S1 Appendix: Proof of limited convergence

We assume that the agent’s local coordinate system is aligned with the world coordinate system. Then let $\mathbf{x} = (x, y)^T$ be the agent’s current location, and $\mathbf{x}_0 = (x_0, y_0)^T$ the agent’s home location. For every landmark $i = 1, \ldots, n$ detected by the agent at location $\mathbf{x}_i = (x_i, y_i)^T$, a landmark vector pointing from the agent’s location to the landmark can be defined as $\mathbf{L}_i(\mathbf{x}) = \frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|}$.

The original ALV model as stated by [1] computes the home vector as

$$H(\mathbf{x}) = \sum_{i=1}^{n} (\mathbf{L}_i(\mathbf{x}_0) - \mathbf{L}_i(\mathbf{x})).$$

(1)

Here we use additional weighting terms $w_i$ and $w^0_i$ to model the influence of each landmark on the home vector computation.

$$H(\mathbf{x}) = \sum_{i=1}^{n} (w^0_i \mathbf{L}_i(\mathbf{x}_0) - w_i \mathbf{L}_i(\mathbf{x})).$$

(2)

For unweighted landmarks, i.e. $w_i = 1$ for all $i$, it can be shown that trajectories following the home vector field $H(\mathbf{x})$ converge on the home location $\mathbf{x}_0$ [2].

Using a distance-variant weight for each landmark — for example the height $h_i$ — the weight $w_i$ at the home location $\mathbf{x}_0$ and the current location $\mathbf{x}$ can be chosen as $w_i^0 := \frac{h_i}{\|\mathbf{x}_i - \mathbf{x}_0\|}$ and $w_i := \frac{h_i}{\|\mathbf{x}_i - \mathbf{x}\|}$.

Substituting these weights into $H(\mathbf{x})$, we obtain

$$H(\mathbf{x}) = \sum_{i=1}^{n} h_i \left( \frac{\mathbf{x}_i - \mathbf{x}_0}{\|\mathbf{x}_i - \mathbf{x}_0\|^2} - \frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|^2} \right).$$

(3)

As shown by the following theorem, all stationary points of $H(\mathbf{x})$ are saddle points. Since $H(\mathbf{x}_0) = 0$, the home location is a stationary point and therefore a saddle point.

**Theorem 1.** Every stationary point in the home vector field $H(\mathbf{x})$ is a saddle point.

**Proof.** The function $U(\mathbf{x})$, which fulfills $\nabla U(\mathbf{x}) = -H(\mathbf{x})$, is called the potential of $H(\mathbf{x})$. To determine sources, saddle points, and sinks in the vector field we can equivalently search for maxima, saddle points, and minima, in its corresponding
potential. For $H(x)$, the potential is given by

$$U(x) = -\sum_{i=1}^{n} h_i \left( \ln(||x_i - x||) + x^T \frac{x_i - x_0}{||x_i - x_0||^2} \right).$$  \hspace{1cm} (4)$$

To compute the Hessian matrix $M(x) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} U(x) & \frac{\partial^2}{\partial x \partial y} U(x) \\ \frac{\partial^2}{\partial x \partial y} U(x) & \frac{\partial^2}{\partial y^2} U(x) \end{pmatrix}$ of $U(x)$, we first have to compute the second derivatives:

$$\frac{\partial^2}{\partial x^2} U(x) = \sum_{i=1}^{n} h_i \frac{(x_i - x)^2 - (y_i - y)^2}{\|x_i - x\|^4}$$ \hspace{1cm} (5)$$

$$\frac{\partial^2}{\partial y^2} U(x) = \sum_{i=1}^{n} h_i \frac{(y_i - y)^2 - (x_i - x)^2}{\|x_i - x\|^4}$$ \hspace{1cm} (6)$$

$$\frac{\partial^2}{\partial x \partial y} U(x) = 2 \sum_{i=1}^{n} h_i \frac{(x_i - x)(y_i - y)}{\|x_i - x\|^4}$$ \hspace{1cm} (7)$$

Since we have $\frac{\partial^2}{\partial x^2} U(x) = -\frac{\partial^2}{\partial y^2} U(x)$ and $\frac{\partial^2}{\partial x \partial y} U(x) = \frac{\partial^2}{\partial y \partial x} U(x)$, the second order principal of $M(x)$ is given by

$$\det(M(x)) = \frac{\partial^2}{\partial x^2} U(x) \frac{\partial^2}{\partial y^2} U(x) - \frac{\partial^2}{\partial x \partial y} U(x) \frac{\partial^2}{\partial y \partial x} U(x)$$ \hspace{1cm} (8)$$

$$= -\left( \frac{\partial^2}{\partial x^2} U(x)^2 + \frac{\partial^2}{\partial y^2} U(x)^2 \right) \leq 0.$$ \hspace{1cm} (9)$$

For $\det(M(x)) < 0$, the Hessian matrix $M(x)$ is indefinite and therefore $x$ a saddle point. For $\det(M(x)) = 0$, it follows that $\frac{\partial^2}{\partial x^2} U(x)^2 = \frac{\partial^2}{\partial y^2} U(x)^2 = 0$, from which we obtain $M(x) = 0$. In this case $x$ is a flat point, which is a special case of a saddle point.

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**Divergence from simulation approach**

As described above, extracting features to use as landmarks is a non-trivial task, which the ASV model avoids by directly using a low-level image statistic, in our case the average pixel intensity, computed from every image column. This process is appealing from a biological perspective as it corresponds to the extraction of a skyline, which is thought to be a process used by many insects for navigational purposes. Hence, in our simulation, there are no fixed landmarks but rather each image column is defined as a
single landmark. We thus write
\[ x_i = w_i \begin{pmatrix} \cos(i\Delta\phi) \\ \sin(i\Delta\phi) \end{pmatrix} \] (10)

with \( \Delta\phi \) being the angular resolution of the panoramic view. We therefore observe some mismatch between the presented proof and our simulation results.

References
