Submodule closed subcategories of module categories

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1 Introduction

1.1 Motivation

We are interested in submodule closed subcategories of abelian length categories $\mathcal{A}$, that is, (full, additive) subcategories, which are closed under subobjects. If $\mathcal{A}$ is a module category, these subcategories are also called submodule closed.

While submodule closed subcategories have not yet been extensively studied, they are a very interesting topic with many connections to different parts of representation theory. For example, if $A$ is a finite dimensional algebra, then every infinite submodule closed subcategory of $\text{mod} \ A$ contains a minimal infinite submodule closed category, see [16].

Submodule closed subcategories can also be used to prove that there is a filtration of the Ziegler spectrum that is indexed by the Gabriel-Roiter filtration, see [12].

Furthermore, if $A$ is a hereditary Artin algebra, then there is a natural bijection between the elements of the Weyl group of $A$ and the full, additive cofinite submodule closed subcategories of $\text{mod} \ A$. This has been proved in [14] for algebras over finite and algebraically closed fields and is proved in general in this thesis.

Another connection arises in the second main part of the thesis: If $A$ is of colocal type, then the lattice formed by full, additive submodule closed subcategories of $\text{mod} \ A$ is distributive. Algebras of colocal type have been studied repeatedly: for example, a first characterization dates back to H. Tachikawa in 1959, see [20]; two gaps in the proof were filled by T. Sumioka in 1984, see [19].

In this thesis, we give a new characterisation for algebras of colocal type, which is especially simple for algebras over an algebraically closed field.

For algebras of colocal type over such fields, we can completely describe the lattice $S(\text{mod} \ A)$. In fact, we get another connection to an important item in representation theory: Young diagrams.
1.2 Main results

In Chapter 3 we take a look at hereditary Artin algebras $A$. Let $\text{mod} \ A$ be the category of finitely generated right modules over $A$.

A full, additive subcategory of $\text{mod} \ A$ is called \textit{cofinite} if it contains all but finitely many indecomposable modules in $\text{mod} \ A$.

We get the following result:

\textbf{Theorem 1.2.1.} Let $A$ be a hereditary Artin algebra. Then there exists a natural bijection between the elements of the Weyl group of $A$ and the full, additive cofinite submodule closed subcategories of $\text{mod} \ A$.

To prove this theorem, we first show the following result, which is important in its own right. We use the notation $X \mid Y$ if the module $X$ is a direct summand of $Y$ and $X \nmid Y$ if $X$ is not a direct summand of $Y$:

\textbf{Proposition 1.2.2.} Let $A$ be a hereditary Artin algebra and $M \in \text{mod} \ A$ indecomposable and preinjective. Suppose that $U \in \text{mod} \ A$, and $M$ is not a direct summand of $U$. There is a monomorphism $M \hookrightarrow U$ if and only if there is some $m \in \mathbb{N}$ with three sequences of modules

\begin{align*}
(X_1, X_2, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_m)
\end{align*}

that fulfill the following conditions:

(S1) There is an Auslander-Reiten sequence

\[ 0 \longrightarrow M \longrightarrow X_1 \oplus X'_1 \longrightarrow Y_1 \longrightarrow 0 \, . \tag{1.1} \]

(S2) For all $1 \leq i < m$, there is some $\alpha_i \in \mathbb{N}$ so that $X_i^{\alpha_i}$ is a direct summand of $X_i \oplus X'_i$, but not of $U$.

(S3) For $1 \leq i < m$, there is an Auslander-Reiten sequence of the form

\[ 0 \longrightarrow X_i \longrightarrow Z_i \longrightarrow \tau^{-1}X_i \longrightarrow 0 \, . \]

Let $Y'_i$ be the maximal module that is both a direct summand of $Y_i$ and $Z_i$. Write $Y_i = Y'_i \oplus Y''_i$ and $Z_i = Y'_i \oplus Z'_i$.

If $\tau^{-1}X_i \mid X'_i$, then let $X''_i$ be the module so that $X'_i = \tau^{-1}X_i \oplus X''_i$ and set $Y''_i := 0$. Otherwise, set $X''_i := X'_i$ and $Y''_i := \tau^{-1}X_i$. 
The following equations hold:

\[ X_{i+1} \oplus X'_{i+1} = X''_i \oplus Z'_{i+1} \]
\[ Y_{i+1} = Y''_i \oplus Y'''_{i+1}. \]

\[(S4) \text{ If for any } 1 \leq i \leq m, \text{ the module } X_i \oplus X'_i \text{ has an injective direct summand } I, \text{ then } I \mid U. \]

\[(S5) X_m \oplus X'_m \text{ is a direct summand of } U. \]

In Chapter 4, we proceed to consider a much broader case: abelian length categories, a generalization of module categories. On the other hand, the question that we answer is much less general: we ask, in which cases such a category is of colocal type (that is, every subobject of an indecomposable object is itself indecomposable). Partly, we can also answer the question, in which cases the lattice of full, additive subobject closed subcategories is distributive.

To state the answers, we first need some notation:

**Definition 1.2.3.** For all simple objects \( S, T \in \mathcal{A} \) let

\[ d^1_S(S, T) := \dim_{\text{End}(S)^{\text{op}}} \text{Ext}^1(S, T) \]

and

\[ d^1_T(S, T) := \dim_{\text{End}(T)} \text{Ext}^1(S, T). \]

Then we can show the following. For simplicity, we are equating objects with isomorphism classes of objects:

**Theorem 1.2.4.** The category \( \mathcal{A} \) is of colocal type if and only if the following conditions hold:

\[(C1) \text{ For all simple objects } S \in \mathcal{A} \]
\[ \sum_{T \text{ simple}} d^1_T(S, T) \leq 1. \]

\[(C2) \text{ For all simple objects } S \in \mathcal{A} \]
\[ \sum_{T \text{ simple}} d^1_T(T, S) \leq 2. \]
(C3) If there is a simple object $S'$ with $\mathsf{Ext}^1(S, S') \neq 0$, let $\mathcal{T}$ be the set of simple objects $T$ for which $d^1_T(T, S) \neq 0$ and there is an indecomposable object $Z$ of length 3 with $\text{top } Z \cong T$ and $\text{soc } Z \cong S'$. Then

$$\sum_{T \in \mathcal{T}} d^1_T(T, S) \leq 1.$$ 

While the last condition is more complicated than the first two, there are several ways to state it. In particular, it is often equivalent to a condition on the 2-extensions between simple objects:

**Proposition 1.2.5.** Suppose that (C1) holds for all simple objects in $\mathcal{A}$.

For fixed simple objects $S$ and $S'$ with $\mathsf{Ext}^1(S, S') \neq 0$, the following classes of objects are the same:

(a) the class of simple objects $T$ so that $d^1_T(T, S) \neq 0$ and there is some indecomposable object $Z$ of length 3 with $\text{soc } Z \cong S'$ and $\text{top } Z \cong T$.

(b) the class of simple objects $T$ so that $d^1_T(T, S) \neq 0$ and there is some indecomposable object $Z$ of length greater or equal 3 with $\text{soc } Z \cong S'$ and $\text{top } Z \cong T$.

If $\mathsf{Ext}(S', S') = 0$, then these classes are the same as

(c) the class of simple objects $T$ so that $d^1_T(T, S) \neq 0$ and there is some indecomposable object $Z$ with $\text{soc } Z \cong S'$ and $\text{top } Z \cong T$.

If $S'$ is not part of an oriented cycle in the $\mathsf{Ext}$-quiver of $\mathcal{A}$, then this class is even the same as

(d) the class of simple objects $T$ so that $d^1_T(T, S) \neq 0$ and $\mathsf{Ext}^2(T, S') = 0$.

For all abelian length categories $\mathcal{A}$ of colocal type, the lattice $S(\mathcal{A})$ is distributive.

Finally, we look at the categories of the form $\mathcal{A} := \mathsf{mod } A$ for some Artin algebra $A$ over an algebraically closed field $k$. If $\mathcal{A}$ is of colocal type, then it is equivalent to $\mathsf{mod } kQ/I$ for an especially simple quiver $Q$ with admissible ideal $I$ and we can completely describe the lattice $S(\mathcal{A})$ up to isomorphism:

For every vertex $m$ in $Q$, consider the paths that end in $m$ and do not contain any relation in $I$. Under these paths, either one or two are maximal.

If there is only one, then we denote its length with $k_m$ and set $l_m := 0$.

If there are two maximal paths, we denote their lengths with $k_m$ and $l_m$.

For a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ of a natural number, the *Young diagram* of $\lambda$ is an array of squares with $n$ rows and exactly $\lambda_i$ squares in the $i$-th row.
1.3. OUTLINE

The partitions of natural numbers form a lattice $Y$, ordered by the inclusion order on the Young diagrams. It is called Young's lattice.

Denote by $Y^{m,n}$ the sublattice of Young's lattice that contains exactly the partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{m'})$ where $\lambda_i \leq n$ for all $1 \leq i \leq m'$ and $m' \leq m$.

Then we get the following lattice isomorphism:

**Theorem 1.2.6.** Suppose $\text{mod } A \equiv \text{mod } kQ/I$ with quiver $Q = (Q_0, Q_1)$ and admissible ideal $I$. If $A$ is of colocal type, then

$$S(\text{mod } A) \cong \prod_{m \in Q_0} Y^{k_{m+1}, l_{m+1}}.$$ 

1.3 Outline

In the second chapter of this thesis, we collect definitions and results needed for the formulation and proofs of our results. We start with an overview of abelian categories, in particular module categories. Then we concentrate on Auslander-Reiten sequences, the main tool to prove the first theorem in this thesis. The next section introduces hereditary algebras and one of their most important examples: quiver algebras.

In the fourth section we define the Weyl group. In a slight deviation of the usual practice, we give (and work with) the definition of the Weyl group as a Coxeter group, thus not defining the reflections on $A$, except as generators of the Weyl group.

We conclude this chapter with a section about string algebras, which are the path algebras of quivers with relations that have a special form. Since the modules over these algebras are well known, they are very useful for some of the proofs in the fourth chapter.

In the third chapter, we prove that there is a natural bijection between the elements of the Weyl group of $A$ and the full, additive cofinite submodule closed subcategories of $\text{mod } A$. Oppermann, Reiten and Thomas have shown this in [14] for algebraically closed fields and finite fields. While we use the same bijection, we give a different method of proof that does not depend on the field.

First, we define an order on the Weyl group and show some properties of this order. The next section is devoted to an algorithm which, given a preinjective module $M$, constructs all modules which contain $M$ as a submodule.

In the third section, we show how the structure of the Weyl group is connected to the submodule relations between preinjective modules. In the next sections, after proving two auxiliary results, we show first that the
bijection is well defined and then that it is surjective. Since the injectivity
is clear by definition, this concludes the proof. In the last section of this
chapter, we draw some corollaries.

The fourth chapter is devoted to abelian length categories $\mathcal{A}$ where $S(\mathcal{A})$
is distributive. In particular, we characterize abelian length categories of
colocal type.

First, we show that the distributivity of the lattice is equivalent to a
simple condition on submodule relations in $\mathcal{A}$. Then we introduce categories
of colocal type and prove that $S(\mathcal{A})$ is distributive if $\mathcal{A}$ is of colocal type.

In the third section, we prove that certain conditions on the $\text{Ext}$-quiver
of $\mathcal{A}$ must be fulfilled if $S(\mathcal{A})$ is distributive; stricter conditions have to be
fulfilled if $\mathcal{A}$ is of colocal type.

In the next section, we collect some auxiliary lemmas about 2-extensions.
We need these to show in the fifth section that several different formulations
of a condition are equivalent under certain assumptions. Afterwards, we
prove that abelian length categories of colocal type fulfil this condition.

In the sixth section, we show that the conditions formulated in the third
and fifth section are even sufficient for $\mathcal{A}$ being of colocal type. This proof
also draws on the auxiliary lemmas in the fourth section.

We complete this chapter with a description of the lattice $S(\mathcal{A})$: First we
show that it is a Cartesian product of certain sublattices. Then, we take a
closer look at categories $\mathcal{A}$ of colocal type which are equivalent to $\text{mod} \ kQ/I$
for some field $k$, quiver $Q$ and admissible ideal $I$, and see that $S(\mathcal{A})$ has an
especially simple form.
2 Abelian length categories and Artin algebras

We are interested in abelian length categories, in particular in the module categories of Artin algebras. For a more detailed introduction into abelian categories, see for example [8]; for an introduction into Artin algebras and Auslander-Reiten theory consult for example [2].

2.1 Abelian categories and module categories

We start by giving the definition of an additive category, which we need to define abelian categories:

Definition 2.1.1. A category $\mathcal{C}$ is called additive if

- For all objects $A, B \in \mathcal{C}$, the morphism set $\text{Hom}(A, B)$ is an abelian group.
- For all objects $A, B, C \in \mathcal{C}$, the composition morphism
  \[ \text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C) \]  
  (2.1)
  is bilinear over $\mathbb{Z}$.
- The category $\mathcal{C}$ has finite sums.

A stronger concept is the following, see [21], Definition 3.3.4:

Definition 2.1.2. An additive category $\mathcal{C}$ is called abelian if the following three conditions hold:

- Every morphism has a kernel and a cokernel.
- Every monomorphism is the kernel of a morphism.
• Every epimorphism is the cokernel of a morphism.

Abelian categories have pushouts and pullbacks:

**Definition 2.1.3.** Let \( V, X_1, X_2 \) be objects in an abelian category \( \mathcal{A} \) and \( f_1 : V \to X_1, f_2 : V \to X_2 \) be morphisms. Then there exist an object \( Y \) and morphisms \( g_1, g_2 \) so that the following diagram is commutative

\[
\begin{array}{ccc}
V & \xrightarrow{f_1} & X_1 \\
f_2 \downarrow & & \downarrow g_1 \\
X_2 & \xrightarrow{g_2} & Y
\end{array}
\]

and for every object \( Z \) with morphisms \( g'_1 : X_1 \to Z, g'_2 : X_2 \to Z \) which form a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f_1} & X_1 \\
f_2 \downarrow & & \downarrow g'_1 \\
X_2 & \xrightarrow{g'_2} & Z
\end{array}
\]

there is a unique morphism \( \phi : Y \to Z \) with \( \phi g_1 = g'_1 \) and \( \phi g_2 = g'_2 \).

If \( f_1 \) is a monomorphism, then \( g_2 \) is also a monomorphism.

The diagram (2.2) is called a *pushout*.

Dually, for two morphisms \( g_1 : X_1 \to Y, g_2 : X_2 \to Y \), there exists a *pullback*, that is, a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f_1} & X_1 \\
f_2 \downarrow & & \downarrow g_1 \\
X_2 & \xrightarrow{g_2} & Y
\end{array}
\]

so that for every object \( U \) with morphisms \( f'_1 : U \to X_1, f'_2 : U \to X_2 \), which form a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f'_1} & X_1 \\
f'_2 \downarrow & & \downarrow g_1 \\
X_2 & \xrightarrow{g_2} & Y
\end{array}
\]

there is a unique morphism \( \phi : U \to V \) with \( f_1 \phi = f'_1 \) and \( f_2 \phi = f'_2 \).

If \( g_1 \) is an epimorphism, then \( f_2 \) is also an epimorphism.
Proposition 2.1.4. Let $V, X_1, X_2, Y \in \text{mod } A$ with morphisms $f_1 : V \to X_1$, $f_2 : V \to X_2$, $g_1 : X_1 \to Y$, $g_2 : X_2 \to Y$. Then the following statements are equivalent:

1. The diagram

$$
\begin{array}{ccc}
V & \xrightarrow{f_1} & X_1 \\
\downarrow{f_2} & & \downarrow{g_1} \\
X_2 & \xrightarrow{g_2} & Y
\end{array}
$$

is a pushout and a pullback.

2. The sequence

$$
0 \longrightarrow V \xrightarrow{[-f_1]} X_1 \oplus X_2 \xrightarrow{[g_1 \ g_2]} Y \longrightarrow 0
$$

is exact.

In fact, if $[-f_1]$ is a monomorphism, then it suffices to demand that (2.3) is a pushout. If $[g_1 \ g_2]$ is an epimorphism, then it suffices to demand that (2.3) is a pullback.

Pushouts and pullbacks have the following important property:

Proposition 2.1.5. If

and

are both pullbacks (pushouts), then the square

$$
\begin{array}{ccc}
V & \xrightarrow{g e_1} & Y_2 \\
\downarrow{e_2} & & \downarrow{h_2} \\
X_2 & \xrightarrow{h_1 f_2} & Z
\end{array}
$$

is itself a pullback (pushout).
Recall that an object is called *simple* if it does not contain proper, non-zero subobjects.

In this thesis, we consider *abelian length categories*, that is, abelian categories where every object is of finite length and the isomorphism classes of objects form a set (see for example [7], p. 81).

**Definition 2.1.6.** Let $A$ be an abelian category. An object $X \in A$ is said to be of *finite length* if there is a filtration $X = X_0 \supset X_1 \supset \cdots \supset X_n = 0$ so that $X_{i-1}/X_i$ is simple for all $1 \leq i \leq n$. We call $l(X) := n$ the *length* of $X$.

**Definition 2.1.7.** An object of finite length is called *indecomposable* if it cannot be written as a direct sum of proper subobjects.

Let $A$ be an Artin algebra. Then $\text{mod } A$, the category of finitely generated right modules over $A$, is an abelian length category.

On the other hand, we have the following, see for example [21], Theorem 3.3.6:

**Theorem 2.1.8.** Every abelian category is a full subcategory of a category $A - \text{Mod}$ for some algebra $A$, where $A - \text{Mod}$ denotes the left modules over $A$.

For an object $X \in A$ define the *Loewy length* of $X$ to be the smallest $n \in \mathbb{N}$ with a filtration

$$X = X_0 \supset X_1 \supset \cdots \supset X_n = 0$$

so that $X_{i-1}/X_i$ is semisimple (i.e. a direct summand of simple objects) for all $1 \leq i \leq n$.

Then the following holds, see [7], 8.2:

**Theorem 2.1.9.** The abelian length category $A$ is equivalent to the module category of an Artin ring if and only if

1. $A$ has only finitely many simple objects.
2. $\dim_{\text{End}(T)}(S, T) < \infty$ for all simple objects $S, T \in A$.
3. The supremum of the Loewy lengths of the objects in $A$ is finite.

Denote by $A^{op}$ the opposite algebra of $A$, that is, the algebra where left and right multiplication are exchanged (see for example [21], Definition 1.1.7).

Then the left modules over $A^{op}$ are just the right modules over $A$:

**Proposition 2.1.10.** There is a duality $D : \text{mod}(A^{op}) \to \text{mod } A$. 
Thus, the following property of finitely generated modules generalizes to abelian length categories:

**Definition 2.1.11.** Let \( \mathcal{A} \) be an abelian length category and \( X \) an object in \( \mathcal{A} \). Then there are indecomposable objects \( X_1, \ldots, X_n \) so that

\[
X \cong \bigoplus_{i=1}^{n} X_i.
\]

The objects \( X_1, \ldots, X_n \) are unique up to order and isomorphism.

If the object \( Y \) is isomorphic to a direct summand of \( X \), we will use the notation \( Y \mid X \). Accordingly, we will write \( Y \nmid X \) if \( Y \) is not isomorphic to a direct summand of \( X \).

Now we define subcategories:

**Definition 2.1.12.** We call \( \mathcal{C}' \) a subcategory of \( \mathcal{C} \) if all objects and morphisms of \( \mathcal{C}' \) are objects and morphisms in \( \mathcal{C} \). Such a subcategory is called full if for all objects \( A, B \in \mathcal{C}' \), the equality \( \text{Hom}_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{C}'}(A, B) \) holds.

We are interested in the following kind of subcategories:

**Definition 2.1.13.** A full, additive subcategory \( \mathcal{C} \) of an abelian length category \( \mathcal{A} \) is called subobject closed if for every object \( X \in \mathcal{C} \) and all subobjects \( X' \) of \( X \), we have \( X' \in \mathcal{C} \).

**Definition 2.1.14.** A full, additive subcategory \( \mathcal{C} \) of the module category \( \text{mod} \ A \) of an Artin algebra \( A \) is called cofinite if all indecomposable modules, except finitely many, lie in \( \mathcal{C} \).

As in [3], Definition 2.6.1, for \( n \in \mathbb{N} \), we can define an \( n \)-fold extension of an object \( X \) by an object \( X' \) as an exact sequence

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X' & \longrightarrow & X_{n-1} & \longrightarrow & \ldots & \longrightarrow & X_0 & \longrightarrow & X & \longrightarrow & 0.
\end{array}
\]

A map between two extensions is a commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X' & \longrightarrow & X_{n-1} & \longrightarrow & \ldots & \longrightarrow & X_0 & \longrightarrow & X & \longrightarrow & 0.
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X'_0 & \longrightarrow & X'_{n-1} & \longrightarrow & \ldots & \longrightarrow & X'_0 & \longrightarrow & X & \longrightarrow & 0.
\end{array}
\]

By adding symmetry and transitivity, this can be completed to an equivalence relation and we can define \( \text{Ext}^n(X, X') \) to be the set of equivalence classes of \( n \)-fold extensions of \( X \) by \( X' \).
These are groups; in particular, for $\text{Ext}^1(X, X')$, the abelian group structure corresponds to the Baer sum, see [21], section 1.8.2:

For

$$
\eta_1 : 0 \longrightarrow X' \xrightarrow{f_1} X_0 \xrightarrow{g_1} X \longrightarrow 0
$$

and

$$
\eta_2 : 0 \longrightarrow X' \xrightarrow{f_2} X'_0 \xrightarrow{g_2} X \longrightarrow 0
$$

there is an object $Z_1$ with a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{f_1} & X' \oplus X' \\
& | & \\
& \uparrow & \\
0 & \xrightarrow{[f_1 \ 0 \ f_2]} & X_0 \oplus X'_0 \\
& | & \\
& \uparrow & \\
0 & \xrightarrow{[g_1 \ 0 \ g_2]} & X \oplus X \\
& | & \\
& \uparrow & \\
0 & \xrightarrow{[id \ id]} & 0
\end{array}
\]

and an object $Z$ with a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{X' \oplus X'} & Z_1 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{X' \oplus X'} & Z \\
\downarrow & & \downarrow \\
0 & \xrightarrow{X \oplus X} & 0
\end{array}
\]

The object $Z_1$ can be found by taking the pullback, while $Z$ can be found via the pushout.

Then

$$
0 \longrightarrow X' \longrightarrow Z \longrightarrow X \longrightarrow 0
$$

is the exact sequence $\eta_1 + \eta_2$.

It is possible to view $\text{Ext}^1(X, X')$ as a module over $\text{End}(X', X')$ and over $\text{End}(X, X)^{op}$.

### 2.2 Auslander-Reiten sequences

An important tool for our proofs are Auslander-Reiten sequences, which we introduce in this section.

Let $A$ be an Artin algebra. First, we need the definition of certain special modules:

**Definition 2.2.1.** A module $P \in \text{mod } A$ is **projective** if for any epimorphism $g : X \rightarrow Y$ with $X, Y \in \text{mod } A$ and morphism $h : P \rightarrow Y$, there is a morphism $s : P \rightarrow X$ such that $gs = h$.

A module $I \in \text{mod } A$ is **injective** if for any monomorphism $f : X \rightarrow Y$ with $X, Y \in \text{mod } A$ and morphism $h : X \rightarrow I$, there is a morphism $s : Y \rightarrow I$ such that $sf = h$. 
Now, we can define projective covers:

**Definition 2.2.2.** Let $X, Y, W \in \mod A$. An epimorphism $f : X \to Y$ is called an *essential epimorphism* if for all morphisms $g : W \to X$ the morphism $fg : W \to Y$ is an epimorphism if and only if $g$ is an epimorphism.

**Definition 2.2.3.** Let $X \in \mod A$. A *projective cover* of $X$ is an essential epimorphism $P \to X$ so that $P$ is a projective module.

An analogous definition exists for injective modules:

**Definition 2.2.4.** For $X \subset Y \in \mod A$, we say that $Y$ is an *essential extension* of $X$ if $W \cap X \neq 0$ for all submodules $W$ of $Y$. A monomorphism $X \to I$ is called an *injective envelope* of $X$ if $I$ is injective and an essential extension of $B$.

For simplicity, when referring to the modules $P$ and $I$, we will also call them projective covers and injective envelopes, respectively.

We have the following:

**Theorem 2.2.5.** Every module $X \in \mod A$ has projective covers and injective envelopes which are unique up to isomorphism.

The indecomposable simple, projective and injective modules are connected:

**Proposition 2.2.6.** Every Artin algebra $A$ has a finite number $n$ of non-isomorphic simple modules $S_1, \ldots, S_n$. Their projective covers $P_1, \ldots, P_n$ are a complete list of non-isomorphic indecomposable projective $A$-modules.

Their injective envelopes $I_1, \ldots, I_n$ are a complete list of non-isomorphic indecomposable injective $A$-modules. For all $1 \leq i \leq n$, we have $\soc I_i = S_i$.

Using the duality $D : \mod (A^{\op}) \to \mod A$, we can define the *Auslander-Reiten translation* $\tau$:

**Proposition 2.2.7.** There is a map $\Tr : \mod A \to \mod (A^{\op})$, so that for $\tau = D\Tr$ the following holds:

1. $\tau \left( \bigoplus_{i=1}^n M_i \right) \cong \bigoplus_{i=1}^n \tau M_i$ for $M_1, \ldots, M_n \in \mod A$.
2. $\tau M = 0$ if and only if $M$ is projective.
3. $\tau^{-1} M = 0$ if and only if $M$ is injective.
4. $\tau M$ is non-injective for all $M \in \mod A$. 
5. If no direct summand of $M, N$ is projective, then $\tau M \cong \tau N$ if and only if $M \cong N$.

Next, we introduce almost split morphisms:

**Definition 2.2.8.** Let $X, Y \in \text{mod } A$. A morphism $f : X \to Y$ is called a *split epimorphism* if the identity $1_Y : Y \to Y$ factors through $f$, that is, if there is some $g : Y \to X$ so that $1_Y = fg$.

The morphism $f$ is called a *split monomorphism* if the identity $1_X : X \to X$ factors through $f$.

**Definition 2.2.9.** A morphism $f : X \to Y$ is called *right almost split* if the following conditions are fulfilled:

1. $f$ is not a split epimorphism.
2. Any morphism $M \to Y$ with $M \in \text{mod } A$, which is not a split epimorphism factors through $f$.

A morphism $f : X \to Y$ is called *left almost split* if the following conditions are fulfilled:

1. $f$ is not a split monomorphism.
2. Any morphism $X \to M$ with $M \in \text{mod } A$, which is not a split monomorphism factors through $f$.

Especially important are exact sequences where the morphisms are almost split:

**Definition 2.2.10.** Let $X, Y, Z \in \text{mod } A$. Then the exact sequence

\[
0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0
\]

is called an *almost split sequence* or *Auslander-Reiten sequence* if $f$ is left almost split and $g$ is right almost split.

For every indecomposable, non-injective module $X$, there is an Auslander-Reiten sequence (2.4). The same holds for every indecomposable, non-projective module $Z$:

**Theorem 2.2.11.** Let $X, Z \in \text{mod } A$. Then the following assertions are equivalent:

1. There is some module $Y \in \text{mod } A$ with an Auslander-Reiten sequence

\[
0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0
\]
2.2. AUSLANDER-REITEN SEQUENCES

2. $X$ is indecomposable, non-injective and $Z \cong \tau^{-1}X$.

3. $Z$ is indecomposable, non-projective and $X \cong \tau Z$.

In fact, we have the following:

**Theorem 2.2.12.** Auslander-Reiten sequences are unique up to isomorphism, that is, if

\[
0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0
\]

and

\[
0 \longrightarrow X \xrightarrow{f'} Y' \xrightarrow{g'} Z \longrightarrow 0
\]

are Auslander-Reiten sequences, then there is a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & X & \xrightarrow{g} & Y & \xrightarrow{g'} & Z & \xrightarrow{0} \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & \\
0 & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \xrightarrow{0} & & \\
\end{array}
\]

There is another important kind of morphisms:

**Definition 2.2.13.** A morphism $f : X \to Y$ is called **irreducible** if the following conditions are fulfilled:

1. $f$ is neither a split epimorphism nor a split monomorphism.

2. If there are $t : X \to M$ and $s : M \to Y$ so that $f = st$, then either $s$ is a split monomorphism or $t$ is a split epimorphism.

An irreducible morphism is either a monomorphism or an epimorphism.

Irreducible morphisms are the components of almost split morphisms:

**Theorem 2.2.14.** (a) Let $X$ be an indecomposable, non-injective module and $f : X \to Y$ a morphism. Then $f$ is irreducible if and only if there are some modules $Y'$, $Z$, morphisms $f' : X \to Y'$, $g : Y \to Z$, $g' : Y' \to Z$ and an Auslander-Reiten sequence

\[
0 \longrightarrow X \xrightarrow{f} Y \oplus Y' \xrightarrow{g, g'} Z \longrightarrow 0 .
\]

(b) Dually, let $Z$ be an indecomposable, non-projective module and $g : Y \to Z$ a morphism. Then $g$ is irreducible if and only if there are some modules $X, Y'$ and morphisms $f : X \to Y$, $f' : X \to Y'$, $g : Y \to Z$, $g' : Y' \to Z$ so that (2.5) is an Auslander-Reiten sequence.
2.3 Hereditary algebras and quivers

In this section, we take $A$ to be a hereditary algebra, that is, an algebra, where all left ideals are projective. For such an algebra, every submodule of a projective module is itself projective. We collect some properties that we need in particular in Chapter 3.

Equivalently, we can give the following definition (see [15], Section 4.1):

**Definition 2.3.1.** An algebra $A$ is *hereditary* if for all simple modules $S, T \in \text{mod } A$, we have $\text{Ext}^2(S, T) = 0$.

We are mainly working with the following kind of modules:

**Definition 2.3.2.** An indecomposable module $X \in \text{mod } A$ is called preprojective if there is some non-zero projective module $P$ and a non-negative integer $n$ so that $\tau^n X = P$.

We set $\nu(X) := n$.

An arbitrary $M \in \text{mod } A$ is called *preprojective* if its indecomposable direct summands are preprojective.

An indecomposable module $X \in \text{mod } A$ is called preinjective if there is some non-zero injective module $I$ and a non-negative integer $n$ so that $\tau^{-n} X = I$.

We set $\mu(X) := n$.

An arbitrary $M \in \text{mod } A$ is called *preinjective* if its indecomposable direct summands are preinjective.

Let $\mathcal{P}$ be the full subcategory of $\text{mod } A$ that consists of the preprojective modules and $\mathcal{I}$ be the full subcategory of $\text{mod } A$ that consists of the preinjective modules.

**Proposition 2.3.3.** The duality $D : \text{mod } A \rightarrow \text{mod } A^{\text{op}}$ induces a duality between $\mathcal{P}$ and $\mathcal{I}$.

So results for preprojective modules induce analogous results for preinjective modules and the other way around.

We need the following properties of preinjective modules:

**Proposition 2.3.4.** Let $M$ be in $\mathcal{P}$ or $\mathcal{I}$. If there are indecomposable modules $M = M_0, M_1, \ldots, M_{n-1}, M_n = M$ and non-zero morphisms $f_i : M_{i-1} \rightarrow M_i$ for $1 \leq i \leq n$, then $f_i$ is an isomorphism for all $1 \leq i \leq n$.

**Lemma 2.3.5.** Let $A$ be a hereditary Artin algebra and $X, Y \in \text{mod } A$ be indecomposable. If there is an irreducible morphism $f : X \rightarrow Y$, then $X \in \mathcal{I}$ if and only if $Y \in \mathcal{I}$.
2.3. HEREDITARY ALGEBRAS AND QUIVERS

Lemma 2.3.6. Let $A$ be a hereditary Artin algebra and $X, Y \in \mathcal{I}$ be indecomposable. If there is an irreducible morphism $f : X \rightarrow Y$, then

1. If $X$ is injective, then $Y$ is injective.

2. If $X$ is not injective, then $0 \leq \mu(X) - 1 \leq \mu(Y) \leq \mu(X)$.

Proposition 2.3.7. Let $f : X \rightarrow Y$ be an irreducible morphism. Then the following holds:

(a) If no direct summand of $X$ is projective, then $\tau f : \tau X \rightarrow \tau Y$ is irreducible.

(b) If no direct summand of $Y$ is injective, then $\tau^{-1}X \rightarrow \tau^{-1}Y$ is irreducible.

(c) The translation $\tau$ preserves monomorphisms.

Lemma 2.3.8. Let $S_1, \ldots, S_n$ be a complete list of simple $A$-modules up to isomorphism. Then for all $1 \leq i, j \leq n$ either $\text{Ext}^1(S_i, S_j) = 0$ or $\text{Ext}^1(S_j, S_i) = 0$.

Furthermore, $\text{Ext}^1(S_i, S_j) \neq 0$ if and only if there is an irreducible morphism $I_j \rightarrow I_i$ for the injective envelopes $I_i, I_j$ of $S_i, S_j$.

Lemma 2.3.9. Let $I_i, I_j$ be the injective envelopes of $S_i, S_j$ with an irreducible morphism $I_j \rightarrow I_i$. Then the multiplicity of $I_j$ in the Auslander-Reiten sequence that ends in $I_i$ is $\dim_{\text{End}_A(S_i)}(\text{Ext}^1(S_i, S_j))$. On the other hand, the multiplicity of $\tau I_i$ in the Auslander-Reiten sequence that ends in $I_j$ is $\dim_{\text{End}_A(S_j)}(\text{Ext}^1(S_i, S_j))$.

An important example for hereditary algebras are path algebras of quivers:

Definition 2.3.10. A quiver $Q = (Q_0, Q_1)$ is an oriented graph where $Q_0$ is the set of vertices, while $Q_1$ is the set of arrows between vertices. If $\alpha : i \rightarrow j$ is an arrow, its start point is $s(\alpha) = i$ and its end point is $e(\alpha) = j$.

For all $i \in Q_0$, there is a trivial path $e_i$ with $s(e_i) = e(e_i) = i$.

A path in this quiver is either a trivial path or a sequence $p = \alpha_n \ldots \alpha_1$ of arrows so that $e(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i < n$. In this case, we define $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_n)$.

An oriented cycle is a non-trivial path so that $e(p) = s(p)$.

A quiver is called finite if $Q_0$ is a finite set.

It is natural to define the following algebra:
Definition 2.3.11. Let $k$ be a field and $kQ$ be the vector space with the paths of $Q$ as a basis. Then there is an algebra structure on $kQ$ induced by the concatenation of paths. That is, the product of $\alpha_n \ldots \alpha_1$ and $\alpha'_n \ldots \alpha'_1$ is $\alpha_n \ldots \alpha_1 \alpha'_n \ldots \alpha'_1$ if $s(\alpha_1) = e(\alpha'_n)$ and 0 otherwise.

Analogously, for every path $p$, we have $e_i p = p$ if $e(p) = i$ and 0 otherwise; $p e_i = p$ if $s(p) = i$ and 0 otherwise.

The module category of every Artin algebra is equivalent to the module category of a basic Artin algebra. So the following result in fact describes the module categories of all hereditary Artin algebras over algebraically closed fields:

**Proposition 2.3.12.** Let $k$ be a field and $Q$ be a finite Quiver without oriented cycles. Then $kQ$ is a hereditary Artin algebra.

If $k$ is algebraically closed, then for every basic hereditary Artin algebra $A$ over $k$, there is some quiver $Q$ so that $A$ is isomorphic to $kQ$.

We can furthermore define representations over quivers:

**Definition 2.3.13.** Let $Q = (Q_0, Q_1)$ be a quiver and $k$ a field. A representation $(V, f)$ of $Q$ over $k$ is a set of vector spaces $\{V_i \mid i \in Q_0\}$ over $k$ together with linear maps $f_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$.

A morphism $h : (V, f) \to (V', f')$ between representations is a collection $\{h_i : V_i \to V'_i\}_{i \in Q_0}$ of linear maps so that the diagrams

\[
\begin{array}{ccc}
V_i & \xrightarrow{h_i} & V'_i \\
\downarrow f_\alpha & & \downarrow f'_\alpha \\
V_j & \xrightarrow{h_j} & V'_j
\end{array}
\]

commute for all arrows $\alpha : i \to j$ in $Q_1$.

The composition of morphisms is of course induced by the composition of linear maps.

Then we get the following:

**Proposition 2.3.14.** The category of representations of $Q$ over $k$ is equivalent to the category of finite dimensional modules over $kQ$.

For every abelian length category $\mathcal{A}$, we can define its Ext-quiver: The vertices are given by a complete set of non-isomorphic simple objects in $\mathcal{A}$. If $S$ and $T$ are simple objects, there is an arrow $S \to T$ if $\text{Ext}^1(S, T) \neq 0$. 
If $\mathcal{A}$ is equivalent to the module category of a hereditary Artin algebra, then its Ext-quiver has no oriented cycle.

Furthermore, for every Artin algebra $A$, one can define its Auslander-Reiten quiver: Its vertices are the isomorphism classes of modules in $\text{mod} \ A$ and there is an arrow between two isomorphism classes $[M]$ and $[N]$ if there is an irreducible morphism $M \to N$.

Attached to such an arrow is the label $(a, b)$, where $a$ is the maximal positive integer with an irreducible morphism $M^a \to N$, while $b$ is the maximal positive integer with an irreducible morphism $M \to N^b$.

Let $A$ be a hereditary algebra and $[M] \to [N]$ have the valuation $(a, b)$. If $\tau^a M$ and $\tau^b N$ are non-zero for some integer $\alpha$, then the valuation of $[\tau^a M] \to [\tau^b N]$ is also $(a, b)$. If $\tau^{b+1} N$ and $\tau^b M$ are non-zero for some integer $\beta$, then $[\tau^{b+1} N] \to [\tau^b M]$ has the valuation $(b, a)$.

The following holds:

**Proposition 2.3.15.** Let $A$ be an Artin algebra. If the Auslander-Reiten quiver of $A$ has a finite component, then $A$ is representation finite.

Moreover, if $A$ is hereditary and representation finite, then every module in $\text{mod} \ A$ is both preinjective and preprojective and the number of components of the Auslander-Reiten quiver of $A$ is the same as the number of blocks of $A$. So if $A$ is indecomposable as an algebra, then the Auslander-Reiten quiver of $A$ consists only of one component.

For simplicity, when drawing an Auslander-Reiten quiver, we will omit all labels $(a, b)$ where $a = b = 1$ and use representatives of the isomorphism classes as vertices.

In the same vein, we will not always differentiate between modules (or objects) and isomorphism classes of modules (or objects) when the meaning is clear. For example, when we have a complete set $I_1, \ldots, I_n$ of non-isomorphic injective modules over a hereditary Artin algebra $A$, we will simply call $I_1, \ldots, I_n$ the injective modules over $A$ and say that every indecomposable, preinjective module $M$ is of the form $\tau^r I_i$ for some $1 \leq i \leq n$ and some $r \in \mathbb{N}$, when it actually is only isomorphic to such a module.

### 2.4 The Weyl group as a Coxeter group

We define words following [13]:

**Definition 2.4.1.** Let $S$ be a set. We call $S$ an alphabet and its elements letters. A word over the alphabet $S$ is a finite sequence 

$$(s_1, s_2, \ldots, s_n), s_i \in S.$$
The product of two words is just the concatenation of the sequences. This product is associative and by identifying a letter \( s \in S \) with the sequence \((s)\), we can write the word \((s_1, s_2, \ldots, s_n)\) as the product \(s_1s_2\ldots s_n\). The neutral element for this product is the empty word, which we accordingly denote as \(1\). Thus, the set of words over \(S\) together with the concatenation forms a monoid \(S^*\).

If \(w := s_1s_2\ldots s_n\) is a word over \(S\), then \(l(w) := n\) is called the length of \(w\). Furthermore, a word of the form \(v = s_{i_1}s_{i_2}\ldots s_{i_m}\) with

\[
1 \leq i_1 < i_2 < \cdots < i_m \leq n
\]

and \(m \leq n\) is a subword of \(w\).

If \(v = s_1s_2\ldots s_m\) with \(m \leq n\), then we say that \(v\) is an initial subword of \(w\).

An introduction into Coxeter groups can be found in [4]. We only need the following properties:

**Definition 2.4.2.** Let \(S\) be a set and \(W\) a group generated by \(S\). Then \(W\) is called a Coxeter group if all relations have the form \((ss')^m(s,s') = 1\) with \(s,s' \in S\) so that

1. \(m(s, s') = 1\) if and only if \(s = s'\).

2. If \(m(s, s')\) exists, then \(m(s', s)\) also exists and \(m(s, s') = m(s', s)\).

If there is no relation between \(s\) and \(s'\), then we write \(m(s, s') = m(s', s) = \infty\).

We can describe the Coxeter group \(W\) through the monoid \(S^*\):

**Proposition 2.4.3.** Let \(S\) be a set and \(S^*\) the monoid of words over \(S\). Let \(W\) be a Coxeter group generated by \(S\) with relations \((ss')^m(s,s') = 1\).

Set \(\equiv\) to be the equivalence relation on \(S^*\) which is generated by allowing the insertion or deletion of words of the form

\[
(ss')^{m(s,s')} = \underbrace{ss'ss'\ldots ss'}_{2m(s,s') \text{ letters}}
\]

for all \(m(s, s') < \infty\). Then \(S^*/\equiv\) is isomorphic to \(W\).

We will use the following notation:

**Definition 2.4.4.** Set \(\{ss'\}^a := \underbrace{ss'ss'\ldots}_{a \text{ letters}}\).

The next lemma makes it easier to work with the relations:
Lemma 2.4.5. Let $S$, $W$ and $\equiv$ be as in Proposition 2.4.3 and $s,s' \in S$. The equivalence of words $\{s's\}^a \equiv \{ss'\}^a$ holds if and only if $m(s,s')$ is a factor of $a$.

Let $S_1, \ldots, S_n$ with $n \in \mathbb{N}$ be a complete list of non-isomorphic simple modules of the Artin algebra $A$.

We can associate to $A$ a Cartan matrix, as in [2], pp 69, 241 and 288:

Definition 2.4.6. To a hereditary Artin algebra $A$ we associate the Cartan matrix $C = (c_{ij})_{nn}$ of the underlying graph of the quiver $A^\text{op}$.

That is, we set $c_{ii} = 2$. If $i \neq j$ and $\text{Ext}^1(S_i, S_j) = \text{Ext}^1(S_j, S_i) = 0$, then $c_{ij} = c_{ji} = 0$. Finally, if $\text{Ext}^1(S_i, S_j) \neq 0$, set

$$c_{ij} = -\dim\text{End}_A(S_i)^{\text{op}} \text{Ext}^1(S_i, S_j)$$

and

$$c_{ji} = -\dim\text{End}_A(S_j) \text{Ext}^1(S_i, S_j).$$

A description of the Weyl group as a Coxeter group can be found in [11], Proposition 3.13:

Proposition 2.4.7. The Weyl group associated to $A$ with the Cartan matrix $(c_{ij})_{nn}$ is a Coxeter group generated by the reflections $s_1, s_2, \ldots, s_n$ with relations $s_i^2 = 1$ for all $1 \leq i \leq n$ and $(s_is_j)^{m_{ij}} = 1$ for all $i \neq j$, where $m_{ij}$ depends on $c_{ij}c_{ji}$ in the following way:

<table>
<thead>
<tr>
<th>$c_{ij}c_{ji}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{ij}$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can write all relations as $(s_is_j)^{m_{ij}}$ if we set $m_{ij} := 1$ for $i = j$.

Every element of the Weyl group is the equivalence class of several different words over the alphabet $S := \{s_1, s_2, \ldots, s_n\}$. To distinguish between the elements of the Weyl group and the words over $S$, we will always use underlined letters to denote words and normal letters for Weyl group elements.

Remark 2.4.8. For $A = kQ$ with a field $k$ and a quiver $Q$ without oriented cycles, the relations depend only on the edges in the underlying graph of $Q$, see e.g. [14], p. 570:

We have $m_{ij} = 2$ if there is no edge between the vertices $i$ and $j$ and $m_{ij} = 3$ if there is exactly one edge between $i$ and $j$. If there are two or more edges between $i$ and $j$, then $m_{ij} = \infty$.

Example 2.4.9. Let $Q$ be the quiver

$$1 \rightarrow 3 \rightarrow 4 \rightarrow 2$$
The Weyl group of $A = kQ$ is a Coxeter group with the relations $(s_is_j)^{m_{ij}} = 1$ and the following values for $m_{ij}$:

<table>
<thead>
<tr>
<th></th>
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<th>2</th>
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<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

### 2.5 String Algebras

To define string algebras, we need quivers with relations:

**Definition 2.5.1.** Let $k$ be a field and $Q$ a quiver. A *relation* on $Q$ over $k$ is a $k$-linear combination $a_1p_1 + \cdots + a_np_n$ of paths $p_1, \ldots, p_n$ with $a_1, \ldots, a_n \in k$, $e(p_1) = \cdots = e(p_n)$ and $s(p_1) = \cdots = s(p_n)$. Let $I$ be an ideal in $kQ$ generated by a set of relations $\rho$. Then $(Q, \rho)$ is called a *quiver with relations* and $kQ/I$ is called the *path algebra* of $(Q, \rho)$. Furthermore, we call $I$ *admissible* if $J^t \subset I \subset J^{2t}$, for some $t \in \mathbb{N}$, where $J$ is the ideal generated by all arrows.

We can define the representations $(V, f)$ of $(Q, \rho)$ analogously to the representations of $Q$ in Definition 2.3.13, but with the following additional condition:

For a path $p = \alpha_1\alpha_2\ldots\alpha_n$ where $\alpha_i$ is an arrow for $1 \leq i \leq n$, we set $f_p := f_{\alpha_1}f_{\alpha_2}\ldots f_{\alpha_n}$. Then for every relation $a_1p_1 + \cdots + a_np_n$ in $\rho$, we demand $a_1f_{p_1} + \cdots + a_nf_{p_n} = 0$.

As before, $\text{mod} \, kQ/I$ is equivalent to the category of representations of $(Q, \rho)$.

**Remark 2.5.2.** Let $Q$ be a finite quiver and $k$ a field. If $Q$ has no oriented cycles, then to every vertex $i$ corresponds a simple module $S_i \in \text{mod} \, kQ$. If the number of arrows $i \to j$ is denoted by $n$, then

$$\dim_{\text{End}(S_i)^{op}} \text{Ext}^1(S_i, S_j) = \dim_{\text{End}(S_j)} \text{Ext}^1(S_i, S_j) = n.$$
Theorem 2.5.3. Every basic, finite dimensional algebra over an algebraically closed field $k$ is isomorphic to $kQ/I$ for some quiver $Q$ and admissible ideal $I$.

A special kind of quiver algebras are string algebras as described in [5]:

**Definition 2.5.4.** Suppose that $Q$ is a quiver and $I$ an ideal in $kQ$ which is generated by a set of zero relations.

Then $A = kQ/I$ is a *string algebra* if and only if

1. Any vertex of $Q$ is starting point of at most two arrows.

2. Any vertex of $Q$ is end point of at most two arrows.

3. Given an arrow $\beta$, there is at most one arrow $\gamma$ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$.

4. Given an arrow $\gamma$, there is at most one arrow $\beta$ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I$.

5. Given an arrow $\beta_1$, there is some bound $n(\beta)$ such that any path $\beta_1\beta_2\ldots\beta_{n(\beta)}$ contains a subpath in $I$.

6. Given an arrow $\beta$, there is some bound $n'(\beta)$ such that any path $\beta_1\beta_2\ldots\beta_{n'(\beta)}$ with $\beta_{n'(\beta)} = \beta$ contains a subpath in $I$.

**Definition 2.5.5.** We can take the formal inverse $\beta^{-1}$ of an arrow $\beta$ by defining $e(\beta^{-1}) := s(\beta_n)$ and $s(\beta^{-1}) := e(\beta)$.

A **string** is a word $w = \beta_1\beta_2\ldots\beta_n$ so that

- $\beta_i$ is either an arrow or the inverse of an arrow for all $1 \leq i \leq n$
- $s(\beta_i) = e(\beta_{i+1})$ for all $1 \leq i \leq n$
- $w$ does not contain a relation in $I$

The multiplication of strings is analogous to the multiplication of paths of a quiver.

A **band** is a string $w = \beta_1\beta_2\ldots\beta_n$ such that every power of $w$ is defined and does not contain a relation in $I$; furthermore $w$ may not be a power of a string $w' \neq w$.

String algebras are especially useful, since their modules are well known, also from [5]:

Definition 2.5.6. Suppose that \( w = \beta_1 \beta_2 \ldots \beta_n \) is a string. Set \( u(i) = e(\beta_{i+1}) \), for \( 0 \leq i < n \), and \( u(n) = s(\beta_n) \). The string module \( M(w) \) is defined as the representation where for every \( v \in Q_0 \), the vector space \( M(w)_v \) has as basis

\[
\{ z_i \mid u(i) = v \}
\]

with \( z_i \neq z_j \) for \( i \neq j \). If \( \beta_i \) is an arrow, then \( f_{\beta_i}(z_{i-1}) = z_i \), otherwise \( f_{\beta_i^{-1}}(z_i) = z_{i-1} \). For all other arrows \( \alpha \), we have \( f_\alpha = 0 \).

Now suppose that \( w \) is even a band and \( \phi : \mathbb{Z} \to \mathbb{Z} \) is an automorphism on a vector space over \( k \). The band module \( M(w, \phi) \) is defined as the representation with

\[
M(w, \phi)_v = \bigoplus_{e(\beta_{i+1}) = v} Z_i
\]

where \( Z_i = \mathbb{Z} \).

If \( \beta_1 \) is an arrow and \( z \in Z_1 \), then \( f_{\beta_1}(z) = \phi(z_1) \in Z_0 \). If \( \beta_1^{-1} \) is an arrow, then for \( z \in Z_0 \), \( f_{\beta_1^{-1}}(z) = \phi^{-1}(z) \in Z_1 \).

Let \( 2 \leq i \leq n \). If \( \beta_i \) is an arrow and \( z \in Z_i \), then \( f_{\beta_i}(z) = z \in Z_{i-1} \); if \( \beta_i^{-1} \) is an arrow and \( z \in Z_{i-1} \), then \( f_{\beta_i^{-1}}(z) = z \in Z_i \).

For all other arrows \( \alpha \), we have \( f_\alpha = 0 \).

Lemma 2.5.7. Let \( A = kQ \) be a string algebra with a string \( w = \beta_1 \beta_2 \ldots \beta_n \).

1. All \( A \)-modules are isomorphic to a string module or a band module

2. Two string modules \( M(w) \) and \( M(w') \) are isomorphic if and only if \( w = w' \) or \( w' = w^{-1} := \beta_n^{-1} \beta_{n-1}^{-1} \ldots \beta_1^{-1} \).

3. Two band modules \( M(w, \phi) \) and \( M(w', \phi') \) are isomorphic if and only if \( \phi \) and \( \phi' \) are similar and \( w \) or \( w^{-1} \) is a cyclic permutation of \( w' \).

4. No band module is isomorphic to a string module.

In [6], p. 34 there is a result about morphisms between tree modules that reduces very nicely to monomorphisms between string modules:

Lemma 2.5.8. \( M(w) \) is a submodule of \( M(w') \) if and only if there are arrows \( \alpha, \beta \) and strings \( w_1, w_2 \) so that \( w' \) is of the form

\[
w_1 \alpha^{-1} w_2 \beta w_2
\]
or

\[
w_1 \beta w_2
\]
or

\[
w_1 \alpha^{-1} w.
\]
In this chapter, let $A$ be a hereditary Artin algebra over an arbitrary field. We aim to prove that there is a natural bijection between the Weyl group and the set of full additive cofinite submodule closed subcategories of the module category. Oppermann, Reiten and Thomas have shown this in [14] for algebraically closed fields and finite fields. While we use the same bijection, we will give a completely different method of proof that does not depend on the field.

First of all, we regard the Weyl group as a Coxeter group. This allows us to regard the Weyl group elements as equivalence classes of words. In Section 3.1, we define a total order on these words and call the smallest word of each equivalence class leftmost. Then we collect some results about this order.

We conclude Section 3.1 by stating the bijection, which is induced by a map between words of Weyl group elements and sets of preinjective modules. In Section 3.4 to Section 3.6, we will prove that a cofinite, full additive subcategory is submodule closed if and only if a leftmost word is mapped to its complement. Since we can assign a unique leftmost word to every element of the Weyl group, this gives a bijection between the full additive cofinite submodule closed subcategories and the Weyl group.

For this proof, we will use the results of Section 3.2, which is devoted to monomorphisms between preinjective modules. In particular, we give a way to construct all modules that contain a given preinjective module as a submodule. This allows us to draw some lemmas in Section 3.3 about the structure of full additive cofinite submodule closed subcategories and how they are related to the words of Weyl group elements.

In the sections 3.4 to 3.6, we use this to prove inductively that the proposed bijection exists. Finally, we conclude this chapter with some corollaries.

Note that in the following, a submodule closed subcategory will always
mean a full additive submodule closed subcategory of \text{mod} A. Furthermore, we are equating modules with isomorphism classes of modules, since submodule closed subcategories are closed under isomorphisms.

### 3.1 Leftmost words

Let \( A \) be a hereditary Artin algebra and \( \text{mod} A \) the category of finitely presented modules over \( A \). Furthermore, let \( \mathcal{I} \) be the subcategory of \( \text{mod} A \) consisting of all preinjective modules.

We order the simple modules \( S_1, \ldots, S_n \) of \( A \) with injective envelopes \( I_1, \ldots, I_n \) in such a way that \( \text{Hom}(I_i, I_j) = 0 \) if \( i < j \). This is possible by Lemma 2.3.8.

Furthermore, let \( W \) be the Weyl group of \( A \). Denote by

\[
S := \{s_1, s_2, \ldots, s_n\}
\]

the set of generators of \( W \) and by

\[
(s_is_j)^{m_{ij}} = 1
\]

the defining relations of \( W \).

**Definition 3.1.1.** Consider \( \mathcal{N} = (\mathbb{N}_0 \times \{1, 2, \ldots, n\}, <) \), where \( < \) is the lexicographic order: for pairs \( (r, i), (r', j) \in \mathcal{N} \), we have \( (r, i) < (r', j) \) if and only if one of the following holds:

1. \( r < r' \)
2. \( r = r' \) and \( i < j \).

Let \( w = s_{i_1} s_{i_2} \ldots s_{i_m} \) be a word over the alphabet \( S \) and \( 0 = r_1 \leq r_2 \leq \cdots \leq r_m \in \mathbb{N}_0 \) the smallest non-negative integers so that

\[
(r_1, i_1) < (r_2, i_2) < \cdots < (r_m, i_m)
\]

is fulfilled. Then we define

\[
\rho(w) := (r_1, i_1)(r_2, i_2)\ldots(r_m, i_m).
\]

**Example 3.1.2.** Consider the Weyl group of the quiver \( Q \) from Example 2.4.9. If we set \( w := s_2 s_3 s_1 s_3 s_4 s_1 \) then

\[
\rho(w) = (0, 2)(0, 3)(1, 1)(1, 3)(1, 4)(2, 1).
\]
Now we can define a total order \(<_l\) on the words of \(W\); this is again a lexicographic order:

**Definition 3.1.3.** Consider two words \(w, w'\) with
\[
\rho(w) = (r_1, i_1)(r_2, i_2) \ldots (r_m, i_m)
\]
and
\[
\rho(w') = (r'_1, i'_1)(r'_2, i'_2) \ldots (r'_m, i'_m).
\]
We write \(w <_l w'\) if one of the following holds:

1. \(m < m'\)
2. \(m = m'\) and there is a \(j \in \mathbb{N}\) so that
   \[ (r_1, i_1) = (r'_1, i'_1), (r_2, i_2) = (r'_2, i'_2), \ldots, (r_{j-1}, i_{j-1}) = (r'_{j-1}, i'_{j-1}) \]
   and
   \[ (r_j, i_j) < (r'_j, i'_j). \]

Now we define the leftmost word; this definition can be found for example in [1], p. 411:

**Definition 3.1.4.** We call a word \(w\) for \(w \in W\) *leftmost* if for every other word \(w'\) for \(w\) the inequality \(w <_l w'\) holds.

**Example 3.1.5.** For the Weyl group from Example 2.4.9, the words
\[
s_3 s_2 s_3 <_l s_3 s_2 s_3 s_2 \quad s_2 s_3 s_2 s_3 <_l s_3 s_2 s_3 s_2 s_3 s_2
\]
are all leftmost words and
\[
s_2 s_3 s_2 s_3 <_l s_3 s_2 s_3 s_2 s_3 s_2 s_2 s_1
\]
are all words for the same element of the Weyl group.

Since \(<_l\) is a total order, every element \(w \in W\) has a unique leftmost word. Obviously, the leftmost word is *reduced*, that is, it has the smallest possible length for a word of \(w\).

We follow with a Lemma about the order \(<_l\) and the relations:

**Lemma 3.1.6.** Suppose that \(w_1 = u s_i s_j v^{m_{ij}}\) for some reflections \(s_i, s_j, i \neq j\), words \(u, v\). Set
\[
\rho(w_1) = \rho(u) (p, i)(q, j)(p+1, i) \ldots \rho_1
\]
for some \(p, q \in \mathbb{N}_0\) and a sequence of pairs \(\rho_1\). Set
\[
w_2 = u s_j s_i v^{m_{ij}}.
\]
Then \(w_2 <_l w_1\) if and only if both of the following conditions are fulfilled:
1. $1 \leq q$.

2. Let $(r, k)$ be a pair in $\rho(u)$. Then $(r, k) < (q - 1, j)$.

**Proof.** Let $\rho(w_2) = \rho(u)(q', j)(p', i) \ldots \rho_2$ for some sequence of pairs $\rho_2$.

Suppose that $w_2 <_l w_1$. Then $(q', j) < (p, i)$ by Definition 3.1.3 and thus $q = q' + 1$. So the first condition is fulfilled.

Now consider a pair $(r, k)$ in $\rho(u)$. Then $(r, k) < (q', j) = (q - 1, j)$ and the second condition is fulfilled.

On the other hand, suppose that the conditions 1 and 2 are fulfilled. Then $q'$ is the smallest integer so that $(q', j)$ is bigger than all $(r, k)$ in $\rho(u)$. By the second condition, $(q', j) \leq (q - 1, j)$.

Furthermore, $q$ is the smallest integer so that $(p, i) < (q, j)$. It follows that $(q - 1, j) < (p, i)$, since $i \neq j$.

Together, $(q', j) \leq (q - 1, j) < (p, i)$ and by Definition 3.1.3, we have $w_2 <_l w_1$.

The following lemmas are important for the induction with which we prove the main theorem of this chapter:

**Lemma 3.1.7.** Let $x, x', y$ be words and $s_i \neq s_j$ reflections. We suppose that the words $\underline{x} = x s_i y$ and $\underline{x'} = x' s_j y$ are equivalent, $\underline{x}$ is reduced and $\underline{w} <_l \underline{w}'$. Let $\tilde{z}$ be the longest initial subword that $\underline{w}$ and $\underline{w}'$ share. If there is no $\tilde{z}' \equiv x'$ that shares an initial subword with $\underline{w}$ which is longer than $\tilde{z}$, then there are pairs $(r, h), (s, i), (t, j)$ and series of pairs $\rho_1, \rho_2, \rho_3, \rho_4$ so that $\rho(\tilde{z}) = \rho_1(r, h)\rho_2$ and $\rho(\tilde{z}') = \rho_1(r, h)\rho_2(s, i)\rho_3$

and there is some word $w'' \equiv w$ with

$$\rho(w'') = \rho_1(\rho_2(s, i)\rho_3(t, j))\rho_4.$$ (3.1)

Either $\rho_3 = \rho_4$, or a pair $(q, g)$ is in $\rho_4$ if and only if $(q - 1, g)$ is in $\rho_3$.

If $v$ is the initial subword of $w$ with $\rho(v) = \rho_1$, then no relation on reflections in $v$ is needed to transform $w$ into $w''$.

**Proof.** We prove this inductively on the number of relations that are needed to transform $w'$ into $w$. Without loss of generality, we assume that $m_{ij}$ is odd. If $m_{ij}$ is even, we only need to relabel $s_i$ and $s_j$ in the arguments below.

If there is some word $x_1$ so that

$$w = x_1 \{s_i s_j\}^{m_{ij}} y <_l x_1 \{s_j s_i\}^{m_{ij}} y = w'$$ (3.2)
then \( w'' := x_1 \{ s_j s_i \}^{m_{ij}} y \) fulfils the assertions by Lemma 3.1.6. To conclude the basis of our induction, we note that for two words
\[
\varepsilon_1 \{ s_j s_i \}^{m_{ij}} y \not< l \varepsilon_1 \{ s_i s_j \}^{m_{ij}} y,
\]
we get a similar result: Then
\[
\rho(\varepsilon_1 \{ s_j s_i \}^{m_{ij}} y) = \rho_1 \rho_2 (s - 1, i)(t - 1, j) \rho_3,
\]
where a pair \((q, g)\) is in \( \rho_3 \) if and only if \((q, g)\) is in \( \rho_2 \). Furthermore, either \( \rho_4 = \rho_3 \) or a pair \((q, g)\) is in \( \rho_4 \) if and only if \((q, g)\) is in \( \rho_3 \).

Now suppose that \( w = u \{ s_{l} s_{k} \}^{m_{kl}} v \) is not of the form in equation (3.2), but the assertion is true for \( w_1 = u \{ s_k s_l \}^{m_{kl}} v \).

If \( w_1 <_l w' \), then there is some word \( w'' \) with
\[
\rho(w_1) = \rho_1(r', h')(s, i)(t, j) \rho_3
\]
and
\[
\rho(w''') = \rho_1 \rho_2 (s, i)(t, j) \rho_4.
\]
for some pair \((r', h')\) and series of pairs \( \rho_1, \rho_2, \rho_3, \rho_4 \). Either \( \rho_3 = \rho_1 \), or a pair \((q, g)\) is in \( \rho_4 \) if and only if \((q, g)\) is in \( \rho_3 \).

There is some \( w' \) so that \( \rho_1 = \rho(w) \).

Furthermore, there is a pair \((q, l)\) and a series of pairs \( \rho \) so that
\[
\rho(w) = \rho(w)(q, l) \rho.
\]
If \( w' = u \) or \( w' = u \{ s_{l} s_{k} \}^{m_{kl}} v \), then we set \((r, h) := (q, l)\). Since \( w <_l w' \), we can assume that \( w <_l w_1 \) and by Lemma 3.1.6, the assertion is true.

Since \( x \) is reduced, there is only one other case: the word \( w'' \) has \( \{ s_k s_l \}^{m_{kl}} \) as a subword and the relation \( \{ s_k s_l \}^{m_{kl}} \equiv \{ s_l s_k \}^{m_{kl}} v \) gives a word \( w'' \) that fulfils (3.1).

It remains to prove the assertion if \( w'' <_l w_1 \).

Then we can inductively assume that there is some \( w'' \) so that
\[
\rho(w''') = \rho_1 \rho_2 (s - 1, i)(t - 1, j) \rho_4,
\]
where a pair \((q - 1, g)\) is in \( \rho_2 \) if and only if \((q, g)\) is in \( \rho_2 \). Furthermore, either \( \rho_4 = \rho_3 \) or a pair \((q - 1, g)\) is in \( \rho_4 \) if and only if \((q, g)\) is in \( \rho_3 \).

But there is no \( x'' \equiv x' \) that shares an initial subword with \( w \) which is longer than \( x \), the longest initial subword that \( w \) and \( w' \) share.

Since \( w'' \) and \( w_1 \) share the initial subword \( w' \) with \( \rho(w_1) = \rho_1 \), this is only possible if \( w' = u \{ s_l s_{k} \}^{m_{kl}} v \).

Again, we set \((r, h) := (q, l)\). Since \( x \) is leftmost, \( w <_l w_1 \) and by Lemma 3.1.6, the assertion is true.
CHAPTER 3. A CONNECTION TO THE WEYL GROUP

Completely analogously, we can prove the following:

**Lemma 3.1.8.** Let \( x, x', y \) be words and \( s_i \neq s_j \) reflections. If the words \( w = x y s_i y \) and \( w' = x' y s_j y \) are equivalent, \( x \) is leftmost and \( w' <_l w \), then there are pairs \((r, h), (s, i)\) and series of pairs \( \rho_1, \rho_2, \rho_3, \rho_4 \) so that \( \rho(x') = \rho(x) \rho_3 \rho_2 \rho_1 \)

and there is some word \( w'' \equiv w \) with

\[
\rho(w'') = \rho_1 \rho_2 (r, h) \rho_2 \rho_4
\]

with \((r, h) \neq (s, i)\) or \( \rho_2 \neq 0 \).

Either \( \rho_3 = \rho_4 \), or a pair \((q, g)\) is in \( \rho_4 \) if and only if \((q + 1, g)\) is in \( \rho_3 \).

If \( v \) is the initial subword of \( w \) with \( \rho(v) = \rho_1 \), then no relation on reflections in \( v \) is needed to transform \( w \) into \( w'' \).

We get the following corollary:

**Corollary 3.1.9.** If \( u\{s_i s_j\}^{m_{ij}} \) and \( u s_j \) are leftmost, then either \( u\{s_i s_j\}^{m_{ij}} \) is leftmost or \( u s_j < u s_i \).

**Proof.** If we have \( u\{s_j s_i\}^{m_{ij}} <_l u\{s_i s_j\}^{m_{ij}} \), then \( u s_j < u s_i \).

Furthermore, if \( u\{s_j s_i\}^{m_{ij}} \) is not leftmost, but \( u\{s_i s_j\}^{m_{ij}} \) is, then we can write \( w = x y s_j y \) and there is some \( w' = x' s_j y \) so that \( w \equiv w' \), \( x \) is leftmost and \( w' <_l w \).

By Lemma 3.1.8, there are some words \( u_1, u_2 \) and a reflection \( s_h \) so that

\[
u = u_1 u_2
\]

\[
w'' = u_1 s_h u_2 \{s_i s_j\}^{m_{ij}} \equiv w
\]

and \( w'' <_l w \).

So \( u s_j \equiv u_1 s_h u_2 \) and \( u_1 s_h u_2 <_l u s_j \). \(\square\)

**Remark 3.1.10.** Note that in Lemma 3.1.8, we do not actually need to assume that \( x \) is leftmost; it is sufficient that \( x \) is reduced and the following holds: let \( x'' \equiv x \) so that \( x'' <_l x \). Furthermore, assume that \( x_2 \) is the maximal initial subword that \( x' \) and \( x \) share. Then there is some \( w' \equiv w \) with \( w' <_l x' s_j y \).

So analogously to 3.1.9, we see: Suppose that there is some word \( w' \) so that \( u\{s_i s_j\}^{m_{ij}} \equiv w' \) and for all \( u' \equiv u\{s_i s_j\}^{m_{ij} - 1} \), we have \( w' <_l u' s_i \) if \( m_{ij} = 3 \) and \( w' <_l u' s_j \) otherwise. Then \( u s_j \) is not leftmost.

**Lemma 3.1.11.** Suppose that \( w \equiv u\{s_i s_j\}^{m_{ij}} \) with \( m_{ij} \geq 3 \) and \( u \) reduced. If there is some \( i \neq j \neq k \) with \( w = u' s_k \{s_i s_j\}^{m_{ij} - m} \) for some even \( m \geq 2 \) or \( w = u' s_k \{s_j s_i\}^{m_{ij} - m} \) for odd \( m \geq 3 \), then \( m_{ik} = 2 \) or \( m_{jk} = 2 \).
3.1. LEFTMOST WORDS

Proof. This is a simple, inductive proof: without loss of generality, we can
assume that \( w = u_1 s_k \{ s_i, s_j \}^{m_{ij} - 2} \) and \( m_{jk} = 3 \). Then there is some \( u_2 \) so that
\[
  w \equiv u_2 s_k s_j s_k \{ s_i, s_j \}^{m_{ij} - 2} \equiv u_2 s_j s_k \{ s_j, s_i \}^{m_{ij} - 1}.
\]
So there is some \( u_3 \) so that
\[
  u_3 s_j s_k \equiv u_3 \{ s_i, s_k \}^{m_{ik}}.
\]
If \( m_{ik} \geq 3 \), then we have the same situation as before, only considering a
shorter word. Since \( w \) is finite, we see with induction on the length of \( w \) that
\( m_{ik} = 2 \) or \( m_{jk} = 2 \). \( \square \)

Similarly, we can prove the following:

Lemma 3.1.12. Suppose that \( u s_j \) is leftmost, but \( u s_j s_i \) is not leftmost for
some word \( u \) and some reflections \( s_i, s_j \) with \( m_{ij} \geq 4 \).

Then there are \( s, t \in \mathbb{N} \) so that \( \rho(u s_j s_i) = \rho(u)(t, j)(s, i) \) and \( \rho(u) \) contains
the pair \( (s - 1, i) \). If \( m_{ij} = 6 \), then \( \rho(u) \) additionally contains the pairs
\( (s - 2, i) \) and \( (t - 1, j) \).

Proof. Suppose that the assertions are not fulfilled.

We can without loss of generality assume that there is some \( u' \) and reflec-
tions \( s_{k_1}, \ldots, s_{k_m}, s_{l_1}, \ldots, s_{l_{m'}} \) so that
\[
  u s_j s_i = u' s_j s_{k_1} \ldots s_{k_m} s_i s_{l_1} \ldots s_{l_{m'}} \{ s_j, s_i \}^{m_{ij} - 2} =: u'' \{ s_j, s_i \}^{m_{ij} - 2}
\]
Then there are \( t', t'', q_1, \ldots, q_m, r_1, \ldots, r_{m'} \in \mathbb{N} \) so that
\[
  \rho(u'' s_j) = \rho(u')(t', j)(q_1, k_1) \ldots (q_m, k_m)(s', i)(r_1, l_1) \ldots (r_{m'}, l_{m'})(t'', j).
\]
If \( (t'' - 1, j) < (s', i) \), then the assertions of the lemma are fulfilled. Otherwise,
we get one of the following cases:

(a) There is some \( 1 \leq o \leq m' \) with \( (t'' - 1, j) < (r_o, l_o) \) with \( m_{l_o, j} \geq 3 \).
(b) The words \( s_i s_{l_1} \ldots s_{l_{m'}} s_j \) and \( u s_i \) are not leftmost, contrary to the as-
sumptions.

So we can assume that the first case is fulfilled. Furthermore, without loss
of generality, we can assume \( o = m' \).

If \( m_{l_1} = \cdots = m_{l_{m'}} = 2 \), then
\[
  u'' \equiv u' s_j s_{k_1} \ldots s_{k_m} s_{l_1} \ldots s_{l_{m'}} s_i =: u^m.
\]
Either $u'' <_l u''$, contrary to the assumptions, or there is some $s'' > s'$ so that

$$\rho(u'') = \rho(u')(l', j)(q_1, k_1) \ldots (q_m, k_m)(r_1, l_1) \ldots (r_{m'}, l_{m'})(s'', i).$$

Since $us_j$ is leftmost, but $us_js_i$ is not, there is some $v_1$ so that

$$us_js_i \equiv v_1 \{s_j s_i\}^{m_{j1}}. \quad (3.3)$$

Thus,

$$u'' \equiv u'(s_j s_{l_m'})^{m_{j1}m'} s_i$$

and we are in an analogous situation to before, only considering a shorter word. Since the length of $w$ is finite, the assertions of the lemma are inductively true under these assumptions.

So we can assume without loss of generality that $m_{d1} \geq 3$. Furthermore, we have $(s' + 1, i) < (t'' - 1, j) < (r_{m'}, l_{m'})$. So there is at least one $1 \leq o \leq m'$ so that $m_{do_{o'}} \geq 3$, since otherwise $s_js_k \ldots s_{km}s_is_i l_1$ is equivalent to a smaller word (that begins with $s_j$), contrary to the assumption that $us_j$ is leftmost.

Then there is some $v_2$ so that

$$u's_js_k \ldots s_{km}s_is_i l_1 \equiv v_2 \{s_is_i l_1\}^{m_{i1}}.$$

Because of (3.3), Lemma 3.1.7, Lemma 3.1.8 and $m_{d1} \geq 3$, we get some $v_3$ so that

$$u'' \equiv v_3 s_js_i s_is_i l_1 \ldots s_{l_m}s_i.$$  

Since $m_{j1} \geq 3$ and $m_{l_m} \geq 3$, we are in the same situation as before, only considering a word of shorter length. Inductively, the proof is complete.

Now we can define an assignment which maps the words of the Weyl group to the cofinite full additive subcategories of $\text{mod } A$. We will show that this map yields a bijection between the Weyl group and the set of cofinite submodule closed subcategories.

Let $\tau = D\text{Tr}$ be the Auslander-Reiten translation, see Proposition 2.2.7. By Definition 2.3.2, every indecomposable preinjective module is isomorphic to $\tau^r I_i$ for some $r \in \mathbb{N}$ and $1 \leq i \leq n$.

**Definition 3.1.13.** We can identify the pairs in $\mathcal{N}$ and the indecomposable preinjective modules by setting $(r, i) = \tau^r I_i$.

Not only does this give us a natural order on the preinjective modules, but this also yields an injective map from the words of the Weyl group to the cofinite full additive subcategories of $\text{mod } A$: If

$$\rho(w) = (r_1, i_1)(r_2, i_2) \ldots (r_{m'}, i_{m'}).$$
then \( w \mapsto C_w \), where \( C_w \) is the full additive category with
\[
\text{ind} C_w = \text{ind} A \setminus \{(r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m)\}.
\]

For a Weyl group element \( w \) with the leftmost word \( \underline{w} \), define \( C_w := C_{\underline{w}} \).

**Example 3.1.14.** Let \( A \) be as in Example 2.4.9 and \( w = s_1 s_2 s_3 s_2 s_4 s_1 \). Then
\[
\rho(\underline{w}) = (0, 1)(0, 2)(0, 3)(1, 2)(1, 4)(2, 1)
\]
and
\[
\text{ind} C_{\underline{w}} = \text{ind} A \setminus \{I_1, I_2, I_3, \tau I_2, \tau I_4, \tau^2 I_1\}.
\]

We will prove that the restriction of this map on the leftmost words is a bijection between those and the cofinite submodule closed subcategories.

Since every element of the Weyl group has a unique leftmost word, this gives a bijection between the elements of the Weyl group and the cofinite submodule closed subcategories.

The same bijection is used in [14].

### 3.2 Monomorphisms between preinjective modules

An observation makes the aim of the chapter much simpler to achieve: the cofinite submodule closed subcategories of the module category correspond naturally to the cofinite submodule closed subcategories of \( \mathcal{I} \), the category of the preinjective modules.

Thus we devote this section to preinjective modules. In particular, we give a way to construct all modules \( U \) that contain a given preinjective, indecomposable module \( M \) as a submodule.

In Section 3.3 we will use this to show the connection to the Coxeter structure of the Weyl group. In Section 3.4 to 3.6, we will use this connection to prove that the bijection that we described exists.

**Proposition 3.2.1.** There is a bijection between full additive cofinite submodule closed subcategories of \( \text{mod} A \) and full additive cofinite submodule closed subcategories of \( \mathcal{I} \). It maps the category \( C \) to the category \( C' = C \cap \mathcal{I} \). Furthermore,
\[
\text{ind} A \setminus C = \text{ind} \mathcal{I} \setminus C.
\]
CHAPTER 3. A CONNECTION TO THE WEYL GROUP

Proof. This is completely analogous to [14], Proposition 2.2:

If \( A \) is representation finite, then \( \text{mod} \ A = \mathcal{I} \) and there is nothing to prove. Suppose that \( A \) is not representation finite. Since \( \mathcal{C} \) is cofinite, there is some \( r \in \mathbb{N} \), so that \( \tau^r I_1, \tau^r I_2, \ldots, \tau^r I_n \in \mathcal{C} \). Now suppose that \( M \) is a preprojective or regular module. Then \( \tau^{-r} M \) exists and has an injective envelope \( I \). Since \( \tau^r \) preserves monomorphisms by 2.3.7, \( M \subseteq \tau^r I \in \mathcal{C} \). So \( M \in \mathcal{C} \) and \( \text{ind} \ A \setminus \mathcal{C} = \text{ind} \mathcal{I} \setminus \mathcal{C} \).

Thus the assignment \( C \mapsto \mathcal{C} \cap \mathcal{I} \) is a bijection between the full additive cofinite submodule closed subcategories of \( \text{mod} \ A \) and the full additive cofinite submodule closed subcategories of \( \mathcal{I} \).

We start the construction of exact sequences with a lemma that holds for all Artin algebras:

**Lemma 3.2.2.** Let \( A \) be an arbitrary Artin algebra and \( M, X \in \text{mod} A \) indecomposable. Let

\[
0 \longrightarrow M \xrightarrow{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}} X \oplus X' \xrightarrow{\begin{bmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{bmatrix}} Y \oplus Y' \longrightarrow 0 \quad (3.4)
\]

be an exact sequence for some \( X', Y, Y' \in \text{mod} A \), so that there is some \( Z \in \text{mod} A \) and an \( AR \)-sequence

\[
0 \longrightarrow X \xrightarrow{\begin{bmatrix} g_{11} \\ f_2 \end{bmatrix}} Y \oplus Z \xrightarrow{\begin{bmatrix} g_1' & g_2' \\ 0 & 0 \end{bmatrix}} \tau^{-1} X \longrightarrow 0 \quad (3.5)
\]

If for some \( U \in \text{mod} A \), a monomorphism \( h : M \rightarrow U \) factors through \( f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \) and \( X \nmid U \), then \( h \) also factors through \( f'' = \begin{bmatrix} -f_2 f_1 \\ f_2 \end{bmatrix} \) and the following sequence is exact:

\[
0 \longrightarrow M \xrightarrow{\begin{bmatrix} g_{11} \\ -f_2 f_1 \end{bmatrix}} Y \oplus Z \oplus Y' \xrightarrow{\begin{bmatrix} g_1' & g_2' & g_{12} \\ 0 & 0 & 0 \end{bmatrix}} \tau^{-1} X \oplus Y' \longrightarrow 0 .
\]

Proof. By (3.5), the sequence

\[
0 \longrightarrow X \xrightarrow{\begin{bmatrix} g_{11} \\ f_2 \end{bmatrix}} Y \oplus Z \oplus Y' \xrightarrow{\begin{bmatrix} g_1' & g_2' & 0 \\ 0 & 0 & \text{id}_Y \end{bmatrix}} \tau^{-1} X \oplus Y' \longrightarrow 0
\]

is also exact. By Proposition 2.1.4, the diagrams

\[
\begin{array}{c}
M \xrightarrow{f_1} X \\
\downarrow \quad \downarrow \\
X' \xrightarrow{g_{12} \\ g_{22}} Y \oplus Y'
\end{array}
\]

\[
\begin{array}{c}
M \xrightarrow{f_2} X \\
\downarrow \quad \downarrow \\
X' \xrightarrow{g_{12} \\ g_{22}} Y \oplus Y'
\end{array}
\]

(3.6)
and

\[
\begin{array}{ccc}
X & \xrightarrow{-f_2} & Z \\
\downarrow^{[g_{11}^0]} & & \downarrow^{[g_{22}^0]} \\
Y \oplus Y' & \xrightarrow{[g_1^0 \ 0 \ \text{id}_{Y'}]} & \tau^{-1}X \oplus Y'
\end{array}
\]

are both pushouts and pullbacks. So the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{-f_2f_1} & Z \\
\downarrow^{f_2} & & \downarrow^{[g_2^0]} \\
X' & \xrightarrow{[g_1^0 \ g_{12}^1 \ g_{22}^0]} & \tau^{-1}X \oplus Y'
\end{array}
\]

is itself a pushout and a pullback by Proposition 2.1.5. Therefore, the sequence

\[0 \rightarrow M \rightarrow Z \rightarrow Y' \rightarrow 0\]

is exact. It remains to show that \( h : M \rightarrow U \) factors through \( f'' = \left[ -f_2f_1 \right] \).

Since we have assumed that \( h \) factors through \( f = \left[ f_1 \ f_2 \right] \), there is a morphism \( s = [s_1 \ s_2] : X \oplus X' \rightarrow U \) so that

\[ h = [s_1 \ s_2] \left[ f_1 \ f_2 \right] = s_1f_1 + s_2f_2 \]

By the Definition 2.2.10, the morphism \( s_1 : X \rightarrow U \) factors through \( \left[ g_1^1 \right] \): there is a morphism \( s' = [s'_1 \ s'_2] : Y \oplus Z \rightarrow U \) so that

\[ s_1 = [s'_1 \ s'_2] \left[ g_1^1 \right] = s'_1g_{11} + s'_2f_1 \]

So we get

\[ h = s'_1g_{11}f_1 + s'_2f_2f_1 + s_2f_2 \]

Since (3.6) is commutative, we have

\[ h = -s'_1g_{12}f_2 + s'_2f_2f_1 + s_2f_2 = \left[ -s'_2 \ s_2 - s'_1g_{12} \right] \left[ -f_2f_1 \right] \]

and \( h \) factors through \( f'' \).

We can even say more:
Lemma 3.2.3. Let $A$ be a hereditary Artin algebra, $M \in \mathcal{I}$ and $U \in \text{mod } A$. Suppose that the sequences of modules

$$(X_1, X_2, \ldots, X_m)$$
$$(X'_1, X'_2, \ldots, X'_m)$$
$$(Y_1, Y_2, \ldots, Y_m)$$

fulfil the following conditions:

(S1) There is an Auslander-Reiten sequence

$$0 \longrightarrow M \longrightarrow X_1 \oplus X'_1 \longrightarrow Y_1 \longrightarrow 0.$$ 

(S2) For all $1 \leq i < m$, there is some $\alpha_i \in \mathbb{N}$ so that $X_i^{\alpha_i} \mid X_i \oplus X'_i$, but $X_i^{\alpha_i} \nmid U$.

(S3) For $1 \leq i < m$, there is an Auslander-Reiten sequence of the form

$$0 \longrightarrow X_i \longrightarrow Z_i \longrightarrow \tau^{-1}X_i \longrightarrow 0.$$ 

Let $Y'_i$ be the maximal module that is a direct summand of both $Y_i$ and $Z_i$. Write $Y_i = Y'_i \oplus Y''_i$ and $Z_i = Y'_i \oplus Z'_i$.

If $\tau^{-1}X_i \mid X'_i$, then let $X''_i$ be the module so that $X'_i = \tau^{-1}X_i \oplus X''_i$ and set $Y''_i := 0$. Otherwise, set $X''_i := X'_i$ and $Y''_i := \tau^{-1}X_i$.

The following equations hold:

$$X_{i+1} \oplus X'_{i+1} = X''_i \oplus Z'_i$$
$$Y_{i+1} = Y''_i \oplus Y''_i.$$ 

Then for all $1 \leq i \leq m$ there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f_i} X_i \oplus X'_i \xrightarrow{g_i} Y_i \longrightarrow 0. \tag{3.8}$$

Furthermore, if a monomorphism $M \to U$ exists, then it factors through all $f_i$.

To prove Lemma 3.2.3, we need the following observation:

Remark 3.2.4. Suppose that

$$X_i \oplus X'_i = X_i \oplus X_{i+1} \oplus B_i$$
and that $C_i$ is the maximal module that is both a direct summand of $Y_i$ and $Z_{i+1}$. Furthermore, write $Y_i = C_i \oplus C''_i$ and $Z_{i+1} = C_i \oplus D_i$. Set $B_i = B'_i$ and $C''_i = \tau^{-1}X_i$ if $\tau^{-1}X_i \mid B_i$ and $B_i = B'_i \oplus \tau^{-1}X_i$ and $C''_i = 0$ if $\tau^{-1}X_i \mid B_i$. Then the sequences of modules

\[
(X_1, X_2, \ldots, X_{i-1}, X_i, X_{i+1}, X_{i+2}, X_{i+3}, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_{i-1}, X_i \oplus B_i, B'_i \oplus D_i, X'_{i+2}, X'_{i+3}, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_{i-1}, Y_i, C_i' \oplus C''_i, Y_{i+2}, Y_{i+3}, \ldots, Y_m)
\]

also fulfil the conditions (S1) - (S3).

Note that only the $i$-th and $(i+1)$-th elements of these sequences differ from the elements in the original sequences.

We can easily generalize this to the following: If $i < j_1 < j_2 < \cdots < j_l$, there is an irreducible morphism $X_{j_k} \rightarrow X_{j_{k+1}}$ for all $1 \leq k \leq l$ and $X_{j_1} \mid X_i$, then there are two sequences with $X'_m$ and $Y_m$ as their $m$-th elements that together with

\[
(X_1, \ldots, X_{i-1}, X_{j_1}, X_{j_2}, \ldots, X_{j_l}, X_i, X_{i+1}, \ldots, X_{j_1-1}, X_{j_1+1}, \ldots, X_{j_2-1}, X_{j_2+1}, \ldots, X_{j_l-1}, X_{j_l+1}, \ldots, X_i)
\]

fulfil (S1) - (S3).

Furthermore, there are sequences of modules that fulfil (S1) - (S3) with $X_m, X'_m, Y_m$ as their $m$-th elements so that $X''_i = X'_i$ for all $1 \leq i \leq m$.

By Definition 3.1.13, if there is a morphism $X_i \rightarrow X_j$, then $X_j < X_i$. So we can use the above to get sequences that fulfils (S1) - (S3) with $X_m, X'_m, Y_m$ as their $m$-th elements so that $X_1 \geq X_2 \geq \cdots \geq X_{m-1}$. Then $X > \tau^{-1}I_i$ for all $X \mid X'_i$ and $X \mid Z_j$ with $j \leq i$ and thus $X''_i = X'_i$ for all $1 \leq i < m$.

\textbf{Proof of Lemma 3.2.3.} We prove the lemma inductively. By Remark 3.2.4, it is sufficient to prove the assertion for all sequences so that $X''_i = X'_i$ for all $1 \leq i < m$.

For these sequences, we additionally show the following: If there is an indecomposable direct summand $X$ of $X_m \oplus X'_m$ and $\tau X_i$ of $Y_m$ so that an irreducible morphism $X \rightarrow \tau X_i$ exists, then one of the following holds:

(a) There is a direct summand $X' \cong X$ of $X_m \oplus X'_m$ so that the component $X' \rightarrow Y_m$ of $g_m$ is irreducible and $g_m(X') \subseteq \tau^{-1}X_i$

(b) Either $X \cong X_j$ for some $i < j < m$ or $X$ is isomorphic to a direct summand of $Y'_i$.

If $m = 1$, the assertion is obvious by definition of the Auslander-Reiten sequence.
Now suppose that it holds for all series of modules of length \( m \in \mathbb{N} \) or smaller. We want to show that it also holds for sequences of length \( m + 1 \) by applying Lemma 3.2.2.

To do this, we need to prove that there is an exact sequence of the form

\[
0 \longrightarrow M \longrightarrow X_m \oplus X'_m \xrightarrow{\left[ \begin{array}{cc}
g_{m1} & g_{m2} \\
0 & g_{m3}
\end{array} \right]} Y'_m \oplus Y''_m \longrightarrow 0.
\]

(3.10)

so that \( g_{m1} \) is irreducible.

Suppose that \( Y'_m \) has some direct summands \( Y''_{m1}, Y''_{m2}, \ldots, Y''_{mk} \) and \( g_m \) has a component

\[
diag(g_{11}, g_{22}, \ldots, g_{kk}) : X^k_m \to \bigoplus_{l=1}^{k} Y''_{ml}
\]

where \( g_{11}, g_{22}, \ldots, g_{kk} \) are irreducible and \( diag(g_{11}, g_{22}, \ldots, g_{kk}) \) is the diagonal matrix with entries \( g_{11}, g_{22}, \ldots, g_{kk} \). Then there is a copy of \( X_m \) on which this restricts to

\[
\begin{bmatrix}
g_{11} \\
g_{22} \\
\vdots \\
g_{kk}
\end{bmatrix} : X_m \to \bigoplus_{l=1}^{k} Y''_{ml},
\]

an irreducible morphism.

By condition (S3) and since \( Y''_1 = 0 \), every indecomposable direct summand of \( Y_m \) has the form \( \tau^{-1}X_i \) for some \( 1 \leq i \leq m \).

If for all \( \tau^{-1}X_i \mid Y''_m \), there is some copy \( X \) of \( X_m \) so that the component \( X \to \tau^{-1}Y_i \) of \( g_m \) is irreducible and \( g_m(X) \subseteq \tau^{-1}X_i \), then the above and the induction hypothesis mean that we can apply Lemma 3.2.2.

Suppose that there is some \( \tau^{-1}X_i \mid Y''_m \), so that the above is not the case. Since \( Y''_m \mid Z_m \), there is an irreducible morphism between \( X_m \) and \( \tau^{-1}X_i \).

By the inductive hypothesis, one of the following holds:

(a) \( X_m \cong X_j \) for some \( i < j \leq m \)

(b) \( X_m \mid Y'_i \).

We show that there are sequences

\[
(X_1^{(1)}, X_2^{(1)}, \ldots, X_m^{(1)})
\]

\[
(X_1^{(2)}, X_2^{(2)}, \ldots, X_m^{(2)})
\]

\[
(Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_m^{(1)})
\]

(3.11)
that fulfil (S1)-(S3) and have $X_m$, $X'_m$, $Y_m$ as their $m$-th elements so that
\( \tau^{-1} X_i = \tau^{-1} X^{(1)}_i \), but $X_m \not\cong X^{(1)}_j$ for all $i' < j \leq m$ and $X_m \not\cong X^{(2)}_j$.

Furthermore, we want to show that $X^{(1)}_l = X^{(1)}_l$ for all $1 \leq l < m$. Since $X_m = X^{(1)}_m$, $X'_m = X^{(1)}_m$, $Y_m = Y^{(1)}_m$, this is already clear for $i = m$.

Obviously, we have $X_m \mid X_m \oplus X'_m$, so either $X_m \mid X'_i$ or $X_m \mid Z'_k$ for some $1 \leq k \leq m$, $k \neq i$.

In the first case, (b) is not possible, since $Y_i$ and $X_i \oplus X'_i$ do not share direct summands. In case (a), Remark 3.2.4 yields a sequence $(X^{(2)}_1, \ldots, X^{(2)}_m)$, where $X_j$ comes before $X_i$.

In the second case, we can get a new sequence where $X_k$ comes before $X_i$, since $Z'_i \mid X_m \oplus X'_m$ (otherwise, $\tau^{-1} X_i$ would not be a direct summand of $Y_m$). In case (b), this sequence is already the one we need; in case (a), we can again get a another sequence by Remark 3.2.4 where $X_j$ comes before $X_i$.

If we call this new exact sequence $(X^{(2)}_1, \ldots, X^{(2)}_m)$, then it is clear by (3.9) that $X^{(2)}_l = X^{(2)}_l$ holds for all $1 \leq l < m$.

Since there are only finitely many $j$ with $i' < j \leq m$, we get sequences of the form (3.11) after finitely many steps.

The inductive assumption gives us an exact sequence
\[
0 \longrightarrow M \xrightarrow{f'_m} X_m \oplus X'_m \xrightarrow{g'_m} Y_m \longrightarrow 0  \tag{3.12}
\]
where the component $X_m \to Y_m$ of $g'_m$ is irreducible and $g'_m(X_m) \subseteq \tau^{-1} X_i$.

If $\tau^{-1} X_i = Y_m$, then it is sufficient to look at the sequence (3.12) instead of
\[
0 \longrightarrow M \xrightarrow{f'_m} X_m \oplus X'_m \xrightarrow{g_m} Y_m \longrightarrow 0  \tag{3.13}
\]
If there is some $\tau^{-1} X_k$ so that $\tau^{-1} X_i \oplus \tau^{-1} X_k \mid Y_m$, then we can assume that $g_m$ induces an indecomposable morphism $X_m \to \tau^{-1} X_k$ and $g_m(X_m) \subset \tau^{-1} X_k$.

So together (3.13) and (3.12) give a new exact sequence
\[
0 \longrightarrow M \xrightarrow{f'_{m+1}} X_{m+1} \oplus X'_{m+1} \xrightarrow{g''_{m+1}} Y_{m+1} \longrightarrow 0 ,
\]
where the induced morphisms $X_m \to \tau^{-1} X_i$ and $X_m \to \tau X_k$ of $g''_m$ are irreducible and $g''_m(X_m) \subset \tau^{-1} X_i \oplus \tau^{-1} X_k$.

Inductively, there is an exact sequence of the form (3.10), where $g_{m1}$ is irreducible and we can use Lemma 3.2.2 to get an exact sequence
\[
0 \longrightarrow M \xrightarrow{f_{m+1}'} X_{m+1} \oplus X'_{m+1} \xrightarrow{g_{m+1}} Y_{m+1} \longrightarrow 0 .
\]
If there is a monomorphism $M \hookrightarrow U$, then it factors through $f_{m+1}$.

This gives us not only the assertion of the lemma but also the additional assumptions we have made:

Let

$$g_m = \begin{bmatrix} g_{m1}' & g_{m2}' \\ g_{m1} & g_{m2} \end{bmatrix} : Y'_m \oplus Z'_m \rightarrow \tau^{-1}X_m$$

be the epimorphism of the AR-sequence. By (3.10) and Lemma 3.2.2, we get

$$g_{m+1} = \begin{bmatrix} g_{m2}' & g_{m1}'g_{m2} \\ 0 & g_{m3} \end{bmatrix} : Z'_m \oplus X'_m \rightarrow \tau^{-1}X_m \oplus Y''_m.$$

Let $X$ be a direct summand of $Z'_m \oplus X'_m$ and $\tau^{-1}X_i$ a direct summand of $\tau^{-1}X_m \oplus Y''_m$ so that there is an irreducible morphism $X \rightarrow \tau^{-1}X_i$.

If $i = m$, then $X$ is a direct summand of $Z_m$. If it is also a direct summand of $Z'_m$, then $g_{m+1}(X) = \begin{bmatrix} g_{m2}' \\ g_{m1}'g_{m2} \end{bmatrix} (X)$ and (a) holds. Otherwise, $X$ is a direct summand of $Y''_m$.

If $i \neq m$ and $X$ is a direct summand of $X'_m$, then either (b) holds or

$$\begin{bmatrix} g_{m2} \\ g_{m3} \end{bmatrix} (X'_m) \subset \tau^{-1}X_i.$$

Thus $g_{m2}(X'_m) = 0$ and $g_{m+1}(X) = \begin{bmatrix} 0 \\ g_{m1}'g_{m2} \end{bmatrix} (X)$. So (a) holds.

Finally, suppose that $i \neq m$ and $X$ is not a direct summand of $X'_m$. Because of the irreducible morphism between $X$ and $\tau^{-1}X_i$, the former is a direct summand of $Z_i$ by (S3), either it is a direct summand of $Y'_i$ or of $X_j$ for some $i < j < m$.

A perhaps simpler way to interpret the sequences of modules used in the lemma above is the following:

Remark 3.2.5. Suppose that $X_i, X'_i, Y_i$ are the $i$-th elements of sequences that fulfil (S1) - (S3). Then $X_{i+1} \oplus X'_{i+1}$ are defined by taking the exact sequence

$$0 \rightarrow M \rightarrow X_i \oplus X'_i \rightarrow Y_i \rightarrow 0.$$ 

and the Auslander-Reiten sequence

$$0 \rightarrow X_i \rightarrow Z_i \rightarrow \tau^{-1}X_i \rightarrow 0.$$ 

We can add these sequences together and get

$$0 \rightarrow M \oplus X_i \rightarrow X_i \oplus X'_i \oplus Z_i \rightarrow Y_i \oplus \tau^{-1}X_i \rightarrow 0.$$
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Then $X_i$ is the maximal module that is a direct summand of both the first and the second term: we still get an exact sequence if we delete it in both terms:

$$0 \longrightarrow M \longrightarrow X'_i \oplus Z_i \longrightarrow Y_i \oplus \tau^{-1}X_i \longrightarrow 0.$$ 

The same holds for $G_i$, the maximal module that is both a direct summand of the middle term and the last term. Deleting this in both terms gives us an exact sequence

$$0 \longrightarrow M \longrightarrow X_{i+1} \oplus X'_{i+1} \longrightarrow Y_{i+1} \longrightarrow 0.$$ 

These modules have some interesting properties:

**Corollary 3.2.6.** If there is a monomorphism $h : M \rightarrow U$, then for sequences of modules

$$(X_1, X_2, \ldots, X_m)$$

$$(X'_1, X'_2, \ldots, X'_m)$$

$$(Y_1, Y_2, \ldots, Y_m),$$

which fulfill (S1) - (S3), there is a monomorphism $X_i \oplus X'_i \rightarrow Y_i \oplus U$ for all $1 \leq i < m$.

Thus, every injective direct summand of $X_i \oplus X'_i$ is a direct summand of $U$.

**Proof.** By Lemma 3.2.3, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f_i} X_i \oplus X'_i \xrightarrow{g_i} Y_i \longrightarrow 0$$

for $1 \leq i \leq m$ so that $h$ factors through $f_i$. Thus, there is some morphism $h_i$ with $h = h_if_i$. So $h_i$ is a monomorphism on $\text{Im} f_i$. Since $\text{Ker} g_i = \text{Im} f_i$, the morphism

$$\begin{bmatrix} g_i \\ h_i \end{bmatrix} : X_i \oplus X'_i \hookrightarrow Y_i \oplus U$$

is a monomorphism.

So every injective direct summand $I$ of $X_i \oplus X'_i$ is a direct summand of $Y_i \oplus U$. Since $X_i \oplus X'_i$ and $Y_i$ do not share any direct summands, $I$ is even a direct summand of $U$. 

We can use the following lemma to show that there is an algorithm that, for given indecomposable, preinjective module $M$ constructs all $U$ with $M \rightarrow U$. 


Lemma 3.2.7. Suppose there is an irreducible morphism between \((s, i) = \tau^s I_i\) and \((t, j) = \tau^t I_j\). Then either \(s = t\) and \(i > j\) or \(s = t + 1\) and \(i < j\).

Proof. By 2.3.6,

\[ s - 1 \leq t \leq s. \]

Furthermore, we have ordered the injective modules so that \(\text{Hom}(I_i, I_j) = 0\) if \(i < j\).

By 2.3.7, we get \(\text{Hom}(\tau^s I_i, \tau^t I_j) = 0\) if \(i < j\). By 2.2.14, there is an irreducible morphism \(\tau^s I_i \to \tau^s I_j\) if and only if there is an irreducible morphism \(\tau^s I_j \to \tau^s I_i\).

So if \(s = t\), then \(i > j\) and if \(s = t + 1\) then \(i < j\).

\[ \square \]

Proposition 3.2.8. Let \(A\) be a hereditary Artin algebra with \(M \in \text{mod}\ A\) indecomposable and preinjective. Let \(U \in \text{mod}\ A\), so that \(M\) is not a direct summand of \(U\). There is a monomorphism \(M \hookrightarrow U\) if and only if for some \(m \in \mathbb{N}\) there are three sequences of modules

\[
(X_1, X_2, \ldots, X_m)
\]

\[
(X'_1, X'_2, \ldots, X'_m)
\]

\[
(Y_1, Y_2, \ldots, Y_m)
\]

that fulfil the conditions \((S1) - (S3)\) and furthermore

\((S4)\) If for some \(1 \leq i \leq m\) the module \(X_i \oplus X'_i\) has an injective direct summand \(I\), then \(I \mid U\).

\((S5)\) \(X_m \oplus X'_m\) is a direct summand of \(U\).

Proof. To prove this, we use Lemma 3.2.3: since the sequences fulfil \((S1)-(S3)\), there are exact sequences of the form

\[
0 \longrightarrow M \mathrel{\overset{f_i}{\longrightarrow}} X_i \oplus X'_i \mathrel{\overset{g_i}{\longrightarrow}} Y_i \longrightarrow 0
\]

for all \(1 \leq i \leq m\). If a monomorphism \(M \hookrightarrow U\) exists, it factors through \(f_i\) for all \(1 \leq i \leq m\).

Thus one direction is obvious: if such sequences of modules exist, \(f_m : M \hookrightarrow U\) is a monomorphism.

On the other hand, suppose that no series of modules fulfil \((S1) - (S5)\).

If \(M\) is injective, then it cannot be a submodule of \(U\). Otherwise, there are series of modules that fulfil \((S1) - (S3)\), since there is an AR-sequence that starts in \(M\) and we can set \(m = 1\).

If \((S4)\) is not fulfilled, then \(M\) cannot be a submodule of \(U\) by Corollary 3.2.6.
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Otherwise, there is some non-injective \( X_{m+1} \) and some \( \alpha_{m+1} \in \mathbb{N} \) so that \( X_{m+1}^{\alpha_{m+1}} \mid X_m \oplus X_m' \), but \( X_{m+1}^{\alpha_{m+1}} \not\mid U \). So we can extend the sequences of modules to

\[
(X_1, X_2, \ldots, X_m, X_{m+1}) \\
(X'_1, X'_2, \ldots, X'_m, X'_{m+1}) \\
(Y_1, Y_2, \ldots, Y_m, Y_{m+1})
\]

so that these series fulfil (S1) - (S3). If these sequences fulfil (S4), we can extend them again to sequences of length \( m + 2 \).

We have \( M = (r, i) \) for some \( r \in \mathbb{N} \) and \( 1 \leq i \leq n \). Every indecomposable direct summand of \( X_1 \oplus X'_1 \) is of the form \( (r', j) < (r, i) \) for some \( r' \in \mathbb{N}_0 \) and \( 1 \leq j \leq n \). Furthermore, if \( X_1 = (r', j) \), then every direct summand of \( Z_1 \) is of the form \( (r'', k) < (r', j) \), and analogously for \( X_2, X_3, \ldots \).

So after finitely many steps, either we find sequences that do not fulfil (S4), or there is some \( m' \) so that every direct summand of \( X_m \oplus X'_m \) is injective. If (S4) is still fulfilled, then (S5) is also fulfilled, a contradiction to our assumption.

The proof of Proposition 3.2.8 shows the following:

**Corollary 3.2.9.** Let \( M \) and \( U \) be preinjective modules over \( A \). If \( M \subset U \), then all sequences of modules that fulfil (S1) - (S3) can be extended to sequences of modules that fulfil (S1) - (S5).

If \( M \not\subset U \), then all sequences of modules that fulfil (S1) - (S3) can be extended to sequences that fulfil (S1) - (S3) so that \( X_m \oplus X'_m \) has an injective direct summand that is not a direct summand of \( U \).

**Remark 3.2.10.** By Corollary 3.2.9, we can use the proposition as an algorithm that finds out for given indecomposable, preinjective \( M \) and modules \( U \), if there is a monomorphism \( M \hookrightarrow U \). Alternatively, we can use it to construct all \( U \) with \( M \subset U \).

Note that it is very simple to generalize this for arbitrary preinjective \( M \):

**Corollary 3.2.11.** Let \( M \) be a preinjective module so that \( M = \bigoplus_{i=1}^m M_i \) with \( M_i \) indecomposable. Let \( U \) be some module in \( \text{mod} \ A \). Denote the middle term of the Auslander-Reiten sequence that starts in \( M_i \) by \( N_i \).

Furthermore, order \( M_1, \ldots, M_m \) so that there is some \( 0 \leq k \leq m \) with \( M_i \mid U \) if and only if \( i \leq k \).

Suppose that the sequences of modules

\[
(X_1, X_2, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_m)
\]
fulfil (S2), (S3) and (S’1) We have

\[
X_1 \oplus X_1' = \bigoplus_{i=1}^{k} M_i \oplus \bigoplus_{i=k+1}^{m} N_i
\]

and

\[
Y_1 = \bigoplus_{i=k+1}^{m} \tau^{-1} M_i.
\]

Then for all \(1 \leq i \leq m\), there is an exact sequence

\[
0 \rightarrow M \xrightarrow{f_i} X_i \oplus X_i' \xrightarrow{g_i} Y_i \rightarrow 0.
\]

There is a monomorphism \(M \rightarrow U\) if and only if there is some \(m' > m\) and modules \(X_{m+1}, \ldots, X_{m'}, X_{m'+1}', \ldots, X_{m'}'\), \(Y_{m+1}, \ldots, Y_{m'}\) so that the sequences

\[
(X_1, X_2, \ldots, X_{m'})
\]

\[
(X_1', X_2', \ldots, X_{m'}')
\]

\[
(Y_1, Y_2, \ldots, Y_{m'})
\]

fulfil (S’1) and (S2) - (S5).

Furthermore, if a monomorphism \(M \rightarrow U\) exists, then it factors through all \(f_i\).

Example 3.2.12. Take \(A\) as in Example 2.4.9. A part of the preinjective component of the AR-quiver of \(A\) is:

\[
\begin{array}{c c c c c c c c c}
& & \cdots & & \tau I_3 & & \tau I_1 & & I_3 & & I_4 & & \cdots \\
& & \times & & \times & & \times & & \times & & \times & & \\
& & \tau I_4 & & \tau I_2 & & I_4 & & \cdots & & \cdots & & \\
& & & & & & & & & & & & \\
& & \tau I_1 & & \tau I_3 & & I_3 & & \cdots & & \cdots & & \\
\end{array}
\]

Suppose that we want to know whether \(M = \tau I_3\) is a submodule of, say, \(U = I_2 \oplus I_3 \oplus I_4\).

Then by (S1), \(X_1 \oplus X_1' = \tau I_1 \oplus \tau I_2\) and \(Y_1 = I_3\). Since neither \(\tau I_1\) nor \(\tau I_2\) is a direct summand of \(U\), we arbitrarily set \(X_1 := \tau I_1\).

The AR-sequence

\[
0 \rightarrow \tau I_1 \rightarrow I_3 \oplus I_4 \rightarrow I_1 \rightarrow 0
\]

and (S3) show that \(X_2 \oplus X_2' = \tau I_2 \oplus I_4\) and \(Y_2 = I_1\). Since \(I_4\) is a direct summand of \(U\), we set \(X_2 := \tau I_2\) to fulfil (S2). Using the AR-sequence

\[
0 \rightarrow \tau I_2 \rightarrow I_3 \oplus I_4 \rightarrow I_2 \rightarrow 0
\]
we get $X_3 \oplus X'_3 = I_3 \oplus I'_4$ and $Y_3 = I_1 \oplus I_2$. Since $I'_4$ is injective, but not a direct summand of $U$, the condition (S4) is not fulfilled and there is no monomorphism between $M$ and $U$.

We have one more lemma:

**Lemma 3.2.13.** Let $M$ be an indecomposable, preinjective module and $U \in \text{mod} \ A$. If the sequences

$$
\begin{align*}
(X_1, X_2, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_m)
\end{align*}
$$

fulfil (S1) - (S3), then for every $1 \leq i \leq m$, there is an exact sequence

$$0 \longrightarrow X_i \oplus X'_i \longrightarrow Y_i \oplus X_m \oplus X'_m \longrightarrow Y_m \longrightarrow 0 \quad . \tag{3.14}$$

Furthermore, if there is an exact sequence

$$0 \longrightarrow X_i \oplus X'_i \longrightarrow Y_i \oplus U \longrightarrow Z \longrightarrow 0 \quad . \tag{3.15}$$

then there is also an exact sequence

$$0 \longrightarrow M_0 \longrightarrow U \longrightarrow Z \longrightarrow 0 \quad . \tag{3.17}$$

**Proof.** Let $Y$ be the maximal module so that $Y \mid Y_i$ and $Y \mid Y_m$. Furthermore, suppose that $Y_i = Y \oplus Y'$ and $Y_m = Y \oplus Y''$.

We use Corollary 3.2.11 on $X_i \oplus X'_i$ and $Y \oplus U$. Take $i < j_1 < j_2 \cdots < j_l$ so that $X_{j_k}$ are those modules in the sequence $(X_{i+1}, \ldots, X_m)$ which are already a direct summand of $X_i \oplus X'_i$. Then

$$(X_{m+1}, \ldots, X_{j_1-1}, X_{j_1+1}, \ldots, X_{j_2-1}, X_{j_2+1}, \ldots, X_{j_l-1}, X_{j_l+1}, \ldots, X_m)$$

is part of a triple of sequences that fulfil (S'1) and (S2) - (S5) with respect to $X_i \oplus X'_i$ and $Y' \oplus U$.

So the same construction that yields an exact sequence

$$0 \longrightarrow M \longrightarrow X_m \oplus X'_m \longrightarrow Y_m \longrightarrow 0 \quad ,$$

also gives an exact sequence

$$0 \longrightarrow X_i \oplus X'_i \longrightarrow Y' \oplus X_m \oplus X'_m \longrightarrow Y'' \longrightarrow 0 \quad ,$$

when used on $X_i \oplus X'_i$ and $Y' \oplus U$ instead of $M$ and $U$. Adding $Y$ to both the middle and the last term gives (3.15).

The exact sequence (3.16) is given by a sequence of modules that fulfil (S’1) and (S2) - (S5). Together with the sequences (3.14), this yields the exact sequence 3.17. \qed
3.3 Preinjective modules and the Weyl group

In this section we connect our results about preinjective modules with the relations of the Weyl group.

First we give a Lemma that shows the connection between the AR-sequences and the relations:

**Lemma 3.3.1.** Fix two integers $1 \leq i, j \leq n$. Set

$$\alpha := \max\{v \mid \text{there are } s, t \text{ with an irreducible morphism } (s, i) \to (t, j)^v\}$$

$$\beta := \max\{v \mid \text{there are } s, t \text{ with an irreducible morphism } (t, j) \to (s, i)^v\}$$

Let $(s, i, s, j)^{m_{ij}}$ be the defining relation of the Weyl group as in Lemma 2.4.7. Then the value of $m_{ij}$ depends on $\alpha \beta$ in the following way:

<table>
<thead>
<tr>
<th>$m_{ij}$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \beta$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\geq 4$</td>
</tr>
</tbody>
</table>

**Proof.** From 2.3.7, we know that $\alpha$ and $\beta$ do not depend on $s$. Let $(c_{ij})_{nn}$ be the Cartan matrix. By Lemma 2.3.9, either $\alpha = c_{ij}$ and $\beta = c_{ji}$ or $\beta = c_{ij}$ and $\alpha = c_{ji}$. Lemma 2.4.7 gives the stated values for $m_{ij}$. \hfill \square

Now we define a recursion that plays a fundamental role in the proof of the bijection:

**Definition 3.3.2.** For given $\alpha, \beta \in \mathbb{N}$ define a recursion formula by

$$E(0) = 1$$

$$E(1) = \alpha$$

$$E(2m) = \max(\beta E(2m - 1) - E(2m - 2), 0)$$

$$E(2m + 1) = \max(\alpha E(2m) - E(2m - 1), 0)$$

for all $m \in \mathbb{N}$.

This recursion is directly linked to the Weyl group:

**Lemma 3.3.3.** Let $\alpha, \beta$ be as in Lemma 3.3.1. Then

$$E(m) = 0 \iff m \geq m_{ij} - 1.$$ 

**Proof.** If $\alpha \beta < 4$, then we get the following values for $m \leq 6$: 

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(m)$</td>
<td>1</td>
<td>$\alpha$</td>
<td>$\beta E(1) - E(0)$</td>
<td>$\beta E(2) - E(1)$</td>
<td>$\alpha E(3) - E(2)$</td>
<td>$\beta E(4) - E(3)$</td>
</tr>
</tbody>
</table>

for $m_{ij} = 5$.
3.3. PREINJECTIVE MODULES AND THE WEYL GROUP

<table>
<thead>
<tr>
<th>$\alpha \beta$</th>
<th>$m_{ij}$</th>
<th>$E(0)$</th>
<th>$E(1)$</th>
<th>$E(2)$</th>
<th>$E(3)$</th>
<th>$E(4)$</th>
<th>$E(5)$</th>
<th>$E(6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
<td>$\alpha$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>1</td>
<td>$\alpha$</td>
<td>2</td>
<td>$\alpha$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(3.18)

Obviously, $E(m) = 0$ if $\alpha \beta < 4$ and $m > 6$.

If $\alpha \beta \geq 4$, then $m_{ij} = \infty$ by Lemma 3.3.1 and we need to show that $E(m) \neq 0$ for all $m \in \mathbb{N}$.

Since $E(2) = \alpha \beta - 1 > E(0) > 0$, we get inductively for $m > 1$:

\[
E(2m) = \beta E(2m - 1) - E(2m - 2) \\
= (\alpha \beta - 1)E(2m - 2) - \beta E(2m - 3) \\
= (\alpha \beta - 1)E(2m - 2) - E(2m - 2) - E(2m - 4) \\
> (\alpha \beta - 3)E(2m - 2) \\
\geq E(2m - 2).
\]

The proof that $E(2m + 1) > E(2m - 1) > 0$ is completely analogous.

Next, we need some notation:

**Definition 3.3.4.** Fix $s \in \mathbb{N}_0$ and $1 \leq i \neq j \leq n$ and let $M_0 := \tau^s I_i$. If $s \geq 1$ or $j < i$, let $t$ be the integer with $(s - 1, i) < (t, j) < (s, i)$. Denote $M_1 := \tau^t I_j$, $M_2 := \tau^{s-1} I_i$, $M_3 := \tau^{t-1} I_j$, \ldots.

The following lemma is a key part in the proof that there is a bijection between cofinite, submodule closed subcategories and the elements of the Weyl group:

**Lemma 3.3.5.** Let $U$ be a module so that $M_k \uparrow U$ for all $M_k \neq 0$ with $0 \leq k \leq m_{ij} - 1$. Furthermore, let $\alpha, \beta$ be as in Lemma 3.3.1. Then for all $m \geq 1$ with $M_{m+1} \neq 0$ and $E(m - 1) > 0$, there are series of modules that fulfil (S1) - (S3) and yield exact sequences

\[
0 \longrightarrow M_0 \overset{f_m}{\longrightarrow} M_m^{E(m)} \oplus U_m \longrightarrow M_m^{E(m-1)} \longrightarrow 0
\]

(3.19)

so that no $M_k \uparrow U_m$ for all $M_k \neq 0$ with $0 \leq k \leq m_{ij} - 1$.

If there is a monomorphism $M_0 \hookrightarrow U$, then it factors through $f_m$ for all $m$. 
Proof. If $M_0$ is injective, there is nothing to show. So we can assume that an AR-sequence starts in $M_0$.

Let $\alpha, \beta$ be as in Lemma 3.3.1. Then there are modules $M', N'$ so that

$$
0 \longrightarrow M_0 \longrightarrow M_1^\alpha \oplus N' \longrightarrow M_2 \longrightarrow 0
$$

(3.20)

and

$$
0 \longrightarrow M_1 \longrightarrow M_2^\beta \oplus M' \longrightarrow M_3 \longrightarrow 0
$$

are AR-sequences. Note that by 2.3.7, for all non-injective $M_{2m-1}$, $m \in \mathbb{N}_0$, there are AR-sequences of the form

$$
0 \longrightarrow M_{2m-1} \longrightarrow M_{2m}^\beta \oplus \tau^m M' \longrightarrow M_{2m+1} \longrightarrow 0.
$$

(3.21)

For all non-injective $M_{2m}$, $m \in \mathbb{N}$ they are of the form

$$
0 \longrightarrow M_{2m} \longrightarrow M_{2m+1}^\beta \oplus \tau^m N' \longrightarrow M_{2m+2} \longrightarrow 0.
$$

If we set $U_1 := N'$, the AR-sequence that starts in $M_0$ is the exact sequence

$$
0 \longrightarrow M_0 \longrightarrow M_1^{E(1)} \oplus U_1 \longrightarrow M_2^{E(0)} \longrightarrow 0.
$$

If $M_1$ is injective, then the proof is complete. So we can assume that an AR-sequence starts in $M_1$ and use Lemma 3.2.3.

Since $M_1 \nmid U$, we set

$$
X_1 := X_2 := X_3 := \cdots := X_{E(1)} := M_1,
$$

Then we get sequences that fulfil conditions (S1) - (S3) by setting

$$
X'_1 := M_1^{E(1)-1} \oplus N', \\
X'_2 := M_1^{E(1)-2} \oplus M_2^{E(1)-1} \oplus N' \oplus M', \\
\cdots \\
X'_{E(1)} := M_2^{E(1)-1} \oplus N' \oplus (M')^{E(1)-1}
$$

and

$$
Y_1 := M_2, \\
Y_2 := M_3, \\
Y_3 := M_3^2, \\
\cdots \\
Y_{E(1)} := M_3^{E(1)-1}
$$
We get
\[ X_{E(1)} \oplus X_{E(1)}' = M_{2}^{E(1)\beta-1} \oplus N' \oplus (M')^{E(1)} \]
and \( Y_{E(1)} = M_{3}^{E(1)} \). Thus by Lemma 3.2.3 there is some \( f_1 \) so that the following sequence is exact:
\[ 0 \longrightarrow M_{0} \xrightarrow{f_1} M_{2}^{E(1)\beta-1} \oplus N' \oplus (M')^{E(1)} \longrightarrow M_{3}^{E(1)} \longrightarrow 0. \]
If there is a monomorphism \( M \rightarrow U \), then it factors through \( f_1 \).
Since \( U_{2} := N' \oplus (M')^{E(1)} \), we can write the exact sequence as
\[ 0 \longrightarrow M_{0} \xrightarrow{f_1} M_{2}^{E(2)} \oplus U_{2} \longrightarrow M_{3}^{E(1)} \longrightarrow 0. \]
We show the rest inductively: Suppose that
\[ 0 \longrightarrow M_{0} \longrightarrow M_{2m-1}^{E(2m-1)} \oplus U_{2m-1} \longrightarrow M_{2m}^{E(2m-2)} \longrightarrow 0 \]
is an exact sequence and \( E(2m-1) \neq 0 \). Furthermore, suppose that this exact sequence is yielded by sequences of modules of the length \( m' - 1 \). Then
\[ X_{m'} := M_{2m-1}, X_{m'+1} := M_{2m-1}^{E(2m-1)-1} \oplus U_{2m-1}, Y_{m'} := M_{2m}^{E(2m-2)} \]
are elements of sequences that fulfill the condition (S1) - (S3) of Lemma 3.2.3.
If \( M_{2m-1} \) is injective, then \( M_{2m+1} = \tau M_{2m-1} = 0 \) and there is nothing to prove. If \( M_{2m-1} \) is not injective, then the AR-sequence (3.21) exists. As above, we set
\[ X_{m'+1} := \cdots := X_{m'+E(2m-1)-1} := M_{2m-1}. \]
This determines \( X_{m'+1}, \ldots, X_{m'+E(2m-1)-1} \) and \( Y_{m'+1}, \ldots, Y_{m'+E(2m-1)-1} \) completely.
Since \( E(2m) = \beta E(2m - 1) - E(2m - 2) \), we get
\[ X_{m'+E(2m-1)} \oplus X_{m'+E(2m-1)}' = M_{2m}^{E(2m)} \oplus U_{2m-1} \oplus (\tau M')^{E(2m-1)} \]
\[ Y_{m'+E(2m-1)} := M_{2m+1}^{E(2m-1)} \]
Together with \( U_{2m} := U_{2m-1} \oplus (\tau M')^{E(2m-1)} \), this yields an exact sequence
\[ 0 \longrightarrow M_{0} \xrightarrow{f_{2m}} M_{2m}^{E(2m)} \oplus U_{2m} \longrightarrow M_{2m+1}^{E(2m-1)} \longrightarrow 0 \]
for some \( f_{2m} \). By Lemma 3.2.3, \( M \xrightarrow{\tau} U \) factors through \( f_{2m} \).
Analogously, we can construct
\[ 0 \longrightarrow M_{0} \xrightarrow{f_{2m+1}} M_{2m+1}^{E(2m+1)} \oplus U_{2m+1} \longrightarrow M_{2m+2}^{E(2m)} \longrightarrow 0 \]
if \( E(2m) \neq 0 \) and \( M_{2m+2} \neq 0 \).

\hfill \square
Corollary 3.3.6. Let $U_m$ be as in Lemma 3.3.5. If $(r,l) \mid U_m$, then $M_0 > (r,l) > M_{m+1}$.
If $M_0 > (r,l) > M_1$, then $(r,l) \mid U_m$ if and only if $m_d \geq 3$.
If $M_1 > (r,l) > M_m$, then $(r,l) \mid U_m$ if and only if $m_d + m_{lj} \geq 5$.
If $M_m > (r,l) > M_{m+1}$ and $m$ is even, then $(r,l) \mid U_m$ if and only if $m_d \geq 3$. If $m$ is odd, then $(r,l) \mid U_m$ if and only if $m_{lj} \geq 3$.

Proof. This is obvious from the proof of Lemma 3.3.5. 

Remark 3.3.7. Note that $m_{ij} = m_{ji}$. If we fix $s,t$ as in Definition 3.3.4, we can set $M'_0 := \tau^t I_j, M'_1 := \tau^s I_i, M'_2 := \tau^{s-1} I_j, \ldots$ and
\[
E'(0) = 1 \\
E'(1) = \beta \\
E'(2m) = \alpha E(2m-1) - E(2m-2) \\
E'(2m + 1) = \beta E(2m) - E(2m-1).
\]

With this definition, we get analogous results to 3.3.3, 3.3.5 and 3.3.6.

3.4 Preliminaries for the main theorem

This section collects some preliminaries which are necessary to prove that there is a bijection between the Weyl group and the cofinite, submodule closed subcategories: First, we show that every cofinite submodule closed subcategory is of the form $\mathcal{C}_w$ for some word $w$.

Then we will prove an auxiliary result that will make the inductions in the next section possible.

Lemma 3.4.1. If a cofinite, full additive subcategory $\mathcal{C}$ of $\text{mod} A$ is submodule closed, then there is a word $w$ over $S = \{s_1, s_2, \ldots, s_n\}$ with $\mathcal{C} = \mathcal{C}_w$.

Proof. By Lemma 3.2.1,
\[
\text{ind } A \setminus \mathcal{C} = \text{ind } \mathcal{I} \setminus \mathcal{C} =: \{(r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m)\}.
\]
for some $m \in \mathbb{N}$ and modules $(r_1, i_1) < (r_2, i_2) < \cdots < (r_m, i_m)$.

Suppose that for all words $w$ over $S$
\[
\rho(w) \neq (r_1, i_1)(r_2, i_2) \cdots (r_m, i_m).
\]
By Definition 3.1.1, either $r_1 > 0$ or there is some $1 \leq j \leq m - 1$ so that
\[
(r_j, i_j) < (r_{j+1}, i_{j+1} - 1).
\]
3.4. PRELIMINARIES FOR THE MAIN THEOREM

In the first case, \( C \) contains the middle term of the AR-sequence that starts in \((r_1, i_1)\) by Lemma 3.2.7. In the second case, \( C \) contains the middle term of the AR-sequence that starts in \((r_{j+1}, i_{j+1})\). In both cases, \( C \) is not submodule closed.

So by Definition 3.1.13, there is some \( w \) with

\[
\rho(w) = (r_1, i_1)(r_2, i_2) \cdots (r_m, i_m)
\]

and \( C = C_w \).

\( \square \)

Recall that \( C_w = C_{\overline{w}} \), where \( \overline{w} \) is the leftmost word for \( w \). So we need to prove that the word \( \overline{w} \) in Lemma 3.4.1 is leftmost. Furthermore, we need the other direction, namely, that \( C_{\overline{w}} \) is submodule closed if \( \overline{w} \) is leftmost.

We will use the following lemma for the proofs of both directions:

**Lemma 3.4.2.** Suppose that the words \( w \) and \( w'' \) are equivalent and there are pairs \((r, h), (s, i), (t + 1, j)\) and series of pairs \( \rho_1, \rho_2, \rho_4 \) so that

\[
\rho(w) = \rho_1(r, h)\rho_2(s, i)\rho_3,
\]

\[
\rho(w'') = \rho_2(s, i)(t + 1, j)\rho_4
\]

and either \( \rho_3 = \rho_4 \), or a pair \((q, g)\) is in \( \rho_3 \) if and only if \((q - 1, g)\) is in \( \rho_3 \).

Furthermore, suppose that the word \( \overline{x} \) with \( \rho(\overline{x}) = \rho_1(r, h)\rho_2 \) is reduced and \( m_{ij} \geq 3 \).

If \( M_0 = (s, i), M_1 = (t, j), M_2, \ldots, M_{m_{ij} - 3} \notin C_{\overline{w}}, \) then there are sequences of modules as in Lemma 3.2.3 (used on \( M_0 \) and any \( U \in C_{\overline{w}} \)) that yield some \( U', Y \in I \) so that

\[
0 \rightarrow M_0 \rightarrow M_{m_{ij} - 2} \oplus U' \rightarrow Y \rightarrow 0
\]  \hspace{1cm} (3.22)

is an exact sequence and either \( Y \in C_w \) or both \( Y = (r, h)^{E(m_{ij} - 3)} \) and \( U' \in C_w \) hold.

**Proof.** We show this by induction on the number \( m \) of Coxeter relations needed to transform \( w \) into \( w'' \).

Furthermore, we show that a few additional assertions hold, which we need for the inductive proof:

(A1) Let \((r', h')\) be a module so that \((r', h') \supset Y \). If \( Z \in C_w \) is a direct summand of the middle term of the AR-sequence that ends in \((r', h')\), then \( Z \supset Y \).

\[
0 \rightarrow M_0 \rightarrow M_{m_{ij} - 2} \oplus U' \rightarrow Y \rightarrow 0
\]  \hspace{1cm} (3.22)
(A2) Let $V$ be the maximal direct summand of $U'$, so that for every $(q, g) \mid V$ there is some $(r', h') \mid Y$ with an indecomposable morphism $(q, g) \to (r', h')$. Furthermore, let $(q', g')$ be the biggest indecomposable direct summand of $V$ and $(r'', h'')$ the smallest indecomposable direct summand of $Y$.

If there is some $o \in \mathbb{N}_0$ so that $Y, \tau^{-1}Y, \ldots, \tau^{-o}Y \in \mathcal{C}_w$ and $\tau^{-o-1}Y \notin \mathcal{C}_w$, then one of the following holds:

(a) $\tau^{-o-1}Y = (r, h)^{E(m_{ij} - 3)}$ and

$$\tau^{-1}V, \tau^{-2}V, \ldots, \tau^{-o-1}V \in \mathcal{C}_w.$$  

(b) Let $V'$ be the maximal direct summand of $V$ so that for every $(q, g) \mid V'$ there is some $0 \leq k \leq o + 1$ so that $(q - k, g) \notin \mathcal{C}_w$. Then there is some module $Y'$ with an exact sequence

$$0 \to \tau^{-o-1}V' \to \tau^{-o-1}Y \oplus U'' \to Y' \to 0. \quad (3.23)$$

Either $Y' \in \mathcal{C}_w$ or both $Y' = (r, h)^{E(m_{ij} - 3)}$ and $U'' \in \mathcal{C}'$ hold, where

$$\text{ind } \mathcal{C}' = \text{ind } \mathcal{C}_w \setminus \mathcal{M}$$

with

$$\mathcal{M} = \left\{ M \in \text{ind } \mathcal{I} \mid \exists 0 \leq k \leq o : \left\{ \begin{array}{l}
\tau^{k+1}M \notin \mathcal{C}_w \\
(r'', h'') < \tau^{k+1}M \\
\tau^{k+1}M < (q' - 1, g')
\end{array} \right\} \right\}.$$  

For all $U \in \mathcal{C}'$, there are sequences of modules that fulfil (S'1), (S2) and (S3) with respect to $\tau^{-o-1}V'$ and $U$ and yield (3.23) as in Corollary 3.2.11.

Furthermore, (A1) and (A2) still hold if we exchange $U'$, $Y$ and $\mathcal{C}_w$ for $U''$, $Y'$ and $\mathcal{C}'$ respectively.

If there are some reflections $s_k, s_l$ and words $u, v$ so that

$$w = u(s_k s_l)^{m_{kl} v} \quad \text{and} \quad w'' = u(s_l s_k)^{m_{kl} v},$$

then this is the result of Lemma 3.3.5.

Now suppose that $w = u(s_k s_l)^{m_{kl} v}$ and the lemma, (A1) and (A2) are proved for the word $w_1 = u(s_l s_k)^{m_{kl} v}$. Furthermore, assume that the transformation of $w$ into $w_1$ is the first step in the transformation of $w$ into $w''$. 

Either there are modules $U'_{(1)}, Y_{(1)}$ so that the exact sequence given by
the inductive assumptions is
\[ 0 \longrightarrow M_0 \longrightarrow M_{m_{ij} - 2} \oplus U'_{(1)} \longrightarrow Y_{(1)} \longrightarrow 0, \]  
(3.24)
or we can write the exact sequence given by the inductive assumption as a
$\tau$-translate or a $\tau^{-1}$-translate of (3.24).

Let $w_1^i \equiv w_1''$ with
\[ \rho(w_1^i) = \rho'_1(r_1, h_1)\rho'_2(s', i)\rho'_3 \]
and
\[ \rho(w_1'') = \rho'_1\rho'_2(s', i)(l' + 1, j)\rho'_4, \]
so that either $\rho'_2 = \rho'_4$, or a pair $(q, g)$ is in $\rho'_4$ if and only if $(q - 1, g)$ is in $\rho'_3$.

Furthermore, we can assume without loss of generality that $m_{kl}$ is even.
Then there are $q_1, q_2 \in \mathbb{N}_0$ and a series of pairs $\rho''$ so that
\[ \rho(w) = \rho(u)(q_1 - \frac{m_{kl}}{2} + 1, k)(q_2 - \frac{m_{kl}}{2} + 1, l) \ldots (q_1, k)(q_2, l)\rho''. \]

We can assume that $(r, h)$ is in the series of pairs $\rho(u)(q_1 - \frac{m_{kl}}{2} + 1, k)(q_2 - \frac{m_{kl}}{2} + 1, l) \ldots (q_1, k)(q_2, l)$, since otherwise there is nothing to show. Analogously, we assume that the pair $(r_1, h_1)$ is in the series of pairs $\rho(u\{s_t s_k\}^{m_{kl}})$. Furthermore, we can assume $w_1'' \neq w''$ and $M_0 > (q_2, l)$: Let $x$ be the word with $\rho(x) = \rho_1(r, h)\rho_2(s, i)$ and $x'$ be the word so that $w = xx'$. Then there is some $x''$ so that $w'' = xx''$.

Analogously to Lemma 3.3.5, if $m_{lk} \geq 3$, then there is some $X \in \mathcal{C}_w$ so that
\[ 0 \longrightarrow (q_2, l) \longrightarrow (q_1, k) \oplus X \longrightarrow (q_1 - \frac{m_{kl}}{2} + 1, k) \]  
(3.25)
is an exact sequence.

We have two different cases to consider:

First, assume that $u$ is also an initial subword of $w''$. Then $u\{s_t s_k\}^{m_{kl} - 1}$
is an initial subword of $w''$, since $x$ with $\rho(x) = \rho_1(r, h)\rho_2$ is reduced. Furthermore, $w <_l w_1$ and $(q_1 - \frac{m_{kl}}{2} + 1, k) = (r, h)$.

If $(r_1, h_1) \in \mathcal{C}_w$, then we can set $Y := Y_{(1)}$ and we have $(q_1, k) = (r_1 - 1, h_1)$.

(A1) holds by the inductive assumption, (A2) holds by (3.25), (A1) and
Lemma 3.2.13.

On the other hand, if $(r_1, h_1) \notin \mathcal{C}_w$, then $(q_1, k) = (r_1, h_1)$. Furthermore, if we have $Y_{(1)} = (r_1, h_1)E^{(m_{ij} - 3)}$, then (A1), (3.25) and Lemma 3.2.13 give
an exact sequence of the form (3.22) with $Y = (r, h)^E(m_{ij} - 3)$. (A1) holds obviously.

If $Y_{(1)} \neq (r_1, h_1)^E(m_{ij} - 3)$, then we set $Y := Y_{(1)}$ and we only need to prove that (A2) holds. Analogously to above, this is the result of Lemma 3.2.13 and (3.25).

It remains to prove the assumption in the case that $\underline{w}$ is not an initial subword of $\underline{w}'$.

If $\underline{w} <_l \underline{w}_1$, then $\rho(w_1)$ contains the series of pairs

$$\rho(\underline{w})(q_2 - \frac{m_{kl}}{2} + 1, l) \ldots (q_1, k)(q_2, l)(q_1 + 1, k)$$

and the exact sequence given by the induction is either (3.24) or the $\tau$-translate of (3.24). We can assume without loss of generality, that some indecomposable direct summand of $Y_{(1)}$ or $\tau Y_{(1)}$ respectively is smaller than $(q_1 + 1, k)$. Otherwise, the arguments below hold analogously for an exact sequence given by (A2).

By Proposition 3.2.8, the exact sequence yielded by the inductive assumption is given by sequences of modules

$$(X_1, X_2, \ldots, X_m)$$

$$(X'_1, X'_2, \ldots, X'_m)$$

$$(Y_1, Y_2, \ldots, Y_m)$$

that fulfil (S1) - (S3). By Remark 3.2.4, we can assume that $X'_\gamma = X'_\gamma$ for all $X'_\gamma$.

In the following we begin with the case where the exact sequence given by the inductive assumption is (3.24).

Since $M_0 > (q_1 + 1, k)$ and by Lemma 3.2.3, these series of modules yield an exact sequence

$$0 \longrightarrow M_0 \longrightarrow X_\gamma \oplus X'_\gamma \longrightarrow Y_\gamma \longrightarrow 0$$

so that for every $(r'_1, h'_1) \mid Y_\gamma$ the inequality $(q_1, k) < (r'_1, h'_1)$ holds. By the inductive assumption, we can even assume that $(q_1, k) < (r'_1, h'_1) \leq (q_1 + 1, k)$ for all $(r'_1, h'_1) \notin C_{w_1}$. So either $(r'_1, h'_1) = (q_2, l)$ or $(r'_1, h'_1) = (q_1 + 1, k)$.

Furthermore, there is an irreducible morphism $X_\gamma \rightarrow (r'_1, h'_1)$ and $(q_1 + 1, k) \leq X_\gamma \notin C_{w}$. Analogously to Lemma 3.3.5, if $m_{lk} \geq 3$, then there is some $X \in C_{w}$ so that

$$0 \longrightarrow (q_1 + 1, k) \longrightarrow (q_2, l) \oplus X \longrightarrow (q_1 - \frac{m_{lk}}{2} + 1, k)$$

(3.27)
with $X \in C_w$. Together with the $\tau$-translate of the exact sequence (3.25), this shows that $X_\gamma \neq (q_2+1, l)$ and $X_\gamma \neq (q_1+1, k)$: Otherwise, by Lemma 3.2.13, we would get an exact sequence where either $(q_1 - \frac{m}{2}, 1, k)$ or $(q_2 - \frac{m}{2}, 1, l)$ is a direct summand of the last term, but every direct summand of the middle term is in $C_w$. This is a contradiction to the inductive assumption.

Since $(q_1 + 2, k) > X_\gamma > (q_1 + 1, k)$, we have $\tau^{-1}X_\gamma \in C_w$. Inductively, $Y_{(1)} \in C_w$ and $(q_1, k) < (r_1', h_1')$ for every $(r_1', h_1') \mid Y_{(1)}$.

If $(q_1 + 1, k) \mid Y_\gamma$ for any exact sequence of the form (3.26), then there is such a sequence so that $Y_\gamma \in C_w$, but $Y_\gamma \notin C_w$.

Otherwise, the sequence (3.24) is already of the form (3.22).

In the latter case, it is easily seen that this sequence fulfills (A1) and (A2): the former holds by the inductive assumption. Define $V, V', C'$ as in (A2) and let $V_{(1)}, V'_{(1)}$ be the corresponding modules, $C'_{(1)}$ the corresponding category for the sequence (3.24). Then $V = V_{(1)}, V' = V'_{(1)}$ and $C' = C'_{(1)}$. Thus, assertion (2) also holds.

So assume that there is some $\gamma$ with $(q_1 + 1, k) \mid Y_\gamma \in C_w$. This sequence is of the form (3.22) and (A1) holds. Let $\alpha \in \mathbb{N}$ be the maximal exponent so that $(q_1 + 1, k)\alpha \mid Y_\gamma$.

By construction, we can write $X_\gamma \oplus X'_\gamma = B_1 \oplus B'_1 \oplus M_{m_{ij}, -2}$ so that there is an exact sequence

$$0 \longrightarrow B_1 \longrightarrow C_1 \oplus (q_1 + 1, k)^{\alpha} \longrightarrow \tau^{-1}B_1.$$  

By Remark 3.2.4 and Lemma 3.2.13, $U_{(1)}' = B'_1 \oplus C_1$. If $Y_\gamma = D \oplus (q_1 + 1, k)^{\alpha}$, then $Y_{(1)} = D \oplus \tau^{-1}B_1$.

So we can write $V_{(1)}' = B''_1 \oplus C''_1$ with $B''_1 \mid B'_1$ and $C''_1 \mid C_1$. We get $V = B''_1 \oplus B_1$. By Proposition 3.2.8, Lemma 3.2.13 and the inductive assumption, assertion (2) is fulfilled.

If we still have $w < l w_1$, but the exact sequence given by the inductive assumption is the $\tau$-translate of (3.24), then analogously we have $\tau Y_{(1)} \in C_w$ and $(q_1, k) < (r_1', h_1')$ for all direct summands $(r_1', h_1')$ of $\tau Y_{(1)}$. Suppose that $\tau Y_{(1)}, Y_{(1)}, \ldots, \tau^{-o+1}Y_{(1)} \in C_w$ and $\tau^{-o}Y \notin C_w$. If $o > 0$, then (3.24) is of the form (3.22). By the inductive hypothesis, (A2) is fulfilled.

If $o = 0$, then we use assertion (A2) of the inductive hypothesis: by Lemma 3.2.13, there is an exact sequence of the form (3.22) that fulfills (A1) and (A2).

It only remains to prove the case $w_1 < l w$. Then $\rho(w_1)$ contains the series of pairs

$$\rho(w)(q_2 - \frac{m_{kl}}{2}, l)(q_1 - \frac{m_{kl}}{2} + 1, k) \ldots (q_2 - 1, l)(q_1, k).$$
CHAPTER 3. A CONNECTION TO THE WEYL GROUP

The exact sequence given by the inductive assumption is either (3.24) or the $\tau^{-1}$-translate. In the first case, we show (analogously to above) that $Y_{(1)} \in \mathcal{C}_{\omega_1}$ and $(q_1, k) < (r'_1, h'_1)$ for all $(r'_1, h'_1) \mid Y_{(1)}$.

If $(q_2, l) \mid Y_{(1)}$, then the sequence (3.24) is already of the form (3.22) and the assertions hold.

Otherwise, $\tau^{-1}Y_{(1)} \notin \mathcal{C}_{\omega_1}$. Analogously to before, $(q_1, k) \mid \tau^{-1}X'_{(1)}$ by Lemma 3.3.5 and (A2): If $(q_1, k) \mid \tau^{-1}X'_{(1)}$ we use Lemma 3.2.13 and get an exact sequence

$$0 \to \tau^{-o-1}V'_{(1)} \to \tau^{-o-1}Y_{(1)} \oplus U''_{(1)} \to Y'_{(1)} \to 0,$$

where $(q_1 - \frac{m}{r}, k) \mid Y'_{(1)}$ or $(q_2 - \frac{m}{r}, l) \mid Y'_{(1)}$ but every direct summand of $U''_{(1)}$ is in $\mathcal{C}'_{(1)}$. This is a contradiction to the inductive assumption.

For every module $B$ with an irreducible morphism $B \to (q_2, l)$, we have $(q_2 - 1, l) < \tau^{-1}B < (q_2, l)$. If $B \mid X'_{(1)}$, then $\tau^{-1}B \in \mathcal{C}_{\omega_1}$: all modules between $(q_1, k)$ and $(q_2, l)$ are in $\mathcal{C}_{\omega_1}$, since the exact sequence given by the inductive assumption is (3.24).

So $B \notin \mathcal{C}_{\omega_1}$ by the definition of $X'_{(1)}$. Thus, we also have $B \notin \mathcal{C}_{\omega_1}$.

Let $\alpha$ be the maximal integer so that $(q_2, l)^\alpha \mid Y_{(1)}$.

Analogously to the arguments above, we can define $B_1 \mid X_{(1)}$ so that there is an exact sequence

$$0 \to B_1 \to C_{1} \oplus (q_1 + 1, k)^\alpha \to \tau^{-1}B_1.$$

As before, the assertion (A2) of the inductive hypothesis and Lemma 3.2.13 show that there is an exact sequence of the form (3.22) and that (A1) and (A2) are fulfilled.

If the exact sequence yielded by the inductive hypothesis is the $\tau^{-1}$-translate of (3.24), then similar to the arguments above we get an exact sequence (3.22) so that $Y \in \mathcal{C}_{\omega_1}$ and $(q_1, k) < (r', h')$ for every direct summand $(r', h')$ of $Y$. This exact sequence fulfills (A1) and (A2).

\[\square\]

3.5 The first direction

In this section we show inductively that for every $w \in W$, the category $\mathcal{C}_w$ is submodule closed. Afterwards, it only remains to show that every cofinite, submodule closed category is of the form $\mathcal{C}_w$.

We begin with the basis of the induction:
Lemma 3.5.1. Let $m_{ij} < \infty$ and $U_1, \ldots, U_{m_{ij}-1} \in C_w$ and $M_0 \notin C_w$, then $C_w$ is not submodule closed and $w$ is not leftmost.

Proof. Since $M_0 \subseteq U_{m_{ij}-1}$, the category $C_w$ is not submodule closed. Let $w$ be a word for the element $w \in W$. By Definition 3.3.4, $M_0 = (s, i)$ and $M_1 = (t, j)$.

Suppose that $M_1 \in C_w$. Let $u$ be the initial subword of $w$ that is defined through the inequality $(r, k) \leq (s - 1, i)$ for every pair $(r, k)$ in $\rho(u)$.

Then there are reflections $s_{k_1}, s_{k_2}, \ldots, s_{k_m}$ and a word $v$ so that

$$w = us_{k_1}s_{k_2}\ldots s_{k_m}v.$$

Since $U_1 \in C_w$, we have

$$m_{k_1} = m_{k_2} = \cdots = m_{k_m} = 2$$

by Lemmas 3.3.1 and 3.2.7. So

$$w' = us_{i}s_{k_1}s_{k_2}\ldots s_{k_m}v$$

is equivalent to $w$ and thus a word for $w$. Since $(r, k) \leq (s - 1, i)$ for all reflections $(r, k)$ in $\rho(u)$, we see that either $w' \ll w$, or $w$ is not reduced.

Clearly, the same argument holds if $M_m \in C_w$ for some $1 \leq m \leq m_{ij} - 1$.

It remains to prove that $C_w$ is not submodule closed if $M_{m_{ij}-1} \notin C_w$. Without loss of generality, we can assume that $M_{m_{ij}-1} = (p, i)$ for some $p \in \mathbb{N}$ and $M_{m_{ij}-2} = (q, j)$ for some $q \in \mathbb{N}$. Suppose that $(r, k)$ is a pair in $w$. If $(q - 1, j) < (r, k) < (t, j)$ then we use that $U_1, U_2, \ldots, U_{m_{ij}-1} \in C_w$ and get $m_{jk} = 2$ by 3.3.6. If $(p, i) < (r, k) < (s, i)$ then $m_{ik} = 2$.

Let $u'$ be the initial subword of $w$ that is defined through the inequality $(r, k) \leq (q - 1, j)$ for every pair $(r, k)$ in $\rho(u)$.

Then there are reflections $s_{k_1}, \ldots, s_{k_m}$ with

$$m_{k_1} = m_{k_2} = \cdots = m_{k_m} = 2$$

so that $w \equiv w'$ for

$$w' = u's_{k_1}\ldots s_{k_m}\{s_is_j\}^{m_{ij}v'}.$$

(3.28)

So $w$ is also equivalent to

$$w'' = u's_{j}s_{k_1}\ldots s_{k_m}\{s_is_j\}^{m_{ij}-1}v'$$

(3.29)

and either $w'' \ll w$ or $w$ is not reduced.

This proof even shows the following:
Corollary 3.5.2. If \( m_{ij} < \infty \), \( M_0, M_1, \ldots, M_{m_{ij}-1} \notin C_w \) and \( w \) is reduced, then there is a word \( w' \equiv w \) with \( w' <_I w \), pairs \((r, h) = M_{m_{ij}-1}, (r', h') \neq (r, h)\) and series of pairs \( \rho_1, \rho_2, \rho_3\) so that

\[
\rho(w') = \rho_1(r, h) \rho_2 \quad \text{and} \quad \rho(w) = \rho_1(r', h') \rho_3.
\]

For the inductive step, we still need some lemmas:

Lemma 3.5.3. Suppose that for some \( w \), we have \( M_0 \notin C_w \) and \( M_0 \) is a submodule of \( U \). Let \( U_{m_{ij}-1} \) be as in Lemma 3.3.5 with modules \( (r_k, l_k) \notin C_w \) for \( 1 \leq k \leq a \) so that \( \bigoplus_{k=1}^a (r_k, l_k) \mid U_{m_{ij}-1} \).

Let

\[
\begin{align*}
(X_1, X_2, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_m)
\end{align*}
\]

be the sequences of modules that yield the exact sequences

\[
\eta_k : 0 \to M_0 \xrightarrow{f_k} M_k^{E(k)} \oplus U_k \xrightarrow{M_k^{E(k+1)}} 0
\]

for all \( 1 \leq k \leq m_{ij} - 1 \).

Then one of the following holds:

(a) There is some \( 1 \leq m' \leq m_{ij} - 1 \) and some \( U' \in C_w \) with a monomorphism \( M_{m'} \to U' \).

(b) If \((X_1, \ldots, X_m, X_{m+1}, \ldots X_{m'})\) is part of a triple of sequences that fulfils (S1) - (S5), there is some \( 1 \leq k \leq m' \) so that \( M_{m_{ij}} \mid X_k \oplus X_k' \). Furthermore, for \( l \in \mathbb{N} \), \( 1 \leq k \leq a \) and \( M_{m_{ij}-1} < (r_k - l, l_k) \), we have \( (r_k - l, l_k) \notin C_w \).

(c) \( a = 1 \), \( m_{il_1} + m_{jl_1} = 5 \) and there is no indecomposable morphism \( M_{m''} \to (r_1, l_1) \) for \( m'' < m_{ij} - 3 \).

(d) \( a = 1 \) and \( (r_1, l_1) \in C_w \) for some \( 1 \leq k \leq a \), then (a) holds.

Proof. By Corollary 3.3.6, for all \( 1 \leq k \leq a \), there are some \( X \in I, \beta_k \in \mathbb{N} \) and \( 2 \leq m_k \leq m_{ij} \) so that the AR-sequence that starts in \((r_k, l_k)\) is of the form

\[
0 \to (r_k, l_k) \to M_{m_k}^{\beta_k} \oplus X \to (r_k - 1, l_k) \to 0
\]

First, suppose that \( a > 1 \). By Lemma 3.3.3 and Lemma 3.3.5, there is an exact sequence

\[
0 \rightarrow M_0 \rightarrow U_{m,ij-1} \rightarrow M_{m,ij} \rightarrow 0.
\]

and we can assume that \( X_m \oplus X_m' = U_m \) and \( Y_m = M_{m,ij} \). By Corollary 3.2.9, we can set \( X_m := (r_1, l_1) \) and \( X_{m+1} := (r_2, l_2) \). If \( M_{m,1} = M_{m,2} = M_{m,ij} \), then \( M_{m,ij} \mid X_{m+2} \oplus X_{m+2}' \) by (S3). Thus (b) is fulfilled.

Otherwise, set \( \gamma = m + m_{ij} + 2 - m_1 \) , \( \delta = \gamma + m_{ij} - m_2 \)

\[
X_{m+3} := M_{m,1}, X_{m+4} := M_{m,1+1}, \ldots, X_{\gamma} := M_{m,ij-1}
\]

and

\[
X_{\gamma+1} := M_{m,2}, X_{\gamma+2} := M_{m,2+1}, \ldots, X_{\delta} := M_{m,ij-1}.
\]

Then \( M_{m,ij} \mid X_{m+1} \oplus X_{m+2} \) and (b) holds.

On the other hand, suppose that there is an indecomposable morphism \( M_{m} \rightarrow (r_1, l_1) \) for some \( m'' < m_{ij} - 3 \). By Corollary 3.3.6, we have \( (r_1-1, l_1) \mid U_{m,ij-1} \). By the construction of \( \eta_{m''+1} \) from \( \eta_{m''+1} \), there must be a module \( U_{m''+1} \) so that we can write \( \eta_{m''+1} \) as the following:

\[
0 \rightarrow M_{0} \rightarrow M_{m''+1} \rightarrow (r_1, l_1) E(m'') \rightarrow 0.
\]

By Corollary 3.2.6 and the monomorphism \( M_0 \rightarrow U \), we get a monomorphism

\[
M_{m,ij-1} \rightarrow U \oplus (r_1 - 1, l_1).
\]

So either (a) is fulfilled or \( (r - 1, l) \notin C_w \). In this case, \( a > 1 \) and (b) is fulfilled.

Finally, suppose that \( m_{d_1} + m_{d_2} > 5 \) and \( a = 1 \). By Lemma 3.3.1, \( \beta_1 > 1 \) if either \( m_{d_1} \geq 4 \) or \( m_{d_2} \geq 4 \). We can set \( X_{m+1} := (r_1, l_1) \) and \( X_{m+2} := X_{m+3} := M_{m,1} \). Analogously to the case \( a > 1 \), this yields some \( m' \in N \) so that \( M_{m,ij} \mid X_{m'} \oplus X_{m'}' \).

If \( m_{d_1} = m_{d_2} = 3 \), then either (d) holds, or we can write the AR-sequence that starts in \((r_1, l_1)\) as

\[
0 \rightarrow (r_1, l_1) \rightarrow M_{m,1} \oplus M_{m,1+1} \oplus X' \rightarrow (r_1 - 1, l_1) \rightarrow 0 \quad (3.31)
\]

for some \( X' \in \mathcal{I} \) and some \( 1 \leq m_1 \leq m_{ij} - 1 \). Since \( X_{m+1} := (r_1, l_1) \), \( X_{m+2} := M_{m,1} \) and \( X_{m+3} = M_{m,1+1} \), we see that (b) holds. \( \Box \)
We can generalize this lemma in the following way, which we will need for an induction:

**Remark 3.5.4.** Let us assume that the assumptions of Lemma 3.5.3 hold, except that \( M_0 \) is not a submodule of some \( U \in \mathcal{C}_w \) but of \( X \oplus U \) for some indecomposable \( X \not\in \mathcal{C}_w \). If \( X = M_k \) for some \( 1 \leq k \leq m_{ij} - 1 \), we alter the assumptions on the sequences (3.30) accordingly so that they fulfil (S1) - (S3) with respect to \( M_0 \) and \( X \oplus U \).

Furthermore, we assume that \( X^2 \mid X_k \oplus X'_k \) for some \( 1 \leq k \leq m \): then for every sequence \((X_1, \ldots, X_m, X_{m+1}, \ldots, X_m')\) which is part of a triple of sequences that fulfils (S1) - (S5), we have some \( 1 \leq k \leq m' \) so that \( X_k = X' \).

So every argument in the proof of 3.5.3 still holds and we still get that one of the conditions (a) - (d) must be fulfilled.

If we take a look at case (b) of 3.5.3 and suppose that \( a = 1 \), then by Lemma 3.2.13, there is a monomorphism \((r_1, l_1) \hookrightarrow M_{m_{ij}} \oplus U\). So there are sequences of modules

\[
\begin{align*}
&\left(\overset{1}{\underset{1}{X_1}}, \overset{1}{\underset{1}{X_2}}, \ldots, \overset{1}{\underset{1}{X_o}}\right) \\
&\left(\overset{1}{\underset{1}{X'_1}}, \overset{1}{\underset{1}{X'_2}}, \ldots, \overset{1}{\underset{1}{X'_o}}\right) \\
&\left(\overset{1}{\underset{1}{Y_1}}, \overset{1}{\underset{1}{Y_2}}, \ldots, \overset{1}{\underset{1}{Y_o}}\right)
\end{align*}
\]

which fulfil (S1) - (S5) with respect to \((r_1, l_1)\) and \(M_{m_{ij}} \oplus U\). Since \(M_{m_{ij}} \mid X_k \oplus X'_k\) for some \( 1 \leq k \leq m' \), there must be some \( k' \) so that \( M_{m_{ij} - 1} \mid X_{k'} \oplus X'_{k'}\).

We still need a result about the case \( a > 1 \):

**Lemma 3.5.5.** Suppose that for some \( w \), we have \( M_0 \not\in \mathcal{C}_w \) and \( M_0 \) is a submodule of \( U \in \mathcal{C}_w \). Let \( U_{m_{ij}-1} \) be as in Lemma 3.3.5 with modules \((r_k, l_k) \not\in \mathcal{C}_w\) for \( 1 \leq k \leq a \) so that \( \bigoplus_{k=1}^{a} (r_k, l_k) \mid U_{m_{ij}-1} \).

Furthermore, suppose that for all \( 1 \leq k \leq a \), there is an irreducible morphism \( M_m \rightarrow (r_k, l_k) \) for some \( 0 \leq m < m_{ij} - 2 \).

If \( a > 1 \), then one of the following holds:

(a) There is some \( N < M, N \not\in \mathcal{C}_w \) that is a submodule of some \( U' \in \mathcal{C}_w \).

(b) We have \( a = 2, m_{i_1, i_2} = 2, m_{j_1, k} = 2 \) for \( 1 \leq k \leq 2 \) and \( m_{ij} = 3 \).

(c) We have \( a = 2, l_1 = l_2, m_{i_1} = 3 \) and \( m_{ij} = 3 \).

In case (b),

\[
M_{m_{ij}}, M_{m_{ij} + 1}, M_{m_{ij} + 2}, M_{m_{ij} + 3} \not\in \mathcal{C}_w.
\]

and

\[
(r_1 - 1, l_1), (r_2 - 1, l_2), (r_1 - 2, l_1), (r_2 - 2, l_2) \not\in \mathcal{C}_w.
\]
3.5. THE FIRST DIRECTION

Proof. The proof is analogous to that of Lemma 3.3.5. First note that if \( m_{ij} = 3 \), we get \( m_{ilk} \geq 3 \) for \( 1 \leq k \leq a \), since there is an irreducible morphism \( M_m \to (r_k, l_k) \) for some \( 0 \leq m < m_{ij} - 2 \).

If \( m_{il_2} \neq 2 \), we can exchange \( j \) and \( l_2 \) in the calculations below.

Define \( \alpha_{ki}, \beta_{ki}, \alpha_{kj}, \beta_{kj} \) analogous to \( \alpha \) and \( \beta \) with \( i, l_k \) and \( j, l_k \) instead of \( i, j \). If \( m_{il_2} = 2 \), but (b) is not fulfilled, then

\[
\alpha \beta + \sum_{k=1}^{a} (\alpha_{ki} + \alpha_{kj})(\beta_{ki} + \beta_{kj}) \geq 4.
\]

By Corollary 3.2.11, we can do completely analogous calculations to the case \( \alpha \beta = 4 \). In these calculations, we construct exact sequences which contain modules of the form \( M_o, (o', l_1) \) and \( (o'', l_2) \) for some \( o, o', o'' \in \mathbb{N} \).

It remains to show that (a) is fulfilled if any of these modules is in \( C_w \).

By Lemma 3.5.3, if \( (r_k - 1, l_k) \in C_w \) for some \( 1 \leq k \leq a \), then (a) holds. The rest follows inductively with the same argument as in 3.5.3.

Now we can show the following, which is the last lemma that we need to prove the first direction of the main theorem:

**Lemma 3.5.6.** Let \( w \) be a word so that \( (s, i) = M_0, M_1, \ldots, M_{m_{ij} - 1} \notin C_w \). Then there are words \( u, v \) so that \( w = u s v \) and there is some \( \rho \) with \( \rho(w) = \rho(u) s \rho(v) \).

Suppose that there is some \( U \in C_w \) with a monomorphism \( M_0 \to U \) and for every \( X < M_0 \) with some \( U' \in C_w \) and a monomorphism \( X \to U' \), we have \( X \in C_w \).

Then there exists some \( u' \) so that

\[
w \equiv u'(s_1 s_2) \ldots \ldots (s_1 s_2) \ldots (s_1 s_2).
\]

**Proof.** By Lemma 3.3.3 and 3.3.5, Proposition 3.2.8 yields an exact sequence

\[
0 \longrightarrow M_0 \longrightarrow U_{m_{ij} - 1} \longrightarrow M_{m_{ij}} \longrightarrow 0.
\]

If \( (r, l) \in C_w \) for all \( (r, l) \) with \( (r, l) \notin U_{m_{ij} - 1} \), then (3.33) is obvious by Corollary 3.3.6.

Otherwise, we get \( M_{m_{ij}} \notin C_w \), since there is a monomorphism \( U_{m_{ij} - 1} \to M_{m_{ij}} \oplus U \) by Lemma 3.2.6.

There are direct summands \( (r_1, l_1), \ldots, (r_a, l_a) \notin C_w \) of \( U_{m_{ij} - 1} \) so that \( M_{m_{ij} - 1} < (r_k, l_k) \) for all \( 1 \leq k \leq a \).

We can assume without loss of generality that \( m_{ij} \) is odd; otherwise we only need to exchange \( s_j \) and \( s_i \) in the arguments below.
If for all $1 \leq k \leq a$, there is no morphism $M_m \to (r_k, l_k)$ for any $0 \leq m < m_{ij} - 2$, then $m_{ii} = 2$ and for some word $x$ we have

$$w = x s_i s_{l_1} \cdots s_{l_a} \{s_j s_i\}^{m_{ij} - 1}$$

So we can assume that for some $(r_k, l_k)$, there is a morphism $M_m \to (r_k, l_k)$ for some $0 \leq m < m_{ij} - 2$.

By Lemma 3.5.3, one of the following cases hold:

(a) $a = 1$, $m_{ii} + m_{jj} = 5$ and there is no indecomposable morphism $M_{m''} \to (r_1, l_1)$ for $m'' < m_{ij} - 3$.

(b) There is some $m'$ so that $M_{m_{ij}} \mid X_{m'} \oplus X'_{m'}$

In case (a) there are words $w'', v$, so that either

$$w = w'' s_i s_{l_1} \{s_j s_i\}^{m_{ij} - 1} v$$

(3.34)

or both $m_{jj} = 2$ and

$$w = w'' s_i s_{s_{l_1}} \{s_j s_i\}^{m_{ij} - 2} v$$

(3.35)

Then there is some word $u_1$ so that $w \equiv u_1 s_i s_{l_1} \{s_i s_j\}^{m_{ij} - 1} v$. If $m_{ii} = 2$, there is nothing to show. If $m_{jj} = 2$, then we use that $M_{m_{ij}}$ is of the form $(q, j)$ for some $q$ and there is a monomorphism $U_{m_{ij} - 1} \to M_{m_{ij}} \oplus U$; Let $u_2$ be the subword of $u_1$ that does not contain the reflection that corresponds to $(q, j)$. By the inductive assumption, $u_2 s_i s_{l_1}$ is equivalent to a word $u_3 \{s_i s_{l_1}\}^{m_{ii}}$.

Now we go back to looking at $u_1$, not $u_2$. Since $m_{ii} = 3$ and $m_{jj}$, there is some word $u_4$ with $u_1 s_i s_{l_1} \equiv u_4 \{s_i s_{l_1}\}^{m_{ij}}$ and thus

$$w \equiv u'' s_i s_{s_{l_1}} \{s_j s_i\}^{m_{ij}} v$$

(3.36)

for some word $w''$.

On the other hand, suppose that

$$M_{m_{ij}} \mid X_{m'} \oplus X'_{m'}.$$  

(3.36)

Since we have $\bigoplus_{k=1}^{a} (r_k, l_k) \mid U_{m_{ij}}$, there is a monomorphism

$$\bigoplus_{k=1}^{a} (r_k, l_k) \to U \oplus M_{m_{ij}}.$$  

(3.37)

Assume that $(r_1, l_1) \leq (r_2, l_2) \leq \cdots \leq (r_a, l_a)$. 


Let the AR-sequence that starts in \((r_a, l_a)\) be
\[
0 \longrightarrow (r_a, l_a) \longrightarrow M_{m_1} \oplus Z_1 \longrightarrow (r_a - 1, l_1) \longrightarrow 0.
\]

If \(a = 1\), then by Lemma 3.5.3, \(w\) must have the form (3.34) or (3.35). The only difference to case (a) is that \(m_{i_1} + m_{j_1} > 5\).

We can use case (a) as the basis of an induction: Instead of the modules \(M_0, M_1, (r_1, l_1)\), we take \((r_1, l_1), M_{m_1}, M_{m_1+1}\) and use the same arguments as before. Since \(M_{m_{ij}} \mid X_{m'} \oplus X_{m'}\), we can use Remark 3.5.4 and either we get the analogue to case (a) above or the analogue to the case (b). In the first case, we get \(m_{i_1} + m_{i_2} = 5\) or \(m_{j_1} + m_{j_2} = 5\) and there is some \(w_1\) so that \(w\) is equivalent to a word with the subword \(\{s_i, s_j\}^{m_{ij}}\) if \(m_1\) is even and \(\{s_i, s_j\}^{m_{ij}}\) is odd. Thus we also get \(w = w_1\{s_i, s_j\}^{m_{ij}}\) for some words \(w_1, w\).

In the case (b), we continue this inductively.

After finitely many steps we get
\[
w = w_1\{s_i, s_j\}^{m_{ij}}w.
\]

If \(a \neq 1\), then \(a = 2\) by Lemma 3.5.5. If \(l_1 = l_2\), then \(l_1 = l_2, m_{i_1} = 3\) and \(m_{i_2} = 3\). We can exchange \(j\) and \(l_1\) to get the case \(a = 1\). Otherwise, \(m_{i_1} = 2, m_{i_2} = 3, m_{j_1} = 2\) for \(1 \leq k \leq 2\) and \(m_{i_2} = 3\). Furthermore,
\[
M_{m_{ij}}, M_{m_{ij}+1}, M_{m_{ij}+2}, M_{m_{ij}+3} \notin C_w.
\]

and
\[
(r_1 - 1, l_1), (r_2 - 2, l_2), (r_1 - 2, l_1), (r_2 - 2, l_2) \notin C_w.
\]

Analogously to before, we see inductively that \(w\) is equivalent to a word with the subword
\[
s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i.
\]

(For the purpose of this induction, we can treat the word above completely analogously to a word of the form \(\{s_i, s_j\}^{m_{ij}}\) with \(m_{ij} = 6\). As in the proof of 3.5.5, all calculations are the same by Corollary 3.2.11.)

We have the following equivalences, where bold reflections denote those which differ from the reflections in the word above:
\[
\begin{align*}
&s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i \\
&\equiv s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i \\
&\equiv s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i \\
&\equiv s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i \\
&\equiv s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i
\end{align*}
\]

\(\equiv s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i, s_1, s_2, s_j, s_i.\)
So \( w \) is equivalent to a word with the subword \( s_is_jsi \) and the assertion is true.  

Finally, we can prove the first direction of our main result:

**Lemma 3.5.7.** If \( w \) is a leftmost word, then \( C_w \) is submodule closed.

**Proof.** Suppose that \( C_w \) is not submodule closed. Then there is some \( M_0 \in \text{ind} \mathcal{I} \setminus C_w \) and some \( U \in C_w \) with a monomorphism \( M_0 \hookrightarrow U \). Furthermore, we can assume that for every \( X < M_0 \) with some \( U' \in C_w \) and a monomorphism \( X \hookrightarrow U' \), we have \( X \in C_w \).

We use induction on the length \( m \) of the sequences of modules in Proposition 3.2.8 applied on \( M_0 \) and \( U \). If \( m = 1 \), then \( \bar{w} \) is not leftmost by Lemma 3.5.1.

Now suppose that \( \bar{w} \) is not leftmost if the sequences have the length \( m \) or smaller. We prove that this is also the case if they have length \( m + 1 \):

We can assume without loss of generality that \( M_1 \notin C_w \), since \( m + 1 > 1 \). On the other hand, by Lemma 3.3.5, the sequences of modules induce an exact sequence

\[
0 \longrightarrow M_0 \longrightarrow M_1^{E(1)} \oplus U_1 \longrightarrow M_2 \longrightarrow 0.
\]

So by Corollary 3.2.6, there is a monomorphism \( M_1^{E(1)} \hookrightarrow M_2 \oplus U \). Since \( M_1 < M_0 \), our assumptions yield \( M_2 \notin C_w \). By the same argument, we get \( M_3, M_4, \ldots, M_{m_{ij}-1} \notin C_w \) and by (S4) this means \( m_{ij} < \infty \).

By Lemma 3.5.6, if we choose \( u \) and \( v \) so that \( w = u s_j v \) and there is some \( \rho \) so that \( \rho(w) = \rho(u)(s, i)\rho \), then \( w' \equiv u' s_j v \) for some word \( u' \).

We still need to show that \( u' s_j v \) is equivalent to a word which is smaller than \( w \).

To do this, we use Lemma 3.1.7. Either there is nothing to show, or there are \( \rho_1, \ldots, \rho_4 \) and a pair \((r, h)\) so that

\[
\rho(w) = \rho_1(r, h)\rho_2(s, i)\rho_3
\]

and there is some \( \bar{w}' \equiv \bar{w} \) with

\[
\rho(w') = \rho_1\rho_2(s, i)(t + 1, j)\rho_3.
\]

We can assume that the word \( \bar{x} \) with \( \rho(\bar{x}) = \rho_1(r, h)\rho_2 \) is reduced, because otherwise there is nothing to prove. So by Lemma 3.4.2, there are some sequences of modules as in Lemma 3.2.3 that yield some \( U' \in \mathcal{I} \) and an exact sequence

\[
0 \longrightarrow M_0 \longrightarrow M_{m_{ij}-2} \oplus U' \longrightarrow Y \longrightarrow 0.
\]
with either $Y \in \mathcal{C}_w$ or $Y = (r, h)^{E(m_{ij}-3)}$.

By Corollary 3.2.6, there is a monomorphism

$$M_{m_{ij}-2} \hookrightarrow U \oplus Y \in \mathcal{C}_{w''}.$$

By (3.38), (3.39) and the induction hypothesis, $w''$ is not leftmost.

So there is some $w_3 \equiv w''$ with $w_3 \prec_l w''$. We still need to show that $w_3 \prec_l w$.

If $w''$ is not reduced, this is obvious. If $Y \in \mathcal{C}_w$, we can use the inductive assumption.

So suppose that $w''$ is reduced and $Y = (r, h)^{E(m_{ij}-3)}$. We denote the sequences of modules that fulfil (S1) - (S5) with respect to $M_0$ and $U$ by

$$(X_1, X_2, \ldots, X_m)$$
$$(X'_1, X'_2, \ldots, X'_m)$$
$$(Y_1, Y_2, \ldots, Y_m).$$

There are sequences of modules

$$(1^{X_1}, 1^{X_2}, \ldots, 1^{X_m})$$
$$(1^{X'_1}, 1^{X'_2}, \ldots, 1^{X'_m})$$
$$(1^{Y_1}, 1^{Y_2}, \ldots, 1^{Y_m})$$

that fulfil (S1) - (S5) with respect to $M_{m_{ij}-2}$ and $U \oplus (r, h)^{E(m_{ij}-3)}$. Let $(r', h')$ be the smallest indecomposable direct summand of $1^{Y_{m'}}$. Then $(r', h') < (r, h)$.

If there is a pair $(r'', h'') \neq (r', h')$ and series of pairs $\rho_1', \rho_2', \rho_3'$ so that we can write

$$\rho(w_3) = \rho_1'(r', h') \rho_2'$$
$$\rho(w) = \rho_1'(r'', h'') \rho_3',$$

then $w_3 \prec_l w''$ implies $w_3 \prec_l w$.

A simple induction on $m'$ shows that this is indeed the case: If $m' = 1$, then $(r, h) = M_{m_{ij}-1}$, $(r', h') = M_{m_{ij}}$ and the assertion is true by Corollary 3.5.2.

By 3.2.13, the smallest direct summand of $1^{Y_{m'}}$ is also the smallest direct summand of $Y_m$ and the inductive step is obvious.

So $w$ is not leftmost and the proof is complete.

\hfill $\Box$

### 3.6 The other direction

In this section we finally conclude the proof that the map between words and full additive cofinite subcategories of $\text{mod} \ A$ introduced in Definition
3.1.13 gives rise to a bijection between the leftmost words and the cofinite submodule closed subcategories. Since every element of the Weyl group has a unique leftmost element, this gives a bijection between the Weyl group elements and the cofinite, submodule closed subcategories.

Again, we start with the basis of an induction:

**Lemma 3.6.1.** (a) Let \( w := u(s_is_j)^{m_{ij}}v \). If \( \mathcal{C}_w \) is submodule closed, then

\[
    w < u(s_is_j)^{m_{ij}}v.
\]

(b) The category \( \mathcal{C}_{u_is_is_j} \) is not submodule closed.

(c) Let \( w' := u(s_is_j)^{m_{ij}+1}v \). Then \( \mathcal{C}_{w'} \) is not submodule closed.

**Proof.** We prove (a) by contraposition. By Definition 3.1.13, \( \text{ind} \mathcal{I} \setminus \mathcal{C}_w \) consists of the modules which correspond to the reflections in \( w \).

Assume that

\[
    u(s_is_j)^{m_{ij}}v < u(s_is_j)^{m_{ij}}v = w
\]

and

\[
    \rho(w) = \rho(u) (p,i)(q,j)(p+1,i) \cdots \rho_1
\]

for a sequence of pairs \( \rho_1 \).

By Lemma 3.1.6, the module \( (q-1,j) \) exists and by Definition 3.1.13, \( \mathcal{C}_w \) contains all indecomposable, preinjective modules \( M \) with \( (q-1,j) < M < (p,i) \) or \( (p,i) < M < (q,j) \).

First, suppose that \( m_{ij} = 2 \). In this case, \( \mathcal{C}_w \) contains the middle term of the AR-sequence that starts in \( (q,j) \) by Lemmas 3.2.7 and 3.3.1. Since \( (q,j) \notin \mathcal{C}_w \), the subcategory is not submodule closed.

Now let \( m_{ij} \geq 3 \). In 3.3.4, we defined \( M_0 := (s,i) \), \( M_1 := (t,j) \), \( \ldots \) for some arbitrary, fixed \( s,t \). By Remark 3.3.7, we can assume without loss of generality that \( m_{ij} \) is odd and we can choose \( s,t \) so that \( M_{m_{ij}-1} = (p,i) \).

Then \( M_{m_{ij}} = (q-1,j) \neq 0 \) and by Lemma 3.3.3, \( E(m_{ij}-2) \neq 0 \). By Lemma 3.3.5, there is an exact sequence

\[
    0 \longrightarrow M_0 \longrightarrow (M_{m_{ij}-1})^{E(m_{ij}-1)} \oplus U_{m_{ij}-1} \longrightarrow M_{m_{ij}}^{E(m_{ij}-2)} \longrightarrow 0
\]

so that no \( M_0, M_1, \ldots, M_{m_{ij}-1} \) is a direct summand of \( U_{m_{ij}-1} \). By Lemma 3.3.3 we have \( E(m_{ij}-1) = 0 \), so there is a monomorphism

\[
    M_0 \hookrightarrow U_{m_{ij}-1}.
\]
It remains to show that $U_{m_{ij}^{-1}} \in C_w$ by Corollary 3.3.6: If $X$ be a direct summand of $U_{m_{ij}^{-1}}$, then $M_{m_{ij}} < X < M_0$ and thus $X \in C_w$.

By Lemma 3.2.7, part (b) is obvious.

The proof of (c) is completely analogous to the proof of (a).

Finally, we are prepared to prove that Definition 3.1.13 gives a bijection. Recall that $C_w = C_w$ if $w$ is the leftmost word for $w$:

**Theorem 3.6.2.** The map $w \mapsto C_w$ is a bijection between the elements of the Weyl group of $A$ and the cofinite submodule closed subcategories of $\mathfrak{mod} A$.

**Proof.** The map is well defined by Lemma 3.5.7 and obviously injective. It remains to prove that it is surjective, i.e. that for all cofinite submodule closed subcategories $C$ of $\mathfrak{mod} A$, there is a $w \in W$, so that $C = C_w$.

We already know that $C = C_w$ for some word from Lemma 3.4.1, so we only need to show that $w$ is leftmost.

Assume that the word $w$ for the element $w \in W$ is not leftmost. We show that $C_w$ is not submodule closed by induction on the number of Coxeter relations that are needed to transform $w$ into a smaller word.

If only one relation is needed, then the theorem is the result of Lemma 3.6.1. Now suppose that the assertion is true if we need $m$ or less relations and that we need $m + 1$ relations to transform $w$ into a smaller word.

Then there are some $1 \leq i, j \leq n$ and some words $x, x', y$ so that

$$w = x s_i y \equiv x' s_j y = w'$$

(3.40)

and

$$w' \leq l w$$

with $i \neq j$.

Thus, there are some words $w'', x''$ so that

$$w = w'' = x'' {s_i s_j}^{m_{ij}} y.$$  

Because of the inductive assumption, we can suppose that $w \leq l w''$ and that $x$ is leftmost. Obviously, we can choose $x''$ to be leftmost.

Let the reflection $s_i$ in (3.40) correspond to $M_0$. We can assume that $m_{ij} \geq 3$, since there is nothing to show if the middle term of the Auslander-Reiten sequence that starts in $M_0$ is contained in $C_w$. We can also assume that there is some word $x''$ so that $x = x'' s_j$: If there are $s_{k_1}, \ldots, s_{k_m}$ with $m_{k_1,i} = \cdots = m_{k_m,i}$ and $x = x'' s_j s_{k_1} \cdots s_{k_m}$, then $x s_i \equiv x'' s_j s_{k_1} \cdots s_{k_m}$ and if there is some $U \in C_{x'' s_j s_i}$ with a monomorphism $M_0 \rightarrow U$, then $U \in C_w$.

Without loss of generality, we can assume that $m_{ij}$ is odd; otherwise we relabel $i$ and $j$ and get the same arguments by Remark 3.3.7.
By Lemma 3.1.12, we can suppose that $M_0, M_1, \ldots, M_{m_{ij}-3}, M_{m_{ij}-2} \in C_w$. We show that there is some $U \in C_w$ with a monomorphism $M_0 \hookrightarrow U$.

Let $(q, j) := M_{m_{ij}-2}$. We use Lemma 3.1.7: Because $m > 1$, there is some $w_3 \equiv w$ with series of pairs $\rho_1, \ldots, \rho_4$ and a pair $(r, h)$ so that

$$\rho(w) = \rho_1(r, h)\rho_2(q, j)\rho_3(s, i)\rho_4$$  \hfill (3.41)

and

$$\rho(w_3) = \rho_1\rho_2(q, j)\rho_3(s, i)(t + 1, j)\rho_4.$$  \hfill (3.42)

By Lemma 3.4.2, if $m_{ij} \geq 3$, then there is an exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_{m_{ij}-2} \oplus U' \longrightarrow Y \longrightarrow 0$$

so that either $Y \in C_w$ or both $Y = (r, h)^E(m_{ij}-3)$ and $U' \in C_w$ hold.

We want to show that there is some $U'' \in C_w$ and a monomorphism

$$M_{m_{ij}-2} = (q, j) \hookrightarrow U'' \oplus Y.$$  \hfill (3.43)

We prove this inductively: First, note that the word $x''s_j$ is not leftmost Lemma 3.1.9 and 3.1.10.

If $w = w''$, then $(r, h) = M_{m_{ij}-1}$. So by the inductive hypothesis, there is a monomorphism $(q, j) \hookrightarrow U'' \oplus (r, h)\gamma$.

Since $E(1) = E(m_{ij} - 3)$ by table (3.18), we see that $Y = (r, h)^E(m_{ij}-3)$ and $\gamma = E(m_{ij} - 3)$.

The inductive step is completely analogous to the one in Lemma 3.4.2.

By our assumptions, $x$ is leftmost and thus $Y \notin C_w$ by Lemma 3.5.7. So $U' \in C_w$.

By Lemma 3.2.13, there is a monomorphism $M_0 \hookrightarrow U' \oplus U'' \in C_w$ and $C_w$ is not submodule closed.

\section{3.7 Some consequences}

We conclude the chapter with a generalization and a corollary:

As in [14], Section 8 we can extend the notion of leftmost words:

\begin{definition}
Define infinite words analogously to words, only as infinite instead of finite sequences. We say that an (infinite) word is leftmost if any initial subword of finite length is leftmost.
\end{definition}

Analogously to [14], Theorem 8.1, we get the following:
Theorem 3.7.2. There is a bijection between the (finite and infinite) leftmost words over $S = \{s_1, s_2, \ldots, s_n\}$ and the submodule closed subcategories of $\mathcal{I}$, the preinjective component of $\text{mod} A$.

Proof. This is completely analogous to [14], 8.1:

Let $\mathcal{C}$ be a submodule closed subcategory of $\mathcal{I}$. Since $\mathcal{I}$ contains at most a countable number of indecomposable modules, we can set

$$\text{ind } \mathcal{I} \setminus \mathcal{C} =: \{(r_1, i_1), (r_2, i_2), \ldots\}$$

and $(r_1, i_1) < (r_2, i_2) < \ldots$. Then the subcategory $\mathcal{C}_m$ with

$$\text{ind } \mathcal{I} \setminus \mathcal{C}_m = \{(r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m)\}$$

is submodule closed for all $m \in \mathbb{N}$. By Lemma 3.2.1 and Theorem 3.6.2, the words $w_m$ with

$$\rho(w_m) = (r_1, i_1)(r_2, i_2)\ldots(r_m, i_m)$$

are leftmost for all $m \in \mathbb{N}$. By Definition 3.7.1, the (infinite) word $w$ with

$$\rho(w) = (r_1, i_1)(r_2, i_2)\ldots$$

is leftmost and $\mathcal{C} = \mathcal{C}_\infty$.

On the other hand assume that the (infinite) word $w$ with

$$\rho(w) = (r_1, i_1)(r_2, i_2)\ldots$$

is leftmost. Then the words with

$$\rho(w_m) = (r_1, i_1)(r_2, i_2)\ldots(r_m, i_m)$$

are leftmost for all $m \in \mathbb{N}$. By 3.6.2, the categories $\mathcal{C}_w$ are submodule closed. Thus $\mathcal{C}_w$ is also submodule closed: if there was a module $M \notin \mathcal{C}_w$ and some module $U \in \mathcal{C}_w$ with a monomorphism $M \hookrightarrow U$, then $U \in \mathcal{C}_w$ for all $m \in \mathbb{N}$ and there is some $m \in \mathbb{N}$ so that $M \notin \mathcal{C}_w$, since $M$ is finitely generated. \qed

We can draw a further corollary. Let $A'$ be a hereditary and let the module category $\text{mod} A'$ be equivalent to the subcategory of $\text{mod} A$ with the simple modules $S_j$, $j \in J$ for some $J \subseteq \{1, 2, \ldots, n\}$. Let $\mathcal{I}_{A'}$ be the subcategory of $\text{mod} A'$ consisting of all preinjective modules.

Corollary 3.7.3. There is a bijection between the submodule closed subcategories of $\mathcal{I}_{A'}$ and the submodule closed subcategories of $\mathcal{I}$ which contain all $r^i I_i$ with $r \in \mathbb{N}_0$ and $i \in \mathbb{N} \setminus J$.

Proof. The words in the Weyl group of $A'$ are exactly the words in the Weyl group of $A$ which only consist of reflections $s_j$ with $j \in J$. \qed
CHAPTER 3. A CONNECTION TO THE WEYL GROUP
In this chapter we consider abelian length categories, a generalization of module categories over Artin algebras. For a category $\mathcal{A}$ we are interested in the lattice $S(\mathcal{A})$ of full additive subobject closed subcategories of $\mathcal{A}$. In particular, we are trying to find cases in which the lattice $S(\mathcal{A})$ is distributive.

In Section 4.1, we show that the distributivity of $S(\mathcal{A})$ is equivalent to a condition on the submodule relations in $\mathcal{A}$, which is much easier to work with. We can show in the next section that the following is an even stronger property: every subobject of an indecomposable object in $\mathcal{A}$ is itself indecomposable. Such categories are said to be of colocal type.

We characterize these categories in Section 4.3 to 4.6: First, we show that two conditions on the Ext-quiver hold. Weaker conditions hold if $S(\mathcal{A})$ is distributive. In Section 4.4, we collect some auxiliary lemmas that are mainly concerned with 2-extensions.

These results are needed in Section 4.5, in which we give different formulations and a proof of the third condition that abelian length categories of colocal type fulfil. Again, we see that a weaker condition is fulfilled if $S(\mathcal{A})$ is distributive.

In Section 4.6, we use the results of Section 4.4 to prove that every abelian length category which fulfils the three conditions is of colocal type.

Returning to the lattice $S(\mathcal{A})$, we prove in the next section that it is the Cartesian product of certain sublattices.

Finally, in the last section, we assume that $\mathcal{A} \equiv \text{mod } kQ/I$ for some field $k$, quiver $Q$ with an admissible ideal $I$. In this case, $\mathcal{A}$ is of colocal type if and only if $\mathcal{A}$ is a string algebra and no vertex in $Q$ is starting point of more than one arrow. For these algebras, we get a complete, explicit description of the lattice $S(\text{mod } \mathcal{A})$.

Note that in this chapter, we are equating objects with isomorphism classes of objects. In particular, all sums over simple objects are actually sums over isomorphism classes of simple objects.
4.1 Conditions on indecomposable objects

Let \( \mathcal{A} \) be an abelian length category. The main result of this section is a characterisation of abelian length categories with distributive lattices \( S(\mathcal{A}) \) in terms of the subobject relations between the objects of \( \mathcal{A} \).

We start with the definition of a distributive lattice, as given for example in [17], p. 69:

**Definition 4.1.1.** A lattice \( L \) is called **distributive** if

\[
(a \lor b) \land c = (a \land c) \lor (b \land c)
\]

for all \( a, b, c \in L \).

Now let \( S(\mathcal{A}) \) be the set of full additive subobject closed subcategories as in [12]. It is partially ordered by inclusion and a complete lattice.

The join \( a \lor b \) for two categories \( a, b \in S(\mathcal{A}) \) is the smallest full additive subobject closed subcategory, which contains both \( a \) and \( b \). The meet \( a \land b \) is of course the largest category in \( S(\mathcal{A}) \) that is contained in both \( a \) and \( b \).

The meet coincides with the intersection \( a \cap b \): all subobjects of direct sums of objects in \( a \cap b \) are again objects in \( a \cap b \), since \( a \) and \( b \) are subobject closed. The join consists of all subobjects of direct sums of objects in \( a \) and \( b \).

Every category in \( S(\mathcal{A}) \) is completely determined by the isomorphism classes of indecomposable objects it contains.

For a class \( \mathcal{X} \) of objects let \( \text{sub} \mathcal{X} \) be the category that consists of all subobjects of direct sums of objects in \( \mathcal{X} \). This is the smallest category in \( S(\mathcal{A}) \) that contains \( \mathcal{X} \). Set \( \text{sub} X := \text{sub}\{X\} \).

In the following case, \( S(\mathcal{A}) \) is not distributive:

**Lemma 4.1.2.** If there exists an indecomposable object \( X \in \mathcal{A} \), and objects \( Y_1, Y_2 \in \mathcal{A} \) so that \( X \in \text{sub} Y_1 \lor \text{sub} Y_2 \) but \( X \notin \text{sub} Y_i \) for all \( 1 \leq i \leq 2 \), then

\[
(\text{sub} Y_1 \lor \text{sub} Y_2) \land \text{sub} X \neq (\text{sub} Y_1 \land \text{sub} X) \lor (\text{sub} Y_2 \land \text{sub} X).
\]

**Proof.** By the assumption

\[
X \in \text{sub} Y_1 \lor \text{sub} Y_2
\]

and by definition

\[
X \in \text{sub} X,
\]
so

$$X \in (\text{sub } Y_1 \lor \text{sub } Y_2) \land \text{sub } X.$$ 

But

$$X \not\in (\text{sub } Y_1 \land \text{sub } X) \lor (\text{sub } Y_2 \land \text{sub } X),$$

since otherwise there were some objects

$$X_i \in \text{sub } Y_i \land \text{sub } X$$

for $1 \leq i \leq 2$ with a monomorphism

$$f : X \hookrightarrow X_1 \oplus X_2.$$ 

But since $X_1 \oplus X_2 \in \text{sub } X$, there is some $\alpha \in \mathbb{N}$ and a monomorphism $g$

$$g : X_1 \oplus X_2 \hookrightarrow X^\alpha.$$ 

Since $gf$ is a monomorphism, its image is isomorphic to $X$ and $g$ induces a morphism $g' : X_1 \oplus X_2 \to X$ so that $g'f$ is an isomorphism on $X$. Thus, $f$ splits.

Because $X$ is indecomposable, there is an $i \in I$, so that $X$ is a direct summand of $X_i$. Since $X_i \in \text{sub } Y_i$, this means that $X \in \text{sub } Y_i$, contrary to the assumption.

So

$$(\text{sub } Y_1 \lor \text{sub } Y_2) \land \text{sub } X \neq (\text{sub } Y_1 \land \text{sub } X) \lor (\text{sub } Y_2 \land \text{sub } X)$$

and the proof is complete. \qed

In fact, we get the following equivalence:

**Proposition 4.1.3.** The following statements are equivalent:

1. The lattice $S(\mathcal{A})$ is distributive

2. If $X \in \mathcal{A}$ is indecomposable and there are objects $Y_1, Y_2 \in \mathcal{A}$, so that $X \in \text{sub } Y_1 \lor \text{sub } Y_2$ then $X \in \text{sub } Y_i$ for some $1 \leq i \leq 2$.

3. For all index sets $I$ and categories $a_i \in S(\mathcal{A})$, $i \in I$ we have

$$\text{ind}\left(\bigvee_{i \in I} a_i\right) = \bigcup_{i \in I} \text{ind } a_i.$$
CHAPTER 4. DISTRIBUTIVE LATTICES

Proof. (1) $\Rightarrow$ (2) is clear from the Lemma 4.1.2.

(2) $\Rightarrow$ (3): The direction

\[ \text{ind}(\bigvee_{i \in I} a_i) \supseteq \bigcup_{i \in I} \text{ind} a_i \]

is clear. For the other direction, we look at an indecomposable object

\[ X \in \bigvee_{i \in I} a_i. \]

There are objects $A_i \in a_i$ with a monomorphism

\[ X \twoheadrightarrow \bigoplus_{i \in I} A_i \]

and thus

\[ X \in \bigvee_{i \in I} \text{sub} A_i. \]

The object $\bigoplus_{i \in I} A_i$ must be of finite length; thus $A_i = 0$ for all except finitely many $i \in I$. By (2) and an induction, we get

\[ X \in \text{sub} A_i \]

for at least one $i \in I$ and thus

\[ X \in a_i. \]

So

\[ X \in \bigcup_{i \in I} \text{ind} a_i \]

and

\[ \text{ind}(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \text{ind} a_i. \]

(3) $\Rightarrow$ (1): Let $a, b, c \in S(A)$. Then

\[
\text{ind}((a \lor b) \land c) = (\text{ind} a \cup \text{ind} b) \cap \text{ind} c \\
= (\text{ind} a \cap \text{ind} c) \cup (\text{ind} a \cap \text{ind} c) \\
= \text{ind}((a \land c) \lor (a \land c)).
\]

Since $a, b, c$ are completely determined by their indecomposable objects,

\[ (a \lor b) \land c = (a \land b) \lor (a \land c) \]

and $S(A)$ is distributive. \qed
We can generalize the notion of a distributive lattice as e.g. in [9], p. 1227:

**Definition 4.1.4.** A complete lattice $\Lambda$ is a *frame* if for all index sets $I$ and elements $a, b_i$ with $i \in I$ the equation

$$a \land \left( \bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \land b_i)$$

holds.

Obviously, every frame is also distributive. But in general, not every distributive lattice is a frame. An exception are lattices of subobject closed categories:

**Corollary 4.1.5.** The lattice $S(\mathcal{A})$ is distributive if and only if it is a frame.

*Proof.* This follows from part (3) of Proposition 4.1.3. \qed

## 4.2 Categories of colocal type

For the following categories, $S(\mathcal{A})$ is always distributive:

**Definition 4.2.1.** We call a category $\mathcal{A}$ of colocal type if any subobject of an indecomposable object is itself indecomposable. If there is some Artin algebra $A$ so that $\mathcal{A} = \text{mod } A$, then we also say that $A$ is of colocal type.

To show this, we need the following lemma:

**Lemma 4.2.2.** (a) If there are objects $V_1, V_2, X$ with a monomorphism

$$f = \begin{bmatrix} f_1 & f_2 \end{bmatrix} : V_1 \oplus V_2 \rightarrow X,$$

then there is also a monomorphism

$$X \rightarrow \text{Coker } f_1 \oplus \text{Coker } f_2.$$

(b) If there are objects $X, Y_1, Y_2$ with a monomorphism

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \rightarrow Y_1 \oplus Y_2,$$

then there is also a monomorphism

$$\text{Ker } f_1 \oplus \text{Ker } f_2 \rightarrow X.$$
Proof. First, we prove that (a) holds: There is an exact sequence

$$0 \to V_1 \oplus V_2 \xrightarrow{\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}} X \oplus X \xrightarrow{g} \text{Coker } f_1 \oplus \text{Coker } f_2 \to 0$$

for some morphism $g$ with $\text{Ker } g \cong V_1 \oplus V_2$. So $g$ induces a monomorphism

$$X \hookrightarrow \text{Coker } f_1 \oplus \text{Coker } f_2.$$

The proof of (b) is similar: We get a morphism $f'$ with an exact diagram

\[
\begin{array}{ccc}
0 & \to & 0 & \to & 0 \\
& \downarrow & & \downarrow & \\
0 & \to & \text{Ker } f_1 \oplus \text{Ker } f_2 & \to & X \oplus X \xrightarrow{\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}} \text{Im}(f_1) \oplus \text{Im}(f_2) & \to & 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& & \text{Ker } f_1 \oplus \text{Ker } f_2 & \to & X & \to & \text{Coker } f' \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0 \\
\end{array}
\]

So

$$0 \to \text{Ker } f_1 \oplus \text{Ker } f_2 \to X \to \text{Coker } f \to 0$$

is an exact sequence and there is a monomorphism $\text{Ker } f_1 \oplus \text{Ker } f_2 \hookrightarrow X$. \qed

**Proposition 4.2.3.** If $\mathcal{A}$ is of colocal type, then $S(\mathcal{A})$ is distributive. Furthermore, for all objects $X, Y_1, Y_2 \in \mathcal{A}$ with a monomorphism $X \to Y_1 \oplus Y_2$, there is a monomorphism $X \hookrightarrow Y_1$ or $X \hookrightarrow Y_2$.

Proof. Assume that every subobject of an indecomposable object is indecomposable. Further suppose that there are objects $X, Y_1, Y_2$, so that $X$ is a subobject of $Y_1 \oplus Y_2$. Then there is a monomorphism

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \to Y_1 \oplus Y_2$$

and by Lemma 4.2.2, we get a monomorphism $\text{Ker } f_1 \oplus \text{Ker } f_2 \hookrightarrow X$.

Since $\mathcal{A}$ is of colocal type, either $\text{Ker } f_1 = 0$ or $\text{Ker } f_2 = 0$. Thus there is a monomorphism $X \hookrightarrow Y_1$ or $X \hookrightarrow Y_2$. By Proposition 4.1.3, the lattice $S(\mathcal{A})$ is distributive. \qed
Lemma 4.2.4. If $A$ is not of colocal type, then there are objects $V_1, V_2$, non-simple objects $Y_1, Y_2$, an indecomposable object $X$ and a simple object $S$ with exact sequences

$$0 \rightarrow V_1 \oplus V_2 \rightarrow X \rightarrow S \rightarrow 0 , \quad (4.1)$$

$$0 \rightarrow X \rightarrow Y_1 \oplus Y_2 \rightarrow S \rightarrow 0 . \quad (4.2)$$

For such objects, the following sequences are exact

$$0 \rightarrow V_j \rightarrow Y_i \rightarrow S \rightarrow 0 \quad (4.3)$$

for $i, j = 1, 2$ and $i \neq j$.

Proof. Suppose that $X$ is an indecomposable object and there is a monomorphism

$$f = \begin{bmatrix} f_1 & f_2 \end{bmatrix} : V_1 \oplus V_2 \rightarrow X$$

with $V_1 \neq 0 \neq V_2$.

Let $S$ be a simple factor module of $\text{Coker} \ f$. Then there is some $V$ with $V_1 \oplus V_2 \subseteq V$ and an exact sequence

$$0 \rightarrow V \rightarrow X \rightarrow S \rightarrow 0 .$$

If $V$ is indecomposable, then it is of smaller length than $X$ and we can regard $V$ instead of $X$. So we can assume that $\text{Coker} \ f = S$ and $V_1 \oplus V_2 = V$. By Lemma 4.2.2, there is a monomorphism

$$g : X \rightarrow \text{Coker} \ f_1 \oplus \text{Coker} \ f_2 .$$

The following diagram is exact for all $i, j \in \{1, 2\}$, $i \neq j$, since all columns and the first and second row are exact:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & V_1 & V_2 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & X & S \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & V_j & \text{Coker} \ f_i & S \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]
By this, we know that the following diagram is exact, since all columns and the first and second row are exact:

\[
\begin{array}{ccc}
0 & \to & V_1 \oplus V_2 \\
\downarrow & & \downarrow \\
0 & \to & X \\
\downarrow & & \downarrow g \\
V_1 \oplus V_2 & \to & \text{Coker } f_1 \oplus \text{Coker } f_2 \to S^2 \to 0 \\
\downarrow & & \downarrow \\
0 & \to & \text{Coker } g \\
\downarrow & & \downarrow \\
0 & \to & S \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

So \( \text{Coker } g = S \). With \( Y_1 = \text{Coker } f_1 \) and \( Y_2 = \text{Coker } f_2 \), we get the exact sequences (4.1) - (4.3).

\[\square\]

### 4.3 Conditions on the Ext-quiver

Using pullbacks and pushouts, we show in this section that every abelian length category \( \mathcal{A} \) of colocal type has to fulfill the conditions (C1) and (C2) in Theorem 1.2.4. Weaker conditions hold if \( S(\mathcal{A}) \) is distributive.

Recall that we defined for simple objects \( S, T \in \mathcal{A} \)

\[d_1^S(S,T) := \dim_{\text{End}(S)^{op}} \text{Ext}^1(S,T)\]

and

\[d_1^T(S,T) := \dim_{\text{End}(T)} \text{Ext}^1(S,T)\]

We begin with an auxiliary lemma:

**Lemma 4.3.1.** Let \( \mathcal{A} \) be an abelian length category and \( S, T_1, T_2 \) simple objects. Then the following holds:

(a) Let there be indecomposable objects \( X_1, X_2 \) with exact sequences

\[
0 \to T_1 \xrightarrow{f_1} X_1 \xrightarrow{g_1} S \to 0
\]

and

\[
0 \to T_2 \xrightarrow{f_2} X_2 \xrightarrow{g_2} S \to 0.
\]
4.3. CONDITIONS ON THE EXT-QUIVER

If $T_1 \cong T_2$, we suppose that these exact sequences are linearly independent over $\text{End}(T_1)$.

Then the object $Y$ in the following pullback is indecomposable:

$$
\begin{array}{ccc}
Y & \longrightarrow & X_1 \\
\downarrow & & \downarrow \text{g}_1 \\
X_2 & \longrightarrow & S \\
g_2 & & \\
\end{array}
$$

(4.4)

Furthermore, there is an exact sequence

$$0 \longrightarrow T_1 \oplus T_2 \longrightarrow Y \longrightarrow S \longrightarrow 0 .$$

(b) Let there be indecomposable objects $X_1$, $X_2$ with exact sequences

$$0 \longrightarrow S \overset{f_1}{\longrightarrow} X_1 \overset{g_1}{\longrightarrow} T_1 \longrightarrow 0$$

and

$$0 \longrightarrow S \overset{f_2}{\longrightarrow} X_2 \overset{g_2}{\longrightarrow} T_2 \longrightarrow 0 .$$

If $T_1 \cong T_2$, we suppose that these exact sequences are linearly independent over $\text{End}(T_1)^{\text{op}}$.

Then the object $Y$ in the following pushout is indecomposable:

$$\begin{array}{ccc}
S & \overset{f_1}{\longrightarrow} & X_1 \\
\downarrow \text{f}_2 & & \downarrow \\
X_2 & \longrightarrow & Y \\
\end{array}
$$

Furthermore, there is an exact sequence

$$0 \longrightarrow S \longrightarrow Y \longrightarrow T_1 \oplus T_2 \longrightarrow 0 .$$

Proof. We only prove (a), since (b) is completely analogous. In this case, the
following diagram is exact by 2.1.4:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_1 \oplus T_2 & Y & S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & T_1 \oplus T_2 & X_1 \oplus X_2 & S^2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0 & X_1 & S & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0 & X_2 & S & \longrightarrow & 0 \\
\end{array}
\]

and we get some morphisms \( f'_1 : T_1 \rightarrow Y \) and \( f'_2 : T_2 \rightarrow Y \) so that

\[
0 \longrightarrow T_1 \oplus T_2 \xrightarrow{[f'_1, f'_2]} Y \longrightarrow S \longrightarrow 0
\]

is an an exact sequence.

Suppose that \( Y \) was decomposable. Since \( X_1 \) and \( X_2 \) are indecomposable, \( S \) cannot be a direct summand of \( Y \).

So we can assume that \( Y = T_1 \oplus X \) for some object \( X \) of length 2. Since (4.4) is commutative, this yields a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f'_i} & X_i \\
\downarrow f'_2 & \swarrow g_1 \\
X_2 & \xrightarrow{g_i} & S
\end{array}
\]

with \( \text{Im}(g_1 f'_i) = S = \text{Im}(g_2 f'_i) \). Furthermore, \( \text{Im}(f'_i) \nless S \), since otherwise \( \text{Im}(f'_i) \rightarrow X_1 \rightarrow S \) splits. Taking the length of \( X_1 \) into consideration, it is obvious that \( \text{Im}(f'_i) = X_1 \) is the only remaining possibility and \( f'_i \) is an isomorphism. Analogously, \( \text{Im}(f'_i) = X_2 \).

Thus, \( X \) is isomorphic to both \( X_1 \) and \( X_2 \), \( T_1 \cong T_2 \) and for \( 1 \leq i \leq 2 \), there are commutative diagrams

\[
\begin{array}{ccc}
0 & \longrightarrow & T_i \\
| & \| & \|
\downarrow f'_i & \downarrow g_i \\
0 & \longrightarrow & X_i \xrightarrow{g_i} S \longrightarrow 0
\end{array}
\]
4.3. CONDITIONS ON THE EXT-QUIVER

Since $g_1 f'_1 = g_2 f'_2$, this is a contradiction to the assumption that $\eta_1$ and $\eta_2$ are linearly independent over $\text{End}(T_1)$.

So $Y$ must be indecomposable. \qed

Now we can prove that condition (C1) holds if $\mathcal{A}$ is of colocal type:

**Lemma 4.3.2.** Let $\mathcal{A}$ be an abelian length category. The following is true for all simple objects $S \in \mathcal{A}$:

(a) If $\mathcal{A}$ is of colocal type, then
\[
\sum_{T \text{ simple}} d^1_T(S, T) \leq 1.
\]

(b) If $S(\mathcal{A})$ is distributive, then there is at most one $T$ with $d^1_T(S, T) \neq 0$.

*Proof.* We begin with the proof of (a). If
\[
\sum_{T \text{ simple}} d^1_T(S, T) \geq 2,
\]
then there are simple objects $T_1, T_2$ and indecomposable objects $X_1, X_2$ with exact sequences
\[
\eta_1 : 0 \longrightarrow T_1 \overset{f_1}{\longrightarrow} X_1 \overset{g_1}{\longrightarrow} S \longrightarrow 0
\]
and
\[
\eta_2 : 0 \longrightarrow T_2 \overset{f_2}{\longrightarrow} X_2 \overset{g_2}{\longrightarrow} S \longrightarrow 0
\]
which are linearly independent over $\text{End}(T_1)$ if $T_1 \cong T_2$.

By Lemma 4.3.1, the object $Y$ in the following pullback is indecomposable:

\[
\begin{array}{ccc}
Y & \longrightarrow & X_1 \\
\downarrow & & \downarrow g_1 \\
X_2 & \longrightarrow & S
\end{array}
\]
and has $T_1 \oplus T_2$ as a subobject.

So $\mathcal{A}$ is not of colocal type.

To prove (b), we suppose that $T_1 \not\cong T_2$. Then $X_1 \not\cong X_2$ and for every epimorphism $h_i : Y \to X_i$, we have $\text{Ker} h_i \cong T_j$ with $1 \leq i \neq j \leq 2$. If $h'_i : Y \to X_i$ is not an epimorphism, then $\text{Im}(h'_i) \cong T_j$ and thus $T_j \mid \text{soc}(\text{ker} h'_i)$. But $T_j^2$ is not a subobject of $Y$ and by Lemma 4.2.2, there is no monomorphism $Y \hookrightarrow X^1$ for any $1 \leq i \leq 2$ and $n \in \mathbb{N}$. So $S(\mathcal{A})$ is not distributive by Proposition 4.1.3. \qed
To show that (C2) holds if $\mathcal{A}$ is of colocal type, we use the following auxiliary lemma:

**Lemma 4.3.3.** Let $\mathcal{A}$ be an abelian category and $S, T_1, \ldots, T_n$ simple objects in $\mathcal{A}$ so that there are exact sequences with indecomposable middle terms

$$
\eta_i : 0 \longrightarrow S \xrightarrow{f_i} X_i \longrightarrow T_i \longrightarrow 0
$$

for $1 \leq i \leq n$. Furthermore, suppose that $2 \leq n$. Let $f := \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} \end{bmatrix}$ and suppose that there is a isomorphism $g : \bigoplus_{i=1}^{n-1} X_i \rightarrow \bigoplus_{i=1}^{n-1} X_i$ and a monomorphism $g_n : X_n \rightarrow \bigoplus_{i=1}^{n-1} X_i$ so that the following diagram is commutative

$$
\begin{array}{ccc}
S & \xrightarrow{f} & \bigoplus_{i=1}^{n-1} X_i \\
\downarrow f_n & & \downarrow g \\
X_n & \xrightarrow{g_n} & \bigoplus_{i=1}^{n-1} X_i
\end{array}
$$

(4.5)

or an isomorphism $g'_n : X_n \rightarrow X_n$ and an epimorphism $g' : \bigoplus_{i=1}^{n-1} X_i \rightarrow X_n$ so that the following diagram is commutative

$$
\begin{array}{ccc}
S & \xrightarrow{f} & \bigoplus_{i=1}^{n-1} X_i \\
\downarrow f_n & & \downarrow g' \\
X_n & \xrightarrow{g'_n} & X_n
\end{array}
$$

(4.6)

Then there are $1 \leq i_1, \ldots, i_m \leq n - 1$ so that $T_{i_1} \cong T_n, \ldots, T_{i_m} \cong T_n$ and $\eta_n$ is linearly dependent of $\eta_{i_1}, \ldots, \eta_{i_m}$ over $T_n$.

**Proof.** We only prove this for (4.5), since the proof for (4.6) is analogous.

Since $S^2$ is not a subobject of $X_n$, the image of the concatenation

$$
\phi : S^{n-1} \rightarrow \bigoplus_{i=1}^{n-1} X_i \rightarrow \text{Coker } g^{-1} g_n
$$

is $S^{n-2}$. Set

$$
F := \begin{bmatrix}
f_1 & 0 & 0 & \cdots & 0 \\
0 & f_2 & 0 & \cdots & 0 \\
\vdots & & & \ddots & \vdots \\
0 & 0 & \cdots & 0 & f_{n-1}
\end{bmatrix}.
$$
Then the following diagram is exact, because all rows and the first and second column are exact:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 \\
\downarrow & & & & \\
0 & S & f_n & X_n & T_n & 0 \\
\downarrow & [\text{id} \ldots \text{id}] & & g^{-1}g_n & & \\
0 & S^{n-1} & F & \bigoplus_{i=1}^{n-1} X_i & \bigoplus_{i=1}^{n-1} T_i & \downarrow & \downarrow \\
\downarrow & & & \phi & \text{Coker }g^{-1}g_n & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Thus, there are \(1 \leq i_1, \ldots, i_m \leq n-1\) so that \(T_{i_1} \cong T_n, \ldots, T_{i_m} \cong T_n\) and \(\eta_n\) is linearly dependent of \(\eta_{i_1}, \ldots, \eta_{i_m}\) over \(T_n\).

Now we prove in particular that (C2) holds if \(\mathcal{A}\) is of colocal type:

**Lemma 4.3.4.** Let \(\mathcal{A}\) be an abelian length category. The following holds for all simple objects \(S \in \mathcal{A}\):

(a) If \(\mathcal{A}\) is of colocal type, then

\[
\sum_{T \text{ simple}} d_1^1(T, S) \leq 2. \tag{4.7}
\]

(b) If \(S(\mathcal{A})\) is distributive, then there are at most two non-isomorphic simple modules \(T\) with \(d_1^1(T, S) \neq 0\).

**Proof.** We start with the proof of (a). If

\[
\sum_{T \text{ simple}} d_1^1(T, S) \geq 3,
\]

then we have three exact sequences

\[
\eta_i : 0 \longrightarrow S \overset{f_i}{\longrightarrow} X_i \longrightarrow T_i \longrightarrow 0
\]
for some indecomposable objects $X_i \in \mathcal{A}$ and simple $T_i$ with $i \in \{1, 2, 3\}$. If $T_i \cong T_j$ for some $i \neq j$, then $\eta_i$ and $\eta_j$ are linearly independent over $\text{End}(T_i)^{\text{op}}$. If $T_1 \cong T_2 \cong T_3$, then over $\text{End}(T_i)^{\text{op}}$ none of the exact sequences is a linear combination of the other two.

There is a pushout

$$
\begin{array}{ccc}
S & \to & X_1 \oplus X_2 \\
\downarrow f_3 & & \downarrow |g_1 \ g_2| \\
X_3 & \to & Y
\end{array}
$$

We show that $Y$ is indecomposable: By 2.1.3, $[g_1 \ g_2]$ and $g_3$ are monomorphisms.

Furthermore, $l(X_i) = 2$, $l(X_1 \oplus X_2) = 4$ and by 2.1.4, $l(Y) = 5$.

Suppose that $Y = Y_1 \oplus Y_2$ for some indecomposable $Y_1$ and write

$$
g_i = \begin{bmatrix} g_{i1} \\ g_{i2} \end{bmatrix}
$$

with $g_{ji} : X_i \to Y_j$ for all $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

Without loss of generality, we can assume that $g_{13}f_3 \neq 0$ and $g_{11}f_1 \neq 0$. Then $g_{11}$ and $g_{13}$ are both monomorphisms; otherwise their image was isomorphic to $S$ and they would split. By Lemma 4.3.3, none of them is an isomorphism and thus $l(Y_1) \geq 3$. If $g_{23}f_3 \neq 0$, we analogously get $l(Y_2) \geq 3$, a contradiction to $l(Y) = 5$.

So we can assume that $g_{23}f_3 = 0$. If $g_{22}f_2 \neq 0$, then $g_{21}f_1 + g_{22}f_2 = 0$. By Lemma 4.3.3, we get again $l(Y_2) \geq 3$ and thus $l(Y_1 \oplus Y_2) = 6$, a contradiction to $l(Y) = 5$.

So $g_{11}, g_{12}, g_{13}$ are all monomorphisms and $g_{23}f_3 = g_{22}f_2 = g_{12}f_1 = 0$. Since $[g_1 \ g_2]$ is a monomorphism by 2.1.3, we get

$$
4 = l(X_1 \oplus X_2) < Y_1 \text{ and } Y_1 \cong Y.
$$

By (4.8), $\mathcal{A}$ is not of colocal type.

To prove (b), we can assume that the objects $T_1, T_2$ and $T_3$ are pairwise non-isomorphic.

Since $[g_1 \ g_2] : X_1 \oplus X_2 \to Y$ is a monomorphism, there is also a monomorphism $Y \to \text{Coker } g_1 \oplus \text{Coker } g_2$ by Lemma 4.2.2.
For $1 \leq i \neq j \neq k \leq 3$, the following diagrams are exact, because all rows and the first and second column are exact:

$$
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow X_k \oplus X_j \\
\downarrow \\
S \rightarrow X_1 \oplus X_2 \oplus X_3 \\
\downarrow \\
0 \rightarrow S \\
\downarrow \\
0 \\
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow X_k \oplus X_j \\
\downarrow \\
Y \\
\downarrow \\
T_i \\
\downarrow \\
0 \\
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\end{array}
$$

and

$$
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow S \\
\downarrow \\
X_i \rightarrow Y \\
\downarrow \\
T_i \\
\downarrow \\
0 \\
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\end{array}
$$

By Lemma 4.3.1, $\text{Coker } g_i$ is indecomposable and there is an exact sequence

$$0 \rightarrow S \rightarrow \text{Coker } g_i \rightarrow T_j \oplus T_k \rightarrow 0. $$

In particular, the socle of $\text{Coker } g_i$ is $S$ and $T_i$ is not a subobject of any indecomposable object $Z \nleq \text{Coker } g_i$ with an epimorphism $(\text{Coker } g_i)^n \rightarrow Z$.

If there was any $n \in \mathbb{N}$ with a monomorphism $\phi : Y \rightarrow (\text{Coker } g_i)^n$, then
there was some $Z'$ so that the following diagram was exact:

$$
\begin{array}{cccccc}
0 & \rightarrow & S^2 & \rightarrow & Y & \rightarrow & T_1 \oplus T_2 \oplus T_3 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \phi & & \downarrow & & \downarrow \\
0 & \rightarrow & S^2 & \rightarrow & (\text{Coker } g_i)^n & \rightarrow & Z' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Coker } \phi & \rightarrow & \text{Coker } \phi & \rightarrow & 0 & & 0 \\
\end{array}
$$

But for the smallest $n$ with such a monomorphism, the component $Y \rightarrow \text{Coker } g_i$ of $\phi$ is non-zero for every copy of $\text{Coker } g_i$, so no direct summand of $Z'$ is isomorphic to $\text{Coker } g_i$, a contradiction.

By Proposition 4.1.3, $S(\mathcal{A})$ is not distributive.

The following lemma is especially important if $\mathcal{A} = \text{mod } A$ for some Artin algebra $A$ over an algebraically closed field $k$, since in this case, $\text{End}(S) \cong \text{End}(T)$ for all simple modules $S, T \in \text{mod } A$, since $\text{mod } A$ is equivalent to the module category of a quiver algebra, see 2.5.3.

**Lemma 4.3.5.** Let $S(\mathcal{A})$ be distributive. Then for all simple modules $S, T$ with $\text{End}(S) \cong \text{End}(T)$, we have

$$d^1_S(S, T) = 1 = d^1_T(S, T). \quad (4.9)$$

**Proof.** Suppose that 4.9 is not fulfilled. Then

$$2 \leq d^1_S(S, T) = d^1_T(S, T) = \dim_{\text{End}(S)^{\text{op}}} \text{Ext}^1(S, T) = \dim_{\text{End}(T)} \text{Ext}^1(S, T).$$

There are $\eta_1', \eta_2' \in \text{Ext}^1(S, T)$ which are linearly independent over $\text{End}(S)^{\text{op}}$ and $\eta_3', \eta_4' \in \text{Ext}^1(S, T)$ which are linearly independent over $\text{End}(T)$. We can choose $\eta_1, \eta_2$ out of these so that they are linearly independent over both $\text{End}(S)^{\text{op}}$ and $\text{End}(T)$: If $\eta_1', \eta_2'$ are linearly dependent over $\text{End}(T)$, then at least one of $\eta_3', \eta_4'$ is linearly independent of both of them over $\text{End}(T)$. We can assume this to be $\eta_3'$. Over $\text{End}(S)^{\text{op}}$, $\eta_3'$ is linearly independent of at least one of $\eta_1'$ and $\eta_2'$.

There are indecomposable objects $X_1, X_2$ so that we can write $\eta_1$ and $\eta_2$ as

$$\eta_i : 0 \rightarrow T \xrightarrow{f_i} X_i \xrightarrow{g_i} S \rightarrow 0$$
with \( 1 \leq i \leq 2 \).

First, assume that \( X_1 \cong X_2 \).

If \( g_1 f_2 \neq 0 \), then \( \text{Im} g_1 f_2 \cong S \cong T \), since \( S \) and \( T \) are simple. Because \( \text{End}(S) \) is a division ring, the identity factors through \( g_1 f_2 \). So these morphisms split, contrary to the assumption that \( X_1 \cong X_2 \) is indecomposable. Analogously, we get \( g_2 f_1 = 0 \).

So \( \text{Im} f_1 = \text{Im} f_2 = \text{Ker} g_1 = \text{Ker} g_2 \). Thus there are isomorphisms \( \phi : T \to T \) and \( \psi : S \to S \) so that \( f_2 = f_1 \phi \) and \( g_2 = \psi g_1 \).

But \( \text{End}(S) \cong \text{End}(T) \), so we can also regard \( \psi \) as an isomorphism over \( T \) and \( \phi \) as an isomorphism over \( S \). Over \( \text{End}(S)^{op} \), we get \( \eta_2 = \eta_1 \phi \psi^{-1} \) and over \( \text{End}(T) \), we have \( \eta_2 = \phi^{-1} \psi \eta_1 \), contrary to the assumption that \( \eta_1 \) and \( \eta_2 \) are linearly independent over these division rings.

So \( X_1 \ncong X_2 \). By Lemma 4.3.1, there is a pullback

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X_1 \\
\downarrow \quad \quad \quad \downarrow g \\
X_2 & \xrightarrow{} & S
\end{array}
\]

so that \( Y \) is indecomposable. Furthermore,

\[ T \oplus T \cong \text{Ker} f \oplus \text{Ker} g \subset Y. \]

Suppose that for \( 1 \leq i \leq 2 \), there is some monomorphism \( h_i : T \hookrightarrow Y \) so that \( X_i \cong \text{Coker} \, h_i \). Then

\[ [h_1 \quad h_2] : T^2 \hookrightarrow Y \]

is a monomorphism.

Since there is only one way to embed \( T \oplus T \) into \( Y \), \( h_1 \) and \( h_2 \) form a basis of \( \text{Hom}(T, Y) \) over \( \text{End}(T) \).

If there are objects \( X'_1, X'_2 \) which are middle terms of exact sequences \( \eta'_1, \eta'_2 \) in \( \text{Ext}^1(S, T) \) and a monomorphism

\[ Y \hookrightarrow X'_1 \oplus X'_2, \]

then there is a monomorphism

\[ [h'_1 \quad h'_2] : T^2 \hookrightarrow Y \]

so that \( X'_1 = \text{Coker} \, h'_1 \) and \( X'_2 = \text{Coker} \, h'_2 \), since \( l(Y) = 3 \), \( l(X'_1) = l(X'_2) = 2 \) and \( \text{soc} \, Y = T^2 \).
There are some $\alpha, \beta, \in \text{End}(T)$ so that $h'_1 = h_1 \alpha + h_2 \beta$. Without loss of generality, we can assume that $\beta \neq 0$.

Since $[h'_1, h'_2]$ is a monomorphism, $h'_1$ and $h'_2$ must be orthogonal to each other: $h'_2 = (h_2 \alpha - h_1 \beta)\gamma$ for some $\gamma \in \text{End}(T)$. Thus, there are exact sequences with $h'_1$ and $h'_2$ as monomorphisms that are linearly independent of each other over both $\text{End}(S')^{\text{op}}$ and $\text{End}(T)$. So $X'_1 \not\subseteq X'_2$ and in particular, $Y \notin \text{sub} X_1$ and $Y \notin \text{sub} X_2$.

By Proposition 4.1.3, the proof is complete. \hfill \qed

## 4.4 2-Extensions

In this section, we collect some results about exact sequences in $\text{Ext}^2(S, T)$, where $S$ and $T$ are simple objects. We will need these results in the later sections.

**Lemma 4.4.1.** Suppose that $S', T$ are simple objects. If

$$
\eta : 0 \to S' \to M \to N \to T \to 0
$$

is an exact sequence in $\text{Ext}^2(T, S')$, then there is some exact sequence

$$
\eta' : 0 \to S' \to M' \to N' \to T \to 0
$$

with a map $\eta' \to \eta$ in $\text{Ext}^2(S', T)$ and $\text{top} N' \cong T$. The induced morphisms of objects $M' \to M$ and $N' \to N$ are monomorphisms. Furthermore, there is an exact sequence

$$
\eta'' : 0 \to S' \to M'' \to N'' \to T \to 0
$$

with a map $\eta \to \eta''$ in $\text{Ext}^2(S', T)$ and $\text{soc} M'' \cong S'$. The induced morphisms of objects $M \to M''$ and $N \to N''$ are epimorphisms.

**Proof.** We only prove the first part, since the second one is completely analogous.

If $\text{top} N = T \oplus T'$ for some non-zero object $T'$, let $h : N \to T'$ be an epimorphism. The concatenation $\text{Ker} h \to N \to T$ is non-zero, so $T \to \text{top}(\text{Ker} h)$. If $\text{top}(\text{Ker} h) \neq T$, we repeat the construction. Since the object $T$ has finite length, we get an object $N'$ with $\text{top} N' = T$ after finitely many steps.
Let \( g \) be the epimorphism of the exact sequence. Then there are morphisms \( f, g' \) with an exact diagram

\[
\begin{array}{c}\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\end{array}
\quad \xymatrix{
0 & \ker g' \ar[r] & N' \ar[r]^{g'} & T \ar[r] & 0 \\
0 & \ker g \ar[r] & N \ar[r]^{g} & T \ar[r] & 0 \\
0 & \coker f \ar[r] & \coker f \ar[r] & 0
\end{array}
\quad (4.10)
\]

Furthermore, there is a concatenation of epimorphisms \( M \to \ker g \to \coker f \) and thus some object \( M' \) with an exact diagram

\[
\begin{array}{c}\begin{array}{c}
0 \\
\downarrow \\
S' \\
\downarrow \\
S' \\
\downarrow \\
0
\end{array}
\end{array}
\quad \xymatrix{
0 & S' \ar[r] & M' \ar[r] & N' \ar[r]^{f} & T \ar[r] & 0 \\
0 & S \ar[r] & M \ar[r] & N \ar[r] & T \ar[r] & 0 \\
0 & \coker f \ar[r] & \coker f \ar[r] & 0
\end{array}
\quad (4.11)
\]

Note that the exact sequences \( \eta' \) and \( \eta'' \) are not unique: in the construction of \( N' \), we have used an arbitrary epimorphism \( h : N \to T' \), where \( \top N = T \oplus T' \). If \( T^2 \mid \top N \), there can be such epimorphisms with non-isomorphic kernels.

But let \( m \) be the maximal integer so that \( T^m \mid \top N \). Then there is some \( T'' \) so that \( \top N = T^m \oplus T'' \). If there are two epimorphisms \( h_1 : N \to T'' \) and \( h_2 : N \to T'' \), then there is some isomorphism \( \chi : T'' \to T'' \) so that \( h_1 = \chi h_2 \). So \( \ker h_1 \cong \ker h_2 \).
Furthermore, the following diagram is exact, since all columns and the second and third row are exact:

\[
\begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 \\
\downarrow & & & & \\
\ker h & \ker h_1 & S^{n-1} & \cdots & 0 \\
\downarrow & & & & \\
0 & N & T' & \cdots & 0 \\
\downarrow h_1 & & & & \\
0 & T'' & T'' & \cdots & 0 \\
\end{array}
\]

Note that neither \( \eta' \) nor \( \eta'' \) in the lemma above are necessarily unique: If \( T^2 \mid \top N \) or \( S^2 \mid \soc M \) respectively, then we can choose different epimorphisms \( N \rightarrow T \) to construct \( N' \) or monomorphisms \( S \rightarrow N \) to construct \( M'' \).

We still need two auxiliary lemmas:

**Lemma 4.4.2.** Let \( A \) be an abelian length category. Let there be simple objects \( S_1, \ldots, S_n \in A \) with \( \Ext^1(S_i, S_{i+1}) \neq 0 \) for all \( 1 \leq i < n \) and for all simple \( S \in A \)

\[
\sum_{T \text{ simple}} d_T^1(S, T) \leq 1.
\]

Suppose that there is some indecomposable object \( Z \) with \( \soc Z = S_n \) and \( S_1 \mid \top Z \) or alternatively \( S_n \mid \soc Z \) and \( S_1 = \top Z \).

Then for all \( 1 \leq i \leq j \leq n \), there is some indecomposable \( X_{ij} \) with \( \soc X_{ij} = S_j \) and \( \top X_{ij} = S_i \) so that for all \( 1 \leq i \leq n \), there is a chain of monomorphisms

\[
X_{ii} \hookrightarrow X_{i-1,i} \hookrightarrow \cdots \hookrightarrow X_{1,i}
\]

and a chain of epimorphisms

\[
X_{in} \twoheadrightarrow X_{i,n-1} \twoheadrightarrow \cdots \twoheadrightarrow X_{1i}.
\]

If \( \soc Z \) is simple, we can choose these objects so that there is a monomorphism \( X_{1,n} \hookrightarrow Z \). If \( \top Z \) is simple, we can choose these objects so that there is an epimorphism \( Z \twoheadrightarrow X_{1,n} \).

**Proof.** We show this inductively. If \( n = 1 \), this is clear. So suppose that the assertion is proved for \( n - 1 \).
We can assume that soc $Z$ is simple, since the proof is completely analogous if top $Z$ is simple. Let

$$S_n = Z_m \implies Z_{m-1} \implies \cdots \implies Z_0 = Z$$

be a filtration of $Z$ so that $Z_i/Z_{i+1}$ is simple. Then soc $Z_i$ is simple and $Z_i$ is indecomposable for all $0 \leq i \leq m$. Furthermore, for every simple $T$ with a morphism $f : Z \to T$, we can choose $Z_1$ to be Ker $f$. If $T \otimes S_1 \mid$ top $Z$, then the concatenation of morphisms $Z_1 \implies Z \to S_1$ is non-zero and thus $S_1 \mid$ top $Z_1$.

So there is some object $Z_i$ with soc $Z_i = S_n$ and top $Z_i = S_1$.

Note that every submodule and every factor module of $Z_i$ is indecomposable.

Then $Z_i/Z_{i+1} = S_1$. There is some simple object $T_1$ and an indecomposable object $X$ so that the following diagram is exact:

$$
\begin{array}{c}
0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & Z_{i+2} & \to & Z_{i+1} & \to & T_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & Z_{i+2} & \to & Z_i & \to & X & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & S_1 & \to & S_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

So Ext$^1(S_1, T) \neq 0$. Analogously, there is some $i_2 > i$ with $T_2 \mid$ top $Z_{i_2}$ and Ext$^1(T, T_2) \neq 0$. Inductively, we get some $k \in \mathbb{N}$ with

$$T_3 \mid$ top $Z_{i_3}, \ldots, T_k \mid$ top $Z_{i_k}$ and $T_{k+1} = S_n$$

and Ext$(T_j, T_{j+1}) \neq 0$. Since the only object $T_k$ with Ext$^1(T_k, S_n) \neq 0$, is $S_{n-1}$, inductively the sequence $(T_1, \ldots, T_{k+1})$ must either be of the form

$$(S_1, \ldots, S_{n-1}, S_n)$$

or (if Ext$^1(S_n, S_n) \neq 0$) of the form

$$(S_1, \ldots, S_{n-1}, S_n, S_n, \ldots, S_n)$$
or (if \( S_1 \to S_2 \cdots \to S_n \to S_1 \) is an oriented cycle in the \( \text{Ext} \)-quiver of \( \mathcal{A} \)) of the form

\[
(S_1, \ldots, S_{n-1}, S_n, S_1, \ldots, S_{n-1}, S_n, \ldots, S_1, \ldots, S_{n-1}, S_n)
\]

Thus \( \text{soc} Z_{i+1} = S_n \) and \( S_2 \mid \text{top} Z_{i+1} \). By the inductive assumption, the objects \( X_{j_1,j_2} \) and the chains of monomorphisms and epimorphisms exist for all \( 2 \leq j_1 \leq j_2 \leq n \). We can choose them so that there is a chain of monomorphisms \( X_{2,n} \hookrightarrow Z_{i+1} \hookrightarrow Z_i =: X_{1,n} \).

We construct the remaining objects \( X_{1,i} \) inductively for all \( i = n - 1, \ldots i = 1 \): We choose them to be the objects that make the middle column in the following diagram exact. Then all columns and the first and second row of the diagram are exact; thus the last row is also exact:

These objects fulfil all assertions made in the lemma and the proof is complete.

From the proof above, we can see:

**Corollary 4.4.3.** Let \( \mathcal{A} \) be an abelian length category with

\[
\sum_{T \text{ simple}} d^1_T(S,T) \leq 1
\]

for all simple objects \( S \in \mathcal{A} \).

Let \( Z \) be an object. If \( \text{top} Z = T \) is simple and there is an object \( Z' \) with an exact sequence

\[
0 \longrightarrow Z' \longrightarrow Z \longrightarrow T \longrightarrow 0,
\]

then for every simple \( T' \mid \text{top} Z' \), we have \( \text{Ext}^1(T,T') \neq 0 \).
Now we can prove the following:

**Lemma 4.4.4.** Let $A$ be an abelian length category. Suppose that there are simple objects $S_1, \ldots, S_n \in A$ with $\text{Ext}^1(S_i, S_{i+1}) \neq 0$ for all $1 \leq i < n$ and for all simple $S \in A$ we have

$$\sum_{T \text{ simple}} d_T^1(S, T) \leq 1. \quad (4.13)$$

Then for every indecomposable object $Z$ with $\text{top} Z = S_1$ and $\text{soc} Z = \bigoplus_{i=1}^n S_i^{m_i}$ we have $\bigoplus_{i=1}^n m_i = 1$.

**Proof.** Suppose that a $Z$ exists with $\bigoplus_{i=1}^n m_i > 1$. By Corollary 4.4.3, there is some $n' \geq n$, so that we can define

$$S_{n+1} = S_1, \ldots, S_{2n} = S_n, S_{2n+1} = S_1, \ldots, S_{n'} = S_n$$

and there is a filtration

$$S_n^{m_{n}} = Z_{n'} \to Z_{n'-1} \to \cdots \to Z_0 = Z$$

so that $Z_{i-1}/Z_i = S_i^{m_i'}$ for all $0 < i \leq n'$ and some $m_i' \in \mathbb{N}$. We prove the assertion by induction on $n'$.

Suppose that $n' = 1$. Then there is some $Y$ with

$$0 \to S \to Z \to Y \to 0.\quad (4.14)$$

The object $Y$ is indecomposable, since $\text{top} Y \cong \text{top} Z \cong T$. Furthermore, $\text{soc} Y = S^{m_n-1}$. Inductively, we can assume that $m_n = 2$. Then $\text{soc} Z \cong S^2$ and the cokernel of $S \to Z$ is some $X$ that is an indecomposable middle term of an exact sequence in $\text{Ext}^1(S_1, S_2)$. By 2.1.4 and Lemma 4.2.2, there are such middle terms $X_1, X_2$ so that the following is both a pushout and a pullback:

$$\begin{array}{ccc}
Z & \xrightarrow{f_1} & X_1 \\
\downarrow f_2 & & \downarrow g_1 \\
X_2 & \xrightarrow{g_2} & T
\end{array} \quad (4.14)$$

By (4.13), $X_1$ and $X_2$ are the middle terms of exact sequences which are linearly dependent over $T$. But then $X_1 \cong X_2$ and there is a commutative diagram

$$\begin{array}{ccc}
X_1 & \xrightarrow{e_1} & X_1 \\
\downarrow e_2 & & \downarrow g_1 \\
X_2 & \xrightarrow{g_2} & T
\end{array}$$
a contradiction to (4.14): If there was any \( \phi \) so that \( e_1 = f_1 \phi \), then \( f_1 \) would split.

Now suppose that the assertion is proved for \( n' - 1 \geq 2 \). By Corollary 4.4.3, \( \text{top } Z_1 = S_2^{m_2'} \) and thus \( m_2' > 1 \). There is some \( Z' \) which makes the following diagram exact:

\[
\begin{array}{ccc}
0 & \rightarrow & Z_2 \\
\downarrow & & \downarrow \\
0 & \rightarrow & Z_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & S_2^{m_2'} \\
\downarrow & & \downarrow \\
0 & \rightarrow & Z' \\
\downarrow & & \downarrow \\
0 & \rightarrow & S_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

Since \( \text{top } Z \) is simple, \( \text{top } Z' = \text{top } Z = S_1 \), the object \( Z' \) is indecomposable and the proof is complete. \( \square \)

We can say even more:

**Lemma 4.4.5.** Let \( \mathcal{A} \) be an abelian length category. Suppose that there are simple objects \( S_1, \ldots, S_n \in \mathcal{A} \) with \( \text{Ext}^1(S_i, S_{i+1}) \neq 0 \) for all \( 1 \leq i < n \) and for all simple \( S \in \mathcal{A} \)

\[
\sum_{T \text{ simple}} d^1_T(S, T) \leq 1. \tag{4.15}
\]

Then the following holds:

(a) If the objects \( X_{ij} \) in Lemma 4.4.2 exist, then we can choose them so that \( l(X_{ij}) = j - i + 1 \) for all \( 1 \leq i \leq j \leq n \).

(b) For every object \( Z \) of length \( n' > n \) with \( \text{top } Z = S_1 \) and \( \text{soc } Z = S_n \), we can define

\[
S_{n+1} = S_1, \ldots, S_{2n} = S_n, S_{2n+1} = S_1, \ldots, S_{n'} = S_n \tag{4.16}
\]

so that \( Z \cong X_{1,n'} \).
Proof. We prove (a) first: From the construction of $X_{ij}$ it is obvious that $l(X_{ij}) \geq j - i + 1$. If $i = j$, we can choose $X_{ij} = S_i$ with length 1. Suppose that $l(X_{ij}) = j - i + 1$ for all $j - i \leq m$. Then we can also find such $X_{ij}$ for $j - i = m + 1$.

If $l(X_{ij}) > j - i + 1$, then the cokernel $Y$ of $S_j \rightarrow X_{ij}$ is indecomposable and not isomorphic to $X_{i,j-1}$, but there is an epimorphism $Y \rightarrow X_{i,j-1}$. By (4.15) and Lemma 4.4.4, the socle of $Y$ is simple; by Corollary 4.4.3, $\text{soc} Y = S_{j-1}$.

We can repeat this construction and since $S_j$ is the only simple object with $\text{Ext}^1(S_{j-1}, S_j) \neq 0$, we get an indecomposable object $Y'$ with an exact sequence

$$0 \rightarrow S_j \rightarrow Y' \rightarrow X_{i,j-1} \rightarrow 0.$$ 

Then $l(Y) = j - i + 1$. Set $X'_{i,j'} := Y'$ for $i' = i$ and $j' = j$. Furthermore, $X'_{i,j'} := X'_{i',j'}$ if $i \leq i' \leq j' \leq j - 1$. Then we can construct $X'_{i,j}$ inductively, analogous to the construction in 4.4.2. These objects fulfil $\text{soc} X'_{ij} = S_j$ and top $X'_{ij} = S_i$ and for all $1 \leq i \leq n$, there is a chain of monomorphisms

$$X'_{ii} \rightarrow X'_{i-1,i} \rightarrow \cdots \rightarrow X'_{1,i}$$

and a chain of epimorphisms

$$X'_{in} \rightarrow X'_{i,n-1} \rightarrow \cdots \rightarrow X'_{ii}.$$

To prove (b), we note that by Lemma 4.4.2, there is an epimorphism $Z \rightarrow X_{1,n}$ and by (a) we can assume that $l(X_{1,n}) = n$. So we can use an analogous construction as in (a) to get a filtration

$$X_{1,n} \rightarrow Z_{n+1} \rightarrow \cdots \rightarrow Z_n = Z$$

where $Z_i/Z_{i-1}$ is simple. Denoting $S_i := Z_i/Z_{i-1}$ for $n < i \leq n'$, we see that this Definition fulfils (4.16) by Corollary 4.4.3 and (4.15).

Now we can show the following:

**Lemma 4.4.6.** For all simple objects $S \in \mathcal{A}$ let

$$\sum_{T \text{ simple}} d_T^1(S, T) \leq 1.$$  \hspace{1cm} (4.17)

Suppose that there are fixed simple objects $S', S, T$ and indecomposable objects $X, Z$ so that $\text{Ext}^1(T, S) \neq 0$ and the following sequences are exact:

$$0 \rightarrow S' \rightarrow X \rightarrow S \rightarrow 0$$

and

$$0 \rightarrow X \rightarrow Z \rightarrow T \rightarrow 0.$$  \hspace{1cm} (4.18)

Then the following holds:
(a) If for all \( S' \cong S_1, S_2, \ldots, S_n \cong T \) with \( \operatorname{Ext}^1(S_i, S_{i+1}) \neq 0 \) for \( 1 \leq i < n \), so that \( X_{1,n-1} \) and \( X_{2,n} \) as in Lemma 4.4.5 exist, we have \( n = 3 \) and \( S_2 \cong S \), then \( \operatorname{Ext}^2(S', T) = 0 \).

(b) All exact sequences of the form (4.18) are equivalent over \( \operatorname{End}(X) \), while all exact sequences of the form

\[
0 \to S' \to Z \to X' \to 0
\]

(4.19)

with indecomposable \( X' \) are equivalent over \( \operatorname{End}(S') \). In particular, all objects \( Z \) so that (4.18) exists are isomorphic.

**Proof.** First, we prove (a): Suppose that

\[
\eta : 0 \to S' \to M \to N \to T \to 0
\]

is an exact sequence in \( \operatorname{Ext}^2(S', T) \) that does not split. By Lemma 4.4.1, there is an exact sequence

\[
\eta' : 0 \to S' \xrightarrow{b} M' \to N' \to T \to 0
\]

with \( \text{top} N' = T \) and \( \eta' \to \eta \). Thus \( \eta \) and \( \eta' \) are equivalent. Again by Lemma 4.4.1, we can assume that \( \text{soc} M' \cong S' \). There is an epimorphism \( M' \to \operatorname{Coker} b \) and a monomorphism \( \operatorname{Coker} b \to N' \). If \( \operatorname{Coker} b = 0 \), then \( \eta' \) splits and there is nothing to show.

Otherwise, there are some objects \( R \) and \( R' \), so that the following diagram is exact, since all columns and the first and second row are exact:

\[
\begin{array}{cccccccc}
0 & \to & S' & \to & M' & \to & N' & \to & T & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & R & \to & M' & \to & \text{top} M' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & R' & \to & \operatorname{Coker} b & \to & \text{top} M' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

Thus, \( \text{top} M' \cong \text{top} \operatorname{Coker} b \) and analogously, \( \text{soc} \operatorname{Coker} b = \text{soc} N' \). By Corollary 4.4.3, there are some \( S' \cong S_1, S_2, \ldots, S_n \cong T \) with \( \operatorname{Ext}^1(S_i, S_{i+1}) \neq 0 \) for
1 ≤ i < n. Furthermore, \( S_{n-1} \mid \text{top } M' \) and \( S_1 \mid \text{soc } N' \). By the assumption, \( \text{Coker } b = \text{top } M' \cong \text{soc } N' \cong S''' \) for some \( m \in \mathbb{N} \).

By Lemma 4.4.4, we get \( m = 1 \) and thus \( M' \cong X \), where \( X \) is the middle term of an exact sequence in \( \text{Ext}(S, S') \) and \( N' \cong X' \), where \( X' \) is the middle term of an exact sequence in \( \text{Ext}(T, S) \).

It remains to show that all exact sequences of the form

\[
\eta_1 : 0 \longrightarrow S' \overset{f}{\longrightarrow} X \overset{g'g}{\longrightarrow} X' \overset{h}{\longrightarrow} T \longrightarrow 0
\]

with \( g : X \twoheadrightarrow S \), \( g' : S \twoheadrightarrow X' \) are equivalent to the split exact sequence

\[
0 \longrightarrow S' \longrightarrow S' \oplus X \overset{d}{\longrightarrow} T \longrightarrow 0.
\]

We show that for all monomorphisms \( c : S' \hookrightarrow Z \) and \( c' : X \hookrightarrow Z \), there is some monomorphism \( b \) and some epimorphism \( d \) with an exact sequence

\[
\eta_2 : 0 \longrightarrow S' \overset{[\text{id} b]}{\longrightarrow} S' \oplus X \overset{[c c']}{\longrightarrow} Z \overset{d}{\longrightarrow} T \longrightarrow 0
\]

and a morphism of exact sequences \( \eta_2 \rightarrow \eta_1 \). Since there is obviously a morphism from \( \eta_2 \) into the split exact sequence, this proves that \( \text{Ext}^2(S', T) = 0 \).

By Lemma 4.4.2, there is an epimorphism \( \phi : Z \twoheadrightarrow X' \). Obviously, \( \text{Im}(\phi c') \cong S \). There is an isomorphism \( \psi \) so that the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \longrightarrow & S' \\
\downarrow{\psi''} & & \downarrow{\psi'} \\
0 & \longrightarrow & S' \\
\end{array}
\]

Since (4.17) holds, there are isomorphisms \( \psi' \), \( \psi'' \) and a monomorphism \( f' \) so that the following diagram is commutative

\[
\begin{array}{ccc}
0 & \longrightarrow & S' \\
\downarrow{\psi''} & & \downarrow{\psi'} \\
0 & \longrightarrow & S' \end{array}
\]

We can set \( d := h \phi \) and get the commutative diagram

\[
0 \longrightarrow S' \overset{f}{\longrightarrow} X \overset{g'g}{\longrightarrow} X' \overset{h}{\longrightarrow} T \longrightarrow 0. \quad (4.20)
\]
So $\Ext^2(S', T) = 0$.

Part (b) is simple to show: By (4.17), every isomorphism on $T$ induces an isomorphism on $X'$ and thus on $S$. On the other hand, every isomorphism on $S$ induces an isomorphism on $X$. So every isomorphism on $T$ induces an isomorphism on $X$ and all exact sequences of the form (4.18) must be equivalent over $\End(X)$.

Analogously, all exact sequences of the form (4.19) are equivalent over $S'$.

It is possible to generalize this to the following:

**Lemma 4.4.7.** For all simple objects $S \in \mathcal{A}$ let

$$
\sum_{T \text{ simple}} d_{T}(S, T) \leq 1. \tag{4.21}
$$

Then the following holds if there are some fixed simple objects $S, T$ with an object $Z$ so that $\soc(Z) = S$ and $\top(Z) = T$:

(a) If there are unique objects $S' \cong S_0, S_1, \ldots, S_n \cong T$ with $\Ext^1(S_i, S_{i+1}) \neq 0$ for $0 \leq i < n$, so that $X_{0,n-1}$ and $X_{1,n}$ as in Lemma 4.4.5 exist, then $\Ext^2(S, T) = 0$.

(b) If $T \cong S_1, S_2, \ldots, S_n \cong S$ and the objects $X_{ij}$ are defined as in 4.4.5, then all exact sequences of the form

$$
0 \to S \to X_{1,n} \to X_{2,n} \to 0
$$

are equivalent over $\End(S)$ and all exact sequences of the form

$$
0 \to X_{2,n} \to X_{1,n} \to T \to 0
$$

are equivalent over $\End(X_{2,n})$; in particular all objects of the form $X_{1,n}$ are isomorphic.

**Proof.** By (4.21), there are up to isomorphism uniquely determined simple objects $T \cong S_1, S_2, \ldots, S_n \cong S$ with $\Ext(S_i, S_{i+1}) \neq 0$. Furthermore, by Lemma 4.4.2, there are $X_{1,n-1}$ and $X_{2,n}$ so that the following sequence is exact:

$$
\eta : 0 \to S \to X_{1,n-1} \to X_{2,n} \to T \to 0.
$$

We prove the lemma with induction on $n$. For $n = 2$, the lemma is obvious; for $n = 3$ it is the result of Lemma 4.4.6. Suppose that the assertion holds for $n - 1$. 


By Lemma 4.4.2, there is a monomorphism $X_{1,n-1} \to X_{1,n}$ and an epimorphism $X_{1,n} \to X_{2,n}$.

Now we can show both (a) and (b) completely analogously to 4.4.6. \qed

Now we show the other direction of 4.4.6:

**Lemma 4.4.8.** Suppose that for all simple objects $S \in \mathcal{A}$

$$\sum_{T \text{ simple}} d^1_T(S, T) \leq 1.$$ 

Let $S, S', T \in \mathcal{A}$ be fixed simple objects with $\text{Ext}^1(S, S') \neq 0 \neq \text{Ext}^1(T, S)$ so that for all $T \cong S_1, S_2, \ldots, S_n \cong S'$ with $\text{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$, we have $n = 3$ and $S_2 \cong S$.

If $\text{Ext}^2(T, S') = 0$, then there is some indecomposable object $Z$ of length 3 with $\text{soc} Z = S'$ and $\text{top} Z = T$.

**Proof.** By the assumptions, there are indecomposable objects $X, X'$ with exact sequences

$$0 \to S' \xrightarrow{f} X \xrightarrow{g} S \to 0 \quad (4.22)$$

$$0 \to S \xrightarrow{g'} X' \xrightarrow{h} T \to 0. \quad (4.23)$$

So the following sequence is also exact:

$$0 \to S' \xrightarrow{f} X \xrightarrow{g'g} X' \xrightarrow{h} T \to 0.$$ 

If $\text{Ext}^2(S', T) = 0$, then $\eta_1$ is equivalent to the split exact sequence

$$0 \to S' \to S' \to T \to 0.$$ 

So there are maps of exact sequences $\eta_1 \to \eta_2 \leftarrow \eta_3 \to \cdots \leftarrow \eta_m$ or $\eta_1 \leftarrow \eta_2 \to \eta_3 \leftarrow \cdots \to \eta_m$ so that $\eta_m$ or $\eta_{m+1}$ with $\eta_m \to \eta_{m+1}$ splits for some $m \in \mathbb{N}$.

Let $\eta$ be an exact sequence with $\eta \to \eta_1$. If we start with an exact sequence $\eta$ with $\eta_1 \to \eta$, the proof is analogous.

Then there are objects $M, N$, morphisms $c, d, e, \phi_1, \psi_1$, and a commutative diagram

$$\begin{array}{ccc}
0 & \to & S' \xrightarrow{f} X \xrightarrow{g'g} X' \xrightarrow{h} T \to 0.
\end{array}$$

\begin{diagram}
0 & \to & S' \xrightarrow{f} M \xrightarrow{d} N \xrightarrow{e} T \to 0.
\end{diagram}
By Lemma 4.4.1, there is some exact sequence

\[ \eta' : 0 \rightarrow S' \rightarrow M' \rightarrow N' \rightarrow T \rightarrow 0 \]

with a map \( \eta' \to \eta \) in \( \text{Ext}^2(S', T) \) and \( \text{top} N' \cong T \). Since \( \eta' \to \eta \to \eta_1 \), we can assume that \( \eta = \eta' \) and thus \( \text{top} N = T \). Then \( \eta'' \) as defined in Lemma 4.4.1 fulfils \( \text{soc} M'' = S' \) and \( \text{top} N'' = T \).

We can assume that some

\[ \eta'' : 0 \rightarrow S' \rightarrow M'' \rightarrow N'' \rightarrow T \rightarrow 0 \]

exists, which is not isomorphic to \( \eta_1 \).

Let \( \phi_2 : M \rightarrow M'' \) and \( \psi_2 : N \rightarrow N'' \) be the morphisms that are induced by \( \eta \to \eta'' \). The morphisms \( \psi_1 \) and \( \psi_2 \) are epimorphisms: since \( \text{top} X' = T = \text{top} N'' \) and \( h \psi_1 = h'' \psi_2 = e \neq 0 \), the cokernels of these morphisms are zero. Thus, \( \phi_1 \) and \( \phi_2 \) are also epimorphisms, since

\[ g' g \phi_1 = \psi_1 d \neq 0 \neq \psi_2 d = g'' \phi_2 \]

and

\[ \phi_1 c = f \neq 0 \neq f'' = \phi_2 c. \]

Similarly to the arguments in Lemma 4.4.1, \( \text{Ker} \phi \) is a subobject of \( \text{Im} d \) and thus there is a monomorphism \( \chi : \text{Ker} \phi \rightarrow N \). Obviously, there is also a monomorphism \( \chi_i : \text{Ker} \phi \rightarrow \text{Ker} \phi_i \cong \text{Ker} \psi_i \) for \( 1 \leq i \leq 2 \). The following diagram is exact, since all rows and the first and second column are exact:

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Ker} \phi \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Ker} \psi_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Ker} \psi_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Ker} \chi_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Ker} \chi \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Im} \psi_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Im} \psi_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Im} \psi_i \\
\end{array}
\]

So there is an exact sequence

\[ \theta : 0 \rightarrow S' \rightarrow \text{Im} \phi \rightarrow \text{Coker} \chi \rightarrow T \rightarrow 0 \]
with maps \( \theta \to \eta_1 \) and \( \theta \to \eta'' \). Thus, we can assume that \( \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \) is a monomorphism.

If \( \text{soc} \ M = S' \), then \( \phi_1 \) is not only an epimorphism, but also monomorphism and thus an isomorphism. Since \( \eta \) is an exact sequence, \( \phi_2 \) is also an isomorphism. But the same holds for \( \phi_2 \) and \( \psi_2 \), so \( \eta'' \cong \eta_1 \), contradictory to our assumptions.

We get \( S'' \mid \text{soc} \ M \) and thus \( S'' \mid \text{soc} \ N \).

By the assumption, we have \( n = 3 \) and \( S_2 \cong S \) for all \( S_i \cong S_1, S_2, \ldots, S_n \cong T \) with \( \text{Ext}^1(S_i, S_{i+1}) \neq 0 \) for \( 1 \leq i < n \). Furthermore, \( d_S(S, S') = 1 = d_T(T, S) \).

By Corollary 4.4.3 and Lemma 4.4.4, we see that either \( N'' \cong X' \) or \( N'' \cong T \), since all exact sequences in \( \text{Ext}^1(T, S) \) are equivalent over \( T \). In the first case, length considerations yield \( M'' \cong X \). Since the morphisms of \( \eta_1 \) are arbitrarily chosen, we can assume that the second case holds. Then \( \eta'' \) splits and \( M'' \cong S' \).

So the epimorphism \( \psi_2 \) splits and we get \( M = X \oplus S' \). We get \( c = \begin{bmatrix} f \\ h \end{bmatrix} \) and its cokernel is isomorphic to \( X \). If we set \( N =: Z \), then there are short exact sequences

\[
0 \to X \to Z \to T \to 0
\]

and

\[
0 \to S' \to Z \to X' \to 0
\]

and \( Z \) fulfills the assertions.

\[ \square \]

4.5 The third condition

In this section, we prove that (C3) holds if \( \mathcal{A} \) is of colocal type.

First note that the following equivalence holds:

**Proposition 4.5.1.** Suppose that for all simple objects \( S \in \mathcal{A} \)

\[
\sum_{T \text{ simple}} d_T^1(S, T) \leq 1.
\]

For fixed simple objects \( S \) and \( S' \) with \( \text{Ext}^1(S, S') \neq 0 \), the following classes of objects are the same:

(a) the class of simple objects \( T \) so that \( d_T(T, S) \neq 0 \) and there is some indecomposable object \( Z \) of length 3 with \( \text{soc} \ Z \cong S' \) and \( \text{top} \ Z \cong T \).
(b) the class of simple objects $T$ so that $d_T(T, S) \neq 0$ and there is some indecomposable object $Z$ of length greater or equal 3 with $\text{soc} \, Z \cong S'$ and $\text{top} \, Z \cong T$

If $\text{Ext}(S', S') = 0$, then these classes are the same as

(c) the class of simple objects $T$ so that $d_T(T, S) \neq 0$ and there is some indecomposable object $Z$ with $\text{soc} \, Z \cong S'$ and $\text{top} \, Z \cong T$

If $S'$ is not part of an oriented cycle in the $\text{Ext}$-quiver of $\mathcal{A}$, then this class is even the same as

(d) the class of simple objects $T$ so that $d_T(T, S) \neq 0$ and $\text{Ext}^2(T, S') = 0$

Proof. The equivalence of (a) and (b) is just Lemma 4.4.2 and Lemma 4.4.5. Part (c) is obvious, since under these assumptions $S \ncong S'$, $\text{Ext}(S, S) = 0$, $T \ncong S$ and $\text{Ext}^1(T, S') = 0$. Part (d) is proved in Lemma 4.4.6 (a) and Lemma 4.4.8: If there is a cycle in the $\text{Ext}$-quiver of $\mathcal{A}$, then it is oriented. So if $S'$ is not part of an oriented cycle in the $\text{Ext}$-quiver of $\mathcal{A}$, then for all $T \cong S_1, S_2, \ldots, S_n \cong S'$ with $\text{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$, we have $n = 3$ and $S_2 \cong S$.

We still need one lemma before we can prove that $(C3)$ holds if $\mathcal{A}$ is of colocal type:

**Lemma 4.5.2.** Let $\mathcal{A}$ be an abelian length category. Suppose that there are simple objects $S, S'$ with $\text{Ext}^1(S, S') \neq 0$ and for $1 \leq i \leq 2$ there are objects $T_i, X_i$ and $Z_i$ with exact sequences

\[
0 \longrightarrow S \xrightarrow{g_i} X_i \xrightarrow{h_i} T_i \longrightarrow 0, \tag{4.24}
\]

which are linearly independent over $\text{End}(T_1)^{\text{op}}$ if $T_1 \cong T_2$ and

\[
0 \longrightarrow S' \longrightarrow Z_i \longrightarrow X_i \longrightarrow 0.
\]

If $T_1 \cong T_2$, we furthermore assume that $d_{S'}(S, S') = 1$.

Then there is a pushout

\[
\begin{array}{ccc}
S' & \longrightarrow & Z_1 \\
\downarrow & & \downarrow \\
Z_2 & \longrightarrow & Y
\end{array}
\]

so that $Y$ is indecomposable.
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Proof. If $Z_1 \not\cong Z_2$, we can choose the monomorphisms $f_1 : S' \hookrightarrow Z_1$ and $f_2 : S' \hookrightarrow Z_2$ arbitrarily.

Otherwise, we denote the kernel of $Z_1 \to T_1$ with $X_3$. Since $l(Z_1) = 3$, there is an exact sequence

$$0 \to S' \xrightarrow{f} X_3 \xrightarrow{g} S \to 0.$$ 

We have $\operatorname{Im} gg_i \cong S \cong \operatorname{soc} X_i$. The image of the concatenation $\phi_i : X_3 \to Z_i \to X_i$ is also $\operatorname{soc} X_i$ and because of $d_{S'}(S, S') = 1$, there is some isomorphism $\psi_i$ so that $\phi_i \psi_i = gg_i$.

So we can choose $f'_1$ and $f'_2$ so that the following diagrams are exact for $1 \leq i \leq 2$:

$$0 \to S' \xrightarrow{f} X_3 \xrightarrow{g} S \to 0.$$  \hspace{1cm} (4.25)

We set $f_1 = f'_1 f$ and $f_2 = f'_2 f$ and form the pushout:

Now, we show that $Y$ is indecomposable:

Since $\operatorname{soc} Z_1 = \operatorname{soc} Z_2 = S'$, there is some indecomposable direct summand $Y'$ of $Y$ with monomorphisms $Z_1 \to Y'$ and $Z_2 \to Y'$. The following diagram
is exact for $1 \leq i \neq j \leq 2$, since its rows and the first two columns are exact:

$$
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & \downarrow & \\
0 & \rightarrow & Z_i & \rightarrow Z_i & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & S' & \rightarrow Z_1 \oplus Z_2 & \rightarrow Y & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & 0 & 0 & & \\
\end{array}
$$

(4.26)

Note that $l(Y) = 5$ and $l(Y') \geq l(Z_1) = 3$. Thus, the cokernel of $\phi_i : Z_i \rightarrow Y'$ cannot be simple; otherwise the cokernel of $Z_i \rightarrow Y$ would be semisimple. So either $Y \cong Y_1$ or $Y_1 \cong Z_1$. But the last case would give us a commutative diagram

$$
\begin{array}{cccc}
S' & \xrightarrow{f_1} & Z_1 & \\
\downarrow & & \downarrow \cong & \\
Z_2 & \xrightarrow{f_2} & Y_1 & \\
\end{array}
$$

So $T_1 \cong T_2$ and with (4.25), this contradicts the assumption that the exact sequences (4.24) are linearly independent over $\text{End}(T_i)^{op}$.

So $Y$ is indecomposable. \hfill \Box

The next Lemma shows in particular that (C3) is fulfilled if $\mathcal{A}$ is colocal.

**Lemma 4.5.3.** Let $\mathcal{A}$ be an abelian length category. The following holds for all simple objects $S \in \mathcal{A}$:

(a) Let $\mathcal{A}$ be of colocal type. If there is a simple object $S'$ with $\text{Ext}(S, S') \neq 0$ and a set $\mathcal{T}$ of simple objects $T$ so that $d_T(T, S) \neq 0$ and there is an indecomposable object $Z$ of length 3 with top $Z \cong T$ and soc $Z \cong S'$, then

$$
\sum_{T \in \mathcal{T}} d_T(T, S) \leq 1.
$$

(b) Let $S(\mathcal{A})$ be distributive. Assume that there is a simple object $S' \cong S$ with $\text{Ext}(S, S') \neq 0$ and a set $\mathcal{T}$ of simple objects $T$ so that $d_T(T, S) \neq 0$
and there is an indecomposable object $Z$ of length 3 with top $Z \cong T$ and soc $Z \cong S'$. If $d_S(S, S') \geq d_{S'}(S, S')$ or there are two non-isomorphic objects in $\mathcal{T}$, then

$$\sum_{T \in \mathcal{T}} d_T(T, S) \leq 1.$$ 

Proof. We show (a) first: Since $\mathcal{A}$ is colocal,

$$d_S(S, S') \geq d_{S'}(S, S') = 1$$

by Lemma 4.3.2. By Lemma 4.5.2, there are some $T_1, T_2 \in \mathcal{T}$, an indecomposable middle term $X_3$ of an exact sequence in $\text{Ext}^1(S, S')$, indecomposable objects $Z_1, Z_2$ with

$$0 \longrightarrow X_3 \longrightarrow Z_i \longrightarrow T_i \longrightarrow 0$$

and an indecomposable object $Y$ so that the following is a pushout:

$$\begin{array}{ccc}
S' & \longrightarrow & Z_1 \\
\downarrow & & \downarrow \\
Z_2 & \longrightarrow & Y
\end{array}$$

In the following, we use the same notation as in Lemma 4.5.2: in its proof, we either have chosen or can choose $f, f_1, f_2$ so that the upper left square of the following diagram is commutative. Thus, the diagram is commutative and exact, since its columns and the first two rows are exact:

$$\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X_3 \oplus X_3 \longrightarrow Z_1 \oplus Z_2 \longrightarrow T_1 \oplus T_2 \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S \oplus X_3 \longrightarrow Y \longrightarrow T_1 \oplus T_2 \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

(4.27)

So soc $Y = S \oplus S'$ and $\mathcal{A}$ is not of colocal type.
To show (b), we assume that \( S \nsubseteq S' \). Let \( k : S \to Y \) and \( l : S' \to Y \) be induced by the last row of the diagram (4.27). Then there is a monomorphism \( Y \to \text{Coker} \, k \oplus \text{Coker} \, l \) by Lemma 4.2.2.

For \( 1 \leq i \neq j \leq 2 \), we get the following exact diagram, since all columns and the first and second row are exact:

\[
\begin{array}{ccccccccc}
0 & 0 & \rightarrow & S & \rightarrow & S & \rightarrow & 0 \\
\downarrow & & \downarrow & \downarrow k & & \downarrow & & \\
0 & \rightarrow & S \oplus Z_i & \rightarrow & Y & \rightarrow & T_j & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z_i & \rightarrow & \text{Coker} \, k & \rightarrow & T_j & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & 0 & 0 & & & & & & \\
\end{array}
\]  

(4.28)

By Lemma 4.2.2, there is an indecomposable direct summand of \( \text{Coker} \, k \) of which \( Z_1 \) is a subobject. The last row of the diagram above cannot split, since \( Y \) is indecomposable. So \( \text{Coker} \, k \) is also indecomposable and has the socle \( S' \). Thus, \( \text{soc} \, Y \cong S' \oplus S \) is not a subobject of any direct sum of copies of \( \text{Coker} \, k \) and neither is \( Y \). Analogously, the next diagram is exact for all \( 1 \leq i \neq j \leq 2 \):

\[
\begin{array}{ccccccccc}
0 & 0 & \rightarrow & S' & \rightarrow & S' & \rightarrow & 0 \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z_i & \rightarrow & Z_1 \oplus Z_2 & \rightarrow & Z_j & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z_i & \rightarrow & Y & \rightarrow & X_j & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & 0 & 0 & & & & & & \\
\end{array}
\]

And finally, the following diagram is exact, because all columns and the
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second and third row are exact:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & Z_1 \oplus Z_2 & Y \oplus Y & X_1 \oplus X_2 \rightarrow 0 \\
0 & Y & 0
\end{array}
\]

Thus, we have \( \text{Coker } l \cong X_1 \oplus X_2 \), \( \text{soc Coker } l \cong S^2 \) and \( Y \) is not a subobject of any direct sum of copies of \( \text{Coker } l \).

\[ \blacksquare \]

4.6 An Equivalence Theorem

We begin with the following special case, from which the general statement follows:

**Lemma 4.6.1.** Let \( A \) be a hereditary Artin algebra and \( \mathcal{A} \equiv \text{mod } A \). Assume that \( \mathcal{A} \) fulfils the following conditions:

(C1) For all simple objects \( S \in \mathcal{A} \)

\[ \sum_{T \text{ simple}} d_T(S, T) \leq 1. \]

(C2) For all simple objects \( S \in \mathcal{A} \)

\[ \sum_{T \text{ simple}} d_T(T, S) \leq 2. \]

(C'3) If there is a simple object \( S \in \mathcal{A} \) with

\[ \sum_{T \text{ simple}} d_T(T, S) = 2, \]

then \( \text{Ext}^1(S, S') = 0 \) for all simple \( S' \in \mathcal{A} \).
Then the following holds:

(a) \( A \) is of colocal type.

(b) \( A \) is representation finite.

Proof. We can assume without loss of generality that \( A \) is indecomposable as an algebra. To show both (a) and (b), we construct the preinjective component of the Auslander-Reiten quiver of \( A \). We will see that this component is finite and thus the complete Auslander-Reiten quiver of \( A \) by Proposition 2.3.15.

The Ext-quiver of \( A \) does not contain oriented cycles, since \( A \) is hereditary. By (C1), the Ext-quiver does not contain other cycles, either. So there are two possible cases:

(a) We have \( d^1_T(S,T) \leq 1 \) and \( d^2_S(S,T) \leq 1 \) for all simple modules \( S,T \in \text{mod} \ A \). Then we can order the simple modules \( S_1, S_2, \ldots, S_n \) of \( \text{mod} \ A \) so that for some \( 1 \leq l \leq n \) we have \( \text{Ext}^1(S_i,S_{i+1}) \neq 0 \) for all \( 1 \leq i < l \) and \( \text{Ext}^1(S_{i+1},S_i) \neq 0 \) for all \( l \leq i < n \). By (C1) - (C2) and (C’3), \( \text{Ext}^1(S,T) = 0 \) for all other simple modules \( S,T \).

(b) We can denote the simple modules by \( S_1, \ldots, S_n \) so that

\[
d^1_S(S_i,S_{i+1}) = 1 = d^2_S(S_i,S_{i+1})
\]

for all \( 1 \leq i \leq n - 2 \), \( d^1_S(S_{n-1},S_n) = 1 \), \( d^2_S(S_{n-1},S_n) = 2 \) and \( \text{Ext}^1(S,T) = 0 \) for all other simple modules \( S,T \).

We will only prove the assertion in case (a). In case (b), we can construct the Auslander-Reiten quiver completely analogously to the first case and thus show the assertion completely analogously. The result of this construction is the following Auslander-Reiten quiver:

\[
\begin{align*}
\tau^{n-1}I_1 & \quad \tau^{n-2}I_1 & \cdots & \tau I_1 & \quad I_1 \\
\tau^{n-1}I_2 & \quad \tau^{n-2}I_2 & \cdots & \cdots & \tau I_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\tau^{n-1}I_{n-1} & \quad \tau^{n-2}I_{n-1} & \cdots & \tau I_{n-1} & \quad I_{n-1} \\
\tau^{n-1}I_n & \quad \tau^{n-2}I_n & \cdots & \tau I_n & \quad I_n
\end{align*}
\]
The arrows of the form $\rightarrow$ denote monomorphisms.

Now suppose that (a) holds. Additionally, we assume without loss of
generality that $l > n - l$. Then the following is part of the preinjective
component of the Auslander-Reiten quiver of $A$ by 2.3.7 and 2.3.9:

$$
\tau^{l-1}I_1 \quad \tau^{l-2}I_1 \quad \ldots \quad \ldots \quad \ldots \quad \tau I_1 \quad I_1
$$

$$
\tau^{l-1}I_2 \quad \tau^{l-2}I_2 \quad \ldots \quad \ldots \quad \ldots \quad \tau I_2 \quad I_2
$$

$$
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
$$

$$
\tau^{n-l}I_l \quad \ldots \quad \ldots \quad \ldots \quad \tau I_l \quad I_l
$$

$$
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
$$

$$
\tau^{n-l}I_n \quad \ldots \quad \ldots \quad \ldots \quad \tau I_n \quad I_n
$$

If the arrows of the form $\leftrightarrow$ correspond to monomorphisms, it is obvious
that the above is a complete component of the Auslander-Reiten quiver of
$A$ and thus the complete Auslander-Reiten quiver: We have $\tau^l I_1 = 0$ and
$\tau^{n-l+1}I_n = 0$, since no Auslander-Reiten sequence can start in these modules.
Inductively,

$$
\tau^l I_2 = \cdots = \tau^l I_{n-l+1} = 0
$$

and

$$
\tau^{n-l+1}I_{n-1} = \cdots = \tau^{n-l+1}I_l = \tau^{n-l+2}I_{l-1} = \cdots = \tau^{l-1}I_{n-l+2} = 0.
$$

To show that the arrows of the form $\rightarrow$ denote indeed monomorphisms, we
use Lemma 3.2.13: There are Auslander-Reiten sequences

$$
0 \longrightarrow \tau^l I_1 \longrightarrow \tau^{l-1}I_2 \longrightarrow \tau^{l-1}I_1 \longrightarrow 0
$$
for \( \tau^i I_1 \neq 0 \) and
\[
0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j+1} \oplus \tau^i I_{j-1} \longrightarrow \tau^{i-1} I_j \longrightarrow 0
\]
for all \( 1 < j \leq l - 1 \) and \( i \) with \( \tau^i I_j \neq 0 \). If there is an exact sequence
\[
0 \longrightarrow \tau^i I_{j-1} \longrightarrow \tau^{i-1} I_j \longrightarrow \tau^{i-1} I_1 \longrightarrow 0
\]
then there is also an exact sequence
\[
0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j+1} \longrightarrow \tau^{i-1} I_1 \longrightarrow 0 . \tag{4.29}
\]
It possible to show that \( \tau^i I_{j-1}, \tau^i I_{j-2}, \ldots, \tau^i I_1 \) are non-zero and thus, we get indeed the exact sequences above. But this is not necessary for our proof, since we are only interested in the monomorphisms of the exact sequences and if \( \tau^i I_{j-1} = 0 \), there is obviously a monomorphism \( \tau^i I_j \hookrightarrow \tau^{i-1} I_{j+1} \), just with a different cokernel.

Analogously, we can assume in the following that all modules in the exact sequences below are non-zero.

There are Auslander-Reiten sequences
\[
0 \longrightarrow \tau^i I_n \longrightarrow \tau^{i-1} I_{n-1} \longrightarrow \tau^{i-1} I_n \longrightarrow 0
\]
for \( \tau^i I_n \neq 0 \) and
\[
0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j-1} \oplus \tau^i I_{j+1} \longrightarrow \tau^{i-1} I_j \longrightarrow 0
\]
for all \( l < j \leq n - 1 \) with \( \tau^i I_j \neq 0 \). We get an exact sequence
\[
0 \longrightarrow \tau^i I_j \longrightarrow \tau^{i-1} I_{j-1} \longrightarrow \tau^{i-1} I_n \longrightarrow 0 , \tag{4.30}
\]
for \( m < j \leq n \) and \( i \) with \( \tau^i I_j \neq 0 \).

The remaining Auslander-Reiten sequences are
\[
0 \longrightarrow \tau^i I_l \longrightarrow \tau^{i-1} I_{l-1} \oplus \tau^i I_{l+1} \longrightarrow \tau^{i-1} I_l \longrightarrow 0
\]
for \( \tau^i I_l \neq 0 \).

Together with (4.29) and (4.30), we get exact sequences
\[
0 \longrightarrow \tau^i I_l \longrightarrow \tau^{i-1} I_{l+1} \longrightarrow \tau^{i-1} I_1 \longrightarrow 0
\]
and
\[
0 \longrightarrow \tau^i I_l \longrightarrow \tau^{i-1} I_{l-1} \longrightarrow \tau^{i-1} I_n \longrightarrow 0 .
\]
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Inductively, we get exact sequences

$$0 \rightarrow \tau^i I_j \rightarrow \tau^i I_{j-1} \rightarrow \tau^{i-l+j-1} I_n \rightarrow 0 \quad (4.31)$$

for all $\tau^i I_j \neq 0$ with $1 < j \leq l$ and $l - j < i$. Analogously,

$$0 \rightarrow \tau^i I_j \rightarrow \tau^i I_{j+1} \rightarrow \tau^{i+l-j-1} I_1 \rightarrow 0 \quad (4.32)$$

is exact for all $\tau^i I_j \neq 0$ with $l \leq j \leq n$ and $j - l \leq i$.

We can find the injective envelope (see 2.2.4) of every indecomposable module $M$ by taking the product of irreducible monomorphisms.

So $M$ has an indecomposable injective envelope $I$ with simple socle (see 2.2.6) and $\text{soc} M = \text{soc} I$. Thus every submodule of $M$ is indecomposable and $A$ is of colocal type.

Now we can prove Theorem 1.2.4:

**Theorem 4.6.2.** The category $\mathcal{A}$ is of colocal type if and only if the following conditions hold:

(C1) For all simple objects $S \in \mathcal{A}$

$$\sum_{T \text{ simple}} d^1_T(S, T) \leq 1.$$  

(C2) For all simple objects $S \in \mathcal{A}$

$$\sum_{T \text{ simple}} d^1_T(T, S) \leq 2.$$  

(C3) If there is a simple object $S'$ with $\text{Ext}^1(S, S') \neq 0$, let $\mathcal{T}$ be the class of simple objects $T$ for which $d^1_T(T, S) \neq 0$ and there is an indecomposable object $Z$ of length 3 with $\text{top} Z \cong T$ and $\text{soc} Z \cong S'$. Then

$$\sum_{T \in \mathcal{T}} d^1_T(T, S) \leq 1.$$  

**Proof.** If $\mathcal{A}$ is of colocal type, then condition (C1) holds by Lemma 4.3.2, condition (C2) holds by Lemma 4.3.4 and condition (C3) by Lemma 4.5.3.

For the other direction, we use Lemma 4.2.4: If $\mathcal{A}$ is not of colocal type, there are objects $X, Y_1, Y_2, S$ so that $X$ is indecomposable, $Y_1, Y_2$ are not simple and there is an exact sequence

$$0 \rightarrow X \rightarrow Y_1 \oplus Y_2 \xrightarrow{[g_1, g_2]} S \rightarrow 0. \quad (4.33)$$
If $Y \mid Y_1 \oplus Y_2$ is indecomposable, we can assume that $	ext{soc} Y$ is simple.

On the other hand, $S \mid \text{top} Y$, since $Y \mid X$.

First, we assume that for every $T \mid \text{soc}(Y_1 \oplus Y_2)$ there are up to isomorphism unique objects $S \cong S_1, S_2, \ldots, S_n \cong T$ with $\text{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$ so that $X_{1,n-1}$ and $X_{2,n}$ as in Lemma 4.4.5 exist. By 4.4.2 and Lemma 4.4.7, these are objects in a full abelian subcategory $\mathcal{A}'$ of $\mathcal{A}$ so that $\text{Ext}^2(S,T) = 0$ for all simple $S,T \in \mathcal{A}'$.

By 2.3.1, we can be embed $\mathcal{A}'$ into the module category of a hereditary Artin algebra $A$. With Lemma 4.6.1, at least one of the conditions $(C1)$, $(C2)$ or $(C'3)$ is not fulfilled by $\text{mod} A$. For a hereditary Artin algebra, the condition $(C'3)$ is equivalent to $(C3)$ by 4.5.1 and thus $\mathcal{A}'$ and $\mathcal{A}$ do not fulfill $(C1)$ - $(C3)$.

Now suppose that for some $T \mid \text{soc}(Y_1 \oplus Y_2)$, there are non-isomorphic objects $S \cong S_1, S_2, \ldots, S_n \cong T$ and $S \cong S'_1, S'_2, \ldots, S'_{n'} \cong T$ so that we have $\text{Ext}^1(S_i, S_{i+1}) \neq 0$ for $1 \leq i < n$, $\text{Ext}^1(S'_i, S'_{i+1}) \neq 0$ for $1 \leq i < n'$ and there are indecomposable objects $X_{1,n-1}, X_{2,n}$ as in Lemma 4.4.5 and $X'_{1,n-1}$ with $\text{soc} X'_{1,n-1} \cong S'_{n-1}$, $\text{top} X'_{1,n-1} \cong S'_1$ and $X'_{2,n}$ with $\text{soc} X'_{2,n} \cong S'_n$, $\text{top} X'_{2,n} \cong S'_2$.

If we assume that $n' \leq n$, then $S_i \cong S'_1, \ldots, S_{n'} \cong S'_n$ by $(C1)$, so $n' < n$. By Lemma 4.4.2 and Lemma 4.4.5, there are objects $X_{i,i+2}$ of length 3 with $\text{soc} X_{i,i+2} = S_{i+2}$ and $\text{top} X_{ij} = S_i$. So by condition $(C3)$, we get

$$d_{S_i}(S_i, S_{i+1}) = 1$$

for all $1 \leq i \leq n$. Analogously, every $T' \mid \text{soc}(Y_1 \oplus Y_2)$ must be of the form $S_i$ for some $1 \leq i \leq n$.

So we can choose $T$ so that for all indecomposable objects $Y \mid Y_1 \oplus Y_2$ there is some $1 \leq i \leq n$ with $\text{soc} Y = S_i$. By $(C1)$, there are some $m_i \in \mathbb{N}$ for $1 \leq i \leq n$ so that $\text{top} Y = \bigoplus_{i=1}^{n} S_i^{m_i}$. Analogously to Lemma 4.4.4, $d_{S_i}(S_i, S_{i+1}) = 1$ means $\text{top} Y = S = S_1$.

By Lemma 4.4.7, all objects with socle $S_i$ and top $S_1$ are isomorphic to $X_{1,i}$ and all exact sequences of the form

$$0 \longrightarrow X_{2,i} \longrightarrow X_{1,i} \longrightarrow S_1 \longrightarrow 0$$

are equivalent over $\mathcal{E}_{\text{nd}}(X_{2,i})$. So for all epimorphisms $g : X_{1,i} \twoheadrightarrow S_1$ and $g' : X_{1,i} \twoheadrightarrow S_1$, there is some isomorphism $\chi$ on $X_{1,i}$ so that $g' = \chi g$.

Since $S = S_1$ and $T \mid \text{soc} Y_1 \oplus Y_2$, we get $X_{1,n} \mid Y_1 \oplus Y_2$ and can assume that $Y_1 = Y'_1 \oplus X_{1,n}$. By Lemma 4.4.2, there is an epimorphism $X_{1,n} \twoheadrightarrow X_{i,n}$.

There are morphisms $f_1, f_2$ so that the following is a commutative dia-
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gram:

$$ X_{1,n} \xrightarrow{\left[ \begin{array}{c} g_1 \\ f_1 \end{array} \right]} X_{1,n} \oplus Y_1'. $$

$$ Y_2 \xrightarrow{g_2} T $$

But by (4.33), there is a pullback

$$ X \xrightarrow{g_1} Y_1' $$

Thus, $X$ cannot be indecomposable, a contradiction to the assumption and the proof is complete.

We can draw the following corollary:

**Corollary 4.6.3.** If $A$ is a colocal Artin algebra, then $A$ is of finite representation type.

**Proof.** Since $A$ is an Artin algebra, it has finitely many non-isomorphic simple modules $S_1, S_2, \ldots, S_n$. The proof of Theorem 4.6.2 shows that every module in $\text{mod} \ A$ can be either be regarded as a module in $\text{mod} \ A'$, where $A'$ is a hereditary subalgebra of $A$ with simple modules $S_{i_1}, S_{i_2}, \ldots, S_{i_m}$ for some $m \leq n$. Or there is a path $S'_1 \to S'_2 \to \cdots \to S'_{n'}$ in the Ext-quiver of $A$ which is part of an oriented cycle so that the module is of the form $X_{1,n'}$ and of length $n'$. Since $A$ is an Artin algebra, there are only finitely many objects of the latter form.

By Lemma 4.6.1, every hereditary subalgebra of $A$ is representation finite. Because there are only finitely many possibilities for $i_1, \ldots, i_m$, the algebra $A$ is also representation finite.

4.7 The lattice $S(A)$

We show in this section that the lattice $S(A)$ is in fact the Cartesian product of certain sublattices.

For Artin algebras $A$ over algebraically closed fields, we will use this in the next section.

We begin with the following lemma:
Lemma 4.7.1. Suppose $X$ is an indecomposable object and there is an index set $I$ so that $X \subseteq \bigoplus_{i \in I} Y_i$. Set

$$I' = \{ i \in I \mid \text{there is a simple object } S \subseteq X \text{ and } S \subseteq X_i \}.$$  

Then $X \subseteq \bigoplus_{i \in I'} X_i$.

Proof. There is a morphism

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}: X \rightarrow \bigoplus_{i \in I'} Y_i \oplus \bigoplus_{i \in I \setminus I'} Y_i$$

with

$$f_1: X \rightarrow \bigoplus_{i \in I'} Y_i \quad \text{and} \quad f_2: X \rightarrow \bigoplus_{i \in I \setminus I'} Y_i.$$  

Furthermore, $\text{Ker}(f_1) \oplus \text{Ker}(f_2) \subseteq X$ and $\text{Ker}(f_1) \subseteq \bigoplus_{i \in I \setminus I'} Y_i$. So there is no simple $S \subset \text{Ker}(f_1)$ and thus $\text{Ker}(f_1) = 0$, which implies that $f_1$ is a monomorphism and $X \subseteq \bigoplus_{i \in I'} Y_i$. \hfill \Box

To simplify the notation, we define:

Definition 4.7.2. For a class $\mathcal{M}$ of indecomposable objects in $\mathcal{A}$ let

$$S(\mathcal{M}) := S(\text{add } \mathcal{M}).$$

Under certain assumptions, $S(\mathcal{M})$ is a sublattice of $S(\mathcal{A})$.

Lemma 4.7.3. Let $\mathcal{M}$ be a class of indecomposable objects in $\mathcal{A}$. If

$$\text{ind sub } \mathcal{M} = \mathcal{M},$$  

then $S(\mathcal{M})$ is a sublattice of $S(\mathcal{A})$.

Proof. We need to show that for $C, C' \in S(\mathcal{M})$, the join and the meet are again in $S(\mathcal{M})$. The first direction is obvious: $C \wedge C' = C \cap C'$ and thus

$$\text{ind } (C \wedge C') = \text{ind } C \cap \text{ind } C' \subseteq \mathcal{M},$$

since $\text{ind } C, \text{ind } C' \subseteq \mathcal{M}$. So $C \wedge C' \in S(\mathcal{M})$.

On the other hand, the join $C \vee C'$ consists of all subobjects of direct sums of objects in $C$ and $C'$. Thus, if $M \in \text{ind } (C \vee C')$ then $M \in \text{sub } \mathcal{M}$. By (4.34), $M \in \mathcal{M}$. So $C \vee C' \in S(\mathcal{M})$ and $S(\mathcal{M})$ is a sublattice of $S(\mathcal{A})$. \hfill \Box

We get the following homomorphism between lattices:
Lemma 4.7.4. Let $\mathcal{A}$ be an abelian length category. Suppose that there is an index set $I$, and classes of indecomposable objects $\mathcal{M}_i, i \in I$ exist, so that

(M1) $\bigcup_{i \in I} \mathcal{M}_i = \text{ind } \mathcal{A}$
(M2) $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $i, j \in I$ with $i \neq j$
(M3) $\text{ind sub } \mathcal{M}_i = \mathcal{M}_i$ for all $i \in I$.

Denote $\mathcal{M} = \{ \mathcal{M}_i \mid i \in I \}$. Then

$$f_{\mathcal{M}} : S(\text{mod } \mathcal{A}) \to \prod_{i \in I} S(\mathcal{M}_i)$$

$$C \to \prod_{i \in I} C_i$$

where $C_i$ is given by $\text{ind } C_i = \text{ind } C \cap \mathcal{M}_i$

is a lattice homomorphism.

Proof. By Lemma 4.7.3 and (M3), $S(\mathcal{M}_i)$ is a lattice for every $i \in I$ and the Cartesian product exists. We have to show that $f_{\mathcal{M}}$ preserves meets and joins. Take $C, C' \in S(\mathcal{M}_i)$. Then $f_{\mathcal{M}}$ preserves meets, since

$$\text{ind}(C \land C') = \text{ind } C \cap \text{ind } C'$$

and

$$\text{ind}(C \land C')_i = \text{ind } (C \land C') \cap \mathcal{M}_i = (\text{ind } C \cap \text{ind } C') \cap \mathcal{M}_i = \text{ind } C_i \cap \text{ind } C'_i.$$ 

Thus $(C \land C')_i = C_i \land C'_i$ and

$$f_{\mathcal{M}}(C \land C') = \prod_{i \in I} (C \land C')_i = \prod_{i \in I} (C_i \land C'_i) = \prod_{i \in I} C_i \land \prod_{i \in I} C'_i = f_{\mathcal{M}}(C) \land f_{\mathcal{M}}(C').$$

The function also preserves joins: For some object $M$, we have $M \in \text{ind}(C \lor C')_i$ if and only if $M \in \mathcal{M}_i$ and there are objects $x_1, \ldots, x_c \in \text{ind } C$ and $x'_1, \ldots, x'_{c'} \in \text{ind } C'$ for some $c, c' \in \mathbb{N}$ so that

$$M \subseteq \bigoplus_{k=1}^c x_k \oplus \bigoplus_{k=1}^{c'} x'_k.$$ 

Suppose that there is some $x_k \notin \mathcal{M}_i$ for some $1 \leq k \leq c$. By (M1), there is some $j \neq i$ with $x_k \in \mathcal{M}_j$. For every simple object $S \subseteq x_k$, we get $S \in \mathcal{M}_j$.
with (M3) and thus \( S \notin \mathcal{M}_i \) by (M2). So \( S \) is not a subobject of \( M \) and by Lemma 4.7.1, we get

\[
M \subseteq \bigoplus_{k=1}^{c} x_k \oplus \bigoplus_{k \neq l}^{c'} x'_k.
\]

So we can assume that \( x_1, \ldots, x_c, x'_1, \ldots, x'_c \in \mathcal{M}_i \) and thus \( M \in \text{ind}(C_i \vee C_i') \).

Since \( C_i \vee C_i' \in S(\mathcal{M}_i) \) and \( C_i \vee C_i' \leq C \vee C' \), the other direction is obvious.

We get \( \text{ind}(C_i \vee C_i') = \text{ind}(C \vee C') \) and \( C_i \vee C_i' = (C \vee C')_i \). So

\[
f_M(C \vee C') = \prod_{i \in I} (C \vee C')_i = \prod_{i \in I} (C_i \vee C_i') = \prod_{i \in I} C_i \vee \prod_{i \in I} C_i' = f_M(C) \vee f_M(C')
\]

and \( f_M \) is a lattice homomorphism.

\[\square\]

Even better, \( f_M \) is an isomorphism:

**Proposition 4.7.5.** Let \( \mathcal{A} \) be an abelian length category and \( \mathcal{M} = \{ \mathcal{M}_i \mid i \in I \} \) be a family of classes of indecomposable objects that fulfil (M1) - (M3). Then \( f_M \), as defined in Lemma 4.7.4, is a lattice isomorphism between \( S(\mathcal{A}) \) and \( \prod_{i \in I} S(\mathcal{M}_i) \).

**Proof.** By Lemma 4.7.4, \( f_M \) is a homomorphism between lattices. To show that \( f_M \) is an isomorphism, we need to prove that \( f \) is injective and surjective.

Suppose that \( f_M(C) = f_M(C') \) for some \( C, C' \in S(\text{mod } A) \). Then

\[
\prod_{i \in I} C_i = \prod_{i \in I} C'_i
\]

and by (M2), we have \( C_i = C'_i \) for all \( i \in I \). This means

\[\text{ind } C \cap \mathcal{M}_i = \text{ind } C' \cap \mathcal{M}_i\]

for all \( i \in I \). By (M1), \( \text{ind } C = \text{ind } C' \) and \( f_M \) is injective.

Now take

\[
\prod_{i \in I} C_i \subseteq \prod_{i \in I} S(\mathcal{M}_i).
\]

Since all \( C_i \) are subobject closed subcategories of \( \mathcal{A} \), we have \( C_i \in S(\mathcal{A}) \) for all \( i \in I \). We will show that

\[
f_M(\bigvee_{i \in I} C_i) = \prod_{i \in I} C_i.
\]

It is obvious that \( C_j \subseteq \left( \bigvee_{i \in I} C_i \right)_j \) for all \( j \in I \) which implies

\[
\prod_{i \in I} C_i \subseteq f_M(\bigvee_{i \in I} C_i).
\]
For the other direction, we need to show that \((\bigvee_{i \in I} C_i)_j \subseteq C_j\) for all \(j \in I\), which is equivalent to

\[
\left( \text{ind} \bigvee_{i \in I} C_i \right) \cap \mathcal{M}_j \subseteq \text{ind} C_j.
\]  

(4.35)

Suppose that \(M \in (\text{ind} \bigvee_{i \in I} C_i) \cap \mathcal{M}_j\). Then there are objects \(N_i \in C_i\), so that

\[M \subseteq \bigoplus_{i \in I} N_i.
\]

Set

\[I' = \{i \in I \mid \text{there is a simple object } S \text{ with } S \subseteq M \text{ and } S \subseteq N_i\}.
\]

By Lemma 4.7.1,

\[M \subseteq \bigoplus_{i \in I'} N_i.
\]

By (M3), we have \(S \in \mathcal{M}_j\) for all simple modules \(S \subseteq M \in \mathcal{M}_j\). On the other hand, if \(S \subseteq N_i \in \mathcal{M}_i\), then \(S \in \mathcal{M}_i\) and by (M2) we get \(I' = \{j\}\). So (4.35) holds, \(f_M\) is surjective and thus a lattice isomorphism.

4.8 The structure of the lattice

Let \(\text{mod } A \equiv \text{mod } kQ/I\) for some quiver \(Q\) and some admissible ideal \(I\). This is always the case if \(A\) is an algebra over a closed field \(k\). If \(A\) is of colocal type, then the lattice \(S(\text{mod } A)\) is relatively simple and can be described completely.

By Theorem 4.6.2 we get the following:

**Proposition 4.8.1.** Let \(A\) be an Artin algebra and \(\text{mod } A \equiv \text{mod } kQ/I\) for some quiver \(Q\) and an admissible ideal \(I\).

(a) The algebra \(A\) is of colocal type if and only if \(S(\text{mod } A)\) is distributive and for every subquiver of \(Q\) of the form

\[
\begin{array}{c}
1 \\
\downarrow \alpha \\
2
\end{array} \quad \begin{array}{c}
\downarrow \beta \\
\circ
\end{array}
\]

\(\alpha \beta \in I\) or \(\alpha^2 \in I\).

(b) The algebra \(A\) is of colocal type if and only if it is a string algebra and no vertex in \(Q\) is starting point of more than one arrow.
**Proof.** First, we prove (a): By Lemma 4.2.3, $\mathcal{S}(\text{mod } A)$ is distributive if $A$ is of colocal type.

On the other hand, suppose that $\mathcal{S}(\text{mod } A)$ is distributive and fulfils the condition above. We have $d_S(S, T) = d_T(S, T)$ for all simple modules $S, T \in \text{mod } A$ by 2.5.2. Since $\text{mod } kQ/I$ is equivalent to the category of representations of $Q$ with the relations that generate $I$, we have $\text{End}(S) \cong \text{End}(T)$. So

$$d_S(S, T) = d_T(S, T) \leq 1$$

for all $S, T$ by Lemma 4.3.5. Thus Lemma 4.3.2 (b), 4.3.4 (b) and 4.5.3 (b) show that $\text{mod } A$ fulfils (C1) - (C3). By Theorem 4.6.2, $A$ is of colocal type.

To show (b), suppose that $A$ fulfils (C1) - (C3). By 2.5.2, this is equivalent to the following:

1. No vertex in $Q$ is starting point of more than one arrow.
2. No vertex in $Q$ is end point of more than two arrows.
3. Given an arrow $\beta$, there is at most one arrow $\gamma$ with $s(\beta) = e(\gamma)$ and $\beta \gamma \notin I$.

Since $A$ is an Artin algebra, the quiver $Q$ must be finite.

Comparing this to Definition 2.5.4, it only remains to show that $I$ is an ideal generated by zero relations. If $Q$ does not contain oriented cycles, then for any given vertices $i$ and $j$, there is at most one path $\rho$ with $s(\rho) = i$ and $e(\rho) = j$.

In fact, any relation which is not a zero relation is of the following form, where $\rho$ is an oriented cycle, $\rho'$ is a subpath of $\rho$ with $s(\rho') = e(\rho') = s(\rho)$, $a_1, \ldots, a_n \in k$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n \in \mathbb{N}$:

$$a_1 \rho' \rho^{\alpha_1} + a_2 \rho' \rho^{\alpha_2} + \cdots + a_n \rho' \rho^{\alpha_n} = 0.$$  \hspace{1cm} (4.36)

Now, we use that $I$ is admissible: there must be some $t \in \mathbb{N}$, so that $\rho^t = 0$. So for every representation $V$ of $Q$, there is some $m$ so that

$$0 = \text{Im } f_{\rho^m} \subsetneq \text{Im } f_{\rho^{m-1}} \subsetneq \cdots \subsetneq \text{Im } f_{\rho},$$

where $f_{\rho^i}$ denotes the map of the representation defined by the path $\rho^i$. We get $\rho^{\alpha_1} = \cdots = \rho^{\alpha_n} = 0$, since otherwise

$$\text{Im } (f_{\rho^i} \rho^{\alpha_2} + \cdots + f_{\rho^i} \rho^{\alpha_n}) \subsetneq \text{Im } f_{\rho^i} \rho^{\alpha_2} \subsetneq \text{Im } f_{\rho^i} \rho^{\alpha_1},$$

a contradiction to (4.36).

Furthermore, we get some useful properties:
4.8. THE STRUCTURE OF THE LATTICE

Lemma 4.8.2. If $A = kQ/I$ for some quiver $Q = (Q_0, Q_1)$ with admissible ideal $I$ and $A$ is of colocal type, then the following holds:

(a) If $Q$ contains a cycle, this cycle is oriented.

(b) At most two paths are maximal under all paths without relations that end in $i$.

(c) Every module in $\text{ind} A$ is a string module.

(d) Every string is of the form

$$w = \alpha_{l_1}^{-1} \alpha_{l_1-1}^{-1} \cdots \alpha_1^{-1} \beta_1 \beta_2 \cdots \beta_{l_2},$$

for some $l_1, l_2 \in \mathbb{N}_0$, and arrows $\alpha_1, \ldots, \alpha_{l_1}, \beta_1, \ldots, \beta_{l_2}$ or of the form $e_m$ for some vertex $m$.

(e) We have $M(w') \subseteq M(w)$ if and only if there are $1 \leq j_1 \leq l_1$ and $1 \leq j_2 \leq l_2$ so that

$$w = \alpha_{j_1}^{-1} \alpha_{j_1-1}^{-1} \cdots \alpha_1^{-1} \beta_1 \beta_2 \cdots \beta_{j_2},$$

or $w = e_m$ with $m = e(\alpha_1) = e(\beta_1)$.

Proof. (a) Every non-oriented cycle contains a vertex which is starting point of two arrows.

(b) Since $A$ is a string algebra, there are at most two arrows which end in $i$ by Definition 2.5.4 (2). By 2.5.4 (3), each of those arrows is part of only one maximal path that ends in $i$.

(c) From definition 2.5.5, it is obvious that every band corresponds to a cycle without relations. Since $I$ is an admissible ideal, every oriented cycle of $Q$ contains a relation in $I$. By (1), $A = kQ/I$ has no band modules and $\text{ind} A$ consists only of string modules.

(d) There are no arrows $\alpha, \beta$ with $e(\beta^{-1}) = s(\beta) = s(\alpha)$.

(e) This follows from Lemma 2.5.8.

We use Proposition 4.7.5 to simplify the problem of describing $S(\text{mod} A)$ and start with the definition of a suitable family $\mathcal{M}$:

By Lemma 4.8.2 (2), there are at most two maximal paths without relation that end in a vertex $m \in Q_0$:

Definition 4.8.3. Suppose that there is at most one arrow $\alpha$ with $e(\alpha) = m$. Then there is only one path that is maximal under the paths without relation that ends in $m$. We denote its length with $k_m$ and set $l_m := 0$.

If there are two arrows that end in $m$, there are two maximal paths. We denote their lengths with $k_m$ and $l_m$. 
Definition 4.8.4. Let $A = kQ/I$ for some quiver $Q = (Q_0, Q_1)$, $m \in Q_0$, and $M_m := M(\ell_w)$ be the module with

$$w_m = \alpha_{k_m}^{-1}\alpha_{i_{1}}^{-1}\ldots\alpha_{1}^{-1}\beta_{1}\beta_{2}\ldots\beta_{i_{m}}$$

so that $\alpha_{k_m} \alpha_1$ and $\beta_{i_m} \ldots \beta_1$ are the maximal paths that end in $m$. By Lemma 2.5.7 (2) and Lemma 4.8.2 (2), this module is well defined.

Furthermore, we define

$$\mathcal{M}_m := \{M \in \text{mod } A \mid M \subset M_m\}.$$

Lemma 4.8.5. If $A$ is of colocal type, then

$$S(\text{mod } A) \cong \prod_{m \in Q_0} S(\mathcal{M}_m).$$

Proof. We need to prove that the sets $\mathcal{M}_m$, $m \in Q_0$ fulfil the conditions of Proposition 4.7.5:

(M1) is fulfilled by Lemma 4.8.2 (3), (4) and (5); (M2) and (M3) are fulfilled by Lemma 4.8.2 (5).

The lattices $S(\mathcal{M}_m)$ for $m \in Q_0$ have a very simple description: they are all sublattices of Young’s lattice, which is defined in [18], p. 58 and Example 3.4.4(b):

Definition 4.8.6. Take a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ of a natural number, ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Young diagram of $\lambda$ is an array of squares with $n$ rows and exactly $\lambda_i$ squares in the $i$-th row.

These partitions form a lattice $Y$, ordered by the inclusion order on the Young diagrams. It is called Young’s lattice.

Let $\lambda' := (\lambda_1', \lambda_2', \lambda_3', \ldots, \lambda_n')$, suppose that $n \leq n'$ and set $\lambda_i := 0$ for $i > n$. Then

$$\lambda \land \lambda' = (\min(\lambda_1, \lambda_1'), \ldots, \min(\lambda_n, \lambda_n'))$$

and

$$\lambda \lor \lambda' = (\max(\lambda_1, \lambda_1), \ldots, \max(\lambda_n, \lambda_n')).$$

Example 4.8.7. The Young diagram of the partition $(5, 3, 2, 1)$ has the following form:

```
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+---+---+---+---+---+
```
4.8. THE STRUCTURE OF THE LATTICE

We will need the following lattices to describe $S(M_m)$ for $m \in \mathbb{Q}_0$:

**Definition 4.8.8.** Denote by $Y^{m,n}$ that sublattice of Young’s lattice that contains exactly the partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{m'})$ where $m' \leq m$ and $\lambda_i \leq n$ for all $1 \leq i \leq m'$. Equivalently, we can define $Y^{m,n}$ as the lattice given by all Young diagrams with at most $m$ rows and at most $n$ columns.

**Example 4.8.9.** The Hasse diagram of the lattice $Y^{3,3}$ is

![Hasse diagram of Y^{3,3}](attachment:image.png)

**Remark 4.8.10.** Note that for $m, n \in \mathbb{N}$, we have $Y^{m,n} \cong Y^{n,m}$ and $Y^{1,n} \cong (\{0, 1, \ldots, n\}, <) \cong Y^{n,1}$.

Now, we can completely describe the distributive lattices $S(\mod A)$:
Theorem 4.8.11. Suppose $A = kQ/I$ with quiver $Q = (Q_0, Q_1)$ and admissible ideal $I$. If $A$ is of colocal type, then

$$S(\text{mod } A) \cong \prod_{m \in Q_0} Y^{k_m+1,l_m+1}.$$ 

Proof. By Lemma 4.8.5,

$$S(\text{mod } A) \cong \prod_{m \in Q_0} S(\mathcal{M}_m).$$

If only one path $\alpha_1 \alpha_2 \ldots \alpha_{k_m}$ ends in $m$, it is obvious from Lemma 4.8.2 (5) that we can order the modules in $\mathcal{M}_m$ the following way:

$$M(e_m) \subseteq M(\alpha_1^{-1}) \subseteq M(\alpha_2^{-1} \alpha_1^{-1}) \subseteq \cdots \subseteq M(\alpha_{k_m}^{-1} \alpha_{k_m-1}^{-1} \cdots \alpha_1^{-1})$$

Thus

$$S(\mathcal{M}_m) \cong (\{0, \ldots, k_m+1\}, <) = Y^{k_m+1,1}.$$ 

If there are two paths without relations $\alpha_1 \alpha_2 \ldots \alpha_{l_2}$ and $\beta_1 \beta_2 \ldots \beta_{l_2}$ are maximal under those that end in $m$, then by 4.8.2 (5) all modules in $\mathcal{M}_m$ are of the form $M(e_m) =: M(w_{0,0})$ or $M(w_{ij})$ with

$$w_{ij} = \alpha_i^{-1} \alpha_{i-1}^{-1} \cdots \alpha_1^{-1} \beta_1 \beta_2 \cdots \beta_j,$$

with $0 \leq i \leq k_m$, $0 \leq j_2 \leq l_m$ and at least one of them non-zero. Furthermore, $M(e_m) \subseteq M(w_{ij})$ and $M(w_{ij}) \subseteq M(i'j')$ if and only if $i \leq i'$ and $j \leq j'$.

For a submodule closed subcategory $C \in \mathcal{M}_m$, there is some $0 \leq c \leq k_m \in \mathbb{N}$ with

$$l_m \geq j_0 \geq j_1 \geq \cdots \geq j_c \geq 0$$

so that

$$\text{ind } C = \{M(w_{ij}) \in \text{ind } A \mid \text{there is some } 0 \leq h \leq c \text{ with } i \leq h, j \leq j_h\}.$$  

We define

$$\lambda_C := (j_0 + 1, j_1 + 1, \ldots, j_c + 1).$$

Then

$$f : S(\mathcal{M}_m) \to Y^{k_m+1,l_m+1}, C \to \lambda_C$$

is obviously injective and surjective. We need to prove that $f$ is a lattice homomorphism, that is, that it preserves joins and meets: Since $S(\mathcal{M}_m)$ is distributive, for any two categories $C_1, C_2 \in S(\mathcal{M}_m)$,

$$\text{ind}(C_1 \wedge C_2) = \text{ind } C_1 \cap \text{ind } C_2$$
and
\[ \text{ind}(C_1 \lor C_2) = \text{ind} C_1 \cup \text{ind} C_2 \]
by Proposition 4.1.3. From the definition of the joins and meets in \( Y^{k_{m+1},l_{m+1}} \), it is clear that \( f \) preserves them. \( \square \)
5 Conclusion and outlook

The first result of this thesis is that for all hereditary Artin algebras $A$, there exists a natural bijection between the elements of the Weyl group associated to $A$ and the cofinite submodule closed subcategories in $\text{mod} A$. So we have shown that a result for algebraically closed fields holds for arbitrary fields.

Next, we get a new characterization of algebras and abelian length categories of colocal type with three conditions that are simple to check, especially if we work over an algebraically closed field, where the module category of an Artin algebra is equivalent to that of a quiver with relations.

In this case, we can completely describe the lattice $S(\text{mod} A)$ for all $A$ of colocal type. It is the Cartesian product of sublattices of Young’s lattice.

While these connections to several topics show the importance of subobject closed subcategories, there are still a myriad of questions that can be asked about them. The most obvious question with regard to the results above is the following: how does a characterization of arbitrary abelian length categories with distributive lattice $S(A)$ looks like?

But we can also ask other questions: Is there a description for the finite submodule closed categories of a hereditary Artin algebra that is similar to the description of the cofinite ones? Is it possible to give a simple description for the lattices of submodule closed categories of all string algebras $A$, not only those with distributive lattice $S(\text{mod} A)$? Can we use submodule relations to characterize string algebras?
CHAPTER 5. CONCLUSION AND OUTLOOK
### Glossary

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<td>( M(w) )</td>
<td>the module defined by the string ( w )</td>
<td>27</td>
</tr>
<tr>
<td>( M(w, \phi) )</td>
<td>the module defined by the band ( w ) and the linear map ( \phi )</td>
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<tr>
<td>( Q_0 )</td>
<td>the set of vertices of the quiver ( Q )</td>
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<tr>
<td>( Q_1 )</td>
<td>the set of arrows of the quiver ( Q )</td>
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</tr>
<tr>
<td>( \rho(w) )</td>
<td>a series of pairs defined by the word ( w )</td>
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<tr>
<td>( (S1) ) - ( (S3) )</td>
<td>conditions on a triple of sequences of modules</td>
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<tr>
<td>( (S4) ) - ( (S5) )</td>
<td>conditions on a triple of sequences of modules</td>
<td>46</td>
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<tr>
<td>( (S'1) )</td>
<td>condition on a triple of sequences of modules</td>
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<tr>
<td>( S(A) )</td>
<td>the lattice of full additive subobject closed subcategories of ( A )</td>
<td>75</td>
</tr>
<tr>
<td>( S(\mathcal{M}) )</td>
<td>for a class of indecomposable objects ( \mathcal{M} ), this is a shorter notation for ( S(\text{add} \ \mathcal{M}) )</td>
<td>118</td>
</tr>
<tr>
<td>( \text{sub} \ \mathcal{X} )</td>
<td>the category that consists of all subobjects of direct sums of objects in the class ( \mathcal{X} )</td>
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</tr>
<tr>
<td>( \text{sub} \ X )</td>
<td>for an object ( X ), this is a shorter notation for ( \text{sub} { X } )</td>
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<tr>
<td>( \tau )</td>
<td>the Auslander-Reiten translation</td>
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<tr>
<td>( Y^{m,n} )</td>
<td>the lattice that contains the partitions whose Young diagrams have ( \leq m ) rows and ( \leq n ) columns</td>
<td>125</td>
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Bibliography


