned} Pin(3) \cap C_3^0 &= \left\{ \pm 1,\ q_1 q_2, : q_1, q_2 \in S^2 \subseteq \mathbb{R}^3 \right\}, \\ &= \left\{ q_1 q_2, : q_1, q_2 \in S^2 \subseteq \mathbb{R}^3 \right\}. \end{aligned}$$

Let $G' \subseteq \mathbb{H}(2)$ denote the image of $Pin(3) \cap C_3^0$ under the isomorphism (1.19) and chose arbitrary $q_1 = \sum_{i=1}^3 \lambda_i e_i \in S^2$ and $q_2 = \sum_{j=1}^3 \mu_j e_j \in S^2$. The product $q_1 q_2$ takes the form

$$q_1 q_2 = -\sum_{i=1}^{3} \lambda_i \mu_i + \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i) e_i e_j.$$

The quaternion that emerges after applying (1.19) has the norm equal[1] to 1. Hence, it is clear that G' is a subgroup of G. The adjoint representation of the spin group[7] determines a double covering $Spin(3) \to SO(3)$. Thus $\dim G' = \dim Spin(3) = \dim SO(3) = 3 = \dim Sp(1) = \dim G$. Since G is connected, there follows $G' = G$. □

Therefore, as previously with the Clifford algebras, we set

$$\begin{aligned} Spin(3) &:= \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in \mathbb{H}(2) : a \in Sp(1) \right\}, \\ Spin^{\mathbb{C}}(3) &:= \left\{ \begin{bmatrix} [a,u] & 0 \\ 0 & [a,u] \end{bmatrix} \in \mathbb{C}(4) : a \in Sp(1),\ u \in U(1) \right\}. \end{aligned}$$

[7] see (1.30)

Obviously, $C_3 \subseteq C_3^{\mathbb{C}}$ and also $Spin(3) \subseteq Spin^{\mathbb{C}}(3)$.

One element of the real Clifford algebra C_n of $\mathbb{R}^n$ will play an important role later. It is the so called *volume element*

$$\omega := e_0 \cdots e_{n-1} \in C_n, \tag{1.25}$$

where $e_0, \ldots, e_{n-1}$ denote the vectors of the canonical basis in $\mathbb{R}^n$. The complex analogue of the real volume element (1.25) is defined as

$$\omega_{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} \cdot_{\mathbb{S}} e_0 \cdots e_{n-1} = e_0 \cdots e_{n-1} \otimes i^{-\lfloor \frac{n+1}{2} \rfloor} \in C_n^{\mathbb{C}}, \tag{1.26}$$

where $e_0, \ldots, e_{n-1}$ denote the vectors of the canonical basis in $\mathbb{C}^n$. It will be called the *complex volume element*[8]. Definitions (1.25) and (1.26) do not depend on the choice of the oriented orthonormal basis of $\mathbb{R}^n$.

Note that, under (1.22), the complex volume element in $C_3^{\mathbb{C}}$ takes the form

$$\omega_{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor} \cdot_{\mathbb{S}} e_1 e_2 e_3 = -e_1 e_2 e_3 \overset{(1.22)}{=} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathbb{C}(4). \tag{1.27}$$

1.4.1 Spin representation

1.4.6. **Convention.** In (1.19) and (1.22), only the upper left entry acts on Δ_3, on which the spinor bundle S is modelled. This yields a map $C_3^{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(\Delta_3) = \mathbb{C}(2)$. For example, this in particular means that $e_1 \in C_3^{\mathbb{C}}$ and $e_2 e_3 \in C_3^{\mathbb{C}}$ represent the same endomorphism. The same goes for the corresponding covectors. For example, e^1 and $e^2 \wedge e^3 = *e^1$ represent the same endomorphism on Δ_3.

More concisely, as suggested by (1.27), an element $e \in C_3^{\mathbb{C}}$ and its "dual" $\omega_{\mathbb{C}} \cdot_{\mathrm{cl}} e$ determine the same endomorphism of Δ_3. Using the relation (A.12) between $\omega_{\mathbb{C}}$ and the Hodge star operator, the same holds for covectors after applying the canonical isomorphism $C_3^{\mathbb{C}} \cong \Lambda^*_{\mathbb{C}}(\mathbb{R}^3)$ (1.5). ◀

The spin representation is defined by

$$\begin{aligned} \rho\colon Spin(3) &\to \mathrm{End}_{\mathbb{H}}(\Delta_3) = \mathbb{H}, \\ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} &\mapsto \left(h \mapsto a \cdot_{\mathbb{H}} h\right). \end{aligned} \tag{1.28}$$

[8] see [LM89, p. 34]

and analogously, the spin$^{\mathbb{C}}$ representation by

$$\begin{aligned} \rho^{\mathbb{C}}\colon \mathit{Spin}^{\mathbb{C}}(3) &\to \mathrm{End}_{\mathbb{C}}(\Delta_3) = \mathbb{C}(2), \\ \begin{bmatrix} [a,u] & 0 \\ 0 & [a,u] \end{bmatrix} &\mapsto (h \mapsto a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{u}). \end{aligned} \tag{1.29}$$

1.4.7. Remark. Up to isomorphism, there are two different irreducible real representations of C_3 ([LM89, §I.5]). The choice in Convention 1.4.6 of the entry which actually acts on spinors corresponds to the choice of irreducible representation. ◄

The adjoint representation of $\mathit{Spin}(3)$ is given by

$$\begin{aligned} \mathrm{Ad}\colon \mathit{Spin}(3) &\to \mathrm{End}_{\mathbb{R}}(\operatorname{im}\mathbb{H}) \equiv \mathrm{End}_{\mathbb{R}}(\mathbb{R}^3) = \mathbb{R}(3), \\ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} &\mapsto (h \mapsto a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{a}). \end{aligned} \tag{1.30}$$

This definition is motivated by the fact that the assignment in (1.30) determines a double covering[2] $Sp(1) \to SO(3)$. The adjoint representation of $\mathit{Spin}^{\mathbb{C}}(3)$ reads:

$$\begin{aligned} \mathrm{Ad}\colon \mathit{Spin}^{\mathbb{C}}(3) &\to \mathrm{End}_{\mathbb{C}}(\operatorname{im}\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}) \equiv \mathbb{C}(3), \\ \begin{bmatrix} [a,u] & 0 \\ 0 & [a,u] \end{bmatrix} &\mapsto \Big((h \otimes \lambda) \mapsto (a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{a}) \otimes (u \cdot_{\mathbb{H}} \lambda \cdot_{\mathbb{H}} \bar{u}) = (a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{a}) \otimes \lambda \Big). \end{aligned} \tag{1.31}$$

Lastly, the Clifford multiplication on Δ_3 is given by

$$\begin{aligned} \mathrm{cl}\colon \Big(\operatorname{im}\mathbb{H}_{\mathrm{Ad}} \otimes_{\mathbb{R}} \mathbb{C}\Big) \otimes_{\mathbb{R}} (\Delta_3)_{\rho^{\mathbb{C}}} &\to (\Delta_3)_{\rho^{\mathbb{C}}} \\ (v \otimes \lambda) \otimes h &\to -\bar{v} \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{\lambda} = v \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{\lambda}, \end{aligned} \tag{1.32}$$

or in terms of matrices

$$\operatorname{im}\mathbb{H} \otimes \mathbb{C} \ni v \otimes \lambda \mapsto \begin{bmatrix} v \otimes \lambda & 0 \\ 0 & -v \otimes \lambda \end{bmatrix} \in \mathbb{C}(4), \tag{1.33}$$

with Convention 1.4.6 in mind. Obviously, (1.32) is a homomorphism of representations, i.e.

$$\begin{aligned} \rho^{\mathbb{C}}([a,u])\Big(\mathrm{cl}\Big((v \otimes \lambda) \otimes h\Big)\Big) &= a \cdot_{\mathbb{H}} \Big(v \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{\lambda}\Big) \cdot_{\mathbb{H}} \bar{u} = (av\bar{a}) \cdot_{\mathbb{H}} (a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{u}) \cdot_{\mathbb{H}} \bar{\lambda} \\ &= \mathrm{cl}\Big(\Big(\mathrm{Ad}(a)(v) \otimes \lambda\Big) \otimes \rho^{\mathbb{C}}([a,u])(h)\Big). \end{aligned} \tag{1.34}$$

1.4.8. Remark. Note the compatibility of (1.20) and (1.33). ◀

1.4.9. Remark. The above definition of the Clifford multiplication (1.32), (1.33) (using left-hand side quaternion multiplication) was the main motivation behind identifications (1.16). ◀

When needed, we will use the symbol $\underset{\mathrm{cl}}{\bullet}$ to denote the above defined Clifford action on Δ_3. The same symbol will also be used for the algebra operation in $C_n^{\mathbb{C}}$.

1.4.10. **Convention.** Convention 1.4.6 yields a map $C_3^{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(\Delta_3) = \mathbb{C}(2)$. However, it will be important to be able go in the other direction too. I.e. it will be important to interpret endomorphisms of Δ_3 as covectors or elements of $C_3^{\mathbb{C}}$. For that, we identify $\mathrm{End}_{\mathbb{C}}(\Delta_3) = \mathbb{C}(2)$ with the subspace $\langle 1, e_1, e_2, e_3\rangle_{\mathbb{C}} \subseteq C_3^{\mathbb{C}}$ via combination of assignments (1.4) and (1.21), i.e.

$$\begin{aligned} 1 &\mapsto I \in \mathbb{C}(2),\\ e_1 &\mapsto i\sigma_3 \in \mathbb{C}(2),\\ e_2 &\mapsto -i\sigma_2 \in \mathbb{C}(2),\\ e_3 &\mapsto -i\sigma_1 \in \mathbb{C}(2). \end{aligned} \tag{1.35}$$

This induces a vector space isomorphism between the above-mentioned spaces, and its inverse will serve as a translation of endomorphisms into covectors[9] using the fact that $\langle 1, e_1, e_2, e_3\rangle_{\mathbb{C}} \subseteq C_3^{\mathbb{C}}$ and $\Lambda^{0,1}_{\mathbb{C}}(\mathbb{R}^3)$ are isomorphic via the canonical isomorphism $C_3^{\mathbb{C}} \overset{(1.5)}{\cong} \Lambda^*_{\mathbb{C}}(\mathbb{R}^3)$. ◀

Before moving on, we mention one more lemma[10]:

1.4.11. Lemma. *The $Pin(2)$ group corresponds to the normaliser of $U(1)$ in $Sp(1)$ and equals $Pin(2) = U(1) \sqcup j\,U(1)$.*

Sketch of proof. In general, we have $C_2 \cong \mathbb{H}$ and $C_2^{\mathbb{C}} \cong \mathbb{C}(2)$ ([LM89, §I.4]). So, C_2 and $C_2^{\mathbb{C}}$ can be seen as a half of C_3 and $C_3^{\mathbb{C}}$ respectively. Similarly as in the proof of Lemma 1.4.5 we have

$$Pin(2) = \left\{\pm 1, q_1, q_1 q_2 \colon q_1, q_2 \in \mathbb{R}^2, \|q_1\| = \|q_2\| = 1\right\}$$

[9] in particular, this will be applied in the second Seiberg-Witten equation, to the quadratic term

[10] we will use this lemma in Proposition 2.3.13

$$= \left\{q_1 \colon q_1 \in \mathbb{R}^2, \|q_1\| = 1\right\} \sqcup \left\{q_1 q_2 \colon q_1, q_2 \in \mathbb{R}^2, \|q_1\| = \|q_2\| = 1\right\}.$$

Assign $(1, e_1e_2, e_1, e_2) \mapsto (1, i, j, k) \in \mathbb{H}^4$. The second part is now obviously isomorphic to $U(1)$. An element q_1 from the first set is clearly of the form $q_1 = e_1 q_1'$, with q_1' being an element of the second set.

The above group is clearly contained in the normaliser N of $U(1)$ in $Sp(1)$. Conversely, let $q = a + jb \in N$, with $a, b \in \mathbb{C}$. Since,

$$qu\bar{q} = |a|^2 u + |b|^2 \bar{u} + j(b\bar{a}u - b\bar{a}\bar{u}) \in U(1), \quad \forall u \in U(1),$$

there follows $b\bar{a} = 0$, i.e. either $q \in U(1)$ or $q \in jU(1)$. □

1.4.2 Other representations of the spin group

For $Spin^{\mathbb{C}}(3)$ there are following short exact sequences of groups[11]:

$$1 \longrightarrow U(1) \xrightarrow{\iota} Spin^{\mathbb{C}}(3) \xrightarrow{\kappa} SO(3) \longrightarrow 1, \tag{1.36}$$

$$1 \longrightarrow Spin(3) \xrightarrow{\iota} Spin^{\mathbb{C}}(3) \xrightarrow{l} U(1) \longrightarrow 1. \tag{1.37}$$

Here, $l \colon Spin^{\mathbb{C}}(3) \to U(1)$ is given by $l : [a, u] \mapsto u^2$ and κ is the composition of the double covering map $Spin^{\mathbb{C}}(3) \to SO(3) \times U(1)$ determined by the double covering[12] $Spin(3) \to SO(3)$ and the map l and the projection onto the first factor.

1.4.3 The associated bundles

For a closed oriented smooth 3-dimensional Riemannian manifold Y equipped with a spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ (i.e. a certain principal $Spin^{\mathbb{C}}(3)$-bundle $P^{Spin^{\mathbb{C}}}$ lifting the frame bundle P^{SO}), the spinor bundle is defined as the bundle S associated to $P^{Spin^{\mathbb{C}}}$ via the spin representation (1.29):

$$S \coloneqq P^{Spin^{\mathbb{C}}} \times_{\rho^{\mathbb{C}}} \Delta_3. \tag{1.38}$$

Its sections will be referred to as spinors and typically denoted by ψ or ϕ. Note that spin representation (1.28) is a special orthogonal representation on Δ_3 with

[11] [Fri97, p. 28]

[12] discussed on page 75

respect to scalar product (1.7), i.e. $\rho\colon Spin(3) \to SO(3)$. Similarly, spin$^{\mathbb{C}}$ representation (1.29) is a unitary representation on Δ_3 with respect to Hermitian product (1.8), i.e. $\rho^{\mathbb{C}}\colon Spin^{\mathbb{C}}(3) \to U(2)$. Hence, spinor bundle (1.38) comes with a natural Hermitian product.

1.4.12. Remark. In the case of spin structure, the spinor bundle is quaternionic line bundle and the Dirac operator preserves the quaternionic structure . ◀

The adjoint representation (1.31) can be used to define the tangent and cotangent bundles of Y:

$$TY := P^{Spin^{\mathbb{C}}} \times_{\mathrm{Ad}} \operatorname{Im}\mathbb{H},$$
$$T^*Y := P^{Spin^{\mathbb{C}}} \times_{\mathrm{Ad}} \operatorname{Im}\mathbb{H},$$

and their complexifications

$$T_{\mathbb{C}}Y := P^{Spin^{\mathbb{C}}} \times_{\mathrm{Ad}} (\operatorname{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}),$$
$$T^*_{\mathbb{C}}Y := P^{Spin^{\mathbb{C}}} \times_{\mathrm{Ad}} (\operatorname{Im}\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}).$$

The (local) identification convention from §1.2 carries over to the present context of complexified tangent and cotangent bundles.

The representation $l\colon Spin^{\mathbb{C}}(3) \to U(1)$ from (1.37) gives rise to the determinant bundle of $\mathfrak{s}$:

$$L = \det \mathfrak{s} = P^{Spin^{\mathbb{C}}} \times_l \mathbb{C}.$$

To the frame bundle P^{SO} we can associate the so called Clifford bundle

$$C_3(Y) := P^{SO} \times_{\mathrm{cl}(\iota)} C_3, \tag{1.39}$$

via the Clifford representation[13]

$$\begin{aligned} &\mathrm{cl}(\iota)\colon SO(3) \to Gl(C_3),\\ &\big[\mathrm{cl}(\iota)(A)\big](v_1 \cdot \ldots \cdot v_k) = \iota(A)v_1 \cdot \ldots \cdot \iota(A)v_k = Av_1 \cdot \ldots \cdot Av_k. \end{aligned} \tag{1.40}$$

Here, $\iota\colon SO(3) \to Gl(3, \mathbb{R})$ denotes the inclusion.

The representation

$$\begin{aligned} &\mathrm{Ad}\colon Spin(3) \to Gl(C_3),\\ &\mathrm{Ad}(a)(v_1 \cdot \ldots \cdot v_k) = a \cdot v_1 \cdot \ldots \cdot v_k \cdot \bar{a} = av_1\bar{a} \cdot \ldots \cdot av_k\bar{a}, \end{aligned}$$

clearly descends (using the double covering $Spin(3) \to SO(3)$) to a representation of $SO(3)$, which coincides[14] with $\mathrm{cl}(\iota)$ in (1.40). Therefore, $C_3(Y)$ can also

[13] cf. [LM89, p. 95]
[14] cf. [LM89, p. 96]

be defined by

$$C_3(Y) = P^{Spin} \times_{\mathrm{Ad}} C_3. \tag{1.41}$$

The complex Clifford bundle is defined analogously as in (1.41)

$$C_3^{\mathbb{C}}(Y) = P^{Spin^{\mathbb{C}}} \times_{\mathrm{Ad}} C_3^{\mathbb{C}}. \tag{1.42}$$

Due to (1.34), Clifford multiplication (1.32) induces well-defined bundle maps

$$\mathrm{cl}\colon T_{\mathbb{C}}Y \otimes S \to S, \tag{1.43}$$
$$\mathrm{cl}\colon T^*_{\mathbb{C}}Y \otimes S \to S, \tag{1.44}$$

and from there also the map

$$\mathrm{cl}\colon C_3^{\mathbb{C}}(Y) \otimes S \to S. \tag{1.45}$$

It is easy to show that all the other local discussions from this chapter carry over to the corresponding bundles by a similar argument.

Note that with the identifications from §1.2 in mind, the maps (1.43) and (1.44) become equal.

1.5 A different Clifford module structure on forms

In the case $n = 3$, the subspace $\Omega^{1,0}(Y) \coloneqq \Omega^1(Y) \oplus \Omega^0(Y) \subseteq \Omega^*(Y)$ of forms of degrees 1 and 0 will be of particular interest in the discussion of the monopole map. On it we can define a slightly different Clifford module structure than usual (cf. §A.3):

$$c(v) = v \underset{\mathrm{cl}}{\bullet} \alpha \coloneqq \begin{cases} v^* \cdot \alpha, & \alpha \in \Lambda^0(\mathbb{R}^3), \\ *(v^* \wedge \alpha) - \iota(v)\alpha, & \alpha \in \Lambda^1(\mathbb{R}^3). \end{cases} \tag{1.46}$$

Clearly, this defines a Clifford module structure on $\Lambda^{1,0}(\mathbb{R}^3) \coloneqq \Lambda^1(\mathbb{R}^3) \oplus \Lambda^0(\mathbb{R}^3)$, which carries over to $\Omega^{1,0}(Y)$.

This modification is motivated by the fact that $c(\alpha) = c(*\alpha)$ for all $\alpha \in \Omega^1(Y)$. Namely, an important consequence of (1.27), (A.12) and Convention 1.4.6 is

$$c(\alpha) = c(*\alpha) \in \mathrm{End}_{\mathbb{C}}(\Delta_3), \quad \forall \alpha \in \Lambda^1(\mathbb{R}^3). \tag{1.47}$$

On the level of spinors and 1-forms on Y, this means

$$c(\alpha) = c(*\alpha) \in \mathrm{End}_{\mathbb{C}}(S), \quad \forall \alpha \in \Omega^1(Y). \tag{1.48}$$

In other words, Clifford multiplication by 2-forms does not introduce new endomorphisms of S, so by staying in the space $\Omega^{1,0}(Y)$ nothing is lost. Therefore, defining a Clifford module structure on the space $\Omega^{1,0}(Y)$ makes sense.

1.5.1. REMARK. Note that (1.46) implies $e^1 \bullet_{\mathrm{cl}} e^2 = e^3$ etc. which via[15] (1.4) translates into standard relations between $i, j, k \in \mathbb{H}$. ◀

1.6 Dirac operator on forms

In general, the usual Clifford module structure[16] defines together with the extension of the Levi-Civita connection to $\Omega^*(M)$ a Dirac operator which equals[17] the Hodge-de Rham operator $D_{HdR} = d + d^*\colon \Omega^*(M) \to \Omega^*(M)$.

In the case of a 3-manifold Y, the following slight modification the Hodge-de Rham operator on $\Omega^{1,0}(Y) = \Omega^1(Y) \oplus \Omega^0(Y)$ appears in the later discussion of the monopole map:

$$D_\Omega := \begin{bmatrix} *d & d \\ d^* & 0 \end{bmatrix} \colon \Omega^{1,0}(Y) \to \Omega^{1,0}(Y). \tag{1.49}$$

The Levi-Civita connection on $\Omega^*(Y)$, together with the Clifford module structure (1.46) determines a Dirac operator on $\Omega^{1,0}(Y)$ which actually equals D_Ω. Thus, D_Ω is an elliptic operator.

The fact that D_Ω is elliptic also follows directly from

$$\begin{bmatrix} *d & d \\ d^* & 0 \end{bmatrix}^2 = \begin{bmatrix} *d*d + dd^* & *d^2 \\ d^**d & d^*d \end{bmatrix} = \begin{bmatrix} d^*d + dd^* & 0 \\ 0 & d^*d \end{bmatrix} = (d + d^*)^2\Big|_{\Omega^{1,0}(Y)}.$$

Here, keep in mind that[18] $*d*\big|_{\Omega^2(Y)} = d^*\big|_{\Omega^2(Y)}$.

[15] and with the use of the canonical isomorphism $\mathbb{R}^3 \cong (\mathbb{R}^3)^*$

[16] see §A.3

[17] [LM89, §II.6]

[18] see (A.8) on p. 72

1.7 The quadratic term

The quadratic term is most commonly defined as an endomorphism of S and then interpreted as a differential form using the inverse of (1.35) from Convention 1.4.10.

For $\psi = \psi_0 + j\psi_1 \in \Delta_3$ we define $\sigma(\psi) \in \mathrm{End}_{\mathbb{C}}(\Delta_3)$ by

$$\sigma(\psi) := \psi \otimes \psi^* - \frac{1}{2}|\psi|^2 I_2. \tag{1.50}$$

Writen as a matrix, the above endomorphism takes the following form

$$\begin{aligned}\sigma(\psi) &= \frac{1}{2} \cdot \begin{bmatrix} |\psi_0|^2 - |\psi_1|^2 & 2\psi_0\bar{\psi}_1 \\ 2\bar{\psi}_0\psi_1 & |\psi_1|^2 - |\psi_0|^2 \end{bmatrix} \\ &\overset{(1.18)}{=} \mathrm{Re}(\bar{\psi}_0\psi_1)\sigma_1 + \mathrm{Im}(\bar{\psi}_0\psi_1)\sigma_2 + \frac{1}{2}(|\psi_0|^2 - |\psi_1|^2)\sigma_3 \in \mathbb{C}(2).\end{aligned}$$

Using (1.22) together with Convention 1.4.6 that e_i acts on ψ with the upper left 2×2 complex matrix we get:

$$\begin{aligned}\sigma(\psi) &= \psi \otimes \psi^* - \frac{1}{2}|\psi|^2 I_2 = \\ &= \mathrm{Re}(\bar{\psi}_0\psi_1)\sigma_1 + \mathrm{Im}(\bar{\psi}_0\psi_1)\sigma_2 + \frac{1}{2}(|\psi_0|^2 - |\psi_1|^2)\sigma_3 \\ &= \mathrm{Re}(\bar{\psi}_0\psi_1)(-i\sigma_1) \cdot i + \mathrm{Im}(\bar{\psi}_0\psi_1)(-i\sigma_2) \cdot i + \frac{1}{2}(|\psi_0|^2 - |\psi_1|^2)i\sigma_3 \cdot (-i) \\ &= -i \cdot_s \mathrm{Re}(\bar{\psi}_0\psi_1)e_3 + -i \cdot_s \mathrm{Im}(\bar{\psi}_0\psi_1)e_2 + i \cdot_s \frac{1}{2}(|\psi_0|^2 - |\psi_1|^2)e_1 \in \mathrm{End}_{\mathbb{C}}(\Delta_3),\end{aligned}$$

which under (1.6) corresponds to the 1-covector

$$\sigma(\psi) = -i \cdot_s \mathrm{Re}(\bar{\psi}_0\psi_1)e^3 + -i \cdot_s \mathrm{Im}(\bar{\psi}_0\psi_1)e^2 + i \cdot_s \frac{1}{2}(|\psi_0|^2 - |\psi_1|^2)e^1. \tag{1.51}$$

The analogous claim holds for a spinor $\psi \in \Gamma(S)$. I.e. for a spinor $\psi \in \Gamma(S)$, its square

$$\sigma(\psi) := \psi \otimes \psi^* - \frac{1}{2}|\psi|^2 I, \tag{1.52}$$

is a well-defined section of $\mathrm{End}_{\mathbb{C}}(S)$, with I denoting the identity map on S. This section can be interpreted as is an imaginary-valued 1-form on Y in the same way as above. The local expression of this form is analogous to (1.51):

$$\sigma(\psi) = -i \cdot_s \mathrm{Re}(\bar{\psi}_0\psi_1)e_3 + -i \cdot_s \mathrm{Im}(\bar{\psi}_0\psi_1)e_2 + i \cdot_s \frac{1}{2}(|\psi_0|^2 - |\psi_1|^2)e_1. \tag{1.53}$$

1.7.1 Two ways of writing the quadratic term

There are two common ways of writing the quadratic term in the literature on Seiberg-Witten theory. Definition (1.52) is the most common, but we will find it more convenient to use a slightly different definition. The quadratic term $\sigma(\psi)$ is related to the following quadratic map[19]

$$q(\psi) = \psi\, i\, \bar{\psi} \;\in \operatorname{End}_{\mathbb{C}}(S), \tag{1.54}$$

which is acting on spinors by quaternionic multiplication on the left-hand side.

To see this, write locally $\psi = \psi_0 + j\psi_1$ and

$$\begin{aligned} q(\psi) &= \psi i \bar{\psi} = (\psi_0 + j\psi_1) i (\bar{\psi}_0 - j\psi_1) \\ &= i(|\psi_0|^2 - |\psi_1|^2) + 2ji\bar{\psi}_0\psi_1 \\ &= i(|\psi_0|^2 - |\psi_1|^2) + 2ji\mathrm{Re}(\bar{\psi}_0\psi_1) - 2j\mathrm{Im}(\bar{\psi}_0\psi_1). \end{aligned}$$

Together with (1.4) we locally have

$$q(\psi) = (|\psi_0|^2 - |\psi_1|^2)e_1 - 2\mathrm{Re}(\bar{\psi}_0\psi_1)e_3 - 2\mathrm{Im}(\bar{\psi}_0\psi_1)e_2 \in \operatorname{End}_{\mathbb{C}}(S),$$

and now it follows from (1.53) that

$$\sigma(\psi) = \frac{i}{2} \cdot_S q(\psi), \tag{1.55}$$

The following calculation confirms the above identity:

$$\frac{1}{2} i \cdot_S (q(\psi) \bullet_{\mathrm{cl}} \psi) = \frac{1}{2}\Big(q(\psi) \bullet_{\mathrm{cl}} \psi\Big)\bar{\imath} = \frac{1}{2}\psi \cdot_{\mathbb{H}} i \cdot_{\mathbb{H}} \bar{\psi} \cdot_{\mathbb{H}} \psi \cdot_{\mathbb{H}} \bar{\imath} = \frac{1}{2}|\psi|^2\psi = \sigma(\psi)\psi = \sigma(\psi) \bullet_{\mathrm{cl}} \psi.$$

In analogy to (1.15) we have for sections of the corresponding bundles the simple but important identity

$$\langle \psi, i\alpha \bullet_{\mathrm{cl}} \psi \rangle_S = \langle \alpha, q(\psi) \rangle_{\operatorname{End}_{\mathbb{C}}(S)} = \langle \alpha, q(\psi) \rangle_\Omega = \langle \alpha, -2i\sigma(\psi) \rangle_\Omega. \tag{1.56}$$

The symbol $\langle\, .\,,\, .\,\rangle_\Omega$ denotes the scalar product on forms, and $\langle\, .\,,\, .\,\rangle_S$ denotes the real part of the Hermitian product on S.

[19] this quadratic map is well-defined because the local definition carries over to bundles, due to commutativity of the appropriate representations (§1.4.1): $q(\rho^{\mathbb{C}}([a,u])(h)) = (a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{u}) \cdot_{\mathbb{H}} i \cdot_{\mathbb{H}} \overline{(a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{u})} = a \cdot_{\mathbb{H}} (hi\bar{h}) \cdot_{\mathbb{H}} \bar{a} = \mathrm{Ad}(a)(q(h))$

1.7.2 Derivation of the quadratic term

Later in the estimates, a close-up analysis of the derivation of the quadratic term will be needed. We include it at this point in the form of the present subsection in order not to disturb the flow later.

Let (e_1, e_2, e_3) be a local orthonormal frame on Y centred at some arbitrary fixed point $y_0 \in Y$ (i.e. all Christoffel symbols vanish at y_0). Unless specified otherwise, all calculations in this section will be done locally, at point y_0.

Locally, D_A is of the form $D_A\psi = \sum_{s=1}^3 e_s \bullet_{\mathrm{cl}} (\nabla_A)_{e_s}\psi$, and at y_0 the following holds[20]:

$$\begin{aligned}
\langle D_A\psi, i\psi\rangle &= \sum_{s=1}^{3}\Big\langle e_s \bullet_{\mathrm{cl}} (\nabla_A)_{e_s}\psi, i\psi\Big\rangle \\
&= -\sum_{s=1}^{3}\Big\langle (\nabla_A)_{e_s}\psi, ie_s\psi\Big\rangle \\
&= -\sum_{s=1}^{3}\partial_s\langle\psi, ie_s\psi\rangle + \sum_{s=1}^{3}\Big\langle\psi, i(\nabla_A)_{e_s}(e_s \bullet_{\mathrm{cl}} \psi)\Big\rangle \\
&= -\sum_{s=1}^{3}\partial_s\langle\psi, ie_s\psi\rangle + \sum_{s=1}^{3}\Big\langle\psi, i(\nabla_{e_s}e_s) \bullet_{\mathrm{cl}} \psi)\Big\rangle + \sum_{s=1}^{3}\Big\langle\psi, ie_s \bullet_{\mathrm{cl}} (\nabla_A)_{e_s}\psi\Big\rangle \\
&= -\sum_{s=1}^{3}\partial_s\langle\psi, ie_s\psi\rangle + 0 + \langle\psi, iD_A\psi\rangle \\
&= -\sum_{s=1}^{3}\partial_s\langle\psi, ie_s\psi\rangle - \langle D_A\psi, i\psi\rangle.
\end{aligned}$$

I.e. $\langle D_A\psi, i\psi\rangle = -\frac{1}{2}\sum_{s=1}^3 \partial_s\langle\psi, ie_s\psi\rangle$. The latter sum, however, equals $\frac{1}{2}d^*q(\psi)$, due to

$$\sum_{s=1}^{3}\partial_s\langle\psi, ie_s\psi\rangle \overset{(1.56)}{=} \sum_{s=1}^{3}\partial_s\langle e_s, q(\psi)\rangle_\Omega = *d*q(\psi)$$

and the fact that $d^*\colon \Omega^1(Y) \to \Omega^0(Y)$ equals[21] $-*d*$. In other words

$$\langle D_A\psi, i\psi\rangle = \frac{1}{2}d^*q(\psi) \overset{(1.55)}{=} -id^*\sigma(\psi). \tag{1.57}$$

[20] cf. [LM89, p. 115]

[21] see (A.8)

On the other hand, for an arbitrary 1-form a and its local expression $a = \sum_{r=1}^{3} a_r e^r$ we have[22] at y_0

$$\begin{aligned}\langle D_A\psi, ia \bullet_{\mathrm{cl}} \psi\rangle &= \sum_{r,s=1}^{3} \Big\langle e_s \bullet_{\mathrm{cl}} (\nabla_A)_{e_s}\psi, ia_r e_r \bullet_{\mathrm{cl}} \psi\Big\rangle \\ &= \sum_{\substack{r,s=1\\ r=s}}^{3} \Big\langle e_s \bullet_{\mathrm{cl}} (\nabla_A)_{e_s}\psi, ia_r e_r \bullet_{\mathrm{cl}} \psi\Big\rangle + \sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle e_s \bullet_{\mathrm{cl}} (\nabla_A)_{e_s}\psi, ia_r e_r \bullet_{\mathrm{cl}} \psi\Big\rangle \\ &= \langle (\nabla_A)_a\psi, i\psi\rangle + \sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle e_s \bullet_{\mathrm{cl}} (\nabla_A)_{e_s}\psi, ia_r e_r \bullet_{\mathrm{cl}} \psi\Big\rangle.\end{aligned}$$

Simple calcuations

$$\begin{aligned}&\sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle e_s(\nabla_A)_{e_s}\psi, ia_r e_r\psi\Big\rangle = -\sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle (\nabla_A)_{e_s}\psi, ia_r e_s e_r\psi\Big\rangle \\ &= -\sum_{\substack{r,s=1\\ r\neq s}}^{3} \partial_s\langle \psi, ia_r e_s e_r\psi\rangle + \sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle \psi, i(\nabla_A)_{e_s}\Big(a_r e_s e_r\psi\Big)\Big\rangle \\ &= -\sum_{\substack{r,s=1\\ r\neq s}}^{3} \partial_s\langle \psi, ia_r e_s e_r\psi\rangle + \sum_{\substack{r,s=1\\ r\neq s}}^{3} \langle \psi, i(\partial_s a_r)e_s e_r\psi\rangle + \sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle \psi, ia_r e_s e_r(\nabla_A)_{e_s}\psi\Big\rangle \\ &= -\sum_{\substack{r,s=1\\ r\neq s}}^{3} \partial_s\Big[a_r\langle \psi, ie_s e_r\psi\rangle\Big] + \sum_{\substack{r,s=1\\ r\neq s}}^{3} \partial_s a_r\langle \psi, ie_s e_r\psi\rangle - \sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle \psi, ia_r e_r e_s(\nabla_A)_{e_s}\psi\Big\rangle \\ &= -\sum_{\substack{r,s=1\\ r\neq s}}^{3} a_r \cdot \partial_s\langle \psi, ie_s e_r\psi\rangle - \sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle ia_r e_r\psi, e_s(\nabla_A)_{e_s}\psi\Big\rangle,\end{aligned}$$

imply

$$\sum_{\substack{r,s=1\\ r\neq s}}^{3} \Big\langle e_s(\nabla_A)_{e_s}\psi, ia_r e_r\psi\Big\rangle = -\frac{1}{2}\sum_{\substack{r,s=1\\ r\neq s}}^{3} a_r \cdot \partial_s\langle \psi, ie_s e_r\psi\rangle,$$

[22] Clifford multiplication by a 1-form is, of course, to be understood as the Clifford multiplication by the corresponding dual vector field, with the appropriate convention from §1.2 in mind

and thus

$$\langle D_A\psi, ia\psi\rangle - \langle(\nabla_A)_a\psi, i\psi\rangle = -\frac{1}{2}\sum_{\substack{r,s=1\\ r\neq s}}^{3} a_r \cdot \partial_s\langle\psi, ie_s e_r\psi\rangle.$$

For the last sum, note that

$$\langle\psi, ie_s e_r \bullet_{\mathrm{cl}} \psi\rangle \overset{(1.56)}{=} \langle e_s e_r, q(\psi)\rangle_{\mathrm{End}_{\mathbb{C}}(S)}.$$

According to Convention[23] 1.4.6, the elements $e_s e_r \in C_3^{\mathbb{C}}$ and their "Hodge duals"[24] $\omega_{\mathbb{C}} e_s e_r \in C_3^{\mathbb{C}}$ determine the same endomorphisms of Δ_3. Using the map $\mathrm{End}_{\mathbb{C}}(\Delta_3) \to \Lambda^{0,1}_{\mathbb{C}}(\mathbb{R}^3)$ from Convention 1.4.10, we conclude that the endomorphism $e_s e_r$ corresponds to the 1-covector $*(e^s \wedge e^r)$. Consequently,

$$\partial_s\langle\psi, ie_s e_r \bullet_{\mathrm{cl}} \psi\rangle = \partial_s\langle *(e^s \wedge e^r), q(\psi)\rangle.$$

Inserting this into the above sum leads to

$$\begin{aligned}
\sum_{\substack{r,s=1\\ r\neq s}}^{3} a_r \cdot \partial_s\langle\psi, ie_s e_r \bullet_{\mathrm{cl}} \psi\rangle &= \sum_{\substack{r,s=1\\ r\neq s}}^{3} a_r \cdot \partial_s\langle *(e^s \wedge e^r), q(\psi)\rangle \\
&= \sum_{r=1}^{3} a_r \cdot \sum_{\substack{s=1\\ r\neq s}}^{3} \partial_s\langle *(e^s \wedge e^r), q(\psi)\rangle \\
&= \quad a_1 \cdot \Big(-\partial_2[q(\psi)]_3 + \partial_3[q(\psi)]_2 \Big) \\
&\quad + a_2 \cdot \Big(+\partial_1[q(\psi)]_3 - \partial_3[q(\psi)]_1 \Big) \\
&\quad + a_3 \cdot \Big(-\partial_1[q(\psi)]_2 + \partial_2[q(\psi)]_1 \Big),
\end{aligned}$$

with the last sum equalling[3] $\langle a, -*dq(\psi)\rangle$. Symbols $[q(\psi)]_l$ denote local component functions of the 1-form $q(\psi)$. In short, we get

$$\langle D_A\psi, ia \bullet_{\mathrm{cl}} \psi\rangle - \langle(\nabla_A)_a\psi, i\psi\rangle = \frac{1}{2}\langle a, *dq(\psi)\rangle.$$

[23] see also (1.27)

[24] see (A.12) for the reason behind this name

Finally, summarising the above calculations brings

$$\begin{aligned} D_\Omega q(\psi) &\overset{(1.49)}{=} (*d + d^*)q(\psi) \\ &= 2\sum_{s=1}^{3}\langle D_A\psi, ie_s \bullet_{\mathrm{cl}} \psi\rangle e^s - 2\langle \nabla_A\psi, i\psi\rangle + 2\langle D_A\psi, i\psi\rangle. \end{aligned} \tag{1.58}$$

Note that the first two summands in the last expression are 1-forms, and the third one is a function. Also, if $a + f \in \Omega^{1,0}(Y)$, the above formula implies

$$\langle D_\Omega q(\psi), a + f\rangle = 2\sum_{s=1}^{3}\langle D_A\psi, i(a + f)\psi\rangle - 2\langle (\nabla_A)_{a^*}\psi, i\psi\rangle. \tag{1.59}$$

1.7.3 Norm of the quadratic term

From (1.53) follows the pointwise equality

$$\begin{aligned} |\sigma(\psi)|^2 &= \mathrm{Re}(\bar{\psi}_0\psi_1)^2 + \mathrm{Im}(\bar{\psi}_0\psi_1)^2 + \frac{1}{4}(|\psi_1|^2 - |\psi_0|^2)^2 \\ &= |\bar{\psi}_0\psi_1|^2 + \frac{1}{4}(|\psi_1|^2 - |\psi_0|^2)^2 = \frac{1}{4}(|\psi_1|^2 + |\psi_0|^2)^2 = \frac{1}{4}|\psi|^4, \end{aligned} \tag{1.60}$$

and so

$$\|i\sigma(\psi)\|_{L^2} = \left(\int_Y |\sigma(\psi)|^2 \,\mathrm{dvol}\right)^{\frac{1}{2}} = \left(\int_Y \frac{1}{4}|\psi|^4 \,\mathrm{dvol}\right)^{\frac{1}{2}} = \frac{1}{2}\|\psi\|^2_{L^4}.$$

For the derivative of $q(\psi)$, the equation (1.58) implies

$$|D_\Omega q(\psi)| \leq 2|D_A\psi||\psi| + 2|\nabla_A\psi||\psi| + 2|D_A\psi||\psi|, \tag{1.61}$$

and thus

$$\begin{aligned} \|D_\Omega q(\psi)\|_{L^2} &\leq 2\|D_A\psi\|_{L^2}\|\psi\|_{C^0} + 2\|\nabla_A\psi\|_{L^2}\|\psi\|_{C^0} + 2\|D_A\psi\|_{L^2}\|\psi\|_{C^0} \\ &\lesssim \|\psi\|_{L^2_1}\|\psi\|_{C^0}. \end{aligned}$$

The identity (1.58) further helps in estimating higher derivations $D^m q(\psi)$ of $q(\psi)$, for $m \geq 1$. Namely, (1.58) indicates that higher derivatives of the quadratic term involve taking repeated exterior derivatives and coderivatives of $\langle D_A\psi, ie_s \bullet_{\mathrm{cl}} \psi\rangle$, $\langle \nabla_A\psi, i\psi\rangle$ and $\langle D_A\psi, i\psi\rangle$. Using the fact that ∇_A is a metric connection, we conclude that the component functions of $D^m q(\psi)$ take one of the following forms

$$\begin{aligned}
&\Big\langle (\nabla_A)_{e_{r_1}} \dots (\nabla_A)_{e_{r_s}} D_A\psi, ie_s \bullet_{\text{cl}} (\nabla_A)_{e_{r_{s+1}}} \dots (\nabla_A)_{e_{r_{t-1}}} \psi \Big\rangle, \\
\text{or} \quad &\Big\langle (\nabla_A)_{e_{r_1}} \dots (\nabla_A)_{e_{r_s}} \psi, ie_s \bullet_{\text{cl}} (\nabla_A)_{e_{r_{s+1}}} \dots (\nabla_A)_{e_{r_t}} \psi \Big\rangle, \\
\text{or} \quad &\Big\langle (\nabla_A)_{e_{r_1}} \dots (\nabla_A)_{e_{r_s}} D_A\psi, i(\nabla_A)_{e_{r_{s+1}}} \dots (\nabla_A)_{e_{r_t}} \psi \Big\rangle,
\end{aligned}$$

where $s + t = m$ and $r_j \in \{1, 2, 3\}$. This leads to a pointwise inequality similar to (1.61)

$$|D_\Omega^m q(\psi)| \lesssim \sum_{\substack{s,t\geq 0\\ s+t=m}} |\nabla_A^s D_A\psi| \left|\nabla_A^{t-1}\psi\right| + \sum_{\substack{s,t\geq 0\\ s+t=m}} |\nabla_A^s \psi| \left|\nabla_A^t \psi\right|. \tag{1.62}$$

Integration gives

$$\begin{aligned}
\|D_\Omega^m q(\psi)\|_{L^2} &\lesssim \sum_{\substack{s,t>0\\ s+t=m}} \|\nabla_A^s D_A\psi\|_{L^2} \left\|\nabla_A^{t-1}\psi\right\|_{L^2} + \sum_{\substack{s,t>0\\ s+t=m}} \|\nabla_A^s \psi\|_{L^2} \left\|\nabla_A^t \psi\right\|_{L^2} \\
&\qquad + \left\|\nabla_A^{m-1} D_A\psi\right\|_{L^2} \|\psi\|_{C^0} + \|\nabla_A^m \psi\|_{L^2} \|\psi\|_{C^0} \\
&\lesssim \|\nabla_A^m \psi\|_{L^2} \|\psi\|_{C^0} + \sum_{\substack{s,t>0\\ s+t=m}} \|\nabla_A^s \psi\|_{L^2} \left\|\nabla_A^t \psi\right\|_{L^2},
\end{aligned}$$

and from that it directly follows for $m \geq 1$

$$\|D_\Omega^m q(\psi)\|_{L^2} \lesssim \|\psi\|_{L_m^2} \|\psi\|_{C^0} + \|\psi\|_{L_m^2}^2. \tag{1.63}$$

For $m = 0$ we simply have $\|q(\psi)\|_{L^2} \lesssim \|\psi\|_{L^2} \|\psi\|_{C^0}$. Thus

$$\begin{aligned}
\|q(\psi)\|_{L_m^2} &\lesssim \sum_{j=0}^m \left\|D_\Omega^j q(\psi)\right\|_{L^2} \\
&\lesssim \|\psi\|_{L^2} \|\psi\|_{C^0} + \sum_{j=1}^m \|\psi\|_{L_j^2} \|\psi\|_{C^0} + \|\psi\|_{L_j^2}^2 \\
&\lesssim \|\psi\|_{L_m^2} \|\psi\|_{C^0} + \|\psi\|_{L_m^2}^2.
\end{aligned}$$

In short

$$\|q(\psi)\|_{L_m^2} \lesssim \|\psi\|_{L_m^2} \|\psi\|_{C^0} + \|\psi\|_{L_m^2}^2, \quad m \geq 0. \tag{1.64}$$

Obviously, the same inequality holds for $\sigma(\psi)$ due to (1.55).

Chapter 2

The monopole map on 3-manifolds

2.1 Assumptions and general context

Unless stated otherwise, Y will denote a closed 3-manifold, and the following will be assumed:

- Y is oriented and equipped with a Riemannian metric.
- Y is equipped with a spin$^{\mathbb{C}}$ structure $\mathfrak{s}$.
- On the determinant bundle $L := \det \mathfrak{s}$ of the chosen spin$^{\mathbb{C}}$ structure $\mathfrak{s}$, a unitary connection A is fixed such that $[F_A] = -2\pi i\, c_1(\mathfrak{s})$. This gives a 1-1 correspondence $Conn(L) \cong i\Omega_1(Y) \cong \Omega_1(Y)$. If the spin$^{\mathbb{C}}$ structure given above is a spin structure, we take A to be the trivial connection (as the natural choice).
- A point $y_0 \in Y$ will be fixed.

2.2 Seiberg-Witten equations on 3-manifolds

After having fixed a spin$^{\mathbb{C}}$ structure on Y, all the definitions and constructions needed for writing down the Seiberg-Witten equations on closed 4-dimensional manifolds can be carried out in the 3-dimensional case. Thus we are able to write down the Seiberg-Witten equations for $(Y, \mathfrak{s})$:

$$D_{A+ia}(\psi) = 0, \tag{2.1a}$$

$$*F_{A+ia} - \sigma(\psi) = 0. \tag{2.1b}$$

The analysis of these equations proceeds in several different directions (e.g. [KM07, Chapter 4], [Nic03]). Here, they will be used to define the monopole map for 3-manifolds.

2.3 Monopole map for closed 3-manifolds

The procedure presented here follows the construction from [BF04].

2.3.1 Definition

In [Fri97, p. 189] the following theorem is stated:

2.3.1. Theorem. (Weyl's Theorem) *Let P be some principal $U(1)$-bundle over a compact n-manifold M with the first Chern class $c_1(P) \in H^2_{dR}(M;\mathbb{R})$, and set $\mathcal{F}(P) := \left\{\omega \in \Omega^2(Y)\colon d\omega = 0, [\omega] = c_1(P)\right\}$. Then the quotient*

$$\Psi\colon Conn(P)/\mathcal{G}(P) \to \mathcal{F}(P)$$

of the map taking a connection A on P to $-\frac{1}{2\pi i} F_A$ is surjective, with each fibre diffeomorphic to the Picard manifold $\operatorname{Pic}(M) = H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$ of M.

Consider the fibre corresponding to the chosen connection A. That is to say, we consider the preimage of $-\frac{1}{2\pi i} F_A$ under the map $Conn(P) \to \mathcal{F}(P)$, $A' \mapsto -\frac{1}{2\pi i} F_{A'}$, which actually equals the space $A + i\ker d$ of all connections on L having the same curvature as A. Division by $\mathcal{G}$ yields the Picard torus, as the above theorem states. In order to have a free action on $A + i\ker d$, we will restrict our attention to the action of $\mathcal{G}_0 := \ker\left(\mathrm{ev}_{y_0}\colon \mathcal{G} \to U(1)\right)$, where ev_{y_0} denotes the evaluation map at the chosen point y_0.

Set

$$\begin{aligned} \tilde{\mathcal{A}}(Y) &:= (A + i\ker d) \times \left(\Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y)\right), \\ \tilde{C}(Y) &:= (A + i\ker d) \times \left(\Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y) \oplus H^1(Y;\mathbb{R})\right). \end{aligned}$$

Anticipating the discussion from Chapter 3, we will fix a parameter $\lambda \in \mathbb{R}$ and include it in the monopole map as a perturbation of the first Seiberg-Witten equation.

A preliminary version of the monopole map is thus given by

$$\tilde{\mu}_\lambda \colon \tilde{\mathcal{A}}(Y) \to \tilde{C}(Y), \tag{2.2}$$

$$(A', \psi, a, f) \mapsto (A',\ D_{A'+ia}\psi + \lambda\psi,\ -i\big(*F_{A'+ia} - \sigma(\psi)\big) + df,\ d^*a + f_h,\ a_h) \tag{2.3}$$

$$\mapsto (A', D_{A'}\psi + \lambda\psi + \frac{1}{2}\, ia\,\psi,\ -*iF_{A'} + i\sigma(\psi) + *da + df,\ d^*a + f_h,\ a_h),$$

where a_h and f_h denote the harmonic parts of a and f respectively. For simplicity, we will omit λ from the notation of the monopole map whenever possible.

2.3.2. Remark. A few words on the terms in the above definition. In its simplest form, the monopole map is defined using only the Seiberg-Witten equations[1], i.e. as a map

$$\begin{aligned} Conn(L) \times \Gamma(S) &\to \Gamma(S) \oplus i\Omega^1(Y), \\ (A', \psi) &\mapsto (D_{A'}\psi, *F_{A'} - \sigma(\psi)). \end{aligned}$$

After fixing a connection A on L we get an identification $Conn(L) \cong \Omega^1(Y)$ and the map translates to

$$\begin{aligned} \Omega^1(Y) \times \Gamma(S) &\to \Gamma(S) \oplus i\Omega^1(Y), \\ (a, \psi) &\mapsto (D_A\psi + \frac{1}{2} ia\psi, *F_A + *ida - \sigma(\psi)). \end{aligned}$$

In order to interpret the monopole map as an element of some stable cohomotopy group, it is desirable to write it fibrewise as a sum of a linear Fredholm map and a compact map ([BF04, Theorem 2.6], [Bau04a, §2]). The linear part of the above version of the monopole map is given by $(\psi, a) \mapsto (D_A\psi, *ida)$. This map has no chance of being Fredholm, because $*d \colon \Omega^1(Y) \to \Omega^1(Y)$ has infinite-dimensional kernel and cokernel.

To remedy this, we include $d \colon \Omega^0(Y) \to \Omega^1(Y)$ and its adjoint $d^* \colon \Omega^1(Y) \to \Omega^0(Y)$ in the definition. The resulting operator $\left[\begin{smallmatrix} *d & d \\ d^* & 0 \end{smallmatrix}\right]$ is elliptic (§1.6), and it has a well-defined index and a Fredholm extension to every Sobolev completion (§A.1.2). The other summands appearing in (2.2) and (2.3) influence the index (2.15) of the linear part l (2.12b) of the monopole map.

Another important role of the additional summands (in particular, the projections onto harmonic forms) is to make the linear part on forms injective. Namely, the linear part on forms corresponds to the restriction of the monopole map onto the set of points fixed by the residual $U(1)$-action discussed shortly. If this restriction (i.e. the linear part on forms) is not injective, the restriction is not proper, and hence the monopole map cannot be proper (i.e. the desired boundedness property[2] cannot hold). ◀

[1] cf. [Sco05, p. 442]

[2] introduced on p. 38

There is an action on $\tilde{\mathcal{A}}(Y)$ and $\tilde{C}(Y)$ of the group $\mathcal{G} = \{u\colon Y \to U(1)\} = \operatorname{map}(Y; U(1))$ of gauge transformations of $L = \det\mathfrak{s}$ which consists of the following actions:

$$\mathcal{G} \times \Gamma(S) \ni (u, \psi) \mapsto u \mathbin{\vdots} \psi \in \Gamma(S), \tag{2.4a}$$

$$u \cdot A' = A' + 2udu^{-1}, \quad A' \in Conn(L), \tag{2.4b}$$

$$u \cdot \nabla_{A'} = \nabla_{A'} + udu^{-1}, \quad A' \in Conn(L). \tag{2.4c}$$

The first action is given, and the others follow from the first. The action is trivial on forms. It is clear that the action of $\mathcal{G}$ on $Conn(L)$ is not free, with stabilisers being the constant functions $Y \to U(1)$. The action becomes free if we restrict to $\mathcal{G}_0 := \ker\left(\mathrm{ev}_{y_0}\colon \mathcal{G} \to U(1)\right)$.

The monopole map $\tilde{\mu}$ (2.3) is equivariant with respect to the action of the group of gauge transformations $\mathcal{G}$, because this is true for the Seiberg-Witten equations.

In particular, it is $\mathcal{G}_0$-equivariant, and we get a map

$$\mu = \tilde{\mu}/_{\mathcal{G}_0}\colon \mathcal{A}(Y) \to C(Y), \tag{2.5}$$

which will be called the *monopole map* of the pair $(Y, \mathfrak{s})$.

2.3.1.1 Picard torus

Seiberg-Witten equations are invariant with respect to the action of $\mathcal{G}$, so the solutions are considered up to gauge equivalence. In order to encode solutions only up to gauge transformations, we discuss the quotient of the space of all Hermitian connections on L with the same curvature by the action of the based gauge group.

2.3.3. Lemma. *For a fixed Hermitian connection $A \in Conn(L)$, the subspace $A + i\ker d \subseteq Conn(L)$ is invariant under the action of the based gauge group $\mathcal{G}_0$. Furthermore,*

$$(A + i\ker d)/_{\mathcal{G}_0} \cong H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z}) \cong \mathrm{Pic}^{\mathfrak{s}}(Y),$$

where $\mathrm{Pic}^{\mathfrak{s}}(Y)$ *denotes the* Picard manifold *of* Y.

Proof. Since $F_{A+ia} = F_A + ida$, the subspace $A + i\ker d \subseteq Conn(L)$ consists precisely of those connections, which have the same curvature as A. Since the curvature of a spin$^{\mathbb{C}}$-connection is invariant with respect to the $\mathcal{G}$-action, the invariance follows.

As mentioned earlier, the action of $\mathcal{G}_0$ on $Conn(L)$ is free. In particular, it acts free on $A + i\ker d$. Let $(\mathcal{G}_0)_0$ denote the connected component of the map $u_0 \equiv 1$. That is, $(\mathcal{G}_0)_0$ is the subgroup of $\mathcal{G}_0$ consisting of all maps u homotopic to u_0 (i.e. null-homotopic). As already mentioned, every $u \in (\mathcal{G}_0)_0$ can be written as $u = e^{if}$ for some smooth function $f\colon Y \to \mathbb{R}$. Hence, $u \in (\mathcal{G}_0)_0$ acts on A by adding $-idf$, i.e. the $(\mathcal{G}_0)_0$-orbit of A is of the form $(\mathcal{G}_0)_0 \cdot A = A + i\,\mathrm{im}\, d$. This implies

$$(A + i\ker\, d) \Big/ (\mathcal{G}_0)_0 \cong \ker d \Big/ \mathrm{im}\, d = H^1(Y;\mathbb{R}).$$

On the other hand, there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow (\mathcal{G}_0)_0 \hookrightarrow \mathcal{G}_0 &\longrightarrow \pi_0(\mathcal{G}_0) \longrightarrow 0, \\ u &\longmapsto [u]_{\mathrm{w}}, \end{aligned}$$

with $[u]_{\mathrm{w}} \in \pi_0(\mathcal{G}_0) = [Y, S^1]$ denoting the path-component of $u \in \mathcal{G}_0$, i.e. its homotopy class. The fact

$$\pi_0(\mathcal{G}_0) = [Y, S^1] = [Y, K(\mathbb{Z}, 1)] \cong H^1(Y;\mathbb{Z}), \tag{2.6}$$

yields a short exact sequence

$$0 \longrightarrow (\mathcal{G}_0)_0 \hookrightarrow \mathcal{G}_0 \longrightarrow H^1(Y;\mathbb{Z}) \longrightarrow 0.$$

In other words

$$\mathcal{G}_0 \Big/ (\mathcal{G}_0)_0 \cong H^1(Y;\mathbb{Z}),$$

and finally

$$\begin{aligned} (A + i\ker\, d) \Big/ \mathcal{G}_0 &\cong \left((A + i\ker\, d) \Big/ (\mathcal{G}_0)_0 \right) \Big/ \left(\mathcal{G}_0 \Big/ (\mathcal{G}_0)_0 \right) \\ &\cong H^1(Y;\mathbb{R}) \Big/ H^1(Y;\mathbb{Z}) \cong \mathrm{Pic}^{\mathfrak{s}}(Y). \end{aligned}$$

□

2.3.4. Remark. The space $\mathrm{Pic}^{\mathfrak{s}}(Y)$ does not really depend on $\mathfrak{s}$ (i.e. for every spin$^{\mathbb{C}}$ structure we get a copy of the same torus). The notation only suggests that we have the copy corresponding to the chosen spin$^{\mathbb{C}}$ structure. ◀

2.3.1.2 Monopole bundles

The domain and codomain of μ are given by:

$$\mathcal{A}(Y) \coloneqq \tilde{\mathcal{A}}(Y)/\mathcal{G}_0,$$
$$C(Y) \coloneqq \tilde{C}(Y)/\mathcal{G}_0.$$

Both $\mathcal{A}(Y)$ and $C(Y)$ are infinite-dimensional vector bundles over $\mathrm{Pic}^{\mathfrak{s}}(Y)$:

$$\pi_{\mathcal{A}} \colon \mathcal{A}(Y) \to \mathrm{Pic}^{\mathfrak{s}}(Y), \tag{2.7a}$$
$$\Big([A + ia', \psi], a, f\Big) \mapsto [A + ia'], \tag{2.7b}$$

$$\pi_{C} \colon C(Y) \to \mathrm{Pic}^{\mathfrak{s}}(Y), \tag{2.7c}$$
$$\Big([A + ia', \psi], b, g, a_h\Big) \mapsto [A + ia']. \tag{2.7d}$$

The bundles $\mathcal{A}(Y) \to \mathrm{Pic}^{\mathfrak{s}}(Y)$ and $C(Y) \to \mathrm{Pic}^{\mathfrak{s}}(Y)$ described in (2.7) are not trivial[1] in general (of course, if $H^1(Y;\mathbb{R}) = 0$, then $\mathrm{Pic}^{\mathfrak{s}}(Y)$ consists of a single point).

2.3.1.3 Monopole map on fibres

The monopole map μ in (2.5) is a fibre-preserving map between infinite-dimensional vector bundles over $\mathrm{Pic}^{\mathfrak{s}}(Y)$ (because the induced map on the base space $\mathrm{Pic}^{\mathfrak{s}}(Y)$ is the identity). However, it is not a vector bundle map because it is not linear on fibres.

In the following, set $A' = A + ia'$, with $a' \in \ker d$. The restriction

$$\mu_{[A']} \colon \pi_{\mathcal{A}}^{\leftarrow}([A']) \to \pi_{C}^{\leftarrow}([A'])$$

to fibres

$$\pi_{\mathcal{A}}^{\leftarrow}([A']) = \Big\{([A', \psi], a, f) : \psi \in \Gamma(S), a \in \Omega^1(Y), f \in \Omega^0(Y;\mathbb{R})\Big\},$$
$$\pi_{C}^{\leftarrow}([A']) = \Big\{([A', \psi], b, g, a_h) \colon \psi \in \Gamma(S), b \in \Omega^1(Y), g \in \Omega^0(Y), a_h \in H^1(Y;\mathbb{R})\Big\},$$

is of the form[3]:

[3] cf. [BF04, p. 11]

$$\begin{aligned}
&\mu_{[A']}\colon \big([A',\psi],a,f\big)\\
&\mapsto \big([A', D_{A'+ia}\psi + \lambda\psi],\ -*iF_{A'+ia} + i\sigma(\psi) + df,\ d^*a + f_h,\ a_h\big) =\\
&\Big([A', D_A\psi + \frac{1}{2}\, ia\,\psi + \frac{1}{2}\, ia'\,\psi + \lambda\psi],\ -*iF_A + i\sigma(\psi) + *da + df,\ d^*a + f_h,\ a_h\Big).
\end{aligned} \tag{2.8}$$

Note that the last equality holds due to $a' \in \ker d$, so $da' = 0$. Also note that the fibres carry the obvious vector space structure: $[A', \psi_1] + \tau[A', \psi_2] := [A', \psi_1 + \tau\psi_2]$, $\tau \in \mathbb{C}$.

Let ι_{H^0} denote the inclusion $H^0(Y) \hookrightarrow \Omega^1(Y)$. Over every point in $\mathrm{Pic}^{\mathfrak{s}}(Y)$ (i.e. in each fibre) the monopole map μ can be written as the sum of the following assignments:

$$\begin{aligned}
&l\colon \pi_{\mathcal{A}}^{\leftarrow}([A']) \to \pi_{\mathcal{C}}^{\leftarrow}([A']),\\
&l\colon \big([A',\psi],a,f\big) \mapsto \big([A', D_A\psi + \lambda\psi],\ *da + df,\ d^*a + f_h,\ a_h\big),
\end{aligned} \tag{2.9a}$$

$$\begin{aligned}
&c\colon \pi_{\mathcal{A}}^{\leftarrow}([A']) \to \pi_{\mathcal{C}}^{\leftarrow}([A']),\\
&c\colon \big([A',\psi],a,f\big) \mapsto \Big([A', \frac{1}{2}\, ia\,\psi + \frac{1}{2}\, ia'\,\psi],\ -*iF_A + i\sigma(\psi),\ 0,\ 0,\ 0\Big).
\end{aligned} \tag{2.9b}$$

The fibre of $\mathcal{A}(Y)$ over $[A'] \in \mathrm{Pic}^{\mathfrak{s}}(Y) = (A + i\ker d)/_{\mathcal{G}_0}$ can be written as follows

$$\begin{aligned}
\pi_{\mathcal{A}}^{\leftarrow}([A']) &= \Big\{(A',\psi,a,f)\colon \psi \in \Gamma(S),\ a \in \Omega^1(Y),\ f \in \Omega^0(Y;\mathbb{R})\Big\}\Big/\mathcal{G}_0\\
&\cong (\{A'\} \times \Gamma(S))/_{\mathcal{G}_0} \times (\Omega^1(Y) \oplus \Omega^0(Y;\mathbb{R})),\\
&\cong \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y;\mathbb{R}),
\end{aligned} \tag{2.10}$$

where the last isomorphism is given by $[A',\psi] \mapsto \psi$ with the obvious inverse. Note that we need to keep the representative A' fixed in the definition of this isomorphism. The isomorphism clearly depends[2] on the choice of a representative of $[A']$.

Similarly, the fibre of $C(Y)$ over $[A'] \in (A + i\ker d)/_{\mathcal{G}_0}$ can be written in the following form

$$
\begin{aligned}
&\pi_C^{\leftarrow}([A']) = \\
&\qquad \left\{(A', \psi, b, g, a_h) \colon \psi \in \Gamma(S),\ b \in \Omega^1(Y),\ g \in \Omega^0(Y), a_h \in H^1(Y;\mathbb{R})\right\} \Big/ \mathcal{G}_0 \\
&\qquad \cong (\{A'\} \times \Gamma(S))/_{\mathcal{G}_0} \times (\Omega^1(Y) \oplus \Omega^0(Y) \oplus H^1(Y;\mathbb{R})), \\
&\qquad \cong \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y) \oplus H^1(Y;\mathbb{R}). \qquad (2.11)
\end{aligned}
$$

In short, after fixing some representative of $[A']$ we get identifications

$$
\begin{aligned}
\pi_{\mathcal{A}}^{\leftarrow}([A']) &\cong \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y;\mathbb{R}), \\
\pi_C^{\leftarrow}([A']) &\cong \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y) \oplus H^1(Y;\mathbb{R}).
\end{aligned}
$$

Using these identifications, the assignments (2.8) and (2.9) now become maps

$$\Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y;\mathbb{R}) \to \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y) \oplus H^1(Y;\mathbb{R}),$$

and are of the following form

$$
\begin{aligned}
&\mu_{[A']} \colon (\psi, a, f) \\
&\mapsto \left(D_A\psi + \frac{1}{2}\, ia\, \psi + \frac{1}{2}\, ia'\, \psi + \lambda\psi,\ -*iF_A + i\sigma(\psi) + *da + df,\ d^*a + f_h,\ a_h\right) \qquad (2.12a)
\end{aligned}
$$

$$l_{[A']} \colon (\psi, a, f) \mapsto (D_A\psi + \lambda\psi,\ *da + df,\ d^*a + f_h,\ a_h), \qquad (2.12b)$$

$$c_{[A']} \colon (\psi, a, f) \mapsto (\frac{1}{2}\, ia\, \psi + \frac{1}{2}\, ia'\, \psi,\ -*iF_A + i\sigma(\psi),\ 0,\ 0,\ 0). \qquad (2.12c)$$

The notation $\mu_{[A']}, l_{[A']}, c_{[A']}$ reflects the dependence of expressions (2.12) on the point in $[A'] \in \mathrm{Pic}^{\mathfrak{s}}(Y)$, as well as on the choice of the representative $A' = A + ia'$ of this point.

How do maps (2.12) vary with the change of a representative of $[A']$? In other words, after applying the same procedure with a different representative for $[A']$, does one get the same maps

$$\Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y;\mathbb{R}) \to \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y) \oplus H^1(Y;\mathbb{R})?$$

The answer is no. With the change of $A' = A + ia'$, all maps (2.12) containing a' change, i.e. $\mu_{[A']}$ and $c_{[A']}$ change as $A' = A + ia'$ varies. On the other hand, the presentation of the linear part $l_{[A']}$ does *not* change[3]. In other words, the change of fibrewise presentation is solely detected by the non-linear part.

2.3.5. REMARK. The fibrewise presentation (2.12b) of the linear part does indeed stay the same. The Dirac operator $D_{A'}$ obviously changes as one travels around $\mathrm{Pic}^{\mathfrak{s}}(Y)$: the dimension of the kernel and the cokernel change (cf. [LM89, p. 206, Eq. (8.4)]), as well as eigenvalues. However, the index stays the same as one travels around $\mathrm{Pic}^{\mathfrak{s}}(Y)$ ([LM89, §III.7 and §III.8]). This is importat in the construction of the stable cohomotopy invariant, where the virtual index bundle of the linear part is used ([BF04, Bau04a]) ◄

2.3.6. LEMMA. *The kernel and the cokernel of the linear part l (2.9a) are finite-dimensional.*

Proof. We take a fibrewise presentation (2.12b) of l. Leaving the projections to harmonic forms out, the operator $l_{[A']}$ is a sum of D_A and the operator

$$D_\Omega = \begin{bmatrix} *d & d \\ d^* & 0 \end{bmatrix} : \Omega^1(Y) \oplus \Omega^0(Y;\mathbb{R}) \to \Omega^1(Y) \oplus \Omega^0(Y). \qquad (a,f) \mapsto (*da + df,\ d^*a). \tag{2.13}$$

The former is an elliptic operator over a closed Riemannian manifold, so its kernel and cokernel are finite-dimensional ([LM89, p. 135]). It therefore has a well-defined index[4].

As we have seen in §1.6 by looking at its square, the latter summand is an elliptic operator on $\Omega^1(Y) \oplus \Omega^0(Y;\mathbb{R})$. □

Note that the operator (2.13) together with the projections onto harmonic forms is injective, so[4]

$$\mathrm{ind}_{\mathbb{R}}\Big(*d + d,\ d^* + \mathrm{pr}_{H^0},\ \mathrm{pr}_{H^1}\Big) = -\dim_{\mathbb{R}} \mathrm{coker}\Big(*d + d,\ d^* + \mathrm{pr}_{H^0},\ \mathrm{pr}_{H^1}\Big) = -b_1, \tag{2.14}$$

and hence

$$\mathrm{ind}_{\mathbb{R}}\, l = \mathrm{ind}_{\mathbb{R}}\, D_A - b_1 = -b_1, \tag{2.15}$$

since $\mathrm{ind}_{\mathbb{R}} D_A = 0$ (D_A is self-adjoint). Therefore, the index of l depends only on b_1 and it is always non-positive.

Note that the preimage of $(A,0,0,0,0)$ under $\tilde{\mu}$ corresponds to the solutions of the Seiberg-Witten equations (2.1). Namely, suppose that $\tilde{\mu}(A,\psi,a,f) = (A,0,0,0,0)$, i.e.

[4] $\mathrm{im}(*d + d) = \mathrm{im}(\Delta) = \mathrm{im}(d) + \mathrm{im}(d^*) \subseteq \Omega^1(Y)$, and $\mathrm{im}(d^*) = \mathrm{im}(\Delta) = \mathrm{im}(d^*) \subseteq \Omega^0(Y)$, and also note that the two coordinates in the image of (2.13) are not dependent, i.e. all elements in the images of Δ are hit

$$\begin{aligned} D_{A+ia}\psi + \lambda\psi &= 0, \\ -*iF_{A+ia} + i\sigma(\psi) + df &= 0, \\ d^*a + f_h &= 0, \\ a_h &= 0. \end{aligned}$$

The third equation implies $d^*a = 0 = f_h$, due to the Hodge decomposition. The first equation and (1.57) give $-id^*\sigma(\psi) = \mathrm{Im}\langle D_{A+ia}\psi, \psi\rangle_{\mathbb{C}} = \langle D_{A+ia}\psi, i\psi\rangle = -\langle \lambda\psi, i\psi\rangle = 0$. Hence, $\sigma(\psi)$ is a coclosed 1-form and second equation now implies $*iF_{A+ia} - i\sigma(\psi) = 0 = df$. From $df = 0$ follows $f = f_h$, and since f_h vanishes, we get $f = 0$. Hence, for $\lambda = 0$ we get the solutions to classical Seiberg-Witten equations. Otherwise, we get the solutions of the Seiberg-Witten equations, with the first of the two equations being slightly modified (perturbed by λ).

The preimage under μ is the same space, only divided by the based gauge group $\mathcal{G}_0$.

Note also that we defined μ as the quotient map of $\tilde{\mu}$ under $\mathcal{G}_0$. So, there is a residual action of the group of constant functions $u\colon Y \to U(1)$, i.e. a residual action of $U(1)$. Since $\tilde{\mu}$ is $\mathcal{G}$-equivariant, the monopole map μ is equivariant with respect to the mentioned residual action of $U(1)$. According to (2.4), $U(1)$ acts on spinors through complex multiplication, and trivially on all other spaces appearing in the definition.

2.3.2 Properties

In what follows, we will consider the fibrewise Sobolev L^2_k-completion $\mathcal{A}_k$ of $\mathcal{A} := \mathcal{A}(Y)$ and the fibrewise Sobolev L^2_{k-1}-completion C_{k-1} of $C := C(Y)$.

The following three properties of the monopole (2.5) map will play a decisive role in the construction of the stable cohomotopy class (cf. [BF04, §2]):

(i) μ is $U(1)$-equivariant;

(ii) μ is the sum of a linear Fredholm map and a (nonlinear) compact map;

(iii) $\mu^{\leftarrow}(B)$ is a bounded disk subbundle of $\mathcal{A}_k$ for every bounded disk subbundle $B \subseteq C_{k-1}$.

The $U(1)$ equivariance is obvious, and was already mentioned. The second property follows from examining the monopole map on fibres (cf. §2.3.1.3).

The third property (from now on referred to as the *boundedness property*), requires a more extensive discussion.

2.3.7. Proposition. *The monopole map extends to a continuous fibre-preserving map* $\mu_k\colon \mathcal{A}_k \to \mathcal{C}_{k-1}$ *over* $\mathrm{Pic}^{\mathfrak{s}}(Y)$ *for* $k \geq 2$.

Proof. The extension of the linear part (2.12b) of the monopole map is a fibrewise bounded linear map for every[5] $k \geq 1$.

It remains to show that the extension

$$c : (a, \psi, f) \mapsto \left(\frac{1}{2}\, ia\,\psi,\; - *iF_A + i\sigma(\psi), 0, 0\right)$$

of the non-linear part (2.12c) of the monopole map to the corresponding Sobolev completions is continuous.

The Sobolev multiplication theorem (Theorem A.1.6) implies that the non-linear part has continuous extension $\mathcal{A}_k \times \mathcal{A}_k \to \mathcal{C}_k$ for $k > \frac{3}{2}$. After composing with the inclusion $\mathcal{C}_k \hookrightarrow \mathcal{C}_{k-1}$, which is a compact map (Theorem A.1.5), we conclude that $c\colon \mathcal{A}_k \times \mathcal{A}_k \to \mathcal{C}_{k-1}$ is a compact map[6]. □

The extension of the monopole map in Proposition 2.3.7 will also be denoted by μ. It has the following important properties:

(i) μ is $U(1)$-equivariant;

(ii) μ is the sum of a linear Fredholm map and a (nonlinear) compact map;

(iii) $\mu^{\leftarrow}(B)$ is a bounded disk subbundle of $\mathcal{A}_k$ for every bounded disk subbundle $B \subseteq \mathcal{C}_{k-1}$.

The unknown terms in the above properties can be found in the following definitions ([BF04, p. 8]).

2.3.8. Definition. A map $f\colon \mathcal{E} \to \mathcal{F}$ between vector bundles $\mathcal{E}$ and $\mathcal{F}$ over some base space B is called a *Fredholm morphism* if it is fibre-preserving and fibrewise a linear Fredholm operator.

2.3.9. Definition. A map $f\colon \mathcal{E} \to \mathcal{F}$ between vector bundles $\mathcal{E}$ and $\mathcal{F}$ over some base space B is called a *compact map* if it is fibre-preserving and fibrewise a continuous compact map. A map $h\colon H' \to H$ between Banach spaces is called *compact* if it is continuous and the closure $\overline{c(B')} \subseteq H$ is a compact subset of H for every bounded $B' \subseteq H'$.

[5] [LM89, Thm III.2.15., p. 176]

[6] in the sense of Definition 2.3.9

2.3.10. Definition. A map $f: \mathcal{E} \to \mathcal{F}$ between vector bundles $\mathcal{E}$ and $\mathcal{F}$ over some base space B is called a *Fredholm map* if it can be written as a sum of a Fredholm morphism and a compact map. In other words, a Fredholm map is a compact perturbation of a Fredholm morphism.

The $U(1)$-equivariance of μ follows from the definition. That the non-linear part of the extension is compact, was proved in Proposition 2.3.7. The next lemma settles the linear part.

2.3.11. Lemma. *The linear part l of the monopole map $\mu: \mathcal{A}_k \to C_{k-1}$ is fibrewise a Fredholm operator for $k \geq 2$. Its index is independent of k and equals*

$$\operatorname{ind}_{\mathbb{R}} l = \operatorname{ind}_{\mathbb{R}}(D_A + \lambda) - b_1(Y) = -b_1(Y). \tag{2.16}$$

Proof. This follows from the discussion on page 37, ellipticity of the linear operator l, and the fact that extensions of elliptic operators to Sobolev spaces are linear Fredholm operators[7].

The statement concerning the index follows from the fact that l is an elliptic operator, the equality (2.15) and the fact that the index of any Fredholm extension of an elliptic operator equals the index of the original operator[8]. □

Finally, we show the boundedness property[9].

2.3.12. Proposition. *Preimages $\mu^{\leftarrow}(B) \subseteq \mathcal{A}_k$ of bounded disk bundles $B \subseteq C_{k-1}$ are contained in bounded disk bundles.*

Proof. We look at the restriction (2.12a) of the monopole map to (completions of) fibres. Let[10]

$$k \geq 4, \tag{2.17}$$

and suppose that there is an L^2_{k-1}-bound on

[7] Theorem A.1.2

[8] cf. [LM89, Corollary III.5.3, p. 194]

[9] the idea of the proof is taken from [Bau12] (see also [BF04, §3])

[10] with Sobolev's embedding theorem in mind, $k \in \mathbb{N}$ is chosen to be greater than or equal to 3 in order to ensure that the inequalities $\|\,.\,\|_{C^0} \lesssim \|\,.\,\|_{L^2_k}$ (i.e. $k - \frac{3}{2} > 0$) and $\|D\,.\,\|_{C^0} \lesssim \|D\,.\,\|_{L^2_{k-1}} \lesssim \|\,.\,\|_{L^2_k}$ (i.e. $k - 1 - \frac{3}{2} > 0$) hold; furthermore, the assumption $k \geq 4$ guarantees that spinors and forms in the L^2_k-completions are at least twice continuously differentiable (since $k - \frac{3}{2} > 2$)

$$\begin{aligned}\mu_{[A]}(\psi, a, f) &= \Big(D_{A+ia}\psi + \lambda\psi, -*iF_{A+ia} + i\sigma(\psi) + df,\ d^*a + f_h,\ a_h\Big) \\ &= \Big(D_A\psi + \lambda\psi + \frac{1}{2}\, ia\,\psi, -*iF_A + i\sigma(\psi) + *da + df,\ d^*a + f_h,\ a_h\Big).\end{aligned}$$

The expression

$$\Big(\Big\langle D^2_{A+ia}\psi, \psi\Big\rangle - |D_{A+ia}\psi|^2\Big) - \Big(\langle\nabla^*_{A+ia}\nabla_{A+ia}\psi, \psi\rangle - |\nabla_{A+ia}\psi|^2\Big),$$

can be written as $d^*\alpha$, for some 1-form α. Combining this with the Weitzenböck formula[11]

$$\begin{aligned}D^2_{A+ia} &= \nabla^*_{A+ia}\nabla_{A+ia} + \frac{s}{4} + \frac{1}{2}c(F_{A+ia}) \\ &= \nabla^*_{A+ia}\nabla_{A+ia} + \frac{s}{4} + \frac{1}{2}c(*F_{A+ia})\end{aligned}$$

yields

$$\begin{aligned}&|D_{A+ia}\psi|^2 - |\nabla_{A+ia}\psi|^2 + d^*\alpha \\ &= \Big\langle D^2_{A+ia}\psi, \psi\Big\rangle - \langle\nabla^*_{A+ia}\nabla_{A+ia}\psi, \psi\rangle \\ &= \frac{s}{4}\,|\psi|^2 + \frac{1}{2}\,\langle *F_{A+ia}\psi, \psi\rangle \\ &= \frac{s}{4}\,|\psi|^2 + \frac{1}{2}\,\langle(*F_{A+ia} - \sigma(\psi) + idf)\psi, \psi\rangle + \frac{1}{2}\,\langle\sigma(\psi)\psi, \psi\rangle - \frac{1}{2}\,\langle idf\psi, \psi\rangle \\ &= \frac{s}{4}\,|\psi|^2 + \frac{1}{2}\,\langle(*F_{A+ia} - \sigma(\psi) + idf)\psi, \psi\rangle + \frac{1}{4}\,|\psi|^4 - \frac{1}{2}\,\langle idf\psi, \psi\rangle.\end{aligned}$$

In other words, we have pointwise

$$\begin{aligned}\frac{1}{4}\,|\psi|^4 = \ &|D_{A+ia}\psi|^2 - |\nabla_{A+ia}\psi|^2 + d^*\alpha - \frac{s}{4}\,|\psi|^2 \\ &- \frac{1}{2}\,\langle(*F_{A+ia} - \sigma(\psi) + idf)\psi, \psi\rangle + \frac{1}{2}\,\langle idf\psi, \psi\rangle \\ \leq \ &|D_{A+ia}\psi|^2 + d^*\alpha + \frac{1}{4}\,\|s\|_{C^0}\,|\psi|^2 \\ &- \frac{1}{2}\,\langle(*F_{A+ia} - \sigma(\psi) + idf)\psi, \psi\rangle + \frac{1}{2}\,\langle idf\psi, \psi\rangle\end{aligned}$$

[11] recall that (1.48) implies $c(F_{A+ia}) = c(*F_{A+ia})$

$$\leq \quad |D_{A+ia}\psi|^2 + d^*\alpha + \frac{1}{4}\,\|s\|_{C^0}\,|\psi|^2 + \frac{1}{2}\,\|{*F_{A+ia}} - \sigma(\psi) + idf\|_{C^0}\cdot|\psi|^2 + \frac{1}{2}\,\langle idf\psi,\psi\rangle.$$

Before proceeding with the proof, recall that pointwise we have[12]

$$\langle idf\psi,\psi\rangle = \langle df, -2i\sigma(\psi)\rangle = 2\langle idf, \sigma(\psi)\rangle$$

so we get

$$|\psi|^4 \lesssim \quad |D_{A+ia}\psi|^2 + d^*\alpha + \|s\|_{C^0}\,|\psi|^2 + \|{*F_{A+ia}} - \sigma(\psi) + idf\|_{C^0}\cdot|\psi|^2 + \langle idf, \sigma(\psi)\rangle.$$

Integrating both sides of the above inequality over Y produces

$$\|\psi\|_{L^4}^4 \lesssim \quad \|D_{A+ia}\psi\|_{L^2}^2 + 0 + \|s\|_{C^0}\,\|\psi\|_{L^2}^2 + \|{*F_{A+ia}} - \sigma(\psi) + idf\|_{C^0}\cdot\|\psi\|_{L^2}^2 + \langle idf, \sigma(\psi)\rangle_{L^2}. \tag{2.18}$$

We pause again to discuss $\langle idf, \sigma(\psi)\rangle_{L^2}$. Note that a Hermitian perturbation of the Dirac operator does not influence the value of $\langle D_{A+ia}\psi, i\psi\rangle_{\mathbb{R}} \overset{(1.57)}{=} -id^*\sigma(\psi)$. In particular, $-id^*\sigma(\psi) = \langle D_{A+ia}\psi, i\psi\rangle_{\mathbb{R}} = \langle D_{A+ia}\psi + \lambda\psi, i\psi\rangle_{\mathbb{R}}$ and we have

$$\begin{aligned}
\langle idf, \sigma(\psi)\rangle_{L^2} &= \int_Y \langle idf, \sigma(\psi)\rangle\,\mathrm{dvol} = \int_Y \langle f, -id^*\sigma(\psi)\rangle\,\mathrm{dvol}\\
&\leq \int_Y |f|\cdot|-id^*\sigma(\psi)|\,\mathrm{dvol} = \int_Y |f|\cdot|\langle D_{A+ia}\psi, i\psi\rangle|\,\mathrm{dvol}\\
&= \int_Y |f|\cdot|\langle D_{A+ia}\psi + \lambda\psi, i\psi\rangle|\,\mathrm{dvol}\\
&\leq \int_Y |f|\cdot|D_{A+ia}\psi + \lambda\psi|\cdot|\psi|\,\mathrm{dvol}\\
&\leq \|D_{A+ia}\psi + \lambda\psi\|_{C^0}\cdot\int_Y |f|\cdot|\psi|\,\mathrm{dvol}\\
&\leq \|D_{A+ia}\psi + \lambda\psi\|_{C^0}\cdot\|f\|_{L^2}\cdot\|\psi\|_{L^2},
\end{aligned} \tag{2.19}$$

which after using Sobolev's embedding theorem[13] implies

$$\langle idf, \sigma(\psi)\rangle_{L^2} \lesssim \|D_{A+ia}\psi + \lambda\psi\|_{L^2_{k-1}}\cdot\|f\|_{L^2}\cdot\|\psi\|_{L^2} \lesssim \|f\|_{L^2}\cdot\|\psi\|_{L^2}.$$

[12] see (1.56) on p. 22

[13] the assumption $k \geq 4$ is used at this point

Plugging the obtained estimate on $\langle idf, \sigma(\psi)\rangle_{L^2}$ into (2.18) yields[14]

$$\begin{aligned}\|\psi\|_{L^4}^4 &\lesssim \|D_{A+ia}\psi\|_{L^2}^2 + \|s\|_{C^0}\|\psi\|_{L^2}^2 \\ &\quad + \|{*F_{A+ia}} - \sigma(\psi) + idf\|_{C^0} \cdot \|\psi\|_{L^2}^2 + \|f\|_{L^2} \cdot \|\psi\|_{L^2} \\ &\lesssim \|D_{A+ia}\psi + \lambda\psi\|_{L^2}^2 + |\lambda|\|\psi\|_{L^2}^2 + \|s\|_{C^0}\|\psi\|_{L^2}^2 \\ &\quad + \|{*F_{A+ia}} - \sigma(\psi) + idf\|_{L^2_{k-1}} \cdot \|\psi\|_{L^2}^2 + \|f\|_{L^2} \cdot \|\psi\|_{L^2} \\ &\lesssim 1 + |\lambda|\|\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|f\|_{L^2} \cdot \|\psi\|_{L^2}.\end{aligned}$$

So in total we have

$$\|\psi\|_{L^4}^4 \lesssim 1 + |\lambda|\|\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|f\|_{L^2} \cdot \|\psi\|_{L^2}. \tag{2.20}$$

The goal is to get on the right side of (2.20) a polynomial in $\|\psi\|_{L^4}$ of degree at most 3. To achieve that, $\|f\|_{L^2}$ has to be bounded by some polynomial in $\|\psi\|_{L^4}$ of degree at most 2.

This can be achieved by the use of the elliptic estimates. Stronger version of elliptic estimate (Lemma A.1.3) for D_Ω gives

$$\begin{aligned}\|(a,f)\|_{L^2_1} &\lesssim \|{*da} + df\|_{L^2} + \|d^*a\|_{L^2} + \|\mathrm{pr}(a,f)\|_{L^2}, \\ &\lesssim \|{*da} + df\|_{L^2} + \|d^*a\|_{L^2} + \|(a_h, f_h)\|_{L^2}, \\ &\lesssim \|{*da} + df\|_{L^2} + \|d^*a\|_{L^2} + \|a_h\|_{L^2} + \|f_h\|_{L^2},\end{aligned}$$

where $\mathrm{pr}\colon \Omega^1(Y) \oplus \Omega^0(Y) \to H^1(Y) \oplus H^0(Y)$ denotes the projection onto the kernel $\ker D_\Omega = H^1(Y) \oplus H^0(Y)$ of D_Ω.

Note that since d^*a and f_h are L^2_{k-1}-orthogonal[15], we have an L^2_{k-1}-bound on d^*a and on f_h. Hence

$$\|(a,f)\|_{L^2_1} \lesssim \|{*da} + df\|_{L^2} + \|d^*a + f_h\|_{L^2} + \|a_h\|_{L^2},$$

and[16]

$$\begin{aligned}\|(a,f)\|_{L^2} \le \|(a,f)\|_{L^2_1} &\lesssim \|{*da} + df\|_{L^2} + \|d^*a + f_h\|_{L^2} + \|a_h\|_{L^2}, \\ \lesssim \|(-{*iF_A} + i\sigma(\psi) + {*da} + df) &+ {*iF_A} - i\sigma(\psi)\|_{L^2} + \|d^*a + f_h\|_{L^2} + \|a_h\|_{L^2}\end{aligned}$$

[14] again, the assumption $k \geq 4$ is used here

[15] see Lemma A.2.1

[16] as argued in §1.7.3, we have $\|i\sigma(\psi)\|_{L^2} = \frac{1}{2}\|\psi\|_{L^4}^2$

$$\begin{aligned}&\lesssim \|-*iF_{A+ia}+i\sigma(\psi)+df\|_{L^2}+\|*iF_A\|_{L^2}+\|i\sigma(\psi)\|_{L^2}+\|d^*a+f_h\|_{L^2}+\|a_h\|_{L^2}\\&\lesssim \|-*iF_{A+ia}+i\sigma(\psi)+df\|_{L^2_{k-1}}+\|*iF_A\|_{L^2}+\|\psi\|^2_{L^4}+\|d^*a+f_h\|_{L^2_{k-1}}+\|a_h\|_{L^2}\\&\lesssim 1+\|\psi\|^2_{L^4}.\end{aligned}\tag{2.21}$$

Inserting (2.21) into (2.20) yields

$$\|\psi\|^4_{L^4}\lesssim 1+|\lambda|\|\psi\|^2_{L^2}+\|\psi\|^2_{L^2}+(1+\|\psi\|^2_{L^4})\cdot\|\psi\|_{L^2},$$

and therefore in particular[17]

$$\|\psi\|^4_{L^4}\lesssim 1+|\lambda|\|\psi\|^2_{L^4}+\|\psi\|^2_{L^4}+(1+\|\psi\|^2_{L^4})\cdot\|\psi\|_{L^4},\tag{2.22}$$

Since the left-hand side grows faster than the right-hand side, $\|\psi\|_{L^4}$ has to be bounded. From (2.21) now also follows that $\|a,f\|_{L^2_1}$ is bounded.

As the next step, we use elliptic estimate together with the Sobolev multiplication theorem to obtain higher order bounds with the help of method called elliptic *bootstrapping*.

For all $j\leq k$ and all[5]

$$\begin{aligned}&1\leq p_j\leq\frac{1}{\frac{1}{2}-\frac{k-j}{3}}=\frac{3}{\frac{3}{2}-(k-j)},\quad \text{if } k-j\leq\frac{3}{2},\\&1\leq p_j,\quad \text{else},\end{aligned}\tag{2.23}$$

we have the following inequality

$$\begin{aligned}&\|\psi,a,f\|^{p_j}_{L^{p_j}_j}-\|\psi,a,f\|^{p_j}_{L^{p_j}_{j-1}}\lesssim\|D_A\psi,*da+df,d^*a\|^{p_j}_{L^{p_j}_{j-1}}\\&\lesssim\|D_{A+ia}\psi+\lambda\psi,-i*F_{A+ia}+i\sigma(\psi)+df,d^*a+f_h\|^{p_j}_{L^{p_j}_{j-1}}+\\&\quad+\|ia\psi\|^{p_j}_{L^{p_j}_{j-1}}+|\lambda|\,\|\psi\|^{p_j}_{L^{p_j}_{j-1}}+\|F_A\|^{p_j}_{L^{p_j}_{j-1}}+\|\sigma(\psi)\|^{p_j}_{L^{p_j}_{j-1}}+\|f_h\|^{p_j}_{L^{p_j}_{j-1}}\\&\lesssim 1+\|a\|^{p_j}_{L^{2p_j}_{j-1}}\|\psi\|^{p_j}_{L^{2p_j}_{j-1}}+|\lambda|\,\|\psi\|^{p_j}_{L^{2p_j}_{j-1}}+\|F_A\|^{p_j}_{L^{p_j}_{j-1}}+\|\psi\|^{2p_j}_{L^{2p_j}_{j-1}}+\|f_h\|^{p_j}_{L^2_{k-1}}\\&\lesssim 1+\|a\|^{p_j}_{L^{2p_j}_{j-1}}\|\psi\|^{p_j}_{L^{2p_j}_{j-1}}+|\lambda|\,\|\psi\|^{p_j}_{L^{2p_j}_{j-1}}+\|\psi\|^{2p_j}_{L^{2p_j}_{j-1}}.\end{aligned}$$

In other words

$$\|\psi,a,f\|^{p_j}_{L^{p_j}_j}\lesssim\|\psi,a,f\|^{p_j}_{L^{p_j}_{j-1}}+1+\|a\|^{p_j}_{L^{2p_j}_{j-1}}\|\psi\|^{p_j}_{L^{2p_j}_{j-1}}+|\lambda|\,\|\psi\|^{p_j}_{L^{2p_j}_{j-1}}+\|\psi\|^{2p_j}_{L^{2p_j}_{j-1}}.\tag{2.24}$$

[17] since $\|\,.\,\|_{L^2}\lesssim\|\,.\,\|_{L^4}$

Setting $j = 1$ and $p_j = 2$ gives

$$\|\psi, a, f\|^2_{L^2_1} \lesssim \|\psi, a, f\|^2_{L^2} + 1 + \|a\|^2_{L^4} \|\psi\|^2_{L^4} + |\lambda| \, \|\psi\|^2_{L^4} + \|\psi\|^4_{L^4}.$$

The previously obtained L^2_1-bound on (a, f) implies an L^6-bound on (a, f), hence $\|\psi, a, f\|_{L^2_1} \lesssim 1$.

The obtained L^2_1-bound on (ψ, a, f) implies an L^p-bound for all $p \leq 6$, which after

$$\begin{aligned}
&\|\psi, a, f\|^3_{L^3_1} - \|\psi, a, f\|^3_{L^3} \lesssim \|D_A\psi, *da + df, d^*a\|^3_{L^3} \\
&\lesssim \|D_{A+ia}\psi + \lambda\psi, -i *F_{A+ia} + i\sigma(\psi) + df, d^*a + f_h\|^3_{L^3} + \\
&\quad + \|ia\psi\|^3_{L^3} + |\lambda| \, \|\psi\|^3_{L^3} + \|F_A\|^3_{L^3} + \|\sigma(\psi)\|^3_{L^3} + \|f_h\|^3_{L^3} \\
&\lesssim 1 + \|a\|^3_{L^6} \|\psi\|^3_{L^6} + |\lambda| \, \|\psi\|^3_{L^3} + 1 + \|\psi\|^6_{L^6} + 1,
\end{aligned}$$

further implies that $\|\psi, a, f\|_{L^3_1}$ is also bounded. Since $\| \,.\, \|_{L^p} \lesssim \| \,.\, \|_{L^3_1}$, for all $p \in \mathbb{R}, p \geq 1$, we get an $\| \,.\, \|_{L^p}$-bound for (ψ, a, f), for every $1 \leq p < \infty$.

Lastly, set $p_j := 2^{k-j+1}$ and note that all p_j satisfy (2.23), since

$$p_j = \begin{cases} 2 \leq 2 = \dfrac{3}{\frac{3}{2} - (k-j)}, & k = j, \\ 4 \leq 6 = \dfrac{3}{\frac{3}{2} - (k-j)}, & k - j = 1. \end{cases}$$

Using (2.24) we obtain an $\| \,.\, \|_{L^2_k}$-bound for (ψ, a, f), from an $\| \,.\, \|_{L^{p_0}}$-bound. □

In the spin case, we have

2.3.13. Proposition. *If Y is a equipped with a spin structure, the monopole map (2.5) is $Pin(2)$ equivariant.*

Proof. In the spin case, the Dirac operator is $\mathbb{H}$-linear. In particular, it is equivariant with respect to the action of the maximal compact connected subgroup of $\mathbb{H}$, that is $Sp(1)$. How much of this $Sp(1)$-action can be transferred to forms?

Let us look at the term $(\psi, a) \mapsto ia\psi$ (for simplicity without the prefactor $\frac{1}{2}$), where a represents a 1-form, as usual. Recall[18] that the scalar multiplication

[18] Convention 1.4.1 on p. 10

by elements of $Sp(1) \subseteq \mathbb{H}$ is given pointwise by $h \cdot_S \psi = \psi \cdot_{\mathbb{H}} \bar{h}$. Suppose there is an action of $Sp(1)$ on $\Lambda^*(\mathbb{R}^3)$. Then we pointwise have

$$\begin{aligned} h\,(\psi, a) \mapsto i(ha)(h\psi) &= (ha) \cdot_{\mathbb{H}} (h\psi) \cdot_{\mathbb{H}} \bar{\imath}, \\ &= (ha) \cdot_{\mathbb{H}} \psi \cdot_{\mathbb{H}} \bar{h} \cdot_{\mathbb{H}} \bar{\imath}, \\ &\overset{\clubsuit}{=} h(ia\psi), \\ &= a \cdot_{\mathbb{H}} \psi \cdot_{\mathbb{H}} \bar{\imath} \cdot_{\mathbb{H}} \bar{h}, \end{aligned}$$

for all $\psi \in \mathbb{H}$, $a \in \mathbb{H}$ and $h \in Sp(1)$. In other words, the action of $Sp(1)$ on forms has to satisfy the equality $\clubsuit$:

$$(ha) \cdot_{\mathbb{H}} \psi \cdot_{\mathbb{H}} \bar{h} \cdot_{\mathbb{H}} \bar{\imath} = a \cdot_{\mathbb{H}} \psi \cdot_{\mathbb{H}} \bar{\imath} \cdot_{\mathbb{H}} \bar{h}, \qquad \forall a, \psi, h.$$

I.e. we have the condition $(ha)\psi\bar{h} = a\psi\bar{\imath}\bar{h}ih$, which suggests $ha := (\bar{h}\bar{\imath}hi) \cdot_S a$ for $h \in Sp(1)$. Clearly, $ha = a$ if and only if $h \in U(1)$. Hence, $U(1)$ is the maximal subgroup of $Sp(1)$ which acts trivially on 1-forms.

Since the action of $U(1) \subseteq Sp(1)$ is trivial (and this is a maximal subgroup which acts trivially on forms), the maximal subgroup of $Sp(1)$ which can act on forms is the normaliser of $U(1)$ in $Sp(1)$ (i.e. the biggest subgroup H of $Sp(1)$ such that $U(1)$ is normal in H). According to Lemma 1.4.11, the normaliser is equal to $Pin(2) = U(1) \sqcup jU(1)$.

So we set the action of $Pin(2) = U(1) \sqcup jU(1)$ on 1-forms to be

$$ha := a, \quad h \in U(1) \tag{2.25a}$$

$$ha := -a, \quad h \in jU(1), \tag{2.25b}$$

and on 0-forms we set the action to be trivial. With this definition, the assignment (2.3) defining the monopole map (2.3) is $Pin(2)$-equivariant. □

2.4 The refined Seiberg-Witten invariant for closed 3-manifolds

As all the requirements are met, the stable homotopy construction analogous to the one in [BF04] is now possible. But there is a problem. The index (2.15) of the linear part (2.12b) of the monopole map is always non-positive. For a non-trivial cohomotopy class, a positive index is needed in general. Therefore, a new map will be investigated.

Chapter 3

New version of the monopole map

After seeing how the usual monopole map fails to yield topological information, we now look at a certain type of its perturbation. As a result of some renormalisation, we obtain a continuous family of monopole maps.

3.1 Definition, assumptions and some notation

As argued in §2.4, a Hermitian perturbation of the Dirac operator D_A does not promise anything interesting. Hence we will allow perturbations of the form if with f being an arbitrary real function on Y and analyse behaviour of the properties of the monopole map discussed in Chapter 2. We will consider the following map

$$\mu\colon \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y) \to \Gamma(S) \oplus \Omega^1(Y) \oplus \Omega^0(Y),$$
$$\mu\colon (\phi, a, f) \mapsto \Big(D_{A+ia+if+\lambda}\phi, -i *F_{A+ia} + i\sigma(\phi) + df, d^*a\Big),$$

with $\lambda \in \mathbb{R}$ as before. In the above definition, the following shorthand notation is used

$$D_{A+ia+if+\lambda} := D_A + \frac{1}{2}ia + \frac{1}{2}if + \lambda.$$

In order to ensure the injectivity of the linear part l of the monopole map on the set of fixed points (i.e. on forms), we restrict the map to the subspace

$\operatorname{im} d^* \subseteq \Omega^0(Y)$ and assume that the first Betti number of Y vanishes[1]:

$$b_1(Y) = 0.$$

The harmonic part of f will, however, be used as a parameter and denoted by $\lambda_1 \in \mathbb{R}$. Since f appears with the prefactor i, so will its harmonic part. As a result, the above monopole map takes the following form:

$$\mu\colon \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^* \to \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^*,$$
$$\mu\colon (\phi, a, f) \mapsto \Big(D_{A+ia+if+\lambda}\phi, -i *F_{A+ia} + i\sigma(\phi) + df, d^* a\Big),$$

with $\lambda = \lambda_0 + i\lambda_1 \in \mathbb{C}$ being a fixed parameter[2]. Hence, we have one monopole map for every $\lambda \in \mathbb{C}$. All of these maps will be denoted by the same symbol μ. In places where it is important to stress the dependency on parameters, the notation will be adapted accordingly.

As discussed in §2.3, individual monopole maps do not give interesting information about Y. Therefore, we will vary λ_1 and, in particular, we will try to understand what happens when $\lambda_1 \to \pm\infty$. In order to explore this limit case, we consider a certain sort of "renormalisation" of the monopole map. Namely, rather than discuss λ_1 directly, we set $\lambda_1 = \tan\theta$ and "renormalise" the map by inserting $\phi = \cos(\theta)\psi$. This will enable us to discuss the limit case $\lambda_1 \to \pm\infty$ by setting $\theta = \pm\frac{\pi}{2}$.

It is worth noting that we choose to renormalise only λ_1 and keep λ_0 fixed. This is because treating λ_0 as a free parameter (rather than a fixed one) would not preserve the spectral decomposition of the Dirac operator D_A.

After setting $\lambda_1 = \tan(\theta)$ and $\phi = \cos(\theta)\psi$ as mentioned above, we finally arrive at the map which will be the central object of interest in the present chapter (and indeed the whole thesis):

$$\mu_\theta\colon \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^* \to \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^*, \tag{3.1a}$$
$$\mu_\theta\colon (\psi, a, f) \mapsto \tag{3.1b}$$
$$\Big(\Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]\psi, -i *F_{A+ia} + i\sigma(\cos(\theta)\psi) + df, d^* a\Big).$$

[1] this is in contrast with the choice made in Chapter 2, where the injectivity of the linear part on forms was achieved by adding projection onto the kernel

[2] there is a slight inconsistency in having the factor $\frac{1}{2}$ next to if and not next to $i\lambda_1$, which represents the harmonic part of if, but since λ_1 is a parameter and it does not appear in the remaining components of the map, this slight inconsistency is inconsequential

Here, θ is taking values in the interval $I_\theta := \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\lambda_0 \in \mathbb{R}$ is arbitrary and fixed. Unless stated otherwise, the term "monopole map" will from now on refer to the map (3.1).

At the extremes $\theta = \pm\frac{\pi}{2}$, the monopole map takes a simple form $(\psi, a, f) \mapsto \left(\pm i\psi, -i *F_A + *da + df, d^*a\right)$. So we have a family of monopole maps starting and ending at almost the same map, which is basically the identity on spinors and D_Ω on forms. With the exception of the boundedness property[3], it is easy to see that the maps in the above family share the properties of the monopole map from §2.3. In particular, there is a continuous extension of μ_θ to the appropriate completions of the domain and the codomain:

$$\mu_\theta\colon L^2_{k+1}(S) \oplus L^2_{k+1}\left(\Omega^1(Y) \oplus \operatorname{im} d^*\right) \to L^2_k(S) \oplus L^2_k\left(\Omega^1(Y) \oplus \operatorname{im} d^*\right). \tag{3.2}$$

However, for $\theta = \pm\frac{\pi}{2}$, the spinor component of the above extension equals $\pm i$ times the canonical inclusion $L^2_{k+1}(S) \to L^2_k(S)$, which is a compact map and therefore does not satisfy the boundedness property.

Achieving the boundedness property simultaneously for all θ is not straightforward and has to be dealt with separately.

3.2 Boundedness property

Due to the additional perturbation terms, the proof of the boundedness property presented in §2.3.2 does not go through[4] in this setup. However, the inequality resulting from the detailed analysis of the quadratic term presented in §1.7 allows (together with the use of L^2-orthogonality of some terms) a modified method of bootstrapping to succeed and yield the desired estimates.

3.2.1 Adaptation of the boundedness property

In order to obtain a continuous family of the desired maps, the boundedness property should hold independently of the value of θ. In other words, for some

[3] formulated on page 38 for bundles, here we are dealing with a map between vector spaces, since the assumption $b_1(Y) = 0$ implies that $\mathrm{Pic}^{\mathfrak{s}}(Y)$ consists of only one point

[4] More precisely, the problems arise in (2.19) on page 42, where we would get the *square* of $\|f\|_{L^2}$ on the right-hand side after substituting the spinor component of the monopole map into the inequality. Applying (2.21) would then yield a polynomial of degree 4 on the right-hand side of (2.22) preventing the argumentation to go through

$k \in \mathbb{N}$ and a fixed arbitrary $R > 0$, a bound on the image

$$\|\mu_\theta(\psi, a, f)\|_{L^2_k} < R,$$

should ideally yield bounds on the preimage

$$\|\psi\|_{L^2_k} < R', \quad \|a, f\|_{L^2_{k+1}} < R',$$

with both $R > 0$ and $R' > 0$ being independent[5] of the value of θ. Note that after requiring R' to be independent of θ, we cannot hope to obtain an L^2_{k+1}-bound on ψ from an L^2_k-bound on the image of the monopole map, since for $\theta = \pm\frac{\pi}{2}$ we have the identity (up to multiplication by $\pm i$) in the spinor component.

As we will see, it is possible to prove an even stronger bound on the preimage, namely

$$\|B_\theta\psi\|_{L^2_k} < R', \quad \|a, f\|_{L^2_{k+1}} < R',$$

where

$$B_\theta := \cos(\theta)D_{A+\lambda_0} + i\sin(\theta). \tag{3.3}$$

In short, we aim to show that for every $\theta \in I_\theta$ and every $R > 0$, there exists an $R' > 0$ which is independent of θ and such that the following implication holds

$$\|\mu_\theta(\psi, a, f)\|_{L^2_k} < R \implies \|B_\theta\psi\|_{L^2_k} < R',\ \|a, f\|_{L^2_{k+1}} < R'.$$

The above implication will be shown to hold for ψ such that $B_\theta\psi \in L^2_k(S)$, $a \in L^2_{k+1}\big(\Omega^1(Y)\big)$ and $f \in L^2_{k+1}\big(d^*\Omega^1(Y)\big)$, with $k \geq 3$.

The first step in the proof is the acquisition of an a priori estimate from an assumed L^2_k-bound on the image of μ.

3.2.2 A priori estimate

Let[6] $k \geq 3$ and consider the set of all spinors ψ such that $B_\theta\psi \in L^2_k(S)$ and L^2_{k+1}-forms a and f satisfying

$$\|\mu_\theta(\psi, a, f)\|_{L^2_k} < R, \tag{3.4}$$

[5] An example of an attempt where this fails would be if we tried to deduce an L^2_{k+1}-bound on ψ by means of bootstrapping using the Dirac operator D_A *alone* (i.e. without the $\cos(\theta)$ prefactor). Namely, the resulting estimates would in this case ”explode” in the limit case

[6] with Sobolev's embedding theorem in mind, $k \in \mathbb{N}$ is chosen to be greater than or equal to 3 in order to ensure that the inequalities $\|\,.\,\|_{C^0} \lesssim \|\,.\,\|_{L^2_k}$ (i.e. $k - \frac{3}{2} > 0$) and $\|D\,.\,\|_{C^0} \lesssim \|D\,.\,\|_{L^2_{k-1}} \lesssim \|\,.\,\|_{L^2_k}$ (i.e. $k-1-\frac{3}{2} > 0$) hold; these inequalities will be used on p. 54; furthermore, the assumption $k \geq 3$ guarantees that spinors and forms in the L^2_{k+1}-completions are at least twice continuously differentiable (since $k + 1 - \frac{3}{2} > 2$)

with $R > 0$ being independent of both θ and λ_0. For the purpose of reducing the amount of writing, we introduce an ever shorter notation for the discussion to come:

$$\rho_\lambda := D_{A+ia+if+\lambda}\psi, \qquad b := -i *F_{A+ia} + i\sigma(\cos(\theta)\psi) + df.$$

Basically, $\cos(\theta)\rho_\lambda$ is the spinor component of μ, and b stands short for the component of the map containing 1-forms. Variables will be omitted from this notation in order to save space.

The goal of this section is to obtain a C^0-bound on ψ, a and f from the above assumption (3.4).

We start by looking at the scalar product[7]

$$\Big\langle \Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]^* \Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]\psi, \psi\Big\rangle.$$

The calculations will for now be performed pointwise.

On the one hand, applying the first operator on the rest of the expression in the first entry of the scalar product gives

$$\begin{aligned}
&\Big\langle \Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]^* \Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]\psi, \psi\Big\rangle \\
&= \cos^2(\theta)\cdot\Big\langle \Big[D_{A+ia+if+\lambda}\Big]^* D_{A+ia+if+\lambda}\psi, \psi\Big\rangle \\
&= \cos^2(\theta)\cdot\Big\langle D_{A+ia-if+\bar\lambda}\, D_{A+ia+if+\lambda}\,\psi, \psi\Big\rangle \\
&= \cos^2(\theta)\Big[\langle D_A\rho_\lambda, \psi\rangle + \frac{1}{2}\langle ia\rho_\lambda, \psi\rangle - \frac{1}{2}\langle if\rho_\lambda, \psi\rangle + \Big\langle \bar\lambda\rho_\lambda, \psi\Big\rangle\Big].
\end{aligned} \tag{3.5}$$

3.2.1. Remark. Since we allow $\cos(\theta) = 0$, factoring $\cos^2(\theta)$ out is clearly a bad practice. However, it has the advantage of reducing the length of the expressions and consequently making the calculations more readable. For that reason we will indeed factor $\cos^2(\theta)$ out and treat the case $\cos(\theta) = 0$ as the limit case when $\cos(\theta) \to 0$.

Also note that the calculations reduce to trivial equalities in the case $\cos(\theta) = 0$, so it is even possible to temporarily assume that $\cos(\theta) \neq 0$ and to simply take the trivial case $\cos(\theta) = 0$ for granted. ◂

On the other hand, after simply expanding the expression we get

[7] $\langle .\,,.\rangle$ denotes the real part of the Hermitian product on S and the star denotes the L^2-dual of the operator in the expression

$$\begin{aligned}
&\Big\langle\Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]^*\Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]\psi,\psi\Big\rangle\\
&= \cos^2(\theta)\cdot\Big\langle\Big[D_{A+ia+if+\lambda}\Big]^* D_{A+ia+if+\lambda}\psi,\psi\Big\rangle\\
&= \cos^2(\theta)\cdot\Big\langle D_{A+ia-if+\bar\lambda}\ D_{A+ia+if+\lambda}\ \psi,\psi\Big\rangle\\
&= \cos^2(\theta)\Big[\Big\langle D^2_{A+ia}\psi,\psi\Big\rangle + \frac{1}{2}\langle D_{A+ia}(if\psi),\psi\rangle + \lambda_0\langle D_{A+ia}\psi,\psi\rangle + \langle i\lambda_1 D_{A+ia}\psi,\psi\rangle\\
&\qquad - \frac{1}{2}\langle ifD_{A+ia}\psi,\psi\rangle + \frac{1}{4}f^2|\psi|^2 - \frac{1}{2}\lambda_0\langle if\psi,\psi\rangle + \frac{1}{2}f\lambda_1|\psi|^2\\
&\qquad + \lambda_0\langle D_{A+ia}\psi,\psi\rangle + \frac{1}{2}\lambda_0\langle if\psi,\psi\rangle + \lambda_0^2|\psi|^2 + \lambda_0\langle i\lambda_1\psi,\psi\rangle\\
&\qquad - \langle i\lambda_1 D_{A+ia}\psi,\psi\rangle + \frac{1}{2}f\lambda_1|\psi|^2 - \lambda_0\langle i\lambda_1\psi,\psi\rangle + \lambda_1^2|\psi|^2\Big]
\end{aligned}$$

Using the equality $D_{A+ia}(if\psi) = idf\psi + ifD_{A+ia}\psi$ and taking certain cancelations[1] into account gives

$$\begin{aligned}
&\Big\langle\Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]^*\Big[\cos(\theta)D_{A+ia+if+\lambda_0} + i\sin(\theta)\Big]\psi,\psi\Big\rangle\\
&= \cos^2(\theta)\cdot\Big[\Big\langle D^2_{A+ia}\psi,\psi\Big\rangle + \frac{1}{2}\langle idf\psi,\psi\rangle + 2\lambda_0\langle D_{A+ia}\psi,\psi\rangle + \lambda_0^2|\psi|^2\\
&\qquad + (\frac{1}{2}f + \lambda_1)^2|\psi|^2\Big].
\end{aligned}$$

The Weitzenböck formula[8] for the operator D_{A+ia}

$$\begin{aligned}
D^2_{A+ia} &= \nabla^*_{A+ia}\nabla_{A+ia} + \frac{s}{4} + \frac{1}{2}c(F_{A+ia})\\
&= \nabla^*_{A+ia}\nabla_{A+ia} + \frac{s}{4} + \frac{1}{2}c(*F_{A+ia})
\end{aligned}$$

and the identity[9]

$$\langle\nabla^*_{A+ia}\nabla_{A+ia}\psi,\psi\rangle = \Delta|\psi|^2 + |\nabla_{A+ia}\psi|^2$$

further yield

[8] recall that (1.48) implies $c(F_{A+ia}) = c(*F_{A+ia})$

[9] see §A.4

$$\Big\langle\Big[\cos(\theta)D_{A+ia+if+\lambda_0}+i\sin(\theta)\Big]^*\Big[\cos(\theta)D_{A+ia+if+\lambda_0}+i\sin(\theta)\Big]\psi,\psi\Big\rangle$$
$$=\cos^2(\theta)\cdot\Big[\Delta|\psi|^2+|\nabla_{A+ia}\psi|^2+\frac{s}{4}|\psi|^2+\frac{1}{2}\langle(*F_{A+ia}+idf)\psi,\psi\rangle$$
$$+2\lambda_0\langle D_{A+ia}\psi,\psi\rangle+\lambda_0^2|\psi|^2+(\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big].$$

Finally, we substitute $*F_{A+ia}+idf=ib+\sigma(\cos(\theta)\psi)$ and use the fact that $\langle\rho_\lambda,\psi\rangle$ and $\langle D_{A+ia}\psi,\psi\rangle$ differ by $\lambda_0|\psi|^2$ to obtain[2]

$$\Big\langle\Big[\cos(\theta)D_{A+ia+if+\lambda_0}+i\sin(\theta)\Big]^*\Big[\cos(\theta)D_{A+ia+if+\lambda_0}+i\sin(\theta)\Big]\psi,\psi\Big\rangle$$
$$=\cos^2(\theta)\cdot\Big[\Delta|\psi|^2+|\nabla_{A+ia}\psi|^2+\frac{s}{4}|\psi|^2+\frac{1}{2}\langle ib\psi,\psi\rangle+\frac{1}{4}\cos^2(\theta)\cdot|\psi|^4$$
$$+2\lambda_0\langle\rho_\lambda,\psi\rangle-\lambda_0^2|\psi|^2+(\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big]. \tag{3.6}$$

Combining (3.5) and (3.6) gives

$$\cos^2(\theta)\Big[\langle D_A\rho_\lambda,\psi\rangle+\frac{1}{2}\langle ia\rho_\lambda,\psi\rangle-\frac{1}{2}\langle if\rho_\lambda,\psi\rangle+\Big\langle\bar{\lambda}\rho_\lambda,\psi\Big\rangle\Big]=$$
$$=\cos^2(\theta)\cdot\Big[\Delta|\psi|^2+|\nabla_{A+ia}\psi|^2+\frac{s}{4}|\psi|^2+\frac{1}{2}\langle ib\psi,\psi\rangle+\frac{1}{4}\cos^2(\theta)\cdot|\psi|^4$$
$$+2\lambda_0\langle\rho_\lambda,\psi\rangle-\lambda_0^2|\psi|^2+(\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big],$$

which (after bringing the term $2\lambda_0\langle\rho_\lambda,\psi\rangle$ to the left-hand side) slightly simplifies to

$$\cos^2(\theta)\Big[\langle D_A\rho_\lambda,\psi\rangle+\frac{1}{2}\langle ia\rho_\lambda,\psi\rangle-\frac{1}{2}\langle if\rho_\lambda,\psi\rangle-\langle\lambda\rho_\lambda,\psi\rangle\Big]=$$
$$=\cos^2(\theta)\cdot\Big[\Delta|\psi|^2+|\nabla_{A+ia}\psi|^2+\frac{s}{4}|\psi|^2+\frac{1}{2}\langle ib\psi,\psi\rangle+\frac{1}{4}\cos^2(\theta)\cdot|\psi|^4-\lambda_0^2|\psi|^2$$
$$+(\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big].$$

Rearranging the terms and applying the Cauchy-Schwarz inequality gives the inequality

$$\frac{1}{4}|\cos(\theta)\psi|^4+\Delta|\cos(\theta)\psi|^2+\Big|\frac{1}{2}f\cos(\theta)+\sin(\theta)\Big|^2|\psi|^2$$

$$
\begin{aligned}
&= \cos^2(\theta)\Big[\langle D_A\rho_\lambda, \psi\rangle + \frac{1}{2}\langle ia\rho_\lambda, \psi\rangle - \Big\langle i\Big(\frac{1}{2}f + \lambda_1\Big)\rho_\lambda, \psi\Big\rangle - \langle \lambda_0\rho_\lambda, \psi\rangle \\
&\qquad - |\nabla_{A+ia}\psi|^2 - \frac{s}{4}|\psi|^2 - \frac{1}{2}\langle ib\psi, \psi\rangle + \lambda_0^2|\psi|^2\Big] \\
&\le \cos^2(\theta)\Big[\langle D_A\rho_\lambda, \psi\rangle + \frac{1}{2}\langle ia\rho_\lambda, \psi\rangle - \Big\langle i\Big(\frac{1}{2}f + \lambda_1\Big)\rho_\lambda, \psi\Big\rangle - \langle \lambda_0\rho_\lambda, \psi\rangle \\
&\qquad + \frac{\|s\|_{C^0}}{4}|\psi|^2 - \frac{1}{2}\langle ib\psi, \psi\rangle + \lambda_0^2|\psi|^2\Big] \\
&\le \cos(\theta)\Big[\langle D_A\cos(\theta)\rho_\lambda, \psi\rangle + \frac{1}{2}\langle ia\cos(\theta)\rho_\lambda, \psi\rangle - \Big\langle i\Big(\frac{1}{2}f + \lambda_1\Big)\cos(\theta)\rho_\lambda, \psi\Big\rangle \\
&\qquad - \langle \lambda_0\cos(\theta)\rho_\lambda, \psi\rangle\Big] + \cos^2(\theta)\Big[\frac{\|s\|_{C^0}}{4}|\psi|^2 - \frac{1}{2}\langle ib\psi, \psi\rangle + \lambda_0^2|\psi|^2\Big] \\
&\lesssim |\cos(\theta)|\Big[\|D_A\cos(\theta)\rho_\lambda\|_{C^0} + \|a\|_{C^0}\|\cos(\theta)\rho_\lambda\|_{C^0} + \Big|\frac{1}{2}f + \lambda_1\Big|\|\cos(\theta)\rho_\lambda\|_{C^0} \\
&\qquad + |\lambda_0|\cdot\|\cos(\theta)\rho_\lambda\|_{C^0}\Big]\,|\psi| + \cos^2(\theta)\Big[\|s\|_{C^0}|\psi|^2 + \|b\|_{C^0}|\psi| + \lambda_0^2|\psi|^2\Big].
\end{aligned}
$$

Due to assumption (3.4), we have in particular[10]

$$
\begin{aligned}
\|\cos(\theta)\rho_\lambda\|_{C^0} &\lesssim \|\cos(\theta)\rho_\lambda\|_{L^2_k} \lesssim 1, \\
\|D_A\cos(\theta)\rho_\lambda\|_{C^0} &\lesssim \|D_A\cos(\theta)\rho_\lambda\|_{L^2_{k-1}} \lesssim \|\cos(\theta)\rho_\lambda\|_{L^2_k} \lesssim 1, \\
\|b\|_{C^0} &\lesssim \|b\|_{L^2_k} \lesssim 1.
\end{aligned}
$$

Inserting these bounds into the inequality above, together with taking into account that $\|s\|_{C^0}$ is a constant, leads to

$$
\begin{aligned}
&\frac{1}{4}|\cos(\theta)\psi|^4 + \Delta|\cos(\theta)\psi|^2 + \Big|\frac{1}{2}f\cos(\theta) + \sin(\theta)\Big|^2|\psi|^2 \\
&\lesssim \Big(1 + \|a\|_{C^0} + \Big|\frac{1}{2}f + \lambda_1\Big| + |\lambda_0|\Big)|\cos(\theta)\psi| + (1 + \lambda_0^2)|\cos(\theta)\psi|^2 \\
&\lesssim \Big(1 + \|a\|_{C^0} + |\lambda_0|\Big)|\cos(\theta)\psi| + \Big|\frac{1}{2}f\cos(\theta) + \sin(\theta)\Big||\psi| + (1 + \lambda_0^2)|\cos(\theta)\psi|^2,
\end{aligned}
$$

and consequently, at the point $p_{\max\psi}$ where $|\psi|$ achieves its maximum[11],

[10] Theorem A.1.5 and $k \ge 3$ imply $\|\,.\,\|_{C^0} \lesssim \|\,.\,\|_{L^2_k}$

[11] $\Delta|\cos(\theta)\psi|^2(p_{\max\psi}) \ge 0$

$$\begin{aligned}
&\|\cos(\theta)\psi\|_{C^0}^4 + \left|\frac{1}{2}f(p_{\max\psi})\cos(\theta) + \sin(\theta)\right|^2 \|\psi\|_{C^0}^2 \\
&\lesssim \Big(1 + \|a\|_{C^0} + |\lambda_0|\Big)\|\cos(\theta)\psi\|_{C^0} + \left|\frac{1}{2}f(p_{\max\psi})\cos(\theta) + \sin(\theta)\right|\|\psi\|_{C^0} \\
&\quad + (1+\lambda_0^2)\|\cos(\theta)\psi\|_{C^0}^2.
\end{aligned}$$

We pause briefly to make the following remark.

3.2.2. Remark. For $j-1 \leq k$ and for p such that $\|\,.\,\|_{L^p_{j-1}} \lesssim \|\,.\,\|_{L^2_k}$ we have

$$\begin{aligned}
\|a, f\|_{L^p_j} &\lesssim \|{*da} + df, d^*a\|_{L^p_{j-1}} \\
&\lesssim \|b, d^*a\|_{L^p_{j-1}} + \|{*F_A}\|_{L^p_{j-1}} + \|\sigma(\cos(\theta)\psi)\|_{L^p_{j-1}} \\
&\lesssim 1 + \|\cos(\theta)\psi\|^2_{L^{2p}_{j-1}}.
\end{aligned}$$

In the first step, we used the stronger version of the elliptic inequality (Lemma A.1.3), together with the fact that D_Ω has trivial kernel. In the last step the Sobolev multiplication theorem (Theorem A.1.6) was used and in the rest the Sobolev embedding theorems (Theorem A.1.5).

In particular (again, using Sobolev's embedding theorem), we have

$$\begin{aligned}
\|a, f\|_{C^0} \lesssim \|a, f\|_{L^4_1} &\lesssim \|{*da} + df, d^*a\|_{L^4} \\
&\lesssim 1 + \|\cos(\theta)\psi\|^2_{L^8} \\
&\lesssim 1 + \|\cos(\theta)\psi\|^2_{C^0}.
\end{aligned}$$

◀

Thus

$$\begin{aligned}
&\|\cos(\theta)\psi\|_{C^0}^4 + \left|\frac{1}{2}f(p_{\max\psi})\cos(\theta) + \sin(\theta)\right|^2 \|\psi\|_{C^0}^2 \\
&\lesssim \Big(1 + \|\cos(\theta)\psi\|_{C^0}^2 + |\lambda_0|\Big)\|\cos(\theta)\psi\|_{C^0} + \left|\frac{1}{2}f(p_{\max\psi})\cos(\theta) + \sin(\theta)\right|\|\psi\|_{C^0} \\
&\quad + (1+\lambda_0^2)\|\cos(\theta)\psi\|_{C^0}^2.
\end{aligned}$$

Both sides of the above inequality are sums of two polynomial expressions in the non-negative terms $\|\cos(\theta)\psi\|_{C^0}$ and $\left|\frac{1}{2}f(p_{\max\psi})\cos(\theta) + \sin(\theta)\right|\|\psi\|_{C^0}$. Since the degrees of the polynomials on the left-hand side are greater than the corresponding degrees on the right-hand side, we conclude that the terms $\|\cos(\theta)\psi\|_{C^0}$ and $\left|\frac{1}{2}f(p_{\max\psi})\cos(\theta)+\sin(\theta)\right|\|\psi\|_{C^0}$ must both be bounded.

From $\|a,f\|_{C^0} \lesssim 1+\|\cos(\theta)\psi\|^2_{C^0}$ it also follows, that $\|a,f\|_{C^0}$ is bounded. This furthermore implies

$$\begin{aligned}\|\sin(\theta)\psi\|_{C^0} &= |\sin(\theta)|\|\psi\|_{C^0}\\ &\leq \left|\frac{1}{2}f(p_{\max\psi})\cos(\theta)+\sin(\theta)\right|\|\psi\|_{C^0} + \left|\frac{1}{2}f(p_{\max\psi})\cos(\theta)\right|\|\psi\|_{C^0}\\ &\lesssim 1+\frac{1}{2}\|f\|_{C^0}\|\cos(\theta)\psi\|_{C^0}\\ &\lesssim 1.\end{aligned}$$

In other words, $\|\sin(\theta)\psi\|_{C^0}$ is bounded as well, and so we obtained the desired a priori bound

$$\|\psi,a,f\|_{C^0} \lesssim 1. \tag{3.7}$$

3.2.3 Bootstrapping

In this section we will show how to obtain a bound on $\|B_\theta\psi\|_{L^2_k}$ and $\|a+f\|_{L^2_{k+1}}$ from (3.7) with the help of an adapted version of bootstrapping.

Assumption (3.4) allows the following chain of inequalities

$$\begin{aligned}&\|\cos(\theta)\psi,a,f\|_{C^1} - \|\cos(\theta)\psi,a,f\|_{C^0}\\ &\lesssim \|D_A\cos(\theta)\psi\|_{C^0} + \|{*da}+df,d^*a\|_{C^0}\\ &\lesssim \left\|\cos(\theta)D_{A+ia+if+\lambda}\psi\right\|_{C^0} + \|-i*F_{A+ia}+i\sigma(\cos(\theta)\psi)+df,d^*a\|_{C^0}\\ &\quad + \|i(a+f)\cos(\theta)\psi\|_{C^0} + |\lambda_0|\,\|\cos(\theta)\psi\|_{C^0} + \|\sin(\theta)\psi\|_{C^0}\\ &\quad + \|F_A\|_{C^0} + \|\sigma(\cos(\theta)\psi)\|_{C^0}\\ &\lesssim 1+\left(\|a+f\|_{C^0}+|\lambda_0|\right)\|\cos(\theta)\psi\|_{C^0} + \|\sin(\theta)\psi\|_{C^0} + \|F_A\|_{C^0} + \|\cos(\theta)\psi\|^2_{C^0}\\ &\lesssim 1,\end{aligned}$$

and after using (3.7), we conclude

$$\|\cos(\theta)\psi,a,f\|_{C^1} \lesssim 1.$$

3.2.3. Remark. The above inequality, and hence all subsequent inequalities, depend on the fixed parameter λ_0. ◀

In particular, we have

$$\|B_\theta\psi\|_{C^0} \lesssim 1 \quad \text{and} \quad \|a, f\|_{C^1} \lesssim 1. \tag{3.8}$$

Since[12] $k \geq 3$, it is straightforward to obtain an estimate for the next higher norm[13]

$$\begin{aligned}
&\|B_\theta\psi, D_\Omega(a, f)\|_{C^1} \\
&\lesssim \|\mu_\theta(\psi, a, f)\|_{C^1} + \|i(a+f)\cos(\theta)\psi, -i *F_A + i\sigma(\cos(\theta)\psi)\|_{C^1} \\
&\lesssim \|\mu_\theta(\psi, a, f)\|_{C^1} + \|(a+f)\cos(\theta)\psi\|_{C^1} + \|F_A\|_{C^1} + \|\sigma(\cos(\theta)\psi)\|_{C^1} \\
&\lesssim \|\mu_\theta(\psi, a, f)\|_{C^1} + \|a+f\|_{C^1}\|\cos(\theta)\psi\|_{C^1} + \|\cos(\theta)\psi\|_{C^1}^2 \\
&\lesssim 1.
\end{aligned}$$

However, even for a large enough k, the above process cannot be continued. On the one hand, in order to get a bound on $\|B_\theta\psi\|_{C^1}$ we need to have $\|\cos(\theta)\psi\|_{C^2}$ under control. On the other hand, for a bound on $\|\cos(\theta)\psi, a, f\|_{C^2}$, a bound on $\|\sin(\theta)\psi\|_{C^1}$ is required. It is not clear how to obtain either of the desired bounds from the bounds proved so far and the ideas and tools used to obtain them.

The crucial observation in circumvention of this problem is the fact that $(D_A + \lambda_0)\psi$ and $i\psi$ are L^2_j-orthogonal for all j. Namely, this implies

$$\left\|D_{A+\lambda_0}\cos(\theta)\psi\right\|_{L^2_1}^2 + \|\sin(\theta)\psi\|_{L^2_1}^2 = \|B_\theta\psi\|_{L^2_1}^2 \lesssim \|B_\theta\psi\|_{C^1}^2 \lesssim 1,$$

and from that follows $\|\sin(\theta)\psi\|_{L^2_1} \lesssim 1$ as well as

$$\begin{aligned}
\|\cos(\theta)\psi\|_{L^2_2} &\lesssim \|\cos(\theta)\psi\|_{L^2_1} + \|D_A\cos(\theta)\psi\|_{L^2_1}, \\
&\lesssim \|\cos(\theta)\psi\|_{C^1} + \left\|D_{A+\lambda_0}\cos(\theta)\psi\right\|_{L^2_1} + |\lambda_0|\|\cos(\theta)\psi\|_{L^2_1} \\
&\lesssim \|\cos(\theta)\psi\|_{C^1} + \left\|D_{A+\lambda_0}\cos(\theta)\psi\right\|_{L^2_1} + |\lambda_0|\|\cos(\theta)\psi\|_{C^1} \\
&\lesssim 1 + |\lambda_0|.
\end{aligned}$$

Note that also $\|a+f\|_{L^2_2} \lesssim \|a+f\|_{C^2} \lesssim 1$.

---

[12] the assumption $k \geq 3$ ensures $\|\,.\,\|_{C^1} \lesssim \|\,.\,\|_{L^2_k}$ (Theorem A.1.5)

[13] recall that D_Ω denotes the operator $\left[\begin{smallmatrix} *d & d \\ d^* & 0 \end{smallmatrix}\right]$, as in §1.6

Next, we try to find a bound for $\|B_\theta\psi, D_\Omega(a,f)\|^2_{L^2_2}$:

$$\begin{aligned}
&\|B_\theta\psi, D_\Omega(a,f)\|^2_{L^2_2} \\
&\lesssim \|\mu_\theta(\psi,a,f)\|^2_{L^2_2} + \|i(a+f)\cos(\theta)\psi, i*F_A - i\sigma(\cos(\theta)\psi)\|^2_{L^2_2} \\
&\lesssim 1 + \|(a+f)\cos(\theta)\psi\|^2_{L^2_2} + \|\sigma(\cos(\theta)\psi)\|^2_{L^2_2}. \qquad (3.9)
\end{aligned}$$

3.2.4. Remark. Before moving on, a remark on the next step in the proof is in order. According to [Pal68, Corollary 9.7], the L^p_k-completion is a Banach algebra for $pk > n$. In the present case ($n = 3$), this implies that L^2_j-completion is a Banach algebra for $j \geq 2$, and we immediately have the inequalities

$$\|\sigma(\cos(\theta)\psi)\|_{L^2_j} \lesssim \|\cos(\theta)\psi\|^2_{L^2_j}, \qquad (3.10a)$$

$$\|(a+f)\cos(\theta)\psi\|_{L^2_j} \lesssim \|a+f\|_{L^2_j}\|\cos(\theta)\psi\|_{L^2_j}. \qquad (3.10b)$$

At the time the proof of the boundedness property was being compiled, the above-mentioned fact somehow managed to escape my attention. Instead, a weaker result (Theorem A.1.6) was considered, causing difficulties explained in the text below, which motivated Section 1.7.3 and inequality (3.11).

In what follows, the version of the proof using the weaker results is presented. Shorter version is obtained by simply using (3.10) in place of (1.64) and (3.11). ◂

Direct application of the Sobolev multiplication theorem (Theorem A.1.6) at this point would produce L^4_2-norms of $a+f$ and $\cos(\theta)$, over which we have no control. This is where the a priori bound (3.7) from the last section crucially comes into play. In fact, the possibility of avoiding the doubling of the exponents of Sobolev norms with the aid of the C^0-norm was the principal motivation behind the analysis of the quadratic term in §1.7 (and in particular its norm in §1.7.3). The manner in which this is achieved is encapsulated by the expression (1.64), which in the present case implies

$$\|\sigma(\cos(\theta)\psi)\|_{L^2_2} \overset{(1.64)}{\lesssim} \|\cos(\theta)\psi\|_{L^2_2}\|\cos(\theta)\psi\|_{C^0} + \|\cos(\theta)\psi\|^2_{L^2_2} \lesssim 1.$$

An inequality similar to (1.64) holds[3] for the Clifford product $(a+f)\cos(\theta)\psi$:

$$\begin{aligned}
\|(a+f)\cos(\theta)\psi\|_{L^2_m} &\lesssim \|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{C^0} + \|a+f\|_{C^0}\|\cos(\theta)\psi\|_{L^2_m} \\
&\quad + \|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{L^2_m}. \qquad (3.11)
\end{aligned}$$

Applied to the term $\|(a+f)\cos(\theta)\psi\|_{L^2_2}$, the above inequality (3.11) implies

$$\begin{aligned}\|(a+f)\cos(\theta)\psi\|_{L^2_2} &\lesssim \|a+f\|_{L^2_2}\|\cos(\theta)\psi\|_{C^0} + \|a+f\|_{C^0}\|\cos(\theta)\psi\|_{L^2_2} \\ &\quad + \|a+f\|_{L^2_2}\|\cos(\theta)\psi\|_{L^2_2} \\ &\lesssim 1,\end{aligned}$$

which means that

$$\|B_\theta\psi\|_{L^2_2} \lesssim 1, \quad \|a+f\|_{L^2_3} \lesssim 1,$$

and, as before, due to orthogonality

$$\|\cos(\theta)\psi\|_{L^2_3} \lesssim 1 + |\lambda_0| \lesssim 1, \quad \|a+f\|_{L^2_3} \lesssim 1.$$

In general, suppose inductively that for some $1 \leq j \leq k$ we have

$$\|\cos(\theta)\psi\|_{L^2_j} \lesssim 1, \quad \|a+f\|_{L^2_j} \lesssim 1.$$

The assumption (3.4) implies

$$\begin{aligned}&\|B_\theta\psi, D_\Omega(a+f)\|^2_{L^2_j} \\ &\lesssim \|\mu_\theta(\psi,a,f)\|^2_{L^2_j} + \|i(a+f)\cos(\theta)\psi, i*F_A - i\sigma(\cos(\theta)\psi)\|^2_{L^2_j} \\ &\overset{(3.4)}{\lesssim} 1 + \|(a+f)\cos(\theta)\psi\|^2_{L^2_j} + \|\sigma(\cos(\theta)\psi)\|^2_{L^2_j},\end{aligned}$$

Using the aforementioned inequalities (1.64) and (3.11) now leads to

$$\begin{aligned}&\|B_\theta\psi, D_\Omega(a+f)\|^2_{L^2_j} \\ &\lesssim 1 + \|a+f\|_{L^2_j}\|\cos(\theta)\psi\|_{C^0} + \|a+f\|_{C^0}\|\cos(\theta)\psi\|_{L^2_j} \\ &\quad + \|a+f\|_{L^2_j}\|\cos(\theta)\psi\|_{L^2_j} + \|\cos(\theta)\psi\|_{L^2_j}\|\cos(\theta)\psi\|_{C^0} + \|\cos(\theta)\psi\|^2_{L^2_j} \\ &\lesssim 1,\end{aligned}$$

where the last inequality follows from the above inductive assumption and the a priori bound (3.7). In other words, we get the desired inequalities:

$$\|B_\theta\psi\|_{L^2_j} \lesssim 1, \quad \|a+f\|_{L^2_{j+1}} \lesssim 1,$$

and furthermore (again due to orthogonality):

$$\|\cos(\theta)\psi\|_{L^2_{j+1}} \lesssim 1, \quad \|a+f\|_{L^2_{j+1}} \lesssim 1.$$

Repeating this step often enough yields the desired estimates

$$\|B_\theta\psi\|_{L^2_k} \lesssim 1, \quad \|a+f\|_{L^2_{k+1}} \lesssim 1,$$

i.e.[4]

$$\|B_\theta\psi\|_{L^2_k} \lesssim 1, \quad \|a\|_{L^2_{k+1}} \lesssim 1, \quad \|f\|_{L^2_{k+1}} \lesssim 1.$$

3.3 Statement of the main result

In conclusion, we proved the following theorem

3.3.1. Theorem. *Fix an arbitrary $k \in \mathbb{N}_0$, $k \geq 3$ and $R > 0$. There exists $R' > 0$ such that the following implication holds*

$$\|\mu_\theta(\psi, a, f)\|_{L^2_k} < R \implies \|B_\theta\psi\|_{L^2_k} < R', \ \|a, f\|_{L^2_{k+1}} < R',$$

for ψ such that $B_\theta\psi \in L^2_k(S)$, $a \in L^2_{k+1}\big(\Omega^1(Y)\big)$ and $f \in L^2_{k+1}\big(d^*\Omega^1(Y)\big)$.

3.4 Renormalisation of the monopole map

Taking the discussion further, we can consider $B_\theta\psi$ as a variable in itself. Namely, suppose $-\lambda_0$ is not an eigenvalue of the Dirac operator D_A:

$$-\lambda_0 \notin \sigma(D_A).$$

Then $D_{A+\lambda_0}$ is an injective operator, and hence an isomorphism[14]. In this case, the operator B_θ is also an isomorphism[15] for all $\theta \in I_\theta$. Therefore, it is possible to consider $\varphi = B_\theta\psi$ as a new variable. In other words, one can renormalise the map (3.1) in the spinor component by precomposing it with B_θ^{-1}. This renormalisation leads to a new map

$$\rho_\theta\colon \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^* \to \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^*, \tag{3.12a}$$

$$\rho_\theta\colon (\varphi, a, f) \mapsto \tag{3.12b}$$
$$\Big(\varphi + \frac{1}{2} i(a+f)\cos(\theta)B_\theta^{-1}\varphi, -i*F_{A+ia} + i\sigma\Big(\cos(\theta)B_\theta^{-1}\varphi\Big) + df, d^*a\Big).$$

[14]since its index equals zero

[15]its inverse can be expressed as $B_\theta^{-1} = B_\theta^* \circ (B_\theta^* B_\theta)^{-1}$, where $B_\theta^* B_\theta = \cos^2(\theta) D^2_{A+\lambda_0} + \sin^2(\theta)$ is a positive operator

Note that

$$\cos(\theta)B_\theta^{-1} = \begin{cases} \Big(\frac{1}{\cos(\theta)}B_\theta\Big)^{-1} = \Big(D_{A+\lambda_0} + i\tan(\theta)\Big)^{-1}, & \cos(\theta) \neq 0, \\ 0, & \cos(\theta) = 0, \end{cases}$$

is a compact operator $L^p_k(S) \to L^p_k(S)$ for all $\theta \in I_\theta$. At the endpoints $\theta = \pm\frac{\pi}{2}$ of the interval I_θ, the above map takes the form

$$\rho_{\pm\frac{\pi}{2}} = (\mathrm{Id}_S, -i * F_A + D_\Omega).$$

I.e., it equals the identity in the spinor component, and is equal (up to a constant) to D_Ω on forms. Due to the assumption $b_1(Y) = 0$, D_Ω is injective, and therefore bijective. Thus, we can renormalise in the forms component as well. From now on, we omit the constant term $-i * F_A$ from the discussion, and consider the following map:

$$\rho_\theta \colon \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^* \to \Gamma(S) \oplus \Omega^1(Y) \oplus \operatorname{im} d^*, \tag{3.13a}$$

$$\rho_\theta \colon (\varphi, a, f) \mapsto \tag{3.13b}$$

$$\Big(\varphi + \frac{1}{2} i [D_\Omega^{-1}(a+f)] \cos(\theta) B_\theta^{-1}\varphi, a + f + i\sigma\Big(\cos(\theta)B_\theta^{-1}\varphi\Big)\Big).$$

If we consider the L^p_k-completion of its domain and codomain, the above maps are a sum of the identity and a compact perturbation. At the endpoints $\theta = \pm\frac{\pi}{2}$, the map is equal to the identity.

Of course, throughout the renormalisation steps the $U(1)$-equivariance of the maps (3.1) is carried over to the maps (3.13).

Theorem 3.3.1 implies the boundedness property for the maps (3.13), so we can extend these maps to the 1-point compactification of the L^2_k-completion of the domain space.

Clearly, by varying the parameter λ_0 we get homotopic loops, unless λ_0 goes through an eigenvalue of D_A. In the case where $-\lambda_0 \in \sigma(D_A)$, the loop is not well-defined at the point where B_θ is not bijective (namely, $\theta = 0$). Eigenvalues of D_A can therefore be seen as singularities which separate different homotopy classes.

Chapter 4

The monopole map on a 3-torus

We conclude the discussion of the monopole map for 3-manifolds with an example of the monopole map (3.1) on a 3-torus using concrete tools developed in Chapter 1. Although not all assumptions of Chapter 3 are satisfied ($b_1(\mathbb{T}) = 3$), the monopole map can nevertheless be written down.

4.1 Notation and setup

Set $\mathbb{T} := \operatorname{Im}\mathbb{H}/2\pi\Lambda$, where $\Lambda \subseteq \operatorname{Im}\mathbb{H}$ is some lattice and fix the splitting

$$\Lambda = \Lambda^+ \sqcup \Lambda^- \sqcup \{0\},$$

determined by the lexicographical order $\Lambda^\pm := \{h \in \Lambda \setminus \{0\} : \pm h > 0\} \subseteq \operatorname{Im}\mathbb{H}$. The symbol $\Lambda_0^\pm$ will be an abbreviation for $\Lambda^\pm \sqcup \{0\}$, respectively.

$\mathbb{T}$ will be equipped with the quotient metric coming from the flat metric on $\mathbb{R}^3$, and with the trivial spin structure. The tangent bundle and the spinor bundle are trivial, and so their sections become functions on $\mathbb{T}$ with values in corresponding fibres. Both the spinor bundle and the bundle $\Lambda^{1,0}(Y)$ are modelled on $\mathbb{H}$ (with $\operatorname{Im}\mathbb{H}$ representing 1-covectors, and $\mathbb{R} \subseteq \mathbb{H}$ representing 0-covectors in the latter's fibre). Since $\mathbb{T}$ is basically a quotient $\mathbb{R}^3/\Lambda$, these functions can be seen as periodic functions on $\mathbb{R}^3$ with values in $\mathbb{H}$.

Functions $x \mapsto \sin\langle n, x\rangle$ and $x \mapsto \cos\langle n, x\rangle$, with $n \in \Lambda$, form a real orthogonal basis of the L^2-completion $L^2(\mathbb{T}; \mathbb{R})$ of the space of smooth real functions on $\mathbb{T}$ (i.e. every such function can be expanded into a Fourier series). Here, $\langle ., .\rangle$ denotes the scalar product (1.7) on $\operatorname{Im}\mathbb{H} \subseteq \mathbb{H}$.

4.1.1. Remark. In an even more general setup, we can fix a basis of Λ, which yields an isomorphism $\Lambda \cong \Lambda^*$ between Λ and its dual lattice

$$\Lambda^* := \{f \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{H}, \mathbb{R}) \colon f(\Lambda) \subseteq \mathbb{Z}\} \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}).$$

If $h^* \in \Lambda^*$ denotes the corresponding dual element of $h \in \Lambda$ the above functions take the form

$$x \mapsto \sin h^*(x), \qquad x \mapsto \cos h^*(x), \quad h^* \in \Lambda^*.$$

However, the discussion is entirely analogous to the one presented here, where the canonical isomorphism $\Lambda \cong \Lambda^*$ determined by the scalar product on $\mathrm{Im}\,\mathbb{H}$ is used. ◀

In general, when considering complex-valued functions on $\mathbb{T}$, functions of the form

$$x \mapsto \exp(i\langle n, x\rangle) = \cos\langle n, x\rangle + i \sin\langle n, x\rangle, \quad n \in \Lambda,$$

are more convenient to use. Presently, we are dealing with $\mathbb{H}$-valued functions on $\mathbb{T}$, so we will use the appropriate analogues of exponential functions:

$$x \mapsto \exp\left(\frac{n}{|n|}\langle n, x\rangle\right), \quad n \in \Lambda.$$

Basically, the imaginary unit quaternion $\frac{n}{|n|}$ takes over the role of the imaginary unit $i \in \mathbb{C}$ in the above functions. In particular, note that since $n^2 = -|n|^2$, we have $\left(\frac{n}{|n|}\right)^2 = -1$, and

$$\begin{aligned}
\exp\left(\frac{n}{|n|}\langle n, x\rangle\right) &= \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{n}{|n|}\right)^l \langle n, x\rangle^l \\
&= \sum_{l=0}^{\infty} \frac{1}{(2l)!}\left(\frac{n}{|n|}\right)^{2l} \langle n, x\rangle^{2l} + \sum_{l=0}^{\infty} \frac{1}{(2l+1)!}\left(\frac{n}{|n|}\right)^{2l+1} \langle n, x\rangle^{2l+1} \\
&= \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}\langle n, x\rangle^{2l} + \frac{n}{|n|}\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!}\left(\frac{n}{|n|}\right)^{2l+1} \langle n, x\rangle^{2l+1} \\
&= \cos\langle n, x\rangle + \frac{n}{|n|}\sin\langle n, x\rangle.
\end{aligned}$$

We will denote the above functions by

$$\begin{aligned}
\psi_n &:= \exp\left(-\frac{n}{|n|}\langle n, x\rangle\right) = \cos\langle n, x\rangle - \frac{n}{|n|}\sin\langle n, x\rangle = \psi_{-n}, \\
\bar{\psi}_n &= \exp\left(\frac{n}{|n|}\langle n, x\rangle\right) = \cos\langle n, x\rangle + \frac{n}{|n|}\sin\langle n, x\rangle = \bar{\psi}_{-n},
\end{aligned} \tag{4.1}$$

for $n \in \Lambda$. Since $\cos\langle n, x\rangle = \frac{1}{2}(\psi_n + \bar{\psi}_n)$ and $\sin\langle n, x\rangle = \frac{n}{2|n|}(\psi_n - \bar{\psi}_n)$, they build an $\mathbb{H}$-basis of $L^2(\mathbb{T};\mathbb{H})$. Furthermore, by using standard trigonometric formulae[1] and the fact

$$\int_{\mathbb{T}} \cos\langle n, x\rangle = 0, \quad \forall n \in \Lambda \setminus \{0\},$$
$$\int_{\mathbb{T}} \sin\langle n, x\rangle = 0, \quad \forall n \in \Lambda,$$

it is clear that $\langle \psi_n, \psi_m\rangle = 0 = \left\langle \bar{\psi}_n, \bar{\psi}_m \right\rangle$ for $n \neq m$, and $\left\langle \psi_n, \bar{\psi}_m \right\rangle = 0$ for all $n, m \in \Lambda$. Thus, the above functions form an orthogonal basis of $L^2(\mathbb{T};\mathbb{H})$.

4.2 The Dirac operators

As already discussed in §1.6, on forms we have the operator

$$D_\Omega = \begin{bmatrix} *d & d \\ d^* & 0 \end{bmatrix} : \Omega^{1,0}(Y) \to \Omega^{1,0}(Y).$$

This is a Dirac operator on $\Omega^{1,0}(Y)$ if the Clifford module structure (1.46) on $\Omega^{1,0}(Y)$ is assumed.

Due to triviality of the exterior bundle, this means that

$$D_\Omega = i \bullet_{\mathrm{cl}} \frac{\partial}{\partial x^1} + j \bullet_{\mathrm{cl}} \frac{\partial}{\partial x^2} + k \bullet_{\mathrm{cl}} \frac{\partial}{\partial x^3},$$

with $\operatorname{Im}\mathbb{H} \equiv \mathbb{R}^3$ (1.4) in mind. A straightforward calculation shows that

$$D_\Omega \psi_n = |n|\psi_n,$$
$$D_\Omega \bar{\psi}_n = -|n|\bar{\psi}_n.$$

Since $\{\psi_n, \bar{\psi}_n : n \in \Lambda_0^+\}$ is an orthogonal $\mathbb{H}$-eigenbasis of $L^2(Y;\mathbb{H})$, it is an $\mathbb{H}$-eigenbasis for D_Ω. In particular, this shows that the spectrum of D_Ω equals $\{\pm|n| : n \in \Lambda_0^+\}$.

The connections on spinors and forms coincide, and are equal to the exterior derivative on $\mathbb{R}^3$. This implies that the corresponding Dirac operators are the same. They will be denoted by D from now on.

4.3 The monopole map

The fibrewise representation analogous to (2.12) of the monopole map (3.1) in this case reads:

$$\begin{aligned}\mu_\theta(\psi, a, f) = \Big(&\cos(\theta)D\psi + \frac{1}{2}(a+f)\cos(\theta)\psi\bar{\imath} + \lambda_0\cos(\theta)\psi + \sin(\theta)\psi\bar{\imath},\\ &D(a+f) + i\sigma(\cos(\theta)\psi)\Big)\\ = \Big(&\cos(\theta)D\psi + \frac{1}{2}(a+f)\cos(\theta)\psi\bar{\imath} + \lambda_0\cos(\theta)\psi + \sin(\theta)\psi\bar{\imath},\\ &D(a+f) - \frac{1}{2}q(\cos(\theta)\psi)\Big).\end{aligned}$$

For simplicity, let us set $\lambda_0 = 0$. If we look at the pointwise norm of $\mu_\theta(\psi, a, f)$ in the spirit of [Wit94], we see that

$$\begin{aligned}|\mu_\theta(\psi, a, f)|^2 = \ &|\cos(\theta)D\psi|^2 + \frac{1}{4}|(a+f)\cos(\theta)\psi|^2 + |\sin(\theta)\psi|^2,\\ &+ \langle \cos(\theta)D\psi, (a+f)\cos(\theta)\psi\bar{\imath}\rangle + 2\langle\cos(\theta)D\psi, \sin(\theta)\psi\bar{\imath}\rangle\\ &+ \langle (a+f)\cos(\theta)\psi\bar{\imath}, \sin(\theta)\psi\bar{\imath}\rangle\\ &+ |D(a+f)|^2 + \frac{1}{4}|\cos(\theta)\psi|^4 - \langle D(a+f), q(\cos(\theta)\psi)\rangle.\end{aligned}$$

Inspection of the possibility of cancelling of some of the above scalar products led to formula (1.58). After an appropriate rescalling of the terms containing forms and application of (1.59), significant portion of the above scalar products cancels out (up to summands which vanish after integration).

In the special case where $\psi = \psi_n$ and $a + f = \psi_m$, for some $n, m \in \Lambda$ we have

$$\begin{aligned}\mu_\theta(\psi_n, a, f) = \Big(&\cos(\theta)|n|\psi_n + \frac{1}{2}\psi_m\cos(\theta)\psi_n\bar{\imath} + \lambda_0\cos(\theta)\psi_n + \sin(\theta)\psi_n\bar{\imath},\\ &|m|\psi_m - \frac{1}{2}q(\cos(\theta)\psi_n)\Big).\end{aligned}$$

The discussion breaks into several cases (depending on the relation of n and m), and can be used for detecting relations and identities that are useful in the analysis of the monopole map on 3-manifolds.

Appendix A

A.1 Sobolev spaces and elliptic operators

In this section we recall some well-known theorems regarding Sobolev norms for the sake of completeness and easier referencing. The stated results and their proofs can be found in [Pal68], [Nic07] and [LM89].

A.1.1 Definition of Sobolev norms

Let M be a closed smooth n-dimensional Riemannian manifold, and E a real (or complex) vector bundle over M equipped with a metric $\langle\ .\ ,\ .\ \rangle$ and a connection ∇^E compatible with it. Let $p \geq 1$ and $k \in \mathbb{N}_0$. For a smooth section ψ of E, we define

$$\|\psi\|_{L^p_k} := \sqrt[p]{\sum_{j=0}^{k} \left\|\nabla^j \psi\right\|^p_{L^p}}, \tag{A.1}$$

where ∇ denotes the connection ∇^E, as well as all the higher covariant derivatives $\Gamma(\otimes^j T^*M \otimes E) \to \Gamma(\otimes^{j+1} T^*M \otimes E)$ determined by the Levi-Civita connection on M and the connection ∇^E on E.

A.1.1. Remark. Of course, the definition in (A.1) and the more natural-looking definition $\|\psi\|_{L^p_k} := \sum_{j=0}^k \left\|\nabla^j \psi\right\|_{L^p}$ yield equivalent norms. The former is chosen because for $p = 2$ this norm is induced by the inner product $\langle\psi, \phi\rangle := \sum_{j=0}^k \left\langle\nabla^j \psi, \nabla^j \phi\right\rangle$. When $k = 0$, we will write L^p instead of L^p_0. ◀

The space of smooth sections of E will be denoted by $C^\infty(E)$ or simply by $\Gamma(E)$ when there is no danger of confusion. The space of C^l-sections of E will be denoted by $C^l(E)$. The completion of $C^\infty(E)$ with respect to the L^p_k-norm will be denoted by $L^p_k(E)$.

A.1.2 Stronger version of the elliptic estimate

One of the most important results about elliptic operators is the so called elliptic estimate or the elliptic inequality[1]:

A.1.2. THEOREM. *Let $P\colon \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator of order m between spaces of sections of some metric vector bundles E and F over M equipped with metric connections. Then P extends to a Fredholm linear map $P\colon L^p_k(E) \to L^p_{k-m}(F)$ whose index is independent of $k \in \mathbb{N}_0$. Furthermore, for all $k \in \mathbb{N}_0$, all $p > 1$ and $u \in L^p_k(E)$ we have*

$$\|u\|_{L^p_k} \lesssim \|Pu\|_{L^p_{k-m}} + \|u\|_{L^p_{k-m}}. \tag{A.2}$$

A slight improvement[2] of the above result is presented in the following lemma, which will prove useful later in the discussion of the monopole map[3]:

A.1.3. LEMMA. *Let $P\colon \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator of order m between spaces of sections of some metric vector bundles E and F. Then the following version of the elliptic estimate holds:*

$$\|u\|_{L^p_k} \lesssim \|Pu\|_{L^p_{k-m}} + \|\mathrm{pr}_{\ker P} u\|, \tag{A.3}$$

where $\mathrm{pr}_{\ker P} u$ denotes the L^2-orthogonal projection of u onto $\ker P$. The norm in $\|\mathrm{pr}_{\ker P} u\|$ can be arbitrarily chosen, since $\ker P$ is finite-dimensional.

Proof. According to Theorem A.1.2, operator P extends to a Fredholm operator $P\colon L^p_k(E) \to L^p_{k-m}(F)$. Let U be the orthogonal complement of $\ker P$ and denote by B the isomorphism $P|_U\colon U \to \operatorname{im} P$. For an arbitrary $v = v_1 + v_2 \in L^p_k(E) = \ker P + U$ we then have

$$\|v\|_{L^p_k} \leq \|v_1\|_{L^p_k} + \|v_2\|_{L^p_k}$$

[1] see [LM89, Theorem III.5.2] and [Nic07, Ch. 10]

[2] cf. [Nic00, Theorem 1.2.18]

[3] more precisely, this result will allow an important estimate of the norms of differential forms appearing in the monopole map in terms of the norm of spinors (see Remark 3.2.2 on p. 55)

$$
\begin{aligned}
&= \|v_1\|_{L^p_k} + \left\|B^{-1}Bv_2\right\|_{L^p_k} \\
&\le \|v_1\|_{L^p_k} + \left\|B^{-1}\right\| \|Bv_2\|_{L^p_{k-m}} \\
&= \|v_1\|_{L^p_k} + \left\|B^{-1}\right\| \|Pv_2\|_{L^p_{k-m}} \\
&= \|\mathrm{pr}_{\ker P} v\|_{L^p_k} + \left\|B^{-1}\right\| \|Pv\|_{L^p_{k-m}} \\
&\lesssim \|\mathrm{pr}_{\ker P} v\| + \|Pv\|_{L^p_{k-m}}.
\end{aligned}
$$

Here, $\left\|B^{-1}\right\|$ denotes the operator norm of the bounded isomorphism

$$B^{-1}\colon L^p_{k-m}(\operatorname{im} P) \to L^p_k(U).$$

□

A.1.3 An equivalent definition of Sobolev norms

One consequence of the elliptic estimate (A.2) is that the Sobolev norms (A.1) can be defined using elliptic operators instead of connections. More concretely, if $P\colon \Gamma(E) \to \Gamma(E)$ is a fixed elliptic operator of order $m = 1$, we can define for $u \in \Gamma(E)$:

$$\|u\|_{L^p_k, P} := \sqrt[p]{\sum_{j=0}^{k} \left\|P^j \psi\right\|^p_{L^p}}. \tag{A.4}$$

A.1.4. Remark. The remark following the definition of the Sobolev norms (A.1) holds here as well. ◂

On the one hand, repeated use of the elliptic estimate (A.2) gives

$$
\begin{aligned}
\|u\|_{L^p_k} &\lesssim \|u\|_{L^p_{k-1}} + \|Pu\|_{L^p_{k-1}} \\
&\lesssim \|u\|_{L^p_{k-2}} + \|Pu\|_{L^p_{k-2}} + \|Pu\|_{L^p_{k-2}} + \left\|P^2 u\right\|_{L^p_{k-2}} \\
&\lesssim \quad \dots \\
&\lesssim \left\|P^k u\right\|_{L^p} + \left\|P^{k-1} u\right\|_{L^p} + \dots + \|Pu\|_{L^p} + \|u\|_{L^p} \\
&\lesssim \|u\|_{L^p_k, P}.
\end{aligned}
$$

On the other hand, using the fact that P extends to a bounded linear (Fredholm) map $P\colon L^p_k(E) \to L^p_{k-1}(E)$ (Theorem A.1.2) leads to

$$\|u\|_{L^p_k, P} \lesssim \left\|P^k u\right\|_{L^p} + \left\|P^{k-1} u\right\|_{L^p} + \dots + \|Pu\|_{L^p} + \|u\|_{L^p}$$

$$
\begin{aligned}
&\lesssim \left\|P^{k-1}u\right\|_{L^p_1} + \left\|P^{k-2}u\right\|_{L^p_1} + \ldots + \|u\|_{L^p_1} + \|u\|_{L^p} \\
&\lesssim \qquad \ldots \\
&\lesssim \|u\|_{L^p_k} + \|u\|_{L^p_{k-1}} + \ldots + \|u\|_{L^p_1} + \|u\|_{L^p} \\
&\lesssim \|u\|_{L^p_k}.
\end{aligned}
$$

Thus, the norms (A.1) and (A.4) are equivalent. For that reason, the two versions will sometimes be used interchangeably (often without explicit mention) after having fixed an elliptic operator.

A.1.4 Sobolev theorems

Now we state (a special case of) the Sobolev embedding theorem ([Pal68, Theorems 9.1 and 9.2]).

A.1.5. THEOREM. (Sobolev embedding theorem) *Let $1 \leq p, q < \infty$ and $k, l \in \mathbb{N}_0$ be such that $l \leq k$ and $l - \frac{n}{q} \leq k - \frac{n}{p}$. Then $L^p_k(E) \subseteq L^q_l(E)$, and the inclusion map is continuous. In particular, in that case we have*

$$\|\,.\,\|_{L^q_l} \lesssim \|\,.\,\|_{L^p_k}.$$

If $l < k$ and $l - \frac{n}{q} < k - \frac{n}{p}$, the inclusion map is compact.

Furthermore, if $l < k - \frac{n}{p}$ then we have $L^p_k(E) \subseteq C^l(E)$ and the inclusion map is compact. In particular, this means

$$\|\,.\,\|_{C^l} \lesssim \|\,.\,\|_{L^p_k}.$$

Due to its frequent appearance in later chapters, the following special case of Theorem A.1.5 is worth emphasising at this point. For $n = 3$ we have the inequality $\|\,.\,\|_{L^{2p}_j} \lesssim \|\,.\,\|_{L^p_{j+1}}$ for all $p \geq 2$ and for all $j \in \mathbb{N}_0$, since $j - \frac{3}{2p} \leq j + 1 - \frac{3}{p}$ corresponds to $p \geq \frac{3}{2}$.

Now we come to another important theorem about Sobolev norms that will be used repeatedly in the subsequent chapters ([Mor98, Theorem 4.4.2.]):

A.1.6. THEOREM. (Sobolev multiplication theorem) *Let E be a bundle as before, and F a bundle with the same kind of structure. If $k, l, m \in \mathbb{N}_0$ and $p, q, r \geq 1$ are such that $m \leq k, l$ and*

$$m - \frac{n}{r} \leq k - \frac{n}{p} + l - \frac{n}{q},$$

then the multiplication map

$$L^p_k(E) \otimes L^q_l(F) \to L^r_m(E \otimes F)$$

is continuous.

A.1.7. Remark. As mentioned in the introduction, a considerably stronger version of Theorem A.1.6 can be found in [Pal68, Theorem 9.6 and Corollary 9.7]. Remark 3.2.4 explains the point where the stronger result can be used to shorten the proof. ◂

A.2 Some facts from Hodge theory

In this section we list some basic calculations in order to pinpoint the sign conventions and definitions for the later discussion. The terms involved in these calculations appear in the literature under the same name, but often with slighly different conventions in mind.

On $\Lambda^*(\mathbb{R}^n)$ there is the Hodge star operator $*\colon \Lambda^*(\mathbb{R}^n) \to \Lambda^*(\mathbb{R}^n)$ defined by the relation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{dvol}, \tag{A.5}$$

where $\alpha, \beta \in \Lambda^p(\mathbb{R}^n)$, for some p. Explicitly

$$*(e^{i_1} \wedge \ldots \wedge e^{i_p}) = \mathrm{sgn}(\tau) \cdot (e^{j_1} \wedge \ldots \wedge e^{j_{n-p}}),$$

where τ denotes the permutation

$$\tau\colon (i_1, \ldots, i_p, j_1, \ldots, j_{n-p}) \to (1, \ldots, n).$$

Easy argumentation gives

$$**|_{\Lambda^p(\mathbb{R}^n)} = (-1)^{p(n-p)} \,\mathrm{Id}_{\Lambda^p(\mathbb{R}^n)}. \tag{A.6}$$

Let d^* denote the formal adjoint of the exterior derivative $d\colon \Omega(M) \to \Omega(M)$ with respect to the L^2 scalar product (cf. [LM89, p. 123]). That is, $d^*\colon \Omega(M) \to \Omega(M)$ is determined by the relation

$$(d\alpha, \beta) = \int_M \langle d\alpha, \beta \rangle \,\mathrm{dvol} = \int_M \langle \alpha, d^*\beta \rangle \,\mathrm{dvol} = (\alpha, d^*\beta), \qquad \alpha, \beta \in \Omega(M). \tag{A.7}$$

For arbitrary $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$ a straightforward calculation yields[1]

$$d^* = (-1)^{np+n+1} * d*\colon \Omega^p(M) \to \Omega^{p-1}(M). \tag{A.8}$$

If v is an arbitrary vector field and α the corresponding 1-form, then $\operatorname{div} v = *d*\alpha$ and hence

$$\operatorname{div} v = -d^*\alpha. \tag{A.9}$$

A.2.1. Lemma. *The L^2-orthogonal decomposition*

$$\Omega^j(M) = \operatorname{im} d \oplus \operatorname{im} d^* \oplus H^j(M),$$

is also orthogonal with respect to the L^2_k norm, for all k.

Proof. On $\Omega^j(M)$ we have the L^2_k norm defined for $k \in \mathbb{N}$ in the usual way using the extension of the Levi-Civita connection on forms. Another possibility is to use the Dirac operator corresponding to the usual Clifford module structure (A.10) on $\Omega^*(M)$ instead of the connection. These yield equivalent norms, which are both induced by a scalar product. In case of the version with the Dirac operator (i.e. the Hodge-de Rham operator) $D_{HdR} = d + d^*$ the scalar product reads $\langle \alpha, \beta \rangle = \sum_{i=0}^k \int_Y \left\langle D^i_{HdR}\alpha, D^i_{HdR}\beta \right\rangle \operatorname{dvol}$.

Suppose $\alpha \in \ker d^*$ and $\beta = d\beta' \in \operatorname{im} d$. Then, clearly, $\langle \alpha, \beta \rangle = \langle \alpha, d\beta' \rangle = \langle d^*\alpha, \beta' \rangle = 0$. Also $D_{HdR}\alpha = d\alpha \in \operatorname{im} d$ and $D_{HdR}\beta = d^*\beta \in \operatorname{im} d^*$, and thus $\langle D_{HdR}\alpha, D_{HdR}\beta \rangle = 0$. Inductively, for every $i \in \mathbb{N}$ we have $\left\langle D^{i+1}_{HdR}\alpha, D^{i+1}_{HdR}\beta \right\rangle = \left\langle D^i_{HdR} D_{HdR}\alpha, D^i_{HdR} D_{HdR}\beta \right\rangle = 0$. In other words, $\langle \alpha, \beta \rangle_{L^2_k} = 0$, for all $k \in \mathbb{N}_0$, so $\ker d^* \perp_{L^2_k} \operatorname{im} d$ for all $k \in \mathbb{N}_0$. □

A.3 Some relations between Clifford and exterior algebras

Clifford module structure on the exterior algebra $\Lambda^*(\mathbb{R}^n)$ is usually[4] defined by

$$c(v)\alpha = v \bullet_{\text{cl}} \alpha := v^* \wedge \alpha - \iota(v)\alpha, \tag{A.10}$$

for $v \in \mathbb{R}^n$ and $\alpha \in \Lambda^*(\mathbb{R}^n)$. Here, $v^* \wedge \alpha$ denotes the exterior multiplication by the covector v^* dual to[5] v and $\iota(v)$ denotes the contraction by the vector v.

[4] e.g. [LM89, p. 25]

[5] with the conventions on dualising from §1.2 in mind

With the above module structure, the complexified exterior algebra $\Lambda^*_{\mathbb{C}}(\mathbb{R}^n)$ and the complex Clifford algebra $C_n^{\mathbb{C}}$ are isomorphic as Clifford modules via the canonical assignment (1.6).

Multiplication with the volume element $\omega \in C_n$ (1.25) is closely related to the Hodge star operator (A.5). Namely, under the canonical isomorphism $C_n \cong \Lambda^*(\mathbb{R}^n)$, we have for $\alpha \in \Lambda^p(\mathbb{R}^n)$

$$\omega \bullet_{\mathrm{cl}} \alpha = (-1)^{p(n-p)+\frac{1}{2}p(p+1)} *\alpha. \tag{A.11}$$

It suffices to show the above identity for elements of the form e^I, with $I = (i_1, \dots, i_p) \subseteq [n]$, with $[n]$ denoting the ordered set $(1, \dots, n)$. Let J be the ordered complement of I in $[n]$. Note that

$$\omega = e_1 \bullet_{\mathrm{cl}} \dots \bullet_{\mathrm{cl}} e_n = e_J \bullet_{\mathrm{cl}} e_I \cdot \operatorname{sgn}(\sigma),$$

where σ is the permutation $\sigma\colon (J, I) \to [n]$. Also

$$e_I = e_{i_1} \bullet_{\mathrm{cl}} \dots \bullet_{\mathrm{cl}} e_{i_p} = (-1)^{\frac{(p-1)p}{2}} e_{i_p} \bullet_{\mathrm{cl}} \dots \bullet_{\mathrm{cl}} e_{i_1},$$

and clearly $e_{i_p} \bullet_{\mathrm{cl}} \dots \bullet_{\mathrm{cl}} e_{i_1} \bullet_{\mathrm{cl}} e^I = (-1)^p$. Hence

$$\begin{aligned} \omega \bullet_{\mathrm{cl}} e^I &= (-1)^{\frac{(p-1)p}{2}} \operatorname{sgn}(\sigma) \cdot e_J \bullet_{\mathrm{cl}} e_{i_p} \bullet_{\mathrm{cl}} \dots \bullet_{\mathrm{cl}} e_{i_1} \bullet_{\mathrm{cl}} e^I \\ &= (-1)^{\frac{(p-1)p}{2}} \operatorname{sgn}(\sigma) \cdot e_J \cdot (-1)^p \\ &= (-1)^{\frac{(p+1)p}{2}} \operatorname{sgn}(\sigma) \cdot e_J. \end{aligned}$$

On the other hand we have $*e^I = \operatorname{sgn}(\tau) \cdot e^J$, with $\tau\colon (I, J) \to [n]$, i.e. $e^J = \operatorname{sgn}(\tau) *e^I$. This gives in total

$$\omega \bullet_{\mathrm{cl}} e^I = (-1)^{\frac{(p+1)p}{2}} \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau) *e^I.$$

Since $\operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma^{-1}) \cdot \operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma^{-1} \circ \tau)$, with $\sigma^{-1} \circ \tau\colon (I, J) \to (J, I)$, and clearly $\operatorname{sgn}(\sigma^{-1} \circ \tau) = (-1)^{p(n-p)}$, we have

$$\omega \bullet_{\mathrm{cl}} e^I = (-1)^{\frac{(p+1)p}{2} + p(n-p)} *e^I.$$

A.3.1. Lemma. *The action of $\omega_{\mathbb{C}}$ on the Clifford module $\Lambda^*_{\mathbb{C}}(\mathbb{R}^n)$ corresponds to*

$$\omega_{\mathbb{C}} \bullet_{\mathrm{cl}} \alpha = i^{\lfloor \frac{n+1}{2} \rfloor} \cdot_S (-1)^{p(n-p)+\frac{1}{2}p(p+1)} *\alpha, \tag{A.12}$$

for $\alpha \in \Lambda^p_{\mathbb{C}}(\mathbb{R}^n)$.

Proof. This follows directly from the definition of $\omega_{\mathbb{C}}$ (1.26) and from (A.11). □

A.4 The connection Laplacian

Let E be a Riemannian (or Hermitian) vector bundle over M, equipped with a metric connection ∇. By ∇^2 we will denote an operator $\Gamma(E) \to \Gamma(T^*M \otimes T^*M \otimes E)$ determined for vector arbitrary fields v, w by

$$\nabla^2_{v,w}\psi = \nabla_v \nabla_w \psi - \nabla_{\nabla_v w}\psi. \tag{A.13}$$

The operator ∇^2 is called invariant second derivative.

A.4.1. DEFINITION ([LM89, p. 154]). If E is a Riemannian (or Hermitian) vector bundle over M, equipped with a metric connection ∇, we can define the *connection Laplacian* $\nabla^*\nabla\colon \Gamma(E) \to \Gamma(E)$ by

$$\nabla^*\nabla\psi = -\mathrm{tr}(\nabla^2_{\cdot,\cdot}\psi), \tag{A.14}$$

or locally, with respect to some orthonormal tangent frame field (e_i)

$$\nabla^*\nabla\psi = -\sum_i \left(\nabla_{e_i}\nabla_{e_i}\psi - \nabla_{\nabla_{e_i}e_i}\psi\right) = -\sum_i \nabla_{e_i}\nabla_{e_i}\psi + \sum_i \nabla_{\nabla_{e_i}e_i}\psi. \tag{A.15}$$

The local definition above does not depend on the choice of the local tangent frame field.

In the present setting, we have the following local identity for $\psi \in \Gamma(E)$:

$$\langle \nabla^*\nabla\psi, \psi\rangle = \frac{1}{2}\Delta(|\psi|^2) + |\nabla\psi|^2.$$

Namely, let v denotes the vector field determined by[6] $\langle v, w\rangle = \langle \nabla_w\psi, \psi\rangle$, for $w \in \Gamma(TM)$. Clearly, $\langle \nabla\psi, \psi\rangle$ is the 1-form corresponding to v. With respect to some orthonormal tangent field we now have

$$\begin{aligned}
\langle \nabla^*\nabla\psi, \psi\rangle &= -\sum_i \left\langle \nabla_{e_i}\nabla_{e_i}\psi, \psi\right\rangle \\
&= -\sum_i d\left\langle \nabla_{e_i}\psi, \psi\right\rangle(e_i) + \sum_i \left\langle \nabla_{e_i}\psi, \nabla_{e_i}\psi\right\rangle \\
&= -\mathrm{div}\, v + \langle \nabla\psi, \nabla\psi\rangle.
\end{aligned}$$

Due to (A.9) and the fact that ∇ is metric, we have

$$\mathrm{div}\, v = d^*\langle \nabla\psi, \psi\rangle = \frac{1}{2}d^*d\langle \psi, \psi\rangle = \frac{1}{2}\Delta(|\psi|^2).$$

[6] in case of a Hermitian bundle (e.g. if E is a spinor bundle), we take the real part of the Hermitian product

Endnotes

Chapter 1: Preliminaries

Endnote 1. (page 13) Namely,

$$\begin{aligned} q_1 q_2 &= -\sum_{i=1}^{3} \lambda_i \mu_i + \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i) e_i e_j \\ &\overset{(1.19)}{=} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \end{aligned}$$

where $a = -\sum_{i=1}^{3} \lambda_i \mu_i + (\lambda_2\mu_3 - \lambda_3\mu_2)i - (\lambda_1\mu_3 - \lambda_3\mu_1)j + (\lambda_1\mu_2 - \lambda_2\mu_1)k \in \mathbb{H}$ and we have

$$\begin{aligned} \|a\|^2 &= \left(\sum_{i=1}^{3} \lambda_i \mu_i\right)^2 + (\lambda_2\mu_3 - \lambda_3\mu_2)^2 + (\lambda_1\mu_3 - \lambda_3\mu_1)^2 + (\lambda_1\mu_2 - \lambda_2\mu_1)^2 \\ &= \sum_{i,j=1}^{3} \lambda_i^2 \mu_j^2 = 1. \end{aligned}$$

Endnote 2. (page 15) We prove the following claim:

4.4.2. Lemma. *The assignment* $\pi\colon Sp(1) \ni a \mapsto (h \mapsto a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{a})$ *defines a 2-fold covering of* $SO(3)$.

Proof. For arbitrary $h \in \mathbb{H}$ we have

$$\|\mathrm{Ad}(a)(h)\| = \|a \cdot_{\mathbb{H}} h \cdot_{\mathbb{H}} \bar{a}\| = \|h\|.$$

Therefore, we have a map $\pi\colon Sp(1) \to SO(3)$, which is actually a Lie group homomorphism. Its kernel consists of all $a \in Sp(1)$ such that $ah\bar{a} = h$, i.e. $ah = ha$ holds for all $h \in \mathbb{H}$. Thus, $\ker \pi$ is a subset of the centre of $\mathbb{H}$ which equals $\{\pm 1\}$. Obviously, $\ker \pi = \{\pm 1\}$. The claim follows from the following general fact about Lie groups (which is a simple consequence of the fact that any Lie group homomorphism has constant rank). □

4.4.3. LEMMA. *Suppose G and H are Lie groups of the same dimension and suppose H is connected. Then every Lie group homomorphism $f : G \to H$ with discrete kernel is a covering map.*

Endnote 3. (page 25) Namely, for a 1-form $\alpha = \sum_{l=1}^{3} \alpha_l e^l$ we have

$$\begin{aligned} d\alpha = & \quad \partial_1\alpha_2 \cdot e^1 \wedge e^2 + \partial_1\alpha_3 \cdot e^1 \wedge e^3 \\ & + \partial_2\alpha_1 \cdot e^2 \wedge e^1 + \partial_2\alpha_3 \cdot e^2 \wedge e^3 \\ & + \partial_3\alpha_1 \cdot e^3 \wedge e^1 + \partial_3\alpha_2 \cdot e^3 \wedge e^2 \\ = & \quad (\partial_1\alpha_2 - \partial_2\alpha_1) \cdot e^1 \wedge e^2 \\ & + (\partial_1\alpha_3 - \partial_3\alpha_1) \cdot e^1 \wedge e^3 \\ & + (\partial_2\alpha_3 - \partial_3\alpha_2) \cdot e^2 \wedge e^3, \end{aligned}$$

i.e.

$$*d\alpha = (\partial_1\alpha_2 - \partial_2\alpha_1) \cdot e^3 + (-\partial_1\alpha_3 + \partial_3\alpha_1) \cdot e^2 + (\partial_2\alpha_3 - \partial_3\alpha_2) \cdot e^1.$$

Hence,

$$\begin{aligned} & a_1 \cdot \Big(- \partial_2[q(\psi)]_3 + \partial_3[q(\psi)]_2 \Big) \\ + \, & a_2 \cdot \Big(+ \partial_1[q(\psi)]_3 - \partial_3[q(\psi)]_1 \Big) \\ + \, & a_3 \cdot \Big(- \partial_1[q(\psi)]_2 + \partial_2[q(\psi)]_1 \Big) = \langle a, - * dq(\psi) \rangle. \end{aligned}$$

Chapter 2: The monopole map on 3-manifolds

Endnote 1. (page 34) The bundles $\mathcal{A}(Y) \to \mathrm{Pic}^{\mathfrak{s}}(Y)$ and $C(Y) \to \mathrm{Pic}^{\mathfrak{s}}(Y)$ described in (2.7) are not trivial in general, but after fibrewise completion to a Hilbert bundle they do become trivial (due to Kuiper's theorem [Kui65]).
Endnote 2. (page 35) Namely, the two isomorphisms

$$\begin{aligned} F_1 &: [A', \psi] && \mapsto \psi, \\ F_2 &: [A' + 2udu^{-1}, \psi] && \mapsto \psi, \end{aligned}$$

induced from different choices of a representative of $[A']$ would be related by some element of $\mathcal{G}_0$:

$$\begin{aligned}\psi &= F_1([A',\psi]) = \left[F_1 \circ F_2^{-1}\right] \circ F_2([A',\psi]) \\ &= \left[F_1 \circ F_2^{-1}\right] \circ F_2([A' + 2udu^{-1}, u\psi]) = \left[F_1 \circ F_2^{-1}\right](u\psi).\end{aligned}$$

In other words, F_1 and F_2 send the same class element to spinors with a difference in phase (which depends solely on the choice of the two representatives of $[A']$):

$$\begin{aligned}&F_1([A',\psi]) = \psi, \\ &F_2([A',\psi]) = F_2([A' + 2udu^{-1}, u\psi]) = u\psi.\end{aligned}$$

Endnote 3. (page 36) The linear part is the same on all fibres (with appropriate identifications (2.10) and (2.11) taken into consideration).

As another way of seeing that, choose two elements $[A_1], [A_2] \in \mathrm{Pic}^{\mathfrak{s}}(Y) = (A + i\ker d)/_{\mathcal{G}_0}$ with some fixed representatives $A_j = A + ib_j$ with $b_j \in \ker d$, $j = 1, 2$. Fixing representatives yields an identification of the corresponding fibres as explained on page 35. With respect to some choice of these identifications, the restriction of μ on the fibre over $[A_j]$ is of the form

$$\begin{aligned}&(\psi, a, f) \mapsto \\ &\quad (D_{A_j+ia}\psi,\ -*iF_{A_j+ia} + i\sigma(\psi),\ d^*a + f,\ a_h) \\ &= (D_{A+ib_j+ia}\psi,\ -*iF_{A+ib_j+ia} + i\sigma(\psi),\ d^*a + f,\ a_h) \\ &= (D_A\psi + \frac{1}{2}\, ib_j\psi + \frac{1}{2}\, ia\psi,\ -*iF_A + *db_j + *da + i\sigma(\psi),\ d^*a + f,\ a_h) \\ &= (D_A\psi + \frac{1}{2}\, ia\psi + \frac{1}{2}\, ib_j\psi,\ -*iF_A + *da + i\sigma(\psi),\ d^*a + f,\ a_h),\end{aligned}$$

respectively. From this it is evident that the linear part does not change in an essential way (i.e. up to a choice of identifications (2.10) and (2.11) of fibres, it is always the same). This observation will be important later, for the definition of the virtual index bundle of the linear part. The non-linear part obviously does change with a change of fibre. The change, relative to the class $[A]$ is given by the addition of a summand $\frac{1}{2}\, ib_j\psi$.

Endnote 4. (page 37) Elliptic operators over closed Riemann manifolds always have finite-dimensional kernel and cokernel [LM89, p. 135], so the definition of index is valid.

Endnote 5. (page 44) We choose all p_j such that $\|\,.\,\|_{L^{p_j}_{j-1}} \lesssim \|\,.\,\|_{L^2_{k-1}}$, i.e. such that

$$j-1-\frac{n}{p_j} \leq k-1-\frac{n}{2}.$$

With $n=3$ this means

$$j-1-\frac{3}{p_j} \leq k-1-\frac{3}{2},$$

or after reshuffling

$$\begin{cases} p_j \leq \dfrac{1}{\frac{1}{2}-\frac{k-j}{3}} = \dfrac{3}{\frac{3}{2}-(k-j)}, & \text{if } k-j<\frac{3}{2}, \\ p_j \geq 1, & \text{else.} \end{cases}$$

Chapter 3: New version of the monopole map

Endnote 1. (page 52)

More explicitly

$$\begin{aligned} \cos^2(\theta)\cdot\Big[&\Big\langle D^2_{A+ia}\psi,\psi\Big\rangle + \frac{1}{2}\langle D_{A+ia}(if\psi),\psi\rangle + \lambda_0\langle D_{A+ia}\psi,\psi\rangle \cancel{+\langle i\lambda_1 D_{A+ia}\psi,\psi\rangle} \\ &\cancel{-\frac{1}{2}\langle ifD_{A+ia}\psi,\psi\rangle} + \frac{1}{4}f^2|\psi|^2 - \cancel{\frac{1}{2}\lambda_0\langle if\psi,\psi\rangle} + \frac{1}{2}f\lambda_1|\psi|^2 \\ &+\lambda_0\langle D_{A+ia}\psi,\psi\rangle + \cancel{\frac{1}{2}\lambda_0\langle if\psi,\psi\rangle} + \lambda_0^2|\psi|^2 + \cancel{\lambda_0\langle i\lambda_1\psi,\psi\rangle} \\ &\cancel{-\langle i\lambda_1 D_{A+ia}\psi,\psi\rangle} + \frac{1}{2}f\lambda_1|\psi|^2 - \cancel{\lambda_0\langle i\lambda_1\psi,\psi\rangle} + \lambda_1^2|\psi|^2\Big] \\ = \cos^2(\theta)\cdot\Big[&\Big\langle D^2_{A+ia}\psi,\psi\Big\rangle + \frac{1}{2}\langle idf\psi,\psi\rangle + 2\lambda_0\langle D_{A+ia}\psi,\psi\rangle \\ &+\lambda_0^2|\psi|^2 + (\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big]. \end{aligned}$$

Endnote 2. (page 53)

Detailed calculation reads

$$\begin{aligned} \cos^2(\theta)\cdot\Big[&\Delta|\psi|^2 + |\nabla_{A+ia}\psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{2}\langle(*F_{A+ia}+idf)\psi,\psi\rangle \\ &+2\lambda_0\langle D_{A+ia}\psi,\psi\rangle + \lambda_0^2|\psi|^2 + (\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big] \\ = \cos^2(\theta)\cdot\Big[&\Delta|\psi|^2 + |\nabla_{A+ia}\psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{2}\langle ib\psi,\psi\rangle + \frac{1}{2}\cos^2(\theta)\langle\sigma(\psi)\psi,\psi\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\lambda_0\langle D_{A+ia}\psi,\psi\rangle + \lambda_0^2|\psi|^2 + (\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big]\\
&\overset{\spadesuit}{=} \cos^2(\theta)\cdot\Big[\Delta|\psi|^2 + |\nabla_{A+ia}\psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{2}\langle ib\psi,\psi\rangle + \frac{1}{4}\cos^2(\theta)\cdot|\psi|^4\\
&\quad + 2\lambda_0\langle\rho_\lambda,\psi\rangle - 2\lambda_0^2|\psi|^2 + \lambda_0^2|\psi|^2 + (\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big].\\
&= \cos^2(\theta)\cdot\Big[\Delta|\psi|^2 + |\nabla_{A+ia}\psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{2}\langle ib\psi,\psi\rangle + \frac{1}{4}\cos^2(\theta)\cdot|\psi|^4\\
&\quad + 2\lambda_0\langle\rho_\lambda,\psi\rangle - \lambda_0^2|\psi|^2 + (\frac{1}{2}f+\lambda_1)^2|\psi|^2\Big].
\end{aligned}
$$

In the penultimate equality (♠) we used the fact that $\langle\rho_\lambda,\psi\rangle$ and $\langle D_{A+ia}\psi,\psi\rangle$ differ by $\lambda_0|\psi|^2$. This is because the anti-self adjoint part of ρ_λ is pointwise orthogonal to the self-adjoint one:

$$
\begin{aligned}
\langle D_{A+ia}\psi,\psi\rangle &= \langle\rho_\lambda,\psi\rangle - \langle if\psi,\psi\rangle - \lambda_0|\psi|^2 - \langle i\lambda_1\psi,\psi\rangle\\
&= \langle\rho_\lambda,\psi\rangle - 0 - \lambda_0|\psi|^2 - 0.
\end{aligned}
$$

Endnote 3. (page 58) Here we prove the inequality (3.11):

$$
\begin{aligned}
\|(a+f)\cos(\theta)\psi\|_{L^2_m} \lesssim \|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{C^0} + \|a+f\|_{C^0}\|\cos(\theta)\psi\|_{L^2_m}\\
+ \|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{L^2_m}.
\end{aligned}
$$

First note that for $m=0$ we simply have

$$\|(a+f)\cos(\theta)\psi\|_{L^2} \lesssim \|a+f\|_{L^2}\|\cos(\theta)\psi\|_{C^0},$$

so (3.11) trivially holds in this case.

As previously with (1.64), for $m \geq 1$ we start by examining the multiple derivation $\nabla_A^m\big((a+f)\cos(\theta)\psi\big)$ of the Clifford product. The spin$^{\mathbb{C}}$ connection ∇_A acts as a derivative with respect to Clifford multiplication[7], hence

$$\nabla_A^m\big((a+f)\cos(\theta)\psi\big) = \sum_{s=0}^{m}\binom{m}{s}\nabla^{m-s}(a+f)\bullet_{\mathrm{cl}}\nabla_A^s\cos(\theta)\psi.$$

The symbol ∇ denotes the extension of the Levi-Civita connection to $\Omega^*(Y)$. The above equality implies

$$\Big\|\nabla_A^m\big((a+f)\cos(\theta)\psi\big)\Big\|_{L^2} \lesssim \sum_{s=0}^{m}\Big\|\nabla^{m-s}(a+f)\nabla_A^s\cos(\theta)\psi\Big\|_{L^2}$$

[7] e.g. [Fri97, p. 65]

$$\begin{aligned}&\lesssim \|\nabla^m(a+f)\cos(\theta)\psi\|_{L^2} + \|(a+f)\nabla_A^m\cos(\theta)\psi\|_{L^2}\\ &\quad+\sum_{s=1}^{m-1}\left\|\nabla^{m-s}(a+f)\nabla_A^s\cos(\theta)\psi\right\|_{L^2}.\end{aligned}$$

For the first two terms we have the straightforward inequalities

$$\begin{aligned}\|\nabla^m(a+f)\cos(\theta)\psi\|_{L^2} &\lesssim \|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{C^0},\\ \|(a+f)\nabla_A^m\cos(\theta)\psi\|_{L^2} &\lesssim \|a+f\|_{C^0}\|\cos(\theta)\psi\|_{L^2_m}.\end{aligned}$$

For the remaining sum we use Sobolev's theorems to conclude

$$\begin{aligned}\sum_{s=1}^{m-1}\left\|\nabla^{m-s}(a+f)\nabla_A^s\cos(\theta)\psi\right\|_{L^2} &\lesssim \sum_{s=1}^{m-1}\left\|\nabla^{m-s}(a+f)\right\|_{L^4}\|\nabla_A^s\cos(\theta)\psi\|_{L^4}\\ &\lesssim \sum_{s=1}^{m-1}\|a+f\|_{L^4_{m-s}}\|\cos(\theta)\psi\|_{L^4_s}\\ &\lesssim \sum_{s=1}^{m-1}\|a+f\|_{L^2_{m-s+1}}\|\cos(\theta)\psi\|_{L^2_{s+1}}\\ &\lesssim \|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{L^2_m}.\end{aligned}$$

In short, we obtained

$$\begin{aligned}\left\|\nabla_A^m\big((a+f)\cos(\theta)\psi\big)\right\|_{L^2} &\lesssim \|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{C^0} + \|a+f\|_{C^0}\|\cos(\theta)\psi\|_{L^2_m}\\ &\quad+\|a+f\|_{L^2_m}\|\cos(\theta)\psi\|_{L^2_m}.\end{aligned}$$

Repeated use of the above inequality now easily leads to (3.11) in a similar fashion as for the quadratic term in §1.7.3.

Endnote 4. (page 60) Obviously $\langle a,f\rangle_{L^2}=0$. That $\left\langle D^j_\Omega a, D^j_\Omega f\right\rangle_{L^2}=0$ holds follows from the Hodge decomposition theorem (Lemma A.2.1). Hence $\langle a,f\rangle_{L^2_j}=0$, for all $j\in\mathbb{N}_0$.

Chapter 4: The monopole map on a 3-torus

Endnote 1. (page 65) More precisely, we use the formulae

$$\begin{aligned}2\cos x\cos y &= \cos(x+y)+\cos(x-y),\\ 2\cos x\sin y &= \sin(x+y)+\sin(-x+y),\\ 2\sin x\sin y &= -\cos(x+y)+\cos(x-y).\end{aligned}$$

Appendix A:

Endnote 1. (page 72) Namely, for $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$ we have

$$\begin{aligned}\langle d\alpha, \beta\rangle \,\mathrm{dvol} &= d\alpha \wedge *\beta \\ &= d(\alpha \wedge *\beta) - (-1)^{|\alpha|}\alpha \wedge d(*\beta) \\ &= d(\alpha \wedge *\beta) + (-1)^{p}\alpha \wedge d(*\beta) \\ &\overset{\spadesuit}{=} d(\alpha \wedge *\beta) + (-1)^{p}(-1)^{(n-p-1)(p+1)}\alpha \wedge **d(*\beta) \\ &= d(\alpha \wedge *\beta) + (-1)^{np+n+1}\langle \alpha, (*d*)\beta\rangle \,\mathrm{dvol}.\end{aligned}$$

In ♠ we used the analogue of (A.6) for forms on M and the fact that $d(*\beta) \in \Omega^{n-p+1}(M)$. The claim now follows from Stokes' theorem and the assumption that M is a closed manifold.

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Notation

$(\mathcal{G}_0)_0$ connected component of $\mathcal{G}_0$ containing the constant map $u_0 \equiv 1$ 33

B_θ abbreviation for $\cos(\theta)D_{A+\lambda_0} + i\sin(\theta)$ 50

D_{HdR} Hodge-de Rham operator 72

D_Ω differential/Dirac operator on forms 20

I identity matrix 12

I_θ abbreviation for the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 49

$L_k^p(E)$ completion of the space $C^\infty(E)$ of smooth sections of the vector bundle E with respect to the L_k^p-norm 68

P^{SO} orthonormal frame bundle of an oriented Riemannian manifold 17

$Pin(2)$ $Pin(2)$ group 16

$[A, B]_\bullet$ space of (pointed) homotopy classes of pointed maps between A and B 33

$[a, u]$ element of $Spin^\mathbb{C}(3)$ 11

$[a]_\mathrm{w}$ path-component containing a 33

$[n]$ ordered set $(1, \ldots, n)$ 73

Δ_3 Dirac spinors in dimension 3 6

$\Delta_3^\mathbb{C}$ complex Dirac spinors in dimension 3 6

Δ_n Dirac spinors 9

$\Gamma(E)$ space of sections of the vector bundle E 67

$\Lambda^*(\mathbb{R}^n)$ exterior algebra of the dual space $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$ 7

$\Lambda^*_{\mathbb{C}}(\mathbb{R}^n)$ complexification of $\Lambda^*(\mathbb{R}^n)$, i.e. $\Lambda^*_{\mathbb{C}}(\mathbb{R}^n) = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ 7

$\Lambda^{1,0}(\mathbb{R}^3)$ abbreviation for $\Lambda^1(\mathbb{R}^3) \oplus \Lambda^0(\mathbb{R}^3)$ 19

$\Omega^*(Y)$ space of differential forms on Y 19

$\Omega^{1,0}(Y)$ abbreviation for $\Omega^1(Y) \oplus \Omega^0(Y)$ 19, 20

$\underset{\mathbb{H}}{\cdot}$ quaternion multiplication 10

$\underset{s}{\cdot}$ multiplication by a scalar 10

$\underset{cl}{\bullet}$ algebra operation in $C_n^{\mathbb{C}}$ or Clifford action on a Clifford module 16

$*$ Hodge star 71

ι_{H^0} the inclusion $H^0(Y) \hookrightarrow \Omega^1(Y)$ 35

$\mathbb{H}$ algebra of quaternions 6

$\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$ of natural numbers together with 0 70

$C^\infty(E)$ space of smooth sections of the vector bundle E 67

$C^l(E)$ space of C^l-sections of the vector bundle E 68

$C_3^{\mathbb{C}}(Y)$ complex Clifford bundle 19

$C_3(Y)$ Clifford bundle of Y 18

$C_n^\times$ set of units in C_n 13

$\mathcal{G}_0$ group of based gauge transformations 30

$\mathfrak{su}(2)$ Lie algebra of $SU(2)$ 10

$\mathrm{Im}(\mathbb{H})$ purely imaginary quaternions 6

∇ Levi-Civita connection, as well as its extension to $\Omega^*(Y)$ 79

$\nabla^*\nabla$ connection Laplacian 74

∇^2 invariant second derivative 74

$\| \,.\, \|_{L^p_k}$ Sobolev L^p_k-norm 67

$\| \,.\, \|_{C^k}$ supremum k-norm, $k \in \mathbb{N}_0$ 53

ω volume element in C_n 14

$\omega_{\mathbb{C}}$ complex volume element in $C_n^{\mathbb{C}}$ 14

$\det \mathfrak{s}$ determinant bundle of the spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ 18

ρ spin representation 14

$\rho^{\mathbb{C}}$ spin$^{\mathbb{C}}$ representation 15

ρ_θ renormalised monopole map 60

ρ_λ abbreviation for $D_{A+ia+if+\lambda}\psi$ 51

$\sigma(\psi)$ quadratic term as an endomorphism 21

$\langle\, , \rangle_{\mathbb{C}}$ Hermitian product 8

$\langle\, , \rangle_{\mathbb{R}}$ (real) scalar product 7

$\sqcup$ disjoint union 63

$a \otimes u$ an element of $C_3^{\mathbb{C}} = C_3 \otimes_{\mathbb{R}} \mathbb{C}$ 11

a_h harmonic part of the 1-form a 31

b abbreviation for $-i * F_{A+ia} + i\sigma(\cos(\theta)\psi) + df$ 51

$c(\alpha)$ Clifford multiplication by $\alpha \in \Omega^*(Y)$ 20

$c(v)$ Clifford multiplication by the vector v 72

$cl(\iota)$ Clifford representation 18

d^* formal adjoint of the exterior derivative d 71

f_h harmonic part of the 0-form f 31

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