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Robust Maximum Detection: Full Information Best Choice Problem under Multiple Priors

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Abstract

We consider a robust version of the *full information best choice* problem (Gilbert and Mosteller (1966)): there is ambiguity (represented by a set of priors) about the measure driving the observed process. We solve the problem under a very general class of multiple priors in the setting of Riedel (2009). As in the classical case, it is optimal to stop if the current observation is a running maximum that exceeds certain thresholds. We characterize the decreasing sequence of thresholds, as well as the (history dependent) minimizing measure. We introduce *locally constant ambiguity neighborhood* (LCAn) which has connections to coherent risk measures. Sensitivity analysis is performed using LCAn and *exponential neighborhood* from Riedel (2009).

1 Introduction

How is one to make the best choice among sequentially presented options when no recall is possible? Many scenarios in economics can be reduced

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to this question. In the well known *secretary problem* the employer is trying to pick the highest ranked among sequentially presented candidates for a position (Ferguson (1989)). In *job search* models, the unemployed agent is choosing among job offers trying to maximize life wealth (Lippman and McCall (1976)). In *house selling problems* the realtor is maximizing the profit in a series of take-it-or-leave-it bids (Porteus (2002)). These admittedly stylized problems are not trivial, and as such represent a useful first step towards more complex models and applications of the theory of optimal stopping (Ferguson (2006), Peskir and Shiryaev (2006)).

We consider the following best choice problem: a venture capitalist (the agent) is looking to invest and her budget allows her to invest in only one of the several sequentially presented start-up companies. She assumes the start-ups are similar and evaluates them by calculating a certain score. Due to the similarity of the start-ups and her familiarity with the matter, she treats the scores as realizations of independent and identically distributed random variables, the distribution of which is known to her. She believes that, given the high competition and failure rate among start-ups, only the company with the highest overall evaluation is the one that will be profitable. There is no recall: the decision not to invest in a start-up cannot be reversed. Hence, she is interested in maximizing the probability of choosing the start-up company with the highest valuation[1] .

This is one of the ways to formulate *full information best choice* (FIBC) problem, one of the best known optimal stopping problems in discrete time (Gilbert and Mosteller (1966), Bojdecki (1978), Samuels (1982), Ferguson (1989)). Formally, the agent is interested in detecting the maximum of a finite sequence of i.i.d. random variables (X_t), i.e. identifying the stopping time τ that maximizes the probability $P(X_\tau = \max(X_1, X_2, \ldots, X_T))$. The solution is elegant: it is optimal to stop only if the current observation is also the current maximum that exceeds some threshold value. The thresholds are decreasing, can be calculated in advance and depend only on the number of remaining observations (Gilbert and Mosteller (1966), Bojdecki (1978)).

The "full information" in the name of the FIBC problem refers to the fact that the agent knows the distribution of start-ups' scores. The reasons to question this strong assumption are numerous. There is no objective way to be certain that the distribution the agent uses is the correct one. Considering a set of measures "around the assumed probability" would make

[1]This formulation is based on a related problem in Bruss, Ferguson, et al. (2002).

the solution more robust. Even if one adopts the subjective probability approach, a single prior is not a reasonable assumption as shown by the Ellsberg paradox (Ellsberg (1961)). Indeed, even rational agents allow for Knightian uncertainty, or ambiguity, and behave in a way that is ambiguity-averse. A well established model of ambiguity aversion is *maxmin expected utility* theory, formulated by Gilboa and Schmeidler (1989). It assumes that the agent considers a set of priors and behaves pessimistically in a certain sense: when choosing the optimal action the agent first considers optimal actions over all of the priors and then chooses the one which has the lowest expected payoff[2].

In this paper we formulate and solve the FIBC problem under multiple priors in the setting of Riedel (2009). We show that the optimal stopping time is of the same form as in the classical case: it is completely characterized by a decreasing sequence of thresholds. We also characterize the measure under which the single prior problem is equivalent to the multiple priors one; it is highly history dependent.

The theory of optimal stopping under multiple priors in discrete time is developed in Riedel (2009). It shows that each adapted optimal stopping problem under multiple priors has a minimizing measure that reduces the problem to a single prior optimal stopping problem. One of the conditions that a set of priors has to fulfill in order to be used in an optimal stopping problem under multiple priors is *time consistency*. It can be viewed as a mechanism that ensures that backward induction procedure gives the same optimal behavior as ex-ante optimization along all possible paths, thus avoiding dynamic inconsistencies.

Among the few optimal stopping problems completely solved in the multiple priors setting is the *secretary problem* (Chudjakow and Riedel (2013)), a better known, yet simpler predecessor to the FIBC problem. After identifying the minimizing measure, the authors proceed to solve the single prior optimal stopping problem. As will be seen, due to the complexity of the FIBC problem, this approach is not viable in our case. Indeed, several advances/generalizations in the theory of optimal stopping under multiple priors were necessary for the problem to be solved in some generality: construction of the set of priors, identifying certain extremal measures and adaptation of non-adapted problems among others.

[2]Extensions and applications are numerous; for a recent review of ambiguity aversion theory see Gilboa and Marinacci (2016).

Time-consistency is explicitly taken into account in the construction of the set of priors that we propose. We start with a set of measures that contains a uniform distribution. It can be thought of as "marginal ambiguity" due to the fact that it describes uncertainty about uniform measure in each period. Using certain predictable processes, we paste these single period measures using a dynamic product of Radon-Nykodim derivatives to obtain a set of priors for the whole process. Random variables X_t are not independent nor identically distributed under each measure in the set of priors. However, as marginal ambiguity remains constant, they can be considered as having *identical and independent ambiguity* in each period.

Identifying the minimizing measure in optimal stopping problems under multiple priors is not always easy. We use ideas from first order stochastic dominance[3] to identify certain *extremal measures* within the set of priors. Extremal measures can facilitate solving the problem and characterizing the minimizing measure, as will be the case in the solution of the FIBC problem.

We note that, to the best of our knowledge, all of the so far solved problems of optimal stopping under multiple priors use sets of priors that can be considered special cases of our construction (in particular the *exponential neighborhood* in Riedel (2009) and the set of multiple priors in Chudjakow and Riedel (2013)). Indeed, our construction is quite general and allows for complex sets of priors that cannot be parametrized by a real parameter, or even countably many of them. Furthermore, the extremal measures we introduce play an important role in all of the already available solutions.

The probability of stopping at the maximum value in the FIBC problem depends on future observations. This means that the problem is not adapted, hence the theory of optimal stopping under multiple priors cannot be applied directly. We formulate an equivalent and adapted version of the problem by conditioning on currently available information and minimizing over all priors. This was already done in Chudjakow and Riedel (2013) and we show that the same approach works with our more general construction, while offering additional details that allow the procedure to be potentially used in other applications.

Even with a deep understanding of the classical FIBC problem it is not immediately clear how multiple priors affect the solution. If one thinks in terms of minimizing measures the "opposing effects of ambiguity" in FIBC appear: if the agent stops, the worst that can happen are high outcomes, and

[3]See Levy (2015).

if she continues, low outcomes would be the worst. This makes identifying the minimizing measure difficult. We initially avoid it altogether by finding suitable representations for values of stopping and continuing. Somewhat surprisingly, the representations are just monotone functions of a single variable. Naturally, it is optimal for the agent to stop once the value of stopping exceeds the value of continuing; this leads to the decreasing thresholds, as in the classical case.

Once we have the solution, we are able to identify the minimizing measure. It is history dependent: the agent's observations and actions up to a certain moment influence what she perceives to be the worst probability measure from that moment on. In particular, under the minimizing measures variables X_t are not independent. This has a technical consequence that the FIBC problem under the minimizing measure is not equivalent to a single prior version of the FIBC problem[4].

Our theoretical results are in accordance with experimental studies of the FIBC problem which is, due to its simple formulation, suitable for behavioral research. The oldest study on the subject (Kahan, Rapoport, and Jones (1967)) shows that agents do not recognize the underlying probability distribution well; this may be another reason to consider multiple priors in models of human behavior. Corbin, Olson, and Abbondanza (1975) shows that agents do not consider the observations as independent, even when they are informed that they actually are. More recently, Lee (2006) demonstrates that the observed behavior of participants in the study is best described by threshold rules. Overall, this is a positive indication that optimal stopping under multiple priors can be a viable model for real human behavior in optimal stopping problems.

We also establish a connection with the theory of coherent measures of risk, which at its core also has multiple priors (Artzner et al. (1999) , Föllmer and Schied (2011)). One could interpret the behavior of the FIBC agent operating in the multiple prior setting as follows: she considers her investment opportunity as a financial position, and chooses behavior that is optimal with respect to a certain risk measure. We introduce a locally constant ambiguity neighborhood (LCAn) that we use as the set of marginal ambiguity. This way we, effectively, describe marginal ambiguity by a risk measure wich turns out to have connections with the well known *Average Value at Risk*[5].

[4]See discussion after the formulation of the theorem 3.1 below.

[5]See ch. 4 in Föllmer and Schied (2011).

We finally investigate the way ambiguity affects the optimal behavior by considering two examples: exponential neighborhood introduced in Riedel (2009) and LCAn. Both examples can be considered robust neighborhoods around the initial measure. By deriving the explicit equations for the values of the optimal thresholds we are able to peform numerical calculations in both cases. Calculations offer two interesting conclusions. First, for "small sets of priors" the threshold values converge to the classical FIBC solution, establishing its robustness. Second, in both settings, the ambiguity averse agent stops earlier. This is different than the conclusions of the similar analysis on the secretary problem Chudjakow and Riedel (2013), where the agent could stop both earlier and later, depending on parameters that describe the set of priors.

We revisit the classical FIBC problem in section 2. In section 3 we first present the generalized way of constructing the set of priors for problems of optimal stopping and identify extremal measures within it. We then formulate and solve the FIBC problem under multiple priors. Examples with numerical calculations can be found in section 4, where we introduce the LCAn neighborhood and identify its extremal measures. Proofs and valuable additional material are in the appendix.

2 The Original FIBC Problem

For the sake of completeness we formulate the classical FIBC problem and briefly revisit its solution.

At each period $t \in \{1, 2, \dots, T\}$ the agent observes the process X_t which consists of random variables independently and, without loss of generality[6], uniformly distributed on the interval $[0, 1]$. Let $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, P_0)$ be a filtered probability space with $\Omega = [0, 1]^T$ being the product space, $\mathcal{F}_t$ being the σ-algebra generated by random variables X_1, X_2,...,X_t, and P_0 being the product of the (given) uniform marginal measures. We denote the set of all stopping times with $\mathcal{T}$ and the running maximum of the process with $M_t = \max(X_1, X_2, \dots, X_t)$.

The agent is interested in detecting the maximum: finding the optimal stopping time τ that maximizes the probability $P_0(X_\tau = M_T)$ of stopping at X_t with the highest valued realization. If we define the reward process

[6] Indeed, if the distribution $F = F_{X_t}$ was not uniform, a simple transformation would suffice: $X'_t = F^{-1}(X_t)$.

$Y_t = P(X_t = M_T)$ we can formulate the FIBC problem as the following optimal stopping problem.

Problem 1 (FIBC problem – non-adapted version)**.** Find $\tau^* \in \mathcal{T}$ such that:

$$E[Y_{\tau*}] = \max_{\tau\in\mathcal{T}} E[Y_\tau] = \max_{\tau\in\mathcal{T}} P_0(X_\tau = M_n).$$

The process Y_t is not adapted to the filtration $\mathcal{F}_t$, so the theory of optimal stopping cannot be applied. We define the adapted payoff process $\widehat{Y}_t = E[Y_t \mid \mathcal{F}_t]$ and the problem can be equivalently formulated as:

Problem 2 (FIBC problem – adapted version)**.** Find $\tau^* \in \mathcal{T}$ such that

$$E[\widehat{Y}_{\tau*}] = \max_{\tau\in\mathcal{T}} E[\widehat{Y}_\tau].$$

The problems are equivalent in the sense that the same stopping time solves both problems. Indeed, using the law of iterated expectation, one can easily prove that $E[\widehat{Y}_\tau] = E[Y_\tau]$ holds for any $\tau \in \mathcal{T}$[7].

In the classical FIBC problem it is optimal to stop if the current value X_t is a candidate (i.e. $X_t = M_t$) that exceeds a certain threshold a_t that decreases with time: the less time remains the lower valued candidate the agent is willing to accept. More precisely the optimal stopping time is given with $\tau* = \min\{t \mid X_t = M_t \geq a_{T-t}\}$ where the numbers a_n satisfy the equations:

$$a_0 = 0; \quad \sum_{j=1}^{n} \frac{1}{j a_n^j} = 1 + \sum_{j=1}^{n} \frac{1}{j},\ n \in \mathbb{N}.$$

For details see equation 1.2 in Samuels (1982) and original papers Gilbert and Mosteller (1966) and Bojdecki (1978).

3 FIBC Problem under Multiple Priors

In the classical formulation of the FIBC problem above (Problems 1 and 2) the agent was maximizing the probability of stopping at the highest value of the process. The probability measure used for calculating the expectation was

[7] It is worth pointing out that, although this equivalence is straightforward in the single prior case, the process of adaptation of payoff will be somewhat more complex in the multiple prior setting, as will be seen in the lemma C.2 below.

the one given prior P_0. In a multiple prior setting the agent performs all her calculations over a set of priors and then makes the most cautious/pessimistic decision.

Before the FIBC problem under multiple priors can be formulated, some technical preparation is needed when it comes to the set of priors. We do so in the first subsection, while the problem's formulation and solution are left for the second subsection.

3.1 The Set of Multiple Priors

We present a relatively general construction of the set of priors for problems of optimal stopping under multiple priors. The idea is to first introduce, for each period, the "marginal set of priors", and then to paste those in a time consistent manner to form the set of priors for the whole proces (X_t).

Let $(S, \mathcal{S}, v_0)$, $S \subset \mathbb{R}$, be a given probability space. Without loss of generality we assume that v_0 is strictly positive; this is clearly the case when v_0 is uniform. We furthermore assume that v_0 has a positive and bounded density. We define a set $\Omega = S^T$, for $T \in \mathbb{N}$, a sigma field $\mathcal{F} = \otimes_{t=0}^{T} \mathcal{S}$ (generated by projections $X_t : \Omega \to S$) and a probability measure $P_0 = \otimes_{t=1}^{T} v_0$ (under which the projections X_t are i.i.d[8]).

Let

$$\mathcal{V}^A = \{v_\alpha : \mathcal{F} \to (0,1) \,|\, \alpha \in A\}$$

be a set of probability measures on S indexed by some fixed set A. Since sets of multiple priors are used to model ambiguity one can, analogously, think of the set $\mathcal{V}^A$ as the set of *marginal ambiguity*. We note that marginal ambiguity will remain constant, i.e. the set $\mathcal{V}^A$ is fixed and does not change with the flow of time[9].

The set of priors on Ω is obtained by pasting together measures from $\mathcal{V}^A$. In order to do the pasting in a time-consistent manner we define a set $\mathcal{A}$ of

[8]Although we use i.i.d. random variables the arguments that follow can readily be adjusted to the case when random variables are not identically distributed (but are still independent).

[9]Again, careful reading of the arguments that follow shows that one does not lose on generality by fixing an identical set of beliefs at every step.

all predictable processes with values in A:

$$\mathcal{A} = \left\{ a = (a_t)_{t \le T} \;\middle|\; \begin{array}{l} a_{t+1} = a_{t+1}(x_1, x_2, \dots, x_t) \in A, \quad x_s \in S, s \le t < T; \\ \dfrac{dv_{a_1}}{dv_0} \cdot \dfrac{dv_{a_2}}{dv_0} \cdot \dots \cdot \dfrac{dv_{a_t}}{dv_0} \in L^0(\Omega, \mathcal{F}_t, P_0|\mathcal{F}_t), \quad t \le T \end{array} \right\},$$

where L^0 is the set of all measurable functions. As can be seen, $\mathcal{A}$ contains all processes that are predictable in a sense that their value at time $t+1$ depends on past realizations of random variables. The second requirement in the definition is technical, but revealing: requiring a "dynamic product" of Radon-Nykodim derivatives to be measurable with respect to $\mathcal{F}_t$ allows us to assign a measure to each process in $\mathcal{A}$; we do so below.

For each $a \in \mathcal{A}$ we can *define* a probability measure P^a on $(\Omega, (\mathcal{F}_t))$ by defining its density process:

$$\left.\frac{dP^a}{dP_0}\right|_{\mathcal{F}_t} = \prod_{s=1}^{t} \frac{dv_{a_s}}{dv_0}, \tag{1}$$

and, finally, we set

$$\mathcal{P} = \mathcal{P}(\mathcal{V}^A) = \{P^a \,|\, a \in \mathcal{A}\}.$$

All measures in the set $\mathcal{P}$ are equivalent; this is due to the definition of the set $\mathcal{V}^A$ within which all the measures are equivalent. We note that random variables X_t are not independent under every measure $P \in \mathcal{P}$. In fact, the only measures under which they are independent are those that correspond to direct products, i.e. processes $a \in \mathcal{A}$ such that for each t the function a_t is constant.

Although we use the set $\mathcal{P}$ as the set of priors to formulate and solve the FIBC problem under multiple priors, the generality of its construction allows for other applications: under mild assumptions on the set $\mathcal{V}^A$ the theory of optimal stopping under multiple priors from Riedel (2009) can be applied. We prove this result in Appendix A, where we also offer some further details on optimal stopping under multiple priors that are relevant for our solution.

Extremal Measures In order to be able to solve and reduce a multiple prior problem to a single prior one, we define "extremal measures" within the set $\mathcal{P}$. The ideas we use are those of the theory of (first order) stochastic dominance (Levy (2015)).

Let us denote by $\underline{v} \in \mathcal{V}^A$ the measure (if it exists) such that:

$$\underline{v}(X_1 \leq x) \geq v_a(X_1 \leq x), \quad \text{for any } x \in \mathbb{R} \text{ and any } v_a \in \mathcal{V}^A. \tag{2}$$

As can be seen the minimizing measure $\underline{v} \in \mathcal{V}^A$ is the one that puts the most weight on the lowest valued outcomes.

This allows us to single out the measure $\underline{P} = \otimes_{t=0}^{n} \underline{v} \in \mathcal{P}$ (under which the variables X_t are independent!). Measures $\overline{v} \in \mathcal{V}^A$ and $\overline{P} = \otimes_{t=0}^{n} \overline{v}$ are defined analogously. Characterizations of extremal measures and a useful lemma are offered in Appendix B.

3.2 FIBC Problem under Multiple Priors

Let $\mathcal{P}$ be a set of multiple priors obtained by pasting together one-period multiple priors sets $\mathcal{V}^A$ indexed by some set A. We assume $\mathcal{V}^A$ satisfies all the conditions of lemma A.1, with v_0 being the uniform distribution. We also assume that the set $\mathcal{V}^A$ contains both a measure $\overline{v}$ that satisfies the lower extremal property and a measure $\overline{v}$ that satisfies the upper extremal property.

Let X, Y, $\mathcal{F}$ be as in section 2. Note that while X_t are independent under the reference measure P_0, they are not independent under every $P \in \mathcal{P}$. The agent is solving the problem:

Problem 3 (FIBC problem under multiple priors – non-adapted version)**.** Find $\tau^* \in \mathcal{T}$ such that

$$\min_{P \in \mathcal{P}} E^P[Y_{\tau^*}] = \max_{\tau \in \mathcal{T}} \min_{P \in \mathcal{P}} E^P[Y_\tau].$$

Since the duality equality

$$\max_{\tau \in \mathcal{T}} \min_{P \in \mathcal{P}} E^P[Y_\tau] = \min_{P \in \mathcal{P}} \max_{\tau \in \mathcal{T}} E^P[Y_\tau] \tag{3}$$

holds[10], the following interpretation is plausible: the agent maximizes the probability of stopping at the maximum of the given process under the "worst possible" measure in the set $\mathcal{P}$.

Similarly as in Problem 1, this problem needs to be reduced to an adapted problem; this will allow us to solve it using the theory of optimal stopping

[10]See theorem 2 in Riedel (2009).

under multiple priors. For that purpose we define the adapted payoff process under multiple priors:

$$Z_t = \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[Y_t | \mathcal{F}_t].$$

Problem 4 (FIBC problem under multiple priors – adapted version)**.** Find $\tau^* \in \mathcal{T}$ such that

$$\inf_{P \in \mathcal{P}} E^P[Z_{\tau*}] = \sup_{\tau \in \mathcal{T}} \inf_{P \in \mathcal{P}} E^P[Z_\tau].$$

Problems 3 and 4 are equivalent – the same stopping time solves both problems. The equivalence in the multiple priors setting is less clear than it was in the single prior setting; we prove it in Appendix C. The proof does not depend on the definition of the payoff process, hence it holds for any non-adapted process under multiple priors.

The following theorem completely characterizes the solution of the FIBC problem under multiple priors.

Theorem 3.1. *1. There is a decreasing sequence of thresholds $(b_t)_{t=1,\ldots T}$ such that the optimal stopping time τ^* that solves the BCIF problem under multiple priors is:*

$$\tau^* = \min\{t \mid X_t = \max(X_1, X_2, \ldots, X_t) > b_t\}. \tag{4}$$

2. Thresholds b_t are the unique solutions of equations $w_t(x) = \overline{r}_t(x)$, $t < T$, where functions $\overline{r}_t$ and w_t are defined recursively:

$$\overline{r}_T(m) \equiv 1, \quad \overline{r}_t(m) = E^{\overline{P}}\left[\prod_{s>t} \mathbb{1}_{X_s \le m}\right]; \quad w_T(m) \equiv 0,$$

$$w_t(m) = \operatorname*{ess\,inf}_{v \in \mathcal{V}^A} \Big(\int_m^1 \max(\overline{r}_{t+1}(x), w_{t+1}(x))\, dv(x) + w_{t+1}(m) \int_0^m dv(x) \Big).$$

Specially, $b_T = 0$.

3. The minimizing measure $P^* = P^{a^*}$ is given by the predictable process

$$a_t^*(x_1, \ldots, x_t) = \begin{cases} a_t^c(x_1, \ldots, x_t), & t < \tau^* \\ \overline{\alpha}, & t \ge \tau^* \end{cases}, \tag{5}$$

where $v_{\overline{\alpha}} = \overline{v}$ *and*

$$a_t^c(x_1,\ldots,x_t) = \arg\min_{a\in A}\Big(\int_{\max(x_1,\ldots,x_t)}^1 \max(\overline{r}_{t+1}(x), w_{t+1}(x))\,dv_a(x) + w_{t+1}(\max(x_1,\ldots,x_t))\int_0^{\max(x_1,\ldots,x_t)} dv_a(x)\Big). \tag{6}$$

The duality equation (3) holds for the process Z_t, too. A reasonable attempt at solving Problem 4 would be to identify the minimizing measure and solve the classical, single prior optimal stopping problem under the minimizing measure. One could even hope that one of the measures $\underline{P}$ or $\overline{P}$ would turn out to be the minimizing measure, thus allowing the problem to be reduced to the classical FIBC problems 1 and 2. The theorem shows that the minimizing measure is significantly more complicated than that. This is due to the fact that the multiple priors setting creates opposing effects about what is "pessimistic": when the agent stops the worst measures are those that put the most weight on high outcomes, and when she continues the worst measures are those that put the most weight on the lowest outcomes, while accounting for future behavior.

As can be seen, the minimizing measure P^* is highly history dependent and even depends on the act of stopping. This implies that random variables X_t are not independent under P^*. Hence, an agent operating in a multiple prior setting views the FIBC problem in a way that is substantially different from that of an agent making decisions under the single prior. This is true on a technical level, too: the reduction to the classical FIBC problem via the probability integral transform (as indicated on pp.51-52 in Gilbert and Mosteller (1966)) is not possible.

The proof ultimately relies on several careful backward inductions and can be considered a multiple priors version of the proof offered in Samuels (1982). The details are available in the appendix. Although tedious, the proof offers significant insight into the FIBC problem.

The proof reveals that, if at time t the agent observes value x_t that is a running maximum, then the expected value (under multiple priors) of continuing is $w_t(x_t)$ while the expected value of stopping is $\overline{r}(x_t)$. Hence, the stopping rule prescribed by τ^* merely says that the agent stops if payoff of stopping exceeds the payoff of continuing; this is in accordance with the theory of optimal stopping (under multiple priors). Furthermore, the multiple prior Snell envelope of the adapted version of the FIBC problem under multiple priors can be expressed in terms of functions $\overline{r}_t$ and w_t:

$$U_t = \max(\overline{r}_t(X_t)\mathbb{1}_{X_t=M_t}, w_t(M_t)).$$

The (classical) FIBC problem is "end-invariant" in the following sense: "optimum decision numbers depend only upon the number of remaining draws", as noted in Gilbert and Mosteller (1966). The proof reveals that, once the set $\mathcal{P}$ is fixed, the same is true for the FIBC problem under multiple priors. Indeed, its solution was derived by backward induction and was shown to depend only on the values of the current observation and the running maximum. Naturally, the cutoff points b_t depend on the on the set of priors $\mathcal{P}$; we explore this dependency numerically in the next section.

4 Examples

What is the effect of introducing multiple priors to the FIBC problem? How does the optimal stopping time change once ambiguity is introduced? We try to give some answers to these and related questions in this section.

The sequence of cutoff points that define the optimal stopping time in the classical version of the problem has been well studied (already in Gilbert and Mosteller (1966)) and their asymptotic behavior is well understood (Samuels (1982)). However, due to the complexity of the minimizing measure and recursive equations in 3.1, that kind of analysis is not trivial in our setting. We focus our attention on the simple case when $T = 3$; it will be seen below that even this case is computationally cumbersome and leads to highly nonlinear equations. Given the comments about the end invariance of the FIBC problem in the previous section, what follows is effectively an analysis of the final three periods of any FIBC problem with the horizon $T \geq 3$; the notations we use reflect this fact.

4.1 Classical Case

For the sake of completeness we briefly review the numerical values of the optimal stopping time in the classical FIBC problem. In our context, it corresponds to the case when $\mathcal{V}^{\mathcal{A}}$ is a singleton with its only element being the uniform measure. Omitting the straightforward calculations we present the interesting parts of the result.

Functions $\overline{r}_t$ and w_t take the following form:

$$\overline{r}_{T-1}(x) = x, \qquad \overline{r}_{T-1}(x_1, \ldots, x_{T-2}) = x^2$$
$$w_{T-1}(m) = 1 - m, \quad w_{T-2}(m) = \int_m^1 \max(x, 1-x)\, dx + m(1-m).$$

The cutoff points that define the stopping time are

$$b_T = 0, \quad b_{T-1} = \frac{1}{2}, \quad b_{T-2} = \frac{2+\sqrt{24}}{10} \approx 0.6899.$$

Note that b_{T-1} and b_{T-2} are the solutions of the equations $m = w_{T-1}(m)$ and $m^2 = w_{T-2}(m)$ respectively.

4.2 Exponential Neighborhood

The *exponential neighborhood* is the set of priors $\mathcal{P}^{\alpha,\beta}$ introduced in Riedel (2009). It has important connections to Girsanov theory and arises naturally in statistics where it is referred to as *exponential family*. It has been used to model uncertainty in optimal stopping problems related to finance (option pricing), as well as ambiguous versions of the *house pricing problem* and the *parking problem*[11].

In the context of this paper it can be introduced by setting $A = [\alpha, \beta]$ and:

$$\mathcal{V}_{EXP}^{A} = \left\{ v_a \,\middle|\, \frac{dv_a}{dv_0}(x) = \frac{e^{ax}}{\int_0^1 e^{at}\, dt},\, a \in A \right\},$$

where v_0 is the uniform measure on the interval $[0,1]$. It is known[12] that $\underline{v} = v_\alpha$ and $\overline{v} = v_\beta$. The exponential neighborhood is simply: $\mathcal{P}^{\alpha,\beta} = \mathcal{P}(\mathcal{V}_{EXP}^{A})$.

We have already seen that $\overline{r}_T(x) = 1$ and $w_T(x) = 0$ for any $x \in [0,1]$. Direct calculations yield:

$$\overline{r}_{T-1}(x) = \frac{e^{\beta x} - 1}{e^{\beta} - 1}, \qquad w_{T-1}(x) = \frac{e^{\beta} - e^{\beta x}}{e^{\beta} - 1}.$$

Equating $\overline{r}_{T-1}(x) = w_{T-1}(x)$ we obtain

$$b_{T-1} = \frac{1}{\beta} \ln \frac{e^{\beta} + 1}{2}.$$

[11] See section 4 in Riedel (2009).

[12] For details see section 4 in Riedel (2009).

α	β	b_{T-1}	b_{T-2}	α	β	b_{T-1}	b_{T-2}
0	0	0.5000	0.6899	-2	2	0.7169	0.8084
-0.01	0.01	0.5013	0.6905	-5	5	0.8627	0.9048
-0.1	0.1	0.5125	0.6958	-10	10	0.9307	0.9518
-0.25	0.25	0.5312	0.7047	-1	2	0.7169	0.8144
-0.5	0.5	0.5619	0.7200	-1	3	0.7851	0.8582
-1	1	0.6201	0.7512	-2	1	0.6201	0.7406
-1.5	1.5	0.6722	0.7811	-3	1	0.6201	0.7335

Table 1: Exponential neighborhood – Values of the cutoff points b_{T-1} and b_{T-2} for different values of α and β.

The expression for w_{T-2} is more cumbersome:

$$w_{T-2}(m) = \frac{1}{e^{\beta}-1} \min_{a\in[\alpha,\beta]} \left(\frac{1}{\int_0^1 e^{at}\,dt} \left(a \int_m^1 e^{ax} \max(e^{\beta x}-1, e^{\beta}-e^{\beta x})\,dx + (e^{\beta}-e^{\beta m})(e^{am}-1) \right) \right),$$

while $\overline{r}_{T-2} = (\overline{r}_{T-1})^2$.

As can be seen, explicitly calculating the cutoff point b_{T-2} (i.e. solving the equation $w_{T-2}(m) = r_{T-2}(m)$) is not computationally easy and to obtain the approximations of its value one can resort to mathematical software[13]. Table 1 provides approximations of values of cutoff points b_{T-1} and b_{T-2} for different values of parameters α and β. If the difference $\beta-\alpha$ is interpreted as the "amount of ambiguity" one can notice that the increase in ambiguity causes later stopping. The last four rows seem to imply greater sensitivity of cutoff point values to the change in β, than in α. This is somewhat expected given the shape of the exponential function.

[13] All the graphs and data for the tables were made using Wolfram Matematica, Research (2015).

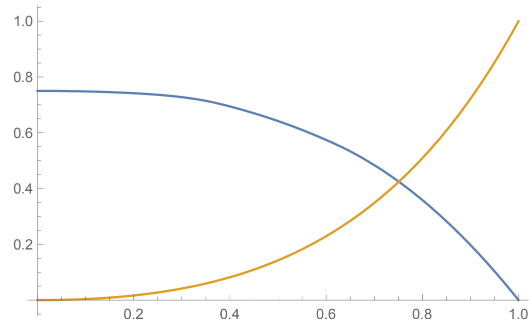

Figure 1: Exponential Neighborhood - Graphs of functions $w_{t-2}(x)$ (decreasing) and $\overline{r}_{T-2}(x)$ (increasing) for $\alpha = -1$ and $\beta = 1$. The point of intersection is b_{T-2}.

4.3 Local Constant Ambiguity Neighborhood

The *locally constant ambiguity neighborhood* (LCAn) is the set whose "marginal ambiguity" is:

$$\mathcal{V}^A_{CLA} = \left\{ v_a \,\middle|\, \frac{1}{\lambda} \leq \frac{dv_a}{dv_0} \leq \lambda \right\}.$$

The constant $\lambda \geq 1$ describes the "amount of ambiguity" – greater values of λ imply greater ambiguity about the "correct measure" that drives the payoff process. Case $\lambda = 1$ corresponds to the case where there is no ambiguity and the set $\mathcal{V}^A_{CLA}$ reduces to a singleton containing v_0. The LCAn is simply: $\mathcal{Q}^\lambda_{CLA} = \mathcal{P}(\mathcal{V}^A_{CLA})$. As can be seen the set $\mathcal{V}^A$ cannot be parametrized by a real parameter, nor even countably many real parameters. In that sense, it differs substantially from the exponential neighborhood, or any other analogously created neighborhood that depends on a fixed family of distributions.

One can interpret the marginal ambiguity of LCAn as follows: the agent is certain about which events are possible/impossible (described by measure v_0), but she allows for the possibility that for any sufficiently "small event" it's probability is up to λ-times overestimated or underestimated by v_0.

LCAn bears some resemblance to the well known ε-contamination from Huber (1981), which was already used in the context of ambiguty in the well known paper Maccheroni, Marinacci, and Rustichini (2006). In our context, ε-contamination could be described as the set of measures, the range of densities of which lies within the interval $[1-\varepsilon, 1+\varepsilon]$. Arguably, this is a less natural model of ambiguity than LCAn when it comes to describing belief by a set of priors. Beyond the obvious fact that ε cannot be greater than one (which discounts for the possibility of any event being more than twice underestimated) there seem to be indications that humans innately think logarithmically, rather than linearly (Dehaene (2003)). In particular, to put it in more plastic terms, this may mean that it is more natural to think of [1/2,2] as a neighborhood around the point 1 than [0.5,1.5]; this corresponds to the way in which the ambiguity around "small events" is modeled by LCAn.

We note that the set V^{λ}_{CLA} is related to certain sets that appear in the theory of risk measures. In particular, the well known risk measure known as *average value at risk* can be characterized by a similarly defined set (chapter 4 in Föllmer and Schied (2011)). It is well known that there are mathematical connections between risk measures and the theory of multiple priors. It is also well known that ambiguity (in the sense of multiple priors) could be viewed as a way to describe model uncertainty. The same is true for risk measures and model uncertainty in finance.

Due to the similarities between the set that characterizes AVaR and LCAn, one could argue that LCAn introduces robustness to the dynamic of the FIBC problem in a way that is closer to robustness in finance. Indeed, at each moment t the agent evaluates the values of all her possible present and future actions, then chooses the least risky one with respect to the risk measure induced by the set LCAn. Arguably, this makes the LCAn an attractive option for future (dynamic) models in economics and finance where uncertainty needs to be introduced.

With the set $\mathcal{Q}^{\lambda}_{CLA}$ defined, we can turn to the question of the existence of extremal measures within it. We answer this question in our context, i.e. with the reference measure v_0 being the uniform measure on the interval $[0,1]$, and we do so by focusing on the monotone function characterization of extremal measures (see equation (9)). It would seem plausible, that measures $\underline{v}$ and $\overline{v}$ are the ones that put the most weight on the right and, respectively, left end of the interval $[0,1]$; we prove this result in lemma 4.1 below. We actually prove that the densities of extremal measures are given with:

$$\frac{d\underline{v}}{dv_0} = \frac{1}{\lambda}\mathbb{1}_{\left[0,\frac{\lambda}{\lambda+1}\right]} + \lambda\mathbb{1}_{\left[\frac{\lambda}{\lambda+1},1\right]} =: \underline{\varphi}, \qquad \frac{d\overline{v}}{dv_0} = \frac{1}{\lambda}\mathbb{1}_{\left[0,\frac{1}{\lambda+1}\right]} + \lambda\mathbb{1}_{\left[\frac{1}{\lambda+1},1\right]} =: \overline{\varphi}.$$

By similar logic as above, the worst measure for U-shaped payoffs should be the one that puts the most weight on an interval that contains the minimum of the payoff function; this result related to extremal measures is included in the lemma. The formulation of the lemma requires us to define the following set of densities of measures in $\mathcal{V}^{\lambda}_{CLA}$:

$$\begin{aligned} D^{\lambda}_{CLA} &= \left\{ \frac{dv}{dv_0} \middle| v \in \mathcal{V}^{\lambda}_{CLA} \right\} \\ &= \left\{ \varphi : [0,1] \to \mathbb{R} \middle| \int_0^1 \varphi(x)\,dx = 1, \tfrac{1}{\lambda} \leq \varphi(x) \leq \lambda \right\}. \end{aligned}$$

Lemma 4.1. *1. For every increasing, bounded measurable function $g : [0,1] \to \mathbb{R}$ and every $\varphi \in D^{\lambda}_{CLA}$ the following inequality holds:*

$$\int_0^1 g(x)\varphi(x)\,dx \geq \int_0^1 g(x)\underline{\varphi}(x)\,dx.$$

2. For every function $h : [0,1] \to \mathbb{R}$ which is decreasing on $[0,k]$ and increasing on $[k,1]$ for some $k \in [0,1]$ and for every function $\varphi \in D^{\lambda}_{CLA}$ there exists a function $\psi \in D^{\lambda}_{U}$ such that:

$$\int_0^1 h(x)\varphi(x)\,dx \geq \int_0^1 h(x)\psi(x)\,dx,$$

where

$$D^{\lambda}_{U} = \left\{ \lambda \mathbb{1}_{\left[c, c+\frac{1}{\lambda+1}\right]} + \frac{1}{\lambda}\mathbb{1}_{[0,c] \cup \left[c+\frac{1}{\lambda+1}, 1\right]} \middle| c \in \left[0, \frac{\lambda}{\lambda+1}\right] \right\}. \tag{7}$$

It can be seen that the set $D^{\lambda}_{U} \subset D^{\lambda}_{CLA}$ is the set of densities that put the most weight on the interval $\left[c, c+\frac{1}{\lambda+1}\right]$, which is in accordance with the considerations preceding the formulation of the lemma.

Analogous results can be formulated about decreasing functions and the inverted-U-shaped functions.

Definition of the function w_{T-2} and the monotonicity of functions w_{T-1} and r_{T-1} imply that the function w_{T-2} is U-shaped. Lemma 4.1 allows us to narrow down the search for the minimizing measure within the set D^{λ}_{U}, which in turn allows for mathematical software to be used to identify the minimizing measure, plot the graph of the function w_{T-2} (see figure 2) and calculate the value of the cutoff point b_{T-2}. Similarly as before, we provide a table with the approximate values of cutoff points b_{T-1} and b_{T-2} for different values of λ.

λ	b_{T-1}	b_{T-2}	λ	b_{T-1}	b_{T-2}
1	0.5	0.6899	2	0.7500	0.8182
1.01	0.5050	0.6916	3	0.8333	0.8754
1.1	0.5455	0.7073	4	0.8750	0.9057
1.25	0.6000	0.7318	8	0.9375	0.9524
1.5	0.6667	0.7671			

Table 2: LCAn – Values of the cutoff points b_{T-1} and b_{T-2} for different values of λ.

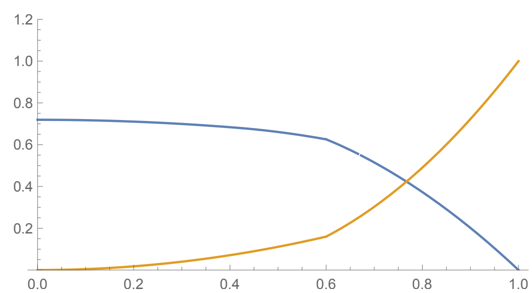

Figure 2: LCAn - Graphs of functions $w_{t-2}(x)$ (decreasing) and $\overline{r}_{T-2}(x)$ (increasing) for $\lambda = 3/2$. The point of interesection is b_{T-2}.

It is worth noting that in both examples we presented the agent stops later than in the classical case. It is not hard to see that this is true for the period $T-1$ for any set of multiple priors, but is less obvious for periods $t < T-1$, hence it remains a conjecture. This is different from the results of Chudjakow and Riedel (2013) where it was found the agent could stop both earlier and later than the agent not facing ambiguity, depending on the shape of the set of multiple priors.

5 Conclusion

We formulated and solved the multiple priors version of the classical *full information best choice* problem under rather general conditions. We showed that the solution can be fully characterized via a set of decreasing thresholds, just as in the classical case. Instead of identifying the minimizing measure and then solving the single prior problem, we solve the problem with a more direct approach using the theory of optimal stopping under multiple priors.

More generally, we have demonstrated that the theory of optimal stopping under multiple priors can accommodate complex problems, hopefully paving the way for even harder problems to come. In this context, of interest is our result about adapting any non-adapted optimal stopping problem under multiple priors.

Our results fit into a wider setting of dynamic problems under multiple priors: we described a construction of a set of priors for the whole process using only a single-period set of priors. The construction ensures that the resulting set of priors is time consistent, thus allowing for "variables with independent and identical ambiguity" to be used practically and in some generality to model uncertainty in multi-period models, even beyond the theory of optimal stopping.

Although the theory of maxmin expected utility is a mature one, non-trivial examples of the sets of multiple priors in dynamic settings are rare. We introduced one such example using ideas from the theory of risk measures: *locally constant ambiguity neighborhood* is a set of priors in which ambiguity of probability about th 'small' events remains constant. The set itself has promising interpretations in terms of model uncertainty and invites future research in the context of risk measures. It also opens possiblities in the other direction – to explicitly use sets of priors related to established risk measures in the context of dynamic economic problems under ambiguity.

As it is becoming increasingly evident that economic models with a single probability measure are not capturing the reality in a satisfactory way, it becomes necessary to investigate robust models that manage to take into account Knightian uncertainty of economic problems; we hope this paper convincingly presents one such model.

Appendix A Applicability of the Theory of Optimal Stopping under Multiple Priors

For the theory of optimal stopping to be applied to processes with bounded payoffs the set of priors $\mathcal{P}$ has to satisfy three assumptions. It should be L^1 weakly closed and all the measures within the set $\mathcal{P}$ should be equivalent. The set $\mathcal{P}$ should also be time consistent: for any two measures, the measure that allows the agent to "switch" between them at some (possibly random) time must also be in the set $\mathcal{P}$; see assumptions $A2-4$ in Riedel (2009). The following lemma shows that the set $\mathcal{P}$ satisfies those assumptions once we impose mild conditions on the set $\mathcal{V}^A$.

Lemma A.1. *Assume the set $\mathcal{V}^A$ satisfies:*

1. $v_0 \in \mathcal{V}^A$
2. *All the densities* $\dfrac{dv_a}{dv_0}$, $a \in A$, *are strictly positive and bounded*
3. *The set $\mathcal{V}^A$ is weakly closed in $L^1(S, \mathcal{S}, v_0)$*

Then the set of measures $\mathcal{P}(\mathcal{V}^A)$ satisfies assumptions $A2$, $A3$ and $A4$ in Riedel (2009).

Proof. The assumption A2 is satisfied because all the densities in $\mathcal{V}^A$ are strictly positive and bounded.

For the weak compactness it is sufficient to show that the set $\mathcal{P}$ is closed and bounded by a uniformly integrable random variable. Since all the densities are bounded, the latter is obvious. Closedness is a consequence of the third assumption in the formulation of the lemma: weakly closed sets are also strongly closed, thus the closedness is inherited from weak closedness in each period by pasting. To see this, it suffices to recall that a sequence of positive functions convergent in L_1 has a subsequence that converges pointwise (almost everywhere). With this the closedness can be proven using the classical argument (that a limit of a sequence of elements of the set also belongs to the set) by exploiting the previous remarks.

It remains to prove the time consistency. Due to the predictability of each of the functions a_k this is straightforward: Let P^a and P^b be two measures with densities $\frac{dP^a}{dP_0}\Big|_{\mathcal{F}_t} = \prod_{s=1}^t \frac{dv_{a_s}}{dv_0}$ and $\frac{dP^b}{dP_0}\Big|_{\mathcal{F}_t} = \prod_{s=1}^t \frac{dv_{b_s}}{dv_0}$ respectably and let τ be a stopping time. Define $c_t = a_t$ when $t \leq \tau$ and $c_t = b_t$ when $t > \tau$. The resulting measure from the property A4 coincides with the measure P^c with density $\frac{dP^c}{dP_0}\Big|_{\mathcal{F}_t} = \prod_{s=1}^t \frac{dv_{c_s}}{dv_0}$ which obviously belongs to $\mathcal{P}$; this is exactly what was supposed to be proven[14]. □

[14]This lemma could alternatively be proven by showing that the set $\mathcal{P}$ coincides with the time-consistent hull "around" the set of all direct product measures from $\mathcal{V}^A$; see pp. 868 in Riedel (2009), or Riedel (2004).

The theory of optimal stopping under multiple priors guarantees the existence of the stopping time $\tau^* \in \mathcal{T}$ such that:

$$\max_{\tau \in \mathcal{T}} \min_{P \in \mathcal{P}} E^P[\mathcal{E}_\tau] = \min_{P \in \mathcal{P}} E^P[\mathcal{E}_{\tau^*}],$$

where $\mathcal{E}_t$ is a bounded payoff process adapted to the filtration $\mathcal{F}_t$. The minimal optimal stopping time τ^* is given with

$$\tau^* = \min \{t \geq 0 \mid U_t = \mathcal{E}_t\}, \quad U_T = \mathcal{E}_T,\, U_t = \max \left(\mathcal{E}_t, \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[U_{t+1} \mid \mathcal{F}_t] \right),$$

where U is the recurcively defined *multiple priors value process*. Furthermore, the theory guarantees the existence of the measure $Q^* \in \mathcal{P}$ such that the value process under multiple priors of the optimal stopping problem under multiple priors coincides with the value process of the (single-prior) optimal stopping problem of the process $\mathcal{E}_t$ under the measure Q^*; this allows the possibility of reducing the multiple priors problems to the classical ones. For further details see Theorems 1 and 2 in Riedel (2009).

Appendix B Details on Extremal measures

It is easy to prove that the inequality:

$$\underline{P}(X_{t+1} \leq x|\mathcal{F}_t) \geq P(X_{t+1} \leq x|\mathcal{F}_t)) \tag{8}$$

holds for any $t > 0$, $x \in \mathbb{R}$ and $P \in \mathcal{P}$, and a characterization in terms of monotone functions is straightforward along the lines of the classical proofs of theorems on first order stochastic dominance (Levy (2015)). Specifically, the measure $\underline{P} \in \mathcal{P}$ satisfies the inequality

$$E^{\underline{P}}\left[h(X_{t+1}) \mid \mathcal{F}_t\right] \leq E^P\left[h(X_{t+1}) \mid \mathcal{F}_t\right] \tag{9}$$

for each $t > 0$, each $P \in \mathcal{P}$ and each bounded, increasing real function h.

We note an immediate consequence of the monotone characterization of the extremal measures (9):

Lemma B.1. *For any function $g_t : S^t \to \mathbb{R}$ that is bounded, measurable and increasing in its last argument the following equality holds:*

$$\operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[g(X_1, ..., X_t, X_{t+1}) \mid \mathcal{F}_t] = E^{\underline{P}}[g(X_1, ..., X_t, X_{t+1}) \mid \mathcal{F}_t]$$

Proof. Since the filtration $\mathcal{F}$ is generated by $X_1, \ldots, X_t$ it suffices to show that, for an arbtirary history $X_1 = x_1$, $X_2 = x_2$, ..., $X_t = x_t$, the following inequality holds:

$$E^P[g(X_1, ..., X_{t+1}) \mid X_1 = x_1, \ldots, X_t = x_t] \\ \geq E^{\underline{P}}[g(X_1, ..., X_{t+1}) \mid X_1 = x_1, \ldots, X_t = x_t].$$

This, however, is true because of the monotone characterization of the extremal measures (9). Indeed, once we fixed the values of random variables X_1, X_2, ..., X_t, the function g_t can be interpreted as a function of a single variable X_{t+1} and the inequality follows directly from the inequality (9). □

An analogous result holds for the decreasing functions.

Appendix C Equivalence of Problems 3 and 4

A version of this result appears in Chudjakow and Riedel (2013); the analysis we offer contains additional details and, due the generality of the construction of the set of priors $\mathcal{P}$, applies to a broader class of problems.

We begin by proving an auxiliary result based on Lemma 8 in Riedel (2009).

Lemma C.1 (Iterated version of Lemma 8 in Riedel 2009 and corollaries)**.** *Let $P_1, P_2, \ldots, P_n$ be measures in $\mathcal{P}$, $\tau \in \mathcal{T}$ a stopping time, and $A_1, A_2, \ldots, A_n$ sets in $\mathcal{F}_\tau$ that form a partition of Ω.*

1. *There exists a measure $P \in \mathcal{P}$ such that for any r.v. Z:*

$$E^P[Z \mid \mathcal{F}_\tau] = \sum_{k=1}^{n} E^{P_k}[Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau]. \tag{10}$$

2. *For any r.v. Z and any $k \in \{1, \ldots, n\}$ the equality $E^P[Z\mathbb{1}_{A_k}] = E^{P_k}[Z\mathbb{1}_{A_k}]$ holds.*

3. *The equality $E^P[Z] = \sum_{k=1}^{n} E^{P_k}[Z\mathbb{1}_{A_k}]$ holds for any r.v. Z and any $k \in \{1, \ldots, n\}$.*

Proof. 1. This claim is just an iterated version of Lemma 8 in Riedel (2009).

2. First, we note that for any set A_k and any r.v. Z, by plugging $Z\mathbb{1}_{A_k}$ in (10) we have:

$$E^P[Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau] = \sum_{i=1}^{n} E^{P_i}[Z\mathbb{1}_{A_k}\mathbb{1}_{A_i} \mid \mathcal{F}_\tau] = E^{P_k}[Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau] \tag{11}$$

In particular for an arbitrary set $B \in \mathcal{F}_\tau$ we have:

$$E^{P_k}[\mathbb{1}_{A_k B}] = E^{P_k}[\mathbb{1}_{A_k B} \mid \mathcal{F}_\tau] = E^P[\mathbb{1}_{A_k B} \mid \mathcal{F}_\tau] = E^P[\mathbb{1}_{A_k B}], \tag{12}$$

Since measures P and P_k are both in $\mathcal{P}$ the Radon-Nykodim derivative $\frac{dP_k}{dP}$ is well defined. Thus, using (11), for an arbitrary set $B \in \mathcal{F}_\tau$, the following holds:

$$E^P\left[E^P\left[\frac{dP_k}{dP}\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right]\mathbb{1}_B\right] = E^P\left[\frac{dP_k}{dP}\mathbb{1}_{A_k}\mathbb{1}_B\right] = E^{P_k}[\mathbb{1}_{A_k B}]. \tag{13}$$

Combining (12) and (13) we obtain[15]:

$$E^P\left[\frac{dP_k}{dP}\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right] = \mathbb{1}_{A_k}. \tag{14}$$

Now, multiplying the well known identity

$$E^P\left[\frac{dP_k}{dP} \mid \mathcal{F}_\tau\right] E^{P_k}\left[Z \mid \mathcal{F}_\tau\right] = E^P\left[\frac{dP_k}{dP}Z \mid \mathcal{F}_\tau\right]$$

with $\mathbb{1}_{A_k}$ (which is $\mathcal{F}_\tau$-measurable) we obtain:

$$E^P\left[\frac{dP_k}{dP}\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right] E^{P_k}\left[Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right] = E^P\left[\frac{dP_k}{dP}Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right].$$

Using the equations (11) and (14) the last equality can be rewritten as:

$$E^P\left[Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right] = E^P\left[Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right] = E^P\left[\frac{dP_k}{dP}Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right].$$

[15] If M is $\mathcal{F}$ measurable then $E[Z \mid \mathcal{F}] = M$ iff for any $B \in \mathcal{F}$ the equality $E[M\mathbb{1}_B] = E[Z\mathbb{1}_B]$ holds.

Finally, taking expectation over P in the last equality, we obtain the desired:

$$E^P\left[Z\mathbb{1}_{A_k}\right] = E^P\left[E^P\left[Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right]\right] = E^P\left[E^P\left[\frac{dP_k}{dP} Z\mathbb{1}_{A_k} \mid \mathcal{F}_\tau\right]\right] = E^P\left[\frac{dP_k}{dP} Z\mathbb{1}_{A_k}\right]$$
$$= E^{P_k}\left[Z\mathbb{1}_{A_k}\right]$$

3. Direct consequence of 2. . □

With this we are prepared for the following lemma:

Lemma C.2. *For Y_t, Z_t and $\mathcal{P}$ as defined above the following equality holds:*

$$\min_{P\in\mathcal{P}} E^P\left[Z_\tau\right] = \min_{P\in\mathcal{P}} E^P\left[Y_\tau\right] \tag{15}$$

Proof of Lemma C.2. Let us, for each t, denote by Q_t and R_t measures that minimize the adapted and non-adapted payoffs at time $\tau = t$, i.e. $Z_t\mathbb{1}_{\{\tau=t\}}$ and $Y_t\mathbb{1}_{\{\tau=t\}}$, respectfully (the existence of these measures is guaranteed by Riedel (2009); see Lemma 10 therein). Using the law of iterated expectation for multiple priors (Lemma 4 in Riedel (2009)) we obtain:

$$\begin{aligned}
E^{Q_t}\left[Z_t\mathbb{1}_{\{\tau=t\}}\right] &= \min_{P\in\mathcal{P}} E\left[Z_t\mathbb{1}_{\{\tau=t\}}\right] \\
&= \min_{P\in\mathcal{P}}\left[\operatorname*{ess\,inf}_{P'\in\mathcal{P}} E^{P'}[Y_t \mid \mathcal{F}_t]\mathbb{1}_{\{\tau=t\}}\right] \\
&= \min_{P\in\mathcal{P}}\left[\operatorname*{ess\,inf}_{P'\in\mathcal{P}} E^{P'}[Y_t\mathbb{1}_{\{\tau=t\}} \mid \mathcal{F}_t]\right] \\
&= \min_{P\in\mathcal{P}}\left[Y_t\mathbb{1}_{\{\tau=t\}}\right] = E^{R_t}\left[Y_t\mathbb{1}_{\{\tau=t\}}\right].
\end{aligned} \tag{16}$$

By the third claim of lemma C.1 above there exist the measures $Q, R \in \mathcal{P}$ such that $\sum_{t=1}^T E^{Q_t}[Z_t\mathbb{1}_{\{\tau=t\}}] = E^Q[Z_\tau]$ and $\sum_{t=1}^T E^{R_t}[Z_t\mathbb{1}_{\{\tau=t\}}] = E^R[Z_\tau]$. Combining the second claim of the same lemma with the equation (16) above we have, for each t:

$$E^Q[Z_t\mathbb{1}_{\{\tau=t\}}] = E^{Q_t}[Z_t\mathbb{1}_{\{\tau=t\}}] = E^{R_t}[Z_t\mathbb{1}_{\{\tau=t\}}] = E^R[Z_t\mathbb{1}_{\{\tau=t\}}] \tag{17}$$

Furthermore, $\arg\min_{P\in\mathcal{P}} E^P[Z_t\mathbb{1}_{\{\tau=t\}}] = Q$ for each t, which allows us to write:

$$\min_{P\in\mathcal{P}} E^P\left[Z_\tau\right] = \min_{P\in\mathcal{P}} \sum_{t=1}^T E^P\left[Z_t\mathbb{1}_{\{\tau=t\}}\right] = \sum_{t=1}^T E^Q\left[Z_t\mathbb{1}_{\{\tau=t\}}\right] = E^Q\left[Z_\tau\right]. \tag{18}$$

Similarly, we conclude that:

$$\min_{P\in\mathcal{P}} E^P\left[Y_\tau\right] = \min_{P\in\mathcal{P}} \sum_{t=1}^{T} E^P\left[Y_t \mathbb{1}_{\{\tau=t\}}\right] = \sum_{t=1}^{T} E^R\left[Y_t \mathbb{1}_{\{\tau=t\}}\right] = E^R\left[Y_\tau\right]. \quad (19)$$

Finally, the left hand side of equation (18) and (19) are equal because of (17), hence the right hand sides are also equal, which completes the proof. □

Appendix D Proof of the Theorem 3.1

For the sake of convenience, we begin by defining a sequence of functions

$$i_{t+1}(x_1,\ldots,x_t,x_{t+1}) = \mathbb{1}_{x_{t+1}\le \max(x_1,\ldots,x_t)}$$

for $t < T$. Note that this allows the random variable $\mathbb{1}_{X_{t+1}\le M_t}$ to be written in terms of the function i_{t+1} as follows:

$$\mathbb{1}_{X_{t+1}\le M_t} = i_{t+1}(X_1,\ldots,X_t,X_{t+1}).$$

As a preparation for the proof of theorem 3.1 we prove a result on the representation of the payoff process Z_t.

Since $X_1,\ldots,X_t$ are independent under $\overline{P}$ we can derive the following representation for functions r_t:

$$\overline{r}_t(m) = \prod_{s>t} E^{\overline{P}}\left[\mathbb{1}_{X_s\le m}\right] = \left(\overline{v}(X_1\le m)\right)^{T-t},$$

where the second equality is due to the definition of the measure $\overline{v}$. It is now obvious that $\overline{r}_t$ is an increasing function.

The next lemma describes the expected (ambiguous) Z_t in terms of the function $\overline{r}_t$:

Lemma D.1. *For each $t\in\{1,\ldots,T\}$ the following representation holds:*

$$Z_t = \mathbb{1}_{\{X_t=M_t\}}\overline{r}_t(X_t) \quad (20)$$

Proof. Note that:

$$Z_t = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\mathbb{1}_{\{X_t=M_t=M_T\}} \mid \mathcal{F}_t\right] = \mathbb{1}_{\{X_t=M_t\}} \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\mathbb{1}_{\{M_t=M_T\}} \mid \mathcal{F}_t\right]. \tag{21}$$

Define the process:

$$R_t = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\mathbb{1}_{\{M_t=M_T\}} \mid \mathcal{F}_t\right],$$

and the function:

$$r_t(x_1,\ldots,x_t) = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\mathbb{1}_{\{M_t=M_T\}} \mid X_1=x_1,\ldots,X_t=x_t\right].$$

Clearly, the following equalities hold:

$$Z_t = R_t\mathbb{1}_{\{X_t=M_t\}} = r_t(X_1,\ldots,X_t)\mathbb{1}_{\{X_t=M_t\}}. \tag{22}$$

Thus it suffices to show the following:

Claim: For each t the equality $R_t = \overline{r}_t(M_t)$ holds almost surely.

The claim is proven by backward induction.

Since $R_T = \overline{r}_T(M_T) = 1$ the claim trivially holds in the last period so we turn to the case $t < T$.

We begin by deriving a recursive expression for R_t (using the law of iterated expectations for multiple priors[16]) as follows:

$$\begin{aligned}
R_t &= \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\mathbb{1}_{M_t=M_{t+1}=M_T} \mid \mathcal{F}_t\right] \\
&= \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\operatorname*{ess\,inf}_{Q\in\mathcal{P}} E^Q \left[\mathbb{1}_{M_{t+1}=M_T} \mid \mathcal{F}_{t+1}\right] \mathbb{1}_{M_t=M_{t+1}} \mid \mathcal{F}_t\right] \\
&= \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[R_{t+1}\mathbb{1}_{M_t\geq X_{t+1}} \mid \mathcal{F}_t\right].
\end{aligned}$$

If we denote the realization of M_t with m_t (i.e. $m_t = \max(x_1,\ldots,x_t)$) we can rewrite the last equality in terms of the functions r_t and $\overline{r}_t$ using (22) and the induction hypothesis as follows:

$$\begin{aligned}
r_t(x_1,\ldots,x_t) &= \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\overline{r}_{t+1}(M_{t+1})\mathbb{1}_{X_{t+1}\leq m_t} \mid X_1=x_1,\ldots,X_t=x_t\right] \\
&= \overline{r}_{t+1}(m_{t+1}) \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P \left[\mathbb{1}_{X_{t+1}\leq m_t} \mid X_1=x_1,\ldots,X_t=x_t\right].
\end{aligned}$$

[16] Lemma 4 in Riedel (2009).

In the last equality above we used the fact that on the set $\{X_{t+1} \leq m_t\}$ the equality $M_{t+1} = M_t$ holds.

Since $\mathbb{1}_{X_{t+1} \leq M_t} = i_{t+1}(X_1, \ldots, X_t, X_{t+1})$ and the function i_{t+1} is decreasing in its last variable we can use lemma B.1 to identify $\overline{P}$ as the minimizing measure in the last expression:

$$\begin{aligned} r_t(x_1, \ldots, x_t) &= \overline{r}_{t+1}(m_t) E^{\overline{P}} \left[\mathbb{1}_{X_{t+1} \leq m_t} \mid X_1 = x_1, \ldots, X_t = x_t \right] \\ &= \left(\prod_{s > t+1} E^{\overline{P}} \left[\mathbb{1}_{X_s \leq m_t} \right] \right) E^{\overline{P}} \left[\mathbb{1}_{X_{t+1} \leq m_t} \right] = \overline{r}_t(m_t); \end{aligned}$$

the last equality is due to the definition of $\overline{r}_t$. □

We note that Lemma D.1 proves that infimum in the definition of the adapted payoff Z_t is attained for $\overline{v}$.

For the sake of convenience we also state a simple result about monotonicity of integral functions in the setting of our problem.

Lemma D.2. *Let $g(x_1, \ldots, x_t, x_{t+1})$, $t < T$, be a function increasing (decreasing) in each of the first t arguments. For any $P \in \mathcal{P}$ the function*

$$h^P(x_1, \ldots, x_t) = E^P[g(X_1, \ldots, X_t, X_{t+1}) \mid X_1 = x_1, \ldots X_t = x_t]$$

is increasing (decreasing) in every argument, as is the function

$$h(x_1, \ldots, x_t) = \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[g(X_1, \ldots, X_t, X_{t+1}) \mid X_1 = x_1, \ldots X_t = x_t].$$

Proof. The elementary proof of the first part of the lemma is omitted. Once one notices that $h = \operatorname{ess\,inf}_{P \in \mathcal{P}} h^P$ the second part follows immediately from the first part and the properties of the essential infimum. □

We turn to proving the core of the theorem and for that purpose we define the value process U of the FIBC optimal stopping problem under multiple priors:

$$U_T = Z_T; \quad U_t = \max(Z_t, \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[U_{t+1} \mid \mathcal{F}_t]), \, t < T.$$

The analysis will focus on the properties of the second argument in the maximum above so we define:

$$W_t = \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[U_{t+1} \mid \mathcal{F}_t], \, 0 \leq t < T.$$

As can be seen from the value process, the random variable W_t describes the expected value (under multiple priors) of the payoff the agent will receive if she does not stop at time t given the available information. The definition above implies:

$$W_{T-1} = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P[Z_T \mid \mathcal{F}_{T-1}], \quad W_t = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P[\max(Z_{t+1}, W_{t+1}) \mid \mathcal{F}_t].$$

If we introduce the sequence of functions:

$$w_t^*(x_1, \ldots, x_t) = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P[U_{t+1} \mid X_1 = x_1, \ldots, X_t = x_t],\ 0 \le t < T.$$

the equality $W_t = w_t^*(X_1, \ldots, X_t)$ clearly holds. Furthermore:

Lemma D.3. *For each $t \in \{0, 1, \ldots, T-1\}$ the function w_t^* is decreasing in every variable.*

Proof. The proof is by backward induction.

We first consider w_{T-1}^*. Notice that:

$$\begin{aligned} w_{T-1}^*(x_1, \ldots, x_{T-1}) &= \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P[Z_T \mid \mathcal{F}_{T-1}] \\ &= \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P[\mathbb{1}_{X_T=M_T} \mid X_1 = x_1, \ldots, X_{T-1} = x_{T-1}] \\ &= \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P[i_T(x_1, \ldots, x_{T-1}, X_T) \mid X_1 = x_1, \ldots, X_{T-1} = x_{T-1}] \end{aligned}$$

Since i_T is obviously decreasing in first $T-1$ variables, we can use the above lemma D.2 to conclude that w_{T-1}^* is decreasing in every variable.

For $t < T-1$ we have:

$$\begin{aligned} w_t^*(x_1, \ldots, x_t) = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P\Big[&\max\big(\mathbb{1}_{X_{t+1}=M_{t+1}}\overline{r}_{t+1}(X_t+1), \\ &w_{t+1}^*(x_1, \ldots, x_t, X_{t+1})\big) \mid X_1 = x_1, \ldots, X_t = x_t\Big] \\ = \operatorname*{ess\,inf}_{P\in\mathcal{P}} E^P\Big[&\max\big(i_{t+1}(x_1, \ldots, x_t, X_{t+1})\overline{r}_{t+1}(X_t+1), \\ &w_{t+1}^*(x_1, \ldots, x_t, X_{t+1})\big) \mid X_1 = x_1, \ldots, X_t = x_t\Big]. \end{aligned}$$

The function i_{t+1} is decreasing in its first t arguments and the function $\overline{r}_{t+1}$ is decreasing in every argument. The function w_{t+1}^* is decreasing in every argument by assumption. Thus, the result now follows from the fact that the maximum of decreasing function is a decreasing function and the lemma D.2.

□

The last result allows us to formulate a simple representation of the process W_t:

Lemma D.4. *For each $t \in \{0, 1, \dots, T-1\}$ there exists a decreasing function $w_t(m)$ such that $W_t = w_t(M_t)$.*

Proof. We begin the proof by backward induction by noting that, since $Z_T = \mathbb{1}_{X_T = M_T}$ and

$$W_{T-1} = \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[Z_T \mid \mathcal{F}_{T-1}] = \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[\mathbb{1}_{X_T = M_T} \mid \mathcal{F}_{T-1}],$$

we have, due to the definition of w^*_{T-1},

$$\begin{aligned} w^*(x_1, \dots, x_{T-1}) &= \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[i_T(x_1, \dots, x_{T-1}, X_T) \mid X_1 = x_1, \dots, X_{T-1} = x_{T-1}] \\ &= E^{\underline{P}}[i_T(x_1, \dots, x_{T-1}, X_T) \mid X_1 = x_1, \dots, X_{T-1} = x_{T-1}] \\ &= \underline{P}(X_T \geq M_{T-1}), \end{aligned}$$

where the second equality is due to lemma B.1. It thus suffices to define $w_{T-1}(m) = \underline{P}(X_T \geq m)$. Indeed, the function w_{T-1} is clearly decreasing and the equality $w_{T-1}(M_t) = W_T$ holds because of the previous considerations.

Suppose that for $t < T$ there exists a decreasing function w_{t+1} such that $w_{t+1}(M_{t+1}) = W_{t+1}$. This allows us to rewrite W_t in terms of w_{t+1} and $\overline{r}_{t+1}$:

$$\begin{aligned} W_t &= \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P[\max(Z_{t+1}, W_{t+1}) \mid \mathcal{F}_t] = \\ &= \operatorname*{ess\,inf}_{P \in \mathcal{P}} E^P\left[\max\left(\overline{r}_{t+1}(M_{t+1}) \cdot \mathbb{1}_{X_{t+1} = M_{t+1}}, w_{t+1}(M_{t+1})\right) \mid \mathcal{F}_t\right] \\ &= \operatorname*{ess\,inf}_{P \in \mathcal{P}} \Big(E^P\left[\max\left(\overline{r}_{t+1}(X_{t+1}), w_{t+1}(X_{t+1})\right) \cdot \mathbb{1}_{X_{t+1} \geq M_t} \mid \mathcal{F}_t\right] \\ &\qquad\qquad + w_{t+1}(M_{t+1}) \cdot E^P[\mathbb{1}_{X_{t+1} < M_t} \mid \mathcal{F}_t]\Big), \end{aligned}$$

where the last equality is due to:

$$\begin{aligned} &\max\left(\overline{r}_{t+1}(M_{t+1}) \cdot \mathbb{1}_{X_{t+1} = M_{t+1}}, w_{t+1}(M_{t+1})\right) = \\ &\max\left(\overline{r}_{t+1}(X_{t+1}), w_{t+1}(X_{t+1})\right) \cdot \mathbb{1}_{X_{t+1} \geq M_t} + w_{t+1}(M_{t+1}) \cdot \mathbb{1}_{X_{t+1} < M_t}. \end{aligned}$$

Since $W_t = w_t^*(X_1, \dots, X_t)$, and $M_t = M_{t+1}$ on the set X_{t+1}, we can write:

$$
\begin{aligned}
&w_t^*(x_1, \dots, x_t) = \\
&\operatorname*{ess\,inf}_{P \in \mathcal{P}} \Big(E^P \left[\max(\overline{r}_{t+1}(X_{t+1}), w_{t+1}(X_{t+1})) \cdot \mathbb{1}_{X_{t+1} \geq M_t} \mid X_1 = x_1, \dots, X_t = x_t \right] \\
&\qquad\qquad\qquad + w_{t+1}(M_t) \cdot E^P [\mathbb{1}_{X_{t+1} < M_t} \mid X_1 = x_1, \dots, X_t = x_t] \Big) \\
&= \operatorname*{ess\,inf}_{v \in \mathcal{V}^A} \Big(\int_{m_t}^{1} \max(\overline{r}_{t+1}(x), w_{t+1}(x)) \, dv + w_{t+1}(m_t) \int_0^{m_t} dv \Big),
\end{aligned}
$$

where $m_t = \max(x_1, \dots, x_t)$ and the last equality is due to the definition of the set $\mathcal{P}$ in section 3. Thus, by setting

$$
w_t(m) = \operatorname*{ess\,inf}_{v \in \mathcal{V}^A} \Big(\int_{m}^{1} \max(\overline{r}_{t+1}(x), w_{t+1}(x)) \, dv + w_{t+1}(m) \int_0^{m} dv \Big), \tag{23}
$$

for $t < T$, we get $w_t(m_t) = w_t^*(x_1, \dots, x_t)$ which, due to the definition of w_t^*, implies $w_t(M_t) = W_t$.

Finally, since $w_t(\max(x_1, \dots, x_t)) = w_t^*(x_1, \dots, x_t)$, the function w_t^* is symmetric; thus, the monotonicity of the function w_t is a consequence of the monotonicity of the function w_t^* as described by the lemma D.3. □

We now turn to proving that the stopping time is of the threshold type.

The proof of the last lemma reveals that the functions w_t are defined by the recursion $w_{T-1}(m) = \underline{P}(X_T \geq m)$ and, for $t < T - 1$, the equation (23). Equivalently, we can expand the definition to include the final period by setting $w_T(m) = 0$ and w_t as defined by the expression in the equation (23) for $t < T$.

It is clear that, for each $t < T$, the equalities $w_t(1) = \overline{r}_t(0) = 0$ hold and that the functions r_t are strictly increasing, while the functions w_t are (weakly) decreasing. Thus, for $t < T$, there exists a unique $b_t \in [0, 1)$ such that $w_t(b_t) = \overline{r}_t(b_t)$. Additionally, we define $b_T = 0$. We record the previous considerations, along with the proof of the monotonicity of the sequence (b_t), in the following lemma.

Lemma D.5. *For each $t < T$ there exists a unique $b_t \in [0, 1]$ such that the equality $w_t(b_t) = \overline{r}_t(b_t)$ holds. Furthermore, for each $t < T$ the inequality $b_t > b_{t+1}$ holds.*

Proof. Suppose $t < T$. Note that, due to the definition of the sequence (b_t) and the fact that the function $\overline{r}_{t+1}$ is strictly increasing, the following (in)equalities hold

$$\max(\overline{r}_{t+1}(x), w_{t+1}(x)) = \overline{r}_{t+1}(x) > \overline{r}_{t+1}(b_{t+1}) = w_{t+1}(b_t),$$

for each $x \in (b_{t+1}, 1]$. Hence:

$$\begin{aligned} w_t(b_{t+1}) &= \operatorname*{ess\,inf}_{v \in \mathcal{V}^A} \left(\int_{b_{t+1}}^{1} \max(\overline{r}_{t+1}(x), w_{t+1}(x))\, dv + w_{t+1}(b_{t+1}) \int_0^{b_{t+1}} dv \right) \\ &> \operatorname*{ess\,inf}_{v \in \mathcal{V}^A} \left(\int_{b_{t+1}}^{1} w_{t+1}(b_{t+1})\, dv + w_{t+1}(b_{t+1}) \int_0^{b_{t+1}} dv \right) = w_{t+1}(b_{t+1}). \end{aligned} \tag{24}$$

Note, also, that the definition of $\overline{r}_t$ implies $\overline{r}_t(x) < \overline{r}_{t+1}(x)$ for any $x \in (0, 1)$. Thus, given the previously obtained inequality (24), we get:

$$w_t(b_{t+1}) > w_{t+1}(b_{t+1}) = \overline{r}_{t+1}(b_{t+1}) > \ \overline{r}_t(b_{t+1}). \tag{25}$$

With the inequality (25) proven we can turn to proving the inequality stated in the formulation of the lemma.

Suppose the opposite: $b_t \leq b_{t+1}$. The definition of b_t and the monotonicity of r_t imply: $w_t(b_t) = \overline{r}_t(b_t) \leq \overline{r}_t(b_{t+1}) < w_t(b_{t+1})$, where the last inequality is due to the previously proven inequality (25). This, however, is in contradiction with the monotonicity of w_t. □

To complete the proof of the first two parts of the theorem it remains to prove the equality (4); we do so in the following lemma:

Lemma D.6.

$$\tau^* = \min\{t \mid X_t = M_t > b_t\}$$

Proof. In the context of FIBC the optimal stopping time is given with:

$$\tau^* = \min\{t \mid Z_t = U_t\} = \min\{t \mid Z_t \geq W_t\}.$$

Using the representations for Z_t and W_t obtained in lemmas D.1 and D.4 respectfully, the inequality $Z_t \geq W_t = w_t(M_t)$ can only be satisfied when $X_t = M_t$ (in which case $Z_t = \overline{r}_t(M_t)$), hence:

$$\tau^* = \min\{t \mid X_t = M_t,\, \overline{r}_t(M_t) \geq w_t(M_t)\}.$$

Finally, due to the monotonicity of $\overline{r}_t$ and w_t and lemma D.5, the inequality $\overline{r}_t(X_t) \geq w_t(X_t)$ is satisfied only when $b_t \leq M_t = X_t$.

□

It remains to note that the essential infimum in (23) is attained (see Lemma 10 in Riedel (2009)). This, with the definitions of W_t and U_t, and Lemma D.1 proves the third part of the theorem. Indeed, before stopping the minimizing measure is the one attained in (23), and once the agent stops the her payoff is Z_t, and lemma D.1 implies that the minimizing measure is $\overline{v}$.

Appendix E Proof of Lemma 4.1

Proof of claim 1 of Lemma 4.1. We define an operator $G : L^1([0,1]) \to \mathbb{R}$ with

$$G\varphi = \int_0^1 g(x)\varphi(x)\, dx,$$

and note that it is (Lipschitz) L^1-continous. Indeed, using the fact that g is increasing and bounded:

$$|G\varphi_1 - G\varphi_2| = \left| \int_0^1 g(x)(\varphi_1(x) - \varphi_2(x))\, dx \right| \leq C \left| \int_0^1 \varphi_1(x) - \varphi_2(x)\, dx \right|,$$

where C is a positive constant that bounds $|g(x)|$.

Let D_S^λ be the set of all the step functions within the set D_{CLA}^λ. We will prove that D_S^λ is dense[17] in D_{CLA}^λ. For an arbitrary $\varphi \in D_S^\lambda$ and an arbitrary $\varepsilon > 0$ one can choose a step function φ_1 such that $\frac{1}{\lambda} \leq \varphi_1(x) \leq \varphi(x) \leq \lambda$ and:

$$\left| \int_0^1 \varphi_1(x) - \varphi(x)\, dx \right| < \frac{\varepsilon}{2}. \tag{26}$$

If one defines $I = \int_0^1 \varphi_1(x)\, dx \leq 1$ and:

$$\gamma = \frac{1 - \int_B \varphi_1(x)\, dx}{\int_A \varphi_1(x)\, dx}$$

[17] With respect to L^1 metric.

for $A = \{\varphi_1(x) \leq I\}$ and $B = [0,1]\backslash A$ it is easy to check that, for sufficiently small ε, the function $\varphi_S = \gamma\varphi_1 \mathbb{1}_A + \varphi_1 \mathbb{1}_B$ is a function that belongs to D^λ_{CLA}. Furthermore, direct calculations show that the inequality

$$\left| \int_0^1 \varphi_1(x) - \varphi_S(x)\, dx \right| < \frac{\varepsilon}{2} \tag{27}$$

holds[18]. Combining the inequalities (26) and (27) gives:

$$\left| \int_0^1 \varphi_S(x) - \varphi(x)\, dx \right| < \varepsilon,$$

which proves the density.

As the operator G is continuous and the set D^λ_S (which contains $\underline{\varphi}$) is dense in D^λ_{CLA}, for the claim to hold it suffices to show that for any $\varphi \in D^\lambda_S$ the inequality $G\varphi \geq G\underline{\varphi}$ holds. We do so in the remainder of the proof.

Let us fix $\varphi \in D^\lambda_S$:

$$\varphi = \sum_{i=1}^{n} d_i \mathbb{1}_{[c_{i-1}, c_i]}, \ \in D^\lambda_{CLA}. \tag{28}$$

Without loss of generality we can assume that there is an index $m \in \{1, \ldots, n\}$ such that $c_m = 1/(1+\lambda)$.

Set $\varphi_0 = \varphi$. The idea is to create a finite sequence of functions (φ_i) in which the last element is $\underline{\varphi}$ with the inequality $G\varphi_i \leq G\varphi_{i-1}$ being satisfied for any $i > 0$.

If $\varphi_0 = \underline{\varphi}$ the proof is done. If not, we choose the step function φ_1 such that it differs from φ_0 by the value it takes on two appropriately chosen intervals. For that purpose we define:

$$j = \min\{i \mid d_i < \lambda\}, \quad j' = \max\{i \mid d_i > 1/\lambda\}.$$

Note that since $\varphi_0 \neq \underline{\varphi}$ we have $j < m < j'$. We now focus on the intervals $[c_{j-1}, c_j]$ and $[c_{j'-1}, c_{j'}]$ and set the value of φ_1 to be λ on the former interval or $1/\lambda$ on the latter by "repositioning the weight" of φ_0.

[18] The inequality (27) is in fact equivalent to (26).

If $(\lambda - d_j)(c_j - c_{j-1}) \leq (d_{j'} - \frac{1}{\lambda})(c_{j'-1} - c_{j'})$ we "reposition the excess weight" from the interval $[c_{j'-1}, c_{j'}]$ to the interval $[c_{j-1}, c_j]$, that is we define:

$$\begin{aligned}\varphi_1 =&\varphi_0 \mathbb{1}_{[0,1]\setminus([c_{j-1},c_j]\cup[c_{j'-1},c_{j'}])} + \\ &\left(d_j + (d_{j'} - \tfrac{1}{\lambda})\frac{c_{j'-1} - c_{j'}}{c_j - c_{j-1}}\right)\mathbb{1}_{[c_{j-1},c_j]} + \frac{1}{\lambda}\mathbb{1}_{[c_{j'-1},c_{j'}]}.\end{aligned}$$

The inequality $G\varphi_o \leq G\varphi_1$ is satisfied. Indeed, direct calculation yields

$$\begin{aligned}G\varphi_o - G\varphi_1 &= \int_{c_{j-1}}^{c_j} g(x)(\varphi_0(x) - \varphi_1(x))\,dx + \int_{c_{j'-1}}^{c_{j'}} g(x)(\varphi_0(x) - \varphi_1(x))\,dx \\ &= -(d_{j'} - \tfrac{1}{\lambda})\frac{c_{j'-1} - c_{j'}}{c_j - c_{j-1}}\int_{c_{j-1}}^{c_j} g(x)\,dx + (d_{j'} - \tfrac{1}{\lambda})\int_{c_{j'-1}}^{c_{j'}} g(x)\,dx,\end{aligned}$$

and one can use the monotonicity of the function g and the inequalities $j < m < j'$ to make the following estimation:

$$\begin{aligned}G\varphi_o - G\varphi_1 \geq\ & (d_{j'} - \tfrac{1}{\lambda})\left(g(c_{j'-1})(c_{j'-1} - c_{j'}) - \frac{c_{j'-1} - c_{j'}}{c_j - c_{j-1}} g(c_j)(c_{j-1} - c_j)\right) \\ =\ & (d_{j'} - \tfrac{1}{\lambda})(c_{j'-1} - c_{j'})(g(c_{j'-1}) - g(c_j) \geq 0.\end{aligned}$$

When the inequality $(\lambda - d_j)(c_j - c_{j-1}) > (d_{j'} - \frac{1}{\lambda})(c_{j'-1} - c_{j'})$ holds one can construct the function φ_1 using an analogous "weight repositioning".

If $\varphi_1 = \underline{\varphi}$ the proof is done. If not, one can create φ_2 from φ_1 as above. As the step function φ has finitely many steps the procedure ends after finitely many iterations.

□

Proof of claim 2 of Lemma 4.1. We begin by fixing $\varphi \in D^{\lambda}_{CLA}$ and defining:

$$\mu_1 = \int_0^k \varphi(x)\,dx, \quad \mu_2 = \int_k^1 \varphi(x)\,dx.$$

We will identify two functions ψ_1 and ψ_2, defined on $[0, k]$ and $[k, 1]$ respectively, such that the function $\psi := \psi_1 \mathbb{1}_{[0,k]} + \psi_2 \mathbb{1}_{(k,1]}$ is the one that satisfies the claim. These will be the functions that "reposition the weight" μ_1 and μ_2 within the intervals $[0, k]$ and $[k, 1]$, such that most weight is on the upper part of the former and the lower part of the latter.

First we focus on the interval $[0, k]$. The first claim showed how to identify the step function $\overline{\varphi}$ that, for a fixed, decreasing and bounded function g, minimizes the integral on the right hand side of (E) among all the functions φ whose range is within the interval $[1/\lambda, \lambda]$ and whose total weight is equal to 1. Note that $\overline{\varphi}$ was simply the function that put the most weight possible on the upper part of the interval $[0, 1]$. Focusing on the interval [0,k], where the function h is decreasing, we are in a similar situation: among all the functions with a range within $[1/\lambda, \lambda]$ and whose integral is equal to μ_1 we are looking for a function ψ_1 that minimizes the integral $\int_0^k h(x)\psi_1(x)\, dx$. Analogous reasoning to the one in the proof of the first claim[19] will lead us to the conclusion that ψ_1 has to be the function that puts the most weight as possible on the upper part of the interval $[0, k]$:

$$\psi_1 = \frac{1}{\lambda}\mathbb{1}_{[0,c_1]} + \lambda\mathbb{1}_{(c_1,k]}, .$$

for an appropriately chosen c_1. Identifying the precise value of c_1 is not difficult: since the inequalities clearly $\frac{k}{\lambda} \leq \mu_1 \leq k\lambda$ hold, there exists $c_1 \in [0, k]$ such that:

$$\frac{c_1}{\lambda} + \lambda(k - c_1) = \mu_1.$$

This proves the inequality:

$$\int_0^k h(x)\varphi(x)\, dx \geq \int_0^k h(x)\psi_1(x)\, dx. \tag{29}$$

Similarly, by focusing on the interval $[k, 1]$ one can identify the function ψ_2 (and the corresponding c_2) which puts the most weight on the lower part of the interval, such that:

$$\int_k^1 h(x)\varphi(x)\, dx \geq \int_k^1 h(x)\psi_2(x)\, dx = \int_k^1 h(x)\left(\lambda\mathbb{1}_{[k,c_2]} + \frac{1}{\lambda}\mathbb{1}_{(c_2,1]}\right) dx. \tag{30}$$

Direct calculations show that $D_U^\lambda \ni \psi := \psi_1\mathbb{1}_{[0,k]} + \psi_2\mathbb{1}_{(k,1]}$ and the claim follows by combining (29) and (30). □

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[19]Note that the first claim could have been formulated for functions on any interval $[a, b]$, and with total weight of densities being equal to any number (as opposed to 1); we chose not to do so for the sake of readability.

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