Hurwitz action in Coxeter groups and elliptic Weyl groups

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

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16.05.2017

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Acknowledgements

First and foremost I would like to thank my supervisor, Barbara Baumeister. I thank her for the support and for helpful discussions. Furthermore I would like to thank her and Henning Krause for coming up with the interesting topic which lead to the results in this thesis.

Parts of this thesis are based on joint work with different people. Therefore I would like to thank them, namely Barbara Baumeister, Matthew Dyer, Thomas Gobet, Kieran Roberts and Christian Stump.

Furthermore I would like to thank Kai-Uwe Bux, Jon McCammond, Dieter Vossieck, Stefan Witzel and Sophiane Yahiatene for discussions and good advices. I am particularly grateful to Thomas Gobet for proofreading parts of the thesis and for the suggestion of Example 6.4.9 as well as to Kieran Roberts for the help with the GAP programs.

I am indebted to Bielefeld University for the financial support by a scholarship. I am also grateful to the CRC 701 'Spectral Structures and Topological Methods in Mathematics' and the Faculty of Mathematics in Bielefeld for their financial support.

Finally, I would like to thank my family and my friends for their ongoing support.

Contents

1. Introduction

Given any group G and an integer $n \geq 2$, the braid group $\mathcal{B}_n$ on n strands acts on G^n as

$$\sigma_i \cdot (g_1, \ldots, g_n) := (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n),$$

where $\sigma_i \in \mathcal{B}_n$ is the standard generator which exchanges the i^{th} and $(i+1)^{\text{th}}$ strands. This action is called Hurwitz action since it was first studied by Hurwitz in 1891 [Hur91] in the case $G = \mathrm{Sym}(n)$. Two elements of G^n are called Hurwitz equivalent if they are in the same orbit under this action. In general, the question whether two elements in G^n are Hurwitz equivalent is undecidable. This has been shown by Liberman and Teicher [LT], see also [Ito13]. However, there are results for particular groups (see for instance [Hou08], [Sia09]). Certainly the Hurwitz action also plays a role in algebraic geometry, more precisely in the braid monodromy of a projective curve (e.g. see [KT00, Bri88]).

Another way of studying the Hurwitz action is given by considering the action for a subset $T \subseteq G$ which is closed under conjugation. More precisely, if G is a group and T a set of generators of G closed under conjugation, the Hurwitz action on G^n restricts to the set $\mathrm{Red}_T(g)$ of reduced T-decompositions for an element $g \in G$. Notice that the Hurwitz action leaves the product of the entries unchanged.

In this thesis we are mainly interested in the case where G is a Coxeter group or an elliptic Weyl group and T is the set of reflections for the corresponding group. Applications arise in the study of Artin groups or representation theory of algebras (see Sections 3.3 and 7.5).

1.1. The main results

First we consider dual Coxeter systems (W, T). That is, W is a Coxeter group and T the set of reflections in W. Dual Coxeter systems were independently introduced by Bessis [Bes03] and by Brady and Watt [BW02]. They are crucial in the theory of dual braid monoids. These are alternative braid monoids embedding in a spherical Artin group and providing an alternative Garside structure of it (see [Bes03]). E.g. having a Garside structure implies that the group has decidable word problem and is torsion-free. The definition of a dual braid monoid depends on a choice of a Coxeter element and the poset of simple elements for this monoid turns out to be precisely the poset of noncrossing partitions with respect to that Coxeter element.

Having replaced the set of simple reflections of a Coxeter group by the whole set of reflections in the Coxeter groups (and in the Artin group), one needs to find new sets of relations between these new generators that define the respective groups. The idea is to take the so-called dual braid relations [Bes03]. Unlike the classical braid relations, a dual braid relation can involve three generators and has the form $ab = ca$ (or $ba = ac$), where a, b and c are reflections.

In the classical case Matsumoto's Lemma [Mat64] allows one to pass from any reduced decomposition of an element to any other one by successive applications of braid relations. The same question can be asked for reduced decompositions with respect to the new set of generators, and can be studied using the Hurwitz action on reduced decompositions.

For finite Coxeter groups, the Hurwitz action was first shown to act transitively on $\mathrm{Red}_T(c)$ for a classical Coxeter element c in a letter from Deligne to Looijenga [Del74]. The first published proof is due to Bessis and can be found in [Bes03]. Igusa and Schiffler generalized this result to classical Coxeter elements in arbitrary Coxeter systems of finite rank (see [IS10, Theorem 1.4]).

We will provide a simple proof of Igusa and Schiffler's theorem, based on results of Dyer (see Sections 2.2 and 2.3). We moreover do this for a parabolic Coxeter element instead of a classical Coxeter element.

Theorem 1.1.1. *Let (W, T) be a dual Coxeter system of finite rank n and let $w = s_1 \cdots s_m$ be a parabolic Coxeter element in W. The Hurwitz action on $\mathrm{Red}_T(w)$ is transitive, that is, for each $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ there is a braid $\sigma \in \mathcal{B}_m$ such that*

$$\sigma(t_1, \ldots, t_m) = (s_1, \ldots, s_m).$$

Apart from its importance in the theory of dual braid monoids, this aspect of the Hurwitz action also has applications in the representation theory of algebras [HK13, IS10, Kra12] and in connection with singularities of isolated hypersurfaces [Bri88, Loo74].

With Theorem 1.1.1 in mind, it seems natural to ask if there are classes of elements besides the class of parabolic Coxeter elements which share this transitivity property. We therefore provide a necessary and sufficient condition on an element of a finite Coxeter group to ensure the transitivity of the Hurwitz action on its set of reduced decompositions. We call an element of a Coxeter group a parabolic quasi-Coxeter element if it admits a reduced decomposition which generates a parabolic subgroup. A quasi-Coxeter element is an element admitting a reduced decomposition which generates the whole Coxeter group.

Theorem 1.1.2. *Let (W, T) be a finite dual Coxeter system of finite rank n and let $w \in W$. The Hurwitz action on $\mathrm{Red}_T(w)$ is transitive if and only if w is a parabolic quasi-Coxeter element for (W, T).*

Unfortunately, the statement of the preceding theorem can not be transferred one to one to infinite dual Coxeter systems (see Remark 6.1.1 (b)). Nevertheless, the necessary condition is still true for a family of infinite Coxeter groups. Namely for affine Coxeter groups.

Theorem 1.1.3. *Let $(\widetilde{W}, \widetilde{T})$ be an irreducible affine dual Coxeter system. Then the Hurwitz action is transitive on $\mathrm{Red}_{\widetilde{T}}(w)$ if w is a parabolic quasi-Coxeter element for $(\widetilde{W}, \widetilde{T})$.*

The other part of this thesis is devoted to the study of the Hurwitz action in elliptic Weyl groups. More precisely, we just deal with the elliptic Weyl groups of types $D_4^{(1,1)}$ and $E_n^{(1,1)}$ ($n \in \{6, 7, 8\}$), which we call tubular elliptic Weyl groups. Elliptic Weyl groups are the Weyl groups of elliptic root systems. These root systems are generalizations of finite and affine root systems and were classified by Saito [Sai85] in terms of so called elliptic Dynkin diagrams. The elliptic Dynkin diagrams for the tubular elliptic root systems are obtained by taking a copy of the vertex of the corresponding affine diagram at the largest exponent (see Figures 2.3 and 7.1). To the simply-laced elliptic root systems (including the tubular elliptic root systems), Saito and Yoshii associated a Lie algebra. For the Lie algebra associated to a tubular elliptic root system, Lin and Peng [LP05] showed that it is isomorphic to the Ringel-Hall Lie algebra of the root category of the tubular algebra with the same type.

Greatly benefiting from work of Kluitmann [Klu87], we obtain the following result in this direction.

Theorem 1.1.4. *Let (Φ, U) be a tubular elliptic root system of rank n, $\Gamma = \Gamma(\Phi, U)$ an elliptic root basis and $c \in W_\Phi$ a Coxeter transformation with respect to Γ. Then the Hurwitz action is transitive on the set*

$$\underline{\mathrm{Red}}_{T_\Phi}(c) = \{(s_{\beta_1}, \dots, s_{\beta_{n+2}}) \mid \beta_i \in \Phi,\ \mathrm{span}_{\mathbb{Z}}(\beta_1, \dots, \beta_{n+2}) = L(\Phi), c = s_{\beta_1} \cdots s_{\beta_{n+2}}\}.$$

The motivation for studying the Hurwitz action in this direction arises in the theory of hereditary categories and will be formulated in Conjecture 7.5.3.

1.2. Outline

The thesis is organized as follows. In Chapter 2 we present the necessary background on root systems and (dual) Coxeter systems. We moreover summarize results of Dyer on the absolute length and reflection decompositions in Coxeter groups. We apply these results to Coxeter elements in Chapter 3 and discuss Theorem 1.1.1 there. In Chapter 4 we present several results about generating sets of finite Coxeter groups and finite root lattices as well as their connections. These results are crucial for the proof of Theorem 1.1.2 at the end of this chapter. In Chapter 5 we discuss the Hurwitz action on nonreduced reflection decompositions in finite Coxeter groups. This is among other things a groundwork for Chapter 6 where we return to the study of the Hurwitz action on reduced reflection decompositions, but now for affine Coxeter groups. In particular we prove Theorem 1.1.3 here. In Chapter 7 we shall first present and summarize all the necessary background on elliptic root systems and elliptic Weyl groups. We then introduce the notion of a tubular elliptic root system and prove Theorem 1.1.4. Finally we state a conjecture about the connection of the Weyl group of a tubular elliptic root system and the derived category of coherent sheaves on a tubular weighted projective line.

Notice that Theorem 1.1.1 appeared in [BDSW14] and Theorem 1.1.2 appeared in [BGRW17].

Explicit calculations were made with [GAP2015]. All programs are listed and shortly described in Appendix A. The programs themselves are stored on a CD, which can be found at the end of the thesis.

1.3. Basic assumptions and conventions

All vector spaces are assumed to be finite dimensional over $\mathbb{R}$. If G is a group and H a subgroup of G we write $H \leq G$. For an element $g \in G$ we denote its order by $o(g)$. If a group is assumed to be finite, affine etc. we state this explicitely. The identity element of a group is denoted by e. We put $\mathbb{N} = \{1, 2, 3, \dots\}$ and for a natural number $n \in \mathbb{N}$ we denote by $[n]$ the set $\{1, 2, \dots, n\}$ resp. by $[\pm n]$ the set $\{\pm 1, \pm 2, \dots, \pm n\}$. The $n \times n$ identity matrix is denoted by I_n.

2. Coxeter groups and dual Coxeter systems

2.1. Root systems and Coxeter systems

We collect some basic facts and definitions about root systems and Coxeter systems. For details we refer to [Hum90].

2.1.1. Coxeter systems

Consider an euclidean vector space V with positive definite symmetric bilinear form $(- \mid -)$. For $\alpha \in V \setminus \{0\}$ the map s_α defined by

$$s_\alpha : V \to V, v \mapsto v - \frac{2(v \mid \alpha)}{(\alpha \mid \alpha)}\alpha$$

is an orthogonal transformation of V of order 2. This map sends α to $-\alpha$ and fixes the hyperplane H_α orthogonal to α pointwise. Therefore we call s_α an orthogonal reflection. A finite group generated by reflections is called finite reflection group.

Coxeter ([Cox35]) and later Tits in his unpublished paper "Groupes et géométries de Coxeter" generalized the concept of finite reflection groups by the following groups.

Definition 2.1.1. A pair (W, S) consisting of a group W and a set $S \subseteq W$ of generators of W is called Coxeter system if W admits the presentation

$$\langle S \mid (st)^{m_{st}} = 1 \rangle,$$

where $m_{ss} = 1$ and $m_{st} = m_{ts} \geq 2$ for $s \neq t$ in S. If there is no relation for the pair s, t we write $m_{st} = \infty$. In this case the group W is called a Coxeter group, the elements of S are called simple reflections and $|S|$ is the rank of (W, S). For $s \neq t$ in S the relation $(st)^{m_{st}} = 1$ can also be written as $sts \cdots = tst \cdots$, where the words on both sides are of length m_{st}. A relation of this kind is also called a braid relation.

Given a Coxeter system (W, S), the presentation given above can be encoded in a diagram $\Gamma(W, S)$, called the Coxeter diagram of (W, S). The vertices of $\Gamma(W, S)$ are the simple reflections S. Two elements $s, t \in S$ are joined by an edge if $m_{st} \geq 3$. If $m_{st} > 3$, the corresponding edge is labeled by m_{st} with the exception that we draw a double edge instead of using the label 4. The Coxeter system (W, S) is called irreducible if $\Gamma(W, S)$ is connected. If $\Gamma(W, S)$ has connected components $\Gamma_1, \ldots, \Gamma_k$ and $S_1, \ldots, S_k$ denote the corresponding subsets of S, then each pair $(\langle S_i \rangle, S_i)$ is a Coxeter system and W is the direct product of the groups $\langle S_1 \rangle, \ldots, \langle S_k \rangle$ (see [Hum90, Proposition 6.1]). Therefore the study of Coxeter systems can be largely reduced to the case when the Coxeter system is irreducible. The set

$$T := \{wsw^{-1} \mid w \in W,\ s \in S\}$$

is called the set of reflections for (W, S). In the following we just consider Coxeter systems of finite rank and call (W, S) a finite Coxeter system if the group W is finite.

Example 2.1.2. If we take W to be the symmetric group $\mathrm{Sym}(n+1)$ on $n+1$ letters and S to be the set of simple transpositions, that is $S = \{(i, i+1) \mid 1 \leq i \leq n\}$, then (W, S) is an irreducible Coxeter system of rank n with set of reflections given by all transpositions.

We say that (W, S) is oddly-laced if m_{st} is odd for all $s, t \in S$ with $m_{st} \geq 3$.

Lemma 2.1.3. *Let (W, S) be an irreducible oddly-laced Coxeter system with set of reflections T. Then all elements in T are conjugated under W.*

Proof. Since (W, S) is irreducible, the Coxeter diagram $\Gamma(W, S)$ is connected. Therefore it is enough to check that two simple reflections $s, t \in S$ which are connected by an edge in $\Gamma(W, S)$ are conjugated. But this is an immediate consequence of the relation $(st)^{m_{st}} = 1$ since m_{st} is odd. □

We continue with a characterization of Coxeter systems. Let (W, S) be a pair consisting of a group W and a set of generators S of W. Since the set S generates W, it induces a length functions ℓ_S. A decomposition $w = s_1 \cdots s_r$ with $s_i \in S$ is called S-reduced if $\ell_S(w) = r$. A well known characterization of Coxeter systems is given by the exchange condition (see [AB08, Theorem 2.49]).

Theorem 2.1.4 (Exchange condition)**.** *Let W be a group and S a set of generators of W consisting of involutions. Then (W, S) is a Coxeter system if and only if the following holds:*

If $w = s_1 \cdots s_r$ $(s_i \in S)$ with $\ell_S(ws) < \ell_S(w)$ for some $s \in S$, then there exists an index i such that $ws = s_1 \cdots \widehat{s_i} \cdots s_r$, where the entry $\widehat{s_i}$ is omitted.

If we replace in the statement of the theorem the simple reflection $s \in S$ by an arbitrary reflection $t \in T$, we obtain the same result (see [Hum90, Theorem 5.8]).

Theorem 2.1.5 (Strong exchange condition)**.** *Let (W, S) be a Coxeter system and $w = s_1 \cdots s_r$ $(s_i \in S)$ with $\ell_S(wt) < \ell_S(w)$ for some $t \in T$. Then there exists an index i such that $wt = s_1 \cdots \widehat{s_i} \cdots s_r$. If $\ell_S(w) = r$, then the index i is unique.*

For later use we give another characterization of Coxeter systems (see again [AB08, Theorem 2.49]), which will appear to be quite natural after having introduced root systems in the next subsection.

Proposition 2.1.6. *Let W be a group and S a set of generators of W consisting of involutions. Put $T := \{wsw^{-1} \mid w \in W,\ s \in S\}$. Then (W, S) is a Coxeter system if and only if there is an action of W on $T \times \{\pm 1\}$ such that $s \in S$ acts as*

$$s(t, \epsilon) = \begin{cases} (sts, \epsilon) & \text{if } t \neq s \\ (s, -\epsilon) & \text{if } t = s. \end{cases}$$

2.1.2. Root systems, geometric representation and finite Coxeter systems

Definition 2.1.7. Let V be an euclidean vector space with positive definite symmetric bilinear form $(- \mid -)$. A finite subset $\Phi \subseteq V$ of nonzero vectors is called root system in V if

(1) $\mathrm{span}_{\mathbb{R}}(\Phi) = V$,

(2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$,

(3) $\Phi \cap \mathbb{R}\,\alpha = \{\pm\alpha\}$ for all $\alpha \in \Phi$.

The root system is called crystallographic if in addition

(4) $\langle \beta, \alpha \rangle := \frac{2(\beta|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The rank $\mathrm{rk}(\Phi)$ of Φ is the dimension of V. The group $W_\Phi = \langle s_\alpha \mid \alpha \in \Phi \rangle$ associated to the root system Φ is a finite reflection group and each finite reflection group arises in this way, but possibly for different choices of Φ ([Hum90, Ch. 1]). If the root system is crystallographic, we call W_Φ a (finite) Weyl group. A subset $\Phi' \subseteq \Phi$ is called a root subsystem if Φ' is a root system in $\mathrm{span}_{\mathbb{R}}(\Phi')$. The root system Φ is reducible if $\Phi = \Phi_1 \dot\cup \Phi_2$ where Φ_1, Φ_2 are nonempty root systems such that $(\alpha \mid \beta) = 0$ whenever $\alpha \in \Phi_1$, $\beta \in \Phi_2$. Otherwise Φ is irreducible.

Since the cardinality of Φ can be large compared with the dimension of V, it would be convenient to have some kind of basis of Φ. Given a total ordering $<$ on V, we call $v \in V$ positive if $v > 0$. A subset Φ^+ is called positive system for Φ if Φ^+ consists of all roots of Φ which are positive with respect to some total ordering of V. If $\alpha \in \Phi$, then also $-\alpha \in \Phi$, thus $\Phi = \Phi^+ \dot\cup -\Phi^+$. A subset $\Delta \subseteq \Phi$ is called a simple system if Δ is a basis for V and if each $\alpha \in \Phi$ is a linear combination of Δ with all coefficients nonnegative or nonpositive. It is easy to see that a total ordering on V and hence a positve system exists. On the other hand each positive system for Φ contains a unique simple system and if Δ is a simple system for Φ, then there is a unique positive system for Φ containing Δ (see [Hum90, Theorem 1.3]).

Let $\Phi \neq \varnothing$ be a root system. Fix a simple system Δ with corresponding positive system Φ^+. Let $\beta \in \Phi$. Then there exists a unique $\mathbb{Z}$-linear combination $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$. We define $\mathrm{ht}(\beta) := \sum_{\alpha\in\Delta} c_\alpha$ and call this the height of β (relative to Δ). If Φ is irreducible and crystallographic, then there exists a unique root of maximal height, called highest root. We will denote this root by $\widetilde{\alpha}$.

For an irreducible crystallographic root system Φ, the set $\{(\alpha \mid \alpha) \mid \alpha \in \Phi\}$ has at most two elements (see [Hum90, Section 2.9]). The roots of greater length are called long. The roots which are not long, are called short. If this set has only one element, we call Φ simply-laced. In the latter case we assume (after possible rescaling) that $(\alpha \mid \alpha) = 2$ for all $\alpha \in \Phi$. The irreducible crystallographic root systems are classified by Dynkin diagrams (see Figure 2.1). To obtain these diagrams let Δ be a simple system of an irreducible crystallographic root system Φ. The vertices of the diagram correspond to the elements of Δ. Given $\alpha, \beta \in \Delta$ there is an undirected edge between them if they make an angle of $\frac{2\pi}{3}$, a directed double edge if they make an angle of $\frac{3\pi}{4}$ and a directed triple edge if they make an angle of $\frac{5\pi}{6}$. Directed edges point towards the shorter root. We define the type of Φ to be the type of the corresponding Dynkin diagram. The irreducible crystallographic root systems have types A_n $(n \geq 1), B_n$ $(n \geq 2), C_n,$ $(n \geq 3), D_n$ $(n \geq 4)$, E_n $(n \in \{6, 7, 8\})$, F_4 and G_2. It follows from

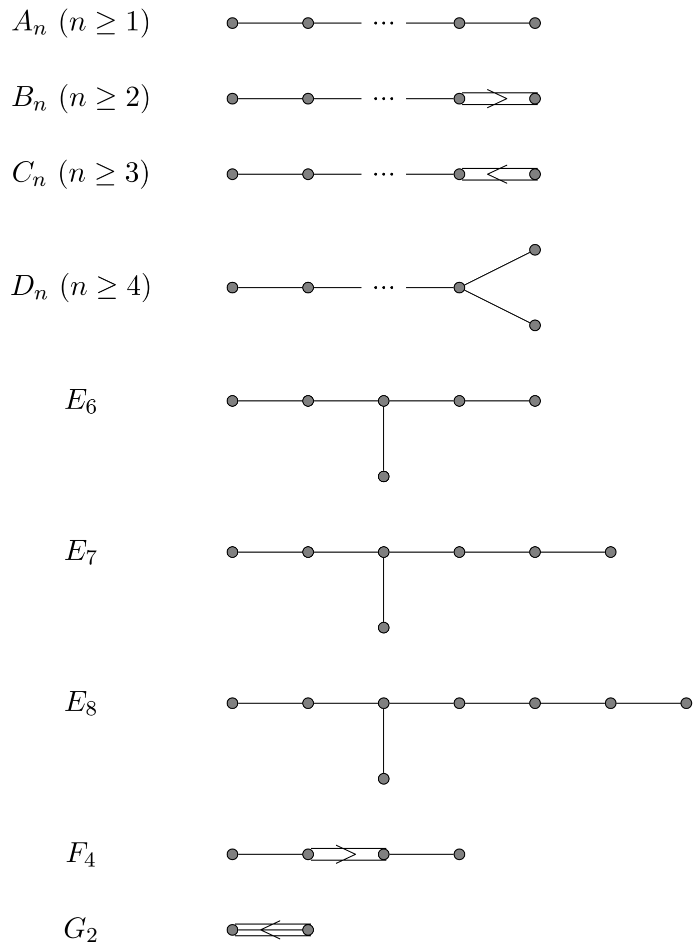

Figure 2.1.: Dynkin diagrams

the classification of irreducible root systems that simply-laced root systems have types A_n $(n \geq 1), D_n$ $(n \geq 4)$ or E_n $(n \in \{6, 7, 8\})$ in the classification. We will sometimes use the notation W_{X_n} for the finite reflection group with corresponding root system of type X_n for convenience.

Since the group W_Φ is a subgroup of $O(V)$, we obtain the following result.

Lemma 2.1.8. *Let Φ be a root system. Then for each $w \in W_\Phi$ and each $\alpha \in \Phi$ we have $(\alpha \mid \alpha) = (w(\alpha) \mid w(\alpha))$. In particular, if $\alpha \in \Phi$ is a short root (resp. a long root), then for each $w \in W$ also $w(\alpha)$ is a short root (resp. a long root).*

Let Φ be a crystallographic root system in V with simple system Δ. The set

$$\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\},$$

where $\alpha^\vee := \frac{2\alpha}{(\alpha|\alpha)}$, is again a crystallographic root system in V with simple system $\Delta^\vee := \{\alpha^\vee \mid \alpha \in \Delta\}$. The root system $\Phi^\vee$ is called dual root system and its elements are called coroots. (Note that in [Bou02] the dual root system is defined in the dual space V^* since they work with a not necessarily euclidean vector space.) Long roots (resp. short roots) in Φ

become short roots (resp. long roots) in $\Phi^\vee$. Note that $(\alpha^\vee)^\vee = \alpha$ and that the root systems of type B_n and C_n are dual to each other, that is, if Φ is of type B_n, then $\Phi^\vee$ is of type C_n and vice versa. Therefore the next result (see [Bou02, Ch. VI, 1.1]) implies that W_{B_n} and W_{C_n} are isomorphic.

Proposition 2.1.9. *The map* $W_\Phi \to W_{\Phi^\vee}, s_\alpha \mapsto s_{\alpha^\vee}$ *is an isomorphism.*

For a set of vectors $\Phi \subseteq V$ we set $L(\Phi) := \operatorname{span}_{\mathbb{Z}}(\Phi)$. If Φ is a root system, then $L(\Phi)$ is a lattice, called the root lattice. If Φ is a crystallographic root system, then $L(\Phi)$ is an integral lattice. In the latter case we call $L(\Phi^\vee)$ the coroot lattice. We will discuss (co)root lattices in more detail in Section 4.2.

Lemma 2.1.10. *For each* $w \in W$ *and each* $\alpha \in \Phi$ *we have* $w(\alpha)^\vee = w(\alpha^\vee)$.

Proof.

$$w(\alpha)^\vee = \frac{2w(\alpha)}{(w(\alpha) \mid w(\alpha))} = \frac{2w(\alpha)}{(\alpha \mid \alpha)} = w(\alpha^\vee),$$

where we used Lemma 2.1.8 to obtain the second equality. □

We already noted that each finite reflection group arises as the group associated to some root system. Conversely to each (finite) Coxeter system (W, S) we can associate a root system. We start with an arbitrary Coxeter system (W, S) and construct a faithful representation for W (see [Hum90, Ch. 5] for details). In general, we can not expect to do this for an euclidean vector space. Therefore we define V to be the vector space over $\mathbb{R}$ with abstract basis $\{\alpha_s \mid s \in S\}$. A linear transformation of V is called linear reflection if it fixes a hyperplane pointwise and sends some nonzero vector to its negative. By setting

$$\beta(\alpha_s, \alpha_t) = \begin{cases} -\cos(\frac{\pi}{m_{st}}) & \text{if } m_{st} < \infty \\ -1 & \text{if } m_{st} = \infty \end{cases}$$

we obtain a bilinear form on V. By setting

$$\sigma_s(v) := v - 2\beta(\alpha_s, v)\alpha_s$$

for each $s \in S$ and $v \in V$, we obtain a linear reflection. The unique homomorphism $\sigma : W \to \mathrm{GL}(V)$ sending s to σ_s is then a faithful representation for W ([Hum90, Corollary 5.4]), called geometric representation. Moreover the form β is positive definite if and only if W is finite ([Hum90, Theorem 6.4]).

Let (W, S) be a finite Coxeter system. We can identify (V, β) with a euclidean vector space. The set

$$\Phi_{(W,S)} := \{\sigma(w)(\alpha_s) \mid w \in W, s \in S\}$$

is a root system in V and $W_{\Phi_{(W,S)}} = \sigma(W) \cong W$. In fact, the finite Coxeter groups turn out to be precisely the finite reflection groups ([Hum90, Theorem 6.4]) and the finite, irreducible Coxeter systems can be classified by their Coxeter diagrams (see Figure 2.2). By abuse of notation we use the same name for both the directed diagram and its underlying undirected diagram. In particular, if X_n is a Dynkin diagram and (W, S) is a Coxeter system with Coxeter diagram X_n, then $W \cong W_{X_n}$. E.g. the Coxeter system given in Example 2.1.2 is of type A_n.

We can identify the set of simple reflections S with the set $\{s_\alpha \mid \alpha \in \Delta\}$ and the set of reflections T for (W, S) with the set $\{s_\alpha \mid \alpha \in \Phi\}$ of reflections for $\Phi = \Phi_{(W,S)}$. Note that $s_\alpha = s_{-\alpha}$ for all $\alpha \in \Phi$, thus T and the set $\{s_\alpha \mid \alpha \in \Phi^+\}$ can be identified.

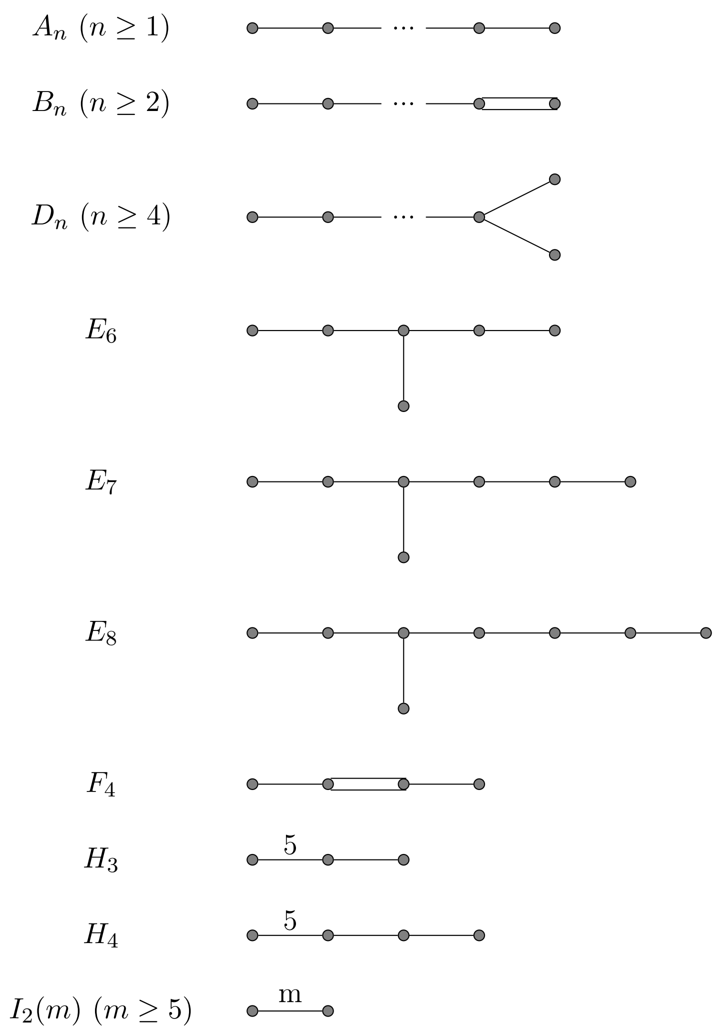

Figure 2.2.: Coxeter diagrams for the finite irreducible Coxeter systems

Remark 2.1.11. Let (W, S) be a finite irreducible Coxeter system of rank n. Then the group W has nice geometric properties. If W is a finite Weyl group (i.e. the associated root system is crystallographic), then it stabilizes a lattice in $\mathbb{R}^n$. For the non-crystallographic types H_3, H_4 and $I_2(m)$ $(m \neq 3, 4, 6)$ the group W is the group of symmetries of a regular polytope. Note that some groups have both properties, that is, they are Weyl groups and groups of symmetries of a regular polytope. Namely, if (W, S) is of type A_n, B_n, F_4 or $G_2 = I_2(5)$.

2.1.3. Affine Coxeter systems

For the content of this section as well as for details and proofs we refer to [Hum90, Chapter 4].

Throughout this section we fix an euclidean vector space V with positive definite symmetric bilinear form $(- \mid -)$ and a crystallographic root system Φ in V. We extend the idea of (orthogonal) reflections in (linear) hyperplanes to affine hyperplanes. For each $\alpha \in \Phi$ and each $k \in \mathbb{Z}$, the set

$$H_{\alpha,k} := \{v \in V \mid (v \mid \alpha) = k\}$$

defines an affine hyperplane. We have $H_{\alpha,k} = H_{-\alpha,-k}$ and $H_{\alpha,0}$ is a (linear) hyperplane. For each $x \in V$ we define the translation in x by

$$\mathrm{tr}(x) : V \to V, v \mapsto v + x.$$

The set $\mathrm{Tr}(V) := \{\mathrm{tr}(x) \mid x \in V\}$ of all translations by elements of V is a group. We have $H_{a,k} = \mathrm{tr}\left(\frac{k}{2}\alpha^\vee\right) H_\alpha$ and define the affine reflection $s_{\alpha,k}$ in the affine hyperplane $H_{\alpha,k}$ by

$$s_{\alpha,k} : V \to V, v \mapsto v - ((v \mid \alpha) - k)\alpha^\vee.$$

Then $s_{\alpha,k}$ fixes $H_{\alpha,k}$ pointwise and sends 0 to $k\alpha^\vee$. Moreover it is $s_{\alpha,k} = s_{-\alpha,-k}$.

For the next proposition see [Hum90, Proposition 4.1].

Proposition 2.1.12. *Let $w \in W_\Phi$, $\alpha \in \Phi$ and $k \in \mathbb{Z}$.*

(a) $wH_{\alpha,k} = H_{w(\alpha),k}$.

(b) $ws_{\alpha,k}w^{-1} = s_{w(\alpha),k}$.

Lemma 2.1.13. *Let $\alpha \in \Phi$ and $k, l \in \mathbb{Z}$.*

(a) $s_{\alpha,k} = \mathrm{tr}(k\alpha^\vee)s_\alpha = s_\alpha \,\mathrm{tr}(-k\alpha^\vee)$. *In particular we have $s_{\alpha,0} = s_\alpha$.*

(b) $s_\alpha s_{\alpha,1} = \mathrm{tr}(-\alpha^\vee)$.

(c) $s_{\alpha,k}s_{\alpha,l} = \mathrm{tr}((k-l)\alpha^\vee)$.

Proof. All assertions can be shown by direct calculations, we just show part (c). By Proposition 2.1.12 we have $s_{\alpha,k}s_\alpha = s_\alpha s_{\alpha,-k}$. Therefore we obtain

$$s_{\alpha,k}s_{\alpha,l} = s_{\alpha,k}s_\alpha \,\mathrm{tr}(-l\alpha^\vee) = s_\alpha s_{\alpha,-k} \,\mathrm{tr}(-l\alpha^\vee) = s_\alpha s_\alpha \,\mathrm{tr}(k\alpha^\vee)\,\mathrm{tr}(-l\alpha^\vee) = \mathrm{tr}((k-l)\alpha^\vee).$$

□

Let $\mathrm{Aff}(V)$ be the semidirect product of the general linear group $\mathrm{GL}(V)$ and $\mathrm{Tr}(V)$ which we call the affine group of V.

Definition 2.1.14. The group

$$W_a = W_{a,\Phi} := \langle s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\rangle \leq \mathrm{Aff}(V)$$

is called affine Weyl group associated to Φ.

Theorem 2.1.15. *([Hum90, Prop. 4.2, Theorem 4.6]) Let Φ be a crystallographic root system with simple system Δ.*

(a) The group W_a is the semidirect product of W_Φ and the group

$$\mathrm{Tr}(\Phi^\vee) := \{\mathrm{tr}(\alpha) \mid \alpha \in L(\Phi^\vee)\},$$

which we fluently identify with $L(\Phi^\vee)$.

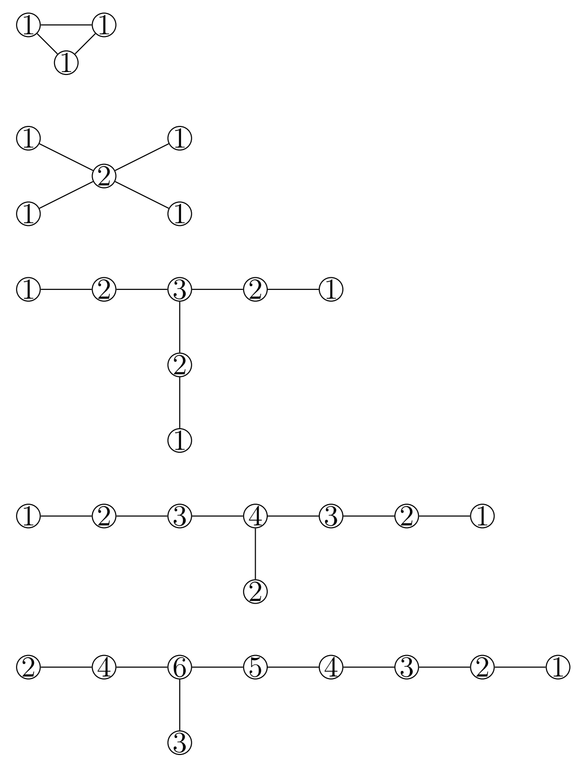

Figure 2.3.: Some extended Dynkin diagrams

(b) (W_a, S_a) *is a Coxeter system, where*

$$S_a = S_{a,\Delta} := \{s_\alpha \mid \alpha \in \Delta\} \cup \{s_{\widetilde{\alpha},1}\}.$$

Therefore we also call (W_a, S_a) *an* affine Coxeter system *and* W_a *an* affine Coxeter group.

Irreducible affine Coxeter systems (W_a, S_a) are classified by so-called extended Dynkin diagrams. They have types $\widetilde{A}_n$ $(n \geq 1)$, $\widetilde{B}_n$ $(n \geq 2)$, $\widetilde{C}_n$ $(n \geq 3)$, $\widetilde{D}_n$ $(n \geq 4)$, $\widetilde{E}_n$ $(n \in \{6, 7, 8\})$, $\widetilde{F}_4$ and $\widetilde{G}_2$. That is, the Coxeter diagram $\Gamma(W_a, S_a)$ is of the corresponding type. See Figure 2.3 for some examples of extended Dynkin diagrams. We refer to [Hum90, Section 2.5] for a complete list. The meaning of the numbers corresponding to the vertices will be explained in Section 7.3.2.

An irreducible affine Coxeter system (W_a, S_a) is said to be of type $\widetilde{X}_n$ if $\Gamma(W_a, S_a)$ is an extended Dynkin diagram of type $\widetilde{X}_n$.

Theorem 2.1.15 provides the following normal form for elements in W_a (see also [McCP11, Prop. 2.11]).

Lemma 2.1.16. *For each element* $\widetilde{w} \in W_a$ *there is a unique factorization* $\widetilde{w} = w_0 \operatorname{tr}(\lambda)$ *with* $w_0 \in W$ *and* $\lambda \in L(\Phi^\vee)$.

We adopt a notation frequently used in the literature and denote in the following an affine Coxeter system by $(\widetilde{W}, \widetilde{S})$ instead of (W_a, S_a).

2.2. Reflection subgroups

Let (W,S) be a Coxeter system with set of reflections T. A subgroup W' of W is called reflection subgroup if $W' = \langle W' \cap T \rangle$. The aim of this section is to show that W' admits a canonical set of generators $\chi(W')$ such that $(W', \chi(W'))$ is a Coxeter system. Furthermore we show that the Bruhat graph of $(W', \chi(W'))$ is given as the full subgraph of the Bruhat graph of (W,S) on vertex set W'. Both statements are due to Dyer and important for our later study of dual Coxeter systems and the Hurwitz action. The content of this section is covered by Dyer's thesis ([Dye87]).

For a subgroup $W' \leq W$ we define

$$\chi(W') := \{t \in W' \cap T \mid \ell_S(t't) > \ell_S(t) \ \forall t' \in W' \cap T \text{ with } t' \neq t\}.$$

We begin with stating the first theorem due to Dyer (see [Dye87, Theorem 1.8]).

Theorem 2.2.1. *Let (W,S) be a Coxeter system with set of reflections T and let W' be a reflection subgroup of W. Then W' is a Coxeter group with simple reflections $\chi(W')$. The set of reflections for $(W', \chi(W'))$ is given by $W' \cap T = \bigcup_{w \in W'} w\chi(W')w^{-1}$.*

Let W be a group and S a set of generators of W consisting of involutions. Put $T := \{wsw^{-1} \mid w \in W,\ s \in S\}$. Regard the power set $\mathcal{P}(T)$ of T as an abelian group via $A + B := (A \cup B) \setminus (A \cap B)$. Since $W = \langle S \rangle$, there exists at most one function $N : W \to \mathcal{P}(T)$ satisfying

(1.1) $N(s) = \{s\}$ for all $s \in S$

(1.2) $N(xy) = y^{-1}N(x)y + N(y)$ for all $x, y \in W$.

Lemma 2.2.2. *Let $N : W \to \mathcal{P}(T)$ be a function satisfying (1.1) and (1.2). Then:*

(a) If $w = s_1 \cdots s_n$ is S-reduced, then $N(w) = \{t_1, \ldots, t_n\}$, where
$t_i := s_n \cdots s_{i+1} s_i s_{i+1} \cdots s_n$.

(b) $|N(w)| = \ell_S(w)$ for all $w \in W$

(c) $t \in N(t)$ for all $t \in T$

(d) $N(w) = \{t \in T \mid \ell_S(wt) < \ell_S(w)\}$

(e) $\ell_S(wt) \neq \ell_S(w)$ for all $t \in T$.

Proof. We start with (a) and (b). Suppose $t_i = t_j$ for $i > j$. Then

$$\begin{aligned} w &= s_1 \cdots s_{j-1} \widehat{s_j} s_{j+1} \cdots s_n t_j \\ &= s_1 \cdots s_{j-1} \widehat{s_j} s_{j+1} \cdots s_n t_i \\ &= s_1 \cdots s_{j-1} \widehat{s_j} s_{j+1} \cdots s_{i-1} \widehat{s_i} s_{i+1} \cdots s_n, \end{aligned}$$

contradicting $w = s_1 \cdots s_n$ to be S-reduced. Therefore all t_i are distinct and thus

$$\begin{aligned} N(w) &\overset{(1.2)}{=} N(s_n) + s_n N(s_{n-1}) s_n + \cdots + s_n \cdots s_2 N(s_1) s_2 \cdots s_n \\ &\overset{(1.1)}{=} \{t_n\} + \{t_{n-1}\} + \cdots + \{t_1\} = \{t_1, \ldots, t_n\}, \end{aligned}$$

which proves (a) and (b).

Let $t = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1 \in T$ with $s_i \in S$ and n minimal. Define

$$(s'_1, \dots, s'_{n-1}, s'_n, s'_{n+1}, \dots, s'_{2n-1}) := (s_1, \dots s_{n-1}, s_n, s_{n-1}, \dots, s_1)$$

and $t_i := s'_{2n-1} \cdots s'_{i+1} s'_i s'_{i+1} \cdots s'_{2n-1}$ $(1 \leq i \leq 2n-1)$. If $1 \leq i \leq n$, then $t t_i t = t_{2n-i}$, thus $t_{2n-i} = t$ if and only if $t = t_i$. By the minimality of n we have $t = t_i$ if and only if $i = n$ $(1 \leq i \leq 2n-1)$. As before we obtain $N(t) = \{t_{2n-1}\} + \cdots + \{t_1\}$, hence $t \in N(t)$, which proves (c).

If $w = s_1 \cdots s_n$ is S-reduced and $t_i := s_n \cdots s_{i+1} s_i s_{i+1} \cdots s_n$, then $\ell_S(wt_i) < \ell_S(w)$. Hence for $t \in N(w) \overset{(a)}{=} \{t_1, \dots, t_n\}$ we have $\ell_S(wt) < \ell_S(w)$. This shows $N(w) \subseteq \{t \in T \mid \ell_S(wt) < \ell_S(w)\}$. Let $t \in T$ with $t \notin N(w)$. Then $t \notin tN(w)t$, but since $t \in N(t)$, we have $t \in tN(w)t + N(t) \overset{(1.2)}{=} N(wt)$. As before we obtain $\ell_S((wt)t) < \ell_S(wt)$, which proves (d) and (e). □

Lemma 2.2.3. *Let W be a group, S a set of generators of W consisting of involutions and $T = \{wsw^{-1} \mid w \in W,\ s \in S\}$. Then (W, S) is a Coxeter system if and only if there exists a function $N : W \to \mathcal{P}(T)$ satisfying (1.1) and (1.2).*

Proof. First let (W, S) be a Coxeter system. We use the characterization given in Proposition 2.1.6. For $w \in W$ put

$$N(w) := \{t \in T \mid w(t, \epsilon) = (wtw^{-1}, -\epsilon) \text{ for } \epsilon = \pm 1\}.$$

By Proposition 2.1.6 condition (1.1) follows immediately. It remains to show (1.2). Therefore note that for $s \in S, y \in W$ we have $N(sy) = \{y^{-1}sy\} + N(y)$ since $sy(y^{-1}sy, \epsilon) = (s, -\epsilon)$. The general assertion for $N(xy)$ with $x \in W$ follows by induction on $\ell_S(x)$.

Now assume that there exists a function $N : W \to \mathcal{P}(T)$ satisfying (1.1) and (1.2). We show that the (strong) exchange condition holds. Therefore let $w = s_1 \cdots s_n$ $(s_i \in S)$ and $t \in T$ such that $\ell_S(wt) < \ell_S(w)$. By Lemma 2.2.2 we have $t \in N(w)$. On the other hand we obtain by repeated application of (1.1) and (1.2) that $N(w) = \{t_n\} + \cdots + \{t_1\}$, where $t_i := s_n \cdots s_{i+1} s_i s_{i+1} \cdots s_n$. Thus $t = t_i$ for some i and $w = s_1 \cdots s_{i-1} s_i s_{i+1} \cdots s_n = s_1 \cdots s_{i-1} \widehat{s}_i s_{i+1} \cdots s_n t$. □

For the rest of this section let (W, S) be a Coxeter system.

Lemma 2.2.4. *Let $t = s_1 \cdots s_{2n+1} \in T$ $(s_i \in S)$ with $\ell_S(t) = 2n+1$. Then*

$$t = s_1 \cdots s_n s_{n+1} s_n \cdots s_1.$$

In particular if $n \geq 1$, that is $t \in T \setminus S$, there exists $s \in S$ with $\ell_S(sts) = \ell_S(t) - 2$.

Proof. For $x := s_n \cdots s_1$ and $y := s_{n+2} \cdots s_{2n+1}$ we have

$$\ell_S(x) = \ell_S(y) = n < n+1 = \ell_S(s_{n+1}x) = \ell_S(s_{n+1}y)$$

and $s_{n+1}y = xt$. Hence we can apply Theorem 2.1.5 to obtain $t = s_{2n+1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{2n+1}$ for some $n+1 \leq i \leq 2n+1$. Thus $x = s_{n+1} \cdots s_{i-1} s_{i+1} \cdots s_{2n+1}$ and this is S-reduced since $\ell_S(x) = n$. Since $\ell_S(s_{n+1}x) > \ell_S(x)$, we have $i = n+1$. Therefore $x = y$ and $t = x^{-1} s_{n+1} x$. □

Proposition 2.2.5. $\chi(W') = \{t \in T \mid N(t) \cap W' = \{t\}\}$

Proof. By Lemma 2.2.2 we have for $t \in T$ that

$$N(t) \cap W' = \{t' \in T \mid \ell_S(tt') < \ell_S(t)\} \cap W'.$$

Therefore the condition $N(t) \cap W' = \{t\}$ yields for $t' \in T \cap W'$ that $\ell_S(tt') < \ell_S(t)$ if and only if $t' = t$. Thus we have $\ell_S(tt') > \ell_S(t)$ for all $t' \in T \cap W'$ with $t \neq t'$. Since $(tt')^{-1} = t't$, we have $\ell_S(tt') = \ell_S(t't)$ and the assertion follows. □

Lemma 2.2.6. *Let (W, S) be a Coxeter system with set of reflections T, $N : W \to \mathcal{P}(T)$ a function satisfying (1.1) and (1.2) and $W' \leq W$ a subgroup.*

(a) If $s \in S \setminus W'$, then $\chi(sW's) = s\chi(W')s$.

(b) If $t \in W' \cap T$, then there exist $m \in \mathbb{Z}_{\geq 0}$ and $t_0, \dots, t_m \in \chi(W')$ such that $t = t_m \cdots t_1 t_0 t_1 \cdots t_m$.

(c) For $w \in W$ let $N'(w) := N(w) \cap W'$. If $x \in W, y \in W'$, then $N'(xy) = y^{-1}N'(x)y + N'(y)$.

Proof. Let $s \in S \setminus W'$ and $t \in \chi(W')$, that is $N(t) \cap W' = \{t\}$. Using (1.2) we obtain

$$N(sts) = sN(st)s + N(s) = s(tN(s)t + N(t))s + N(s) = \{ststs\} + sN(t)s + \{s\}.$$

This yields

$$N(sts) \cap sW's = s((\{tst\} + N(t) + \{s\}) \cap W')s \overset{(s \notin W')}{=} s(N(t) \cap W')s = \{sts\}.$$

Therefore $sts \in \chi(sW's)$, which shows that $s\chi(W')s \subseteq \chi(sW's)$. But since $s \in S \setminus sW's$, we can use the same argument as before (with $sW's$ instead of W') to obtain $s\chi(sW's)s \subseteq \chi(W')$. Thus $\chi(sW's) \subseteq s\chi(W')s$ which shows (a).

For part (b) let $t \in W' \cap T$ and proceed by induction on $\ell_S(t)$. If $\ell_S(t) = 1$, we can put $t_0 = t$ since $t \in W' \cap S$. Therefore let $\ell_S(t) > 1$. By Lemma 2.2.4 there exists $s \in S$ such that $\ell_S(sts) < \ell_S(t)$. Put $W'' := sW's$ and $t'' := sts$. By induction there exist $m \in \mathbb{Z}_{\geq 0}$, $t_0, \dots, t_m \in \chi(W'')$ such that $t'' = t_m \cdots t_1 t_0 t_1 \cdots t_m$. We distinguish two cases. If $s \in W''$, then $W' = W''$ and $N(s) \cap W'' = \{s\}$. The decomposition $t = st''s = st_m \cdots t_1 t_0 t_1 \cdots t_m s$ yields the assertion. Therefore let $s \notin W''$. Put $t'_i := st_i s$ for $0 \leq i \leq m$. By part (a) we obtain $t'_i \in s\chi(W'')s = \chi(sW''s) = \chi(W')$. Since $t = st''s = t'_m \cdots t'_1 t'_0 t'_1 \cdots t'_m$, the assertion follows. To show part (c) let $x \in W$ and $y \in W'$. Since $W' = y^{-1}W'y$, we obtain

$$\begin{aligned} N'(xy) &= (y^{-1}N(x)y + N(y)) \cap W' \\ &= (y^{-1}N(x)y \cap y^{-1}W'y) + (N(y) \cap W') \\ &= y^{-1}(N(x) \cap W')y + (N(y) \cap W') \\ &= y^{-1}N'(x)y + N'(y). \end{aligned}$$

□

Proof of Theorem 2.2.1. Set $S' := \chi(W')$, $W'' := \langle S' \rangle$ and $T' := \cup_{w \in W''} wS'w^{-1}$. It is $S' \subseteq T \cap W'$, thus $W'' \subseteq W'$ and $T' \subseteq T \cap W'$. By Lemma 2.2.6 (*b*) we have $T \cap W' \subseteq T'$, hence $T \cap W' = T'$. Since T' is the union of W''-conjugates of S' and $W'' = \langle S' \rangle$, we have $\langle T' \rangle \subseteq \langle S' \rangle$. It follows $W' = \langle T' \rangle \subseteq \langle S' \rangle = W'' \subseteq W'$, thus $W' = \langle S' \rangle = W''$. Therefore

$$T \cap W' = T' = \cup_{w \in W''} wS'w^{-1} = \cup_{w \in W'} w\chi(W')w^{-1}.$$

It remains to show that (W', S') is a Coxeter system. We use the characterization as given in Lemma 2.2.3. Since $T' \subseteq T$, the group $\mathcal{P}(T')$ is a subgroup of $\mathcal{P}(T)$. Define the function

$$N' : W' \to \mathcal{P}(T'), w \mapsto N(w) \cap W'.$$

Let $s' \in S'$. By definition of S' we obtain $N'(s) = N(s) \cap W' = \{s\}$, thus (1.1) holds for N'. By Lemma 2.2.6 (c) condition (1.2) holds for N'. Lemma 2.2.3 yields that (W', S') is a Coxeter system. □

As an immediate consequence of Theorem 2.2.1 and Lemma 2.2.2 we obtain:

Corollary 2.2.7. *Let $W' \leq W$ be a reflection subgroup and $S' := \chi(W')$ be its set of Coxeter generators. For $w \in W'$ we have $N(w) \cap W' = \{t \in W' \cap T \mid \ell_{S'}(wt) < \ell_{S'}(w)\}$, where $\ell_{S'}$ is the length function of (W', S').*

Definition 2.2.8. Let (W, S) be a Coxeter system with set of reflections T. We define the Bruhat graph of (W, S) to be the directed graph $\Omega_{(W,S)}$ on vertex set W and there is a directed edge from x to y if there exists $t \in T$ such that $y = xt$ and $\ell_S(x) < \ell_S(y)$.

For a subset $X \subseteq W$ we denote by $\Omega_{(W,S)}(X)$ the full subgraph of $\Omega_{(W,S)}$ on vertex set X.

Moreover, we denote by $\overline{\Omega}_{(W,S)}$ the corresponding undirected graph. Since this graph does not depend on S, but on T, we sometimes denote it by $\overline{\Omega}_{(W,T)}$.

Proposition 2.2.9. *Let $W' \leq W$ be a reflection subgroup and $\chi(W')$ be its set of Coxeter generators. Then $\Omega_{(W',\chi(W'))} = \Omega_{(W,S)}(W')$.*

Proof. Put $T' := T \cap W'$, $S' := \chi(W')$ and $N'(x) := N(x) \cap W'$ for $x \in W$. Since both graphs $\Omega_{(W',S')}$ and $\Omega_{(W,S)}(W')$ share by definition the same vertex set, it remains to show that they have the same set of edges. The set of edges of the graph $\Omega_{(W,S)}(W')$ is given by

$$\begin{aligned} &\{(x,y) \in W' \times W' \mid x^{-1}y \in T,\ \ell_S(x) < \ell_S(y)\} \\ =&\{(x,y) \in W' \times W' \mid x^{-1}y \in T',\ x^{-1}y \notin N(x)\} \\ =&\{(x,y) \in W' \times W' \mid x^{-1}y \in T',\ x^{-1}y \notin N'(x)\} \\ =&\{(x,y) \in W' \times W' \mid x^{-1}y \in T',\ \ell_{S'}(x) < \ell_{S'}(y)\}, \end{aligned}$$

which is precisely the set of edges of the graph $\Omega_{(W',S')}$. □

Theorem 2.2.10. *Let (W, S) be a Coxeter system and W' a reflection subgroup of W. For any $x \in W$ the coset xW' contains a unique element x_0 with $\ell_S(x_0)$ minimal, and the map*

$$\theta : W' \to xW', v \mapsto x_0 v$$

induces an isomorphism between the Bruhat graph $\Omega_{(W',\chi(W'))}$ and the full subgraph of $\Omega_{(W,S)}$ on vertex set xW'.

Proof. Choose $x_0 \in xW'$ with $\ell_S(x_0)$ minimal and put $T' = W' \cap T$, $S' = \chi(W')$. For any $t \in T'$ we have $x_0 t \in xW'$. By the minimality of x_0 we get $\ell_S(x_0 t) \geq \ell_S(x_0)$. Therefore we obtain

$$N'(x_0) = N(x_0) \cap W' = \{t \in T \mid \ell_S(x_0 t) < \ell_S(x_0)\} \cap W' = \varnothing.$$

For any $w \in W'$ we have

$$N'(\theta(w)) = N'(x_0 w) \overset{2.2.6\ (c)}{=} w^{-1} N'(x_0) w + N'(w) = N'(w). \tag{2.1}$$

Since θ is a bijection between the sets W' and xW', it remains to show that, if $y, z \in W'$ then (y, z) is an edge in $\Omega_{(W,S)}(W') = \Omega_{(W',S')}$ if and only if $(\theta(y), \theta(z))$ is an edge in $\Omega_{(W,S)}(xW') = \Omega_{(W,S)}(x_0 W')$. The set of edges for $\Omega_{(W',S')}$ is given by (see also the proof of Proposition 2.2.9)

$$\begin{aligned} &\{(y,z) \in W' \times W' \mid y^{-1}z \in T,\ \ell_S(y) < \ell_S(z)\} \\ =\ &\{(y,z) \in W' \times W' \mid y^{-1}z \in T', y^{-1}z \notin N'(y)\} \\ \overset{(2.1)}{=} &\{(y,z) \in W' \times W' \mid y^{-1}z \in T', y^{-1}z \notin N'(\theta(y))\} \end{aligned}$$

Since $y^{-1}z = (x_0 y)^{-1}(x_0 z) = \theta(y)^{-1}\theta(z)$, we obtain that (y, z) is an edge in $\Omega_{(W',S')}$ if and only if $\theta(y)^{-1}\theta(z) \in T'$ and $\theta(y)^{-1}\theta(z) \notin N'(\theta(y))$. The latter being equivalent to $\theta(y)^{-1}\theta(z) \in T$ and $\ell_S(\theta(y)) < \ell_S(\theta(z))$, which is precisely the definition for $(\theta(y), \theta(z))$ being an edge of $\Omega_{(W,S)}(xW')$.

It remains to show that x_0 is unique. Therefore we show that $\ell_S(x_0) < \ell_S(x_0 w)$ for all $w \in W' \setminus \{e\}$. Let $w \in W' \setminus \{e\}$ and choose $w' = s'_1 \cdots s'_n$ to be S'-reduced. Put $w_0 := e$ and $w_i := s'_1 \cdots s'_i$ for $1 \leq i \leq n$. Therefore we get that (w_{i-1}, w_i) is an edge of $\Omega_{W',S'}$ for all $i \in \{1, \ldots, n\}$. Hence also an edge of $\Omega_{(W,S)}(W')$ by Proposition 2.2.9. Using what we have observed above, this is equivalent to $(\theta(w_{i-1}), \theta(w_i)) = (x_0 w_{i-1}, x_0 w_i)$ being an edge of $\Omega_{(W,S)}(xW')$ for all $i \in \{1, \ldots, n\}$. Thus

$$\ell_S(x_0) = \ell_S(x_0 w_0) < \ell_S(x_0 w_1) < \ldots < \ell_S(x_0 w_n) = \ell_S(x_0 w),$$

which shows the uniqueness of x_0. □

2.3. The absolute length

Let (W, S) be a Coxeter system. Since the set of reflections T contains the set of simple reflections S, it also generates W. Therefore T induces a length function ℓ_T on W, called absolute length. Let $w \in W$. As before a decomposition $w = t_1 \cdots t_r$ with $t_i \in T$ is called T-reduced if $\ell_T(w) = r$. We moreover define for an element $w \in W$ with $\ell_T(w) = r$ its set of reduced T-decompositions, that is

$$\mathrm{Red}_T(w) := \{(t_1, \ldots, t_r) \mid t_i \in T,\ w = t_1 \cdots t_r\}.$$

We give two criteria for a T-decomposition to be reduced. One is a geometric criterion for finite Coxeter groups due to Carter, the other criterion is of combinatorial nature. It is valid for arbitrary Coxeter groups and due to Dyer. For a proof of the first statement we refer to [Car72, Lemma 3].

Lemma 2.3.1. *Let (W, S) be a finite Coxeter system with set of reflections T and corresponding root system $\Phi = \Phi_{(W,S)}$. Let $\alpha_1, \dots, \alpha_k \in \Phi$. Then $s_{\alpha_1} \cdots s_{\alpha_k}$ is T-reduced if and only if the roots $\alpha_1, \dots, \alpha_k$ are linearly independent.*

As a direct conclusion we obtain the following statement.

Corollary 2.3.2. *Let (W, S) be a finite Coxeter system of rank n with set of reflections T. Then $\ell_T(w) \leq n$ for all $w \in W$.*

The rest of this subsection is based on [Dye01].

Theorem 2.3.3. *Let (W, S) be a Coxeter system with set of reflections T. If $w = s_1 \cdots s_n$ is S-reduced, then $\ell_T(w)$ is the minimum of natural numbers p for which there exists $1 \leq i_1 < \ldots < i_p \leq n$ such that*

$$e = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_p}} \cdots s_n,$$

where a factor $\widehat{s_{i_j}}$ is omitted.

For the rest of this subsection we fix a Coxeter system (W, S) with set of reflections T. As a direct consequence of the definition of the Bruhat graph (Definition 2.2.8) we obtain the following.

Proposition 2.3.4. *For $w \in W$ the absolute length $\ell_T(w)$ is precisely the minimal length of a path in $\overline{\Omega}_{(W,S)}$ from e to w.*

Definition 2.3.5. The partial order on W defined by $x \leq y$ if there is a path from x to y in $\Omega_{(W,S)}$ is called Bruhat order.

Let $y = s_1 \cdots s_n$ be S-reduced. As an immediate consequence of the strong exchange condition (Theorem 2.1.5) we obtain that $x \leq y$ if and only if $x = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_p}} \cdots s_n$ for some $1 \leq i_1 < \ldots < i_p \leq n$.

Proposition 2.3.6. *Let $w \in W$ and $t_1, t_2 \in T$ with $t_1 \neq t_2$ such that*

$$w \longrightarrow wt_1 \longleftarrow wt_1t_2$$

in $\Omega_{(W,S)}$. Then there exist $t_1', t_2' \in \langle t_1, t_2 \rangle \cap T$ with $t_1t_2 = t_1't_2'$ such that one of the following cases hold:

1. $w \longrightarrow wt_1' \longrightarrow wt_1't_2' = wt_1t_2$
2. $w \longleftarrow wt_1' \longleftarrow wt_1't_2' = wt_1t_2$
3. $w \longleftarrow wt_1' \longrightarrow wt_1't_2' = wt_1t_2$

Proof. Let $W' := \langle t_1, t_2 \rangle$ and $S' := \chi(W')$. We consider the coset wW' since $w, wt_1, wt_1t_2 \in wW'$. By Theorem 2.2.10 we have

$$\Omega_{(W,S)}(W') \cong \Omega_{(W,S)}(wW') \cong \Omega_{(W',S')},$$

where (W', S') is dihedral and we can check the claim there directly. Inside W' any reflection (element of odd S'-length) and any rotation (element of even S'-length) are joined by an edge in $\overline{\Omega}_{(W',S')}$ which in $\Omega_{(W',S')}$ is oriented towards the element of greater S'-length. For $x \in W'$ there are three possible situations:

- $\ell_{S'}(x) < \ell_{S'}(xt_1t_2)$
- $\ell_{S'}(x) > \ell_{S'}(xt_1t_2)$
- $\ell_{S'}(x) = \ell_{S'}(xt_1t_2)$ (in particular $x \neq e$ since $t_1 \neq t_2$).

Hence we can choose $t_1', t_2' \in W' \cap T$ with $t_1't_2' = t_1t_2$ in the three situations such that we have one of the following situations:

- $x \longrightarrow xt_1' \longrightarrow xt_1't_2'$
- $x \longleftarrow xt_1' \longleftarrow xt_1't_2'$
- $x \longleftarrow xt_1' \longrightarrow xt_1't_2'$

To see this note that x and xt_1t_2 are both either reflections or rotations. Therefore both are either of odd or even S'-length. Thus $\ell_{S'}(x) < \ell_{S'}(xt_1t_2)$ implies $\ell_{S'}(x)+2 \leq \ell_{S'}(xt_1t_2)$ and we find t_1' with $\ell_{S'}(x) < \ell_{S'}(xt_1') < \ell_{S'}(xt_1t_2)$. By setting $t_2' := t_1't_1t_2$ we obtain $x \longrightarrow xt_1' \longrightarrow xt_1't_2'$ and $t_1't_2' = t_1t_2$ (see also the example below). The remaining cases are similar. □

Example 2.3.7. Consider a dihedral Coxeter system (W, S) with $S = \{s_1, s_2\}$ and s_1s_2 of order $n \geq 8$. Let $x = s_1s_2s_1s_2$, $t_1 = s_2$ and $t_2 = s_2s_1s_2s_1s_2$. Then $xt_1t_2 = s_1s_2s_1s_2s_1s_2s_1s_2$ is S-reduced and

$$\ell_S(x) < \ell_S(xt_1t_2) \leq n.$$

Hence we have

$$x \longleftarrow xt_1 \longrightarrow xt_1t_2.$$

Setting $t_1' := s_1$ and $t_2' := t_1't_1t_2 = s_2s_1s_2$, we obtain $t_1t_2 = t_1't_2'$ and

$$x \longrightarrow xt_1' \longrightarrow xt_1't_2'.$$

Lemma 2.3.8. *Let $x \in W$. If there exists a path from e to x in $\overline{\Omega}_{(W,S)}$ of length n, then there exists a path from e to x in $\Omega_{(W,S)}$ of length n' for some $n' \leq n$. In particular, for any $w \in W$, $\ell_T(w)$ is the minimal length of a path in $\Omega_{(W,S)}$ from e to w.*

Proof. We proceed by induction on n. Let $n = 1$. Since $\ell_S(e) = 0$ an edge between e and $x \neq e$ has to be directed towards x. Therefore consider a path

$$e = x_0 - x_1 - \ldots - x_{n-1} - x_n = x$$

of length $n > 1$ between e and x. By the induction assumption there exists a directed path from e to x_{n-1} of length $\leq n - 1$. Replacing n by a smaller integer if necessary, we can still assume that $n > 1$ and that

$$x_0 \longrightarrow x_1 \longrightarrow \ldots \longrightarrow x_{n-1}$$

in $\Omega_{(W,S)}$. If $x_n = x_{n-2}$ or $\ell_S(x_n) > \ell_S(x_{n-1})$, we are already done. Therefore assume $x_n \neq x_{n-2}$ and $\ell_S(x_n) < \ell_S(x_{n-1})$. The actual situation is as follows

$$x_{n-2} \longrightarrow x_{n-1} \longleftarrow x_n.$$

Hence by definition of $\Omega_{(W,S)}$ and since $x_{n-2} \neq x_n$ there exist $t_1 \neq t_2$ in T such that

$$x_{n-2} \longrightarrow x_{n-1} = x_{n-2}t_1 \longleftarrow x_n = x_{n-2}t_1t_2.$$

We apply Proposition 2.3.6. If the first case therein holds, we are done. If the second or third case holds, then there are $t_1', t_2' \in \langle t_1, t_2\rangle \cap T$ with $t_1t_2 = t_1't_2'$ such that

$$\begin{aligned} x_{n-2} &\leftarrow x_{n-2}t_1' \leftarrow x_{n-2}t_1t_2 = x_{n-2}t_1't_2' \\ \text{or } x_{n-2} &\leftarrow x_{n-2}t_1' \rightarrow x_{n-2}t_1t_2 = x_{n-2}t_1't_2'. \end{aligned}$$

If we are in the second situation, we can apply the induction assumption to find a directed path from e to $x_{n-2}t_1'$ and hence from e to $x_{n-2}t_1't_2' = x_n$. Therefore assume that we are in the first situation. Put $x_{n-1}' := x_{n-2}t_1'$. Since there is an edge $x_{n-1}' \rightarrow x_{n-2}$, we have $\ell_S(x_{n-1}') < \ell_S(x_{n-2})$ and since there is an edge $x_{n-2} \rightarrow x_{n-1}$, we have $\ell_S(x_{n-2}) < \ell_S(x_{n-1})$, thus $\ell_S(x_{n-1}') < \ell_S(x_{n-1})$. We apply the induction assumption to the path

$$e \rightarrow x_1 \rightarrow \cdots \rightarrow x_{n-2} \leftarrow x_{n-1}' \leftarrow x_n$$

to obtain a path

$$e \rightarrow x_1' \rightarrow \cdots \rightarrow x_{n-2}' \rightarrow x_{n-1}' \leftarrow x_n.$$

Again we can apply Proposition 2.3.6 to the subpath $x_{n-2}' \rightarrow x_{n-1}' \leftarrow x_n$. Without loss of generality we assume that we find $t_1'', t_2' \in \langle t_1', t_2'\rangle \cap T$ with $t_1't_2' = t_1''t_2''$ such that

$$x_{n-2}' \leftarrow x_{n-1}'' := x_{n-2}'t_1'' \leftarrow x_{n-2}'t_1''t_2'' = x_n.$$

This yields $\ell_S(x_{n-1}'') < \ell_S(x_{n-1}') < \ell_S(x_{n-1})$. Since ℓ_S is bounded from below, continuing in this manner eventually yields a directed path from e to x_n. □

Proof of Theorem 2.3.3. To see the minimality of $\ell_T(w)$ let $w = s_1 \cdots s_n$ be S-reduced and set $t_i := s_1 \cdots s_{i-1}s_is_{i-1} \cdots s_1$. If $e = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_p}} \cdots s_n$. then also $e = t_{i_1} \cdots t_{i_p}w$ and hence $w = t_{i_1} \cdots t_{i_p}$. Therefore $p \geq \ell_T(w)$.

To see existence let $\ell_T(w) = p$. By Lemma 2.3.8 there is a path

$$x_0 = e \rightarrow x_1 \rightarrow \ldots \rightarrow x_p = w$$

of length p in $\Omega_{(W,S)}$. If we have $y \rightarrow x$ in $\Omega_{(W,S)}$ then, by the strong exchange condition, a reduced S-decomposition of y is obtained from a reduced S-decomposition of x by deleting one simple reflection. By descending induction on i we conclude that x_i has a reduced decomposition obtained by deleting $p-i$ simple reflections from the reduced decomposition $s_1 \ldots s_n$ for w. For $i = 0$, thus deleting p simple reflections, the assertion follows. □

2.4. Dual Coxeter systems

2.4.1. The dual set up

Let (W, T) be a pair consisting of a group W and a generating subset T of W. In the sense of [Bes03] we call (W, T) a dual Coxeter system of finite rank n if there is a subset $S \subseteq T$ with $|S| = n$ such that (W, S) is a Coxeter system, and $T = \{wsw^{-1} \mid w \in W, s \in S\}$ is the set of reflections for the Coxeter system (W, S). We then call (W, S) a simple system for (W, T). If $S' \subseteq T$ is such that (W, S') is a Coxeter system, then $\{wsw^{-1} \mid w \in W, s \in S'\} = T$ (see [BMMN02, Lemma 3.7]). Hence a set $S' \subseteq T$ is a simple system for (W, T) if and only if

(W, S') is a Coxeter system. The rank of (W, T) is defined as $|S|$ for a simple system $S \subseteq T$. This is well-defined by [BMMN02, Theorem 3.8].

So far, there is no formalism which justifies the name *dual* Coxeter system. It is based on some numerology (see [Bes03, Section 5]) which provides an indication that there actually might be some kind of duality.

Simple systems for (W, T) have been studied by several authors (see [FHM06]). Clearly, if S is a simple system for (W, T), then so is wSw^{-1} for any $w \in W$. Moreover, it is shown in [FHM06] that for an important class of infinite Coxeter groups including the irreducible affine Coxeter groups, all simple systems for (W, T) are conjugate to one another in this sense.

The following result is well-known and follows from [Dye87] or [Dye90].

Proposition 2.4.1. *Let (W, S) be a (not necessarily finite) Coxeter system of rank n. Then W cannot be generated by less than n reflections.*

Proof. Assume that $W = \langle t_1, \dots, t_k \rangle$, with $k \leq n$. It follows from [Dye90, Corollary 3.11 (i)] that $|\chi(W)| \leq k$, where $\chi(W)$ is the set of simple reflections defined in Section 2.2. But by definition the set $\chi(W)$ contains S. Hence we have $|\chi(W)| \geq n$. ☐

Let W' be a reflection subgroup of W. By Theorem 2.2.1 $(W', W' \cap T)$ is again a dual Coxeter system. The reflection subgroup generated by $\{s_1, \dots, s_m\} \subseteq T$ is called a parabolic subgroup for (W, T) if there is a simple system $S = \{s_1, \dots, s_n\}$ for (W, T) with $m \leq n$. This definition differs from the usual notion of a parabolic subgroup generated by a conjugate of a subset of a fixed simple system S (see [Hum90, Section 1.10]) which we call a classical parabolic subgroup for (W, S). Obviously a parabolic subgroup as defined here is a classical parabolic subgroup. But the two definitions are not equivalent in general (see [Gob, Example 2.2]). However we prove in Section 2.4.2 the equivalence of the definitions for a class of Coxeter groups which in particular includes the finite and affine Coxeter groups.

Definition 2.4.2. The partial order on W defined by

$$u \leq_T v \text{ if and only if } \ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v)$$

for $u, v \in W$ is called absolute order.

Definition 2.4.3. Let (W, T) be a dual Coxeter system and $S = \{s_1, \dots, s_n\}$ be a fixed simple system for (W, T).

(a) We say that $c \in W$ is a classical Coxeter element if c is conjugate to some $s_{\pi(1)} \cdots s_{\pi(n)}$ for π an element of the symmetric group $\mathrm{Sym}(n)$. An element $w \in W$ is a classical parabolic Coxeter element if $w \leq_T c$ for some classical Coxeter element c.

(b) An element $c \in W$ is a Coxeter element if there exists a simple system $S' = \{s'_1, \dots, s'_n\}$ for (W, T) such that $c = s'_1 \cdots s'_n$. An element $w \in W$ is a parabolic Coxeter element if there exists a simple system $S' = \{s'_1, \dots, s'_n\}$ for (W, T) such that $w = s'_1 \cdots s'_m$ for some $m \leq n$. The element w is moreover called a standard parabolic Coxeter element for the Coxeter system (W, S').

Remark 2.4.4. (a) The definition of classical Coxeter element depends on the choice of a simple system (W, S). If the Coxeter diagram $\Gamma(W, S)$ is a forest, then the set of classical Coxeter elements forms a single conjugacy class (see [Bou02, V.6, Lemma 1]). In particular this holds for all finite Coxeter systems.

(b) If (W,T) is finite, then an element $w \in W$ is a parabolic Coxeter element if and only if $w \leq_T c$ for some Coxeter element c (see [DG17, Corollary 3.6]). It is clear that a classical Coxeter element is a Coxeter element and hence that a classical parabolic Coxeter element is a parabolic Coxeter element. The difference between classical Coxeter elements and Coxeter elements is somewhat subtle. For finite Weyl groups the two definitions are equivalent, as a consequence of [RRS17, Theorem 1.8(ii) and Remark 1.10]. It seems however that no case-free proof of this fact is known. An example where these two definitions differ is the dihedral group $I_2(5)$ (see [BDSW14, Remark 1.1]).

In Section 2.1.2 we introduced the notion of irreducibility for a Coxeter system (W,S) and observed that finite irreducible and affine irreducible Coxeter systems are classified by certain types. We would like to do the same for dual Coxeter systems.

Definition 2.4.5. Let (W,S) be a Coxeter system with set of reflections T. If for each simple system $S' \subseteq T$ the Coxeter diagram $\Gamma(W,S')$ equals the Coxeter diagram $\Gamma(W,S)$, then the Coxeter system (W,S) is called reflection rigid. If moreover S and S' are conjugated under W, the Coxeter system (W,S) is called strongly reflection rigid. A dual Coxeter system (W,T) is called (strongly) reflection rigid if (W,S) is (strongly) reflection rigid for one (equivalently each) simple system $S \subseteq T$.

In fact, finite Coxeter systems are reflection rigid (see [BMMN02, Theorem 3.10]). For affine Coxeter groups there is even more to say. Affine Coxeter groups are strongly rigid (see [CD00], [FHM06], [Müh05, Theorem 3.2] or Section 2.4.2 for more details), that is, any two simple systems for W are conjugate. In particular affine Coxeter systems are (strongly) reflection rigid. Therefore we call (W,T) a finite dual Coxeter system (resp. an affine dual Coxeter system) if (W,S) is finite (resp. affine) for one (equivalently each) simple system $S \subseteq T$. In both cases we define the type of (W,T) to be the type of (W,S) for some (equivalently each) simple system $S \subseteq T$.

A reflection rigid dual Coxeter system (W,T) is called irreducible if (W,S) is irreducible for one (equivalently each) simple system $S \subseteq T$.

Considering finite Coxeter groups, in most cases, the type is determined by the group itself. There are only two exceptions, namely $W_{B_{2k+1}} \cong W_{A_1} \times W_{D_{2k+1}}$ ($k \geq 1$) and $W_{I_2(4k+2)} \cong W_{A_1} \times W_{I_2(2k+1)}$ ($k \geq 1$), which follows from the classification of the finite irreducible Coxeter groups and [Nui06, Theorem 2.17, Lemma 2.18 and Theorem 3.3].

Notice that (W,T) is strongly reflection rigid if and only if every Coxeter element is a classical Coxeter element. Hence Remark 2.4.4 (b) implies that (W,T) is strongly reflection rigid if W is a finite Weyl group.

We give some equivalent characterizations of the absolute order.

Proposition 2.4.6. *Let (W,T) be a dual Coxeter system, $t \in T$ and $w \in W$ with $\ell_T(w) = k$. Then the following conditions are equivalent:*

(a) $t \leq_T w$

(b) $\ell_T(tw) = k - 1$

(c) $\ell_T(wt) = k - 1$

(d) There exists a T-reduced decomposition $w = tt_2 \cdots t_k$

(e) There exists a T-reduced decomposition $w = t'_2 \cdots t'_k t$

Proof. By definition of the absolute order, (b) follows from (a). That (b) implies (c) is a consequence of the fact, that the length function ℓ_T is invariant under conjugation. Assume (c). Then there exist $t'_2, \ldots, t'_k \in T$ with $wt = t'_2 \cdots t'_k$, thus $w = t'_2 \cdots t'_k t$, which shows (e). If $w = t'_2 \cdots t'_k t$, then also $w = t(tt'_2 t) \cdots (tt'_k t)$, which shows that (e) implies (d). That (d) implies (a) is a direct consequence of the definition of the absolute order. □

2.4.2. Equivalent definitions of parabolic subgroups

The aim of this subsection is to show that the definitions of parabolic subgroups and classical parabolic subgroup are equivalent for finite and infinite irreducible 2-spherical Coxeter systems (see [BGRW17, Corollary 4.4, Proposition 4.6]). An infinite irreducible Coxeter system (W, S) is called 2-spherical if S is finite, and ss' has finite order for every $s, s' \in S$.

As an immediate consequence of the definition of strongly reflection rigidity, we obtain the following statement.

Proposition 2.4.7. *Let (W, T) be a strongly reflection rigid dual Coxeter system, $S \subseteq T$ a simple system and $W' \leq W$ a subgroup. Then W' is a parabolic subgroup for (W, T) if and only if W' is a classical parabolic subgroup for (W, S).*

We already noted that affine Coxeter systems are strongly reflection rigid and hence both definitions of parabolic subgroups are equivalent. However, Caprace and Mühlherr showed that infinite irreducible 2-spherical Coxeter systems are strongly reflection rigid (see [CM07]). Hence:

Corollary 2.4.8. *Let (W, S) be an infinite irreducible 2-spherical Coxeter system. Then a subgroup of W is parabolic if and only if it is parabolic in the classical sense.*

In fact, each irreducible affine Coxeter system, except $\widetilde{A}_1$, is 2-spherical. For (W, S) of type $\widetilde{A}_1$, i.e W is a dihedral group of infinite order, the equivalence of both definitions is immediate since the non-trivial parabolic subgroups are precisely those subgroups that are generated by a single reflection.

We also already noted that (W, T) is strongly reflection rigid if W is a finite Weyl group. However, finite Coxeter systems of type H_3 or $I_2(5)$ are reflection rigid, but not strongly reflection rigid. Therefore and as an important part of our considerations in Section 4, we prove the equivalence of both definitions for finite Coxeter systems directly.

Let (W, S) be a finite Coxeter system with root system Φ and $V := \mathrm{span}_{\mathbb{R}}(\Phi)$. The classical parabolic subgroups are exactly the subgroups of the form

$$C_W(E) := \{w \in W \mid w(v) = v \text{ for all } v \in E\}$$

where $E \subseteq V$ is any subset of vectors (see for instance [Kan01, Section 5-2]).

Definition 2.4.9. Given a subset $\mathcal{A} \subseteq W$, the parabolic closure $P_{\mathcal{A}}$ of $\mathcal{A}$ is the intersection of all the parabolic subgroups in the classical sense containing $\mathcal{A}$. It is again a parabolic subgroup in the classical sense (see [Qi07]).

We denote by $\mathrm{Fix}(\mathcal{A})$ the subspace of vectors in V which are fixed by every element in $\mathcal{A}$. If $\mathcal{A} = \{w\}$, then we simply write $\mathrm{Fix}(w)$ for $\mathrm{Fix}(\mathcal{A}) = \ker(w - 1)$ and P_w for $P_{\mathcal{A}}$. For convenience we also set $\mathrm{Mov}(w) := \mathrm{im}(w-1)$. Note that $V = \mathrm{Fix}(w) \oplus \mathrm{Mov}(w)$ (see [Arm09, Definition 2.4.6]). It follows from the above description that $P_{\mathcal{A}} = C_W(\mathrm{Fix}(\mathcal{A}))$.

By [Bes03, Lemma 1.2.1(i)] we have the following.

Lemma 2.4.10. *Let (W,T) be a finite dual Coxeter system, $w \in W$ and $t \in T$. Then*

$$\mathrm{Fix}(w) \subseteq \mathrm{Fix}(t) \text{ if and only if } t \leq_T w.$$

Proposition 2.4.11. *Let (W,S) be a finite Coxeter system, $T = \bigcup_{w\in W} wSw^{-1}$ and $w \in W$. If the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive, then the subgroup generated by the reflections in any reduced decomposition of w is equal to P_w.*

Proof. We prove the contrapositive of the statement. Let $(t_1,\dots,t_m) \in \mathrm{Red}_T(w)$ and assume that $W' := \langle t_1,\dots,t_m\rangle$ is not equal to P_w. Since $t_i \leq_T w$ for each i, we have $t_i \in P_w$ for all $i = 1,\dots,m$ by Lemma 2.4.10. It follows that $W' \subsetneq P_w$. Since both W' and P_w are reflection subgroups of W, there exists a reflection $t \in P_w$ with $t \notin W'$. It follows that $\mathrm{Fix}(w) \subseteq \mathrm{Fix}(t)$, hence also that $t \leq_T w$ by Lemma 2.4.10. In particular there exists $(q_1,\dots,q_m) \in \mathrm{Red}_T(w)$ with $q_1 = t$. Since the Hurwitz orbit of $(t_1,\dots,t_m)$ remains in W' and $t \notin W'$, the Hurwitz action on $\mathrm{Red}_T(w)$ can therefore not be transitive. □

Corollary 2.4.12. *Let (W,S) be a finite Coxeter system and $T = \bigcup_{w\in W} wSw^{-1}$. A subgroup $P \subseteq W$ is parabolic if and only if it is parabolic in the classical sense. In particular, if $S' \subseteq T$ is such that (W,S') is a simple system, then the parabolic subgroups in the classical sense defined by S coincide with those defined by S'.*

Proof. If P is parabolic, then $P = \langle s_1,\dots,s_m\rangle$ where $\{s_1,\dots,s_n\} = S' \subseteq T$ is a simple system for W and $m \leq n$. By [BDSW14, Theorem 1.3], the Hurwitz action on $\mathrm{Red}_T(w)$ where $w = s_1 s_2 \cdots s_m$ is transitive. By Proposition 2.4.11, it follows that P is parabolic in the classical sense.

Conversely, if P is parabolic in the classical sense, then P is generated by a conjugate of a subset of S, and a conjugate of S is again a simple system for W. Hence P is parabolic. □

2.5. Noncrossing partitions

The poset of noncrossing partitions attached to a Coxeter group W, in particular in the case when W is finite, has gained a lot of attention in the recent years from different areas of mathematics (see [McC06]). For a detailed treatment of this topic we refer to Armstrong's memoir ([Arm09]).

Definition 2.5.1. Let (W,T) be a dual Coxeter system and $c \in W$. The poset

$$\mathrm{Nc}(W,c) := \{w \in W \mid e \leq_T w \leq_T c\}$$

is called poset of noncrossing partitions.

For an explanation of the name noncrossing see also [Arm09]. In particular, this poset has nice properties and is well understood for finite Coxeter systems and Coxeter elements. Therefore let (W,T) be a finite dual Coxeter system and $c \in W$ a Coxeter element. Combining [DG17, Corollary 3.6] and [Arm09, Proposition 2.6.11] we obtain the following result.

Proposition 2.5.2. *Every interval in* $\mathrm{Nc}(W, c)$ *is isomorphic to* $\mathrm{Nc}(W', c')$*, where* $W' \leq W$ *is a parabolic subgroup and* $c' \leq_T c$ *is a parabolic Coxeter element.*

The poset $\mathrm{Nc}(W, c)$ for W a finite Coxeter group and c a Coxeter element, was independently introduced and studied by Bessis in [Bes03] and Brady and Watt in [BW02]. While Bessis showed the following result in a case-by-case analysis, Brady and Watt gave a uniform proof in [BW08].

Theorem 2.5.3. *The poset* $\mathrm{Nc}(W, c)$ *is a lattice.*

Based on previous work of Digne, McCammond precisely determined in [McC15] for W an affine Coxeter group if $\mathrm{Nc}(W, c)$ is a lattice or if it is not.

3. Hurwitz action on Coxeter elements

The results presented in Sections 3.1 and 3.2 are taken from [BDSW14].

3.1. The Hurwitz action

Let (W, T) be a dual Coxeter system. For an element $w \in W$ denote by $\mathrm{Red}_T(w)$ the set of all reduced T-decompositions of w. The aim of this section is to show that the braid group acts transitively on this set if w is a parabolic Coxeter element.

The braid group on n strands is the group $\mathcal{B}_n$ with generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

It acts on the set T^n of n-tuples of reflections as

$$\sigma_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1},\ t_i t_{i+1} t_i,\quad t_i\quad,\ t_{i+2}, \ldots, t_n),$$
$$\sigma_i^{-1}(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1},\quad t_{i+1}\quad,\ t_{i+1} t_i t_{i+1},\ t_{i+2}, \ldots, t_n).$$

We use the notation

$$(t_1, \ldots, t_n) \sim (t'_1, \ldots, t'_n)$$

to indicate that both n-tuples are in the same orbit under this action.

Example 3.1.1. If $n = 2$, then the action of σ_1 is described by

$$\ldots \sim (srs, srsrs) \sim (s, srs) \sim (r, s) \sim (rsr, r) \sim (rsrsr, rsr) \sim \ldots$$

for any $r, s \in T$. Note that in this case, the $\mathcal{B}_2$-orbit of (r, s) is the set of all pairs (t_1, t_2) of reflections of the subgroup $\langle r, s \rangle$, such that $t_1 t_2 = rs$.

Definition 3.1.2. For an element $w \in W$ denote by $[w]$ the corresponding conjugacy class in W. Two tuples $(w_1, \ldots, w_n), (v_1, \ldots, v_n) \in W^n$ are said to share the same multiset of conjugacy classes if $\{[w_1], \ldots, [w_n]\}$ and $\{[v_1], \ldots, [v_n]\}$ are equal as multisets.

The following lemmata are direct consequences of the definition.

Lemma 3.1.3. *Let (W, T) be a dual Coxeter system and $(t_1, \ldots, t_n), (t'_1, \ldots, t'_n) \in T^n$ such that $(t_1, \ldots, t_n) \sim (t'_1, \ldots, t'_n)$. Then:*

- $\langle t_1, \ldots, t_n \rangle = \langle t'_1, \ldots, t'_n \rangle$.
- *$(t_1, \ldots, t_n)$ and $(t'_1, \ldots, t'_n)$ share the same multiset of conjugacy classes.*

Lemma 3.1.4. *Let W' be a reflection subgroup of W and let $T' = T \cap W'$ be the set of reflections in W'. For an element $w \in W'$ with $\ell_{T'}(w) = m$, the braid group $\mathcal{B}_m$ acts on* $\mathrm{Red}_{T'}(w)$.

Proof. Let $w = t_1 \cdots t_m$ be a reduced T'-decomposition. and $\sigma_i(t_1, \ldots, t_m) = (t'_m, \ldots, t'_m)$. The assertion follows from the two observations that $t_1 \cdots t_m = t'_1 \cdots t'_m$ and $\{t_1, \ldots, t_m\} \subseteq T'$ if and only if $\{t'_1, \ldots, t'_m\} \subseteq T'$. □

We call this action on $\mathrm{Red}_{T'}(w)$ also the Hurwitz action.

3.2. The proof of Theorem 1.1.1

We will now come to the proof of Theorem 1.1.1. So let $w = s_1 \cdots s_m$ be a parabolic Coxeter element in a dual Coxeter system (W, T). By definition of the Hurwitz action, this theorem has the direct consequence that the parabolic subgroup $\langle s_1, \ldots, s_m \rangle$ of W does indeed not depend on the particular T-decomposition $c = s_1 \cdots s_m$ but only on the parabolic Coxeter element w itself. We thus denote this parabolic by $W_w := \langle t_1, \ldots, t_m \rangle$ for any $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$. We moreover obtain that $\mathrm{Red}_T(w) = \mathrm{Red}_{T'}(w)$ with $T' = W_w \cap T$ the set of reflections in W_w. The main argument in the proof of Theorem 1.1.1 (see Proposition 3.2.3 below) will also imply the following theorem that extends this direct consequence to all elements in a parabolic subgroup.

Theorem 3.2.1. *Let W' be a parabolic subgroup of W. Then for any $w \in W'$,*

$$\mathrm{Red}_T(w) = \mathrm{Red}_{T'}(w),$$

where $T' = W' \cap T$ is the set of reflections in W'.

Notice that Lemma 2.4.10 already implies this Theorem in the finite case.

For the proof of the two theorems, we fix a dual Coxeter system (W, T) of rank n and a simple system (W, S) for (W, T). The following lemma provides an alternative description of standard parabolic Coxeter elements.

Lemma 3.2.2. *An element $w \in W$ is a standard parabolic Coxeter element for (W, S) if and only if $\ell_T(w) = \ell_S(w)$.*

Proof. Let $w = s_{i_1} \ldots s_{i_k}$ be a reduced S-decomposition. By Theorem 2.3.3 we have $\ell_T(w) = k = \ell_S(w)$ if and only if $s_{i_1} \ldots s_{i_k}$ does not contain any generator twice. □

Notice that the definition of Bruhat graph given in 2.2.8 depends on the chosen simple system S. In contrast to this the definition of the underlying undirected Bruhat graph does not. It only depends on T, but for simplicity we keep the notation $\overline{\Omega}_{(W,S)}$.

The proofs of Theorem 1.1.1 and Theorem 3.2.1 are based on the case $x = e$ of the following proposition.

Proposition 3.2.3. *Let (W, S) be a Coxeter system. Moreover, let $w = t_1 \cdots t_m \in W$ be a reduced T-decomposition of an element in W, and let*

$$x - xt_1 - xt_1t_2 - \ldots - xt_1 \cdots t_m = xw$$

be the corresponding path in $\overline{\Omega}_{(W,S)}$ *starting at an element* $x \in W$. *Then there is a* T-*decomposition* $w = t_1' \cdots t_m'$ *in the Hurwitz orbit of the* T-*decomposition* $t_1 \cdots t_m$ *such that the corresponding path in* $\Omega_{(W,S)}$ *starting at* x *is first decreasing, then increasing; more precisely, it is of the form*

$$x \leftarrow xt_1' \leftarrow xt_1't_2' \leftarrow \ldots \leftarrow xt_1' \cdots t_i' \rightarrow \ldots \rightarrow xt_1' \cdots t_m' = xw$$

for some (unique) integer i *with* $0 \le i \le m$. *In the special case* $x = e$, *this gives a directed path*

$$e \rightarrow t_1' \rightarrow t_1't_2' \rightarrow \ldots \rightarrow t_1' \cdots t_m' = w$$

in $\Omega_{(W,S)}$.

Proof. First consider two distinct reflections t_1 and t_2 and an element $z \in W$ such that $z \rightarrow zt_1 \leftarrow zt_1t_2$ in $\Omega_{(W,S)}$. By Proposition 2.3.6 there exist reflections $t_1', t_2' \in \langle t_1, t_2 \rangle$ with $t_1t_2 = t_1't_2'$ such that $z \rightarrow zt_1' \rightarrow zt_1't_2'$ or $z \leftarrow zt_1' \leftarrow zt_1't_2'$ or $z \leftarrow zt_1' \rightarrow zt_1't_2'$. This implies, by Example 3.1.1, that one can get from the decomposition t_1t_2 to the decomposition $t_1't_2'$ inside $W' = \langle t_1, t_2 \rangle$ by the Hurwitz action, and hence in particular that $W' = \langle t_1', t_2' \rangle$. Moreover, one has $\ell(zt_1') < \max(\ell(z), \ell(zt_1t_2)) < \ell(zt_1)$.

Consider the path in $\overline{\Omega}_{(W,S)}$ attached to $w = t_1 \cdots t_m$ and beginning at x. Any subpath $z — zt_1 — zt_1t_2$ may be replaced by a path $z — zt_1' — zt_1't_2'$ as above, to give a new path from x to xw of the same length m; we call this a "replacement". Apply to the original path a sequence of successive replacements. Any path so obtained corresponds to the path beginning at x attached to some reduced T-decomposition of w in the same Hurwitz orbit as $t_1 \cdots t_m$, and is a shortest path in $\overline{\Omega}_{(W,S)}$ from x to xw. Note that a replacement of any subpath $z \rightarrow zt_1' \leftarrow zt_1't_2'$ of such a path is possible since the path's minimal length implies that $t_1' \neq t_2'$. Each replacement decreases the total sum of the ℓ-lengths of the vertices of the path, so eventually one obtains a path in which no further replacements are possible i.e. of the desired decreasing-then-increasing form. Finally, if $x = e$, then $i = 0$ since there are no paths $e \leftarrow t$. □

Proof of Theorem 3.2.1. By Theorem 2.2.10 we have $\Omega_{(W',\chi(W'))} \cong \Omega_{(W,S)}(W')$. Let $w \in W'$. Then Lemma 3.1.4, Proposition 2.3.4 and Proposition 3.2.3 (with $x = e$) imply that $\mathrm{Red}_T(w) = \mathrm{Red}_{T'}(w)$ if and only if every shortest directed path from e to w in $\Omega_{(W,S)}$ lies inside $\Omega_{(W,S)}(W')$.

Now assume that W' is a standard parabolic subgroup generated by some subset S' of S. Then it is well known that every reduced S-decomposition for $w \in W'$ is actually inside W' (by [AB08, Theorem 2.33] all reduced S-decompositions for w are related by braid relations). Let

$$e \rightarrow t_1 \rightarrow \ldots \rightarrow t_1 \cdots t_{m-1} \rightarrow t_1 \cdots t_m = w$$

be a shortest directed path from e to w in $\Omega_{(W,S)}$. Let $w = s_1' \cdots s_r'$ be an S'-decomposition of w. Since $wt_m = t_1 \cdots t_{m-1}$, we have $\ell_S(wt_m) < \ell_S(w)$. Using the strong exchange condition (see Theorem 2.1.5) we find an index i such that $wt_m = s_1' \cdots \widehat{s_i'} \cdots s_r'$. In particular, $wt_m = t_1 \cdots t_{m-1} \in W'$. Continuing in this manner, it follows that the above directed path is indeed a path inside $\Omega_{(W,S)}(W')$. The assertion follows by the above equivalence. □

Lemma 3.2.4. *Let* (W,T) *be a dual Coxeter system of finite rank* n *and let* $w = s_1 \cdots s_m$ *be a parabolic Coxeter element in* W. *If* s_i *and* s_{i+1} *do not commute, then* $s_{i+1}s_i \not\le_T w$.

Proof. Without loss of generality we assume that $i = 1$ and that s_1 and s_2 do not commute. Assume to the contrary that $s_2 s_1 \leq_T w$. Then $\ell_T(s_1 s_2 w) = m - 2$. But since $s_1 s_2 w = (s_1 s_2 s_1) s_2 s_3 \cdots s_m$ and since $s_2 s_3 \cdots s_m$ is T-reduced, we obtain $s_1 s_2 s_1 \leq_T s_2 \cdots s_m =: w'$. This yields that there exists a T-reduced decomposition for w' beginning with $s_1 s_2 s_1$. By Theorem 3.2.1 we have $\mathrm{Red}_T(w') = \mathrm{Red}_{T'}(w')$, where $T' = T \cap \langle s_2, \ldots, s_m \rangle$. Since s_1 and s_2 do not commute, we have $s_1 s_2 s_1 \notin T'$, a contradiction. □

Proof of Theorem 1.1.1. Fix a parabolic Coxeter element $c = s_1 \cdots s_m \in W$ which is a standard parabolic Coxeter element for the simple system (W, S). By Proposition 3.2.3, it is left to show that any two directed paths from e to c in $\Omega_{(W,S)}$ are in the same Hurwitz orbit. Let therefore

$$e \longrightarrow t_1 \longrightarrow t_1 t_2 \longrightarrow \ldots \longrightarrow t_1 \cdots t_m = c$$

be such a path. It remains to show that this path is in the same orbit as the path

$$e \longrightarrow s_1 \longrightarrow s_1 s_2 \longrightarrow \ldots \longrightarrow s_1 \cdots s_m = c.$$

We give two different arguments. The first argument is a direct one using induction. The second one is constructive. It yields a braid which transforms the first path into the second.

First argument: We proceed by induction on m. For $m = 1$ the assertion is obvious. Therefore let $m > 1$. We have seen in Lemma 3.2.2 that $\ell_S(c) = \ell_T(c) = m$. It thus follows that $\ell_S(t_1 \cdots t_i) = \ell_T(t_1 \cdots t_i) = i$ for any i and therefore, again by Lemma 3.2.2, $t_1 \cdots t_i$ is a standard parabolic Coxeter element for any i. Put $c' = c t_m = t_1 \cdots t_{m-1}$. Since $\ell_S(c') < \ell_S(c)$, the strong exchange condition (see Theorem 2.1.5) yields that there exists $1 \leq i \leq m$ such that $c' = s_1 \cdots \widehat{s_i} \cdots s_m$, hence $t_m = s_m \cdots s_{i+1} s_i s_{i+1} \cdots s_m$. Using Theorem 3.2.1, we obtain by induction that

$$(t_1, \ldots, t_{m-1}) \sim (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m).$$

Thus it follows

$$\begin{aligned} (t_1, \ldots t_{m-1}, t_m) &\sim (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m, t_m) \\ &= (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m, s_m \cdots s_{i+1} s_i s_{i+1} \cdots s_m) \\ &\overset{\sigma_i \cdots \sigma_{m-1}}{\sim} (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_m). \end{aligned}$$

Second argument: As in the previous argument we obtain by the strong exchange condition that $t_1 \cdots t_{i+1}$ is obtained from $t_1 \cdots t_i$ by adding a single simple generator into its position within $s_1 \cdots s_m$ (see also Example 3.2.5 below). Therefore and by Lemma 3.2.4, the path corresponding to the decomposition $t_1 \cdots t_m$ is (bijectively) encoded by a permutation $\pi = [\pi_1, \ldots, \pi_m]$ where π_i is the index of the simple generator added at the i-th step. Given the decomposition corresponding to such a path, we claim that the embedding of the permutation into the braid group (by sending a simple transposition $(i, i+1)$ to the generator σ_i of $\mathcal{B}_m$) yields a braid that turns the given decomposition $(t_1, \ldots, t_m)$ into the decomposition $(s_1, \ldots, s_m)$. First of all note that this does not depend on the chosen reduced decomposition of π into simple transpositions. By [AB08, Theorem 2.33] all such decompositions are related by braid relations. Hence they yield the same braid. Denote by $\ell(-)$ the length function on the symmetric group with respect to the generating set consisting of all simple transpositions.

We proceed by induction on $\ell(\pi)$. If $\ell(\pi) = 1$, say $\pi = (i, i+1)$, then the corresponding path is given by

$$e \longrightarrow s_1 \longrightarrow \ldots \longrightarrow s_1 \cdots s_{i-1} \longrightarrow s_1 \cdots s_{i-1} s_{i+1} \longrightarrow s_1 \cdots s_{i-1} s_i s_{i+1} \longrightarrow \ldots \longrightarrow s_1 \cdots s_m = c.$$

It is straightforward to see that the braid σ_i turns the corresponding decomposition into the decomposition $(s_1, \ldots, s_m)$. Therefore let $\ell(\pi) = k > 1$ and write $\pi = \tau_1 \cdots \tau_k$, where each τ_i is a simple transposition. Put $\pi' = \pi\tau_k$ and observe that $\ell(\pi') < \ell(\pi)$. By induction the embedding of $\tau_1 \cdots \tau_{k-1}$ into the braid group yields a braid that turns the decomposition $(t'_1, \ldots, t'_m)$ corresponding to π' into the decomposition $(s_1, \ldots, s_m)$. Let $\tau_k = (j, j+1)$. It remains to show that $\sigma_j(t_1, \ldots, t_m) = (t'_1, \ldots, t'_m)$. Note that $t_i = t'_i$ for $i \neq j, j+1$. If the path corresponding to $(t_1, \ldots, t_m)$ looks like

$$e \longrightarrow \ldots \longrightarrow s_{k_1} \cdots \widehat{s_{k_p}} \cdots \widehat{s_{k_q}} \cdots s_{k_j} \xrightarrow{t_j} s_{k_1} \cdots \widehat{s_{k_p}} \cdots s_{k_q} \cdots s_{k_j} \xrightarrow{t_{j+1}} s_{k_1} \cdots s_{k_p} \cdots s_{k_q} \cdots s_{k_j} \longrightarrow \ldots \longrightarrow c,$$

where $1 \leq k_1 < \ldots < k_p < \ldots < k_q < \ldots < k_j \leq m$, the path corresponding to $(t'_1, \ldots, t'_m)$ looks like

$$e \longrightarrow \ldots \longrightarrow s_{k_1} \cdots \widehat{s_{k_p}} \cdots \widehat{s_{k_q}} \cdots s_{k_j} \xrightarrow{t'_j} s_{k_1} \cdots s_{k_p} \cdots \widehat{s_{k_q}} \cdots s_{k_j} \xrightarrow{t'_{j+1}} s_{k_1} \cdots s_{k_p} \cdots s_{k_q} \cdots s_{k_j} \longrightarrow \ldots \longrightarrow c.$$

Therefore we see that

$$\begin{aligned} t_j &= s_{k_j} \cdots s_{k_{q+1}} s_{k_q} s_{k_{q+1}} \cdots s_{k_j}, \\ t_{j+1} &= s_{k_j} \cdots s_{k_{p+1}} s_{k_p} s_{k_{p+1}} \cdots s_{k_j} \end{aligned}$$

and $t'_j = t_j t_{j+1} t_j,\ t'_{j+1} = t_j$. Thus $\sigma_j(t_1, \ldots, t_m) = (t'_1, \ldots, t'_m)$.

□

Example 3.2.5. As an example of the construction in the previous proof, consider the path

$$e \longrightarrow \quad s_2 \qquad \longrightarrow \quad s_2 \quad s_5 \longrightarrow \quad s_2 s_3 \quad s_5 \longrightarrow s_1 s_2 s_3 \quad s_5 \longrightarrow s_1 s_2 s_3 s_4 s_5 = c.$$

The corresponding factorization of c is given by

$$c = s_2 \cdot s_5 \cdot s_5 s_3 s_5 \cdot s_5 s_3 s_2 s_1 s_2 s_3 s_5 \cdot s_5 s_4 s_5,$$

and the permutation is $\pi = [2, 5, 3, 1, 4] = (1, 2)(2, 3)(4, 5)(3, 4)(2, 3)$. On the other hand,

$$\sigma_1 \sigma_2 \sigma_4 \sigma_3 \sigma_2 (s_2, s_5, s_5 s_3 s_5, s_5 s_3 s_2 s_1 s_2 s_3 s_5, s_5 s_4 s_5) = (s_1, s_2, s_3, s_4, s_5),$$

as desired.

3.3. Applications of the Hurwitz action

In this section we want to emphasize two applications of the Hurwitz action and its transitivity on decpompositions of Coxeter elements as described in the preceding section. For definitions as well as details we refer to the respective specified references.

3.3.1. Noncrossing partitions and thick subcategories

As a reference for this subsection see [Kra12]. Let A be a connected hereditary Artin algebra and $S_1, \dots, S_n$ representatives of the simple A-modules. Since A is connected, its center is a field k. For $i \neq j$ we have $\mathrm{Ext}^1_A(S_i, S_j) = 0$ or $\mathrm{Ext}^1_A(S_j, S_i) = 0$. Assume the latter one and define

$$C_{ij} := -\ell_{\mathrm{End}_A(S_i)}(\mathrm{Ext}^1_A(S_i, S_j)),\ C_{ji} := -\ell_{\mathrm{End}_A(S_j)}(\mathrm{Ext}^1_A(S_i, S_j)).$$

For each i put $C_{ii} := 2$ and $d_i := \ell_k(\mathrm{End}_A(S_i))$. The matrix $C(A) := (C_{ij})_{1 \leq i,j \leq n}$ is a symmetrizable generalized Cartan matrix in the sense of [Kac83]. Let $e_1, \dots e_n$ be the standard basis of $\mathbb{R}^n$. We obtain a symmetric bilinear form on $\mathbb{R}^n$ by setting $(e_i \mid e_j) := d_i C_{ij}$. For a non-isotropic $\alpha \in \mathbb{R}^n$ we define the reflection in α as before by the map

$$s_\alpha : \mathbb{R}^n \to \mathbb{R}^n, v \mapsto v - \frac{2(v \mid \alpha)}{(\alpha \mid \alpha)}\alpha.$$

Putting $s_i := s_{e_i}$, one obtains that $W := \langle s_1, \dots, s_n \rangle$ is the Coxeter group associated to $C(A)$. Based on work of Ingalls and Thomas ([IT09]) as well as Igusa, Schiffler and Thomas ([IS10]) and using the transitivity of the Hurwitz action as stated in Theorem 1.1.1 as a major ingredient in his proof, Krause obtains the following result (see [Kra12, Theorem 6.10]).

Theorem 3.3.1. *Let A be a connected hereditary Artin algebra with simple modules $S_1, \dots S_n$ satisfying $\mathrm{Ext}^1_A(S_j, S_i) = 0$ for all $i < j$. Denote by W the associated Weyl group and fix the Coxeter element $c = s_1 \cdots s_n$. Then there exists an order preserving bijection between*

- *the set of thick subcategories of $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}(A))$ generated by an exceptional sequence in* $\mathrm{mod}(A)$
- *the poset of noncrossing partitions* $\mathrm{Nc}(W, c)$.

The question whether it is possible to find a similar statement for other hereditary categories, will be adressed in Section 7.5.

3.3.2. Dual Artin groups and Garside structures

As a reference for this subsection see [McC15]. Let (W, S) be a Coxeter system, that is, we have a presentation

$$W = \langle S \mid (st)^{m_{st}} = 1 \rangle$$

as defined in Definition 2.1.1. In particular, among all the relations, we have the braid relations $sts \cdots = tst \cdots$ for $s \neq t$ and $m_{st} < \infty$. We define the Artin group $\mathrm{Art}(W, S)$ associated to (W, S) to be the group with generators S, but relations are just given by the braid relations. Thus

$$\mathrm{Art}(W, S) = \langle S \mid sts \cdots = tst \cdots \ (s \neq t) \rangle.$$

As before, if $m_{st} = \infty$, then there is no relation. E.g. if (W, S) is a Coxeter system of type A_n, then $\mathrm{Art}(W, S)$ is the braid group $\mathcal{B}_{n+1}$.

Consider a dual Coxeter system (W, T) and a Coxeter element $c \in W$. Put $T_0 = \{t \in T \mid t \in \mathrm{Nc}(W, c)\}$. The dual Artin group $\mathrm{Art}^*(W, T, c)$ is defined to be the group generated by T_0 and subject only to those relations that are visible inside the interval $\mathrm{Nc}(W, c)$. More precisely, the relations are given by the closed loops inside the full subgraph of $\overline{\Omega}_{(W,T)}$ on vertex set $\mathrm{Nc}(W, c)$.

Let $S \subseteq T$ be a simple system for (W, T). Then it is always possible to find a homomorphism from $\mathrm{Art}(W, S)$ to $\mathrm{Art}^*(W, S)$ (see [McC15, Remark 2.6]). If (W, T) is irreducible of finite or affine type, then Bessis [Bes03] for the finite case and McCammond [McC15] for the affine case showed, by using the transitivity of the Hurwitz action as stated in Theorem 1.1.1, that $\mathrm{Art}(W, S)$ and $\mathrm{Art}^*(W, S)$ are isomorphic. The importance of the question whether an Artin group and its dual Artin group are isomorphic, arises in the theory of Garside structures (see [Bes03, Section 0] for more on Garside structures). Namely, McCammond showed (see [McC15, Proposition 2.7]) that a dual Artin group $\mathrm{Art}^*(W, T, c)$ has a Garside presentation provided $\mathrm{Nc}(W, c)$ is a lattice. Groups with Garside presentation have nice properties, e.g. they have decidable word problem and are torsion-free. Unfortunately, $\mathrm{Nc}(W, c)$ is not a lattice in general and not every dual Artin group has a Garside presentation. Based on work of Digne [Dig06, Dig12], McCammond precisely determined for which Artin groups of affine type the dual presentation yields a Garside structure (see [McC15, Theorem A]).

4. Hurwitz action in finite Coxeter groups

The results of this chapter are taken from [BGRW17], but they were sometimes slightly generalized.

We have seen in Chapter 3 that the Hurwitz action is transitive on the set of reduced decompositions of parabolic Coxeter elements as products of reflections. Not each element of a Coxeter group has this property. As an example consider a Coxeter system $(W, \{s, t\})$ of type B_2. The element $w := stst$ has absolute length 2. Since st has order 4, the reflections sts and t resp. the reflections tst and s commute. We have two Hurwitz orbits inside $\mathrm{Red}_T(w)$:

$$\mathcal{B}_2(sts, t) = \{(sts, t), (t, sts)\}$$
$$\mathcal{B}_2(tst, s) = \{(tst, s), (s, tst)\}.$$

Consider the reflection subgroups corresponding to these orbits, that is, $\langle sts, t\rangle$ and $\langle tst, s\rangle$. We observe that (in contrast to parabolic Coxeter elements), none of them is parabolic. It turns out that this is the only obstacle for Hurwitz transitivity in the finite case (see Theorem 1.1.2). In fact, this example also shows that Theorem 3.2.1 is not true if we assume W' to be an arbitrary reflection subgroup.

The aim of this section is to determine those elements w in a finite Coxeter group W such that the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive. That is, we prove Theorem 1.1.2.

4.1. Quasi-Coxeter elements

Definition 4.1.1. Let (W, T) be a dual Coxeter system of rank n. An element $w \in W$ is called quasi-Coxeter element for (W, T) if there exists $(t_1, \ldots, t_n) \in \mathrm{Red}_T(w)$ such that $W = \langle t_1, \ldots, t_n\rangle$. An element $w \in W$ is called parabolic quasi-Coxeter element for (W, T) if there is a simple system $S = \{s_1, \ldots, s_n\}$ and $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ such that $\langle t_1, \ldots, t_m\rangle = \langle s_1, \ldots, s_m\rangle$ for some $m \leq n$.

Remark 4.1.2. This definition is a generalization of Voigt's original definition in [Voi85], see also Remark 4.1.4.

Example 4.1.3. In type D_4 with simple system $\{s_1, s_2, s_3, s_4\}$ where s_3 does not commute with any other simple reflection, the element

$$w := s_2(s_3s_2s_3)(s_3s_1s_3)s_4$$

is a quasi-Coxeter element. It has a reduced decomposition generating the whole group since if writing

$$(t_1, \ldots, t_4) = (s_2, s_3s_2s_3, s_3s_1s_3, s_4)$$

we have that $t_1t_2t_1 = s_3$ and $s_3t_3s_3 = s_1$. Using the permutation model for a group of type D_4 (see Section 4.4), it can be shown that there is no reduced decomposition of this element yielding a simple system for the group. By computer we checked that the poset $\mathrm{Nc}(W, w)$ has 54 elements and is not a lattice. There is a single conjugacy class of quasi-Coxeter elements which are not Coxeter elements in that case. Therefore we can not define a new Garside structure on the Artin-Tits group of type D_4 by replacing the Coxeter element by a quasi-Coxeter element.

Remark 4.1.4. For the case where (W, T) is simply-laced and w is a quasi-Coxeter element in W, Voigt observed in his thesis [Voi85] that the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive. His definition of a quasi-Coxeter element is slightly different. Choose a simple system $S \subseteq T$ and let Φ be the root system associated to (W, S). Then Voigt defined $w = s_{\alpha_1} \cdots s_{\alpha_n} \in W$ to be quasi-Coxeter if $\mathrm{span}_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_n)$ is equal to the root lattice of Φ. The connection with our definition will be made clear by Theorem 4.2.12.

4.2. Root lattices

In this section, we study (co)root lattices and their sublattices. The results will be needed for the better understanding of quasi-Coxeter elements. For the rest of this section we fix an euclidean vector space V with positive definite symmetric bilinear form $(- \mid -)$.

Definition 4.2.1. A lattice L in V is the integral span of a basis of V. The lattice L is called integral if $(\alpha \mid \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in L$ and an integral lattice is called even if $(\alpha \mid \alpha) \in 2\,\mathbb{Z}$ for all $\alpha \in L$.

Proposition 4.2.2. *For an even lattice L the set*

$$\Phi(L) := \{\alpha \in L \mid (\alpha \mid \alpha) = 2\}$$

is a simply-laced, crystallographic root system in $\mathrm{span}_{\mathbb{R}}(L)$.

Proof. The set $\Phi(L)$ is contained in the ball around 0 with radius 2, therefore bounded, thus finite. The rest of the proof is straightforward. $\square$

Definition 4.2.3. Let Φ be a crystallographic root system in V. The weight lattice $P(\Phi)$ of Φ is defined by

$$P(\Phi) := \{x \in V \mid (x \mid \alpha^\vee) \in \mathbb{Z} \;\; \forall \alpha \in \Phi\}.$$

Similarly, the coweight lattice $P(\Phi^\vee)$ is defined by

$$P(\Phi^\vee) := \{x \in V \mid (x \mid \alpha) \in \mathbb{Z} \;\; \forall \alpha \in \Phi\}.$$

By [Bou02, VI, 9, Prop. 26] $P(\Phi)$ (resp. $P(\Phi^\vee)$) is again a lattice containing $L(\Phi)$ (resp. $L(\Phi^\vee)$). The groups $P(\Phi)/L(\Phi)$ and $P(\Phi^\vee)/L(\Phi^\vee)$ are finite of equal order. We call this order the connection index of Φ and denote it by $i(\Phi)$.

Remark 4.2.4. (a) If Φ is simply-laced, then $L(\Phi) = L(\Phi^\vee)$ and $P(\Phi) = P(\Phi^\vee)$.

(b) In terms of lattice theory, the lattice $P(\Phi^\vee)$ is dual to the lattice $L(\Phi)$. If M is the Gram matrix of $L(\Phi)$, then M^{-1} is the Gram matrix of $P(\Phi^\vee)$ and $|P(\Phi^\vee) : L(\Phi)| = \det(M)$ (see for instance [Ebe02, Section 1.1]).

Proposition 4.2.5. *Let Φ be a simply-laced root system and let C be the Cartan matrix of Φ. Then*

$$i(\Phi) = \det(C).$$

Proof. Let $\Delta = \{\alpha_1, \ldots, \alpha_m\} \subseteq \Phi$ be a simple system for the root system Φ. Then Δ is a basis of $L(\Phi)$. Denote by M the Gram matrix of $L(\Phi)$ with respect to Δ. By Remark 4.2.4 we have

$$|P(\Phi^\vee) : L(\Phi)| = \det(M).$$

Since Φ is simply-laced, we have $P(\Phi^\vee) = P(\Phi)$ and hence $i(\Phi) = |P(\Phi^\vee) : L(\Phi)|$. Again since Φ is simply-laced, we have $C = M$, which concludes the proof. □

We list $i(\Phi)$ for the irreducible, simply-laced root systems. These can be found in [Bou02, Plates I, IV,V,VI,VII].

Type of Φ	A_n	D_n	E_6	E_7	E_8
$i(\Phi)$	$n+1$	4	3	2	1

Table 4.1.: Connection indexes

As a consequence we obtain the following result.

Proposition 4.2.6. *Let Φ be an irreducible, simply-laced root system. Then Φ is determined by the pair* $(\mathrm{rk}(\Phi), i(\Phi))$.

The following lemma seems to be folklore, but we could not find a proof in the literature, hence we state it here.

Lemma 4.2.7. *Let Φ be a simply-laced root system. Then the root lattice determines the root system, that is,*

$$\Phi(L(\Phi)) = \Phi.$$

Proof. By the previous proposition, we have to show that the ranks and connection indices of Φ and $\Phi(L(\Phi))$ coincide. Since $\Phi \subseteq \Phi(L(\Phi))$, we have $\mathrm{rk}(\Phi) \leq \mathrm{rk}(\Phi(L(\Phi)))$. On the other hand, the rank of $\Phi(L(\Phi))$ is bounded above by the dimension of the ambient vector space which equals $\mathrm{rk}(\Phi)$.

In the proof of Proposition 4.2.5, we used the fact that the Cartan matrix of a root system and the Gram matrix of the corresponding root lattice coincide. Denote by C_Φ the Cartan matrix with respect to Φ. Then

$$\sqrt{\det(C_\Phi)} = \mathrm{vol}(L(\Phi)) = \mathrm{vol}(L(\Phi(L(\Phi)))) = \sqrt{\det(C_{\Phi(L(\Phi))})},$$

which yields $i(\Phi) = i(\Phi(L(\Phi)))$ by Proposition 4.2.5. □

Remark 4.2.8. Notice that the condition in Lemma 4.2.7 on Φ to be simply-laced is necessary. For example, if Φ is of type B_2, then one can choose two orthogonal short roots α, β generating a proper root subsystem of type $A_1 \times A_1$ while one has $L(\{\alpha, \beta\}) = L(\Phi)$.

Lemma 4.2.9. *Let Φ be as in Lemma 4.2.7 and $\Phi' \subseteq \Phi$ be a root subsystem. Then $L(\Phi') \cap \Phi = \Phi'$.*

Proof. We have the equations

$$L(\Phi') \cap \Phi \overset{4.2.7}{=} L(\Phi') \cap \{\alpha \in L(\Phi) \mid (\alpha \mid \alpha) = 2\} = \{\alpha \in L(\Phi') \mid (\alpha \mid \alpha) = 2\} \overset{4.2.7}{=} \Phi'.$$

□

Notation 4.2.10. Let Φ be a root system and $R \subseteq \Phi$ a set of roots. We denote by W_R the group $\langle s_\alpha \mid \alpha \in R \rangle$ and set $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$.

Note that this is consistent with the notation introduced in Definition 2.1.7 in case $R = \Phi$.

By [Bou02, Ch. VI, 3, Corollary to Theorem 1] we have the following.

Lemma 4.2.11. *Let Φ be a crystallographic root system and $\alpha, \beta \in \Phi$ with $(\alpha \mid \beta) < 0$ and $\alpha \neq -\beta$. Then $\alpha + \beta \in \Phi$.*

The statements of Lemma 4.2.7 and Lemma 4.2.9 do not hold in general. For an arbitrary crystallographic root system, one also has to consider the coroot lattice as the following Theorem demonstrates. It is a generalization of [Voi85, Proposition 1.6.1].

Theorem 4.2.12. *Let Φ be a crystallographic root system, $\Phi' \subseteq \Phi$ be a root subsystem, $R := \{\beta_1, \dots, \beta_k\} \subseteq \Phi'$ be a non-empty set of roots. The following statements are equivalent:*

- *(a) The root subsystem Φ' is the smallest root subsystem of Φ containing R (i.e., the intersection of all root subsystems containing R).*
- *(b) $\Phi' = W_R(R)$.*
- *(c) $W_{\Phi'} = W_R$.*
- *(d) $L(\Phi') = L(R)$ and $L((\Phi')^\vee) = L(R^\vee)$.*

Proof. Obviously $W_R(R)$ is a root system with $R \subseteq W_R(R)$. Thus if (a) holds, then $\Phi' \subseteq W_R(R)$. As $W_R(R) \subseteq W_{\Phi'}(\Phi') = \Phi'$, it follows that (a) implies (b). The converse direction follows from the definition of a root subsystem.

Statement (b) implies that $R \subseteq \Phi'$ and therefore we have $W_R \subseteq W_{\Phi'}$. We show that in this case $W_{\Phi'} \subseteq W_R$. To this end, let $\alpha \in \Phi' = W_R(R)$. Then $\alpha = w(\beta_i)$ for some $w \in W_R$ and $i \in \{1, \dots, k\}$. Then

$$s_\alpha = s_{w(\beta_i)} = w s_{\beta_i} w^{-1} \in W_R,$$

which shows the claim. Thus (b) implies (c).

Assume (c) and let Φ'' be the smallest root subsystem of Φ containing R. Then $W_{\Phi'} = W_R \subseteq W_{\Phi''}$. By definition of Φ'' we have $\Phi'' \subseteq \Phi'$. If $\Phi'' \subsetneq \Phi'$ then there exists $\alpha \in \Phi' \setminus \Phi''$, hence a reflection $s_\alpha \in W_{\Phi'}$ with $s_\alpha \notin W_{\Phi''}$. Thus $W_{\Phi''} \subsetneq W_\Phi$, a contradiction. Hence $\Phi'' = \Phi'$, which shows (a).

Next we show that (c) implies (d). So assume (c) and let $t_i := s_{\beta_i}$, $1 \leq i \leq k$. Let $T_{\Phi'}$ be the set of reflections in $W_{\Phi'}$. By [Dye90, Corollary 3.11 (ii)], we have $T_{\Phi'} = \{w t_i w^{-1} \mid 1 \leq i \leq k, w \in W_{\Phi'}\}$. In particular any root in Φ' has the form $w(\beta_i)$ for some $w \in W_{\Phi'}$, $1 \leq i \leq k$. Since $W_{\Phi'} = \langle t_1, \dots, t_k \rangle$, we can write $w = t_{i_1} \cdots t_{i_m}$ with $1 \leq i_j \leq k$ for each $1 \leq j \leq m$. Since

Φ' is crystallographic it follows that $w(\beta_i) = t_{i_1} \cdots t_{i_m}(\beta_i)$ is an integral linear combination of the β_j's, hence that $\Phi' \subseteq L(R)$. Since $R \subseteq \Phi'$ we get that $L(R) = L(\Phi')$. By Proposition 2.1.9 there is an isomorphism $\varphi : W_{\Phi'} \xrightarrow{\sim} W_{(\Phi')^\vee}$ with $\varphi(s_\alpha) = s_{\alpha^\vee}$. Thus

$$\langle s_{\beta_1^\vee}, \dots, s_{\beta_k^\vee} \rangle = \varphi(\langle s_{\beta_1}, \dots, s_{\beta_k} \rangle) = \varphi(W_{\Phi'}) = W_{(\Phi')^\vee}.$$

Using the same argumentation as before (now for $(\Phi')^\vee$ and $R^\vee$ instead of Φ' and R) we obtain $L((\Phi')^\vee) = L(R^\vee)$, which shows (d).

It remains to show that (d) implies one of the other statements. So assume $L(R) = L(\Phi')$ and $L(R^\vee) = L((\Phi')^\vee)$. We show (a), that is, if Φ'' denotes the smallest root susbsystem of Φ' containing R, then $\Phi'' = \Phi'$. Since $R \subseteq \Phi'' \subseteq \Phi'$ and $L(R) = L(\Phi')$, we have $L(\Phi'') = L(R)$. Let $\gamma \in \Phi'$. It remains to show that $\gamma \in \Phi''$. Since $L(\Phi') = L(R) = L(\Phi'')$, we have

$$\gamma = \sum_{i=1}^{m} \mu_i \beta_i$$

with $\mu_i \in \mathbb{Z}$ and $\beta_i \in \Phi''$. As $\beta_i \in \Phi''$ implies $-\beta_i \in \Phi''$, we may assume $\mu_i \in \mathbb{Z}_{>0}$. Therefore we can write

$$\gamma = \sum_{i=1}^{m} \beta_i$$

with $\beta_i \in \Phi''$ and we may assume that m is minimal with that property (note that $\beta_i = \beta_j$ for $i \neq j$ is possible and that m might have changed). We obtain

$$(\gamma \mid \gamma) = \sum_{i=1}^{m} (\beta_i \mid \beta_i) + \sum_{i \neq j} (\beta_i \mid \beta_j),$$

thus

$$1 = \sum_{i=1}^{m} \frac{(\beta_i \mid \beta_i)}{(\gamma \mid \gamma)} + \sum_{i \neq j} \frac{(\beta_i \mid \beta_j)}{(\gamma \mid \gamma)}.$$

Assume $\gamma \notin \Phi''$. This implies $m \geq 2$. If the root γ is short, then $\sum_{i=1}^{m} \frac{(\beta_i|\beta_i)}{(\gamma|\gamma)} \geq 2$, hence $(\beta_i \mid \beta_j) < 0$ for some $i \neq j$. By the minimality of m we have $\beta_i \neq -\beta_j$. Therefore $\beta_i + \beta_j \in \Phi''$ by Lemma 4.2.11, contradicting the minimality of m. Thus $\gamma \in \Phi''$.

If the root γ is long, then $\gamma^\vee$ is short. Since $L((\Phi')^\vee) = L(R^\vee)$, we can argue as before and obtain $\gamma^\vee \in (\Phi'')^\vee$. Thus $\gamma \in \Phi''$. □

Remark 4.2.13. Both conditions in part (d) of Theorem 4.2.12 are necessary. To see this let Φ be the root system of type B_n and R the set of short roots. Then $L(\Phi) = L(R)$, but W_R is a proper subgroup of W_Φ as well as $L(R^\vee) \subsetneq L(\Phi^\vee)$. Note that none of the short roots in $\Phi^\vee$ is contained in $L(R^\vee)$.

The following two results will be useful tools for the next section. The following proposition is a generalization of [Voi85, Prop. 1.5.2] to the crystallographic case.

Proposition 4.2.14. *Let Φ be a crystallographic root system and $\Phi' \subseteq \Phi$ be a root subsystem with* $\mathrm{rk}(\Phi') = m = \mathrm{rk}(\Phi)$. *Then*

$$\left| L(\Phi) : L(\Phi') \right| \left| L(\Phi^\vee) : L((\Phi')^\vee) \right| = i(\Phi') i(\Phi)^{-1}.$$

Proof. Since Φ and Φ' have the same rank, there is a lattice isomorphism $\mathcal{L} : L(\Phi) \xrightarrow{\sim} L(\Phi')$, hence $|L(\Phi) : L(\Phi')| = |\det(\mathcal{L})|$ (see [Ebe02, Section 1.1]). Furthermore we have

$$\begin{aligned} P((\Phi')^\vee) &= \{x \in V \mid (x \mid y) \in \mathbb{Z},\ \forall y \in L(\Phi')\} \\ &= \{x \in V \mid (x \mid \mathcal{L}(y)) \in \mathbb{Z},\ \forall y \in L(\Phi)\} \\ &= \{x \in V \mid (\mathcal{L}^t(x) \mid y) \in \mathbb{Z},\ \forall y \in L(\Phi)\} \\ &= \{x \in V \mid \mathcal{L}^t(x) \in P(\Phi^\vee)\} \\ &= \left(\mathcal{L}^t\right)^{-1}(P(\Phi^\vee)). \end{aligned}$$

Thus

$$\left[P((\Phi')^\vee) : P(\Phi^\vee)\right] = |\det(\mathcal{L}^t)| = |\det(\mathcal{L})| = |L(\Phi) : L(\Phi')|.$$

If we take a lattice isomorphism $\mathcal{L} : L(\Phi^\vee) \xrightarrow{\sim} L((\Phi')^\vee)$ then a similar argumentation yields

$$\left[P(\Phi') : P(\Phi)\right] = |L(\Phi^\vee) : L((\Phi')^\vee)|.$$

We have

$$L(\Phi') \subseteq L(\Phi) \subseteq P(\Phi) \subseteq P(\Phi').$$

It follows that

$$|L(\Phi) : L(\Phi')| \underbrace{|P(\Phi) : L(\Phi)|}_{=i(\Phi)} \underbrace{|P(\Phi') : P(\Phi)|}_{=|L(\Phi^\vee):L((\Phi')^\vee)|} = \underbrace{|P(\Phi') : L(\Phi')|}_{=i(\Phi')},$$

which concludes the proof. □

Consider a crystallographic root system Φ and an arbitrary subset $R \subseteq \Phi$. Let Φ' be the smallest root subsystem of Φ containing R. We extend the definition of connection index by defining

$$i(R) := |P(\Phi') : L(\Phi')| = |P((\Phi')^\vee) : L((\Phi')^\vee)|.$$

By Theorem 4.2.12 this is well-defined.

The following theorem is part of the Diploma thesis of Kluitmann for Φ a simply-laced root system (see also [Voi85]). We extend it to crystallographic root systems.

Theorem 4.2.15. *Let Φ be a crystallographic root system. Let $w \in W_\Phi$ and let $(s_{\alpha_1}, \ldots, s_{\alpha_k})$, $(s_{\beta_1}, \ldots, s_{\beta_k}) \in \mathrm{Red}_T(w)$, where $\alpha_i, \beta_i \in \Phi$, for $1 \leq i \leq k$. Then for $R := \{\alpha_1, \ldots, \alpha_k\}$ and $Q := \{\beta_1, \ldots, \beta_k\}$ we have*

$$i(R) = i(Q).$$

Proof. Let Φ' (resp. Φ'') be the smallest root subsystem of Φ containing R (resp. Q). Consider $L(\Phi')$ and $P((\Phi')^\vee)$ (respectively $L(\Phi'')$ and $P((\Phi'')^\vee)$) as lattices in $V' := \mathrm{span}_{\mathbb{R}}(R)$ (respectively in $V'' := \mathrm{span}_{\mathbb{R}}(Q)$). By Lemma 2.3.1 the set R is linearly independent, thus a basis for $L(\Phi')$. Let $\{\alpha_1^*, \ldots, \alpha_k^*\}$ be the basis of $P((\Phi')^\vee)$ dual to R, that is $(\alpha_i \mid \alpha_j^*) = \delta_{ij}$. Then

$$\begin{aligned}(w - \mathrm{id})(\alpha_i^*) &= s_{\alpha_1} \cdots s_{\alpha_i} \cdots s_{\alpha_k}(\alpha_i^*) - \alpha_i^* \\ &= s_{\alpha_1} \cdots s_{\alpha_i}(\alpha_i^*) - \alpha_i^* \\ &= s_{\alpha_1} \cdots s_{\alpha_{i-1}} \left(\alpha_i^* - \frac{2}{(\alpha_i|\alpha_i)} \alpha_i \right) - \alpha_i^* \\ &= \alpha_i^* - \frac{2}{(\alpha_i|\alpha_i)} s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i) - \alpha_i^* \\ &= -\frac{2}{(\alpha_i|\alpha_i)} s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i).\end{aligned}$$

Since $(\alpha_i \mid \alpha_i) = (s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i) \mid s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i))$, we obtain

$$(w - \mathrm{id})(\alpha_i^*) = -s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)^\vee \in (\Phi')^\vee.$$

It follows by Lemma 4.2.16 that $\{(w - \mathrm{id})(\alpha_i^*) \mid 1 \leq i \leq k\}$ is a basis of $L(R^\vee) = L((\Phi')^\vee)$ and in particular of V'. Hence the map

$$(w - \mathrm{id})_{|V'} : P((\Phi')^\vee) \to L((\Phi')^\vee)$$

is bijective.

Thus $i(R) = |\det(w - 1)_{|V'}|$. The same argument with Q instead of R yields that $i(Q) = |\det(w-1)_{|V''}|$. By the proof of [Arm09, Theorem 2.4.7] we have that R and Q are both bases for $\mathrm{Mov}(w)$, hence $V' = V'' = \mathrm{Mov}(w)$ and hence $i(R) = i(Q)$. □

Lemma 4.2.16. *Let Φ be a crystallographic root system and $R = \{\alpha_1, \ldots, \alpha_k\} \subseteq \Phi$ be a set of linearly independent roots. Then*

$$\begin{aligned} L(R) &= \mathrm{span}_{\mathbb{Z}}(\alpha_1, s_{\alpha_1}(\alpha_2), \ldots, s_{\alpha_1} \cdots s_{\alpha_{k-1}}(\alpha_k)) \\ \text{and } L(R^\vee) &= \mathrm{span}_{\mathbb{Z}}(\alpha_1^\vee, s_{\alpha_1}(\alpha_2)^\vee, \ldots, s_{\alpha_1} \cdots s_{\alpha_{k-1}}(\alpha_k)^\vee).\end{aligned}$$

Proof. Let Φ' be the smallest root susbsystem of Φ containing R. By Theorem 4.2.12 we have

$$W_{\Phi'} = \langle s_{\alpha_1}, \ldots, s_{\alpha_k} \rangle.$$

We consider the Hurwitz orbit, that is

$$\begin{aligned}(s_{\alpha_k}, \ldots, s_{\alpha_1}) &\sim (s_{\alpha_1}, s_{s_{\alpha_1}(\alpha_k)}, \ldots, s_{s_{\alpha_1}(\alpha_2)}) \\ &\sim \big(s_{\alpha_1}, s_{s_{\alpha_1}(\alpha_2)}, s_{s_{\alpha_1}s_{\alpha_2}(\alpha_k)}, \ldots, s_{s_{\alpha_1}s_{\alpha_2}(\alpha_3)}\big) \\ &\cdots \\ &\sim \big(s_{\alpha_1}, s_{s_{\alpha_1}(\alpha_2)}, \ldots, s_{s_{\alpha_1} \cdots s_{\alpha_{k-1}}(\alpha_k)}\big).\end{aligned}$$

Therefore we obtain by Lemma 3.1.3 that

$$W_{\Phi'} = \langle s_{\alpha_1}, s_{s_{\alpha_1}(\alpha_2)}, \ldots, s_{s_{\alpha_1} \cdots s_{\alpha_{k-1}}(\alpha_k)} \rangle.$$

By Theorem 4.2.12 we obtain

$$\begin{aligned} L(\Phi') &= \mathrm{span}_{\mathbb{Z}}(\alpha_1, s_{\alpha_1}(\alpha_2), \ldots, s_{\alpha_1} \cdots s_{\alpha_{k-1}}(\alpha_k)) \\ \text{and } L((\Phi')^\vee) &= \mathrm{span}_{\mathbb{Z}}(\alpha_1^\vee, s_{\alpha_1}(\alpha_2)^\vee, \ldots, s_{\alpha_1} \cdots s_{\alpha_{k-1}}(\alpha_k)^\vee),\end{aligned}$$

which yields the assertion. □

4.3. Reflection subgroups related to prefixes of quasi-Coxeter elements

In this section we prove that each reduced T-decomposition of a parabolic quasi-Coxeter element generates the same parabolic subgroup. Notice that by definition, there is a priori just one decomposition which ensures this. Further we show that the reflections in a reduced T-decomposition of an element $w \in W$ generate a parabolic subgroup whenever $w \leq_T w'$ for some quasi-Coxeter element w'. Last but not least we demonstrate that parabolic quasi-Coxeter elements coincide with parabolic Coxeter elements in types A_n, B_n and $I_2(m)$.

Recall that for $w \in W$, we denote by P_w the parabolic closure of w (see Definition 2.4.9) and that $P_w = C_W(\mathrm{Fix}(w))$ (see Section 2.4.2).

Theorem 4.3.1. *Let (W, T) be a crystallographic dual Coxeter system of rank n. If $w \in W$ is a parabolic quasi-Coxeter element, then the reflections in any reduced T-decomposition of w generate the parabolic subgroup P_w. That is, for each $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$ we have $P_w = \langle t_1, \dots, t_m \rangle$.*

Proof. By the definition of a parabolic quasi-Coxeter element, there exists $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$ such that $P := \langle t_1, \dots, t_m \rangle$ is a parabolic subgroup. By Lemma 2.4.10, we have $P \subseteq C_W(\mathrm{Fix}(w)) = P_w$. Since $w \in P$, we have by definition of the parabolic closure that $P = P_w$. Let $(q_1, \dots, q_m) \in \mathrm{Red}_T(w)$. Then for all $1 \leq i \leq m$ we have that $q_i \leq_T w$, which yields that q_i is in $C_W(\mathrm{Fix}(w)) = P_w$. Thus $W' := \langle q_1, \dots, q_m \rangle$ is a subgroup of P_w. Let Φ be the root system of P_w and $\beta_i \in \Phi$ be such that $q_i = s_{\beta_i}$, for $1 \leq i \leq m$. Then $L(\{\beta_1, \dots, \beta_m\}) = L(\Phi')$ is a sublattice of $L(\Phi)$, where Φ' is the smallest root subsystem of Φ that contains $\beta_1, \dots, \beta_m$. Therefore Theorem 4.2.15 yields that $i(\Phi') = i(\Phi)$. By Proposition 4.2.14 we obtain

$$\left|L(\Phi) : L(\Phi')\right| = \left|L(\Phi^\vee) : L((\Phi')^\vee)\right| = 1$$

Thus $L(\Phi) = L(\Phi')$ and $L(\Phi^\vee) = L((\Phi')^\vee)$, which yields $W' = W_{\Phi'} = W_\Phi = W$ by Theorem 4.2.12. □

We will show in Corollary 4.3.12 that the following property of parabolic quasi-Coxeter elements does in fact characterize them.

Proposition 4.3.2. *Let (W, T) be a finite dual Coxeter system. If $w \in W$ is a parabolic quasi-Coxeter element, then there exists a quasi-Coxeter element $w' \in W$ such that $w \leq_T w'$.*

Proof. Let $w \in W$ be a parabolic quasi-Coxeter element. By definition, there exists a simple system $S = \{s_1, \dots, s_n\}$ for W and a T-reduced decomposition $w = t_1 \cdots t_m$ such that $\langle t_1, \dots, t_m \rangle = \langle s_1, \dots, s_m \rangle$, with $m \leq n$. Set $w' := t_1 \cdots t_m s_{m+1} \cdots s_n$. It is clear that

$$\langle t_1, \dots, t_m, s_{m+1}, \dots, s_n \rangle = W.$$

Moreover we have $\ell_T(w') = n$, hence w' is a quasi-Coxeter element with $w \leq_T w'$. □

Lemma 4.3.3. *Let (W, T) be a dual Coxeter system of type A_n. Then each $w \in W$ is a classical parabolic Coxeter element.*

Proof. In type A_n, the set of $(n+1)$-cycles forms a single conjugacy class. Hence the set of classical Coxeter elements is exactly the set of $(n+1)$-cycles (see Remark 2.4.4 (a)). The assertion follows with Remark 2.4.4 (b) as for every element $w \in W$, we have $w \leq_T w'$ for at least one $(n+1)$-cycle w'. □

Lemma 4.3.4. *Let (W, T) be a dual Coxeter system of type B_n. Then every parabolic quasi-Coxeter element $w \in W$ for (W, T) is a classical parabolic Coxeter element.*

Proof. For the proof we use the combinatorial description of W_{B_n} as given in [BB05, Section 8.1]. Let $S_{-n,n}$ be the group of permutations of $[\pm n] = \{-n, -n+1, \dots, -1, 1, \dots, n\}$ and define

$$W = W_{B_n} := \{w \in S_{-n,n} \mid w(-i) = -w(i),\ \forall i \in [\pm n]\},$$

also known as a *hyperoctahedral group*. Then (W, S) is a Coxeter system of type B_n with

$$S = \{(1, -1), (1, 2)(-1, -2), \dots, (n-1, n)(-n+1, -n)\}.$$

The set of reflections T for this choice of S is given by

$$T = \{(i, -i) \mid i \in [n]\} \cup \{(i, j)(-i, -j) \mid 1 \leq i < |j| \leq n\}.$$

We show that every quasi-Coxeter element for (W, T) is a classical Coxeter element. If w is a parabolic quasi-Coxeter element, then by Proposition 4.3.2, there exists a quasi-Coxeter element $w' \in W$ such that $w \leq_T w'$. Hence if w' is a classical Coxeter element, then $w \leq_T w'$ implies that w is a classical parabolic Coxeter element.

Let $R = \{r_1, \dots, r_n\} \subseteq T$ be such that $\langle R \rangle = W$. It suffices to show that $r_1 r_2 \cdots r_n$ is in fact a classical Coxeter element.

The group W cannot be generated only by reflections of type $(i, j)(-i, -j)$, $i \neq \pm j$. Therefore there exists $i \in [n]$ with $(i, -i) \in R$. If there exists $j \in [n], j \neq i$ with $(j, -j) \in R$, then R cannot generate the whole group W. Since classical Coxeter elements are closed under conjugation, we can conjugate the set R with $(1, i)(-1, -i)$ (if necessary) and assume $(1, -1) \in R$.

Since R generates the whole group W, there does not exist $j \in [n]$ which is fixed by each $r \in R$. Thus for each $k \in [n], k \neq 1$, we can find $i_k \in [\pm n]$ with $k \neq \pm i_k$ such that $(k, i_k)(-k, -i_k) \in R$. Therefore

$$R = \{(1, -1), (2, i_2)(-2, -i_2), \dots, (n, i_n)(-n, -i_n)\}.$$

Note that some i_j has to equal ± 1, because otherwise $(1, -1)$ would commute with any element of W. By conjugating R with $(j, 2)(-j, -2)$ resp. $(j, -2)(-j, 2)$ (if necessary) we can assume that $i_2 = 1$. Hence after rearrangement we can assume that R is of the form

$$R = \{(1, -1), (2, 1)(-2, -1), (3, i_3)(-3, -i_3), \dots, (n, i_n)(-n, -i_n)\}.$$

Similarly to what we did above, there exists $j \geq 3$ with $i_j \in \{\pm 1, \pm 2\}$. By conjugating R with $(j, 3)(-j, -3)$ resp. $(j, -3)(-j, 3)$ (if necessary) we can assume that $i_3 \in \{1, 2\}$. Continuing in this manner we obtain

$$R = \{(1, -1), (2, i_2)(-2, -i_2), \dots, (n, i_n)(-n, -i_n)\}$$

with $i_j \in \{1, \dots, j-1\}$ for each $j \in \{2, \dots, n\}$. A direct computation shows that $c := r_1 r_2 \cdots r_n$ is a $2n$-cycle and thus a classical Coxeter element. Indeed, there is a single conjugacy class of $2n$-cycles in W. □

Remark 4.3.5. Notice that by Remark 2.4.4 (b), it is already known that classical Coxeter elements and Coxeter elements must coincide in type B_n. Moreover it follows from [Car72, Lemma 8, Theorem A] that every quasi-Coxeter element is actually a Coxeter element. Hence one can derive Lemma 4.3.4 from these two observations. However since both of them rely on sophisticated methods, we preferred to give here a direct proof using the combinatorics of the hyperoctahedral group.

Remark 4.3.6. In type A_n, we even have that every element w such that $\ell_T(w) = n$ is a classical Coxeter element (thus quasi-Coxeter), because such an element is necessarily an $(n+1)$-cycle. Notice that this fails in type B_n. For instance, the product $(1,-1)(2,-2)\cdots(n,-n)$ in W_{B_n} has absolute length equal to n, but it is not a quasi-Coxeter element.

The following is well-known (see [Bou02, IV, 1.2, Proposition 2]):

Proposition 4.3.7. *A group W is a dihedral group if and only if it is generated by two elements s, t of order 2, in which case $\{s, t\}$ is a simple system for W.*

Corollary 4.3.8. *Let (W, T) be a dual Coxeter system of type $I_2(m)$, $m \geq 3$. Then w is a quasi-Coxeter element in W if and only if w is a Coxeter element in W. It follows that $w \in W$ is a parabolic quasi-Coxeter element if and only if w is a parabolic Coxeter element.*

Note that Coxeter elements and classical Coxeter elements do not coincide in general in dihedral type (see Remark 2.4.4 (b)).

Theorem 4.3.9. *Let w be a quasi-Coxeter element in a finite dual Coxeter system (W, T) of rank n and $(t_1, \ldots, t_n) \in Red_T(w)$ such that $W = \langle t_1, \ldots, t_n \rangle$. Then the reflection subgroup $W' := \langle t_1, \ldots, t_{n-1} \rangle$ is parabolic.*

Proof. The reduction to the case where (W, T) is irreducible is easy. The proof is uniform for the crystallographic types and case-by-case for the non crystallographic types.

(Crystallographic types) Let Φ be a root system for (W, T) with ambient vector space V. Let $P_{W'}$ be the parabolic closure of W'. For $1 \leq i \leq n$, let $\beta_i \in \Phi$ be a root corresponding to t_i and let Φ' be the smallest subsystem of Φ containing $R := \{\beta_1, \ldots, \beta_{n-1}\}$ so that $W_R = W_{\Phi'} = W'$ (see Theorem 4.2.12). Let $\Psi \subseteq \Phi$ be the root subsystem of Φ associated to $P_{W'}$. We have $W' \leq P_{W'} = W_\Psi$. By Lemma 2.3.1 the set $R \cup \{\beta_n\}$ is a basis of V and thus so is $R \cup \{\beta_n^\vee\}$.

Let U be the ambient vector space for Ψ. As the linearly independent set R is a subset of Ψ, the dimension of U is at least $n-1$. Since $P_{W'}$ is the parabolic closure of $\langle t_1, \ldots, t_{n-1} \rangle$ it has to be the centralizer of a line in V and therefore $\dim U = n-1$. It follows that $U = \operatorname{span}_\mathbb{R}(\beta_1, \ldots, \beta_{n-1})$.

By Theorem 4.2.12 we have that $L(\Phi') = L(\{\beta_1, \ldots, \beta_{n-1}\})$ and $L(\Phi) = L(\{\beta_1, \ldots, \beta_n\})$ as well as $L((\Phi')^\vee) = L(\{\beta_1^\vee, \ldots, \beta_{n-1}^\vee\})$ and $L(\Phi^\vee) = L(\{\beta_1^\vee, \ldots, \beta_n^\vee\})$. Since $V = U \oplus \mathbb{R}\,\beta_n = U \oplus \mathbb{R}\,\beta_n^\vee$ we have

$$L(\Phi) \cap U = L(\{\beta_1, \ldots, \beta_{n-1}\}) = L(\Phi')$$
$$\text{and } L(\Phi^\vee) \cap U = L(\{\beta_1^\vee, \ldots, \beta_{n-1}^\vee\}) = L((\Phi')^\vee)$$

As $L(\Psi) \subseteq U$ (resp. $L(\Psi^\vee) \subseteq U$), it follows that $L(\Psi) \subseteq L(\Phi')$ (resp. $L(\Psi^\vee) \subseteq L((\Phi')^\vee)$). But since $\Phi' \subseteq \Psi$ and $(\Phi')^\vee \subseteq \Psi^\vee$ we get that $L(\Phi') \subseteq L(\Psi)$ and $L((\Phi')^\vee) \subseteq L(\Psi^\vee)$. Therefore $L(\Phi') = L(\Psi)$ and $L((\Phi')^\vee) = L(\Psi^\vee)$. Thus $W' = W_{\Phi'} = W_\Psi = P_{W'}$ by Theorem 4.2.12.

(Types H_3 and H_4) We refer to [DPR14, Tables 8 and 9], where the reflection subgroups of W_{H_3} and W_{H_4} and their parabolic closures are determined.

1. Each rank 2 reflection subgroup of the group W_{H_3} is already parabolic.

2. The only rank 3 reflection subgroup of W_{H_4} that is not parabolic, is of type $A_1 \times A_1 \times A_1$. Taking a set of three reflections generating such a reflection subgroup, we checked using [GAP2015] that this set cannot be completed to obtain a generating set for W_{H_4} by adding a single reflection.

(Dihedral type) The claim is obvious in that case. The non-trivial parabolic subgroups are precisely those subgroups that are generated by a reflection

□

Remark 4.3.10. Theorem 4.3.9 is not true in general. It can even fail if w is a Coxeter element, as the following example borrowed from [HK13, Example 5.7] shows: Let $W = \langle s, t, u \rangle$ be of affine type $\widetilde{A}_2$, and let $w = stu$. Then $s(tut) \leq_T c$, but $W' = \langle s, tut \rangle$ is an infinite dihedral group, hence it is not a classical parabolic subgroup since proper classical parabolic subgroups of W are finite. We mentioned in Subsection 2.4.2 that for affine Coxeter groups classical parabolic subgroups coincide with parabolic subgroups (see Subsection 2.4.1 for definitions).

Corollary 4.3.11. *Let (W, T) be a finite dual Coxeter system of rank n and W' a reflection subgroup of rank $n - 1$. Then W' is a parabolic subgroup if and only if there exists $t \in T$ such that $\langle W', t \rangle = W$.*

Proof. The necessary condition is clear by the definition of parabolic subgroup. The sufficient condition is a direct consequence of Theorem 4.3.9. □

We also derive a characterization of parabolic quasi-Coxeter elements analogous to that of parabolic Coxeter elements (see Remark 2.4.4 (b)).

Corollary 4.3.12. *Let (W, T) be a finite dual Coxeter system and $w \in W$. Then w is a parabolic quasi-Coxeter element if and only if there exists a quasi-Coxeter element $w' \in W$ such that $w \leq_T w'$.*

Proof. The forward direction is given by Proposition 4.3.2. Now let $w \leq_T w'$, where $w' \in W$ is a quasi-Coxeter element. Using Theorem 4.3.9 inductively we get that w is a parabolic quasi-Coxeter element in W. □

Remark 4.3.13. Corollary 4.3.12 does not hold for infinite Coxeter groups as it fails for the Coxeter element given in Remark 4.3.10.

4.4. Intersection of maximal parabolic subgroups in type D_n

The aim of this section is to show the following result which will be needed in the next section in the proof of Theorem 1.1.1.

Proposition 4.4.1. *Let (W, S) be a Coxeter system of type D_n ($n \geq 6$). Then the intersection of two maximal parabolic subgroups is non-trivial.*

Remark 4.4.2. This statement is not true in general, not even in the simply-laced case. In particular, it fails in types D_4, D_5, E_7 and E_8. For example:

(a) Let (W, S) be of type D_4 where $S = \{s_0, s_1, s_2, s_3\}$ with s_2 commuting with no other simple reflection, then both $W' := \langle s_0, s_1, s_3\rangle$ and $s_2 W' s_2$ are maximal parabolic subgroups of type $A_1 \times A_1 \times A_1$ and have trivial intersection.

(b) Let (W, S) be of type E_7 where $S = \{s_1, \ldots, s_7\}$ labelled as in [Bou02, Plate VI]. Let $I = \{s_1, s_2, s_3, s_4, s_6, s_7\}$ and $J = \{s_1, s_2, s_3, s_5, s_6, s_7\}$. Then the non-conjugate parabolic subgroups W_J and $wW_I w^{-1}$ intersect trivially, where

$$w = s_6 s_2 s_4 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_6 s_7 s_4 s_5 s_6 s_4 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_4 s_3 s_2 s_4 s_1.$$

This was checked using [GAP2015].

For the rest of this section we work with the combinatorial realization of W as a subgroup (which we denote by W_{D_n}) of the hyperoctahedral group W_{B_n} (see Section 4.3). To this end, set

$$\begin{aligned} s_0 &= (1, -2)(-1, 2) \\ s_i &= (i, i+1)(-i, -(i+1)) \text{ for } i \in [n-1]. \end{aligned}$$

Then $\{s_0, s_1, \ldots, s_{n-1}\}$ is a simple system for a Coxeter group W_{D_n} of type D_n. The set of reflections is given by $T = \{(i, j)(-i, -j) \mid i, j \in [\pm n], i \neq \pm j\}$. Notice that W is a subgroup of the group W_{B_n} of type B_n; indeed, the above generators are clearly contained in W_{B_n}. Given $A \subset [\pm n]$, write $\mathrm{Stab}(A)$ for the subgroup of W_{D_n} of elements preserving the set A. Notice that since $W_{D_n} \subseteq W_{B_n}$, we have $\mathrm{Stab}(A) = \mathrm{Stab}(-A)$. The maximal standard parabolic subgroups of W_{D_n} are then described as follows (see [BB05, Prop. 8.2.4]). Let $i \in \{0, 1, \ldots, n-1\}$ and $I = S \setminus \{s_i\}$. Then $W_I = \mathrm{Stab}(A_I)$, where

$$A_I := \begin{cases} [i+1, n] & \text{if } i \neq 1 \\ \{-1, 2, 3, \ldots, n\} & \text{if } i = 1. \end{cases}$$

Since W_I stabilizes both A_I and $-A_I$, it stabilizes also the complement A_I^0 of $A_I \cup (-A_I)$ in $[\pm n]$. Notice that $A_I^0 = -A_I^0$.

From this description we can easily achieve a description of maximal parabolic subgroups:

Lemma 4.4.3. *If $W_J = \mathrm{Stab}(A_J)$ is a maximal standard parabolic subgroup and $w \in W$, then $wW_J w^{-1} = \mathrm{Stab}(w(A_J))$.*

Proof of Proposition 4.4.1. It is enough to show that $W_I \cap wW_J w^{-1} \neq \{\mathrm{id}\}$ for $I, J \subseteq S$ with $|I| = |J| = n-1$ and $w \in W$. We therefore assume that $W_I \cap wW_J w^{-1} = \{\mathrm{id}\}$ for some $I, J \subseteq S$ with $|I| = |J| = n-1$ and $w \in W$ and show that this implies that $n \leq 5$. Consider the intersections $A_I \cap w(A_J)$ and $A_I \cap (-w(A_J))$. If one of these intersections contains at least two elements, say k and l, then $(k, l)(-k, -l) \in W_I \cap wW_J w^{-1}$ since $A_I \cap (-A_I) = \varnothing = w(A_J) \cap (-w(A_J))$. Therefore we can assume that $|A_I \cap w(A_J)| \leq 1$ and $|A_I \cap -w(A_J)| \leq 1$. Now if $|A_I| \geq 4$, then $|A_I \cap w(A_J)^0| \geq 2$, and since $A_I \cap (-A_I) = \varnothing$ it follows that there exist $k, \ell \in A_I \cap w(A_J)^0$ with $k \neq \pm\ell$, and we then have that $(k, \ell)(-k, -\ell) \in W_I \cap wW_J w^{-1}$. Hence we can furthermore assume that $|A_I| < 4$. It follows that $|A_I^0| \geq 2n - 6$.

But arguing similarly we can also assume that $|A_I^0 \cap w(A_J)^0| < 4$, hence $|A_I^0 \cap w(A_J)^0| \leq 2$ since it has to be even and $|A_I^0 \cap w(A_J)| \leq 1$. It follows that $|A_I^0| \leq 4$. Together with the inequality above we get $2n - 6 \leq |A_I^0| \leq 4$, hence $n \leq 5$. $\square$

4.5. The proof of Theorem 1.1.2

The aim of this section is to prove Theorem 1.1.2. Therefore let (W, T) be a finite, dual Coxeter system of rank n and let $w \in W$.

We first prove the necessary condition. Assume that the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive. Let $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$. By Proposition 2.4.11, $W' := \langle t_1, \ldots, t_m \rangle$ is a classical parabolic subgroup. By Corollary 2.4.12, it follows that W' is parabolic and hence w is a parabolic quasi-Coxeter element.

To prove the sufficient condition we first consider (W, T) of crystallographic type and then treat the remaining types. Note that if w is a parabolic quasi-Coxeter element in (W, T), then each conjugate of w is also a parabolic quasi-Coxeter element in (W, T). Since the Hurwitz action commutes with conjugation, we can restrict ourselves to check transitivity for one representative of each conjugacy class of parabolic quasi-Coxeter elements of W. The proof of the following is easy:

Lemma 4.5.1. *Let (W_i, T_i), $i = 1, 2$ be dual Coxeter systems and let $w_i \in W_i$, $i = 1, 2$. Then $(W_1 \times W_2, T := (T_1 \times \{1\}) \cup (\{1\} \times T_2))$ is a dual Coxeter system. Furthermore, if the Hurwitz action is transitive on $\mathrm{Red}_{T_i}(w_i)$, $i = 1, 2$, then the Hurwitz action is transitive on $\mathrm{Red}_T((w_1, w_2))$.*

4.5.1. The crystallographic types.

We first treat the parabolic quasi-Coxeter elements in an irreducible, finite, crystallographic dual Coxeter system (W, T) of rank n. First of all notice that in types A_n and B_n every parabolic quasi-Coxeter element is already a parabolic Coxeter element by Lemmata 4.3.3, 4.3.4 and Corollary 4.3.8. Therefore the assertion follows with Theorem 1.1.1.

For the remaining types we only need to show the assertion for quasi-Coxeter elements. Indeed, let w be a parabolic quasi-Coxeter element in (W, T) and let $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$. Then $W' = \langle t_1, \ldots, t_m \rangle$ is by Theorem 4.3.1 a parabolic subgroup of (W, T), in fact $W' = P_w$.

Therefore, it follows from Lemma 2.4.10 that all the reflections in any reduced factorization of w are in W'. The latter group is a direct product of irreducible Coxeter groups of crystallographic type. If we know that the Hurwitz action is transitive on $\mathrm{Red}_T(\widetilde{w})$ for all the quasi-Coxeter elements $\widetilde{w}$ in these irreducible Coxeter groups of crystallographic type, then the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive as well by Lemma 4.5.1 and Theorem 3.2.1.

The strategy to prove the theorem is as follows: we first show by induction on the rank n (with $n \geq 4$) that the Hurwitz action is transitive on the set of reduced decompositions of quasi-Coxeter elements of type D_n; for this we will need to use the result for parabolic subgroups, but since they are (products) of groups of type A with groups of type D of smaller rank, the result holds for groups of type A by Lemma 4.3.3 and they hold for groups of type D_k, $k < n$ by induction.

Using the fact that it holds for type D_n, $n \geq 4$, we then prove the result for the groups E_6, E_7 and E_8. Similarly as for type D_n, parabolic subgroups of type E are of type A, D or E. It was previously shown to hold for types A and D and holds for type E by induction. We proceed in the same way for F_4 since parabolic subgroups are of type A or B.

Let w be a quasi-Coxeter element and let $(t_1, \ldots, t_n) \in \mathrm{Red}_T(w)$.

Type D_n. For types D_4 and D_5 the assertion is checked directly using [GAP2015]. Therefore assume $n \geq 6$. Let $(r_1, \dots, r_n) \in \mathrm{Red}_T(w)$ be a second reduced decomposition of w. By Theorem 4.3.1 and Theorem 4.3.9 the groups $\langle t_1, \dots, t_{n-1} \rangle$ and $\langle r_1, \dots, r_{n-1} \rangle$ are maximal parabolic subgroups and since $\ell_T(wt_n) = n-1 = \ell_T(wr_n)$ it follows that $C_W(V^{wt_n}) = P_{wt_n} = \langle t_1, \dots, t_{n-1} \rangle$, $C_W(V^{wr_n}) = P_{wr_n} = \langle r_1, \dots, r_{n-1} \rangle$. By Proposition 4.4.1 there exists a reflection t in their intersection. It follows by Lemma 2.4.10 that $t \leq_T wt_n, wr_n$. Hence there exists $t_2', \dots, t_{n-1}', r_2', \dots, r_{n-1}' \in T$ such that $(t, t_2', \dots, t_{n-1}') \in \mathrm{Red}_T(wt_n)$ and $(t, r_2', \dots, r_{n-1}') \in \mathrm{Red}_T(wr_n)$. In particular we get

$$(t_2', \dots, t_{n-1}', t_n), (r_2', \dots, r_{n-1}', r_n) \in \mathrm{Red}_T(tw).$$

By Theorem 4.3.9 the element tw is quasi-Coxeter in the parabolic subgroup

$$P_{tw} = \langle t_2', \dots, t_{n-1}', t_n \rangle.$$

It follows from Lemma 2.4.10 that the reflections $r_2', \dots, r_{n-1}', r_n$ are in P_{tw} since $r_i' <_T tw$ for each i. As P_{tw} is a direct product of irreducible Coxeter groups of type A and D of smaller rank, we have by induction together with Lemma 4.5.1 that

$$(t_2', \dots, t_{n-1}', t_n) \sim (r_2', \dots, r_{n-1}', r_n),$$

as well as

$$(t, t_2', \dots, t_{n-1}') \sim (t_1, \dots, t_{n-1}) \text{ and } (t, r_2', \dots, r_{n-1}') \sim (r_1, \dots, r_{n-1}).$$

This implies

$$(t_1, \dots, t_n) \sim (t, t_2', \dots, t_{n-1}', t_n) \sim (t, r_2', \dots, r_{n-1}', r_n) \sim (r_1, \dots, r_n) \in \mathrm{Red}_T(w),$$

which concludes the proof.

Types E_6, E_7 and E_8. First we calculated representatives of the conjugacy classes of quasi-Coxeter elements using [GAP2015], see also Remark 4.5.2 (b) below. Then given a quasi-Coxeter element w we checked using again [GAP2015] that there is a reduced decomposition $(t_1, \dots, t_n)$ of w such that for every reflection t in T there exists $(t_1', \dots, t_{n-1}', t) \in \mathrm{Red}_T(w)$ with $(t_1, \dots, t_n) \sim (t_1', \dots, t_{n-1}', t)$.

Let $(r_1, \dots, r_n) \in \mathrm{Red}_T(w)$. By our computations in GAP there exists $(t_1', \dots, t_{n-1}', r_n) \in \mathrm{Red}_T(w)$ with $(t_1, \dots, t_n) \sim (t_1', \dots, t_{n-1}', r_n)$. Then

$$wr_n = t_1' \cdots t_{n-1}' = r_1 \cdots r_{n-1}$$

are reduced decompositions. Furthermore wr_n is a quasi-Coxeter element in (W', T') where $W' := \langle t_1', \dots, t_{n-1}' \rangle$ and $T' := T \cap W'$. By Theorem 4.3.9 we have that W' is a equal to the parabolic closure $P_{t_1' \cdots t_{n-1}'}$ of $t_1' \cdots t_{n-1}'$ and therefore $r_1, \dots, r_{n-1} \in W'$ by Lemma 2.4.10. Thus $(t_1', \dots, t_{n-1}')$ and $(r_1, \dots, r_{n-1})$ are reduced decompositions of a quasi-Coxeter element in a dual, simply-laced Coxeter system of rank $n-1$. By induction and by Lemma 4.5.1 we get $(t_1', \dots, t_{n-1}') \sim (r_1, \dots, r_{n-1})$, thus

$$(t_1, \dots, t_n) \sim (t_1', \dots, t_{n-1}', r_n) \sim (r_1, \dots, r_n).$$

Type F_4. The proof is analogous to the cases E_6, E_7 and E_8.

4.5.2. The types H_3 and H_4

For these cases we calculated representatives of the conjugacy classes of quasi-Coxeter elements using [GAP2015] and then we checked Theorem 1.1.2 directly for each representative using [GAP2015].

4.5.3. The dihedral types $I_2(m)$ $(m \geq 3)$

In type $I_2(m)$, parabolic quasi-Coxeter elements and parabolic Coxeter elements coincide, but parabolic Coxeter elements and classical parabolic Coxeter elements do not (see Remark 2.4.4 (b)). Nevertheless the assertion follows by Theorem 1.1.1.

Remark 4.5.2. For the convenience of the reader we shortly describe Carter's classification of the conjugacy classes in finite Weyl groups by means of so called admissible diagrams [Car72]. Due to [Car72, Lemma 8, Theorem A], we obtain the following description of conjugacy classes of quasi-Coxeter elements (in the notation of [Car72]):

- For the infinite families the conjugacy classes correspond to the admissible diagrams
$$A_n, B_n, D_n, D_n(a_1), D_n(a_2), \dots D_n(a_{\lfloor \frac{1}{2}n-1 \rfloor}).$$
In particular in types A_n and B_n the conjugacy class of the Coxeter element is the only quasi-Coxeter class (see Lemmata 4.3.3 and 4.3.4).
- For the exceptional types the conjugacy classes correspond to the admissible diagrams
$$E_6, E_6(a_1), E_6(a_2), E_7, E_7(a_1), \dots, E_7(a_4), E_8, E_8(a_1), \dots, E_8(a_8), F_4, F_4(a_1), G_2.$$

For the remaining non-crystallographic types we found by computer:

- For the type H_3 resp. H_4 there are 3 resp. 11 conjugacy classes of quasi-Coxeter elements.

Note that there might be more than one admissible diagram for the same conjugacy class (e.g. the class $E_7(a_2)$ might also be parameterized by the diagram $E_7(b_2)$). For (W,T) irreducible and crystallographic, the conjugacy classes of quasi-Coxeter elements are precisely described by the connected admissible diagrams with number of nodes equal to the rank of (W,T). In [CE72] such a class is called semi-Coxeter class.

Finally, let us state two direct consequences of Theorem 1.1.2.

Corollary 4.5.3. *Let (W,T) be a dual Coxeter system and let $w \in W$. Further let $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$, set $W' := \langle t_1, \dots, t_m \rangle$ and $T' := W' \cap T$. If W' is finite, then the Hurwitz action on $\mathrm{Red}_{T'}(w)$ is transitive.*

Corollary 4.5.4. *Let (W,T) be a finite dual Coxeter system of rank n and $w \in W$ quasi-Coxeter. Then for each $t \in T$ and each $(t_1, \dots, t_n) \in \mathrm{Red}_T(w)$ there exists $(t, t_2', \dots t_n') \in Red_T(w)$ with*
$$(t_1, \dots, t_n) \sim (t, t_2', \dots, t_n').$$

Proof. By Lemma 2.3.1 we have $t \leq_T w$ for each $t \in T$ and therefore the assertion follows by Theorem 1.1.2 □

5. Hurwitz action on nonreduced decompositions in finite Coxeter groups

5.1. Results for nonreduced decompositions

In the preceding chapters we focused on the Hurwitz action on the set $\mathrm{Red}_T(w)$ for an element w in a dual Coxeter system (W,T). At the beginning of Chapter 4 we considered the example of the element $w := stst$ in a Coxeter system $(W,\{s,t\})$ of type B_2 with the two Hurwitz orbits $\{(sts,t),(t,sts)\}$ and $\{(tst,s),(s,tst)\}$ inside $\mathrm{Red}_T(w)$. In particular we see that the requirements of Lemma 3.1.3 are not fulfilled. We have both $\langle sts,t\rangle \neq \langle tst,s\rangle$ and decompositions from different orbits have different multisets of conjugacy classes. To get rid of the first obstacle, we define for $w \in W$ with $\ell_T(w) = n$ the set

$$\mathrm{Fac}_{T,n+2}(w) = \{(t_1,\dots,t_{n+2}) \in T^{n+2} \mid w = t_1\cdots t_{n+2},\ \langle t_1,\dots,t_{n+2}\rangle = W\}.$$

Note that $\mathrm{Fac}_{T,n+2}(w) = \varnothing$ is possible. E.g. if we chose (W,T) to be of type F_4 and w to be a Coxeter element in a reflection subgroup of type $4A_1$, then $\mathrm{Fac}_{T,n+2}(w)$ turns out to be empty.

Example 5.1.1. Consider again the element $w = stst$ in the Coxeter system of type B_2 as above, then direct calculations in [GAP2015] show that the Hurwitz action is transitive on $\mathrm{Fac}_{T,4}(w)$. In this example all decompositions in $\mathrm{Fac}_{T,4}(w)$ share the same multiset of conjugacy classes. In general this fails to be true. Let $c := st$ be a Coxeter element in the Coxeter system of type B_2. Then $(s,t,s,s),(s,t,t,t) \in \mathrm{Fac}_{T,4}(c)$, but both decompositions do not share the same multiset of conjugacy classes.

In fact, the obstacle for Hurwitz transitivity on $\mathrm{Fac}_{T,n+2}(c)$ for c a (quasi-)Coxeter element as described in the previous example is the only one as the following result of Lewis and Reiner shows (see [LR16, Theorems 1.1 and 6.1]).

Theorem 5.1.2. *Let (W,T) be a finite dual Coxeter system and $c \in W$ a (quasi-)Coxeter element. Then two reflection decompositions of c lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

In particular, the Hurwitz action is transitive on $\mathrm{Fac}_{T,n+2}(c)$ for c a (quasi-)Coxeter element if and only if all decompositions in $\mathrm{Fac}_{T,n+2}(c)$ share the same multiset of conjugacy classes. For c a (quasi-)Coxeter element we have

$$\mathrm{Fac}_{T,n+2}(c) = \{(t_1,\dots,t_{n+2}) \in T^{n+2} \mid c = t_1\cdots t_{n+2}\}$$

as Theorem 1.1.2 and the following result imply. It is also due to Lewis and Reiner (see [LR16, Corollary 1.4]).

Theorem 5.1.3. *Let (W, T) be a finite dual Coxeter system and $w \in W$ with $\ell_T(w) = n$. Then every decomposition of w into m reflections lies in the Hurwitz orbit of some $(t_1, \ldots, t_m)$ such that*

$$\begin{aligned} t_1 &= t_2, \\ t_3 &= t_4, \\ &\vdots \\ t_{m-n-1} &= t_{m-n}, \end{aligned}$$

and $(t_{m-n+1}, \ldots, t_m) \in \mathrm{Red}_T(w)$.

Corollary 5.1.4. *Let w be quasi-Coxeter for (W, T) and $w = t_1 \cdots t_m$ an arbitrary (not neccesarily reduced) T-decomposition of w. Then $\langle t_1, \ldots, t_m \rangle = W$.*

Let us first point that all reflection decompositions of a fixed element have same parity.

Proposition 5.1.5. *Let (W, S) be a Coxeter system with reflections T. Let $w \in W$ and $w = t_1 \cdots t_k = r_1 \cdots r_l$ be two (not necessarily reduced) T-decompositions for w, then k and l differ by a multiple of 2.*

Proof. Consider the geometric representation $\sigma : W \to \mathrm{GL}(V)$ of W as introduced in Section 2.1.2. The reflections have determinant -1 with respect to this representation. Consider the decompositions $w = t_1 \cdots t_k$ and $w = r_1 \cdots r_l$ under the sign representation $w \mapsto \det(\sigma(w))$. Hence $(-1)^k = (-1)^l$, which yields the assertion. □

The rest of this section will be devoted to showing the following result.

Theorem 5.1.6. *Let (W, T) be a finite crystallographic irreducible dual Coxeter system of rank n, but not of type E_8, and let $w \in W$ with $\ell_T(w) = n$ which is not quasi-Coxeter and such that $\mathrm{Fac}_{T,n+2}(w) \neq \varnothing$. Then two elements of $\mathrm{Fac}_{T,n+2}(w)$ are in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

Remark 5.1.7. (a) In particular, if w is as in Theorem 5.1.6, then the Hurwitz action is transitive on $\mathrm{Fac}_{T,n+2}(w)$ for the oddly laced types A_n, D_n, E_6, E_7 and G_2 since in these cases all reflections are conjugated (see Lemma 2.1.3). In contrast to (quasi-)Coxeter elements the further assumption on the multiset of conjugacy classes (see Theorem 5.1.2) for type B_n seems not to be necessary. The reason seems to be that a situation as described in Example 5.1.1 for a Coxeter element in W_{B_2} cannot occur for non quasi-Coxeter elements since they do not have a reduced reflection decomposition which yields a generating set for W. In fact, the only example of an element w which is not quasi-Coxeter, but yields different multisets of conjugacy classes inside $\mathrm{Fac}_{T,n+2}(w)$, is in type F_4. More precisely it is the Coxeter element for a reflection subgroup of type $A_2 + A_2'$, where A_2 consists of long roots while A_2' consists of short roots. We will describe this in more detail in Example 5.1.8.

(b) By Remark 4.3.6 there is no element w in type A_n with $\ell_T(w) = n$ which is not quasi-Coxeter. In general this is not true for the remaining types.

Example 5.1.8. Consider the root system $\Phi = \Phi_{F_4}$ of type F_4 as described in [Bou02, Plate VIII]. Let (W, S) be the corresponding Coxeter system and $T = \{s_\alpha \mid \alpha \in \Phi\}$. A possible choice for the simple roots are

$$\alpha_1 = e_2 - e_3,\ \alpha_2 = e_3 - e_4,\ \alpha_3 = e_4,\ \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

The highest root is $\widetilde{\alpha} = e_1 + e_2$. Let Φ' the smallest subroot system containing $\{\widetilde{\alpha}, \alpha_1, \alpha_3, \alpha_4\}$. It is of type $A_2 + A_2'$ and $w := s_{\widetilde{\alpha}} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4}$ is a Coxeter element for $W_{\Phi'}$. Obviously $\langle s_{\widetilde{\alpha}}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}, s_{\alpha_2} \rangle = W$ and therefore $(s_{\widetilde{\alpha}}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}, s_{\alpha_2}, s_{\alpha_2}) \in \mathrm{Fac}_{T,6}(w)$. Put $\alpha := e_3 \in \Phi$. A direct calculation yields that

$$s_\alpha s_{\alpha_1} s_{\widetilde{\alpha}} s_{\alpha_4} s_\alpha(\alpha_1) = \alpha_2,$$

thus $\langle s_{\widetilde{\alpha}}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}, s_\alpha \rangle = W$ and therefore $(s_{\widetilde{\alpha}}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}, s_\alpha, s_\alpha) \in \mathrm{Fac}_{T,6}(w)$, but this decomposition does not have the same multiset of conjugacy classes as the decomposition $(s_{\widetilde{\alpha}}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}, s_{\alpha_2}, s_{\alpha_2})$.

5.2. From parabolic subgroups to the whole group: The gap

The aim of this section is to provide some auxiliary results for the proof of Theorem 5.1.6 (and for later use in Section 6). Namely we adress the following question: Given a finite dual Coxeter system (W, T) and a parabolic subgroup $W' \leq W$, what can be said about the reflections which have to be added to W' to obtain the whole group W?

Lemma 5.2.1. *Let Φ be a root system of type B_n and $\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi$ a simple system, where the roots α_i $(1 \leq i \leq n-1)$ are long and α_n is short. If $\alpha \in \Phi$ is any other short root, then $W_\Phi = \langle s_{\alpha_1}, \dots, s_{\alpha_{n-1}}, s_\alpha \rangle$.*

Proof. Let $R := \{\alpha_1, \dots, \alpha_{n-1}, \alpha\}$. Using the realizations of the root systems of type B_n and C_n given in [Bou02, Plates I,II], it is straightforward to check that $L(R) = L(\Phi)$ and $L(R^\vee) = L(\Phi^\vee)$, where $\Phi^\vee$ is of type C_n. Using Theorem 4.2.12, the assertion follows. □

Proposition 5.2.2. *Let (W, T) be a finite dual Coxeter system of type $X_n \in \{A_n, B_n, D_n\}$ and $S \subseteq T$ a simple system. Let $s \in S$ and $W' := \langle S \setminus \{s\} \rangle$. If $W = \langle W', t_1, t_2 \rangle$ for some reflections $t_1, t_2 \in T$, then $W = \langle W', t_1 \rangle$ or $W = \langle W', t_2 \rangle$.*

Proof. Let Φ be the root system associated to (W, S) and $\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi$ a simple system. We choose the numbering such that $s = s_{\alpha_1}$. Furthermore let $t_1 = s_\alpha$ and $t_2 = s_\beta$ for positive roots α and β. Let $R := \{\alpha, \beta\} \cup \{\alpha_1, \dots, \alpha_{n-1}\}$. By assumption we have $W = W_\Phi = W_R$. By Theorem 4.2.12 we conclude $L(\Phi) = L(R)$ and $L(\Phi^\vee) = L(R^\vee)$. Note that $\Delta^\vee$ is a simple system for $\Phi^\vee$ (see for instance [Hal15, Prop. 8.18]).

If $s_\alpha \in W'$, then $W = \langle W', s_\beta \rangle$ and vice versa. Therefore assume $s_\alpha, s_\beta \notin W'$. If $W = \langle W', s_\alpha \rangle$, we are done again. Therefore assume that $\langle W', s_\alpha \rangle$ is a proper subgroup of W.

Write $\alpha = \sum_{i=1}^n a_i \alpha_i$ (resp. $\alpha^\vee = \sum_{i=1}^n a_i' \alpha_i^\vee$) and $\beta = \sum_{i=1}^n b_i \alpha_i$ (resp. $\beta^\vee = \sum_{i=1}^n b_i' \alpha_i^\vee$) with $a_i, a_i', b_i, b_i' \in \mathbb{Z}_{\geq 0}$. Note that $a_i' = \frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} a_i$ and $b_i' = \frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} b_i$. Furthermore let Φ' be the root system associated to W'. Then $\Delta' := \Delta \setminus \{\alpha_1\}$ is a simple system for Φ'. Since $s_\alpha, s_\beta \notin W' = W_{\Phi'}$, none of the coefficients a_1, a_1', b_1, b_1' can be zero. By [Bou02, Plates I-IV] we conclude $a_i, a_i', b_i, b_i' \in \{1, 2\}$. If $a_1 = 2 = b_1$, then $\alpha_1 \notin L(R) = L(\Phi)$ which is not possible.

Therefore $a_1 = 1$ or $b_1 = 1$. A similar argument shows $a_1' = 1$ or $b_1' = 1$. First we consider the case where $a_1 = 1 = a_1'$. Then $L(\Phi) = L(\Phi' \cup \{\alpha\})$ and $L(\Phi^\vee) = L((\Phi')^\vee \cup \{\alpha^\vee\})$, thus $W = \langle W', s_\alpha \rangle$ by Theorem 4.2.12, contradicting the assumption that $\langle W', s_\alpha \rangle$ is a proper subgroup of W. Therefore we can assume that $b_1 = 1$ or $b_1' = 1$. Without loss of generality we assume the latter one. Assume that $b_1 = 2$, thus $\beta = 2\alpha_1 + \cdots$ and $\beta^\vee = \frac{2}{(\beta|\beta)} \cdot 2\alpha_1 + \cdots$. Note that

$$\frac{2}{(\beta \mid \beta)} \cdot 2\alpha_1 = \begin{cases} \frac{2}{(\beta|\beta)}\alpha_1^\vee, & \alpha_1 \text{ short} \\ \frac{4}{(\beta|\beta)}\alpha_1^\vee, & \alpha_1 \text{ long.} \end{cases}$$

Since the coefficient of $\alpha_1^\vee$ is $b_1' = 1$, the only possibility is that α_1 is short and β is long. Clearly this is not possible for $X_n \in \{A_n, D_n\}$. If $X_n = B_n$ and α_1 is short, the remaining simple roots in Δ' have to be long. Since $W = W_{\Delta' \cup \{\alpha, \beta\}}$, the root α has to be short. Note that W cannot be generated just by reflections in short roots. But then it follows by Lemma 5.2.1 that $\langle W', s_\alpha \rangle = W$, a contradiction. Hence $b_1 = 1 = b_1'$. Therefore we obtain

$$L(\Phi) = L(\Delta) = L(\Delta' \cup \{\beta\}) \text{ and } L(\Phi^\vee) = L(\Delta^\vee) = L((\Delta')^\vee \cup \{\beta^\vee\}),$$

which yields $W = \langle W', s_\beta \rangle$ by Theorem 4.2.12. □

Remark 5.2.3. The statement of Proposition 5.2.2 is still valid if we consider an arbitrary parabolic subgroup W' of rank $n-1$. To see this note that the notions of parabolic subgroups and classical parabolic subgroups are equivalent for finite (W, T) and since (W, T) is crystallographic, that is W is a finite Weyl group, the dual Coxeter system (W, T) is strongly reflection rigid. Therefore it is enough to check the assertion only for standard parabolic subgroups of rank $n-1$, that is subgroups of the form $\langle J \rangle$ with $J \subseteq S$ for some simple system $S \subseteq T$ and $|J| = n-1$. This is done in Proposition 5.2.2.

The following proposition provides an additional property to what we have already observed in Corollary 4.3.11.

Proposition 5.2.4. *Let (W, T) be a finite crystallographic dual Coxeter system of rank n, W' a reflection subgroup of rank $n-1$ and $t \in T$ such that $\langle W', t \rangle = W$. Then t is unique up to conjugacy with elements in W'.*

Proof. By Corollary 4.3.11 the reflection subgroup W' is parabolic. As described in Remark 5.2.3 it is enough to check the assertion for standard parabolic subgroups of rank $n-1$. We do this case-by-case.

Type A_{n-1}. Choose $S = \{(1,2), (2,3), \ldots, (n-1, n)\}$. Then

$$T = \{(i,j) \mid 1 \leq i < j \leq n\}.$$

Consider $W_J := \langle J \rangle$ with $J = S \setminus \{(i, i+1)\}$. Then

$$T \setminus (W_J \cap T) = \{(j,k) \mid j \leq i, k \geq i+1\}.$$

It is easy to see that for each reflection r in this set there exists $x \in W_J$ with $r^x = (i, i+1)$.

Type B_n. Consider the simple system $S = \{s_0, s_1, \ldots, s_{n-1}\}$ as given in [BB05]. If s_0 is the simple reflection missing in W', then a reflection t added to W' generates the whole W if and only if t is of the form $(i, -i)$. But such a reflection is a conjugate of $s_0 = (1, -1)$.

If s_i is the simple reflection missing in W', where $i > 0$, then a reflection t added to W' generates the whole W if and only if it is of the form $(k,m)(-k,-m)$ where $|k| \leq i$, $|m| \geq i+1$. Conjugating by elements in W' this reflection t can be transformed into either $s_i = (i,i+1)(-i,-i-1)$ or $t' := (-i,i+1)(i,-i-1)$. In the first case we are done. Assume we are in the second case. The reflection $(i,-i)$ lies in W' and conjugating t' by it, we get s_i, hence we are done again.

Type D_n. Choose the simple system $S := \{s_0, s_1, \ldots, s_{n-1}\}$, where

$$s_0 := (1,-2)(2,-1)$$
$$s_i := (i,i+1)(-i,-(i+1)),\ 1 \leq i \leq n-1.$$

Then $T = \{(i,j)(-i,-j) \mid 1 \leq |i| < j \leq n\}$ is the set of reflections. For $J := S \setminus \{s_i\}$ with $0 \leq i \leq n-1$ the corresponding standard parabolic subgroup $W_J = \langle J \rangle$ is given by (see [BB05])

$$W_J = \begin{cases} \mathrm{Stab}([i+1,n]) & i \neq 1 \\ \mathrm{Stab}(\{-1,2,3,\ldots,n\}) & i = 1 \end{cases}$$

We will show that each reflection $r \in T$ with $\langle W_J, r\rangle = W$ is conjugated to s_i under W_J. We consider two cases:

Case 1: $i \neq 1$. In this case we have

$$T \setminus (W_J \cap T) = \{(j,k)(-j,-k) \mid 1 \leq |j| \leq i < k \leq n\} \cup \{(-j,k)(j,-k) \mid j < k,\ j,k \in [i+1,n]\}.$$

First consider a reflection $r = (-j,k)(j,-k)$ with $j < k$ and $j,k \in [i+1,n]$ (note that all reflections in $T \setminus (W_J \cap T)$ are of this type if $i = 0$). Then $\langle W_J, r\rangle = W$. To see this, we put $t_1 := (-j,i)(j,-i) \in W_J$ and $t_2 := (k,i+1)(-k,-(i+1)) \in W_J$. We have $r^{t_2 t_1} = s_i$, thus $\langle W_J, r\rangle = W$.

Therefore let $r = (j,k)(-j,-k)$ with $1 \leq |j| \leq i < k \leq n$ and $i \neq 0,1$. Define

$$x_1 := \begin{cases} \mathrm{id} & k = i+1 \\ s_{i+1} \cdots s_{k-2} s_{k-1} & k > i+1 \end{cases} \in W_J$$

Then $r^{x_1} = (j,i+1)(-j,-(i+1))$. We distinguish whether j is negative or not.

(a) $j \leq -1$. Since $|j| \leq i$, we have

$$x_2 := \begin{cases} \mathrm{id} & j = -1 \\ s_1 \cdots s_{|j|-1} & j \neq -1 \end{cases} \in W_J$$

We already excluded the case $i = 0$, thus $r^{s_1 s_0 x_2 x_1} = (1,i+1)(-1,-(i+1))$. This reflection is conjugated to s_i under W_J.

(b) $j \geq 1$. Setting $x_2 := s_{i-1} s_{i-2} \cdots s_j$, we get $r^{x_2 x_1} = s_i$.

Case 2: $i = 1$. Consider $r = (j,k)(-j,-k) \in T$. If $j,k \geq 2$ (or equivalently $j,k \leq -2$), then $r \in W_J$, but we are only interested in those r with $\langle W_J, r\rangle = W$. Since $|j| < k$ we have $k > 1$. There are two possibilities left:

(a) $|j| = 1, k \geq 2$. If $j = 1$, define $x_1 := s_2 s_3 \cdots s_{k-1} \in W_J$. Thus $r^{x_1} = s_1$. The case $j = -1$ can be omitted, since then $r \in W_J$.

(b) $j \leq -2, k \geq 2$. Keeping in mind that $|j| < k$, we define

$$x_1 := \begin{cases} \mathrm{id} & k = |j|+1 \\ s_{|j|+1} \cdots s_{k-2} s_{k-1} & k > |j|+1 \end{cases} \in W_J$$

Thus $r^{x_1} = (j, |j|+1)(-j, -(|j|+1))$. Defining

$$\begin{aligned} x_2 &:= s_2 \cdots s_{|j|-2} s_{|j|-1} \\ x_3 &:= s_2 s_3 \cdots s_{|j|}, \end{aligned}$$

we get (since j is negative) that

$$\begin{aligned} r^{x_2 x_1} &= (-2, |j|+1)(2, -(|j|+1)) \\ r^{s_0 x_2 x_1} &= (1, |j|+1)(-1, -(|j|+1)). \end{aligned}$$

For $x := x_3 s_0 x_2 x_1 \in W_J$ we obtain $r^x = s_1$.

Types E_6, E_7, E_8, F_4, G_2. [GAP2015] □

Lemma 5.2.5. *Let (W,T) be a finite irreducible dual Coxeter system of type $X_n \in \{E_6, E_7\}$, W' a parabolic subgroup of rank $n-1$ and $t \in T$ such that $\langle W', t\rangle = W$. Let $t_1, t_2 \in T$ such that $\langle W', t_1, t_2\rangle = W$. Then there exist $x \in \langle W', t_1\rangle$ and $y \in \langle W', t_2\rangle$ such that $t_2^x = t$ and $t_1^y = t$.*

Proof. [GAP2015]. □

5.3. The proof of Theorem 5.1.6

Lemma 5.3.1. *Let (W,T) be a dual Coxeter system and $t_1, \ldots, t_n, t \in T$. Then*

$$(t_1, \ldots, t_n, t, t) \sim (t_1, \ldots, t_n, t^x, t^x)$$

for each $x \in \langle t_1, \ldots, t_n\rangle$.

Proof. We have

$$\begin{aligned} (t_1, \ldots, t_n, t, t) &\sim (t_1, \ldots, \hat{t_i}, t_{i+1}^{t_i} \ldots, t_n^{t_i}, t^{t_i}, t^{t_i}, t_i) \\ &\sim (t_1, \ldots, \hat{t_i}, t_{i+1}^{t_i} \ldots, t_n^{t_i}, t_i, t^{t_i}, t^{t_i}) \\ &\sim (t_1, \ldots, t_n, t^{t_i}, t^{t_i}), \end{aligned}$$

where the entry $\hat{t_i}$ is omitted. □

Proof of Theorem 5.1.6. If two elements of $\mathrm{Fac}_{T,n+2}(w)$ are in the same Hurwitz orbit, they share the same multiset of conjugacy classes by Lemma 3.1.3. It remains to show that this condition is sufficient.

Let $(t_1, \dots, t_{n+2}) \in \mathrm{Fac}_{T,n+2}(w)$. Since we are interested in the Hurwitz orbit we can assume that $t_{n+1} = t_{n+2}$ by Theorem 5.1.3. Then $W = \langle t_1, \dots, t_{n+1} \rangle$. By Corollary 2.3.2 the decomposition $t_1 \cdots t_{n+1}$ is not T-reduced. We apply again Theorem 5.1.3 to obtain $(t_1, \dots, t_{n+1}) \sim (t'_1, \dots, t'_{n-1}, t'_n, t'_n)$. Therefore $W = \langle t'_1, \dots, t'_n \rangle$, that is $t'_1 \cdots t'_n$ is quasi-Coxeter.

By the preceding arguments it is enough to choose elements $(t_1, \dots, t_n, t_n, t_{n+1})$ and $(r_1, \dots, r_n, r_n, r_{n+1})$ in $\mathrm{Fac}_{T,n+2}(w)$ such that $t_1 \cdots t_n$ and $r_1 \cdots r_n$ are quasi-Coxeter and both tuples share the same multiset of conjugacy classes. It remains to show that these tuples are in the same Hurwitz orbit.

By Corollary 4.5.4 we have $(r_1, \dots, r_n) \sim (r'_1, \dots, r'_n)$, where $r'_n = t_{n+1}$. Thus

$$\begin{aligned}(r_1, \dots, r_n, r_n, r_{n+1}) &\sim (r'_1, \dots, r'_n, r_n, r_{n+1}) \\ &\sim (r'_1, \dots, r_n^{r'_n}, r_{n+1}^{r'_n}, r'_n) \\ &\overset{5.1.3}{\sim} (r''_1, \dots, r''_n, r''_n, r'_n)\end{aligned}$$

for reflections $r''_1, \dots, r''_n \in T$. We have $t_1 \cdots t_{n-1} \leq_T t_1 \cdots t_n$, hence $w' := t_1 \cdots t_{n-1} = r''_1 \cdots r''_{n-1}$ is a parabolic quasi-Coxeter element by Corollary 4.3.12. Therefore $(t_1, \dots, t_{n-1}) \sim (r''_1, \dots, r''_{n-1})$ by Theorem 1.1.2. Let

$$W' := \langle t_1, \dots, t_{n-1} \rangle = \langle r''_1, \dots, r''_{n-1} \rangle.$$

As $t_1 \cdots t_n$ is quasi-Coxeter, it is $W = \langle W', t_n \rangle$. We continue with a case-by-case analysis.

(Types A_n, B_n and D_n) Assume that $\langle W', r''_n \rangle$ is a proper subgroup of W. By Hurwitz equivalence we have $\langle W', r''_n, r'_n \rangle = W$. By Proposition 5.2.2 we obtain $\langle W', r'_n \rangle = W$, thus $w = r''_1 \cdots r''_{n-1} r'_n$ is quasi-Coxeter, which we have excluded. Therefore $\langle W', r''_n \rangle = W$. By Proposition 5.2.4 there exists $x \in W'$ with $t_n^x = r''_n$. Overall we get

$$(t_1, \dots, t_{n-1}, t_n, t_n, t_{n+1}) \sim (r''_1, \dots, r''_{n-1}, t_n, t_n, t_{n+1}) \sim (r''_1, \dots, r''_{n-1}, t_n^x, t_n^x, t_{n+1}),$$

where we used Lemma 5.3.1 in the last step. Since $t_n^x = r''_n$ and $t_{n+1} = r'_n$ the assertion follows.

(Types E_6 and E_7) We have $(t_1, \dots, t_{n-1}) \sim (r''_1, \dots, r''_{n-1})$ and since $r'_n = t_{n+1}$ we obtain

$$(t_1, \dots, t_{n-1}, t_n, t_n, t_{n+1}) \sim (t_1, \dots, t_{n-1}, t_{n+1}, t_n, t_n) \sim (r''_1, \dots, r''_{n-1}, r'_n, t_n, t_n).$$

We have $\langle W', r'_n, r''_n \rangle = W = \langle W', t_n \rangle$. By part (a) of Lemma 5.2.5 there exists $x \in \langle W', r'_n \rangle$ such that $(r''_n)^x = t_n$ resp. $r''_n = t_n^{x^{-1}}$. By Lemma 5.3.1 we obtain

$$\begin{aligned}(r''_1, \dots, r''_{n-1}, r'_n, t_n, t_n) &\sim (r''_1, \dots, r''_{n-1}, r'_n, t_n^{x^{-1}}, t_n^{x^{-1}}) = (r''_1, \dots, r''_{n-1}, r'_n, r''_n, r''_n) \\ &\sim (r''_1, \dots, r''_{n-1}, r''_n, r''_n, r'_n) \\ &\sim (r_1, \dots, r_n, r_n, r_{n+1}).\end{aligned}$$

(Type F_4) [GAP2015] □

Remark 5.3.2. In fact, in the proof of Theorem 5.1.6 for the cases A_n, B_n, D_n, E_6 and E_7, we have made no use of the fact that the elements $(t_1, \dots, t_n, t_n, t_{n+1}), (r_1, \dots, r_n, r_n, r_{n+1}) \in \mathrm{Fac}_{T,n+2}(w)$ share the same multiset of conjugacy classes.

Conjecture 5.3.3. Let (W, T) be a finite irreducible dual Coxeter system of rank n and $w \in W$ with $\ell_T(w) = n$ which is not quasi-Coxeter and such that $\mathrm{Fac}_{T,n+2}(w) \neq \varnothing$. Then two elements of $\mathrm{Fac}_{T,n+2}(w)$ are in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.

By Theorem 5.1.6 it remains to check the conjecture for E_8 and the non-crystallographic cases. In the case H_3 we were able to verify the conjecture using [GAP2015].

6. Hurwitz action in affine Coxeter groups

For details on affine Coxeter groups see Section 2.1.3 and the references therein.

6.1. Quasi-Coxeter elements in affine Coxeter groups

We fix a finite crystallographic Coxeter system (W, S) of rank n with set of reflections T and associated crystallographic root system Φ in an euclidean vector space V. Then $\widetilde{W} := \langle s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\rangle$ is the associated affine Coxeter group. By regarding the reflections s_α as reflections $s_{\alpha,0}$, the group W becomes a proper subgroup of $\widetilde{W}$. In particular we have an epimorphism

$$p : \widetilde{W} \to W, s_{\alpha,k} \mapsto s_\alpha.$$

In Theorem 2.1.15 we saw how to obtain a simple system $\widetilde{S}$ for $\widetilde{W}$. Denote by $\widetilde{T}$ the set of reflections for $(\widetilde{W}, \widetilde{S})$. In particular $(\widetilde{W}, \widetilde{T})$ is an affine dual Coxeter system of rank $n+1$ (see Section 2.4.1) and

$$\widetilde{T} = \{s_{\alpha,k} \mid \alpha \in \Phi,\ k \in \mathbb{Z}\}.$$

Therefore we have $p(\widetilde{T}) = T$.

We showed in Section 2.4.2 that the notions of parabolic subgroups and classical parabolic are equivalent for affine Coxeter groups.

The aim of this chapter is to prove Theorem 1.1.3, which partially generalizes Theorem 1.1.2.

Remark 6.1.1.

(a) Quasi-Coxeter elements in affine Coxeter groups indeed extend the class of Coxeter elements, that is, the class of Coxeter elements for $(\widetilde{W}, \widetilde{T})$ is in general a proper subclass of the class of quasi-Coxeter elements for $(\widetilde{W}, \widetilde{T})$. To see this, we use the same notation as above. Note that finite Weyl groups and affine Coxeter groups are strongly reflection rigid, thus the notion of classical Coxeter element and Coxeter element are equivalent. We assume that the Coxeter diagram corresponding to the type of $(\widetilde{W}, \widetilde{T})$ is a tree (e.g. $(\widetilde{W}, \widetilde{T})$ is of type $\widetilde{D}_n$). In particular, up to conjugacy, the Coxeter element for $(\widetilde{W}, \widetilde{T})$ is $c := s_{\alpha_1} \cdots s_{\alpha_n} s_{\widetilde{\alpha},1}$, where $\alpha_1, \ldots, \alpha_n$ are simple roots for Φ and $\widetilde{\alpha}$ the corresponding highest root. The element $c' := s_{\alpha_1} \cdots s_{\alpha_n}$ is Coxeter for (W, T). Using Theorem 1.1.1 and the fact that $t \leq_T c'$ for all $t \in T$, we can assume (up to Hurwitz equivalence) that $\alpha_n = \widetilde{\alpha}$. It is

$$p(c) = s_{\alpha_1} \cdots s_{\alpha_{n-1}} \leq_T c',$$

thus $p(c)$ is a parabolic Coxeter element by Remark 2.4.4 (b). For all $x \in \widetilde{W}$ we have $p(c^x) = p(x)p(c)p(x)^{-1}$. Since all Coxeter elements are conjugate, we see that the projection of a Coxeter element under p is always a parabolic Coxeter element. If we substitute the Coxeter element $c' = s_{\alpha_1} \cdots s_{\alpha_n}$ by an arbitrary quasi-Coxeter element w' for (W, T), then $w's_{\widetilde{\alpha},1} \in \widetilde{W}$ is quasi-Coxeter and $p(w's_{\widetilde{\alpha},1})$ is a parabolic quasi-Coxeter element by Corollary 4.3.12. The example of the parabolic quasi-Coxeter element $D_4(a_1)$ in $D_5(a_1)$ shows that $p(w's_{\widetilde{\alpha},1})$ needs not to be a parabolic Coxeter element. The notation $D_n(a_1)$ refers to the conjugacy class of quasi-Coxeter elements introduced in Remark 4.5.2.

(b) In contrast to Theorem 1.1.2, there exist elements w which are not parabolic quasi-Coxeter elements for $(\widetilde{W}, \widetilde{T})$, but the Hurwitz action is transitive on $\mathrm{Red}_{\widetilde{T}}(w)$. Consider again the example of the Coxeter element $w = stu$ in an affine Coxeter group of type $\widetilde{A}_2$ (see Remark 4.3.10). Then $s(tut) \leq_T w$. It can be checked directly resp. the results in [HK13] imply that the Hurwitz action is transitive on $\mathrm{Red}_{\widetilde{T}}(s(tut))$, but as pointed out in Remark 4.3.10, the element $s(tut)$ cannot be quasi-Coxeter.

6.2. Conjugation and Hurwitz moves in affine Coxeter groups

The aim of this section is to give some auxiliary results about conjugation in affine Coxeter groups which supplement the results already stated in Section 2.1.3. These results turn out to be helpful when working with the Hurwitz action. At the end of this subsection we obtain a characterization of quasi-Coxeter elements for $(\widetilde{W}, \widetilde{T})$ in Proposition 6.2.8. See Definition 4.1.1 for the definition of quasi-Coxeter element. We maintain the notation of the previous section.

Lemma 6.2.1. *Let $\alpha \in \Phi$, $\lambda \in P(\Phi^\vee)$, $w \in W$ and $k \in \mathbb{Z}$.*

(a) $\mathrm{tr}(\lambda)H_{\alpha,k} = H_{\alpha,k+(\lambda|\alpha)}$

(b) $\mathrm{tr}(\lambda)s_{\alpha,k}\,\mathrm{tr}(-\lambda) = s_{\alpha,k+(\lambda|\alpha)}$.

In particular, the coweight lattice $P(\Phi^\vee)$ normalizes $\widetilde{W}$.

(c) $\mathrm{tr}(\alpha^\vee)^w = \mathrm{tr}(w(\alpha)^\vee)$.

Proof. For parts (a) and (b) see [Hum90, Prop. 4.1]. For part (c) it is sufficient to consider $w = s_\beta$ for some $\beta \in \Phi$. By Lemma 2.1.13 and Lemma 2.1.10 we obtain

$$\mathrm{tr}(\alpha^\vee)^{s_\beta} = s_\beta\,\mathrm{tr}(\alpha^\vee)s_\beta = s_\beta s_{\alpha,1}s_\alpha s_\beta = s_{s_\beta(\alpha),1}s_{s_\beta(\alpha)} = \mathrm{tr}(s_\beta(\alpha)^\vee).$$

□

Lemma 6.2.2. *The restriction of the action of the group W on V yields an action of W on $P(\Phi^\vee)$.*

Proof. Let $\lambda \in P(\Phi^\vee)$, thus $(\lambda \mid \alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. Therefore and since $W = W_\Phi \leq O(V)$ we obtain

$$(w(\lambda) \mid \alpha) = (\lambda \mid w^{-1}(\alpha)) \in \mathbb{Z} \text{ for all } \alpha \in \Phi,$$

because $w^{-1}(\alpha) \in \Phi$. Hence $w(\lambda) \in P(\Phi^\vee)$. □

Lemma 2.1.16 provides a normal form for each element in an affine Coxeter group. When we have a decomposition of an element in $\widetilde{W}$ as a product of reflections, the following Lemma tells us how this normal form is achieved.

Lemma 6.2.3. *For $\beta_i \in \Phi$ and $k_i \in \mathbb{Z}$ $(1 \leq i \leq m)$ we have*

$$s_{\beta_1,k_1} \cdots s_{\beta_m,k_m} = s_{\beta_1} \cdots s_{\beta_m} \operatorname{tr}(\sum_{i=1}^{m} -k_i s_{\beta_m} \cdots s_{\beta_{i+1}}(\beta_i)^\vee).$$

Proof. By Lemma 2.1.13 we have for $\alpha \in \Phi, k \in \mathbb{Z}$ that

$$s_{\alpha,k} = \operatorname{tr}(k\alpha^\vee)s_\alpha = s_\alpha \operatorname{tr}(-k\alpha^\vee) \tag{6.1}$$

and by Proposition 2.1.12 that

$$w s_{\alpha,k} w^{-1} = s_{w(\alpha),k} \text{ for all } w \in W. \tag{6.2}$$

We show the assertion by induction on m. It is clear for $m = 1$. By induction it follows

$$s_{\beta_1,k_1} \cdots s_{\beta_m,k_m} = s_{\beta_1,k_1} s_{\beta_2} \cdots s_{\beta_m} \operatorname{tr}(\sum_{i=2}^{m} -k_i s_{\beta_m} \cdots s_{\beta_{i+1}}(\beta_i)^\vee).$$

Put $w := s_{\beta_m} \cdots s_{\beta_2}$. Then

$$\begin{aligned} s_{\beta_1,k_1} s_{\beta_2} \cdots s_{\beta_m} &= s_{\beta_1,k_1} w^{-1} \\ &\overset{(6.2)}{=} w^{-1} s_{w(\beta_1),k_1} \\ &\overset{(6.1)}{=} w^{-1} s_{w(\beta_1)} \operatorname{tr}(-k_1 w(\beta_1)^\vee) \\ &= s_{\beta_1} w^{-1} \operatorname{tr}(-k_1 w(\beta_1)^\vee) \\ &= s_{\beta_1} s_{\beta_2} \cdots s_{\beta_m} \operatorname{tr}(-k_1 s_{\beta_m} \cdots s_{\beta_2}(\beta_1)^\vee), \end{aligned}$$

which yields the assertion. □

Lemma 6.2.4. *For $\alpha, \beta \in \Phi$ and $k, l \in \mathbb{Z}$ we have*

$$s_{\alpha,k} s_{\beta,l} s_{\alpha,k} = s_{s_\alpha(\beta), l - k\frac{2(\alpha|\beta)}{(\alpha|\alpha)}}.$$

Proof. By Lemma 6.2.3 we have

$$s_{\alpha,k} s_{\beta,l} s_{\alpha,k} = s_{s_\alpha(\beta)} \operatorname{tr}(-k s_\alpha s_\beta(\alpha)^\vee - l s_\alpha(\beta)^\vee - k\alpha^\vee).$$

It is

$$\begin{aligned} s_\alpha s_\beta(\alpha)^\vee + \alpha^\vee &= \frac{2 s_\alpha s_\beta(\alpha)}{(\alpha \mid \alpha)} + \frac{2\alpha}{(\alpha \mid \alpha)} \\ &= -\frac{2}{(\alpha \mid \alpha)} \cdot \frac{2(\alpha \mid \beta)}{(\beta \mid \beta)} s_\alpha(\beta) \\ &= -\frac{2(\alpha \mid \beta)}{(\alpha \mid \alpha)} s_\alpha(\beta)^\vee, \end{aligned}$$

which yields the assertion. □

By the previous Lemma and the definition of the Hurwitz action we obtain the following.

Lemma 6.2.5. *The i-th Hurwitz move is given as*

$$\sigma_i(\ldots, s_{\beta_i,k_i}, s_{\beta_{i+1},k_{i+1}}, \ldots) = (\ldots, s_{\beta_i,k_i} s_{\beta_{i+1},k_{i+1}} s_{\beta_i,k_i}, s_{\beta_i,k_i}, \ldots),$$

where

$$s_{\beta_i,k_i} s_{\beta_{i+1},k_{i+1}} s_{\beta_i,k_i} = s_{s_{\beta_i}(\beta_{i+1}),k_{i+1}-\frac{2(\beta_i,\beta_{i+1})}{(\beta_i|\beta_i)}k_i}.$$

Proposition 6.2.6. *Let $(\widetilde{W}, \widetilde{T})$ be an affine dual Coxeter system of rank $n+1$, $x \in \widetilde{W}$ with $\ell_{\widetilde{T}}(x) = m \geq n+1$ and $x = s_{\beta_1,k_1} \cdots s_{\beta_m,k_m}$ a reduced $\widetilde{T}$-decomposition. Then there exist $\beta_i' \in \Phi, k_i' \in \mathbb{Z}$ $(1 \leq i \leq m)$ such that*

$$(s_{\beta_1,k_1}, \ldots, s_{\beta_m,k_m}) \sim (s_{\beta_1',k_1'}, \ldots, s_{\beta_{m-1}',k_{m-1}'}, s_{\beta_{m-1}',k_m'}).$$

Proof. By Corollary 2.3.2 the decomposition $s_{\beta_1} \cdots s_{\beta_m}$ can not be reduced. Hence we can apply Theorem 5.1.3 to obtain

$$(s_{\beta_1}, \ldots, s_{\beta_m}) \sim (s_{\beta_1'}, \ldots, s_{\beta_{m-1}'}, s_{\beta_{m-1}'}).$$

This Hurwitz equivalence is given by some braid τ. Apply the braid τ to the decomposition $(s_{\beta_1,k_1}, \ldots, s_{\beta_m,k_m})$. □

Let $\widetilde{w} = s_{\gamma_1',k_1} \cdots s_{\gamma_{n+1}',k_{n+1}}$ $(\gamma_i' \in \Phi, k_i \in \mathbb{Z})$ be a reflection decomposition of an element $\widetilde{w} \in \widetilde{W}$ such that $\widetilde{W} = \langle s_{\gamma_1',k_1}, \ldots, s_{\gamma_{n+1}',k_{n+1}} \rangle$. Note that $\widetilde{W}$ cannot be generated by less than $n+1$ reflections (see Proposition 2.4.1). Using Theorem 5.1.3, we obtain

$$(s_{\gamma_1'}, \ldots, s_{\gamma_{n+1}'}) \sim (s_{\gamma_1}, \ldots, s_{\gamma_{n-1}}, s_{\gamma_n}, s_{\gamma_n}) \tag{6.3}$$

for roots $\gamma_1, \ldots, \gamma_n \in \Phi$. By Lemma 3.1.3 and since $p(\widetilde{W}) = W$, we have

$$W = \langle s_{\gamma_1}, \ldots, s_{\gamma_{n-1}}, s_{\gamma_n} \rangle.$$

Therefore the element $w := s_{\gamma_1} \cdots s_{\gamma_{n-1}} s_{\gamma_n}$ is quasi-Coxeter. Furthermore we have

$$w' := p(\widetilde{w}) = s_{\gamma_1} \cdots s_{\gamma_{n-1}} \leq_T s_{\gamma_1} \cdots s_{\gamma_{n-1}} s_{\gamma_n} = w.$$

By Proposition 4.3.2 it follows that w' is a parabolic quasi-Coxeter element with $\ell_T(w') = n-1$. Let $W' := \langle s_{\gamma_1}, \ldots, s_{\gamma_{n-1}} \rangle$ be the corresponding parabolic subgroup. As described in (the proof of) Proposition 6.2.6 we have

$$(s_{\gamma_1',k_1}, \ldots, s_{\gamma_{n+1}',k_{n+1}}) \sim (s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_n,l_{n+1}}) \tag{6.4}$$

for $l_1, \ldots, l_{n+1} \in \mathbb{Z}$. Note that we can assume the roots γ_i to be positive since $s_{\alpha,k} = s_{-\alpha,-k}$. We conclude

$$\widetilde{W} = \langle s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_n,l_{n+1}} \rangle \tag{6.5}$$

and $l_n \neq l_{n+1}$, because otherwise $\widetilde{W}$ is generated by less than $n+1$ reflections, contradicting Proposition 2.4.1.

Lemma 6.2.7. *Let Φ be a crystallographic root system and $\{\beta_1, \dots, \beta_n\} \subseteq \Phi$ be linearly independent. Then*

$$s_{\beta_n} \cdots s_{\beta_2}(\beta_1)^\vee \in \operatorname{span}_{\mathbb{Z}}(\beta_1^\vee, \dots, \beta_n^\vee).$$

Proof. We have

$$\begin{aligned} s_{\beta_2}(\beta_1)^\vee &= \frac{2s_{\beta_2}(\beta_1)}{(\beta_1 \mid \beta_1)} \\ &= \frac{2}{(\beta_1 \mid \beta_1)}(\beta_1 - (\beta_1 \mid \beta_2^\vee)\beta_2) \\ &= \beta_1^\vee - \underbrace{\frac{2(\beta_1 \mid \beta_2)}{(\beta_1 \mid \beta_1)}}_{\in\mathbb{Z}}\beta_2^\vee. \end{aligned}$$

The general assertion follows by induction. □

Proposition 6.2.8. *An element $\widetilde{w}$ is quasi-Coxeter for an affine dual Coxeter system $(\widetilde{W}, \widetilde{T})$ of rank $n+1$ if and only if there exists a reflection decomposition $\widetilde{w} = s_{\gamma_1,l_1} \cdots s_{\gamma_{n+1},l_{n+1}}$ $(\gamma_i \in \Phi, l_i \in \mathbb{Z})$ such that $\widetilde{W} = \langle s_{\gamma_1,l_1}, \dots, s_{\gamma_{n+1},l_{n+1}} \rangle$.*

Proof. The forward direction is clear by the definition of quasi-Coxeter element. For the other direction it remains to show that $s_{\gamma_1,l_1} \cdots s_{\gamma_{n+1},l_{n+1}}$ is $\widetilde{T}$-reduced. By what we have observed before (in particular in (6.4)), we can assume that $\gamma_n = \gamma_{n+1}$.

Under the map $p : \widetilde{W} \to W, s_{\alpha,k} \mapsto s_\alpha$ the element $\widetilde{w}$ is mapped to an element of absolute length $n-1$. Thus by Proposition 5.1.5 there are just two possibilities: $\ell_{\widetilde{T}}(\widetilde{w}) = n+1$ or $\ell_{\widetilde{T}}(\widetilde{w}) = n-1$. Assume the latter one. Let $(s_{\beta_1,m_1}, \dots, s_{\beta_{n-1},m_{n-1}}) \in \operatorname{Red}_{\widetilde{T}}(\widetilde{w})$. It is

$$w' = p(\widetilde{w}) = s_{\gamma_1} \cdots s_{\gamma_{n-1}} = s_{\beta_1} \cdots s_{\beta_{n-1}}$$

a parabolic quasi-Coxeter element. Therefore by Theorem 1.1.2 we have

$$(s_{\gamma_1}, \dots, s_{\gamma_{n-1}}) \sim (s_{\beta_1}, \dots, s_{\beta_{n-1}}).$$

Hence we can assume (up to Hurwitz equivalence) that $\gamma_i = \beta_i$ for $1 \leq i \leq n-1$. Note that the integers $l_1, \dots, l_{n+1}$ might have changed. By applying Lemma 6.2.3 we obtain

$$\begin{aligned} \widetilde{w} &= w' \operatorname{tr}\left(\sum_{i=1}^{n-1} -l_i s_{\beta_{n-1}} \cdots s_{\beta_{i+1}}(\beta_i)^\vee + (l_n - l_{n+1})\gamma_n^\vee\right) \\ &= w' \operatorname{tr}\left(\sum_{i=1}^{n-1} -m_i s_{\beta_{n-1}} \cdots s_{\beta_{i+1}}(\beta_i)^\vee\right). \end{aligned}$$

Hence

$$\sum_{i=1}^{n-1}(m_i - l_i)s_{\beta_{n-1}} \cdots s_{\beta_{i+1}}(\beta_i)^\vee + (l_n - l_{n+1})\gamma_n^\vee = 0.$$

Since $\{\beta_1, \dots, \beta_{n-1}, \gamma_n\}$ is linearly independent and by Lemma 6.2.7 we obtain $l_n - l_{n+1} = 0$, hence $s_{\gamma_n,l_n} = s_{\gamma_n,l_{n+1}}$, contradicting Proposition 2.4.1. □

6.3. Generating affine Coxeter groups by reflections

Let Φ be an irreducible crystallographic root system of rank n. If Φ is not simply-laced, that is Φ consists of long and short roots (i.e. Φ is of type B_n, C_n, F_4 or G_2), we decompose $\Phi = \Phi_s \cup \Phi_l$, where Φ_s is the set of short roots and Φ_l is the set of long roots. Let

$$W := W_\Phi = \langle s_{\alpha_1}, \dots, s_{\alpha_n} \rangle$$

for some roots $\alpha_i \in \Phi$. As before $\widetilde{W}$ denotes the associated affine Coxeter group. Furthermore we assume

$$\widetilde{W} = \langle s_{\alpha_1}, \dots, s_{\alpha_n}, s_{\alpha_n,1} \rangle. \tag{6.6}$$

Note that we always find such a generating set. To see this, we start with a generating set $\{s_{\alpha_1}, \dots, s_{\alpha_n}\}$ of W, that is $s_{\alpha_1} \cdots s_{\alpha_n}$ is quasi-Coxeter, we can assume by Theorem 1.1.2 and Corollary 4.5.4 that $\alpha_n = \widetilde{\alpha}$. Since the set $\{s_{\alpha_1}, \dots, s_{\alpha_n}\}$ generates W, the set $\{s_{\alpha_1}, \dots, s_{\alpha_n}, s_{\alpha_n,1}\}$ generates $\widetilde{W}$ by Theorem 2.1.15.

Lemma 6.3.1. *Let Φ be an irreducible crystallographic root system which consists of long and short roots. Let $\alpha, \beta \in \Phi$ be of different lengths with $(\alpha \mid \beta) \neq 0$ and let δ be the ratio between long and short roots. Then*

$$\frac{2(\alpha \mid \beta)}{(\beta \mid \beta)} = \begin{cases} \pm 1 & \text{if } \alpha \in \Phi_s, \beta \in \Phi_l \\ \pm \delta & \text{if } \alpha \in \Phi_l, \beta \in \Phi_s. \end{cases}$$

Proof. Case by case resp. see [Bou02, Ch. VI, 1.3] (note that in this source also non-reduced root systems are considered). □

Next we are going to show that the root α_n in equation (6.6) cannot be short. Therefore note the following.

Remark 6.3.2. If we have in Equation (6.6) that $\alpha_n \in \Phi_s, \alpha_i \in \Phi_l$ for some $i < n$ and $(\alpha_n \mid \alpha_i) \neq 0$, then by Lemma 6.2.4 and Lemma 6.3.1 we have

$$s_{\alpha_i}^{s_{\alpha_n,1}} = s_{s_{\alpha_n}(\alpha_i), \pm\delta},$$

where δ is the ratio between long and short roots.
If $\alpha_n \in \Phi_l, \alpha_i \in \Phi_s$ we have

$$s_{\alpha_i}^{s_{\alpha_n,1}} = s_{s_{\alpha_n}(\alpha_i), \pm 1}.$$

Lemma 6.3.3. *Let Φ be an irreducible crystallographic root system which consists of long and short roots and let δ be the ratio between long and short roots. Then*

$$\operatorname{span}_{\mathbb{Z}}(\{\delta\alpha \mid \alpha \in \Phi_s\} \cup \Phi_l) \subsetneq L(\Phi).$$

Proof. We show that no short root is contained in $\operatorname{span}_{\mathbb{Z}}(\{\delta\alpha \mid \alpha \in \Phi_s\} \cup \Phi_l)$. Therefore let $\alpha \in \Phi_s$ be arbitrary. Assume that there exist $\alpha_1, \dots, \alpha_n \in \Phi_s$ and $\beta_1, \dots, \beta_m \in \Phi_l$ such that

$$\alpha = \sum_{i=1}^{n} \delta\alpha_i + \sum_{j=1}^{m} \beta_j.$$

Here we allow $\alpha_i = \alpha_j$ as well as $\beta_i = \beta_j$ for $i \neq j$ so that all coefficients in the above linear combination are equal to one. Thus we have

$$(\alpha \mid \alpha) = \sum_i \delta^2(\alpha_i \mid \alpha_i) + \sum_{i<k} 2\delta^2(\alpha_i \mid \alpha_k) + \sum_i \sum_j 2\delta(\alpha_i \mid \beta_j) + \sum_j (\beta_j \mid \beta_j) + \sum_{j<l} 2(\beta_j \mid \beta_l). \tag{6.7}$$

Let $\alpha' \in \Phi_s, \beta' \in \Phi_l$, thus $\delta = \frac{(\beta' \mid \beta')}{(\alpha' \mid \alpha')}$. Let $r \in \{1, \ldots, m\}$, then

$$\frac{2\delta^2(\alpha_i \mid \alpha_k)}{(\beta_r \mid \beta_r)} = \frac{2\delta(\beta' \mid \beta')(\alpha_i \mid \alpha_k)}{(\alpha' \mid \alpha')(\beta_r \mid \beta_r)} = \delta \cdot \underbrace{\frac{2(\alpha_i \mid \alpha_k)}{(\alpha' \mid \alpha')}}_{\in \mathbb{Z}} \in \mathbb{Z}\,. \tag{6.8}$$

Note that this is true for $\alpha_i = \alpha_k$ as well as for $\alpha_i \neq \alpha_k$. Therefore, if we divide equation (6.7) by $(\beta_r \mid \beta_r)$, we obtain

$$\begin{aligned} \frac{(\alpha \mid \alpha)}{(\beta_r \mid \beta_r)} &= \sum_i \underbrace{\delta^2 \frac{(\alpha_i \mid \alpha_i)}{(\beta_r \mid \beta_r)}}_{=\delta} + \sum_{i<k} \underbrace{\frac{2\delta^2(\alpha_i \mid \alpha_k)}{(\beta_r \mid \beta_r)}}_{\in \mathbb{Z} \text{ by } (6.8)} + \sum_i \sum_j \delta \cdot \underbrace{\frac{2(\alpha_i \mid \beta_j)}{(\beta_r \mid \beta_r)}}_{\in \mathbb{Z}} \\ &+ \sum_j \underbrace{\frac{(\beta_j \mid \beta_j)}{(\beta_r \mid \beta_r)}}_{=1} + \sum_{j<l} \underbrace{\frac{2(\beta_j \mid \beta_l)}{(\beta_r \mid \beta_r)}}_{\in \mathbb{Z}}\,. \end{aligned}$$

Hence the right hand side of this equation is an integer, while the left hand side is not; a contradiction. □

Remark 6.3.4. If we consider in the situation of Lemma 6.3.3 the dual root system $\Phi^\vee$ instead of Φ, then we obtain that $\operatorname{span}_{\mathbb{Z}}(\{\delta\alpha^\vee \mid \alpha \in \Phi_l\} \cup \{\alpha^\vee \mid \alpha \in \Phi_s\})$ is a proper sublattice of $L(\Phi^\vee)$.

Lemma 6.3.5. *With the assumptions as in Lemma 6.3.3. Let* $L' := \operatorname{span}_{\mathbb{Z}}(\{\delta\alpha^\vee \mid \alpha \in \Phi_l\} \cup \{\alpha^\vee \mid \alpha \in \Phi_s\})$ *be the proper sublattice of* $L(\Phi^\vee)$*. Then it is* $w(\lambda) \in L'$ *for all* $w \in W$ *and for all* $\lambda \in L'$*.*

Proof. It is sufficient to show the assertion for the generators of L'. If $\alpha \in \Phi_l$, then $w(\delta\alpha^\vee) = \delta w(\alpha^\vee) = \delta w(\alpha)^\vee$, where we used Lemma 2.1.10 to obtain the last equality. By Lemma 2.1.8 we have $w(\alpha) \in \Phi_l$ and therefore $w(\delta\alpha^\vee) \in L'$. Similarly, we obtain $w(\alpha^\vee) = w(\alpha)^\vee \in \Phi_s$ if $\alpha \in \Phi_s$. □

Proposition 6.3.6. *Let* Φ *be an irreducible crystallographic root system which consists of long and short roots. If there exist roots* $\alpha_1, \ldots, \alpha_n \in \Phi$ *such that*

$$\widetilde{W} = \langle s_{\alpha_1}, \ldots, s_{\alpha_n}, s_{\alpha_n,1} \rangle,$$

then $\alpha_n \in \Phi_l$*.*

Proof. Assume the root α_n to be short. By Lemma 2.1.13 we have $s_{\alpha_n} s_{\alpha_n,1} = \operatorname{tr}(-\alpha_n^\vee)$ and therefore

$$\widetilde{W} = \langle s_{\alpha_1}, \dots, s_{\alpha_n}, \operatorname{tr}(\alpha_n^\vee) \rangle.$$

As before we let $L' := \operatorname{span}_{\mathbb{Z}}(\{\delta\alpha^\vee \mid \alpha \in \Phi_l\} \cup \{\alpha^\vee \mid \alpha \in \Phi_s\})$ be the proper sublattice of $L(\Phi^\vee)$. By Lemma 6.3.5 we obtain $\operatorname{tr}(\alpha_n^\vee) \in W \ltimes L'$, thus $W \ltimes L' = \widetilde{W}$. But by Lemma 6.3.3 we have that L' is a proper sublattice of $L(\Phi^\vee)$ and therefore $W \ltimes L'$ is a proper subgroup of $W \ltimes L(\Phi^\vee) = \widetilde{W}$, a contradiction.

$\square$

Proposition 6.3.7. *Let Φ be an irreducible crystallographic root system of rank n and $W = W_\Phi$. If there exist roots $\{\beta_1, \dots, \beta_n\} \subseteq \Phi$ such that $W = \langle s_{\beta_1}, \dots, s_{\beta_n} \rangle$, then for any integers $k_1, \dots, k_n \in \mathbb{Z}$ we have*

$$W \cong \langle s_{\beta_1,k_1}, \dots, s_{\beta_n,k_n} \rangle.$$

Proof. Since the roots $\beta_1, \dots, \beta_n$ have to be linearly independent, the corresponding hyperplanes intersect in one point. Therefore the group $W' := \langle s_{\beta_1,k_1}, \dots, s_{\beta_n,k_n} \rangle$ is finite by [Bou02, Ch.V, 3, Prop. 4]. We consider the map $p : \widetilde{W} \to W, s_{\alpha,k} \mapsto s_\alpha$. Let $w = s_{\alpha_{i_1},k_{i_1}} \cdots s_{\alpha_{i_m},k_{i_m}} \in \ker(p_{|W'})$, thus $p(w) = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_m}} = e$. Considering w in its normal form $w = w_0 \operatorname{tr}(\lambda)$ with $w_0 \in W$ and $\lambda \in L(\Phi^\vee)$, we have $w_0 = p(w) = e$. Hence w has to be a translation. Therefore $\lambda = 0$ since W' is finite. Thus $w = e$. By the first isomorphism theorem we obtain $W \cong W'$. $\square$

Proposition 6.3.8. *Let Φ be an irreducible crystallographic root system of rank n which consists of long and short roots. If there exist roots $\alpha_1, \dots, \alpha_n \in \Phi$ and integers $k_1, \dots, k_{n-1} \in \mathbb{Z}$ such that*

$$\widetilde{W} = \langle s_{\alpha_1,k_1}, \dots, s_{\alpha_{n-1},k_{n-1}}, s_{\alpha_n}, s_{\alpha_n,1} \rangle,$$

then $\alpha_n \in \Phi_l$.

Proof. First of all note that $W = p(\widetilde{W}) = \langle s_{\alpha_1}, \dots, s_{\alpha_n} \rangle$. Therefore the roots $\alpha_1, \dots, \alpha_n$ are linearly independent. Hence the system of equations

$$\begin{aligned} (\lambda \mid \alpha_1) &= l_1 \\ (\lambda \mid \alpha_2) &= l_2 \\ &\vdots \\ (\lambda \mid \alpha_n) &= l_n, \end{aligned}$$

where $l_1, l_2, \dots, l_n \in \mathbb{Z}$ are fixed, has a unique solution $\lambda \in V$. By Theorem 4.2.12 the roots $\alpha_1, \dots, \alpha_n$ are a basis of $L(\Phi)$. Since $(\lambda \mid \alpha_i) = l_i \in \mathbb{Z}$ for all $1 \le i \le n$, we have by definition $\lambda \in P(\Phi^\vee)$. Thus

$$\begin{aligned} \langle s_{\alpha_1,k_1}, \dots, s_{\alpha_{n-1},k_{n-1}}, s_{\alpha_n}, s_{\alpha_n,1} \rangle &= \operatorname{tr}(\lambda) \langle s_{\alpha_1,k_1}, \dots, s_{\alpha_{n-1},k_{n-1}}, s_{\alpha_n}, s_{\alpha_n,1} \rangle \operatorname{tr}(-\lambda) \\ &= \langle s_{\alpha_1,k_1+l_1}, \dots, s_{\alpha_{n-1},k_{n-1}+l_{n-1}}, s_{\alpha_n,l_n}, s_{\alpha_n,1+l_n} \rangle, \end{aligned}$$

where we used part (b) of Lemma 6.2.1 to obtain both equations. If we choose $l_i = -k_i$ for $1 \le i \le n-1$ and $l_n = 0$, the assertion follows by Proposition 6.3.6. $\square$

Lemma 6.3.9. *Let Φ be an irreducible crystallographic root system of rank n. If*

$$\widetilde{W} = \langle s_{\alpha_1,l_1}, \dots, s_{\alpha_{n-1},l_{n-1}}, s_{\alpha_n,l_n}, s_{\alpha_n,l_{n+1}} \rangle$$

for roots $\alpha_1, \dots, \alpha_n \in \Phi$ and integers $l_1, \dots, l_n, l_{n+1} \in \mathbb{Z}$, then $|l_{n+1} - l_n| = 1$.

Proof. Consider the subgroup $H := \langle s_{\alpha_1,l_1}, \dots, s_{\alpha_{n-1},l_{n-1}}, s_{\alpha_n,l_n} \rangle$ of $\widetilde{W}$. By Lemma 2.1.13 we have $s_{\alpha_n,l_n} s_{\alpha_n,l_{n+1}} = \mathrm{tr}((l_n - l_{n+1})\alpha_n^\vee)$. Hence

$$\widetilde{W} = \langle H, \mathrm{tr}((l_n - l_{n+1})\alpha_n^\vee) \rangle = \langle s_{\alpha_1,l_1}, \dots, s_{\alpha_{n-1},l_{n-1}}, s_{\alpha_n,l_n}, \mathrm{tr}((l_n - l_{n+1})\alpha_n^\vee) \rangle. \tag{6.9}$$

Using the same argument as in the proof of Proposition 6.3.8, we take $\lambda \in P(\Phi^\vee)$ to be the solution of the system of equations

$$\begin{aligned} (\lambda \mid \alpha_1) &= -l_1 \\ (\lambda \mid \alpha_2) &= -l_2 \\ &\vdots \\ (\lambda \mid \alpha_n) &= -l_n. \end{aligned}$$

Thus by part (b) of Lemma 6.2.1 we obtain

$$\mathrm{tr}(\lambda) s_{\alpha_i,l_i} \mathrm{tr}(-\lambda) = s_{\alpha_i,l_i+(\lambda|\alpha_i)} = s_{\alpha_i} \ (1 \leq i \leq n).$$

Since two translations commute, we obtain again by part (b) of Lemma 6.2.1 and by (6.9) that

$$\widetilde{W} = \mathrm{tr}(\lambda) \langle s_{\alpha_1,l_1}, \dots, s_{\alpha_n,l_n}, \mathrm{tr}((l_n - l_{n+1})\alpha_n^\vee) \rangle \mathrm{tr}(-\lambda) = \langle s_{\alpha_1}, \dots, s_{\alpha_n}, \mathrm{tr}((l_n - l_{n+1})\alpha_n^\vee) \rangle. \tag{6.10}$$

Put $l := l_n - l_{n+1}$. By Lemma 2.1.13 we have $s_{\alpha_n,l} s_{\alpha_n} = \mathrm{tr}(l\alpha_n^\vee)$ and therefore (6.10) yields

$$\widetilde{W} = \langle s_{\alpha_1}, \dots, s_{\alpha_n}, \mathrm{tr}(l\alpha_n^\vee) \rangle = \langle s_{\alpha_1}, \dots, s_{\alpha_n}, s_{\alpha_n,l} \rangle. \tag{6.11}$$

Put $R := \{s_{\alpha_1}, \dots, s_{\alpha_n}, s_{\alpha_n,l}\}$ and $T' := \cup_{w \in \widetilde{W}} wRw^{-1}$. Since we can write each element of $\widetilde{W}$ as a product of elements in R, Lemma 6.2.4 yields, that if $s_{\alpha,k} \in T'$ ($\alpha \in \Phi$, $k \in \mathbb{Z}, k \neq 0$), then k has to be a multiple of l. Using (6.11) we obtain by [Dye90, Corollary 3.11] that

$$T' = \widetilde{T} = \{s_{\alpha,k} \mid \alpha \in \Phi,\ k \in \mathbb{Z}\}.$$

Hence, if we assume that $|l| > 1$, we arrive at a contradiction, because in this case we have $T' \subsetneq \widetilde{T}$. Therefore $|l| \leq 1$ and Proposition 2.4.1 yields $1 = |l| = |l_n - l_{n+1}|$.

□

Combining Proposition 6.3.8 and Lemma 6.3.9, we obtain

Corollary 6.3.10. *Let Φ be an irreducible crystallographic root system of rank n. If*

$$\widetilde{W} = \langle s_{\alpha_1,l_1}, \dots, s_{\alpha_{n-1},l_{n-1}}, s_{\alpha_n,l_n}, s_{\alpha_n,l_{n+1}} \rangle$$

for roots $\alpha_1, \dots, \alpha_n \in \Phi$ and integers $l_1, \dots, l_n, l_{n+1} \in \mathbb{Z}$, then $|l_{n+1} - l_n| = 1$ and $\alpha_n \in \Phi_l$.

6.4. The proof of Theorem 1.1.3

We fix an affine irreducible dual Coxeter system $(\widetilde{W}, \widetilde{T})$ of rank $n+1$ and a simple system $\widetilde{S} \subseteq \widetilde{T}$. Let (W, S) be the underlying finite Coxeter system. Denote by Φ the corresponding irreducible crystallographic root system and by T the set of reflections for (W, S).

Let $\widetilde{w}$ be a quasi-Coxeter element for $(\widetilde{W}, \widetilde{T})$. We already noted in Section 6.2 that the element $w' := p(\widetilde{w})$ is a parabolic quasi-Coxeter element for (W, T) of absolute length $n-1$. Our main tool to prove Theorem 1.1.3, that is, to prove that the Hurwitz action is transitive on the set $\mathrm{Red}_{\widetilde{T}}(\widetilde{w})$, will be the investigation of the Hurwitz action on the set

$$\mathrm{Fac}_{T,n+1}(w') = \{(t_1, \ldots, t_{n+1}) \in T^{n+1} \mid \; t_1 \cdots t_{n+1} = w', \; \langle t_1, \ldots, t_{n+1} \rangle = W\}.$$

Note that this set does not contain reduced decompositions for w'. Consider the map

$$\pi : \mathrm{Red}_{\widetilde{T}}(\widetilde{w}) \to \mathrm{Fac}_{T,n+1}(w'), (r_1, \ldots, r_{n+1}) \mapsto (p(r_1), \ldots, p(r_{n+1})).$$

The three main steps to prove Theorem 1.1.3 will be to show the following assertions:

- The map π is well-defined.
- The Hurwitz action is transitive on $\mathrm{Fac}_{T,n+1}(w')$.
- For each $\underline{r} = (r_1, \ldots, r_{n-1}, r_n, r_n) \in \mathrm{Fac}_{T,n+1}(w')$ there exists a subgroup of the isotropy subgroup $\mathrm{Stab}_{\mathcal{B}_{n+1}}(\underline{r})$ which acts transitively on the fibre $\pi^{-1}(\underline{r})$.

Proposition 6.4.1. *The map π is well-defined, that is, if $(s_{\beta_1,k_1}, \ldots, s_{\beta_{n+1},k_{n+1}}) \in \mathrm{Red}_{\widetilde{T}}(\widetilde{w})$, then $\langle s_{\beta_1}, \ldots, s_{\beta_{n+1}} \rangle = W$.*

Proof. By equation (6.4) in Section 6.2 there exists a decomposition of $\widetilde{w}$ of the form

$$(s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_{n+1},l_{n+1}}),$$

where $\gamma_{n+1} = \gamma_n$ and $\widetilde{W} = \langle s_{\gamma_1,l_1}, \ldots, s_{\gamma_n,l_n}, s_{\gamma_{n+1},l_{n+1}} \rangle$. Let $(s_{\beta_1,k_1}, \ldots, s_{\beta_{n+1},k_{n+1}})$ be an arbitrary element of $\mathrm{Red}_{\widetilde{T}}(\widetilde{w})$ and put $w' := p(\widetilde{w})$. By Proposition 6.2.6 we can assume that $\beta_{n+1} = \beta_n$. We have

$$w := w' s_{\gamma_n} = \underbrace{s_{\gamma_1} \cdots s_{\gamma_{n-1}}}_{=w'} s_{\gamma_n} = \underbrace{s_{\beta_1} \cdots s_{\beta_{n-1}}}_{=w'} s_{\gamma_n}.$$

Since w is quasi-Coxeter we have

$$W = \langle s_{\gamma_1}, \ldots, s_{\gamma_n} \rangle = \langle s_{\beta_1}, \ldots, s_{\beta_{n-1}}, s_{\gamma_n} \rangle.$$

Since w' is a parabolic quasi-Coxeter element, Theorem 1.1.2 yields

$$(s_{\beta_1}, \ldots, s_{\beta_{n-1}}) \sim (s_{\gamma_1}, \ldots, s_{\gamma_{n-1}}).$$

Therefore we assume without loss of generality that $\gamma_i = \beta_i$ for $1 \leq i \leq n-1$. By Theorem 4.2.12 it remains to show:

(i) $\gamma_n \in \mathrm{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_n) = \mathrm{span}_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_{n-1}, \beta_n)$.

(ii) $\gamma_n^\vee \in \operatorname{span}_{\mathbb{Z}}(\beta_1^\vee, \ldots, \beta_n^\vee) = \operatorname{span}_{\mathbb{Z}}(\gamma_1^\vee, \ldots, \gamma_{n-1}^\vee, \beta_n^\vee)$.

Writing $\widetilde{w}$ in its normal form as described in Lemma 6.2.3, we obtain

$$\begin{aligned}\widetilde{w} &= w' \operatorname{tr}\left(\sum_{i=1}^{n+1} -l_i s_{\gamma_{n+1}} \cdots s_{\gamma_{i+1}}(\gamma_i)^\vee\right)\\ &= w' \operatorname{tr}\left(\sum_{i=1}^{n+1} -k_i s_{\beta_{n+1}} \cdots s_{\beta_{i+1}}(\beta_i)^\vee\right).\end{aligned}$$

Hence both translation parts must be equal. By Corollary 6.3.10 we have $l_n - l_{n+1} = \pm 1$ and γ_n is a long root. Using the facts that $\gamma_i = \beta_i$ for $1 \leq i \leq n-1$ and $\gamma_{n+1} = \gamma_n$, $\beta_{n+1} = \beta_n$, we obtain

$$\sum_{i=1}^{n-1} -l_i s_{\gamma_{n-1}} \cdots s_{\gamma_{i+1}}(\gamma_i)^\vee + \underbrace{(l_n - l_{n+1})}_{=\pm 1} \gamma_n^\vee = \sum_{i=1}^{n-1} -k_i s_{\gamma_{n-1}} \cdots s_{\gamma_{i+1}}(\gamma_i)^\vee + (k_n - k_{n+1})\beta_n^\vee,$$

thus

$$\pm\gamma_n^\vee = \sum_{i=1}^{n-1} (l_i - k_i) s_{\gamma_{n-1}} \cdots s_{\gamma_{i+1}}(\gamma_i)^\vee + (k_n - k_{n+1})\beta_n^\vee.$$

It follows by Lemma 6.2.7 that $\gamma_n^\vee \in \operatorname{span}_{\mathbb{Z}}(\gamma_1^\vee, \ldots, \gamma_{n-1}^\vee, \beta_n^\vee) = \operatorname{span}_{\mathbb{Z}}(\beta_1^\vee, \ldots, \beta_n^\vee)$, hence (ii). It remains to show (i). The root $\gamma_{n+1} = \gamma_n$ is a long root. By (ii) we have

$$\gamma_n^\vee = \sum_{i=1}^{n} \lambda_i \beta_i^\vee$$

for some integer coefficients $\lambda_i \in \mathbb{Z}$. Therefore we obtain

$$\gamma_n = \sum_{i=1}^{n} \underbrace{\frac{(\gamma_n \mid \gamma_n)}{(\beta_i \mid \beta_i)}}_{\in\{1,2,3\}} \lambda_i \beta_i \in \operatorname{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_n).$$

□

Proposition 6.4.2. *The map π is equivariant with respect to the Hurwitz action.*

Proof. By Lemma 6.2.5 we have

$$\sigma_i(\ldots, s_{\alpha_i,k_i}, s_{\alpha_{i+1},k_{i+1}}, \ldots) = \left(\ldots, s_{s_{\alpha_i}(\alpha_{i+1}), k_{i+1} - \frac{2(\alpha_i \mid \alpha_{i+1})}{(\alpha_i \mid \alpha_i)} k_i}, s_{\alpha_i}, \ldots\right),$$

hence

$$\pi(\sigma_i(\ldots, s_{\alpha_i,k_i}, s_{\alpha_{i+1},k_{i+1}}, \ldots)) = (\ldots, s_{s_{\alpha_i}(\alpha_{i+1})}, s_{\alpha_i}, \ldots) = \sigma_i(\ldots, s_{\alpha_i}, s_{\alpha_{i+1}}, \ldots).$$

□

Theorem 6.4.3. *The Hurwitz action is transitive on $\operatorname{Fac}_{T,n+1}(w')$.*

Proof. Let $(t_1, \ldots, t_{n+1}), (r_1, \ldots, r_{n+1}) \in \mathrm{Fac}_{T,n+1}(w')$ be arbitrary. By Theorem 5.1.2 we can assume (up to Hurwitz equivalence) that $t_{n+1} = t_n$ and $r_{n+1} = r_n$. It is $w' = t_1 \cdots t_{n-1} = r_1 \cdots r_{n-1}$ a parabolic quasi-Coxeter element with corresponding parabolic subgroup $W' = \langle t_1, \ldots, t_{n-1} \rangle$. In particular we have

$$(t_1, \ldots, t_{n-1}) \sim (r_1, \ldots, r_{n-1})$$

by Theorem 1.1.2. Therefore

$$(r_1, \ldots, r_{n-2}, r_{n-1}, r_{n-1}) \sim (t_1, \ldots, t_{n-2}, r_{n-1}, r_{n-1})$$

and the assertion follows by Proposition 5.2.4 and Lemma 5.3.1. □

Direct calculations yield the following statement.

Lemma 6.4.4. *Let $\alpha \in \Phi$. Then*

$$(s_{\alpha,1}, s_{\alpha,0}) \sim (s_{\alpha,k+1}, s_{\alpha,k}) \text{ and } (s_{\alpha,0}, s_{\alpha,1}) \sim (s_{\alpha,k}, s_{\alpha,k+1}) \text{ for all } k \in \mathbb{Z}\,.$$

Lemma 6.4.5. *Let $\widetilde{w}$ be quasi-Coxeter for $(\widetilde{W}, \widetilde{T})$. Then $\langle \sigma_n \rangle \subseteq \mathcal{B}_{n+1}$ acts transitively on the fibre $\pi^{-1}(p(\underline{r}))$ for each $\underline{r} = (s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_n,l_{n+1}}) \in \mathrm{Red}_{\widetilde{T}}(\widetilde{w})$*

Proof. Fix $\underline{r} = (s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_n,l_{n+1}}) \in \mathrm{Red}_{\widetilde{T}}(\widetilde{w})$. Clearly an element in the fibre has to be of the form

$$(s_{\gamma_1,m_1}, \ldots, s_{\gamma_{n-1},m_{n-1}}, s_{\gamma_n,m_n}, s_{\gamma_n,m_{n+1}}) \in \mathrm{Red}_{\widetilde{T}}(\widetilde{w})$$

for integers $m_1, \ldots, m_{n+1} \in \mathbb{Z}$. Considering the normal form of the element corresponding to this decomposition and the element corresponding to $\underline{r}$, we have equality of the translation parts. By the same calculations as in the proof of Proposition 6.4.1 we obtain

$$\sum_{i=1}^{n-1} -l_i s_{\gamma_{n-1}} \cdots s_{\gamma_{i+1}}(\gamma_i)^\vee + (l_n - l_{n+1})\gamma_n^\vee = \sum_{i=1}^{n-1} -m_i s_{\gamma_{n-1}} \cdots s_{\gamma_{i+1}}(\gamma_i)^\vee + (m_n - m_{n+1})\gamma_n^\vee.$$

By Proposition 6.4.1 we have $W = \langle s_{\gamma_1}, \ldots, s_{\gamma_n} \rangle$. Hence by Theorem 4.2.12 the roots $\gamma_1^\vee, \ldots, \gamma_{n-1}^\vee, \gamma_n^\vee$ are a basis of $L(\Phi^\vee)$. Therefore by Lemma 6.4.6 the set

$$\{s_{\gamma_{n-1}} \cdots s_{\gamma_2}(\gamma_1)^\vee, \ldots, s_{\gamma_{n-1}}(\gamma_{n-2})^\vee, \gamma_{n-1}^\vee, \gamma_n^\vee\}$$

is another base and we obtain that $m_i = l_i$ for all $i \in \{1, \ldots, n-1\}$ and $m_n - m_{n+1} = l_n - l_{n+1} = \pm 1$ by Lemma 6.3.9. By these properties and by Lemma 6.4.4 we conclude that $\langle \sigma_n \rangle \subseteq \mathcal{B}_{n+1}$ acts transitively on $\pi^{-1}(\underline{r})$. □

Lemma 6.4.6. *Let Φ be an irreducible crystallographic root system of rank $n \geq 2$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ such that $W = \langle s_{\alpha_1}, \ldots, s_{\alpha_n} \rangle$. Then the set*

$$\{s_{\alpha_{n-1}} \cdots s_{\alpha_{i+1}}(\alpha_i)^\vee \mid 1 \leq i \leq n-1\} \cup \{\alpha_n^\vee\}$$

is a basis of $L(\Phi^\vee)$.

Proof. This is a direct consequence of Theorem 4.2.12 and Lemma 4.2.16 □

Remark 6.4.7. In fact, the fibre we considered in Lemma 6.4.5 can be described completely in terms of the translation part of s_{γ_n,l_n} and $s_{\gamma_n,l_{n+1}}$ (resp. its coefficients l_n and l_{n+1}). For $l_n - l_{n+1} = 1$, we have

$$\ldots \sim (-2,-1) \sim (-1,0) \sim (0,1) \sim (1,2) \sim (2,3) \sim \ldots$$

For $l_n - l_{n+1} = -1$, we have

$$\ldots \sim (-1,-2) \sim (0,-1) \sim (1,0) \sim (2,1) \sim (3,2) \sim \ldots$$

We are finally in the position to prove the main result of this chapter.

Proof of Theorem 1.1.3. First let $\widetilde{w} \in \widetilde{W}$ be a quasi-Coxeter element. We fix

$$(s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_n,l_{n+1}}) \in \mathrm{Red}_{\widetilde{T}}(\widetilde{w})$$

obtained as in Equation (6.4) in Section 6.2. Let $(t_1, \ldots, t_{n+1}) \in \mathrm{Red}_{\widetilde{T}}(\widetilde{w})$ be arbitrary. By Theorem 6.4.3 and Proposition 6.4.2 there exists a braid $\sigma \in \mathcal{B}_{n+1}$ such that

$$\pi(\sigma(t_1, \ldots, t_{n+1})) = \pi(s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_n,l_{n+1}}).$$

Hence $\sigma(t_1, \ldots, t_n)$ and $(s_{\gamma_1,l_1}, \ldots, s_{\gamma_{n-1},l_{n-1}}, s_{\gamma_n,l_n}, s_{\gamma_n,l_{n+1}})$ are in the same fibre and the assertion follows by Lemma 6.4.5.

If $\widetilde{w} \in \widetilde{W}$ is a parabolic quasi-Coxeter element, but not a quasi-Coxeter element, then the assertion follows by Theorem 1.1.2 and Theorem 3.2.1 since all proper parabolic subgroups of $(\widetilde{W}, \widetilde{T})$ are finite. □

Remark 6.4.8. Let (W, T) be a dual Coxeter system of rank n. By Proposition 2.4.1 a quasi-Coxeter element has to be at least of absolute length n. If (W, T) is finite, then by Lemma 2.3.1 the absolute length is bounded by n. Hence it is canonical to demand a quasi-Coxeter element to be of absolute length n. If (W, T) is not finite, then the absolute length is in general not bounded by above, except for the case where (W, T) is affine. In that case it is bounded by $2n - 2$ (see [McCP11]). Therefore it makes sense to ask whether one can extend the definition of quasi-Coxeter element to elements of absolute length greater than n. Namely if it is possible to find an element $w \in W$ with $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ and $m > n$ such that $W = \langle t_1, \ldots, t_m \rangle$ and the Hurwitz action is transitive on $\mathrm{Red}_T(w)$. We give a positive answer in the following example, proposed by Thomas Gobet.

Example 6.4.9. Let $(\widetilde{W}, \widetilde{T})$ be affine of type $\widetilde{A}_2$. We choose $\widetilde{S} \subseteq \widetilde{T}$ such that $\widetilde{S} = \{s_{\alpha_1}, s_{\alpha_2}, s_{\widetilde{\alpha},1}\}$, where α_1, α_2 are simple roots for the corresponding root system of type A_2 and $\widetilde{\alpha} = \alpha_1 + \alpha_2$ is the highest root. We consider the element $w := (s_{\alpha_1} s_{\alpha_2} s_{\widetilde{\alpha},1})^2$. Since this is the power of a Coxeter element, we have $\ell_{\widetilde{S}}(w) = 6$ by [Spe09]. Using Theorem 2.3.3 we see that $\ell_{\widetilde{T}}(w) = 4$. Note that 4 is precisely the upper bound for the absolute length in $\widetilde{A}_2$. A reduced $\widetilde{T}$-decomposition is given by $w = s_{s_{\alpha_1}(\alpha_2)} s_{s_{\alpha_1}(\widetilde{\alpha}),1} s_{\alpha_2} s_{\widetilde{\alpha},1}$ and

$$(s_{s_{\alpha_1}(\alpha_2)}, s_{s_{\alpha_1}(\widetilde{\alpha}),1}, s_{\alpha_2}, s_{\widetilde{\alpha},1}) \sim (s_{\widetilde{\alpha},1}, s_{\widetilde{\alpha}}, s_{\alpha_2,1}, s_{\alpha_2}).$$

Therefore we see that $w = \mathrm{tr}(\alpha_1 + 2\alpha_2)$. By Theorem 5.1.3 a reduced decomposition of w (up to Hurwitz equivalence) looks like $w = s_{\alpha,k_1} s_{\alpha,l_1} s_{\beta,k_2} s_{\beta,l_2}$ with $\alpha, \beta \in \{\alpha_1, \alpha_2, \widetilde{\alpha}\}$. Note that

$\alpha \neq \beta$ since otherwise $s_{\alpha,k_1} s_{\alpha,l_1} s_{\beta,k_2} \in \widetilde{T}$ by Lemma 6.2.3, which would contradict $\ell_{\widetilde{T}}(w) = 4$. We have

$$(s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}) \overset{\sigma_2\sigma_1\sigma_3\sigma_2}{\sim} (s_{\alpha_2}, s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_1})$$

and

$$(s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}) \sim (s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}) \sim (s_{\alpha_2}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_3}) \overset{\sigma_2\sigma_1\sigma_3\sigma_2}{\sim} (s_{\alpha_3}, s_{\alpha_3}, s_{\alpha_2}, s_{\alpha_2})$$

and

$$(s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}) \sim (s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}, s_{\alpha_1}) \sim (s_{\alpha_3}, s_{\alpha_3}, s_{\alpha_1}, s_{\alpha_1}) \overset{\sigma_2\sigma_1\sigma_3\sigma_2}{\sim} (s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_3}).$$

By comparison of coefficients and Lemma 6.4.4 we obtain Hurwitz transitivity on $\mathrm{Red}_{\widetilde{T}}(w)$ by using similar arguments as before in this section.

7. Hurwitz action in elliptic Weyl groups

In Sections 7.1, 7.2 and 7.3 we collect some facts about elliptic root systems and its Weyl groups from [Sai85]. We just state proofs if these are relevant for the later understanding. Note that the notation in this chapter differs from the notation used by Saito in [Sai85]. We often work with further assumptions compared to [Sai85], e.g. we assume the root systems to be simply-laced. We do this for simplicity since these assumptions are sufficient for our purposes.

7.1. Extended affine and elliptic root systems

In this section we summarize some facts about extended affine root systems. For details see [Sai85].

Before we considered a finite dimensional euclidean vector space. We generalize the setting by considering a finite dimensional vector space V over $\mathbb{R}$ equipped with a quadratic form q. The form q induces a symmetric bilinear form

$$\begin{aligned}(- \mid -) := (- \mid -)_q : V \times V &\to \mathbb{R} \\ (x, y) &\mapsto q(x+y) - q(x) - q(y).\end{aligned}$$

We are now going to generalize the notion of root system. For $\alpha \in V$ non isotropic (i.e. $(\alpha \mid \alpha) \neq 0$) we define as before

$$\alpha^\vee := \frac{1}{q(\alpha)}\alpha = \frac{2}{(\alpha \mid \alpha)}\alpha \in V.$$
$$s_\alpha(v) := v - (v \mid \alpha^\vee)\alpha = v - \frac{2(v \mid \alpha)}{(\alpha \mid \alpha)}\alpha.$$

It is immediate to check that $s_\alpha^2 = \mathrm{id}$ and that s_α is an isometry of V with respect to the bilinear form induced by q, i.e. $s_\alpha \in O(V) = \{\varphi \in \mathrm{GL}(V) \mid (\varphi(v) \mid \varphi(v)) = (v \mid v)\}$. For a subset $R \subseteq V$ consisting of non isotropic elements, we define

$$W_R := \langle s_\alpha \mid \alpha \in R \rangle \leq O(V).$$

To the bilinear form $(- \mid -) = (- \mid -)_q$ we associate the signature (μ_+, μ_0, μ_-), where μ_* is the number of eigenvalues of $(- \mid -)$ (i.e the eigenvalues of the associated symmetric matrix), which are positive resp. zero resp. negative. We obtain the symmetric matrix A associated to $(- \mid -)$ if we choose a basis $\{v_1, ..., v_n\}$ of V and define

$$A := ((v_i \mid v_j))_{1 \leq i,j \leq n}.$$

The matrix A determines the form $(- \mid -)$. If $x, y \in V$ we have $(x \mid y) = x^T A y$. Furthermore $(- \mid -)$ is non degenerate, if the determinant of A does not equal zero.

Definition 7.1.1. A non isotropic subset $\Phi \subseteq V$ is called generalized root system with respect to $(- \mid -) = (- \mid -)_q$ if

(a) $L(\Phi) := \mathrm{span}_{\mathbb{Z}}(\Phi)$ is a full lattice of V (in particular $\mathrm{span}_{\mathbb{R}}(\Phi) = V$), called root lattice of Φ,

(b) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$,

(c) $(\alpha \mid \beta^\vee) = \frac{2(\alpha|\beta)}{(\beta|\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Furthermore Φ is called reducible if $\Phi = \Phi_1 \dot{\cup} \Phi_2$ where Φ_1, Φ_2 are nonempty generalized root systems such that $(\alpha \mid \beta) = 0$ whenever $\alpha \in \Phi_1, \beta \in \Phi_2$. Otherwise Φ is called irreducible. The generalized root system Φ is called reduced if $\alpha, \lambda\alpha \in \Phi$, where $\lambda \in \mathbb{R}$ and $\alpha \in \Phi$, implies $\lambda = \pm 1$.

Proposition 7.1.2. *[Sai85, (1.2) Assertion] Let Φ be a reduced generalized root system with respect to a symmetric bilinear form $(- \mid -)$. Then there exists a unique disjoint decomposition $\Phi = \Phi_1 \dot{\cup} \Phi_2 \dot{\cup} \ldots \dot{\cup} \Phi_m$ such that*

- *Φ_i is a non-empty irreducible reduced generalized root system with respect to $(- \mid -)$, $1 \leq i \leq m$,*
- *Φ_i and Φ_j are orthogonal with respect to $(- \mid -)$ for $i \neq j$.*

For the rest of this section we fix an irreducible reduced generalized root system Φ with respect to a symmetric bilinear form $(- \mid -) = (- \mid -)_q$.

Note that, by definition, the group W_Φ preserves $(- \mid -)$. Conversely we have the following by [Sai85, (1.2) Note 4].

Proposition 7.1.3. *The symmetric bilinear form $(- \mid -)$ is the unique form (up to a constant factor) that is invariant under W_Φ.*

Let us first point out that the definition of generalized root system indeed generalizes our previous definition of a root system (see Definition 2.1.7).

Example 7.1.4. If the form $(- \mid -)$ is positive definite, then Φ is a root system in the sense of Definition 2.1.7. To see this, we first assume the form $(- \mid -)$ to be positive semi-definite. Therefore the Cauchy-Schwarz inequality holds, that is $|(\alpha \mid \beta)| \leq \sqrt{(\alpha \mid \alpha)(\beta \mid \beta)}$ for all $\alpha, \beta \in \Phi$. Thus $(\alpha|\beta)^2 \leq (\alpha \mid \alpha)(\beta \mid \beta)$, which implies

$$(\alpha^\vee \mid \beta)(\alpha|\beta^\vee) = \frac{4(\alpha \mid \beta)^2}{(\alpha \mid \alpha)(\beta \mid \beta)} \leq 4.$$

Since $(\alpha^\vee \mid \beta) \in \mathbb{Z}$ and $(\alpha \mid \beta^\vee) \in \mathbb{Z}$, we obtain $(\alpha \mid \beta^\vee) \in \mathbb{Z} \cap [-4, 4]$ for all $\alpha, \beta \in \Phi$. Let $(n, 0, 0)$ be the signature of $(- \mid -)$ and let $\alpha_1, \ldots, \alpha_n$ be a basis of V. To show that Φ is a root system we only need to show that Φ is finite. Since $(- \mid -)$ is non degenerate, the map

$$\Phi \to (\mathbb{Z} \cap [-4, 4])^n, \alpha \mapsto \left((\alpha \mid \alpha_i^\vee)\right)_{1 \leq i \leq n}$$

is injective.

Example 7.1.5. [Sai85, (1.3)] If the form $(- \mid -)$ is positive semi-definite of signature $(n, 1, 0)$, then Φ is an affine root system in the sense of [Mac72].

Motivated by the previous examples we make the following definitions.

Definition 7.1.6. A generalized root system Φ with respect to $(- \mid -)$ is called extended affine root system or elliptic root system if the form $(- \mid -)$ has signature $(n, 2, 0)$. In that case we define n to be the rank of Φ. The group W_Φ is called elliptic Weyl group.

Remark 7.1.7. The name elliptic root system was introduced in [SY00]. The authors give the following reason for the name (see [SY00, Remark 1]):

> The two-extended affine root systems describe the (transcendental) lattices generated by vanishing cycles for simple elliptic singularities. This is the reason why we call them the elliptic root systems. In fact, the radical of the root system corresponds to the lattice of an elliptic curve, and a rank one subspace of the radical, called a marking, corresponds to a choice of a primitive form for the elliptic singularities.

Lemma 7.1.8. *[Sai85, (1.4) Lemma] Let Φ (resp. Φ') be a generalized root system in V with respect to $(- \mid -)$ (resp. $(- \mid -)'$) and let φ be an isomorphism between Φ and Φ', i.e. φ is an automorphism of V with $\varphi(\Phi) = \Phi'$. Then there exists $c \in \mathbb{R}$ with $(- \mid -) = c(- \mid -)' \circ (\varphi \times \varphi)$.*

Lemma 7.1.9. *[Sai85, (1.7) Assertion] There exists $c \in \mathbb{R} \setminus \{0\}$ such that $c(- \mid -)$ is an integral bilinear form on $L(\Phi) \times L(\Phi)$.*

Definition 7.1.10. [Sai85, (1.7) Definition] A subspace U of V is said to be defined over $\mathbb{Q}$ if $L(\Phi) \cap U$ is a full lattice of U.

Definition 7.1.11. Relative to the form $(- \mid -)$ (resp. the root lattice $L(\Phi)$) we define the radical

$$R_{(-|-)} := \{x \in V \mid (x \mid y) = 0 \text{ for all } y \in V\}$$
$$(\text{resp. } R_{L(\Phi)} := \{x \in L(\Phi) \mid (x \mid y) = 0 \text{ for all } y \in L(\Phi)\}).$$

Note that the subspace $R_{(-|-)}$ is defined over $\mathbb{Q}$ by Lemma 7.1.9 and that $R_{L(\Phi)} \otimes_{\mathbb{Z}} \mathbb{R} \cong R_{(-|-)}$.

Let U be a subspace of the radical $R := R_{(-|-)}$ which is defined over $\mathbb{Q}$ and let $p := p_U : V \to V/U, v \mapsto v + U$ be the canonical map. Furthermore let $(- \mid -)_U$ be the induced form on U, that is $(p(x) \mid p(y))_U = (x \mid y)$ for all $x, y \in V$. Then it is

$$L(p(\Phi)) = p(L(\Phi)) \cong L(\Phi)/(L(\Phi \cap U)).$$

Therefore we observe:

Proposition 7.1.12. *[Sai85, (1.8) Assertion] If Φ is a generalized root system with respect to $(- \mid -)$, then $p(\Phi)$ is a generalized root system with respect to $(- \mid -)_U$, called* quotient root system *of Φ by U.*

Lemma 7.1.13. *[Sai85, (1.9) Lemma]*

(a) There exists $c \in \mathbb{R} \setminus \{0\}$ such that the set $\{c(\alpha \mid \alpha) \mid \alpha \in \Phi\} \subseteq \mathbb{Z}$ is finite.

(b) It is $\frac{(\alpha|\alpha)}{(\beta|\beta)} \in \mathbb{Q}$ *for all* $\alpha, \beta \in \Phi$.

Let $\Phi \subseteq V$ be an irreducible reduced elliptic root system with respect to $(- \mid -)$ of rank n, i.e. $(- \mid -)$ is a positive semi-definite bilinear form on a $(n+2)$-dimensional real vector space V such that the rank of $R := R_{(-|-)}$ is two.

Definition 7.1.14. [Sai85, (2.1) Definition 1] A linear subspace U of R of rank 1, which is defined over $\mathbb{Q}$, is called a marking for Φ. The pair (Φ, U) is called marked elliptic root system.

An isomorphism of marked elliptic root systems (Φ, U) and (Φ', U') is an isomorphism $\varphi : V \to V'$ of elliptic root systems such that $\varphi(U) = U'$.

Let (Φ, U) be an irreducible reduced marked elliptic root system. Then $L(\Phi) \cap R$ is a full lattice of R. We regard it as a $\mathbb{Z}$-module and choose a $\mathbb{Z}$-basis a, b as follows

$$L(\Phi) \cap U = \mathbb{Z}\, a \tag{7.1}$$
$$L(\Phi) \cap R = \mathbb{Z}\, a + \mathbb{Z}\, b. \tag{7.2}$$

Lemma 7.1.15. *[Sai85, (3.1) Lemma 1] With notations as above.*

(i) $p_R(\Phi)$ *is a (finite) root system with respect to* $(- \mid -)_R$.

(ii) $p_U(\Phi)$ *is an affine root system with respect to* $(- \mid -)_U$.

We call $p_R(\Phi)$ quotient finite root system *and* $p_U(\Phi)$ quotient affine root system.

Proof. We just show (i). By Proposition 7.1.12 $p_R(\Phi)$ is a generalized root system with respect to $(- \mid -)_R$. Since $(- \mid -)_R$ is positive definite, the assertion follows by Example 7.1.4. □

By Lemma 7.1.9 there exists a constant c such that $c(- \mid -)$ takes integral values on $L(\Phi) \times L(\Phi)$. We choose such a constant c such that $\gcd\{c \cdot q(\alpha) \mid \alpha \in \Phi\} = 1$ and identify $(- \mid -)$ and $c(- \mid -)$ in the following. In particular we can define

$$t(\Phi) := \frac{\operatorname{lcm}\{q(\alpha) \mid \alpha \in \Phi\}}{\gcd\{q(\alpha) \mid \alpha \in \Phi\}},$$

called total tier number of Φ (see [Sai85, Section (4.1)]).

Definition 7.1.16. An irreducible elliptic root system Φ is called homogenous if $t(\Phi) = 1$. In particular it is $(\alpha \mid \alpha)_q = (\alpha \mid \alpha) = \pm 2$ for all $\alpha \in \Phi$. The elliptic root system Φ is called simply-laced if $(\alpha \mid \alpha) = 2$ for all $\alpha \in \Phi$.

7.2. The elliptic Weyl group

The goal of this section is to give a description of the elliptic Weyl group similar to that one for affine Weyl groups in Theorem 2.1.15. As before we refer to [Sai85] for details.

For the rest of this section let $\Phi \subseteq V$ be an irreducible reduced elliptic root system with respect to $(- \mid -)$ of rank n and assume that Φ is simply-laced. Let $R := R_{(-|-)}$ be the radical of the form. As in [Sai85, Section (1.14)] we define a semi-group structure $\circ$ on $V \otimes (V/R)$ by

$$\varphi_1 \circ \varphi_2 := \varphi_1 + \varphi_2 - (\varphi_1 \mid \varphi_2),$$

where $(\varphi_1 \mid \varphi_2) := \sum_{i_1,i_2} f^1_{i_1} \otimes I(g^1_{i_1}, f^2_{i_2}) g^2_{i_2}$ for $\varphi_j = \sum_{i_j} f^j_{i_j} \otimes g^j_{i_j}$.

Definition 7.2.1. The map

$$E : V \otimes (V/R) \to \operatorname{End}(V), \sum_i f_i \otimes g_i \mapsto \left(v \mapsto v - \sum_i f_i (g_i \mid v) \right)$$

is called Eichler-Siegel map for V with respect to $(- \mid -)$.

Proposition 7.2.2. *[Sai85, Sections (1.14) and (1.15)]*

(a) E is injective.

(b) E is a homomorphism of semi-groups, i.e. $E(\varphi \circ \psi) = E(\varphi)E(\psi)$.

(c) For non-isotropic $\alpha \in V$, it is $s_\alpha = E(\alpha \otimes \alpha)$.

(d) The inverse of E on W_Φ is well-defined:

$$E^{-1} : W_\Phi \to V \otimes (V/R).$$

(e) The subspace $R \otimes (V/R)$ is closed under $\circ$ and the semi-group structure coincides with the additive structure of the vector space on this subspace.

(f) $E^{-1}(W_\Phi) \subseteq L(\Phi) \otimes_{\mathbb{Z}} (L(\Phi)/(L(\Phi) \cap R))$.

(g) Let U be a subspace of R defined over $\mathbb{Q}$. Then

$$T_U := E^{-1}(W_\Phi) \cap (U \otimes (V/R))$$

is a lattice of $U \otimes (V/R)$.

For the rest of this section let U be a subspace of R defined over $\mathbb{Q}$. Recall that we denoted by p_U the canonical map $p_U : V \to V/U$. It induces a group homomorphism

$$p_* : W_\Phi \to W_{p_U(\Phi)}, s_\alpha \mapsto s_{p_U(\alpha)}.$$

Lemma 7.2.3. *[Sai85, (1.15) Assertion (i)] The following sequence is exact*

$$1 \to T_U \xrightarrow{E} W_\Phi \xrightarrow{p_*} W_{p_U(\Phi)} \to 1. \tag{7.3}$$

Lemma 7.2.4. *[Sai85, (1.15) Assertion (ii)] The group W_Φ acts on T_U by*

$$wE\left(\sum_i f_i \otimes g_i\right)w^{-1} = E\left(\sum_i f_i \otimes w(g_i)\right)$$

for $w \in W_\Phi$ and $\sum_i f_i \otimes g_i \in U \otimes (V/R)$.

Lemma 7.2.5. *[Sai85, (1.15) Lemma] Let $\mathcal{L}$ be a linear subspace of V such that*

(i) $V = \mathcal{L} \oplus U$,

(ii) $p_{*|W_{\Phi\cap\mathcal{L}}} : W_{\Phi\cap\mathcal{L}} \to W_{p_U(\Phi)}$ *is surjective.*

Then:

(a) The homomorphism $p_{*|W_{\Phi\cap\mathcal{L}}}$ is an isomorphism. Hence the sequence (7.3) splits into a semi direct product

$$W_\Phi = W_{\Phi\cap\mathcal{L}} \ltimes T_U.$$

(b) T_U is a full lattice of $U \otimes (V/R)$, which is generated by $\alpha_U \otimes \alpha_{\mathcal{L}}$ for $\alpha \in \Phi$. Here $\alpha = \alpha_U + \alpha_{\mathcal{L}}$ is the splitting as assumed in (i).

Remark 7.2.6. [Sai85, (1.15) Note] Let $\mathcal{L}$ be as in Lemma 7.2.5.

(a) $\Phi \cap \mathcal{L}$ is an elliptic root system with respect to the form $(- \mid -)_{|\mathcal{L}}$.

(b) The isomorphism $\mathcal{L} \cong V/U$ induces an injection $\Phi \cap \mathcal{L} \to p_U(\Phi)$, which induces an isomorphism $W_{\Phi\cap\mathcal{L}} \cong W_{p_U(\Phi)}$.

7.3. Splitting of the elliptic Weyl group and elliptic root basis

For details and proofs we refer to [Sai85] and [SY00].

7.3.1. Splitting of the elliptic Weyl group

For this subsection we fix an irreducible reduced simply-laced marked elliptic root system (Φ, U) and the radical $R = R_{(-|-)}$ of the form. As before we have the natural map $p_R : V \to V/R$.

Theorem 7.3.1.

$$W_\Phi \cong W_{p_R(\Phi)} \ltimes T_R.$$

Proof. By Lemma 7.2.5 we have to show the existence of a linear subspace $\mathcal{L} \subseteq V$ complementary to R such that $p_{*|W_{\Phi\cap\mathcal{L}}} : W_{\Phi\cap\mathcal{L}} \to W_{p_R(\Phi)}$ is surjective. By Lemma 7.1.15 the root system $p_R(\Phi)$ is finite. Choose a simple system $\{\beta_1, \dots, \beta_n\} \subseteq p_R(\Phi)$ and elements $\alpha_i \in \Phi \cap p_R^{-1}(\beta_i)$ for all $i \in \{1, \dots, n\}$. Put $\mathcal{L} := \oplus_{i=1}^n \mathbb{R}\,\alpha_i$. Since any root in $p_R(\Phi)$ is a nonnegative or a nonpositive integral linear combination of the β_i and since the reflections s_{β_i} generate $W_{p_R(\Phi)}$, we have

$$W_\Phi = W_{\Phi\cap\mathcal{L}} \ltimes T_R$$

by Lemma 7.2.5. By part (b) of Remark 7.2.6 the assertion follows. □

Remark 7.3.2. Choose $\mathcal{L} = \oplus_{i=1}^n \mathbb{R}\,\alpha_i$ as in the proof of Theorem 7.3.1. The splitting $V = \mathcal{L} \oplus R$ induces the following splitting of $L(\Phi)$ over $\mathbb{Z}$:

$$\begin{aligned} L(\Phi) &= (L(\Phi) \cap \mathcal{L}) \oplus (L(\Phi) \cap R) \\ &= (\oplus_{i=1}^n \mathbb{Z}\,\alpha_i) \oplus (\mathbb{Z}\,a \oplus \mathbb{Z}\,b) \end{aligned}$$

7.3.2. Elliptic root basis

In contrast to finite or affine root systems, the Weyl group W_Φ of an elliptic root system does not act anywhere properly on the ambient vector space. Hence there is no analogous of a Weyl chamber. Nevertheless Saito introduced in [Sai85] the notion of a basis for Φ and classified the irreducible elliptic root systems in terms of so called elliptic Dynkin diagrams.

Throughout this section let (Φ, U) be an irreducible reduced simply-laced marked elliptic root system in V and let $R := R_{(-|-)}$ be the radical of the form. Denote by p_U resp. p_R the canonical maps onto V/U resp. V/R. We already noted in Lemma 7.1.15 that $\Phi_{\mathrm{af}} := p_U(\Phi)$ is an affine root system. For our purpose it is sufficient to assume that Φ_{af} is reduced and of type $X_n^{(1)}$ (see [Mac72]), where X_n is one of the simply-laced types A_n $(n \geq 1)$, D_n $(n \geq 4)$ or E_n $(n \in \{6, 7, 8\})$. It is $\mathrm{L}_{\mathrm{af}} := p_U(L(\Phi)) = L(\Phi_{\mathrm{af}})$ the corresponding affine root lattice.

Choose a set $\Delta_{\mathrm{af}} \subseteq \Phi_{\mathrm{af}}$ of simple roots and a set $\Gamma_{\mathrm{af}} \subseteq \Phi \cap p_U^{-1}(\Delta_{\mathrm{af}})$ such that p_U induces a bijection between Δ_{af} and Γ_{af}. The set Γ_{af} is unique up to isomorphism of (Φ, U).

We list some facts about Φ_{af} as can be found in [Kac83, Chapter 6]:

- Δ_{af} is a basis of L_{af} such that Φ_{af} is contained in ${\mathrm{L}_{\mathrm{af}}}^+ \cup {\mathrm{L}_{\mathrm{af}}}^-$, where

$$ {\mathrm{L}_{\mathrm{af}}}^\pm = \left(\pm \sum_{\alpha \in \Delta_{\mathrm{af}}} \mathbb{Z}_{\geq 0}\,\alpha \right) \setminus \{0\}.$$

- There exist positive integers n_α ($\alpha \in \Delta_{\mathrm{af}}$) such that $\delta := \sum_{\alpha\in\Delta_{\mathrm{af}}} n_\alpha \alpha$ is a generator of $R_{\mathrm{L_{af}}}$, the radical of $\mathrm{L_{af}}$. For certain affine root systems, the coefficients n_α are listed as the values of the vertices in Figure 2.3.
- There exists $\alpha_0 \in \Delta_{\mathrm{af}}$ with $n_{\alpha_0} = 1$ such that $\Delta_{\mathrm{af}} \setminus \{\alpha_0\}$ yields the Dynkin diagram of the finite root system $p_R(\Phi)$.
- Since Φ_{af} is assumed to be of type $X_n^{(1)}$, we have $\widetilde{\alpha} = \delta - \alpha_0$, where $\widetilde{\alpha}$ is the highest root for the finite root system $p_R(\Phi)$ of type X_n.

Put $\mathcal{L}' := \oplus_{\alpha\in\Gamma_{\mathrm{af}}} \mathbb{R}\,\alpha$. As described in [Sai85, Section (3.3)] the subspace $\mathcal{L}'$ satisfies the conditions (i) and (ii) of Lemma 7.2.5. In particular this yields the following splitting of $L(\Phi)$ over $\mathbb{Z}$

$$
\begin{aligned}
L(\Phi) &= \big(L(\Phi) \cap \mathcal{L}'\big) \oplus (L(\Phi) \cap U) &(7.4)\\
&= (\oplus_{\alpha\in\Gamma_{\mathrm{af}}} \mathbb{Z}\,\alpha) \oplus \mathbb{Z}\,a. &(7.5)
\end{aligned}
$$

In the following we identify b with $-\delta$, i.e. $b = -(\widetilde{\alpha} + \alpha_0)$.
For $\alpha \in \Phi$ put

$$
\begin{aligned}
k(\alpha) &:= \inf\{k \in \mathbb{Z}_{>0} \mid \alpha + ka \in \Phi\},\\
\alpha^* &:= \alpha + k(\alpha)a.
\end{aligned}
$$

For $\alpha \in \Gamma_{\mathrm{af}}$ put

$$
\begin{aligned}
m_\alpha &:= \frac{q(\alpha)}{k(\alpha)} n_\alpha,\\
m_{\max} &:= \max\{m_\alpha \mid \alpha \in \Gamma_{\mathrm{af}}\},\\
\Gamma_{\max} &:= \{\alpha \in \Gamma_{\mathrm{af}} \mid m_\alpha = m_{\max}\},\\
\Gamma^*_{\max} &:= \{\alpha^* \mid \alpha \in \Gamma_{\max}\}
\end{aligned}
$$

and define $\mathrm{cod}(\Phi, U) = |\Gamma_{\max}|$, called the codimension of (Φ, U) (see [Sai85, (8.1) Definition]).

Definition 7.3.3. The set $\Gamma(\Phi, U) := \Gamma_{\mathrm{af}} \cup \Gamma^*_{\max}$ is called elliptic root basis for (Φ, U).

Theorem 7.3.4. *([Sai85, Theorem 9.6]) Let* $\Gamma := \Gamma(\Phi, U)$ *be the elliptic root basis of* (Φ, U). *Then:*

(a) $W_\Phi = W_\Gamma$

(b) $\Phi = W_\Gamma(\Gamma)$

Remark 7.3.5. The matrix $((\alpha^\vee \mid \beta))_{\alpha,\beta\in\Gamma(\Phi,U)}$, called elliptic Cartan matrix, is not a generalized Cartan matrix in the sense of [Kac83] since $(\alpha \mid \alpha^*) = 2$ implies that this matrix has positive off-diagonal entries.

Definition 7.3.6. Let (Φ, U) be an irreducible marked elliptic root system with elliptic root basis $\Gamma = \Gamma(\Phi, U)$. The elliptic Dynkin diagram for (Φ, U) is defined as the undirected graph on vertex set in correspondence with Γ. If for $\alpha, \beta \in \Gamma$ it is $(\alpha \mid \beta^\vee) = 0$, then there is no edge between α and β. Otherwise the type of the edge between them is defined as follows:

$$\begin{aligned}
&\alpha \,\text{---}\, \beta \text{ if } (\alpha \mid \beta^\vee) = (\beta \mid \alpha^\vee) = -1 \\
&\alpha \xleftarrow{t} \beta \text{ if } (\alpha \mid \beta^\vee) = -1,\ (\beta \mid \alpha^\vee) = -t, \text{ where } t \in \{2,3,4\} \\
&\alpha \overset{\infty}{\text{---}} \beta \text{ if } (\alpha \mid \beta^\vee) = (\beta \mid \alpha^\vee) = -2
\end{aligned}$$

and there is a dotted double-edge between α and β if $(\alpha \mid \beta^\vee) = (\beta \mid \alpha^\vee) = 2$.

In particular, the marked elliptic root system (Φ, U) is simply-laced if $(\alpha \mid \beta^\vee) = (\beta \mid \alpha^\vee) = -1$ or $(\alpha \mid \beta^\vee) = (\beta \mid \alpha^\vee) = 2$ for all $\alpha, \beta \in \Gamma$, that is, the elliptic diagram for (Φ, U) only consists of simply-laced edges and dotted double-edges.

In terms of the elliptic Dynkin diagrams, Saito obtains in [Sai85] a complete classification of irreducible marked elliptic root systems.

Theorem 7.3.7. *[Sai85, (9.6) Theorem] Let (Φ, U) be an irreducible marked elliptic root system. Then the elliptic Dynkin diagram for (Φ, U) is uniquely determined by the isomorphism class of (Φ, U). Conversely, the elliptic Dynkin diagram for (Φ, U) uniquely determines the isomorphism class of (Φ, U) together with an elliptic root basis which is identified with the vertices of the elliptic Dynkin diagram for (Φ, U).*

We give the following description of the lattice T_R introduced in Proposition 7.2.2.

Lemma 7.3.8. *Let (Φ, U) be an irreducible reduced simply-laced marked elliptic root system, $R = R_{(-|-)}$ the radical of the form and let $p_R : V \to V/R$ be the canonical map. Then*

$$T_R = R_{L(\Phi)} \otimes L(p_R(\Phi)).$$

Proof. Let $\alpha \in \Phi$. Since we assume Φ to be simply-laced, we have $\alpha^* = \alpha + a \in \Phi$ by Saito's classification of irreducible elliptic root systems. Since $s_\alpha s_{\alpha^*}$ is an element of W_Φ, we can calculate its image under E^{-1}. It is

$$\begin{aligned}
E^{-1}(s_\alpha s_{\alpha^*}) &= E^{-1}(s_\alpha) \circ E^{-1}(s_{\alpha^*}) \\
&= (\alpha \otimes \alpha) \circ (\alpha^* \otimes \alpha^*) \\
&= \alpha \otimes \alpha + \alpha^* \otimes \alpha - \alpha \otimes (\alpha^* \mid \alpha)\alpha \\
&= (\alpha + \alpha^*) \otimes \alpha + (-2)\alpha \otimes \alpha \\
&= (\alpha + \alpha^* - 2\alpha) \otimes \alpha \\
&= a \otimes \alpha.
\end{aligned}$$

Similar arguments show that $b \otimes \alpha$ is in the image of E^{-1} for each $\alpha \in \Phi$. In particular, if we identify a set of roots $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ with a set of simple roots for $p_R(\Phi)$, we see that

$$\{x \otimes \alpha_i \mid x \in \{a, b\},\ 1 \leq i \leq n\} \subseteq E^{-1}(W_\Phi).$$

Since $\{a, b\}$ is a basis of $R_{L(\Phi)}$ and $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of $L(p_R(\Phi))$, we obtain

$$R_{L(\Phi)} \otimes L(p_R(\Phi)) \subseteq E^{-1}(W_\Phi).$$

By Proposition 7.2.2 (in particular parts (f) and (g)) the assertion follows. □

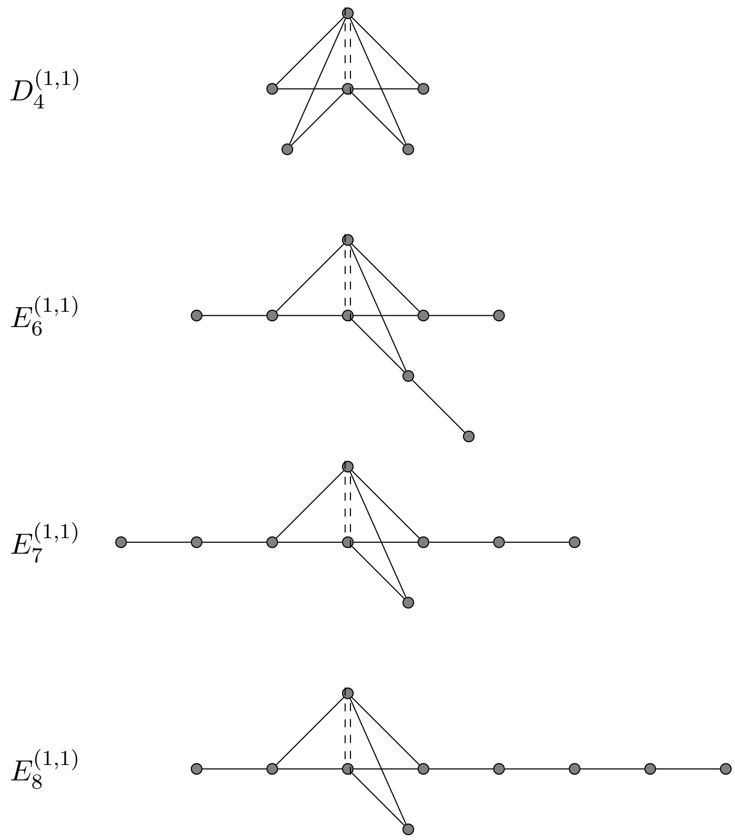

Figure 7.1.: Elliptic Dynkin diagrams for the tubular elliptic root systems

Combining Theorem 7.3.1 and Lemma 7.3.8 and with the same assumptions as in Lemma 7.3.8, we obtain the following description of W_Φ

Corollary 7.3.9. $W_\Phi \cong W_{p_R(\Phi)} \ltimes \big(R_{L(\Phi)} \otimes L(p_R(\Phi))\big)$.

In Section 7.5 we will shortly introduce the notion of a weighted projective line. Inspired by the notion of a weighted projective line of tubular type (e.g. see [Mel04, Definition 3.1.11]), we make the following definition.

Definition 7.3.10. An irreducible reduced simply-laced marked elliptic root system (Φ, U) is called tubular if $\mathrm{cod}(\Phi, U) = 1$.

In Saito's classification of marked elliptic root systems, the simply-laced ones are of types $A_n^{(1,1)}$ $(n \geq 2)$, $D_n^{(1,1)}$ $(n \geq 4)$ and $E_n^{(1,1)}$ $(n \in \{6, 7, 8\})$. In particular, our assumption that $p_U(\Phi)$ is reduced of type $X_n^{(1)}$ is true for these cases. The elliptic Dynkin diagrams for the tubular marked elliptic roots systems can be found in Figure 7.1.

Remark 7.3.11. Let (Φ, U) be an irreducible reduced simply-laced marked elliptic root system.

(a) From Saito's classification, one obtains the following decomposition of the root system:

$$\Phi = p_U(\Phi) + \mathbb{Z}\, a = p_R(\Phi) + \mathbb{Z}\, b + \mathbb{Z}\, a.$$

(b) If we remove in the elliptic Dynkin diagram for (Φ, U) the vertices corresponding to $\Gamma_{\max} \cup \Gamma^*_{\max}$ we obtain a disjoint union of diagrams of type A_{l_i}. An irreducible marked elliptic root system (Φ, U) is tubular if and only if

$$\chi(\Phi, U) := 2 + \sum_i \left(-1 + \frac{1}{l_i + 1}\right) = 0.$$

7.4. The Coxeter transformation

Definition 7.4.1. Let (Φ, U) be an irreducible reduced simply-laced marked elliptic root system. The Coxeter transformation c for (Φ, U) with respect to the elliptic root basis $\Gamma = \Gamma(\Phi, U)$ is defined as

$$c = \prod_{\alpha \in \Gamma \setminus (\Gamma_{\max} \cup \Gamma^*_{\max})} s_\alpha \cdot \prod_{\alpha \in \Gamma_{\max}} s_\alpha s_{\alpha^*}.$$

We already noted that the elliptic Dynkin diagram for (Φ, U) decomposes as a union of diagrams of type A_{l_i} $(1 \leq i \leq k)$ if we remove the vertices corresponding to $\Gamma_{\max} \cup \Gamma^*_{\max}$. Put

$$l_{\max} := \max\{l_i \mid 1 \leq i \leq k\}.$$

For the root lattice $L = L(\Phi)$ put

$$O(L) := \{\varphi \in O(V) \mid \varphi(L) = L\}.$$

Proposition 7.4.2. *Let (Φ, U) be an irreducible reduced simply-laced marked elliptic root system.*

(a) The conjugacy class of a Coxeter transformation c in W_Φ depends only on the subspace $\oplus_{\alpha \in \Gamma_{af}} \mathbb{R}\, \alpha$ of V and the sign of the generator a of $L(\Phi) \cap U$, but neither on the order of the product for the expression of c nor on the choice of the simple system Δ_{af}.

(b) The sign change $a \mapsto -a$ brings the conjugacy class of c to the conjugacy class of c^{-1}.

(c) The set of all Coxeter transformations is precisely one $O(L)$-conjugacy class.

(d) A Coxeter transformation is semi-simple of finite order $l_{\max} + 1$.

Proof. Except for part (c) this is precisely [Sai85, Section 9, Lemma A]. Regarding part (c), Saito shows that if φ is an (outer) automorphism of (Φ, U) and c is a Coxeter transformation with respect to the basis $\Gamma_{\mathrm{af}} = \{\alpha_0, \ldots, \alpha_n\}$, then $\varphi c \varphi^{-1}$ is the Coxeter transformation with respect to the basis $\varphi(\Gamma_{\mathrm{af}}) = \{\varphi(\alpha_0), \ldots, \varphi(\alpha_n)\}$. By [Sai85, Corollary (6.2)] and its proof it follows $\varphi \in O(V)$. Furthermore, since $\varphi(\Phi) = \Phi$, it follows $\varphi(L) = L$. Hence indeed $\varphi \in O(L)$. □

Let $c \in W_\Phi$ be a Coxeter transformation and $R = R_{(-|-)}$ the radical of the form. We have $c_{|R} = \mathrm{id}$, c is of finite order $o(c)$ and $\langle c \rangle$ acts on $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$c^j \left(\sum_i \alpha_i \otimes \lambda_i\right) := \sum_i c^j(\alpha_i) \otimes \lambda_i.$$

The next statement is a direct consequence of a Theorem of Maschke. For later use we state a proof here.

Proposition 7.4.3. *There exists a c-invariant subspace V_c of V such that*

$$V = R \oplus V_c.$$

Proof. Let V' be an arbitrary subspace of V complementary to R, i.e $V = R \oplus V'$. Define

$$\rho : V = R \oplus V' \to R,\ v = w + w' \mapsto w$$

and

$$\varrho : V \to R,\ v \mapsto \frac{1}{o(c)} \sum_{m=1}^{o(c)} (c^m \rho c^{-m}(v)).$$

Let $v \in V$. Then $c^{-m}(v) \in V$ and $\rho c^{-m}(v) \in R$. Since R is c-invariant we have $c^m \rho c^{-m}(v) \in R$ and hence ϱ is well-defined. Next we show $\varrho_{|R} = \mathrm{id}$. Therefore let $w \in R$.

$$\begin{aligned}
\varrho(w) &= \frac{1}{o(c)} \sum_{m=1}^{o(c)} (c^m \rho \underbrace{c^{-m}(w)}_{\in R}) \\
&= \frac{1}{o(c)} \sum_{m=1}^{o(c)} c^m (c^{-m}(w)) \\
&= \frac{1}{o(c)} \sum_{m=1}^{o(c)} w = w.
\end{aligned}$$

Hence we showed that ϱ projects V onto R. Next we claim that ϱ lies in the centralizer $C_V(c)$ of c. To see this choose an arbitrary $v \in V$.

$$\begin{aligned}
c^j \varrho(v) &= c^j \frac{1}{o(c)} \sum_{m=1}^{o(c)} (c^m \rho c^{-m}(v)) \\
&= \frac{1}{o(c)} \sum_{m=1}^{o(c)} (c^j c^m \rho c^{-m}(v)) \\
&= \frac{1}{o(c)} \sum_{m=1}^{o(c)} c^j c^m \rho((c^j c^m)^{-1} c^j(v)) \\
&= \frac{1}{o(c)} \sum_{m=1}^{o(c)} c^{j+m} \rho c^{-(j+m)}(c^j(v)) = \varrho(c^j(v)).
\end{aligned}$$

Finally note that $\ker(\varrho)$ is c-invariant. For this let $v \in \ker(\varrho)$. Then we have

$$0 = c^j(\varrho(v)) = \varrho(c^j(v))$$

and hence $c^j(v) \in \ker(\varrho)$. By combining these results we get

$$V = \mathrm{im}(\varrho) \oplus \ker(\varrho) = R \oplus \ker(\varrho),$$

which proves the claim by setting $V_c := \ker(\varrho)$. □

Consider the map $p_R : V \to V/R$ and put $L' := p_R(L)$. Thus p_R induces by restriction a map

$$p_R : L \to L'.$$

We have seen in (7.4) that L splits as $L = L' \oplus R_{L(\Phi)}$. Put $O(L') := \{\varphi \in O(V/R) \mid \varphi(L') = L'\}$ and define

$$\text{can} : O(L) \to O(L'), \varphi \mapsto \overline{\varphi}, \tag{7.6}$$

where $\overline{\varphi}(x) = p_R(\varphi(x))$ for all $x \in L'$.

Remark 7.4.4.

(a) Let (Φ, U) be a tubular elliptic root system and $c \in W_\Phi$ be a Coxeter transformation. Since $s_\alpha s_{\alpha^*} \in \ker(\text{can})$ and $\text{can}(b) = -\widetilde{\alpha}$, where b is chosen as in Section 7.3.2, we obtain a description of $\overline{c}$ as given in Table 7.1.

Type of (Φ, U)	Type of $\overline{c}$
$D_4^{(1,1)}$	$2D_2$
$E_6^{(1,1)}$	$3A_2$
$E_7^{(1,1)}$	$2A_3 + A_1$
$E_8^{(1,1)}$	$A_5 + A_2 + A_1$

Table 7.1.: Types of projections

E.g. $p_R\left(D_4^{(1,1)}\right)$ is a root system of type D_4 and $\overline{c}$ to be of type $2D_2$ means that $\overline{c}$ is a Coxeter element in a reflection subgroup of W_{D_4} of type $2D_2$. The notation D_2 is adopted from [Car72].

(b) It is $L(\Phi) = L(p_R(\Phi)) \oplus R_{L(\Phi)}$ and $V = L(\Phi) \otimes_{\mathbb{Z}} \mathbb{R}$. By [Car72, Lemma 2] the element $\overline{c}$ acts without fixed points on V/R. Hence $R = C_V(c)$ and $V_c = \text{span}_{\mathbb{R}}(\{v - c(v) \mid v \in V\})$.

In the remaining part of this chapter we will investigate the Coxeter transformations for the Weyl group of a tubular elliptic root system. This is done by a case-by-case analysis. The cases $E_n^{(1,1)}$ ($n \in \{6, 7, 8\}$) are already treated in [Klu87]. We adopt most of the proofs given there to treat the remaining case $D_4^{(1,1)}$. That is, we just give proofs for the case $D_4^{(1,1)}$, but all statements are true for the tubular case. Nevertheless the ideas and strategies we present here for the case $D_4^{(1,1)}$ might be a good reference how to handle the general case, including the types $E_n^{(1,1)}$ ($n \in \{6, 7, 8\}$).

For this purpose we describe an elliptic root basis for (Φ, U) of type $D_4^{(1,1)}$ following Section 7.3: Let

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4,\ \alpha_4 = e_3 + e_4$$

be the standard simple system for the root system of type D_4 and put

$$\alpha_0 = \widetilde{\alpha} + b,\ \alpha_2^* = \alpha_2 + a,$$

where a is a generator induced by the marking. Then $\{\alpha_0, \dots, \alpha_4, \alpha_2^*\}$ is an elliptic root basis for the elliptic root system of type $D_4^{(1,1)}$ in the vector space $V = \text{span}_{\mathbb{R}}(\{e_1, \dots, e_4, a, b\})$.

Lemma 7.4.5. *Let (Φ, U) be a tubular elliptic root system and $c \in W_\Phi$ a Coxeter transformation. Then $\varphi \in C_{O(L)}(c) = \{\phi \in O(L) \mid \phi \circ c = c \circ \phi\}$ if and only if*

(i) $\varphi \in O(L)$

(ii) $\varphi(V_c) = V_c$

(iii) $\overline{\varphi} \circ \overline{c} = \overline{c} \circ \overline{\varphi}$

Proof. Let $\varphi \in C_{O(L)}(c)$. Then $\varphi(v - c(v)) = \varphi(v) - c(\varphi(v))$, thus $\varphi(v - c(v)) \in V_c$ by part (b) of Remark 7.4.4. The other direction is also clear by Remark 7.4.4. □

Next we are going to explicitly calculate a basis for the subspace V_c. Analogous to the proof of Proposition 7.4.3 we consider the projection $\pi : V \to V_c$ and the induced map

$$\varpi : V \to V_c,\ v \mapsto \frac{1}{o(c)} \sum_{m=1}^{o(c)} (c^m \pi c^{-m}(v)),$$

where $c = s_{\alpha_0} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_2^*}$ is the Coxeter transformation of type $D_4^{(1,1)}$. We have to compute the image of the elliptic root basis for $D_4^{(1,1)}$ under ϖ. Note that $o(c) = 2$ and hence $c = c^{-1}$. As examples we calculate $\varpi(\alpha_1)$ and $\varpi(\alpha_2)$:

- It is $c(\alpha_1) = -\alpha_1 + a$ and therefore $\varpi(\alpha_1) = \alpha_1 - \frac{1}{2}a$.
- It is $c(\alpha_2) = -\alpha_2 - 2a + b$ and therefore $\varpi(\alpha_2) = \alpha_2 + a - \frac{1}{2}b$.

As a basis we obtain

$$\alpha_1 - \frac{1}{2}a, \alpha_2 + a - \frac{1}{2}b, \alpha_3 - \frac{1}{2}a, \alpha_4 - \frac{1}{2}a. \tag{7.7}$$

Definition 7.4.6. Let Φ be an elliptic root system. Then the set $T_\Phi = \{s_\alpha \mid \alpha \in \Phi\}$ is called the set of reflections for Φ.

Since the elliptic root basis is contained in Φ, the group W_Φ is generated by T. We denote the corresponding length function again by ℓ_T.

Proposition 7.4.7. *Let (Φ, U) be a tubular elliptic root system of rank n and $c \in W_\Phi$ a Coxeter transformation. Then $\ell_T(c) = n + 2$.*

Proof. For the cases $E_n^{(1,1)}$ ($n \in \{6, 7, 8\}$) see [Klu87, Korollar 2.3]. It remains to consider the case $D_4^{(1,1)}$. By Remark 7.4.4 the Coxeter transformation c is projected to an element $\overline{c}$ in $W_{p_R(\Phi)}$ of absolute length 4. Therefore it has to be $\ell_T(c) = 6$ or $\ell_T(c) = 4$. Assume the latter one. Then the corresponding 4 roots from Φ will span a c-invariant sublattice of $L(\Phi)$, which is complementary to $R_{L(\Phi)}$. This yields a contradiction since $V_c \cap \Phi = \varnothing$ by (7.7). □

Remark 7.4.8. The last observation in the preceding proof is true in a more general setting. Namely, if (Φ, U) is an elliptic root system, then one of the main results in [Sai85] is that $\mathrm{im}(c - \mathrm{id}) \cap \Phi = \varnothing$ for a Coxeter transformation $c \in W_\Phi$.

Lemma 7.4.9. *Let $\phi \in O(V)$ with $\overline{\phi}_{|L(p_R(\Phi))} = \mathrm{id}_{L(p_R(\Phi))}$ and $\phi_{|V_c} = \mathrm{id}_{V_c}$. Then $\phi \in O(L)$ if and only if*

(i) $\phi(R_{L(\Phi)}) = R_{L(\Phi)}$

(ii) $\phi_{|R}$ *acts trivial on* $(\frac{1}{2}R_{L(\Phi)})/R_{L(\Phi)}$.

Proof. Let $\phi \in O(L)$. Then (i) is clear. Consider the vector $\alpha_1 - \frac{1}{2}a$ of the basis (7.7) of V_c.

$$\alpha_1 - \frac{1}{2}a = \phi\left(\alpha_1 - \frac{1}{2}a\right) = \phi(\alpha_1) - \phi\left(\frac{1}{2}a\right).$$

Since $\overline{\phi}_{|L(p_R(\Phi))} = \mathrm{id}_{L(p_R(\Phi))}$, it is $\phi(\alpha_1) = \alpha_1 + x$ for some $x \in R_{L(\Phi)}$. Hence $\phi(\frac{1}{2}a) = \frac{1}{2}a + x$ and therefore $\phi(\frac{1}{2}a + R_{L(\Phi)}) = \frac{1}{2}a + R_{L(\Phi)}$. This can be shown similarly for the remaining vectors of the basis of V_c given in (7.7). Thus ϕ acts trivial on $(\frac{1}{2}a + R_{L(\Phi)})/R_{L(\Phi)}$ and $(\frac{1}{2}b + R_{L(\Phi)})/R_{L(\Phi)}$ and therefore trivial on $(\frac{1}{2}R_{L(\Phi)})/R_{L(\Phi)}$, yielding (ii).

Now assume that (i) and (ii) hold for ϕ. We have to show that $\phi(L) = L$. Let us first show that $\phi(L) \subseteq L$. By (i) it remains to show that $\phi(L_{p_R(\Phi)}) \subseteq L$. This can be done by showing that $\phi(\alpha_i) \in L$ for $1 \leq i \leq 4$. We show this for $i = 1$. The remaining cases are similar. By the assumption we obtain

$$\phi(\alpha_1) = \alpha_1 + \phi\left(\frac{1}{2}a\right) - \frac{1}{2}a,$$

where $\alpha_1 \in L(p_R(\Phi))$ and $\phi(\frac{1}{2}a) - \frac{1}{2}a \in R_{L(\Phi)}$ by (i). Thus $\phi(\alpha_1) \in L$.

For the remaining inclusion let $\alpha = \alpha' + x \in L$ with $\alpha' \in L(p_R(\Phi))$ and $x \in R_{L(\Phi)}$. Since $\overline{\phi}_{|L(p_R(\Phi))} = \mathrm{id}_{L(p_R(\Phi))}$, it is $\phi(\alpha') = \alpha' + x'$ for some $x' \in R_{L(\Phi)}$. Furthermore we have $\phi(x) = x''$ for some $x'' \in R$ by (i). Thus

$$\phi(\alpha) = \alpha' + (x' + x'') \in L(p_R(\Phi)) \oplus R_{L(\Phi)} = L.$$

□

Definition 7.4.10. For $k \in \mathbb{Z}$ the subgroup

$$\Gamma(k) := \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a_1 \equiv a_4 \equiv 1, a_2 \equiv a_3 \equiv 0 \bmod k \right\}$$

is called the k-th congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 7.4.11. *Let* (Φ, U) *be a tubular elliptic root system,* $c \in W_\Phi$ *a Coxeter transformation and* $\varphi \in C_{O(L)}(c)$ *such that* $\overline{\varphi}_{|L(p_R(\Phi))} = \mathrm{id}_{L(p_R(\Phi))}$. *Then*

(a) $\varphi_{|R_{L(\Phi)}} \in \Gamma(k)$, *where* $k = o(c)$.

(b) *If* ψ *is a linear transformation of* R *with* $\psi(R_{L(\Phi)}) = R_{L(\Phi)}$ *and* $\psi \in \Gamma(k)$, *where* $k = o(c)$, *then there exists a uniquely determined* $\phi \in C_{O(L)}(c)$ *with* $\phi_{|R_{L(\Phi)}} = \psi_{|R_{L(\Phi)}}$ *and* $\overline{\phi}_{|L(p_R(\Phi))} = \mathrm{id}_{L(p_R(\Phi))}$.

Proof. For (a) let $\varphi \in O(V)$ with $\overline{\varphi}_{|L(p_R(\Phi))} = \mathrm{id}_{L(p_R(\Phi))}$ and $\varphi \circ c = c \circ \varphi$. Thus $\varphi(v - c(v)) = \varphi(v) - c(\varphi(v)) \in V_c$ and therefore $\varphi(V_c) = V_c$. By Lemma 7.4.5 we have $\varphi \in C_{O(L)}(c)$ if and only if $\varphi \in O(L)$. By Lemma 7.4.9 the latter condition is equivalent to $\varphi(R_{L(\Phi)}) = R_{L(\Phi)}$ and $\varphi_{|R}$ acts trivial on $(\frac{1}{2}R_{L(\Phi)})/R_{L(\Phi)}$. These conditions are equivalent to $\varphi_{|R_{L(\Phi)}} \in \Gamma(2)$.

For (b) define ϕ by putting $\phi_{|V_c} = \mathrm{id}_{V_c}$ and $\phi_{|R_{L(\Phi)}} = \psi_{|R_{L(\Phi)}}$. Thus $\phi \in O(V)$. The assertion follows by Lemma 7.4.5 and Lemma 7.4.9. □

7.4.1. The element of type $2D_2$ in D_4

Let (Φ, U) be a tubular elliptic root system of type $D_4^{(1,1)}$ and let

$$c = s_{\alpha_0} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_2^*}$$

be the Coxeter transformation in W_Φ as defined in section 7.4. We already noted in Remark 7.4.4 that

$$\overline{c} = s_{\widetilde{\alpha}} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} \in W_{p_R(\Phi)}$$

is of type $2D_2$, that is, $\overline{c}$ is a Coxeter element in the reflection subgroup $\langle s_{\widetilde{\alpha}}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4} \rangle$. Note that the root subsystem corresponding to this reflection subgroup indeed has $\{\widetilde{\alpha}, \alpha_1, \alpha_3, \alpha_4\}$ as a simple system. We use the realization $\{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}$ of the root system $p_R(\Phi)$ of type D_4. Put $\Phi_{ij} := \{\pm e_i \pm e_j\}$ with $i, j \in \{1, 2, 3, 4\}$, $i \neq j$ and define

$$\begin{aligned}
\Phi_1 &:= \Phi_{12} \cup \Phi_{34} \\
\Phi_2 &:= \Phi_{13} \cup \Phi_{24} \\
\Phi_3 &:= \Phi_{14} \cup \Phi_{23}.
\end{aligned}$$

We see that each of these root subsystems is of type $2D_2$. Furthermore we have

$$p_R(\Phi) = \Phi_1 \cup \Phi_2 \cup \Phi_3,$$

that is, $p_R(\Phi)$ is the disjoint union of the $2D_2$ root subsystems it contains and $\overline{c} = \prod_{\alpha \in \Phi_1} s_\alpha$.

Proposition 7.4.12. *For each $\alpha \in p_R(\Phi)$ it is $\overline{c}(\alpha) = -\alpha$.*

Proof. We only have to check the assertion for the simple system $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of $p_R(\Phi)$. It is

$$\begin{aligned}
&\overline{c}(\alpha_1) = s_{\widetilde{\alpha}}(-\alpha_1) = -\alpha_1 \\
&\overline{c}(\alpha_2) = s_{\widetilde{\alpha}} s_{\alpha_1} s_{\alpha_3}(\alpha_2 + \alpha_4) = s_{\widetilde{\alpha}} s_{\alpha_1}(\alpha_2 + \alpha_3 + \alpha_4) = s_{\widetilde{\alpha}}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = -\alpha_2.
\end{aligned}$$

The remaining two cases are also done by direct calculations. □

By [Bou02, Ch. VI, 1, 6, Corollary 3] we obtain the following.

Corollary 7.4.13. *The element $\overline{c}$ is the unique longest element (with respect to the simple system $\{s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}\}$) in $W_{p_R(\Phi)}$. In particular $C_{W_{p_R(\Phi)}}(\overline{c}) = W_{p_R(\Phi)}$, where $C_{W_{p_R(\Phi)}}(\overline{c})$ is the centralizer of $\overline{c}$ in $W_{p_R(\Phi)}$.*

Direct calculations show that $W_{p_R(\Phi)}$ acts transitively on the set $\{\Phi_i \mid i \in \{1, 2, 3\}\}$. Therefore Corollary 7.4.13 implies

$$\overline{c} = \prod_{\alpha \in \Phi_i} s_\alpha \text{ for each } i \in \{1, 2, 3\}.$$

7.4.2. Root lattices of type $2D_2$ in D_4

We keep the notation from Section 7.4.1, put $L_i := \operatorname{span}_{\mathbb{Z}}(\Phi_i)$ $(1 \leq i \leq 3)$ and $W := W_{p_R(\Phi)}$, that is, W is a Coxeter group of type D_4.

Example 7.4.14. We have

$$\operatorname{span}_{\mathbb{Z}}(\Phi_{12}) = \{k_1e_1 + k_2e_2 \mid k_1, k_2 \in \mathbb{Z},\ k_1 + k_2 \in 2\,\mathbb{Z}\}$$

and therefore

$$L_1 = \operatorname{span}_{\mathbb{Z}}(\Phi_1) = \left\{ \sum_{i=1}^{4} k_ie_i \mid k_i \in \mathbb{Z},\ k_1 + k_2, k_3 + k_4 \in 2\,\mathbb{Z} \right\}.$$

The analogous statements are valid for L_2 and L_3.

Proposition 7.4.15.

(a) $L_1 \cap L_2 = \{\sum_{i=1}^{4} k_ie_i \mid k_i \in \mathbb{Z},$ *either all* k_i *are even or all* k_i *are odd*$\}$.

(b) $L_1 \cap L_2 = L_1 \cap L_3 = L_2 \cap L_3 = L_1 \cap L_2 \cap L_3$.

Proof. Let $x = \sum_{i=1}^{4} k_ie_i \in L_1 \cap L_2$. By Example 7.4.14 we have

$$k_1 + k_2 \in 2\,\mathbb{Z} \tag{7.8}$$
$$k_3 + k_4 \in 2\,\mathbb{Z} \tag{7.9}$$
$$k_1 + k_3 \in 2\,\mathbb{Z} \tag{7.10}$$
$$k_2 + k_4 \in 2\,\mathbb{Z}\,. \tag{7.11}$$

If $k_1 \in 2\,\mathbb{Z}$, then by (7.8) we have $k_2 \in 2\,\mathbb{Z}$, hence by (7.11) we have $k_4 \in 2\,\mathbb{Z}$ and therefore by (7.9) we have $k_3 \in 2\,\mathbb{Z}$. If we assume that $k_1 \notin 2\,\mathbb{Z}$, we obtain analogously $k_i \notin 2\,\mathbb{Z}$ $(2 \leq i \leq 4)$. This shows that $L_1 \cap L_2$ is contained in the right hand site. Now let $x = \sum_{i=1}^{4} k_ie_i$ with $k_i \in \mathbb{Z}$ and either all k_i are even or all k_i are odd. Therefore $k_i + k_j \in 2\,\mathbb{Z}$ for all $i, j \in \{1, 2, 3, 4\}$, thus $x \in L_1 \cap L_2$, which proves (a). But since in particular we also have $k_1 + k_4, k_2 + k_3 \in 2\,\mathbb{Z}$, we obtain $x \in L_3$. Hence $L_1 \cap L_2 \subseteq L_1 \cap L_2 \cap L_3$, which shows (b). □

The proofs of the next two statements are given by straightforward computations.

Lemma 7.4.16. *Let* $i, j, k \in \mathbb{Z}$ *such that* $\{i, j, k\} = \{1, 2, 3\}$ *and let* $\alpha \in \Phi_i, \beta \in \Phi_j$. *Then* $s_\beta(\alpha) \in \Phi_k$. *In particular* W *acts transitively on the set* $\{\Phi_i \mid i \in \{1, 2, 3\}\}$.

Lemma 7.4.17. *It is* $2e_i \in L_j$ *for all* $i \in \{1, 2, 3, 4\}$ *and all* $j \in \{1, 2, 3\}$.

Lemma 7.4.18. *Let* $x \in L(p_R(\Phi))$. *Then* $x \in L_i$ $(1 \leq i \leq 3)$, *but* $x \notin L_1 \cap L_2 \cap L_3$ *if and only if* $x = \alpha + x'$ *with* $\alpha \in \Phi_i$ *and* $x' \in L_1 \cap L_2 \cap L_3$.

Proof. Let $x = \sum_{i=1}^{4} k_ie_i \in L(p_R(\Phi))$. By Lemma 7.4.16 we can assume without loss of generality that $x \in L_1$. Then $x \in L_1$, but $x \notin L_1 \cap L_2 \cap L_3$ if and only if $k_1 + k_2 \in 2\,\mathbb{Z}$, $k_3 + k_4 \in 2\,\mathbb{Z}$, that is k_1 and k_2 have the same parity as well as k_3 and k_4, but k_1 and k_3 have different parity. This is equivalent to $\alpha := e_1 + e_2 \in \Phi_1$ and $x' := (k_1 - 1)e_1 + (k_2 - 1)e_2 + k_3e_3 + k_4e_4 \in L_1 \cap L_2 \cap L_3$. □

Remark 7.4.19. In the preceding proof, either k_1 and k_2 or k_3 and k_4 are even. Therefore we always find a decomposition $\alpha = \alpha' + x'$ and $x' = \sum_{i=1}^{4} k_i' e_i \in L_1 \cap L_2 \cap L_3$ with all k_i' even. In particular $x' \in 2L(p_R(\Phi))$.

Lemma 7.4.20. *If $x \in L_1 \cap L_2 \cap L_3$, then also $w(x) \in L_1 \cap L_2 \cap L_3$ for all $w \in W$.*

Proof. Let $x = \sum_{i=1}^{4} k_i e_i$ ($k_i \in \mathbb{Z}$). It is enough to show the assertion for $w = s_\alpha$ ($\alpha \in \Phi$). But s_α just permutes the set $\{e_1, e_2, e_3, e_4\}$ up to sign. Hence $s_\alpha(x) = \sum_{i=1}^{4} k_i s_\alpha(e_i) \in L_1 \cap L_2 \cap L_3$. □

Proposition 7.4.21. *Let $i, j \in \{1, 2, 3\}$ with $i \neq j$ and let $x \in L_i, y \in L_j$, but $x, y \notin L_1 \cap L_2 \cap L_3$. Then for all $w \in W$ there exist $i', j' \in \{1, 2, 3\}$ with $i' \neq j'$ such that $w(x) \in L_{i'}, w(y) \in L_{j'}$, but $w(x), w(y) \notin L_1 \cap L_2 \cap L_3$.*

Proof. By Lemma 7.4.18 we have $x = \alpha + x',\ y = \beta + y'$ with $\alpha \in \Phi_i,\ \beta \in \Phi_j$ and $x', y' \in L_1 \cap L_2 \cap L_3$. Again it is enough to show the assertion just for $w = s_\gamma$ ($\gamma \in \Phi$). We have $\gamma \in \Phi_k$ for some $k \in \{1, 2, 3\}$. Thus $s_\gamma(x) = s_\gamma(\alpha) + s_\gamma(x')$ and $s_\gamma(y) = s_\gamma(\beta) + s_\gamma(y')$. By Lemma 7.4.20 we have $s_\gamma(x'), s_\gamma(y') \in L_1 \cap L_2 \cap L_3$. By Lemma 7.4.16 we have $s_\gamma(\alpha) \in \Phi_{i'}, s_\gamma(\beta) \in \Phi_{j'}$ for some $i', j' \in \{1, 2, 3\}$ with $i' \neq j'$. Using again Lemma 7.4.18, the assertion follows. □

Proposition 7.4.22. *Let $w \in W$ and $x_1, x_2, y_1, y_2 \in L(p_R(\Phi))$ such that $x_1 \in L_i,\ y_1 \in L_j$ with $i, j \in \{1, 2, 3\}$, $i \neq j$, but $x_1, y_1 \notin L_1 \cap L_2 \cap L_3$. Then there exist $i', j' \in \{1, 2, 3\}$ with $i' \neq j'$ such that $w(x_1) + 2x_2 \in L_{i'}$, $w(x_2) + 2y_2 \in L_{j'}$, but $w(x_1) + 2x_2, w(x_2) + 2y_2 \notin L_1 \cap L_2 \cap L_3$.*

Proof. By Proposition 7.4.21 there exist $i', j' \in \{1, 2, 3\}$ with $i' \neq j'$ such that $w(x_1) \in L_{i'}, w(y_1) \in L_{j'}$, but $w(x_1), w(y_1) \notin L_1 \cap L_2 \cap L_3$. By Lemma 7.4.17 we have $2x_2, 2y_2 \in L_1 \cap L_2 \cap L_3$. Hence $w(x_1) + 2x_2 \in L_{i'}$. If we assume that $w(x_1) + 2x_2 \in L_k$ with $k \in \{1, 2, 3\}$, $k \neq i'$, then we obtain $w(x_1) \in L_k$, contradicting $w(x_1) \notin L_1 \cap L_2 \cap L_3$. □

Let $\eta : L(p_R(\Phi)) \to L(p_R(\Phi))/2L(p_R(\Phi))$ be the natural map. Direct calculations yield the following two Lemmata.

Lemma 7.4.23. *For all $\alpha \in L(p_R(\Phi))$ it is $\eta(\alpha) = \eta(-\alpha)$.*

Lemma 7.4.24. *Let $(i, j), (k, l) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. Then for $\alpha \in \Phi_{ij}$ and $\beta \in \Phi_{kl}$, there exists $w \in W_{\Phi_1}$ such that $\eta(w(\alpha)) = \eta(\beta)$, that is, up to sign the roots $w(\alpha)$ and β are equal.*

Proof. We show the assertion for the case $(i, j) = (1, 3)$ and $(k, l) = (2, 3)$. The remaining cases are similar. Up to sign, the only roots in Φ_{13} are $e_1 - e_3$ and $e_1 + e_3$, while up to sign the only roots in Φ_{23} are $e_2 - e_3$ and $e_2 + e_3$. We have $s_{e_1 - e_2}, s_{e_1 + e_2} \in W_{\Phi_1}$ and

$$\begin{aligned}
s_{e_1 - e_2}(e_1 - e_3) &= e_2 - e_3 \\
s_{e_1 + e_2}(e_1 - e_3) &= -(e_2 + e_3) \\
s_{e_1 - e_2}(e_1 + e_3) &= e_2 + e_3 \\
s_{e_1 + e_2}(e_1 + e_3) &= -(e_2 - e_3).
\end{aligned}$$

By Lemma 7.4.23 the assertion follows. □

7.4.3. Characterization of Coxeter transformations

Let (Φ, U) be a tubular elliptic root system. We have seen in Corollary 7.3.9 that

$$W_\Phi \cong W_{p_R(\Phi)} \ltimes \left(R_{L(\Phi)} \otimes L(p_R(\Phi))\right). \tag{7.12}$$

We fix the basis a, b of $R_{L(\Phi)}$ as introduced above. By (7.12) each element of W_Φ can be written uniquely as

$$w(a \otimes x + b \otimes y) =: \begin{bmatrix} w \\ x \\ y \end{bmatrix},$$

where $w \in W_{p_R(\Phi)}$ and $x, y \in L(p_R(\Phi))$. Since

$$w_1(a \otimes x_1 + b \otimes y_1) \cdot w_2(a \otimes x_2 + b \otimes y_2) = w_1 w_2(a \otimes (w_2^{-1}(x_1) + x_2) + b \otimes (w_2^{-1}(y_1) + y_2)),$$

the group operation is given by

$$\begin{bmatrix} w_1 \\ x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} w_2 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 \\ w_2^{-1}(x_1) + x_2 \\ w_2^{-1}(y_1) + y_2 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} w \\ x \\ y \end{bmatrix}^{-1} = \begin{bmatrix} w^{-1} \\ -w(x) \\ -w(y) \end{bmatrix}$$

and

$$\begin{bmatrix} w_2 \\ x_2 \\ y_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} w_1 \\ x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} w_2 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} w_2^{-1} w_1 w_2 \\ w_2^{-1}(x_1) + (\mathrm{id} - w_2^{-1} w_1^{-1} w_2)(x_2) \\ w_2^{-1}(y_1) + (\mathrm{id} - w_2^{-1} w_1^{-1} w_2)(y_2) \end{bmatrix}.$$

It is $E^{-1}(W_\Phi) \subseteq V \otimes V/R$ and by Lemma 7.3.8 we have

$$R_{L(\Phi)} \otimes L(p_R(\Phi)) = E^{-1}(W_\Phi) \cap (R \otimes V/R).$$

By Proposition 7.4.3 we have $V = R \oplus V_c$, hence we can write each $v \in V$ as $v = r + v_c$ with $r \in R$ and $v_c \in V_c$. Consider the induced map $\xi : V \otimes V/R \to R \otimes V/R, v \otimes u \mapsto r \otimes u$. We already saw in the proof of Lemma 7.3.8 that $E^{-1}(s_\alpha s_{\alpha^*}) = a \otimes \alpha$. Therefore we obtain

$$\xi(E^{-1}(c)) = b \otimes \widetilde{\alpha} + a \otimes \alpha_2,$$

where $c = s_{\alpha_0} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_2^*}$ is chosen as above. Hence we have the following description of c in the notation introduced above:

$$c = \begin{bmatrix} \bar{c} \\ \alpha_2 \\ \widetilde{\alpha} \end{bmatrix}.$$

Theorem 7.4.25. *Let (Φ, U) be a tubular elliptic root system of type $D_4^{(1,1)}$. Then $c = \begin{bmatrix} w \\ x \\ y \end{bmatrix} \in W_\Phi$ is a Coxeter transformation if and only if*

(i) *$w \in W_{p_R(\Phi)}$ is of type $2D_2$. In this case w distinguishes the three $2D_2$ root subsystems Φ_1, Φ_2, Φ_3 of $p_R(\Phi)$.*

(ii) *The elements $x, y \in L(p_R(\Phi))$ do not lie in a common sublattice $L(\Phi_i)$ $(i \in \{1,2,3\})$.*

Since the Coxeter transformation $c = s_{\alpha_0}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_2^*}$ fulfills the properties (i) and (ii) of Theorem 7.4.25 and since the set of all Coxeter transformations for W_Φ is one $O(L)$-conjugacy class (see Proposition 7.4.2), the following three Lemmata will prove Theorem 7.4.25.

Lemma 7.4.26. *The statement of Theorem 7.4.25 does neither depend on the splitting $L(\Phi) = R_{L(\Phi)} \oplus L(p_R(\Phi))$ nor on the chosen basis a, b of $R_{L(\Phi)}$.*

Lemma 7.4.27. *The set of elements defined by the conditions (i) and (ii) of Theorem 7.4.25 is closed under conjugacy with elements of $O(L)$.*

Lemma 7.4.28. *The set of elements defined by the conditions (i) and (ii) of Theorem 7.4.25 is one orbit under the conjugacy action of W_Φ.*

We need a short preparation before we can prove these Lemmata. Let (Φ, U) be a tubular elliptic root system of rank n. Let $L(p_R(\Phi)) = \operatorname{span}_\mathbb{Z}(\epsilon_1, \ldots, \epsilon_n)$ and choose another splitting $L(\Phi) = R_{L(\Phi)} \oplus \widetilde{L}$, where

$$\widetilde{L} = \operatorname{span}_\mathbb{Z}(\epsilon_1 + \iota_1(\epsilon_1)a + \iota_2(\epsilon_1)b, \ldots, \epsilon_n + \iota_1(\epsilon_n)a + \iota_2(\epsilon_n)b)$$

with $\iota_j(\epsilon_i) \in \mathbb{Z}$. By linear extension we obtain $\iota_1, \iota_2 \in \operatorname{Hom}_\mathbb{Z}(L(p_R(\Phi)), \mathbb{Z})$. Note that

$$\operatorname{Hom}_\mathbb{Z}(L(p_R(\Phi)), \mathbb{Z}) = L(p_R(\Phi))^* \cong P(p_R(\Phi)^\vee). \tag{7.13}$$

Lemma 7.4.29. *If an element $w \in W_\Phi$ is written as $\begin{bmatrix} c \\ x \\ y \end{bmatrix}$ with respect to the splitting $L(\Phi) = R_{L(\Phi)} \oplus L(p_R(\Phi))$, then it is written as $\begin{bmatrix} c \\ x + (\operatorname{id} - c^{-1})(\iota_1) \\ y + (\operatorname{id} - c^{-1})(\iota_2) \end{bmatrix}$ with respect to the splitting $L(\Phi) = R_{L(\Phi)} \oplus \widetilde{L}$.*

Proof. First we represent the element $w = c \cdot (a \otimes x + b \otimes y)$ with respect to the basis $B := \{\epsilon_1, \ldots, \epsilon_n, a, b\}$ by a matrix. Under the Eichler-Siegel map (see Definition 7.2.1) we have

$$E(a \otimes x + b \otimes y)(\epsilon_i) = \epsilon_i + (\epsilon_i \mid x)a + (\epsilon_i \mid y)b.$$

Therefore we obtain as a matrix representation of w:

$$w = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & c & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline (\epsilon_1 \mid x) & \cdots & (\epsilon_n \mid y) & 1 & 0 \\ (\epsilon_1 \mid y) & \cdots & (\epsilon_n \mid y) & 0 & 1 \end{array}\right).$$

Put $\widetilde{\epsilon_i} := \epsilon_i + \iota_1(\epsilon_i)a + \iota_2(\epsilon_i)b$ ($1 \le i \le n$) and $\widetilde{B} := \{\widetilde{\epsilon_1}, \dots, \widetilde{\epsilon_n}, a, b\}$. The base change matrix is given by

$$M_B^{\widetilde{B}}(\mathrm{id}) = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & I_n & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline \iota_1(\epsilon_1) & \cdots & \iota_1(\epsilon_n) & 1 & 0 \\ \iota_2(\epsilon_1) & \cdots & \iota_2(\epsilon_n) & 0 & 1 \end{array}\right).$$

If we denote the columns of $c = (c_{ij})$ as $c^{(i)}$ ($1 \le i \le n$) and put $\underline{\epsilon}_j := (\iota_j(\epsilon_1), \dots, \iota_j(\epsilon_n))$ ($j \in \{1, 2\}$), a direct calculation yields

$$\begin{aligned} M_{\widetilde{B}}^{\widetilde{B}}(w) &= M_B^{\widetilde{B}}(\mathrm{id})^{-1} \cdot w \cdot M_B^{\widetilde{B}}(\mathrm{id}) \\ &= \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & c & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline -\underline{\epsilon}_1 c^{(1)} + (\epsilon_1 \mid x) + \iota_1(\epsilon_1) & \cdots & -\underline{\epsilon}_1 c^{(n)} + (\epsilon_n \mid x) + \iota_1(\epsilon_n) & 1 & 0 \\ -\underline{\epsilon}_2 c^{(1)} + (\epsilon_1 \mid y) + \iota_2(\epsilon_1) & \cdots & -\underline{\epsilon}_2 c^{(n)} + (\epsilon_n \mid y) + \iota_2(\epsilon_n) & 0 & 1 \end{array}\right). \end{aligned}$$

Under the isomorphism (7.13) we identify ι_j with a vector in V such that $\iota_j(\epsilon_i) = (\epsilon_i \mid \iota_j)$. We have

$$\underline{\epsilon}_j c^{(i)} = \sum_{k=1}^{n} c_{ki} \iota_j(\epsilon_k) = \sum_{k=1}^{n} c_{ki} (\epsilon_k, \iota_j) = (c(\epsilon_i), \iota_j) = (\epsilon_i, c^{-1}(\iota_j)).$$

Therefore we obtain

$$\begin{aligned} -\underline{\epsilon}_1 c^{(i)} + (\epsilon_i \mid x) + \iota_1(\epsilon_i) &= -(\epsilon_i \mid c^{-1}(\iota_1)) + (\epsilon_i \mid x) + (\epsilon_i \mid \iota_1) = (\epsilon_i \mid x + (\mathrm{id} - c^{-1})(\iota_1)), \\ -\underline{\epsilon}_2 c^{(i)} + (\epsilon_i \mid y) + \iota_2(\epsilon_i) &= -(\epsilon_i \mid c^{-1}(\iota_2)) + (\epsilon_i \mid y) + (\epsilon_i \mid \iota_2) = (\epsilon_i \mid y + (\mathrm{id} - c^{-1})(\iota_2)). \end{aligned}$$

□

Proof of Lemma 7.4.26. First let a', b' be another basis of $R_{L(\Phi)}$. Then there exists

$$A = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \text{ such that } A \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Hence, if $c = \begin{bmatrix} w \\ x \\ y \end{bmatrix}$ is as described in Theorem 7.4.25, we obtain

$$\begin{aligned} c &= w(a \otimes x + b \otimes y) \\ &= w((\lambda_{11} a' + \lambda_{12} b') \otimes x + (\lambda_{21} a' + \lambda_{22} b') \otimes y) \\ &= w(a' \otimes \underbrace{(\lambda_{11} x + \lambda_{21} y)}_{=:x'} + b' \otimes \underbrace{(\lambda_{12} x + \lambda_{22} y)}_{=:y'}) \end{aligned}$$

Assume that $x', y' \in L(\Phi_i)$ for some $1 \le i \le 3$. Since $(x, y)A = (x', y')$, or equivalently $(x, y) = (x', y')A^{-1}$, we obtain $x, y \in L(\Phi_i)$, a contradiction to our assumption on x and y.

It remains to show that the statement of Theorem 7.4.25 does not depend on the splitting of $L(\Phi)$. Therefore let $L(\Phi) = R_{L(\Phi)} \oplus \widetilde{L}$ be another splitting, where

$$\widetilde{L} = \operatorname{span}_{\mathbb{Z}}(\alpha_1 + \iota_1(\alpha_1)a + \iota_2(\alpha_1)b, \ldots, \alpha_4 + \iota_1(\alpha_4)a + \iota_2(\alpha_4)b)$$

with $\iota_i(\alpha_j) \in \mathbb{Z}$ and $\{\alpha_1, \ldots, \alpha_4\}$ a basis for $L(p_R(\Phi))$. By linear extension we obtain $\iota_1, \iota_2 \in \operatorname{Hom}_{\mathbb{Z}}(L(p_R(\Phi)), \mathbb{Z})$. By Lemma 7.4.29 we have that if a transformation is written as $\begin{bmatrix} w \\ x \\ y \end{bmatrix}$ with respect to the splitting $L(\Phi) = R_{L(\Phi)} \oplus L(p_R(\Phi))$, then it is written as $\begin{bmatrix} w \\ x + (\operatorname{id} - w^{-1})(\iota_1) \\ y + (\operatorname{id} - w^{-1})(\iota_2) \end{bmatrix}$ with respect to the splitting $L(\Phi) = R_{L(\Phi)} \oplus \widetilde{L}$. We want to show that the statement of Theorem 7.4.25 is still valid for the new splitting chosen above. Therefore let $\begin{bmatrix} w \\ x \\ y \end{bmatrix}$ fulfill the conditions (i) and (ii) of Theorem 7.4.25 and let Φ_1, Φ_2, Φ_3 be the distinguished $2D_2$ root subsystems of $p_R(\Phi)$. Note that $w = w^{-1}$. As in the proof of Theorem 7.4.25 we obtain isomorphisms

$$(\operatorname{id} - w) : L(\Phi_i)^* \xrightarrow{\sim} L(\Phi_i) \ (1 \leq i \leq 4).$$

Since

$$\cap_{i=1}^3 L(\Phi_i)^* \cong \cap_{i=1}^3 \operatorname{Hom}_{\mathbb{Z}}(L(\Phi), \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(\cup_{i=1}^3 L(\Phi), \mathbb{Z}) \cong L(\Phi)^*,$$

these isomorphisms induce an isomorphism

$$(\operatorname{id} - w) : L(\Phi)^* \cong \cap_{i=1}^3 L(\Phi_i)^* \xrightarrow{\sim} \cap_{i=1}^3 L(\Phi_i). \tag{7.14}$$

By assumption x and y lie in different distinguished root lattices, say $x \in L(\Phi_i)$, $y \in L(\Phi_j)$ with $i \neq j$. By (7.14) we have $x + (\operatorname{id} - w)(\iota_1) \in L(\Phi_i)$, but $x + (\operatorname{id} - w)(\iota_1) \notin L(\Phi_j)$, since otherwise it would follow that $x \in L(\Phi_j)$. The same argument works for $y + (\operatorname{id} - w)(\iota_2)$. Hence $x + (\operatorname{id} - w)(\iota_1)$ and $y + (\operatorname{id} - w)(\iota_2)$ do not lie in a common distinguished $2D_2$ sublattice. □

For the proof of Lemma 7.4.27 we need the following result.

Lemma 7.4.30. *The canonical exact sequence*

$$0 \longrightarrow R_{L(\Phi)} \longrightarrow L(\Phi) \longrightarrow L(p_R(\Phi)) \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow R_{L(\Phi)} \otimes \operatorname{Hom}_{\mathbb{Z}}(L(p_R(\Phi)), \mathbb{Z}) \xrightarrow{\zeta_1} O(L(\Phi)) \xrightarrow{\zeta_2} \operatorname{GL}_2(\mathbb{Z}) \times O(L(p_R(\Phi))) \longrightarrow 0.$$

A splitting $L(p_R(\Phi)) \to L(\Phi)$ *naturally induces a splitting* $\operatorname{GL}_2(\mathbb{Z}) \times O(L(p_R(\Phi))) \to O(L(\Phi))$.

Proof. We will just state the maps ζ_1 and ζ_2. To check that the sequence is exact, is straightforward. As before we fix the basis a, b of $R_{L(\Phi)}$ and define the map ζ_1 as

$$\zeta_1 : R_{L(\Phi)} \otimes \operatorname{Hom}_{\mathbb{Z}}(L(p_R(\Phi)), \mathbb{Z}) \to O(L(\Phi))$$
$$a \otimes \iota_1 + b \otimes \iota_2 \mapsto (L \to L, x \mapsto x + \iota_1(p_R(x))a + \iota_2(p_R(x))b)\,.$$

To define ζ_2 we define for an element $\phi \in O(L(\Phi))$ a matrix $A_\phi \in \mathrm{GL}_2(\mathbb{Z})$. We have $(\phi(a), x) = (a, \phi(x)) = 0$ since $\phi(x) \in L(\Phi)$, hence $\phi(a) \in R_{L(\Phi)}$. Analogous we get $\phi(b) \in R_{L(\Phi)}$. Since $\phi(a), \phi(b)$ is again a basis of $R_{L(\Phi)}$, we obtain a matrix $A_\phi := \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ with

$$\begin{aligned} \phi(a) &= \lambda_{11} a + \lambda_{12} b \\ \phi(b) &= \lambda_{21} a + \lambda_{22} b. \end{aligned}$$

We define ζ_2 by

$$\zeta_2 : O(L(\Phi)) \to \mathrm{GL}_2(\mathbb{Z}) \times O(L(p_R(\Phi))),\ \phi \mapsto (A_\phi, \overline{\phi}),$$

where $\overline{\phi} = \mathrm{can}(\phi)$ (see (7.6)). □

Proof of Lemma 7.4.27. By Lemma 7.4.30 we have

$$O(L(\Phi)) \cong \left(R_{L(\Phi)} \otimes \mathrm{Hom}_{\mathbb{Z}}(L(p_R(\Phi)), \mathbb{Z})\right) \ltimes (\mathrm{GL}_2(\mathbb{Z}) \times O(L(p_R(\Phi)))).$$

Let $\begin{bmatrix} w \\ x \\ y \end{bmatrix}$ fulfill the properties (*i*) and (*ii*) of Theorem 7.4.25. We can write an element of $R_{L(\Phi)} \otimes \mathrm{Hom}_{\mathbb{Z}}(L(p_R(\Phi))$ as $\begin{bmatrix} \mathrm{id} \\ \iota_1 \\ \iota_2 \end{bmatrix}$, where $\iota_1, \iota_2 \in \mathrm{Hom}_{\mathbb{Z}}(L(p_R(\Phi)) \cong L(p_R(\Phi))^*$. We have

$$\begin{bmatrix} \mathrm{id} \\ \iota_1 \\ \iota_2 \end{bmatrix}^{-1} \begin{bmatrix} w \\ x \\ y \end{bmatrix} \begin{bmatrix} \mathrm{id} \\ \iota_1 \\ \iota_2 \end{bmatrix} = \begin{bmatrix} w \\ x + (\mathrm{id} - w)\iota_1 \\ y + (\mathrm{id} - w)\iota_2 \end{bmatrix}.$$

The assertion follows as in the proof of Lemma 7.4.26. Likewise we can argue with the proof of Lemma 7.4.26 for conjugation with a matrix in $\mathrm{GL}_2(\mathbb{Z})$.

It remains to consider conjugation with $O(L(p_R(\Phi)))$. We can write an element $\phi \in O(L(p_R(\Phi)))$ as $\begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix}$. Then conjugation with ϕ is given by

$$\begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} w \\ x \\ y \end{bmatrix} \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \phi^{-1} w \phi \\ \phi^{-1}(x) \\ \phi^{-1}(y) \end{bmatrix}.$$

Note that $\phi(L(p_R(\Phi))) = L(p_R(\Phi))$ and $p_R(\Phi) = \{\alpha \in L(p_R(\Phi)) \mid (\alpha \mid \alpha) = 2\}$ by Lemma 4.2.7. Since $(\alpha \mid \alpha) = (\phi^{-1}(\alpha) \mid \phi^{-1}(\alpha))$, we have that $\phi^{-1}(p_R(\Phi))$ is again a root system of type D_4. Likewise the decomposition of $p_R(\Phi)$ into three disjoint $2D_2$ root subsystems is preserved under ϕ^{-1}. Therefore conjugation with ϕ^{-1} preserves the properties (*i*) and (*ii*) of Theorem 7.4.25. □

Proof of Lemma 7.4.28. Let $\begin{bmatrix} w_1 \\ x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ x_2 \\ y_2 \end{bmatrix} \in W_\Phi$ be two elements fulfilling the conditions (*i*) and (*ii*) of Theorem 7.4.25. Hence $w_1 = w_2$ (see the end of Section 7.4.1) and

$C_{W_{p_R(\Phi)}}(w_1) = W_{p_R(\Phi)}$ by Corollary 7.4.13. Therefore conjugating with an element $\begin{bmatrix} w \\ x \\ y \end{bmatrix} \in$ W_Φ yields

$$\begin{bmatrix} w \\ x \\ y \end{bmatrix}^{-1} \cdot \begin{bmatrix} w_1 \\ x_1 \\ y_1 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} w_1 \\ w^{-1}(x_1) + 2x \\ w^{-1}(y_1) + 2y \end{bmatrix}.$$

and this element fulfills again conditions (i) and (ii) of Theorem 7.4.25 by Proposition 7.4.22. We have to show that we can pick $\begin{bmatrix} w \\ x \\ y \end{bmatrix} \in W_\Phi$ such that

$$\begin{bmatrix} w_1 \\ w^{-1}(x_1) + 2x \\ w^{-1}(y_1) + 2y \end{bmatrix} = \begin{bmatrix} w_2 \\ x_2 \\ y_2 \end{bmatrix}.$$

Let us first assume that $x_1, x_2, y_1, y_2 \in p_R(\Phi)$. By Lemma 2.1.3 we find $w' \in W_{p_R(\Phi)}$ such $(w')^{-1}(x_1) = x_2$. By Lemma 7.4.16 we can assume that $x_2 \in \Phi_1$. By Proposition 7.4.22 we have $(w')^{-1}(y_1), y_2 \in \Phi_2 \cup \Phi_3$. By Lemma 7.4.24 we find $w'' \in W_{\Phi_1} \subseteq \mathrm{Stab}_{W_{p_R(\Phi)}}(\{\pm x_2\})$ such that $(w'')^{-1}((w')^{-1}(y_1)) = \pm y_2$. If we put $w := w'w''$, $x := 0$ and (if necessary) $y := 2y_2$, the assertion follows. Using Lemma 7.4.18 and Remark 7.4.19, the general assertion follows. □

7.4.4. Transitive Hurwitz action

We are finally in the position to prove Theorem 1.1.4.

Let (Φ, U) be a tubular elliptic root system of rank n. Let $c \in W_\Phi$ be a Coxeter transformation and put

$$\underline{\mathrm{Red}}_{T_\Phi}(c) := \{(s_{\beta_1}, \ldots, s_{\beta_{n+2}}) \mid \beta_i \in \Phi,\ \mathrm{span}_\mathbb{Z}(\beta_1, \ldots, \beta_{n+2}) = L(\Phi), c = s_{\beta_1} \cdots s_{\beta_{n+2}}\},$$

where $T_\Phi = \{s_\alpha \mid \alpha \in \Phi\}$. The natural map $p_R : V \to V/R$ induces the map $p_R : L(\Phi) \to L(p_R(\Phi))$. Therefore we have the following map

$$\pi : \underline{\mathrm{Red}}_{T_\Phi}(c) \to \mathrm{Fac}_{T,n+2}(\overline{c}), (s_{\beta_1}, \ldots, s_{\beta_{n+2}}) \mapsto (s_{p_R(\beta_1)}, \ldots, s_{p_R(\beta_{n+2})}),$$

where $T = \{s_{p_R(\alpha)} \mid \alpha \in \Phi\}$. Note that $L(p_R(\Phi)) = \mathrm{span}_\mathbb{Z}(p_R(\beta_1), \ldots, p_R(\beta_n))$ by Theorem 4.2.12 and $W_{p_R(\Phi)} = \langle s_{p_R(\beta_1)}, \ldots, s_{p_R(\beta_{n+2})} \rangle$, which shows that π is well-defined. In particular, $(W_{p_R(\Phi)}, T)$ is a dual Coxeter system of type D_4 or E_n ($n \in \{6, 7, 8\}$).

The idea of the proof of Theorem 1.1.4 is as follows: The Hurwitz action on the set $\mathrm{Fac}_{T,n+2}(\overline{c})$ is transitive by Theorem 5.1.6 resp. by [Klu87] for the case where $p_R(\Phi)$ is of type E_8. The map π is equivariant with respect to the Hurwitz action. Analogous to the proof of Theorem 1.1.3 it remains to show that there exists a fibre of π and a subgroup of $\mathcal{B}_{n+2}$ acting transitively on this fibre. A first step is the following result (see also [Klu87, Kap. VI, Satz 1.3]).

Theorem 7.4.31. *Let (Φ, U) be a tubular elliptic root system, $x \in W_\Phi$ a Coxeter transformation of order k and $\underline{t} \in \mathrm{Fac}_{T,n+2}(\overline{c})$. Then:*

(a) *The congruence subgroup*

$$\Gamma(k) = \{\phi \in O(L) \mid \phi \circ c = c \circ \phi,\ \overline{\phi}_{|L(p_R(\Phi))} = \mathrm{id}\}$$

acts simply transitive on $\pi^{-1}(\underline{t})$ *via*

$$\phi(s_{\beta_1}, \ldots, s_{\beta_{n+2}}) = (s_{\phi(\beta_1)}, \ldots, s_{\phi(\beta_{n+2})})$$

for all $(s_{\beta_1}, \ldots, s_{\beta_{n+2}}) \in \pi^{-1}(\underline{t})$.

(b) *For each* $t \in \pi^{-1}(\underline{t})$ *there exists a canonical anti-homomorphism*

$$a_t : Stab_{\mathcal{B}_{n+2}}(\underline{t}) \to \Gamma(k), \sigma \mapsto a_t(\sigma),$$

where $a_t(\sigma)$ *is defined as follows: If* $\sigma \in Stab_{\mathcal{B}_{n+2}}(\underline{t})$, *then* $\sigma(t) \in \pi^{-1}(\underline{t})$ *since* π *is equivariant with respect to the Hurwitz action. By* (a) *there exists a unique* $a_t(\sigma) \in \Gamma(k)$ *such that* $\sigma(t) = a_t(\sigma)(t)$.

We proceed as before and just give proofs for the case $D_4^{(1,1)}$ while referring to [Klu87] for the proofs for the cases $E_n^{(1,1)}$ $(n \in \{6, 7, 8\})$.

Proof of Theorem 7.4.31. Let $(s_{\beta_1}, \ldots, s_{\beta_6}), (s_{\beta'_1}, \ldots, s_{\beta'_6}) \in \pi^{-1}(\underline{t})$. Thus $\beta_i = \beta'_i + x_i$ with $x_i \in R_{L(\Phi)}$ for $1 \leq i \leq 6$. We have $\mathrm{span}_{\mathbb{Z}}(\beta_1, \ldots, \beta_6) = L(\Phi) = \mathrm{span}_{\mathbb{Z}}(\beta'_1, \ldots, \beta'_6)$, thus there exists $\phi \in \mathrm{GL}(V)$ such that

$$(\phi(\beta_1), \ldots, \phi(\beta_6)) = (\beta'_1, \ldots, \beta'_6).$$

In particular $\phi \in O(L)$. Furthermore

$$\begin{aligned} c\phi &= s_{\beta'_1} s_{\beta'_2} \cdots s_{\beta'_6} \phi \\ &= s_{\phi(\beta_1)} s_{\phi(\beta_2)} \cdots s_{\phi(\beta_6)} \phi \\ &= (\phi s_{\beta_1} \phi^{-1})(\phi s_{\beta_2} \phi^{-1}) \cdots (\phi s_{\beta_6} \phi^{-1}) \phi \\ &= \phi s_{\beta_1} s_{\beta_2} \cdots s_{\beta_6} \\ &= \phi c, \end{aligned}$$

thus $\phi \in C_{O(L)}(c)$. We have $p_R(\beta_i) = p_R(\beta'_i) = p_R(\phi(\beta_i))$, hence $\overline{\phi}_{|L(p_R(\phi))} = \mathrm{id}$. Therefore we can apply Theorem 7.4.11 to obtain assertion (a). For part (b) note that the action of $\Gamma(2)$ on $\underline{\mathrm{Red}}_{T_\Phi}(c)$ and the Hurwitz action of $\mathcal{B}_6$ on $\underline{\mathrm{Red}}_{T_\Phi}(c)$ commute. This can be checked directly on the generators of $\mathcal{B}_6$. In particular the action of $\Gamma(2)$ on $\underline{\mathrm{Red}}_{T_\Phi}(c)$ and the action of $\mathrm{Stab}_{\mathcal{B}_6}(\underline{t})$ on $\underline{\mathrm{Red}}_{T_\Phi}(c)$ commute. Let $\sigma_1, \sigma_2 \in \mathrm{Stab}_{\mathcal{B}_6}(\underline{t})$. Then

$$a_t(\sigma_1\sigma_2)(t) = \sigma_1\sigma_2(t) = \sigma_1 a_t(\sigma_2)(t) = a_t(\sigma_2)\sigma_1(t) = a_t(\sigma_2)a_t(\sigma_1)(t).$$

□

Lemma 7.4.32. *There exist* $\underline{t} \in \mathrm{Fac}_{T,n+2}(c)$ *and* $t \in \pi^{-1}(\underline{t})$ *such that the anti-homomorphism* a_t *is surjective.*

Before we prove this, we need some preparation. For a root $\alpha = \alpha' + k_1 a + k_2 b \in \Phi$ with $\alpha' \in p_R(\Phi)$ we have

$$s_\alpha = s_{\alpha'}(a \otimes k_1\alpha' + b \otimes k_2\alpha') = \begin{bmatrix} s_{\alpha'} \\ k_1\alpha' \\ k_2\alpha' \end{bmatrix}.$$

Let $\beta = \beta' + l_1 a + l_2 b \in \Phi$ with $\beta' \in p_R(\Phi)$ be another root. By what we have observed in Section 7.4.3 we obtain

$$s_\beta s_\alpha s_\beta = \begin{bmatrix} s_{s_{\beta'}(\alpha')} \\ s_{\beta'}(k_1\alpha') + (\mathrm{id} - s_{s_{\beta'}(\alpha')})(l_1\beta') \\ s_{\beta'}(k_2\alpha') + (\mathrm{id} - s_{s_{\beta'}(\alpha')})(l_2\beta') \end{bmatrix} = \begin{bmatrix} s_{s_{\beta'}(\alpha')} \\ (k_1 - l_1(\alpha' \mid \beta'))s_{\beta'}(\alpha') \\ (k_2 - l_2(\alpha' \mid \beta'))s_{\beta'}(\alpha') \end{bmatrix}.$$

We shortly write $\begin{bmatrix} \alpha' \\ k_1 \\ k_2 \end{bmatrix}$ for the reflection in $\alpha = \alpha' + k_1 a + k_2 b \in \Phi$, thus

$$s_\beta s_\alpha s_\beta = \begin{bmatrix} s_{\beta'}(\alpha') \\ k_1 - l_1(\alpha' \mid \beta') \\ k_2 - l_2(\alpha' \mid \beta') \end{bmatrix}.$$

Proof of Lemma 7.4.32. We fix the Coxeter transformation $c = s_{\alpha_0}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_2^*}$ in the elliptic Weyl group W_Φ of type $D_4^{(1,1)}$. Thus

$$\underline{t} := (s_{\widetilde{\alpha}}, s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}, s_{\alpha_2}, s_{\alpha_2}) \in \mathrm{Fac}_{T,6}(\overline{c})$$

and

$$t := \left(\begin{bmatrix} \alpha_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \widetilde{\alpha} \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_4 \\ 0 \\ 0 \end{bmatrix} \right) \in \pi^{-1}(\underline{t})$$

The strategy of the proof is the following: We take an element $\tau_i \in \mathrm{Stab}_{\mathcal{B}_6}(\underline{t})$ and compute $\tau_i(t)$. By Theorem 7.4.11 this yields a matrix $a_t(\tau_i) \in \Gamma(2)$. In this way we will find matrices $a_t(\tau_1), a_t(\tau_2), a_t(\tau_3)$ which also generates $\Gamma(2)$ and therefore a_t will be surjective.

To find the braids τ_i $(1 \leq i \leq 3)$ we used [Sage], but for sake of completeness we will state them here explicitly.

- $\tau_1 = \tau_{11}^{-1}\sigma_3\tau_{11}$, where $\tau_{11} = \sigma_2^{-1}\sigma_1\sigma_2\sigma_4\sigma_5^2\sigma_4\sigma_2^2\sigma_3\sigma_2\sigma_1\sigma_2$. It is

$$\begin{aligned} &\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right) \\ \overset{\tau_{11}}{\to} &\left(\begin{bmatrix} 12 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 1234 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 2 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 2 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 23 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 24 \\ * \\ * \end{bmatrix} \right) \\ \overset{\tau_{11}^{-1}\sigma_3}{\to} &\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right). \end{aligned}$$

For this case we explicitly state how to calculate the induced matrix. Let $\phi := a_t(\tau_1)$. Then

$$\begin{aligned}
\phi(b) &= \phi(\alpha_2 + b - \alpha_2) = \phi(\alpha_2 + b) - \phi(\alpha_2) = (\alpha_2 - a - 2b) - (\alpha_2 + a + 3b) \\
&= -2a - 5b \\
\phi(\widetilde{\alpha}) &= \phi(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) \\
&= (\alpha_1 + a + 3b) + 2(\alpha_2 + a + 3b) + (\alpha_3 + a + 3b) + (\alpha_4 + a + 3b) \\
&= \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + 5a + 15b = \widetilde{\alpha} + 5a + 15b \\
\phi(\widetilde{\alpha} - a) &= \widetilde{\alpha} - 2a - 3b \\
\phi(a) &= \phi(\widetilde{\alpha} - (\widetilde{\alpha} - a)) = (\widetilde{\alpha} + 5a + 15b) - (\widetilde{\alpha} - 2a - 3b) = 7a + 18b.
\end{aligned}$$

The induced matrix is

$$a_t(\tau_1) = \begin{pmatrix} 7 & -2 \\ 18 & -5 \end{pmatrix}.$$

- $\tau_2 = \tau_{21}^{-1}\sigma_3\tau_{21}$, where $\tau_{21} = \sigma_2\sigma_3^{-1}\sigma_4\sigma_5^{-1}\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}$. It is

$$\begin{aligned}
&\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right) \\
\stackrel{\tau_{21}}{\to} &\left(\begin{bmatrix} 124 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 123 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 0 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 0 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 234 \\ * \\ * \end{bmatrix}, \begin{bmatrix} 2 \\ * \\ * \end{bmatrix}\right) \\
\stackrel{\tau_{21}^{-1}\sigma_3}{\to} &\left(\begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}\right).
\end{aligned}$$

The induced matrix is

$$a_t(\tau_2) = \begin{pmatrix} 9 & -2 \\ 32 & -7 \end{pmatrix}.$$

- $\tau_3 = \sigma_1$.

$$\begin{aligned}
&\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right) \\
\stackrel{\tau_3}{\to} &\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right).
\end{aligned}$$

The induced matrix is

$$a_t(\tau_3) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is known that

$$\Gamma(2) = \left\langle C_1 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, C_2 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, C_3 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

We have

$$\begin{aligned} C_1 &= a_t(\tau_3)^{-2} a_t(\tau_2)^{-1} a_t(\tau_1)^{-1} a_t(\tau_2)^{-1} a_t(\tau_1)^{-1} \\ C_2 &= a_t(\tau_3) \\ C_3 &= a_t(\tau_3)^{-1} a_t(\tau_2)^{-1} a_t(\tau_1)^{-1}, \end{aligned}$$

which yields the assertion. □

Proof of Theorem 1.1.4. Since all Coxeter transformations for the elliptic Weyl group of type $D_4^{(1,1)}$ are conjugated and since the Hurwitz action commutes with conjugation, it is enough to show the assertion for the fixed Coxeter transformation $c = s_{\alpha_0} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_2^*}$. We also fix $\underline{t} \in \mathrm{Fac}_{T,6}(\overline{c})$ and $t \in \pi^{-1}(\underline{t}) \subseteq \underline{\mathrm{Red}}_{T_\Phi}(c)$ as in the proof of Lemma 7.4.32. Let $t' := (s_{\beta_1}, \ldots, s_{\beta_6}) \in \underline{\mathrm{Red}}_{T_\Phi}(c)$ be arbitrary. It is $\pi(t') \in \mathrm{Fac}_{T,6}(\overline{c})$. By Theorem 5.1.6 the Hurwitz action on $\mathrm{Fac}_{T,6}(\overline{c})$ is transitive, thus there exists $\sigma \in \mathcal{B}_6$ such that

$$\sigma(\pi(t')) = \underline{t} = \pi(t).$$

Since π is equivariant with respect to the Hurwitz action, we have

$$\pi(\sigma(t')) = \underline{t} = \pi(t).$$

Therefore $\sigma(t'), t \in \pi^{-1}(\underline{t})$. By part (a) of Theorem 7.4.31 there exists $\phi \in \Gamma(2)$ such that $\phi(\sigma(t')) = t$, hence $\sigma(t') = \phi^{-1}(t)$. By Lemma 7.4.32 the map a_t is surjective, that is, there exists $\tau \in \mathrm{Stab}_{\mathcal{B}_6}(\underline{t})$ such that $\phi^{-1} = a_t(\tau)$, thus $\sigma(t') = \tau(t)$. □

7.5. Weighted projective lines and Hurwitz action in elliptic Weyl groups

Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{P}^1_{\mathbb{K}}$ the projective line over $\mathbb{K}$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ a (possibly empty) tuple of distinct closed points of $\mathbb{P}^1_{\mathbb{K}}$ and $\mathbf{p} = (p_1, \ldots, p_n)$ a sequence of positive integers, called weight sequence. The triple $\mathbb{X} = (\mathbb{P}^1_{\mathbb{K}}, \lambda, \mathbf{p})$ is called weighted projective line. Geigle and Lenzing associated to $\mathbb{X}$ the category of coherent sheaves $\mathrm{coh}(\mathbb{X})$. For details and definitions we refer to their paper [GL87] as well as to [CK] for a more detailed treatment of this topic. The significance of the category $\mathrm{coh}(\mathbb{X})$ is exhibited by the following theorem of Happel (see [Hap01, Theorem 3.1]).

Theorem 7.5.1. *Let $\mathcal{H}$ be a connected hereditary abelian $\mathbb{K}$-category with tilting object. Then $\mathcal{H}$ is derived equivalent to* $\mathrm{mod}(A)$ *for some finite dimensional hereditary k-algebra A or derived equivalent to* $\mathrm{coh}(\mathbb{X})$ *for some weighted projective line $\mathbb{X}$.*

In view of this theorem it seems natural to ask whether it is possible to find a similar statement to that of Theorem 3.3.1 if we replace therein the category $\mathrm{mod}(A)$ by $\mathrm{coh}(\mathbb{X})$.

First note that there is an action of the braid group on the set of exceptional sequences in $\mathrm{coh}(\mathbb{X})$ (see [KM02, Section 3]). Kussin and Meltzer obtained for this action the following transitivity result (see [KM02, Theorem 1.1, Corollary 1.2]).

Theorem 7.5.2. *Let $\mathbb{X}$ be a weighted projective line over $\mathbb{K}$ and n be the rank of the Grothendieck group $K_0(\mathbb{X})$. Then:*

(a) The braid group $\mathcal{B}_n$ acts transitively on the set of complete exceptional sequences in $\text{coh}(\mathbb{X})$.

(b) The group $\mathbb{Z}^n \ltimes \mathcal{B}_n$ acts transitively on the set of complete exceptional sequences in $\mathcal{D}^{\text{b}}(\text{coh}(\mathbb{X}))$.

From now on let $\mathbb{K}$ be algebraically closed of characteristic zero. Shiriashi et al ([STW16]) associate to a derived category $\mathcal{D}$ which fulfills some additional properties (in particular $\mathcal{D}^{\text{b}}(\text{coh}(\mathbb{X}))$ fulfills this properties) a so-called simply-laced generalized root system $\Phi_{\mathcal{D}} = (K_0(\mathcal{D}), I_{\mathcal{D}}, \Delta_{\text{re}}(\mathcal{D}), c_{\mathcal{D}})$.

Let (Φ, U) be a tubular elliptic root system with respect to some symmetric bilinear form $(- \mid -)$. Then it is straightforward to see that $(L(\Phi), (- \mid -), \Phi, c)$ is a simply-laced generalized root system. Here c denotes the Coxeter transformation with respect to an elliptic root basis $\Gamma(\Phi, U)$.

A weighted projective line $\mathbb{X} = (\mathbb{P}^1_{\mathbb{K}}, \lambda, \mathbf{p})$ is said to be of tubular type if the weight sequence $\mathbf{p}$ is (up to permutation) given by $(2,2,2,2), (3,3,3), (2,4,4)$ or $(2,3,6)$. If $\mathbb{X}$ is a weighted projective line of tubular type and $\mathcal{D} = \mathcal{D}^{\text{b}}(\text{coh}(\mathbb{X}))$, then $\Phi_{\mathcal{D}}$ is isomorphic to a tubular elliptic root system (see [STW16, Section 2.6]). We list the isomorphism types in Table 7.2.

weight sequence of $\mathbb{X}$	isomorphism type of $\Phi_{\mathcal{D}}$
$(2,2,2,2)$	$D_4^{(1,1)}$
$(3,3,3)$	$E_6^{(1,1)}$
$(2,4,4)$	$E_7^{(1,1)}$
$(2,3,6)$	$E_8^{(1,1)}$

Table 7.2.: Weight sequences and associated root systems

Let (Φ, U) be a tubular elliptic root system, $T = T_\Phi$ be the set of reflections for Φ and $c \in W_\Phi$ a Coxeter transformation. Then we can define analogously to Definition 2.4.2 the absolute order $\leq_T$ on W_Φ.

Considering Theorem 1.1.4, Theorem 7.5.2 and the proof of Theorem 3.3.1 given in [Kra12], we formulate the following conjecture. It is strongly influenced by the ideas of Henning Krause.

Conjecture 7.5.3. Let $\mathbb{X}$ be a weighted projective line of tubular type over an algebraically closed field $\mathbb{K}$ of characteristic zero, $\mathcal{D} = \mathcal{D}^{\text{b}}(\text{coh}(\mathbb{X}))$ the bounded derived category, $\Phi = \Phi_{\mathcal{D}}$ the associated simply-laced generalized root system and $c = c_{\mathcal{D}}$ the corresponding Coxeter transformation. Then there exists an order preserving bijection between

- the set of thick subcategories of $\mathcal{D}^{\text{b}}(\text{coh}(\mathbb{X}))$ generated by an exceptional sequence in $\text{coh}(\mathbb{X})$
- the poset $\{w \in W_\Phi \mid e \leq_T w \leq_T c\}$.

A. GAP programs

The aim of this appendix is to explain the structure and use of the individual GAP programs used in this thesis. E.g. if the program `HelloWorld.g` is stored in the directory `/homes`, you can load the program within GAP with the command `Read("/homes/HelloWorld.g")`.

A.1. Basic programs

The programs discussed in this section are fundamental for the other programs which we will discuss in the next sections. These are:

- `CoxeterGroups.g`
- `QuasiCoxeterClasses.g`
- `Hurwitz.g`

First load the program `CoxeterGroups.g` as described above. It provides a set of simple reflections (as permutations) for each of the finite irreducible Coxeter systems. E.g. the command `CoxGrpGensB(5)` provides a set of simple reflections for the Coxeter system of type B_5. The commands are also explained within the file. Using the GAP function `GroupByGenerators()` you will obtain the corresponding Coxeter group. The permutation degrees used here are proven to be minimal in [Sau14, Table 1]. Furthermore `CoxeterGroups.g` contains the function `Reflections()` which takes a Coxeter group W as an input and will give you the set of reflections as an output. E.g. the commands

```
W:=GroupByGenerators(CoxGrpGensB(5));
Reflections(W);
```

returns the set of reflections for the Coxeter system of type B_5.

The program `QuasiCoxeterClasses.g` provides the function `QuasiCoxeterClasses()`. It returns a list of pairs (w, x) where w is a representative from the conjugacy class of a quasi-Coxeter element and x is an element of $\mathrm{Red}_T(w)$. The input for `QuasiCoxeterClasses()` is a crystallographic Coxeter group W, the set of reflections T, the rank n of the corresponding Coxeter system and the number of conjugacy classes of quasi-Coxeter elements. The latter one can be found in the associated file `qcclasses.txt`. E.g.

```
W:=GroupByGenerators(CoxGrpGensE(7));
T:=Reflections(W);
QuasiCoxeterClasses(W,T,7,5);
```

returns a list of pairs (w, x) where we obtain for each conjugacy class of quasi-Coxeter elements such a pair as described above. For the Coxeter groups of type H_3 and H_4, representatives of the conjugacy classes of quasi-Coxeter elements are stored in the file `qcclasses.txt`.
The program `Hurwitz.g` provides the function `HurwitzOrbit()`. The input is a list of reflections and the output is the corresponding Hurwitz orbit. E.g.

```
W:=GroupByGenerators(CoxGrpGensE(7));
```

```
T:=Reflections(W);
qcc:=QuasiCoxeterClasses(W,T,7,5);
HurwitzOrbit(qcc[3][2]);
```

computes the Hurwitz orbit for one of the quasi-Coxeter elements in the Coxeter group of type E_7.

A.2. The proof of Theorem 4.3.9

The assertion for this theorem was checked for (W, T) of type H_4 by GAP. The corresponding code is stored in the file `PrefixQuasiCoxIsParabolicH4.g`.

A.3. The proof of Theorem 1.1.2 in Section 4.5

Here it is left to show that for a quasi-Coxeter element w in a Coxeter group of type E_n ($n \in \{6, 7, 8\}$) there exists a reduced decomposition $(t_1, \ldots, t_n)$ of w such that for every reflection t in T there exists $(t'_1, \ldots, t'_{n-1}, t) \in \mathrm{Red}_T(w)$ with $(t_1, \ldots, t_n) \sim (t'_1, \ldots, t'_{n-1}, t)$. First load the programs `CoxeterGroups.g` and `QuasiCoxeterClasses.g` to calculate T and a reduced decomposition $(t_1, \ldots, t_n)$ for a quasi-Coxeter element. Next load the program `RefOrbit.g`. The function `allReflections()` within this program applies randomly generated Hurwitz moves to the decomposition $(t_1, \ldots, t_n)$ until each reflection occured once as a factor in the Hurwitz orbit.

For the quasi-Coxeter elements in H_3 and H_4 representatives from each conjugacy class of quasi-Coxeter elements can be found in `qcclasses.txt`. Using the function `HurwitzOrbit()` in `Hurwitz.g`, the assertion of Theorem 1.1.2 can be checked directly.

A.4. The proof of Proposition 5.2.4

The proof of this proposition for the cases E_6, E_7, E_8, F_4 and G_2 was done by explicit calculations in GAP. The program `ParPlusRefUnique.g` provides the function `IsUnique()`. This function takes as an input the set of simple reflections S, the set of reflections T and the rank n and checks the assertion of Proposition 5.2.4 directly for standard parabolic subgroups of rank $n - 1$. Namely, if the function returns a list of zeros of length n, then the assertion is true. E.g.

```
S:=CoxGrpGensF();
W:=GroupByGenerators(S);
T:=Reflections(W);
IsUnique(S,T,4);
```

yields the output `[0,0,0,0]`, hence the assertion is true for the Coxeter system of type F_4.

A.5. The proof of Lemma 5.2.5

The program `ParPlusTwoRef.g` provides the function `IsConjBoth()`. This function takes as an input the set of simple reflections S, the set of reflections T and the rank n and checks the assertion of Lemma 5.2.5 directly for standard parabolic subgroups of rank $n - 1$. Namely, if the function returns a list of zeros of length n, then the assertion is true. E.g.

```
S:=CoxGrpGensE(6);
W:=GroupByGenerators(S);
T:=Reflections(W);
IsUnique(S,T,6);
```

yields the output `[0,0,0,0,0,0]`, hence the assertion is true for the Coxeter system of type E_6.

A.6. The proof of Theorem 5.1.6 in Section 5.3

Here it is left to show the assertion for the case F_4. You first have to load the programs `CoxeterGroups.g` and `Hurwitz.g`. The program `NonReducedF4.g` provides amongst other things the functions `LengthFour()` and `HurwitzNonRed()`. The first function expects the group W (so here the Coxeter group of type F_4) and the set of reflections T as an input. The output is a list of conjugacy classes of all elements of absolute order 4. The function `HurwitzNonRed()` expects as an input W, T and a conjugacy class x as computed with the function `LengthFour()`. The output is a tuple of integers. If all of them are zero, then the assertion of Theorem 5.1.6 is true for the elements belonging to the conjugacy class x.

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