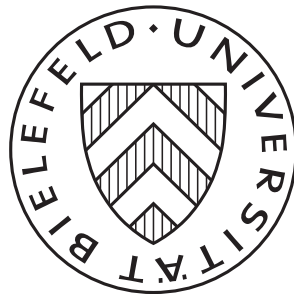


July 2017

Viability and Arbitrage under Knightian Uncertainty

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July 21, 2017

Abstract

We reconsider the microeconomic foundations of financial economics under Knightian Uncertainty. In a general framework, we discuss the absence of arbitrage, its relation to economic viability, and the existence of suitable nonlinear pricing expectations. Classical financial markets under risk and no ambiguity are contained as special cases, including various forms of the Efficient Market Hypothesis. For Knightian uncertainty, our approach unifies recent versions of the Fundamental Theorem of Asset Pricing under a common framework.

Keywords: Robust Finance, No Arbitrage, Viability, Knightian Uncertainty

JEL subject classification: D53, G10

AMS 2010 subject classification. Primary 91B02; secondary 91B52, 60H30

1 Introduction

Recently, a large and increasing literature discusses decisions, markets, and economic interactions under uncertainty. The pioneering work of Knight (1921) distinguishes *risk*, a situation which allows for an objective probabilistic description, from (Knightian) *uncertainty*, a situation that cannot be modeled by one single probability distribution.

In this paper, we reconsider economic viability, arbitrage and pricing under Knightian Uncertainty. Under risk, it is (frequently implicitly) assumed that all potential agents in the economy agree on some probabilistic description of future events. More precisely, it is assumed that there exists a reference probability which determines the null sets and the topology of the model. The seminal paper Harrison and Kreps (1979) discuss economic viability in such a probabilistic setting. A model of asset prices is viable if it is consistent with an economic equilibrium in the sense that one can construct an economy consisting of agents (selected from a given class of potential agents) and suitable endowments such that the financial market is in equilibrium. The reference measure plays another important role in the probabilistic setting as it leads to a natural

*Frank Riedel acknowledges gratefully financial support by the German Research Foundation (DFG) via CRC 1283. Matteo Burzoni and H. Mete Soner also acknowledge support by the ETH Foundation, Swiss Finance Institute and the Swiss National Foundation through SNF 200020_172815.

common order on which all potential agents agree: if a random payoff X is greater or equal than Y with probability one under the reference measure, then every possible agent prefers X to Y .

Under Knightian uncertainty, it is no longer true that all potential agents agree on some reference probability. Therefore, it is necessary to reconsider the concepts of negligible contract, continuity of preferences and unanimous order of contracts.

Our analysis is based on a general weak order which is agreed upon by all potential agents and it is not necessarily induced by any probability measure. The set of potential agents consists of market participants with weakly monotone, convex, and weakly continuous preferences. The notion of weak continuity is just a very weak notion of consistency with the order on real numbers: if there is a vanishing sequence of fees such that a contract is not chosen by an agent when the fee has to be paid, then the contract is also not chosen in the absence of fees. In addition, we suppose that a class of relevant contracts is given; these are the contracts that are considered desirable by any potential agent of the market. Conceptually, these are the two pillars of our market models, which obviously include the framework of risk as a special case.

We ask what can be said about the absence of arbitrage, its relation to economic equilibrium, and the existence of suitable pricing measures in this general setting. We adapt the notion of arbitrage to incorporate the Knightian uncertainty setting. With the help of relevant contracts, we define an arbitrage opportunity as a trade that has no cost and dominates a relevant contract. We also introduce the weaker notion of a free lunch with vanishing risk following in spirit the mathematical literature on the Fundamental Theorem of Asset Pricing in Finance (Delbaen and Schachermayer (1998)); a free lunch with vanishing risk consists of a sequence of trades that dominate a relevant contract with arbitrarily small cost.

We show that absence of arbitrage is equivalent to the economic viability of the model in the sense that one can construct an economy from the class of potential agents such that the given asset market prices support an equilibrium. Compared to classical results, we weaken the monotonicity assumption of the preference relation which supports the given asset prices in equilibrium (compare Definition 3.1 below). Indeed, this relaxation is not only necessary to account for Knightian uncertainty, but it also allows to characterize viable markets as arbitrage-free markets. The latter is achieved only partially in Kreps (1981) and Harrison and Kreps (1979). In particular, we provide an independent definition of arbitrage and show its equivalence to viability and also an appropriate version of extendability of the associated linear pricing mapping (in Theorems 3.5 and 4.5 below).

In contrast to risk, it is no longer possible to characterize the absence of arbitrage by the existence of a single linear pricing measure (or equivalent martingale measure). Instead, one has to use a suitable *nonlinear* pricing expectation which we call a sublinear martingale expectation. The nonlinearity appears naturally for *preferences* in decision-theoretic models of ambiguity-averse preferences (Gilboa and Schmeidler (1989), Maccheroni, Marinacci, and Rustichini (2006)). It is interesting to see that a similar nonlinearity arises here for the *pricing* functional which supports viable financial markets

(compare also the discussion of equilibrium with nonlinear prices in Beissner and Riedel (2016)).

Properly discounted asset prices are *symmetric* martingales under the nonlinear pricing expectation. Note that under nonlinear expectations, one has to distinguish martingales from symmetric martingales; a symmetric martingale has the property that the process itself and its negative are martingales¹.

When there is a reference probability, the martingale measures need to be equivalent to the original measure in order to preclude arbitrage. In other words, the martingale measures share the null sets of the reference measure. As no such reference measure exists in our framework, we have to replace equivalence by another property which reflects the fact that the market needs to assign positive prices to desirable contracts. This is here a full support property of the pricing expectation. All relevant contracts, i.e. the contracts which are considered as desirable by all agents, have a positive expectation under the nonlinear pricing expectation.

Our main theorems contain the existing results under risk as a special case and lead to new insights for more specific models of Knightian uncertainty.

The original (strong) version of the Efficient Market Hypothesis (Fama (1970)) states that the expected returns of all assets are equal. We show that the Efficient Market Hypothesis holds true in equilibrium and under no arbitrage conditions whenever the common order of the economy is given by the expected value under a common probability measure. This corresponds, of course, to the “risk-neutral world” interpretation.

When the common order is given by the almost sure ordering under some measure, we obtain the weak form of the Efficient Market Hypothesis which states that expected returns are equal under some equivalent probability measure. This is the classic version of the FTAP and the viability theorem (Dalang, Morton, and Willinger (1990); Delbaen and Schachermayer (1998); Duffie and Huang (1985); Harrison and Kreps (1979); Harrison and Pliska (1981)).

Under conditions of Knightian uncertainty, our theorem leads to some new conclusions and unifies various other results which have recently appeared in the literature.

If we take the order of the economy to be given by a multiple prior expectation in the spirit of Gilboa and Schmeidler (1989), we obtain a new version of the strong Efficient Market Hypothesis under Knightian uncertainty. In this version, absence of arbitrage and consistency with economic equilibrium is equivalent to the fact that discounted asset prices are martingales under each element of the given set of multiple priors.

If we just assume that agents agree on the natural *quasi-sure* order induced by a set of priors², we obtain a weak version of the efficient market hypothesis under

¹In the context of volatility uncertainty, symmetric martingales are related to the G -expectation constructed by Peng (2006, 2007) and the corresponding martingale representation theorem is proved in Soner, Touzi, and Zhang (2011). Also Beißner (2013) employed the symmetric martingales in his study of the fundamental theorem in the context of uncertain volatility.

²Under Knightian uncertainty, one is naturally led to study sets of probability measures which are not dominated by one common reference measure (Epstein and Ji (2014), Vorbrink (2014), e.g.). It is then natural to take the quasi-sure ordering as the common order of the market. A claim dominates quasi-surely another claim if it is almost surely greater or equal under all considered probability

Knightian uncertainty. Bouchard and Nutz (2015) and Burzoni, Frittelli, and Maggis (2016) discuss the absence of arbitrage in such a setting. We thus complement their analysis by giving a precise economic equilibrium foundation. There exists a sublinear martingale expectation which can be written as a supremum of expectations over a set of priors which is equivalent to the original set of priors in a suitable sense³. The nonlinear pricing expectation shares the set of negligible contracts with the sublinear martingale expectation induced by the given set of priors.

Riedel (2015) works in a setting of complete Knightian uncertainty under suitable topological assumptions. Absence of arbitrage is equivalent to the existence of full support martingale measures in this context. We show that one can obtain this result from our main theorem when all agents use the pointwise order and consider contracts as relevant if they are nonnegative and positive in some state of the world. Several different notions given in robust finance are also covered in our setting by choosing the weak order the set of relevant sets properly. Indeed, the definition given in the initial paper of Acciaio, Beiglböck, Penkner, and Schachermayer (2016) uses a small class of relevant contracts and Bartl, Cheridito, Kupper, and Tangpi (2017) considers only the contracts that are uniformly positive as relevant. A comparative summary of these studies is given in the subsection 5.2 below. Hence our approach provides a unification of different notions in this context as well.

Our main results in the text derive the sublinear martingale expectation as a supremum over boundedly additive measures. In applications, one usually needs countably additive measures in order to profit from the powerful convergence theorems of measure theory. In Appendix E, we show how to obtain such a representation in general discrete time markets. The appendix also discusses further extensions as, e.g., the equivalence of absence of arbitrage and absence of free lunches with vanishing risk, or the question if an optimal superhedge for a given claim exists.

The paper is set up as follows. The next section introduces the general model, the class of potential preferences, and the notion of relevant contracts. Section 3 proves equivalence of a suitable notion of arbitrage and viability under Knightian uncertainty. Section 4 introduces the notion of sublinear martingale expectation with full support. With the help of this concept, we introduce a new version of the Fundamental Theorem of Asset Pricing: absence of arbitrage is equivalent to the existence of a sublinear martingale expectation with full support. Section 5 shows how to apply the general theorems to various environments, ranging from finite models over probabilistic environments to Knightian uncertainty. The remaining sections discuss the proofs of the main theorems and provide further extensions.

measures. If the class of probability measures describing Knightian uncertainty is not dominated by a single probability measure, the quasi-sure ordering is more incomplete than any almost sure ordering.

³Duality in this context and also the related backward stochastic differential equations were introduced in a series of papers by Soner, Touzi, and Zhang (2012, 2013).

2 The Financial Market

2.1 Notations.

Let (Ω, \mathcal{F}) be any measurable space and $\mathcal{L} := \mathcal{L}^0(\Omega, \mathcal{F})$ be the set of all real-valued, measurable random variables on (Ω, \mathcal{F}) . Any financial contract X then takes values in the set \mathcal{L} representing the sum of all future payments in terms of a real-valued numéraire minus its initial cost. Usually, we use the money–market account as the numéraire.

For a given constant $c \in \mathbb{R}$, we let \mathbf{c} be the contract that is identically equal to the constant c . The notation \geq_Ω is reserved for the pointwise order on \mathcal{L} , namely, $X \leq_\Omega Y$ if and only if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. Set

$$\mathcal{B}_b := \{X \in \mathcal{L} : \|X\|_\infty < \infty\}, \quad \text{where} \quad \|X\|_\infty := \sup_{\omega \in \Omega} |X_t(\omega)|.$$

Let ba be the topological dual of \mathcal{B}_b equipped with the sup-norm. Then, $\varphi \in ba$ implies that φ is a bounded, finitely additive measure on the sigma algebra \mathcal{F} .

We follow the standard notation (see for instance Aliprantis and Border (1999)) and write \mathcal{B}' for the topological dual of a Banach space \mathcal{B} . We let \mathcal{B}'_+ be the set of all positive functionals, i.e., $\varphi \in \mathcal{B}'_+$ provided that $\varphi(X) \geq 0$ for every $X \geq_\Omega 0$ and $X \in \mathcal{B}$. For $\varphi \in \mathcal{B}'_+$ with $\mathcal{B}_b \subset \mathcal{B}$, the value $\varphi(X)$ is well defined for $X \geq_\Omega 0$, with values in $[0, \infty]$. Indeed,

$$\varphi(X) = \lim_{K \uparrow \infty} \varphi(X \wedge K), \quad \forall X \geq_\Omega \mathbf{0}, X \in \mathcal{L}.$$

For a general $X \in \mathcal{L}$, set $X^+ := X \vee 0$, $X_- := -X \vee 0$ and define

$$\varphi(X) := \varphi(X^+) - \varphi(X_-),$$

where in above and what follows we use the convention

$$\infty - \infty = -\infty.$$

If both terms are finite we say $X \in \mathcal{L}^1(\Omega, \varphi)$. For a measurable set $A \subset \Omega$, we write $\varphi(A)$ instead of $\varphi(\chi_A)$.

2.2 Outcomes and Contracts

The set Ω represents all possible *uncertain outcomes or states of the world*.

The set of all (*financial*) *contracts or net trades* (also *contingent claims*) is a given vector space $\mathcal{H} \subset \mathcal{L}$ containing all constant functions. We assume that the space of contingent claims \mathcal{H} is an *ordered vector space* with a partial order \leq . By definition this order is compatible with the vector space operations, i.e.,

$$X \leq Y \quad \Rightarrow \quad \begin{array}{l} X + F \leq Y + F \\ \lambda X \leq \lambda Y \end{array}, \quad \forall F \in \mathcal{H} \quad \text{and} \quad \forall \lambda \geq 0.$$

We also assume \leq coincides with the usual order on $\mathbb{R} \subset \mathcal{H}$.

The order \leq is interpreted as the objective order; every conceivable agent agrees that Y is to be preferred to X if $X \leq Y$. By definition, the order \leq is *convex* and *transitive*.

We write $X \sim Y$ whenever $X \leq Y$ and $Y \leq X$, and write $X < Y$ if $X \leq Y$ and $Y \not\leq X$. As \mathcal{H} contains all constant contracts, the order \leq is *strictly increasing in positive constant contracts*.

We then say:

- ▷ $Z \in \mathcal{H}$ is *negligible* if $Z \sim \mathbf{0}$;
- ▷ $P \in \mathcal{H}$ is *non-negative* if $P \geq \mathbf{0}$ and *positive* if $P > \mathbf{0}$.

We let \mathcal{Z} be the set of all negligible contracts, \mathcal{P} denotes the set of all non-negative contracts and \mathcal{P}^+ is the set of positive ones. It is clear that the zero contract $\mathbf{0}$ belongs to \mathcal{Z} . Also, since \leq coincides with usual order on \mathbb{R} , $\mathbf{c} \in \mathcal{P}$ for every $c \geq 0$. Moreover, $\mathbf{c} \in \mathcal{P}^+$ when $c > 0$. It is clear that \mathcal{P} , \mathcal{P}^+ are convex cones and \mathcal{Z} is a subspace. Moreover, $Z \in \mathcal{Z}$ if and only if $Z, -Z \in \mathcal{P}$.

2.3 Preferences

Let \mathcal{A} be a set of monotone (with respect to \leq), weakly continuous and convex preference relations on \mathcal{H} , namely, \mathcal{A} is the set of all complete and transitive binary relations \preceq on \mathcal{H} satisfying,

- ▷ $X \leq Y$ implies $X \preceq Y$;
- ▷ the upper contour set of \preceq is convex, i.e.,

$$F \preceq X \text{ and } F \preceq Y \quad \Rightarrow \quad F \preceq \lambda X + (1 - \lambda)Y, \quad \forall \lambda \in [0, 1];$$

- ▷ \preceq is *weakly continuous*, i.e., for every sequence $\{c_n\} \subset \mathbb{R}_+$ with $c_n \downarrow 0$ we have

$$X - \mathbf{c}_n \preceq Y, \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad X \preceq Y, \quad X, Y \in \mathcal{H}.$$

The set \mathcal{A} can be interpreted as the set of all conceivable preferences for the given space of contingent claims or contracts \mathcal{H} . As we interpret \leq as the order which is unanimously agreed upon by all agents, it is clear that any contract that is negligible for all preference relations in \mathcal{A} should also be negligible for \leq .

Note that we could also revert the reasoning. From a given set of preferences \mathcal{A} , one can construct a set of negligible contracts \mathcal{Z}_u as follows:

$$\mathcal{Z}_u := \bigcap_{\preceq \in \mathcal{A}} \mathcal{Z}_{\preceq}, \quad \text{where } Z \in \mathcal{Z}_{\preceq} \Leftrightarrow X \preceq Z + X \preceq X, \quad \forall X \in \mathcal{H}.$$

Then, one defines a partial order \leq'_u on \mathcal{H} by,

$$X \leq'_u Y \quad \Leftrightarrow \quad \exists Z \in \mathcal{Z}_u \text{ such that } X \leq_{\Omega} (Y + Z). \quad (2.1)$$

(\mathcal{H}, \leq'_u) is an ordered vector space and the negligible set of \leq'_u is exactly⁴ \mathcal{Z}_u .

⁴In the same spirit, one could define a partial order $X \leq'_u Y \Leftrightarrow X \preceq Y$ for any $\preceq \in \mathcal{A}$. In general

2.4 Attainable claims

The set of contracts achievable with zero initial cost or in short, *achievable* contracts is a given convex cone \mathcal{I} . We sometimes refer to them as *zero cost investment opportunities*. We assume that all contracts are properly discounted with respect to a suitable numéraire.

The set \mathcal{I} models the liquidly traded contracts with zero initial cost. Indeed, when in a market a certain contract X is liquidly traded at time zero, for the price p_X , then the contract $\ell_X(\omega) := X(\omega) - p_X$ belongs to \mathcal{I} . Conversely, for any $\ell \in \mathcal{I}$, one may consider, for any constant $c \in \mathbb{R}$, c as the price of the contract $X_\ell(\omega) := \ell(\omega) + c$.

Example 2.1. In a finite discrete time financial model with M stocks with discounted prices process $S_0, S_1, \dots, S_N \in \mathbb{R}_+^M$, one may take the set of achievable contracts to be,

$$\mathcal{I} = \{(H \cdot S)_N : H \text{ is a predictable process}\}, \quad (2.2)$$

where $(H \cdot S)_0 = 0$ and for $t = 1, \dots, N$,

$$(H \cdot S)_t := \sum_{k=1}^t H_k \cdot (S_k - S_{k-1}).$$

In continuous time one can define,

$$\mathcal{I} = \left\{ \int_0^T \theta_u \cdot dS_u : \theta \in \mathcal{A}_{adm} \right\},$$

where \mathcal{A}_{adm} is a suitable set of strategies that one might call *admissible* strategies. There are several possible choices of such a set. When the stock price process S is a semimartingale one example of \mathcal{A}_{adm} is the set of all S -integrable, predictable processes whose integral is bounded from below.⁵ Other choices for \mathcal{A}_{adm} are studied in the literature as well. When S is a continuous process and \mathcal{A}_{adm} is the set of process with finite variation then the above integral can be defined through integration by part (see Dolinsky and Soner (2014a, 2015)). Also, in addition to dynamic trading, in recent studies of robust hedging, static hedges are also used.

2.5 Relevant Contracts

In the previous section, we defined the notion of negligibility through the partial order. To complete the theory, we also need to identify a set of contracts that are unanimously considered to be desirable. We call these contracts *relevant*. It is clear that all constant contracts \mathbf{c} with $c > 0$ should be relevant. It is also clear that any relevant contract

this will not define an ordered vector space (\mathcal{H}, \leq') . The analysis of the paper carries over with minor modifications.

⁵In continuous time, to avoid doubling strategies a lower bound (maybe more general than above) has to be imposed on the stochastic integrals. In such cases, the set \mathcal{I} is not a linear space.

should be positive. In fact, one possible choice of relevant sets would be the set \mathcal{P}^+ . However, it is quite possible that not every positive contract would be assessed as relevant by the market. Indeed, the set of relevant contracts are determined by the preferences of the participating agents. These observations lead us to postulate the existence of a set of *relevant* contracts, which we assume to be non-empty, convex, subset $\mathcal{R} \subset \mathcal{P}^+$ and such that contains all positive constant contracts⁶.

We call a pair (Θ, \mathcal{R}) with $\Theta := (\mathcal{H}, \leq, \mathcal{I})$ a *financial market*.

3 Viability and Arbitrage

In this section, we introduce the definitions of viability and arbitrage under Knightian uncertainty.

3.1 Viability

The definition of viability is an extension of the definition given by Harrison and Kreps (1979, p. 384) in the context of a dominating measure. Let \mathcal{A} be the set of preference relations introduced in the subsection 2.3. A price system in a financial market is viable if it can be derived from an economic equilibrium in which agents have preferences from \mathcal{A} .

Definition 3.1. Let (Θ, \mathcal{R}) be a financial market. We say that (Θ, \mathcal{R}) is *viable*, if there exists $\preceq' \in \mathcal{A}$ and a net trade vector $\ell^* \in \mathcal{I}$ satisfying

$$\ell + X \preceq' \ell^* + X, \quad \forall \ell \in \mathcal{I}, X \in \mathcal{H}, \quad (3.1)$$

$$\ell^* - R \prec' \ell^*, \quad \forall R \in \mathcal{R}. \quad (3.2)$$

Clearly, the first condition (3.1) is an equilibrium condition. Indeed, we require the optimality of ℓ^* among all other achievable contracts at all levels $X \in \mathcal{H}$. So the existence of such an optimal contract ℓ^* is a necessary condition for equilibrium. The second condition (3.2) replaces and weakens the classical monotonicity condition assumed in Harrison and Kreps (1979); Kreps (1981), where the constructed preference relation \preceq' is required to be strictly monotone in the direction of \mathcal{R} . Here, the strict monotonicity, is only required at the optimal ℓ^* and only in the direction $-R$. This is the main relaxation that allows for inclusion of Knightian uncertainty.

In a viable financial market, one can assume without loss of generality that ℓ^* is the zero contract by appropriately adjusting the preferences. Define a new preference relation \preceq' on \mathcal{H} by

$$X \preceq Y \quad \Leftrightarrow \quad X + \ell^* \preceq' Y + \ell^*.$$

⁶The set of relevant sets is in direct analogy with the cone K used in the seminal paper of Kreps (1981). We further comment on this in Remark 3.2 below. A similar notion is used in Burzoni, Frittelli, and Maggis (2016) where the notion of arbitrage is given using a chosen set \mathcal{S} and it is called “*de la classe S*”.

Additionally, we note that for any $\ell \in \mathcal{I}$, $\ell + \ell^* \in \mathcal{I}$ and hence by (3.1),

$$\ell + \ell^* + X \preceq' \ell^* + X, \quad \Leftrightarrow \quad \ell + X \preceq' X, \quad \ell \in \mathcal{I}, X \in \mathcal{H}.$$

Therefore, a market is viable if and only there is $\preceq \in \mathcal{A}$ satisfying,

$$\ell + X \preceq X \quad \text{and} \quad -R \prec \mathbf{0}, \quad \forall \ell \in \mathcal{I}, R \in \mathcal{R}, X \in \mathcal{H}. \quad (3.3)$$

So in what follows, without loss of generality, in a viable market we always take ℓ^* to be the zero contract.

The above definition of viability extends the one given by Kreps (1981). We continue by discussing this connection.

Remark 3.2 (Connection to Kreps (1981)). In Kreps (1981) one starts with a topological space \mathcal{X} and a linear map π defined on a subspace M . An instance of this framework is studied in Harrison and Kreps (1979), where $\mathcal{X} := \mathcal{L}^2(\Omega, \mathbb{P})$ for some reference probability \mathbb{P} . The additional important object is a cone K with the origin deleted.

In our setting, the space \mathcal{X} is the subspace \mathcal{H} and we do not make use of the topology. The cone K induces a partial order induced via

$$X \prec_K Y \quad \Leftrightarrow \quad Y - X \in K.$$

K is the positive cone with respect to this partial order.

In Kreps' paper, the set of tradable contracts with zero initial cost is given by

$$\mathcal{I} = \{m \in M : \pi(m) = 0\}.$$

Suppose now that the market (\mathcal{X}, K, M, π) is viable in the sense of Kreps (1981). Then, it is clear that the financial market $(\mathcal{X}, \preceq_K, \mathcal{I}, \mathcal{R})$ with \mathcal{I} as above and $\mathcal{R} = K = \mathcal{P}^+$ is also viable in the sense defined in this manuscript. However, since we relax the strict monotonicity condition on the constructed preference relation, the opposite implication does not hold in general.

Hence, our structure directly extends definitions of Kreps (1981) when \mathcal{X} is a set of real-valued random variables⁷.

The strict monotonicity condition is the crucial change in our extension of viability.

Remark 3.3 (Non-linear extension). The viability with a strictly monotone preference relation as defined in Kreps (1981) is equivalent to the extension property. Indeed, one says that (π, M) has the extension property if there exists a continuous, linear functional φ that extends π to whole space \mathcal{X} and is strictly monotone in all directions $k \in K$, i.e.,

$$\varphi(m) = \pi(m), \quad \forall m \in M, \quad \text{and} \quad \varphi(k) > 0, \quad \forall k \in K.$$

⁷Kreps' set-up, however, is more general as it considers a general topological vector space \mathcal{X} . Our constructions and definitions extend in a straightforward manner to the more abstract Kreps' framework as well. We chose to present the theory in the more concrete framework for clarity.

When $\mathcal{X} = \mathcal{L}^2(\Omega, \mathbb{P})$, for some fixed probability measure, as in Harrison and Kreps (1979), this property leads to the existence of an equivalent martingale measure. However, as illustrated in Example 5.2 below, such a linear extension is not always possible. Indeed, in markets with Knightian uncertainty, the extension property holds only with a non-linear expectation. This is the reason for the weaker strict monotonicity used in our definition above.

3.2 Arbitrage

We continue by defining two notions of arbitrage.

Definition 3.4. Let (Θ, \mathcal{R}) be a financial market.

▷ We say that an achievable contract $\ell \in \mathcal{I}$, is an *arbitrage*, if there exists a relevant contract $R^* \in \mathcal{R}$ so that $\ell \geq R^*$.

▷ We say that a sequence of achievable contracts $\{\ell^n\}_{n=1}^\infty \subset \mathcal{I}$ is a *free lunch with vanishing risk*, if there exists a relevant contract $R^* \in \mathcal{R}$ and a sequence $\{c_n\}_{n=1}^\infty \subset \mathbb{R}_+$ with $c_n \downarrow 0$ satisfying,

$$c_n + \ell^n \geq R^*, \quad n = 1, 2, \dots$$

We say the financial market is *strongly free of arbitrages* and write $NA(\Theta, \mathcal{R})$ when (Θ, \mathcal{R}) has no free lunches with vanishing risk.

When $\mathcal{R} = \mathcal{P}^+$ the above definitions simplify. Indeed, in this case $\ell \in \mathcal{I}$ is an arbitrage if and only if $\ell \in \mathcal{P}^+$. Moreover, when the investment set \mathcal{I} represents a discrete-time market with finite horizon then the above two arbitrage conditions are equivalent as proved in Theorem D.6, below.

Clearly, the second definition is motivated by the notion of free lunch with vanishing risk introduced and completely characterized by Delbaen and Schachermayer (1998). Indeed, in the classical set-up (see Example 5.4 below) the weak order is given through \mathbb{P} almost sure inequalities with a fixed probability measure \mathbb{P} . Then, the above definition is exactly the same as the one given in Delbaen and Schachermayer (1998); see Example 5.5 for further discussion of this equivalence.

Our first main result establishes the equivalence between absence of arbitrage in our sense and economic viability.

Theorem 3.5. *A financial market (Θ, \mathcal{R}) is viable if and only if it is strongly free of arbitrage.*

The theorem is proved in Section A.

4 Sublinear Martingale Expectations with Full Support

Under risk, viability is equivalent to the existence of an equivalent martingale measure. In this section, we characterize viability by the existence of a suitable *sublinear* pricing

functional. The notion of equivalence is replaced by a full support property. Indeed, when the weak order is defined through a given probability measure \mathbb{P} , the equivalence of a linear pricing measure \mathbb{Q} to \mathbb{P} means that every set of positive probability with respect to \mathbb{P} needs to have a positive \mathbb{Q} probability as well. In the language of this paper, this property can be reformulated as: every relevant contract has a positive price. This property extends easily to sublinear expectations and in Definition 4.3 below, we say that a sublinear expectation has full support if every relevant contract has a positive sublinear expectation. Then, in Theorem 4.5 below, we show that the viability is equivalent to the existence of a sublinear martingale expectation with full-support.

In this section, to simplify the arguments we impose the following assumption⁸.

Assumption 4.1. The unanimous partial order \leq is consistent with \leq_Ω , i.e.,

$$X \leq_\Omega Y \quad \Rightarrow \quad X \leq Y, \quad \forall X, Y \in \mathcal{H}.$$

4.1 Sublinear Expectations

For the further characterization of viability through pricing functionals, we first need to recall and define several notions. Consider a general functional

$$\mathcal{E} : \mathcal{H} \rightarrow \mathbb{R} \cup \{-\infty, \infty\},$$

and set

$$U(X) = U_\mathcal{E}(X) := -\mathcal{E}(-X), \quad \forall X \in \mathcal{H}.$$

Definition 4.2. We say that a functional \mathcal{E} is a *sublinear expectation* if it is monotone with respect to \leq , translation-invariant, i.e.

$$\mathcal{E}(X + c) = \mathcal{E}(X) + c$$

for all constant contracts $c \in \mathbb{R}$, and if $U_\mathcal{E}$ is super-additive⁹. If, in addition, \mathcal{E} is positively homogeneous of degree one, we say that \mathcal{E} is a *coherent sublinear expectation*.

Sublinear expectations \mathcal{E} (resp. the corresponding concave version $U_\mathcal{E}$) arise naturally in the analysis of preferences under Knightian uncertainty (see Lemma 3.3 in Gilboa and Schmeidler (1989); translation-invariance is called c-independence therein).

The definition of a sublinear expectation uses only the structure of the weak order and not the financial market (Θ, \mathcal{R}) . Next, we use the sets \mathcal{Z} , \mathcal{R} and \mathcal{I} to define the notion of a sublinear martingale measure with full-support.

⁸We believe that it is possible to develop the theory without this assumption. However, such a theory would require routine but technical constructions such as the quotient space of \mathcal{B}_b modulo the negligible contracts, and its dual.

⁹When \mathcal{E} is finite valued, the super-additivity of $U_\mathcal{E}$ is equivalent to the sub-additivity of \mathcal{E} . However, when \mathcal{E} may take the values $\pm\infty$, they are not necessarily equivalent. Indeed, when $\mathcal{E}(X) = \infty$ and $\mathcal{E}(Y) = -\infty$, then $U_\mathcal{E}(-X) + U_\mathcal{E}(-Y) = -\infty + \infty = -\infty$ and the inequality $U_\mathcal{E}(-X) + U_\mathcal{E}(-Y) \leq U_\mathcal{E}(-X - Y)$ is immediate. This, however, is not the case with \mathcal{E} . Indeed, $\mathcal{E}(X) + \mathcal{E}(Y) = -\infty$ and one needs to verify the inequality $\mathcal{E}(X) + \mathcal{E}(Y) \geq \mathcal{E}(X + Y)$.

Definition 4.3. For a given financial market (Θ, \mathcal{R}) , we say that a functional \mathcal{E}

▷ is *absolutely continuous*, if $\mathcal{E}(Z) = 0$, for every $Z \in \mathcal{Z}$.

▷ has *full support*, if $\mathcal{E}(R) > 0$, for every $R \in \mathcal{R}$.

▷ has the *martingale property*, if $\mathcal{E}(\ell) \leq 0$ for every $\ell \in \mathcal{I}$.

We denote by $\mathcal{M}(\Theta, \mathcal{R})$ the class of sublinear expectations, which satisfies the properties listed above. $\mathcal{M}^c(\Theta, \mathcal{R})$ those which are, in addition, positively homogeneous.

We say that a given set of bounded linear functionals $\mathcal{Q} \subset ba(\Omega)$ is *absolutely continuous*, has *full support* or has the *martingale property*, if the induced Choquet capacity $\mathcal{E}_{\mathcal{Q}}(\cdot) := \sup_{\varphi \in \mathcal{Q}} \varphi(\cdot)$ satisfies the corresponding properties.

For any coherent sublinear expectation, absolute continuity with respect to \mathcal{Z} is always satisfied. This simple fact is proven in Lemma F.3 of the Appendix. Absolute continuity implies that $\mathcal{E}(\mathbf{0}) = 0$. In conjunction with monotonicity, we obtain $\mathcal{E}(X) \geq 0$ for all non-negative contracts $X \geq \mathbf{0}$.

In the above definitions, we use suggestive terminology. With a given reference probability measure, absolute continuity with respect to \mathcal{Z} is equivalent to the martingale measure being absolutely continuous with respect to the reference measure. In this case, the full support property is equivalent to the converse: the reference measure is absolutely continuous with respect to the risk-neutral measure when the set of relevant contracts is given by the almost surely positive contracts \mathcal{P}^+ .

Moreover, in the classical examples, \mathcal{I} is usually the set of stochastic integrals. In this context, the condition $\mathcal{E}(\ell) \leq 0$ is equivalent to the martingale property.

4.2 Viability and the Fundamental Theorem

To simplify the exposition of the main results, we first discuss viability and the fundamental theorem of asset pricing for bounded contracts. The extension to more general lower bounded contracts is discussed in Section C.

Let ba_+^1 be the set of all positive linear functionals φ on \mathcal{B}_b which are probabilities, i.e., $\varphi(\Omega) = 1$. We first define the natural generalization of risk neutral measures.

Definition 4.4. We say that $\varphi \in ba_+^1$ is a *martingale measure* if it satisfies

- ▷ $\varphi(P) \geq 0$, for all $P \in \mathcal{P}$,
- ▷ $\varphi(\ell) \leq 0$ for all $\ell \in \mathcal{I}$.

We denote by $\mathcal{Q}(\Theta)$ the set of all martingale measures and define the induced Choquet capacity by,

$$\mathcal{E}_{\Theta}(X) := \sup_{\varphi \in \mathcal{Q}(\Theta)} \varphi(X), \quad X \in \mathcal{B}_b.$$

Since any $Z \in \mathcal{Z}$ satisfies $Z, -Z \in \mathcal{P}$, we conclude that for any risk neutral measure $\varphi(Z) = 0$. Hence, any linear functional $\varphi \in \mathcal{Q}(\Theta)$ is also absolutely continuous, positively homogeneous and has the martingale property. However, any single element of

$\mathcal{Q}(\Theta)$ may not have full support and to assure it, one needs to consider the sublinear expectation, namely the Choquet \mathcal{E}_Θ capacity generated by $\mathcal{Q}(\Theta)$.

The following is the characterization of viability and strong no-arbitrage when all contracts are bounded. An extension to lower bounded contracts is given in Theorem C.7 below.

Theorem 4.5 (Fundamental Theorem of Asset Pricing). *Suppose $\mathcal{H} = \mathcal{B}_b$. Then, the following are equivalent:*

1. (Θ, \mathcal{R}) is viable.
2. (Θ, \mathcal{R}) is strongly free of arbitrage.
3. There exists a sublinear martingale expectation with full support.
4. The set of martingale measures $\mathcal{Q}(\Theta)$ is non-empty and the Choquet capacity \mathcal{E}_Θ is a coherent sublinear martingale expectations with full support.

The proof of the above theorem is given in Section B.2 below.

5 The Efficient Market Hypothesis and Robust Finance

In this section, we discuss several examples of $\Theta = (\leq, \mathcal{I})$. We start with two simple examples to illustrate the definitions. Then, in the two subsections, we show how the standard examples of the literature fit into this framework.

Example 5.1 (The atom of finance). This simple one-step binomial model, consists of two states of the world, $\Omega = \{1, 2\}$. Then any element $X \in \mathcal{L}$ is a real-valued function of $\{1, 2\}$. Hence, \mathcal{L} is isomorphic to \mathbb{R}^2 . Let \leq be the usual partial order of \mathbb{R}^2 . Then, $\mathcal{Z} = \{\mathbf{0}\}$ and $p \in \mathcal{P}$ if and only if $p \geq_\Omega 0$.

We assume that there is a riskless asset B and a risky asset S . Both assets have value $B_0 = S_0 = 1$ at time zero. The riskless asset yields $B_1 = 1 + r$ for an interest rate $r > -1$ at time one, whereas the risky asset takes the values u in state 1 and respectively d in state 2 with $u > d$.

We use the riskless asset B as numéraire. The discounted return on the risky asset is $\hat{\ell} := S_1/(1+r) - 1$. \mathcal{I} is the linear space spanned by $\hat{\ell}$. One can directly check there is no arbitrage if and only if the unique candidate for a full support martingale probability of state one

$$p^* = \frac{1 + r - d}{u - d}$$

belongs to $(0, 1)$ which is equivalent to $u > 1 + r > d$. The market is viable with the preference relation induced by the linear utility

$$U(X) := \mathbb{E}^*[X] = p^* X(1) + (1 - p^*) X_1(2).$$

Then, $X \preceq Y$ if and only if $U(X) \leq U(Y)$. Indeed, under this preference $\ell \sim \mathbf{0}$ for any $\ell \in \mathcal{I}$ and $X - R \prec X$ for any $X \in \mathcal{L}$ and $R \in \mathcal{P}^+$. In particular, any $\ell \in \mathcal{I}$ is an optimal portfolio and the market is viable.

The preceding analysis carries over to all finite Ω and complete financial markets.

Example 5.2 (Highly incomplete one-period models). In contrast to the preceding example of finite complete markets, we consider now an incomplete model.

Let $\Omega = [0, 1]$ and \leq be the usual pointwise partial order. Then, X is any Borel function on Ω . Again as in the previous example, $\mathcal{Z} = \{0\}$ and $P \in \mathcal{P}$ if and only if $P \geq_{\Omega} 0$. Assume that there is a riskless asset with interest rate $r \geq 0$ and one risky asset with $S_1(\omega) = 2\omega$ and $S_0 = 1$. We define \mathcal{I} as in the previous example as well. It is clear that there are uncountably many risk neutral measures on this market. Any probability measure \mathbb{Q} satisfying $\int_{\Omega} 2\omega \mathbb{Q}(d\omega) = 1 + r$ defines a risk neutral measure. The market is viable with the preference relation induced by the utility function,

$$U(X) := \inf_{\mathbb{Q} \in \mathcal{Q}(\Theta)} \mathbb{E}^{\mathbb{Q}}[X], \quad \forall X \in \mathcal{L}.$$

Here, $\mathcal{Q}(\Theta)$ is the set of all martingale measures. Indeed, $U(\ell) = U(-\ell) = 0$ for any $\ell \in \mathcal{I}$. Hence, as in the previous example, $\ell \sim \ell^*$ for any $\ell^*, \ell \in \mathcal{I}$. Then, for any $R \geq 0$,

$$\ell^* + \ell - R \preceq \ell^* + \ell \sim \ell^*.$$

Therefore, (3.1) holds with any $\ell^* \in \mathcal{I}$.

Moreover, $R \in \mathcal{P}^+$ if and only if $R \geq 0$ and there is $\omega^* \in \Omega$ so that $R(\omega^*) > 0$. Given such a R , we define \mathbb{Q}^* by

$$\mathbb{Q}^* := \frac{1}{2} [\delta_{\{\omega^*\}} + \delta_{\{1-\omega^*\}}].$$

Then, $\mathbb{Q}^* \in \mathcal{Q}(\Theta)$ and for any $\ell^* \in \mathcal{I}$,

$$\begin{aligned} U(\ell^* - R) &\leq \mathbb{E}^{\mathbb{Q}^*}[\ell^* - R] = \mathbb{E}^{\mathbb{Q}^*}[\ell^*] - \mathbb{E}^{\mathbb{Q}^*}[R] = -\mathbb{E}^{\mathbb{Q}^*}[R] \\ &= -\frac{1}{2}R(\omega^*) - \frac{1}{2}R(1 - \omega^*) < 0 = U(\ell^*). \end{aligned}$$

Hence, $\ell^* - R \prec \ell^*$ and the monotonicity condition (3.2) is satisfied with any $\ell^* \in \mathcal{I}$.

This example shows the necessity to work with nonlinear expectations to characterise no arbitrage. Indeed, with $\mathcal{R} = \mathcal{P}^+$ there is no single linear martingale probability measure which is strictly increasing in \mathcal{P}^+ . Indeed, such a measure would have to assign a non-zero value to every point. Hence, the equivalence “no arbitrage” to “there is a martingale measure with some property” does not hold true if one insists on having only one martingale measure.

5.1 The Efficient Market Hypothesis

In this subsection, we consider examples in which the preferences are given through a single probability measure or a family of probability measures together with Bernoulli utility functions. In all examples, we use the riskless asset as numéraire.

Example 5.3 (The Strong Efficient Market Hypothesis). In its original version, the efficient market hypothesis postulates that the “real world probability” or historical measure P is itself a martingale measure. The “efficient market hypothesis” goes back to Fama (1970). We can obtain this conclusion if we consider the following partial order.

Let \mathcal{F} be a sigma algebra on Ω and \mathbb{P} be a probability measure on (Ω, \mathcal{F}) . Set $\mathcal{H} = \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. We say $X \leq Y$ if

$$\mathbb{E}^{\mathbb{P}}[X] \leq \mathbb{E}^{\mathbb{P}}[Y].$$

Then, \mathcal{Z} is the set of all functions with mean zero. Also, $P \in \mathcal{P}$ if $\mathbb{E}^{\mathbb{P}}[P] \geq 0$.

If $\mathbb{Q} \in \mathcal{Q}(\Theta)$, we have $\mathbb{E}_{\mathbb{Q}}[X] \geq 0$, whenever $\mathbb{E}_{\mathbb{P}}[X] \geq 0$. This is a strong condition and implies that $\mathbb{Q} = \mathbb{P}$. Hence the only possible risk neutral measure is the historical measure itself.

In this setting, absence of arbitrage is equivalent to \mathbb{P} being the only martingale measure. We thus obtain the strong version of Fama’s efficient market hypothesis if we are willing to make the strong assumption that all potential agents of the economy order contracts by the expected return under the real world measure.

Example 5.4 (Weak Efficient Market Hypothesis). In its weak form, the efficient market hypothesis just states that expected returns of all portfolios all are equal under some probability measure. It can be derived in our framework as follows.

Let $(\mathcal{H}, \Omega, \mathbb{P})$ be as in the previous example. In this example, we assume that the agents are risk averse and the preference relation given by the partial order induced by \mathbb{P} , i.e.,

$$X \leq_{\mathbb{P}} Y \quad \Leftrightarrow \quad \mathbb{P}(X \leq Y) = 1.$$

This order can be derived from the von Neumann-Morgenstern utilities. Indeed, for any non-decreasing and concave function U , define

$$X \preceq_U Y \quad \Leftrightarrow \quad \mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)].$$

It is well known that this order coincides with second order stochastic dominance. A random variable Y dominates 0 in the sense of second order stochastic dominance if and only if it is P -almost surely nonnegative.

In this example, the notions of negligibility and positivity are also given by the order on \mathcal{H} induced by the probability measure \mathbb{P} .

The typical choice for \mathcal{R} in this and the previous example is the following,

$$\mathcal{R} = \{R \in \mathcal{L}^1(\Omega, \mathcal{R}, \mathbb{P}) : \mathbb{P}(R > 0) > 0\}.$$

Depending on the tradable set \mathcal{I} , one recovers the frameworks of all classical papers with one dominating measure, in particular, the results of Harrison and Kreps (1979), Dalang, Morton, and Willinger (1990) and Delbaen and Schachermayer (1998). In non-technical terms the absence of arbitrage is equivalent to the existence of an equivalent martingale measure.

Example 5.5 (Continuous Time). In Delbaen and Schachermayer (1998), the ordered vector space is given by $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. A locally bounded, semimartingale S is given as stock price process. Then, \mathcal{I} is the set of all stochastic integrals $(H \cdot S)_t := \int_0^t H_u dS_u$ with a predictable integrand H so that $H \cdot S$ is uniformly bounded from below.

Delbaen & Schachermayer calls a sequence of stochastic integrals $f_n := (H \cdot S)_\infty$ a *free lunch with vanishing risk* if f_n^- converges to zero uniformly and f_n converges \mathbb{P} -almost surely to a random variable f that satisfies $f \geq 0$, \mathbb{P} -almost surely but not equal to zero, i.e., $f \in \mathcal{R}$ as defined in Example 5.4.

We claim that this definition is the same as the one given in Definition 3.4 above.

Indeed, for $\epsilon > 0$ and a positive integer m set,

$$A_{n,\epsilon} := \bigcap_{m=n}^{\infty} \{f_n \geq f - \epsilon\}, \quad R_{n,\epsilon} := (f - \epsilon) \chi_{A_{n,\epsilon}} \chi_{\{f \geq 2\epsilon\}}.$$

Since $f \geq 0$, \mathbb{P} -almost surely and $\mathbb{P}(f > 0) > 0$, and since f_n converges to f , \mathbb{P} -almost surely, there is $\epsilon^* > 0$, sufficiently small and n^* sufficiently large so that

$$R^* := R_{n^*,\epsilon^*} \quad \text{satisfies} \quad R^* \geq 0, \quad \text{and} \quad \mathbb{P}(R^* > \epsilon^*) > 0.$$

Set $c_n := \|f_n^-\|_\infty$. Then, for all $n \geq n^*$, $c_n + f_n \geq R^*$. Hence f_n is a free lunch with vanishing in the sense defined in Definition 3.4. The other implication follows directly. Hence our notion and the definition given in Delbaen and Schachermayer (1998) agree.

The deep analysis of Delbaen and Schachermayer (1998) shows that the intersection of $\mathcal{Q}(\Theta)$ with countably additive measures is non-empty.

Example 5.6 (Strong Efficient Market Hypothesis under Knightian Uncertainty). Fix a sigma algebra \mathcal{F} and let \mathcal{M} be a given set of probability measures on (Ω, \mathcal{F}) . Set

$$\mathcal{H} := \bigcap_{\mathbb{P} \in \mathcal{M}} \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \quad \mathcal{E}_{\mathcal{M}}[X] := \inf_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}}[X], \quad X \in \mathcal{H}.$$

The partial order is induced by the Choquet capacity (or nonlinear expectation) $\mathcal{E}_{\mathcal{M}}$, i.e.,

$$X \geq Y \quad \Leftrightarrow \quad \mathcal{E}_{\mathcal{M}}(X - Y) \geq 0.$$

Then, $Z \in \mathcal{Z}$ if $\mathbb{E}^{\mathbb{P}}[Z] = 0$ for every $\mathbb{P} \in \mathcal{M}$. A contract P is positive if $\mathbb{E}^{\mathbb{P}}[P] \geq 0$ for every $\mathbb{P} \in \mathcal{M}$ or equivalently

$$P \in \mathcal{P} \quad \Leftrightarrow \quad 0 \leq \inf_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}}[P].$$

As in Example 5.3, one can directly show that the Choquet capacity induced by \mathcal{M} and $\mathcal{Q}(\Theta)$ are equal.

If in addition, we assume that \mathcal{I} is a vector space and not just a cone, then there is absence of arbitrage if and only if all relevant contracts are symmetric martingales under $\mathcal{E}_{\mathcal{Q}(\Theta)} = \mathcal{E}_{\mathcal{M}}$, i.e.

$$\mathcal{E}_{\mathcal{M}}(\ell) = \mathcal{E}_{\mathcal{M}}(-\ell) = 0.$$

A possible choice for \mathcal{R} is,

$$R \in \mathcal{R} \Leftrightarrow 0 \leq \inf_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}}[R] \text{ and } 0 < \sup_{\mathbb{P} \in \mathcal{M}} \mathbb{E}^{\mathbb{P}}[R].$$

Then, $\mathcal{E}_{\mathcal{M}}$ satisfies

$$\mathcal{E}_{\mathcal{M}}(\ell - R) < \mathcal{E}_{\mathcal{M}}(\ell), \quad \forall \ell \in \mathcal{I}.$$

Therefore, the Knightian uncertainty as described by the set of priors \mathcal{M} induces a sublinear expectation. With the weak order as defined above, we thus obtain that expected returns are the same under all priors in \mathcal{M} .

Example 5.7 (Weak Efficient Market Hypothesis under Knightian Uncertainty). This example is the analogue of the extension of Example 5.3 to Example 5.4. Indeed, let the basic structure to be as in the previous example. The partial order is given by the \mathcal{M} quasi-sure ordering, i.e.,

$$X \leq Y \Leftrightarrow \mathbb{P}(X \leq Y) = 1, \quad \forall \mathbb{P} \in \mathcal{M}.$$

Similarly as in Example 5.4, this partial order is also induced, through the construction (2.1), by the family of Gilboa Schmeidler utilities,

$$X \preceq_U Y \Leftrightarrow \mathcal{E}_{\mathcal{M}}[U(X)] \leq \mathcal{E}_{\mathcal{M}}[U(Y)].$$

Indeed, one can analogously prove that $0 \preceq_U Y$ for any concave and non-decreasing U if and only if $0 \leq Y$ \mathcal{M} -quasi-surely, that is, \mathbb{P} -almost surely for every $\mathbb{P} \in \mathcal{M}$.

The multi-step financial market with \mathcal{I} as in (2.2) is studied in Bouchard and Nutz (2015). In that paper, the set of relevant contracts are given by,

$$\mathcal{R} = \{R \in \mathcal{P} : \exists \mathbb{P} \in \mathcal{M} \text{ such that } \mathbb{P}(R > 0) > 0 \}.$$

In this setting, there exists a set of probability measures $\mathcal{Q} = \mathcal{Q}(\Theta)$ which is equivalent to \mathcal{M} such that all traded contracts are symmetric martingales under the sublinear expectation induced by \mathcal{Q} .

Example 5.8 (Smooth Ambiguity). In this example, we consider the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005).¹⁰

Let \mathcal{F} be a sigma algebra on Ω and $\mathfrak{P} = \mathfrak{P}(\Omega)$ the set of all probability measures on (Ω, \mathcal{F}) . In this example, we assume that the agents are risk averse and ambiguity averse. The preference relation is given by,

$$X \leq Y \Leftrightarrow \mu(\{P \in \mathfrak{P} : \mathbb{P}(X \leq Y) = 1\}) = 1,$$

¹⁰A recent manuscript of Cuchiero, Klein, and Teichmann (2017) also discusses a similar model but also with a nontrivial information structure.

where μ is a probability measure on \mathfrak{P} representing how plausible are the different priors in the market, according to the agents' preferences.

Then, a risk neutral measure would have the form

$$\mathcal{Q}(A) = \int_{\mathfrak{P}} \mathbb{P}(A) \nu(d\mathbb{P}), \quad A \in \mathcal{F},$$

for some measure ν absolutely continuous with respect to μ . Moreover, suppose the market has a discounted stock price process. Then, under suitable integrability conditions on S , ν has to satisfy the martingale condition,

$$\int_{\mathfrak{P}} \mathbb{E}^{\mathbb{P}}(S_t \mid \mathcal{F}_u) \nu(d\mathbb{P}) = S_u,$$

for every $0 \leq u \leq t \leq T$.

5.2 On Recent Results in Mathematical Finance

In this subsection, Ω is a metric space and $\mathcal{H} = \mathcal{L}$. We say $X < Y$ if

$$\inf_{\Omega} X < \inf_{\Omega} Y, \quad (5.1)$$

which implies $\mathcal{Z} = \{\mathbf{0}\}$. Also, a contract is non-negative, $P \in \mathcal{P}$, if $P(\omega) \geq 0$ for every $\omega \in \Omega$ and $R \in \mathcal{P}^+$ if $R \in \mathcal{P}$ and there exists $\omega_0 \in \Omega$ such that $R(\omega_0) > 0$. This approach is called *model-independent* as it considers all points of Ω likely events as the only negligible set is the constant zero.

In the literature several different notions of arbitrage have been used. Indeed, in our context this corresponds different choices of the set of relevant contracts. It is our view that all these different definitions simply depend on the agents perception of relevance. In particular, this approach in the model-independent setting is analogous to the relevant sets considered in Burzoni, Frittelli, and Maggis (2016) and called there “de la class \mathcal{S} ”.

We continue by outlining some of these choices and briefly discussing their consequences. We start with the following large set of relevant contracts

$$\mathcal{R}_{op} := \mathcal{P}^+ = \{P \in \mathcal{P} : \exists \omega_0 \in \Omega \text{ such that } P(\omega_0) > 0\}.$$

Then, an investment opportunity ℓ is an arbitrage if $\ell(\omega) \geq 0$ for every ω and is strictly positive at least at one point. This agrees with the notion of *one point arbitrage* considered in Riedel (2015). In this setting, no arbitrage is equivalent to the existence a set of martingale measures \mathcal{Q}_{op} so that for each point there exists $\mathbb{Q} \in \mathcal{Q}_{op}$ putting positive mass to that point.

In a second example, one requires the relevant contracts to be continuous, i.e.,

$$\mathcal{R}_{open} := \{P \in C_b(\Omega) \cap \mathcal{P} : \exists \omega_0 \in \Omega \text{ such that } P(\omega_0) > 0\}.$$

It is clear that when $p \in \mathcal{R}$ then it is non-zero on an open set. Hence, in this example the empty set is the only small set and the large sets are the ones that contain a non-empty open set.

Then, $\ell \in \mathcal{I}$ is an arbitrage opportunity if it is nonnegative and is strictly positive on an open set. This agrees with the notion of *open arbitrage* appeared in Burzoni, Frittelli, and Maggis (2016); Dolinsky and Soner (2014b); Riedel (2015).

Acciaio, Beiglböck, Penkner, and Schachermayer (2016) defines a contract to be an arbitrage when it is positive everywhere. In our context, this defines the relevant contracts to be the ones which are positive everywhere, i.e.,

$$\mathcal{R}_+ := \{P \in \mathcal{P} : P(\omega) > 0, \forall \omega \in \Omega\}.$$

Bartl, Cheridito, Kupper, and Tangpi (2017) considered a slightly stronger notion of relevant contracts. Their choice is

$$\mathcal{R}_u = \{P \in \mathcal{P} : \exists c \in (0, \infty) \text{ such that } P \equiv c\}. \quad (5.2)$$

Hence, $\ell \in \mathcal{I}$ is an arbitrage if it is uniformly positive. This is sometimes called *uniform arbitrage*. Notice that with the choice \mathcal{R}_u the notions of arbitrage and free lunch with vanishing risk are equivalent.

The no arbitrage condition with \mathcal{R}_u is the weakest while the one with \mathcal{R}_{op} is the strongest. The first one is equivalent to the existence of one sublinear martingale expectation. On the other hand, the strongest no arbitrage is equivalent to the existence of a sublinear expectation which puts positive measure to all points.

Also, in general, the no-arbitrage condition with \mathcal{R}_+ is not equivalent to no uniform arbitrage. However, the no-uniform arbitrage implies the existence of a linear bounded functional consistent with the market. In particular, the action of the risk neutral measures on \mathcal{R}_u are positive. Hence, they are positive and also have total mass one. Moreover, if the set \mathcal{I} is “large” enough then one can show that the risk neutral measures are in fact countably additive. Hence, their action on \mathcal{R}_+ is also positive. This fact implies the weaker no-arbitrage with the set \mathcal{R}_+ . In Acciaio, Beiglböck, Penkner, and Schachermayer (2016), this is achieved by using the so-called “power-option” placed in the set \mathcal{I} as a static hedging possibility. This implication, namely uniform no-arbitrage implying no-arbitrage with \mathcal{R}_+ has already proved in Bartl, Cheridito, Kupper, and Tangpi (2017).

Example 5.9. Suppose Ω be an uncountable set and let $\mathcal{H} = \mathcal{H}_0 = \mathcal{B}_b$. Let \mathcal{Z} be set the set of all countable sets. We define a weak order by

$$X \leq Y \quad \Leftrightarrow \quad \{\omega \in \Omega : X(\omega) > Y(\omega)\} \in \mathcal{Z}.$$

Then, for $X \in \mathcal{H}$ define

$$U(X) := \sup \{c \in \mathbb{R} : \{\omega \in \Omega : X(\omega) < c\} \text{ is countable}\}.$$

In this example, a set $N \subset \mathcal{H}$ is negligible if and only if N is countable. Then, \mathcal{P} is the set of all functions that are nonnegative except on a countable set and $P \in \mathcal{R}$ if

$P \in \mathcal{P}$ and the set $\{\omega \in \Omega : P(\omega) > 0\}$ is uncountable. Here one may simply take $\mathcal{R} = \mathcal{R}^+$. In this example, if we a choice of \mathcal{I} there is no arbitrage, then a sublinear expectation with full-support exists. However, it is not known if the set \mathcal{Z} is the polar set of a set of countably additive measures. So the intersection of the risk neutral measures $\mathcal{Q}(\Theta)$ with the intersection of countably additive measures may not have full-support property.

In summary, this example shows the necessity to extend the notion of a risk neutral measure to sublinear expectation possibly generated by finitely additive measures.

6 Conclusion

This paper reviews the idea of economic viability of a given financial market, the absence of arbitrage, and the existence of suitable nonlinear pricing expectations under Knightian uncertainty. Our setup is broad enough to cover all existing models, including finite state space models, and probabilistic models.

We show that one can understand the absence of arbitrage based on a common notion of “more” which is shared by all potential agents of the economy. The stronger the assumptions are that we are willing to make on this common order, the stronger are the consequences. If all agents order contracts by looking at the expected return under some fixed probability measure, economic viability and the absence of arbitrage require equality of expected returns. We thus get Fama’s Efficient Market Hypothesis. If agents only agree that contracts are to be preferred whenever they can be ordered almost surely under some probability measure, we obtain the weaker form of the efficient market hypothesis which leads to equal expected returns under some (martingale) measure which shares the same null sets as the reference probability. In situations of Knightian uncertainty, when agents can only agree on a class of probability measures, we have to replace the linear expectation by a sublinear expectation which has full support.

A Proof of Theorem 3.5

We start with the properties of the super-replication functional.

A.1 Super-replication Functional

The following functional plays a central role in our analysis. For any $X \in \mathcal{H}$, define the super-replication functional by,

$$\begin{aligned} \mathcal{D}(X, \Theta) &:= \inf \{ c \in \mathbb{R} : \exists \ell \in \mathcal{I}, \text{ such that } \mathbf{c} + \ell - X \in \mathcal{P} \} \\ &= \inf \{ c \in \mathbb{R} : \exists \ell \in \mathcal{I}, \text{ such that } \mathbf{c} + \ell \geq X \}. \end{aligned} \tag{A.1}$$

When the context is clear, we may omit the dependence of \mathcal{D} on Θ and write $\mathcal{D}(X)$. Also, following the standard convention, we set $\mathcal{D}(X)$ to plus infinity, when the above

set is empty. Note that \mathcal{D} is extended real valued. In particular, it takes the value $+\infty$ when there are no super-replicating portfolios. Also it can take the value $-\infty$ if there is no lower bound.

We first observe that the no free lunch with vanishing risk condition is completely characterized through this functional.

Proposition A.1. *The financial market (Θ, \mathcal{R}) has no free lunch with vanishing risk if and only if*

$$\mathcal{D}(R) > 0, \quad \forall R \in \mathcal{R}. \quad (\text{A.2})$$

Proof. Suppose $\{\ell^n\}_{n=1}^\infty \subset \mathcal{I}$ is a free lunch with vanishing risk. Then, there is $R^* \in \mathcal{R}$ so that $\mathbf{c}_n + \ell^n \geq R^*$, for some $c_n \downarrow 0$. In view of the definition, the super-replication functional $\mathcal{D}(R^*) \leq 0$.

To prove the converse suppose that $\mathcal{D}(R^*) \leq 0$ for some $R^* \in \mathcal{R}$. Then, the definition of $\mathcal{D}(R^*)$ implies that for each positive integer n , there is $\ell^n \in \mathcal{I}$ so that $\mathbf{1}/n + \ell^n \geq R^*$. Hence, $\{\ell^n\}_{n=1}^\infty$ is a free lunch with vanishing risk. \square

The condition of no-arbitrage, however, is not characterized through the super-replication functional. We only have the following implication,

$$(\Theta, \mathcal{R}) \text{ has no arbitrages} \quad \Rightarrow \quad \mathcal{D}(R, \Theta) \geq 0, \quad \forall R \in \mathcal{R}. \quad (\text{A.3})$$

Recall that $\mathcal{M}^c(\Theta, \mathcal{R})$ represents the set of full support, coherent, sublinear expectations with the martingale property, as in Definition 4.3.

Proposition A.2. *The super-replication functional \mathcal{D} defined in (A.1) is a coherent, sublinear expectation. If the financial market (Θ, \mathcal{R}) is strongly free of arbitrages, then \mathcal{D} has full-support with respect to \mathcal{R} , $\mathcal{D}(\mathbf{c}) = c$ for every $c \in \mathbb{R}$ and,*

$$\mathcal{D}(X + \ell) \leq \mathcal{D}(X), \quad \forall \ell \in \mathcal{I}, X \in \mathcal{H}. \quad (\text{A.4})$$

In particular, $\mathcal{D} \in \mathcal{M}^c(\Theta, \mathcal{R})$.

Proof. We prove this result in two steps.

Step 1. In this step we prove that \mathcal{D} is a sublinear expectation. Let $X, Y \in \mathcal{H}$ such that $X \leq Y$. Suppose that there are $c \in \mathbb{R}, \ell \in \mathcal{I}$ satisfying, $Y \leq \mathbf{c} + \ell$. Then, from the transitivity of \leq , we also have $X \leq \mathbf{c} + \ell$. Hence, $\mathcal{D}(X) \leq \mathcal{D}(Y)$ and consequently \mathcal{D} is monotone with respect to \leq .

Translation-invariance, $\mathcal{D}(\mathbf{c} + g) = c + \mathcal{D}(g)$, follows directly from the definitions.

We next show that the functional $U_{\mathcal{D}}(X) := -\mathcal{D}(-X)$ is super-additive, i.e., we claim that

$$U_{\mathcal{D}}(X) + U_{\mathcal{D}}(Y) \leq U_{\mathcal{D}}(X + Y), \quad \forall X, Y \in \mathcal{H}. \quad (\text{A.5})$$

Indeed, suppose that either $U_{\mathcal{D}}(X) = -\infty$ or $U_{\mathcal{D}}(Y) = -\infty$. Then, by our convention, $U_{\mathcal{D}}(X) + U_{\mathcal{D}}(Y) = -\infty$ and (A.5) follows directly. Now we consider the case

$U_{\mathcal{D}}(X), U_{\mathcal{D}}(Y) > -\infty$. Then, $\mathcal{D}(-X), \mathcal{D}(-Y) < \infty$. Hence, there are $c_X, c_Y \in \mathbb{R}$, $\ell^X, \ell^Y \in \mathcal{I}$ satisfying,

$$\mathbf{c}_X + \ell^X \geq -X, \quad \mathbf{c}_Y + \ell^Y \geq -Y.$$

Set $\bar{c} := c_X + c_Y$, $\bar{\ell} := \ell^X + \ell^Y$. Since \mathcal{I}, \mathcal{P} is a positive cone, $\bar{\ell} \in \mathcal{I}$ and

$$\bar{\mathbf{c}} + \bar{\ell} \geq -X - Y \quad \Rightarrow \quad \mathcal{D}(-X - Y) \leq \bar{c}.$$

Since this holds for any such c_X, c_Y , we conclude that

$$\mathcal{D}(-X - Y) \leq \mathcal{D}(-X) + \mathcal{D}(-Y), \quad \Rightarrow \quad U_{\mathcal{D}}(X) + U_{\mathcal{D}}(Y) \leq U_{\mathcal{D}}(X + Y).$$

Hence (A.5) holds in all cases and $U_{\mathcal{D}}$ is super-additive.

Finally we show that \mathcal{D} positively homogeneous of degree one. Suppose that $\mathbf{c} + \ell \geq X$ for some constant c , $\ell \in \mathcal{I}$ and $X \in \mathcal{H}$. Then, for any $\lambda > 0$, $\lambda\mathbf{c} + \lambda\ell \geq \lambda X$. Since $\lambda\ell \in \mathcal{I}$, this implies that

$$\mathcal{D}(\lambda X) \leq \lambda \mathcal{D}(X), \quad \lambda > 0, \quad X \in \mathcal{H}.$$

Notice that above holds trivially when $\mathcal{D}(X) = +\infty$. Conversely, if $\mathcal{D}(\lambda X) = +\infty$ we are done. Otherwise,

$$\mathcal{D}(X) = \mathcal{D}\left(\frac{1}{\lambda}\lambda X\right) \leq \frac{1}{\lambda}\mathcal{D}(\lambda X), \quad \Rightarrow \quad \lambda\mathcal{D}(X) \leq \mathcal{D}(\lambda X).$$

Hence, \mathcal{D} positively homogeneous and it is a coherent sublinear expectation.

Step 2. In this step, we assume that (Θ, \mathcal{R}) is strongly free of arbitrages. Since $\mathbf{0} \in \mathcal{I}$, we have $\mathcal{D}(\mathbf{0}) \leq 0$. If the inequality is strict we obviously have a strong arbitrage, hence $\mathcal{D}(\mathbf{0}) = 0$ and from translation-invariance the same applies to every $c \in \mathbb{R}$. Moreover, by Proposition A.1, \mathcal{D} has full support, as in Definition 4.3. Thus, we only need to prove (A.4).

Suppose that $X \in \mathcal{H}, \ell \in \mathcal{I}$ and $\mathbf{c} + \ell^* \geq X$. Since \mathcal{I} is a convex cone, $\ell^* + \ell \in \mathcal{I}$ and $\mathbf{c} + (\ell + \ell^*) \geq X + \ell$. Therefore, $\mathcal{D}(X + \ell) \leq c$. Since this holds for all such constants, we conclude that $\mathcal{D}(X + \ell) \leq \mathcal{D}(X)$ for all $X \in \mathcal{H}$. In particular $\mathcal{D}(\ell) \leq 0$ and, in view of the previous step, we also conclude that $\mathcal{D} \in \mathcal{M}^c(\Theta, \mathcal{R})$. \square

A.2 Proof of Theorem 3.5

Set $\mathcal{D}(\cdot) := \mathcal{D}(\cdot, \Theta)$.

The implication \Rightarrow .

Define a utility function U on \mathcal{H} by,

$$U(X) := U_{\mathcal{D}}(X) = -\mathcal{D}(-X), \quad X \in \mathcal{H}.$$

Let \preceq be the preference relation defined on \mathcal{H} through U , i.e.,

$$X \preceq Y \quad \Leftrightarrow \quad U(X) \leq U(Y).$$

We first show that $\preceq \in \mathcal{A}$. Recall that, from Proposition A.2, \mathcal{D} is a coherent, sub-linear expectation. It is then clear that U is monotone with respect to \leq and concave. Consequently, \preceq is also monotone and convex. Moreover, it is also clear that $U(\mathbf{c} + X) = c + U(X)$ for any constant $c \in \mathbb{R}$ and $X \in \mathcal{H}$. Now, suppose that $X, Y \in \mathcal{H}$, $\{c_n\} \subset \mathbb{R}_+$ with $c_n \downarrow 0$ satisfy $Y - \mathbf{c}_n \preceq X$. Then,

$$U(Y) - c_n = U(Y - \mathbf{c}_n) \leq U(X) \quad \forall n \quad \Rightarrow \quad U(Y) \leq U(X) \quad \Rightarrow \quad Y \preceq X.$$

Hence, \preceq is weakly continuous. This shows that $\preceq \in \mathcal{A}$.

Next we show viability. In view of (A.4), for any $X \in \mathcal{H}, \ell \in \mathcal{I}$,

$$U(X + \ell) = -\mathcal{D}(-[X + \ell]) \leq -\mathcal{D}(-[X + \ell] + \ell) = -\mathcal{D}(-X) = U(X).$$

Hence, $X + \ell \preceq X$ for any $X \in \mathcal{H}$ and $\ell \in \mathcal{I}$. Also strong no arbitrage assumption implies that $\mathcal{D}(R) > 0$ (Proposition A.1). Also, by Proposition A.2, $\mathcal{D}(\mathbf{0}) = 0$. Therefore,

$$U(-R) = -\mathcal{D}(R) < 0 = U(\mathbf{0}) \quad \Rightarrow \quad -R \prec \mathbf{0}.$$

We conclude that \preceq satisfies (3.3) and thus, (Θ, \mathcal{R}) is viable.

The implication \Leftarrow .

Suppose the market is viable. Towards a contradiction, let $\{\ell^n\}_{n=1}^\infty$ be a free lunch with vanishing risk. Then, there is a sequence of real numbers $c_n \in \mathbb{R}_+$ converging to zero and $R^* \in \mathcal{R}$, satisfying $R^* \leq \mathbf{c}_n + \ell^n$. This also implies that $-\mathbf{c}_n \leq \ell^n - R^*$. Since \preceq is monotone with respect to \leq , this implies $-\mathbf{c}_n \preceq \ell^n - R^*$. We now use (3.3) to arrive at

$$-\mathbf{c}_n \preceq \ell^n - R^* \preceq -R^*, \quad \forall n, \quad \Rightarrow \quad (\text{weak continuity}) \quad \mathbf{0} \preceq -R^*.$$

However, by (3.3), we also have $-R^* \prec \mathbf{0}$ which yields a contradiction. Hence, the financial market (Θ, \mathcal{R}) is strongly free of arbitrages. \square

B Proof of Theorem 4.5

The main tool in the proof is the dual representation of the super-replication functional.

B.1 Convex Duality

In this section $\mathcal{H} = \mathcal{B}_b$ with the uniform norm.

Lemma B.1. *Suppose that (Θ, \mathcal{R}) is strongly free of arbitrage. Then, the super-replication functional is Lipschitz continuous. In fact,*

$$|\mathcal{D}(X) - \mathcal{D}(Y)| \leq \|X - Y\|_\infty, \quad \forall X, Y \in \mathcal{H}$$

Moreover,

$$|\mathcal{D}(X)| \leq \|X\|_\infty, \quad \forall X \in \mathcal{H}.$$

Proof. For $X, Y \in \mathcal{B}_t$,

$$X \leq_{\Omega} Y + \|X - Y\|_{\infty}.$$

Hence,

$$\mathcal{D}(X) \leq \mathcal{D}(Y + \|X - Y\|_{\infty}) \leq \mathcal{D}(Y) + \|X - Y\|_{\infty}.$$

All of these imply the Lipschitz estimate. The second estimate follows from this by taking $Y = \mathbf{0}$. \square

Assume now that the financial market (Θ, \mathcal{R}) is strongly free of arbitrage. Then, by Proposition A.2, \mathcal{D} is an equivalent, coherent sublinear martingale with full-support. Moreover, Lemma B.1 implies that the super-replication functional,

$$\mathcal{D} : \mathcal{H} = \mathcal{B}_b \rightarrow \mathbb{R},$$

is a regular convex function in the language of convex analysis, Rockafellar (2015). Then, by the classical Fenchel-Moreau theorem Aliprantis and Border (1999), we have the following dual representation of \mathcal{D} ,

$$\begin{aligned} \mathcal{D}(X) &= \sup_{\varphi \in \mathcal{H}'} \{ \varphi(X) - \mathcal{D}^*(\varphi) \}, \quad X \in \mathcal{H}, \quad \text{where,} \\ \mathcal{D}^*(\varphi) &= \sup_{Y \in \mathcal{H}} \{ \varphi(Y) - \mathcal{D}(Y) \}, \quad \varphi \in \mathcal{H}'. \end{aligned}$$

Since $\varphi(\mathbf{0}) = \mathcal{D}(\mathbf{0}) = 0$, $\mathcal{D}^*(\varphi) \geq \varphi(\mathbf{0}) - \mathcal{D}(\mathbf{0}) = 0$ for every $\varphi \in \mathcal{H}'$. However, it may take the value plus infinity. Set,

$$\text{dom}(\mathcal{D}^*) := \{ \varphi \in \mathcal{H}' : \mathcal{D}^*(\varphi) < \infty \}.$$

We show below that the positive homogeneity of \mathcal{D} implies that \mathcal{D}^* is zero whenever it is finite.

Lemma B.2. *Suppose Θ satisfies Assumptions 4.1. Then, $\text{dom}(\mathcal{D}^*)$ is given by,*

$$\begin{aligned} \text{dom}(\mathcal{D}^*) &= \{ \varphi \in \mathcal{H}'_+ : \mathcal{D}^*(\varphi) = 0 \} \\ &= \{ \varphi \in \mathcal{H}'_+ : \varphi(X) \leq \mathcal{D}(X), \quad \forall X \in \mathcal{H} \}. \end{aligned}$$

In particular,

$$\mathcal{D}(X, \Theta) = \sup_{\varphi \in \text{dom}(\mathcal{D}^*)} \varphi(X), \quad X \in \mathcal{H}. \quad (\text{B.1})$$

Furthermore, there are free lunches with vanishing risk in (Θ, \mathcal{R}) , whenever $\text{dom}(\mathcal{D}^)$ is empty.*

Proof. The inclusion \supset is obvious. Let now $\varphi \in \text{dom}(\mathcal{D}^*)$ and suppose $X \in \mathcal{H}$ satisfies $X \geq_{\Omega} 0$. Since \leq is monotone with respect to \leq_{Ω} , $-X \leq 0$. Then, by the monotonicity of \mathcal{D} , $\varphi(-X) \leq \mathcal{D}(\mathbf{0}) \leq 0$. Hence, $\varphi \in \mathcal{H}'_+$.

The definition of \mathcal{D}^* implies that

$$\varphi(X) \leq \mathcal{D}(X) + \mathcal{D}^*(\varphi), \quad \forall X \in \mathcal{H}, \varphi \in \mathcal{H}'.$$

By homogeneity,

$$\varphi(\lambda X) \leq \mathcal{D}(\lambda X) + \mathcal{D}^*(\varphi), \quad \Rightarrow \quad \varphi(X) \leq \mathcal{D}(X) + \frac{1}{\lambda} \mathcal{D}^*(\varphi),$$

for every $\lambda > 0$ and $X \in \mathcal{H}$. Suppose that $\varphi \in \text{dom}(\mathcal{D}^*)$. We then let λ go to infinity to arrive at $\varphi(X) \leq \mathcal{D}(X)$ for all $X \in \mathcal{B}_b$. Since $\mathcal{D}^* \geq 0$, then, $\mathcal{D}^*(\varphi) = 0$.

Now suppose that $\text{dom}(\mathcal{D}^*)$ is empty or, equivalently, $\mathcal{D}^* \equiv \infty$. Then, the dual representation implies that $\mathcal{D} \equiv -\infty$. This holds, in particular for every constant contract \mathbf{c} with $c > 0$. In view of Proposition A.1, there are free lunches with vanishing risk in the market (Θ_l, \mathcal{R}) . \square

We continue by showing that $\text{dom}(\mathcal{D}^*)$ is indeed equal to the set of martingale measures $\mathcal{Q}(\Theta)$ defined in Definition 4.4. Recall that \mathcal{R}_u is defined in (5.2) and any other relevant set \mathcal{R} , by our assumption, contains \mathcal{R}_u . In particular, if (Θ, \mathcal{R}) is strongly free of arbitrages and so is (Θ, \mathcal{R}_u) .

Proposition B.3. *Suppose (Θ, \mathcal{R}_u) is strongly free of arbitrages. Then, the set of risk neutral measures $\mathcal{Q}(\Theta)$ is non-empty and it is equal to $\text{dom}(\mathcal{D}^*)$.*

Proof. Fix an arbitrary $\varphi \in \text{dom}(\mathcal{D}^*)$. By Proposition A.2, $\mathcal{D}(\mathbf{c}) = c$ for every constant $c \in \mathbb{R}$. In view of the dual representation of Lemma B.2,

$$c\varphi(\Omega) = \varphi(\mathbf{c}) \leq \mathcal{D}(\mathbf{c}) = c, \quad \forall c \in \mathbb{R}.$$

Hence, $\varphi(\Omega) = 1$.

We continue by proving the monotonicity property. Suppose that $D \in \mathcal{P}$. Since $\mathbf{0} \in \mathcal{I}$, we obviously have $\mathcal{D}(-D) \leq 0$. The dual representation implies that $\varphi(-D) \leq \mathcal{D}(-D) \leq 0$. Thus, $\varphi(D) \geq 0$.

We now prove the supermartingale property. Let $\ell \in \mathcal{I}$. Obviously $\mathcal{D}(\ell) \leq 0$. By the dual representation, $\varphi(\ell) \leq \mathcal{D}(\ell) \leq 0$.

Since $\varphi \in \text{dom}(\mathcal{D}^*)$ is arbitrary, these prove that for every $\varphi \in \text{dom}(\mathcal{D}^*)$, φ must satisfy the conditions of Definition 4.4.

To prove the converse, fix an arbitrary that $\varphi \in \mathcal{Q}(\Theta)$. Suppose that $X \in \mathcal{B}_b$, $c \in \mathbb{R}$, $\ell \in \mathcal{I}$ satisfy, $\mathbf{c} + \ell - X \in \mathcal{P}$. From the properties of φ ,

$$0 \leq \varphi(\mathbf{c} + \ell - X) = \varphi(\mathbf{c} - X) + \varphi(\ell) \leq c - \varphi(X).$$

Hence, $\varphi(X) \leq \mathcal{D}(X)$ for every $X \in \mathcal{B}_b$. Therefore, $\varphi \in \text{dom}(\mathcal{D}^*)$. \square

We have the following immediate corollary, which can be seen as the fundamental theorem in this context.

Corollary B.4. (Θ, \mathcal{R}) is strongly free of arbitrage if and only if $\mathcal{Q}(\Theta)$ is non-empty and has the full support property with respect to \mathcal{R} .

Proof. When $\mathcal{Q}(\Theta)$ is empty, or the full support property fails, Proposition A.1, Lemma B.2 and Proposition B.3 show that there are free lunches with vanishing risk in the market (Θ, \mathcal{R}) .

In the other direction, let $R \in \mathcal{R}$. By Proposition A.1, $\mathcal{D}(R) > 0$. Also, by Proposition B.3, $\text{dom}(\mathcal{D}^*) = \mathcal{Q}(\Theta)$. These imply that there exists $\varphi_R \in \mathcal{Q}(\Theta)$ satisfying $\varphi_R(R) > 0$. \square

Remark B.5. The set of positive probability measures $\mathcal{Q}(\Theta) \subset ba_+$ is the analogue of the set of local martingale measures of the classical setting. Indeed, all elements of $\varphi \in \mathcal{Q}(\Theta)$ can be regarded as a martingale, since $\varphi(\ell) \leq 0$ for every $\ell \in \mathcal{I}$. Moreover, the property $\varphi(Z) = 0$ for every $Z \in \mathcal{Z}$ can be regarded as absolute continuity with respect to null sets. The full support property can be regarded as the converse absolute continuity which gives the equivalence. However, the full-support property cannot be achieved by a single element of $\mathcal{Q}(\Theta)$.

Indeed, Bouchard and Nutz (2015) consider a set of priors \mathcal{M} . The absolute continuity and the full support properties then translate to the statement that “ \mathcal{M} and \mathcal{Q} have the same polar sets”. In the paper by Burzoni, Frittelli, and Maggis (2016), a class of relevant sets \mathcal{S} is given and the two properties can summarised by the statement “the set \mathcal{S} is not contained in the polar sets of \mathcal{Q} ”.

Also, it is a classical question whether one can restrict $\mathcal{Q}(\Theta)$ to the set of countable additive measures $ca_r(\Omega)$. In several of the examples described in the next section this is proved. However, there are simple examples for which this is not true.

B.2 Proof of Theorem 4.5

The implication $\mathbf{1} \Leftrightarrow \mathbf{2}$ is already proven in Theorem 3.5. We continue by proving the remaining implications.

$\mathbf{2} \Rightarrow \mathbf{3}$. Consider the super-replication cost as the nonlinear functional. Then, by Proposition A.2, it is a full support sublinear martingale expectation on (Θ, \mathcal{R}) .

$\mathbf{3} \Rightarrow \mathbf{2}$. Let $\mathcal{E} \in \mathcal{M}(\Theta, \mathcal{R})$. Suppose that a sequence of non-negative real numbers $\{c_n\}$, a sequence of investment opportunities $\{\ell^n\} \subset \mathcal{I}$ and a relevant contract $R^* \in \mathcal{R}$ which satisfy, $\mathbf{c}_n + \ell^n \geq R^*$, for $n = 1, 2, \dots$. Therefore, by monotonicity and translation-invariance,

$$\mathcal{E}(R^*) \leq \mathcal{E}(\mathbf{c}_n + \ell^n) = c_n + \mathcal{E}(\ell^n).$$

Since \mathcal{E} has the martingale property, $\mathcal{E}(\ell^n) \leq 0$ for each n . On the other hand, the full-support property of \mathcal{E} implies that $\mathcal{E}(R^*) > 0$. Hence, for any such sequence, $\liminf_n c_n > 0$ and consequently there are no free lunches in (Θ, \mathcal{R}) .

$\mathbf{2} \Rightarrow \mathbf{4}$. By Proposition B.3 and Corollary B.4, $\mathcal{Q}(\Theta)$ has the listed properties.

$\mathbf{4} \Rightarrow \mathbf{3}$. This is immediate. \square

C Extension to Lower Bounded Contracts

In this section we show the characterization of viability through sublinear pricing functionals when \mathcal{H} is not necessarily a subspace of \mathcal{B}_b . Although the proof of the statements follow exactly the same ideas, some technical considerations are required.

Since no *a priori* historical probability measure is assumed, we will typically work with bounded additive measures and some integrability conditions are clearly required. We consider linear functionals which are defined on a convex cone $\mathcal{B}_l \subset \mathcal{H}$, defined below, which in particular includes all bounded contracts in \mathcal{H} .

To define the set of lower bounded contracts we use tradable contract $\hat{\ell}$ that is the analogue of the stock price process. There could be many such contracts but we assume that this contract satisfies the following.

Assumption C.1. Let $\hat{\ell} \in \mathcal{I}$ be such that there exists $c^* \in \mathbb{R}^+$ satisfying,

$$L^* \geq_{\Omega} \mathbf{1} \quad \text{where} \quad L^*(\omega) := 1 + c^* + \hat{\ell}(\omega), \quad \omega \in \Omega.$$

We fix a contract $\hat{\ell} \in \mathcal{I}$ satisfying the above assumption and set

$$\mathcal{B}_l := \{X \in \mathcal{H} : \exists \alpha \in \mathbb{R}^+ \text{ such that } |X| \leq_{\Omega} \alpha L^*\},$$

equipped with the norm,

$$\|X\|_* := \inf\{\alpha \in \mathbb{R}^+ : |X| \leq_{\Omega} \alpha L^*\}.$$

Note that if $L^* = 1$ (i.e. $\hat{\ell} = \mathbf{0}$, $c^* = 0$), then $\mathcal{B}_l = \mathcal{B}_b$.

We now define \mathcal{H}_l , \mathcal{I}_l and Θ_l by,

$$\begin{aligned} \mathcal{H}_l &:= \{X \in \mathcal{H} : \exists \alpha \in \mathbb{R}^+ \text{ such that } X \geq_{\Omega} -\alpha L^*\}, \\ \mathcal{I}_l &:= \{\ell \in \mathcal{I} : \exists Z \in \mathcal{Z}, \text{ such that } \ell + Z \in \mathcal{H}_l\}, \\ \mathcal{R}_l &:= \{R \in \mathcal{R} : \exists Z \in \mathcal{Z}, \text{ such that } R + Z \in \mathcal{B}_l\}, \\ \Theta_l &:= (\mathcal{H}, \leq, \mathcal{I}_l). \end{aligned} \tag{C.1}$$

Notice that above sets depend on the choice of $\hat{\ell}$.

Remark C.2. In continuous time models one usually needs to assume that elements of \mathcal{I} are bounded from below pointwise (up to negligible contracts). In that case, one can take $\hat{\ell}$ to be the zero contract and $c^* = 0$. In finite discrete time markets however, a pointwise lower bound could be too restrictive.¹¹ In such markets, with non-negative stock values,

¹¹Technically, one has to either allow portfolio positions (or equivalently the random integrands in the gains process) to be a general predictable processes and not only the simple functions or allow for some static hedges as in Dolinsky and Soner (2014a, 2015). In finite discrete time, all integrands are simple functions and that is why the pointwise lower bound is restrictive in these markets when no static hedges are included.

one can construct $\hat{\ell}$ from the stock process as follows. For each $k = 1, \dots, N$, and $i = 1, \dots, M$, the cash flow

$$\ell^{k,i}(\omega) := -S_0^i + S_k^i(\omega), \quad t = 0, \dots, N,$$

belongs to \mathcal{I} . It corresponds to buying one share of the i -th stock at time zero and selling it at time k . In particular, the following contract is in \mathcal{I} ,

$$\hat{\ell} := \sum_{k=1}^N \sum_{i=1}^M \ell^{k,i}. \quad (\text{C.2})$$

Since stock values are non-negative,

$$\hat{\ell} \geq_{\Omega} -N \sum_{i=1}^M S_0^i =: -c^*.$$

Hence, $L^* := 1 + c^* + \hat{\ell}$ satisfies the Assumption C.1. Moreover, for all bounded H ,

$$(H \cdot S)_N \geq_{\Omega} -\|H\|_{\infty} \left[c^* + \hat{\ell} \right].$$

Thus,

$$\mathcal{I}_l \supseteq \{H \cdot S : H \text{ is a bounded predictable process}\}.$$

Note that, in classical discrete-time model, this set is enough to describe martingale measures.

Assumptions. We collect several technical assumptions which will be used in the next section. These are needed for technical integrability reasons. However, if one assumes that all contracts in \mathcal{H} are bounded, all of them are trivially satisfied.

We make the following natural structural assumption on the set of desirable claims which is satisfied by all examples.

Assumption C.3. For every $\varepsilon > 0$ and $P \in \mathcal{P}$, there is a constant $K_{\varepsilon,P} \in \mathbb{R}$ such that $(P + \varepsilon) \wedge K_{\varepsilon,P} \in \mathcal{P}$.

For the ease of reference, we collect all the above assumptions into the following.

Definition C.4. We say that Θ is *consistent and bounded* if they satisfy the Assumptions 4.1, C.1 and C.3.

C.1 Convex duality in \mathcal{B}_l

We observe here that the considerations of section B.1 extends to $(\mathcal{B}_l, \|\cdot\|_*)$ and Θ_l . Set

$$\mathcal{D}_l(X) := \mathcal{D}(X, \Theta_l), \quad \forall X \in \mathcal{H}.$$

We have $\mathcal{D} \leq \mathcal{D}_l$ because of $\mathcal{I}_l \subset \mathcal{I}$. In many examples, these two functionals agree on the set \mathcal{B}_l .

Lemma C.5. *Suppose that Θ satisfies the Assumptions 4.1, C.1 and (Θ_l, \mathcal{R}) is strongly free of arbitrage. Then, the super-replication functional is Lipschitz continuous on $(\mathcal{B}_l, \|\cdot\|_*)$. In fact,*

$$|\mathcal{D}_l(X) - \mathcal{D}_l(Y)| \leq (1 + c^*) \|X - Y\|_*, \quad \forall X, Y \in \mathcal{B}_l,$$

where c^* is the constant in Assumption C.1. Moreover,

$$|\mathcal{D}_l(X)| \leq (1 + c^*) \|X\|_*, \quad \forall X \in \mathcal{B}_l.$$

Proof. For $X, Y \in \mathcal{B}_l$,

$$X \leq_\Omega Y + \|X - Y\|_* L^*.$$

Hence,

$$\mathcal{D}_l(X) \leq \mathcal{D}_l(Y + \|X - Y\|_* L^*) \leq \mathcal{D}_l(Y) + \|X - Y\|_* \mathcal{D}_l(L^*).$$

It is clear that, $\mathcal{D}_l(L^*) \leq 1 + c^*$. All of these imply the Lipschitz estimate. The second estimate follows from this by taking $Y = \mathbf{0}$. \square

It is clear that the map

$$\psi : \mathcal{B}_b \rightarrow \mathcal{B}_l, \quad \psi(X) = \frac{X}{L^*}$$

is an isometric isomorphism. Moreover, an analogous map can be defined on the dual spaces, i.e.,

$$\tilde{\varphi} \in ba \rightarrow \varphi \in (\mathcal{B}_l)', \quad \varphi(X) := \tilde{\varphi} \left(\frac{X}{L^*} \right)$$

so that all elements of ba are embedded in $(\mathcal{B}_l)'$. On the other hand, given $\varphi \in (\mathcal{B}_l)'$ the restriction of φ to $\mathcal{B}_b \subset \mathcal{B}_l$ is obviously an element in ba .

Assume now that the financial market (Θ_l, \mathcal{R}) is strongly free of arbitrage. Then, in view of Proposition A.2, \mathcal{D}_l is an equivalent, coherent sublinear martingale. Also, under the Assumption C.1, Lemma C.5 implies that the super-replication functional restricted to \mathcal{B}_l ,

$$\mathcal{D}_l : \mathcal{B}_l \rightarrow \mathbb{R},$$

is a regular convex function and we can apply the same techniques of section B.1 to obtain the following.

Lemma C.6. *Suppose Θ satisfies Assumptions 4.1 and C.1. Then, $\text{dom}(\mathcal{D}_l^*)$ is given by,*

$$\begin{aligned} \text{dom}(\mathcal{D}_l^*) &= \{ \varphi \in (\mathcal{B}_l)'_+ : \mathcal{D}_l^*(\varphi) = 0 \} \\ &= \{ \varphi \in (\mathcal{B}_l)'_+ : \varphi(X) \leq \mathcal{D}_l(X), \quad \forall X \in \mathcal{B}_l \}. \end{aligned}$$

In particular,

$$\mathcal{D}_l(X, \Theta) = \sup_{\varphi \in \text{dom}(\mathcal{D}_l^*)} \{ \varphi(X) \}, \quad X \in \mathcal{B}_l. \quad (\text{C.3})$$

Furthermore, there are free lunches with vanishing risk in (Θ_l, \mathcal{R}) , whenever $\text{dom}(\mathcal{D}_l^)$ is empty.*

Proof. The proof follows the same arguments of Lemma B.2. \square

C.2 Characterization

The sets $\mathcal{M}(\Theta, \mathcal{R})$ from Definition 4.3. and $\mathcal{Q}(\Theta)$ from Definition 4.4 directly extend to Θ_l . The following is then a generalization of Theorem 4.5.

Theorem C.7. *Suppose that Θ is consistent and bounded in sense of Definition C.4. Then, the following are equivalent:*

1. $(\Theta_l, \mathcal{R}_l)$ is viable.
2. $(\Theta_l, \mathcal{R}_l)$ is strongly free of arbitrage.
3. The set of sublinear martingale expectations with full support $\mathcal{M}(\Theta_l, \mathcal{R}_l)$ is non-empty.
4. The set of martingale measures $\mathcal{Q}(\Theta_l)$ is non-empty and the Choquet capacity \mathcal{E}_{Θ_l} is a coherent sublinear martingale expectations with full support.

Remark C.8. In the above characterization, we restrict ourselves to the lower bounded achievable contracts \mathcal{I}_l and to \mathcal{R}_l . However, typically this is not a restriction as in many examples one can prove that $NA(\Theta, \mathcal{R})$ is equivalent to $NA(\Theta_l, \mathcal{R}_l)$. Indeed, a trivial case is when $\mathcal{I}_l = \mathcal{I}$ as for continuous-time markets. Also in the discrete-time markets of Example 2.1 this equivalence holds. However, in the generality of our structure, one needs to restrict to \mathcal{I}_l for the result to hold.

The proof of Theorem C.7 follows the same argument of the proof of Theorem 4.5. However, an extension of Proposition B.3 is needed. This is the content of the following result.

Proposition C.9. *Suppose Θ satisfies the Assumptions C.1 and C.3, and $(\Theta_l, \mathcal{R}_l)$ is strongly free of arbitrages. Then, a bounded linear functional $\varphi \in (\mathcal{B}_l)'_+$ belongs to $\text{dom}(\mathcal{D}_l^*)$ if and only if satisfies all the following conditions,*

1. $\varphi(\Omega) = 1$,
2. $\varphi(P) \geq 0$ for every $P \in \mathcal{P}$,
3. $\varphi(\ell) \leq 0$ for every $\ell \in \mathcal{I}_l$.

In particular, for every $\varphi \in \text{dom}(\mathcal{D}_l^)$ and $Z \in \mathcal{Z} \cap \mathcal{L}^1(\Omega, \varphi)$, $\varphi(Z) = 0$.*

Proof. Fix $\varphi \in \text{dom}(\mathcal{D}_l^*)$. By Proposition A.2, $\mathcal{D}_l(\mathbf{c}) = c$ for every constant $c \in \mathbb{R}$. In view of Lemma C.6,

$$c\varphi(\Omega) = \varphi(\mathbf{c}) \leq \mathcal{D}_l(\mathbf{c}) = c, \quad \forall c \in \mathbb{R}.$$

Hence, $\varphi(\Omega) = 1$.

We continue by proving the monotonicity property. Suppose that $D \in \mathcal{P} \cap \mathcal{B}_b$. Since $\mathbf{0} \in \mathcal{I}$, we obviously have $\mathcal{D}_l(-D) \leq 0$. The dual representation implies that

$\varphi(-D) \leq \mathcal{D}_l(-D) \leq 0$, for any $\varphi \in \text{dom}(\mathcal{D}_l^*)$. Now we fix $D \in \mathcal{P}$ which is bounded from above. For every $K > 0$, $D \vee (-K) \geq_\Omega D$ and hence $D \vee (-K) \in \mathcal{P}$ and clearly $D \vee (-K) \in \mathcal{B}_b$. Then, by the previous arguments $\varphi(D \vee (-K)) \geq 0$ for every K . Since D is bounded from above, by definition of $\varphi(D)$,

$$\varphi(D) = \lim_{K \rightarrow \infty} \varphi(D \vee (-K)) \geq 0.$$

Next let $D \in \mathcal{P}$ be general. Then, by (C.3), for every ε there is $K_\varepsilon > 0$ so that

$$D^\varepsilon := (D + \varepsilon) \wedge K_\varepsilon \in \mathcal{P}.$$

It is clear that D^ε is bounded from above. Hence, by the previous arguments we have $\varphi(D^\varepsilon) \geq 0$ for every $\varphi \in \text{dom}(\mathcal{D}_l^*)$. Since, $D + \varepsilon \geq_\Omega D^\varepsilon$ and since $\varphi \in (\mathcal{B}_l)_+'$, $\varphi(D + \varepsilon) \geq \varphi(D^\varepsilon)$. Consequently,

$$\varphi(D) + \varepsilon = \varphi(D + \varepsilon) \geq \varphi(D^\varepsilon) \geq 0.$$

This proves the second property.

Let $Z \in \mathcal{Z}$ and $\varphi \in \text{dom}(\mathcal{D}_l^*)$. Then, $\varphi(Z) \geq 0$. Therefore, $\varphi(Z^-) < \infty$ and equivalently, $Z^- \in \mathcal{L}^1(\Omega, \varphi)$. Since $-Z \in \mathcal{P}$, we also conclude that $\varphi(-Z) \geq 0$ and $(-Z)^- \in \mathcal{L}^1(\Omega, \varphi)$. However, $(-Z)^- = Z^+$. Hence, $Z^+ \in \mathcal{L}^1(\Omega, \varphi)$ also. This implies that $Z \in \mathcal{L}^1(\Omega, \varphi)$. Therefore, $0 \leq \varphi(-Z) = -\varphi(Z)$ and consequently, $\varphi(Z) \leq 0$. Combining all these, we conclude that $\varphi(Z) = 0$.

Let $\ell \in \mathcal{I}_l$. By Assumption C.1, there are $\alpha_\ell \in \mathbb{R}^+$ and a negligible contract $Z^\ell \in \mathcal{Z}$ so that $\ell + Z^\ell \geq_\Omega -\alpha_\ell L^*$. Hence, $[\ell + Z^\ell] \wedge K \in \mathcal{B}_l$ for any $K \in \mathbb{R}$. Moreover,

$$\mathcal{D}_l([\ell + Z^\ell] \wedge K) \leq \mathcal{D}_l(\ell + Z^\ell) = \mathcal{D}_l(\ell) \leq 0.$$

Then, by the dual representation on \mathcal{B}_l , $\varphi([\ell + Z^\ell] \wedge K) \leq \mathcal{D}_l([\ell + Z^\ell] \wedge K) \leq 0$, for every $\varphi \in \text{dom}(\mathcal{D}_l^*)$. Since $\ell + Z^\ell \geq_\Omega -\alpha_\ell L^*$, by Lemma F.10 in the Appendix,

$$\varphi(\ell + Z^\ell) = \lim_{K \rightarrow \infty} \varphi([\ell + Z^\ell] \wedge K) \leq 0.$$

Since $Z^\ell \in \mathcal{L}^1(\Omega, \varphi)$ and $\varphi(Z^\ell) = 0$, by Lemma F.9 in the Appendix, we conclude that

$$0 \geq \varphi(\ell + Z^\ell) = \varphi(\ell) + \varphi(Z^\ell) = \varphi(\ell).$$

These prove that for every $\varphi \in \text{dom}(\mathcal{D}_l^*)$, φ must satisfy the conditions stated above.

To prove the converse, suppose that a bounded, linear functional $\varphi \in (\mathcal{B}_l)_+'$ satisfies the three conditions. Suppose that $X \in \mathcal{B}_l$, $c \in \mathbb{R}$, $\ell \in \mathcal{I}$ satisfy, $\mathbf{c} + \ell - X \in \mathcal{P}$. Since $\mathbf{c} - X \in \mathcal{B}_l$, by Lemma F.9 of the Appendix and the properties of φ ,

$$0 \leq \varphi(\mathbf{c} + \ell - X) = \varphi(\mathbf{c} - X) + \varphi(\ell) \leq c - \varphi(X).$$

Hence, $\varphi(X) \leq \mathcal{D}_l(X)$ for every $X \in \mathcal{B}_l$. Therefore, $\varphi \in \text{dom}(\mathcal{D}_l^*)$. \square

D No Arbitrage versus No Free-Lunch-with-Vanishing-Risk

From Definition 3.4 it is clear that an arbitrage opportunity is always a free lunch with vanishing risk. The purpose of this section is to investigate when these two notions are equivalent.

D.1 Attainment

We first show that the attainment property is useful in discussing the connection between two different notions of arbitrage. We start with a definition.

Definition D.1. We say that Θ has the *attainment property*, if for every $X \in \mathcal{H}$ there exists a minimizer in (A.1), i.e., there exists $\ell_X \in \mathcal{I}$ satisfying,

$$\mathcal{D}(X) + \ell_X \geq X.$$

Proposition D.2. *Suppose Θ has the attainment property. Then, it is strongly free of arbitrage if and only if it has no arbitrages.*

Proof. Let $R^* \in \mathcal{R}$. By hypothesis, there exist $\ell \in \mathcal{I}^*$ so that $\mathcal{D}(R^*) + \ell^* \geq R^*$. If the market has no arbitrage, then we conclude that $\mathcal{D}(R^*) > 0$. In view of (A.2), this proves that Θ is also strongly free of arbitrage. Since no arbitrage is weaker condition, they are equivalent. \square

D.2 Finite discrete time markets

In this subsection and in the next section, we restrict ourselves to finite discrete-time markets.

We start by introducing a discrete filtration $\mathbb{F} := (\mathcal{F}_t)_{t=0}^T$ on subsets of Ω . Let $S = (S_t)_{t=0}^T$ be an adapted stochastic process^{12,13} with values in \mathbb{R}_+^M for some M .

We next describe the set \mathcal{I} . We say that $\ell \in \mathcal{H}$ is in \mathcal{I} provided that there exists predictable integrands $H_t \in \mathcal{L}^0(\Omega, \mathcal{F}_{t-1})$ for all $t = 1, \dots, T$ such that,

$$\ell = (H \cdot S)_T := \sum_{t=1}^T H_t \cdot \Delta S_t, \quad \text{where } \Delta S_t := (S_t - S_{t-1}).$$

Denote by $\ell_t := (H \cdot S)_t$ for $t \in \mathcal{I}$ and $\ell := \ell_T$.

¹²When working with N stocks, a canonical choice for Ω would be

$$\Omega = \{\omega = (\omega_0, \dots, \omega_T) : \omega_i \in [0, \infty)^N, i = 0, \dots, T\}.$$

Then, one may take $S_t(\omega) = \omega_t$ and \mathbb{F} to be the filtration generated by S .

¹³Note that we do not specify any probability measure.

We let $\hat{\ell}$ be as in (C.2). Then, as argued in Section C, (C.1) is satisfied with an appropriate c^* . We then define the sets \mathcal{B}_ℓ using $\hat{\ell}$ and denote by \mathcal{I}_t the subset of \mathcal{I} with H_t bounded for every $t = 1, \dots, T$.

We next prescribe the equivalence relation and the relevant sets. Our starting point is the set of negligible sets \mathcal{Z} which we assume is given. We also make the following structural assumption.

Assumption D.3. Let \mathcal{I} be given as above and let \mathcal{Z} be a lattice which is closed with respect to pointwise convergence.

We also assume that $\mathcal{R} = \mathcal{P}^+$ and the weak order is given by,

$$X \leq Y \quad \Leftrightarrow \quad \exists Z \in \mathcal{Z} \quad \text{such that} \quad X \leq_\Omega Y + Z.$$

In particular, $D \in \mathcal{P}$ if and only if there exists $Z \in \mathcal{Z}$ such that $Z \leq_\Omega D$.

An example of the above structure is the Example 5.7. In that example, \mathcal{Z} is polar sets of a given class \mathcal{M} of probabilities. Then, in this context all inequalities should be understood as \mathcal{M} quasi-surely. Also note also that the assumptions on \mathcal{Z} are trivially satisfied when $\mathcal{Z} = \{0\}$. In this latter case, inequalities are pointwise.

Observe that in view of the definition of \leq and the fact $\mathcal{R} = \mathcal{P}^+$, $\ell \in \mathcal{I}$ is an arbitrage if and only if there is $R^* \in \mathcal{P}^+$ and $Z^* \in \mathcal{Z}$, so that $\ell \geq_\Omega R^* + Z^*$. Hence, $\ell \in \mathcal{I}$ is an arbitrage if and only if $\ell \in \mathcal{P}^+$. We continue by showing the equivalence of the existence of an arbitrage to the existence of a one-step arbitrage.

Lemma D.4. *Suppose that Assumption D.3 holds. Then, there exists arbitrage if and only if there exists $t \in \{1, \dots, T\}$, $h \in \mathcal{L}^0(\Omega, \mathcal{F}_{t-1})$ such that*

$$\ell := h \cdot \Delta S_t$$

is an arbitrage.

Proof. The sufficiency is clear. To prove the necessity, suppose that $\ell \in \mathcal{I}$ is an arbitrage. Then, there is a predictable process H so that $\ell = (H \cdot S)_T$. Also $\ell \in \mathcal{P}^+$, hence, $\ell \notin \mathcal{Z}$ and there exists $Z \in \mathcal{Z}$ such that $\ell \geq Z$. Define

$$\hat{t} := \min\{t \in \{1, \dots, T\} : (H \cdot S)_t \in \mathcal{P}^+\} \leq T.$$

First we study the case where $\ell_{\hat{t}-1} \in \mathcal{Z}$. Define

$$\ell^* := H_{\hat{t}} \cdot \Delta S_{\hat{t}},$$

and observe that $\ell_{\hat{t}} = \ell_{\hat{t}-1} + \ell^*$. Since $\ell_{\hat{t}-1} \in \mathcal{Z}$, we have that $\ell^* \in \mathcal{P}^+$ iff $\ell_{\hat{t}} \in \mathcal{P}^+$ and consequently the lemma is proved.

Suppose now $\ell_{\hat{t}-1} \notin \mathcal{Z}$. If $\ell_{\hat{t}-1} \geq_\Omega 0$, then $\ell_{\hat{t}-1} \in \mathcal{P}$ and, thus, also in \mathcal{P}^+ , which is not possible from the minimality of \hat{t} . Hence the set $A := \{\ell_{\hat{t}-1} <_\Omega 0\}$ is non empty and $\mathcal{F}_{\hat{t}-1}$ -measurable. Define, $h := H_{\hat{t}} \chi_A$ and

$$\ell^* := h \cdot \Delta S_{\hat{t}}.$$

Note now that,

$$\begin{aligned} \ell^* = \chi_A(\ell_{\hat{t}} - \ell_{\hat{t}-1}) &\geq_{\Omega} \chi_A \ell_{\hat{t}} \\ &\geq_{\Omega} \chi_A Z \in \mathcal{Z}. \end{aligned}$$

This implies $\ell^* \in \mathcal{P}$. Towards a contradiction, suppose that $\ell^* \in \mathcal{Z}$. Then,

$$\ell_{\hat{t}-1} \geq_{\Omega} \chi_A \ell_{\hat{t}-1} \geq \chi_A (Z - \ell^*) \in \mathcal{Z},$$

Since, by assumption, $\ell_{\hat{t}-1} \notin \mathcal{Z}$ we have $\ell_{\hat{t}-1} \in \mathcal{P}^+$ from which \hat{t} is not minimal. \square

Corollary D.5. *The financial market (Θ, \mathcal{P}^+) has no arbitrage if and only if there are none in $(\Theta_t, \mathcal{P}_t^+)$.*

Proof. From Lemma D.4 there exists $\hat{h} \in \mathcal{L}^0(\Omega, \mathcal{F}_{\hat{t}-1})$ such that

$$\hat{h} \cdot \Delta S_t \geq_{\Omega} Z,$$

for some $Z \in \mathcal{Z}$. Since, by Lemma F.2, \mathcal{Z} is stable under multiplication, it is clear that $\hat{h}/\|\hat{h}\|$ satisfies the same. \square

The following is the main result of this section. For the proof we follow the approach of Kabanov and Stricker (2001) which is also used in Bouchard and Nutz (2015).

Theorem D.6. *Under the Assumption D.3, the following are equivalent:*

1. *The financial market (Θ, \mathcal{P}^+) has no arbitrages.*
2. *The attainment property holds and (Θ, \mathcal{P}^+) is free of arbitrage.*
3. *The financial market (Θ, \mathcal{P}^+) is strongly free of arbitrages.*

Proof. In view of Proposition D.2 we only need to prove the implication 1 \Rightarrow 2.

For $X \in \mathcal{H}$ such that $\mathcal{D}(X)$ is finite we have that

$$c_n + \mathcal{D}(H) + \ell^n \geq_{\Omega} X + Z^n,$$

for some $c_n \downarrow 0$, $\ell^n \in \mathcal{I}$ and $Z^n \in \mathcal{Z}$. Note that since \mathcal{Z} is a lattice we assume, without loss of generality, that $Z^n = (Z^n)^-$ and denote by $\mathcal{Z}^- := \{Z^- \mid Z \in \mathcal{Z}\}$.

We show that $\mathcal{C} := \mathcal{I} - (\mathcal{L}_+^0(\Omega, \mathcal{F}) + \mathcal{Z}^-)$ is closed under pointwise convergence where $\mathcal{L}_+^0(\Omega, \mathcal{F})$ denotes the class of pointwise non-negative random variables. Once this result is shown, by observing that $X - c_n - \mathcal{D}(X) = W^n \in \mathcal{C}$ and by the pointwise closure of \mathcal{C} we obtain the attainment property..

We proceed by induction on the number of time steps. Suppose first $T = 1$. Let

$$W^n : \ell^n - K^n - Z^n \rightarrow W, \tag{D.1}$$

where $\ell^n \in \mathcal{I}$, $K^n \geq_\Omega 0$ and $Z^n \in \mathcal{Z}^-$. We need to show $W \in \mathcal{C}$. Note that any ℓ^n can be represented as $\ell^n = H_1^n \cdot \Delta S_1$ with $H_1^n \in \mathcal{L}^0(\Omega, \mathcal{F}_0)$.

Let $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. From Lemma 2 in Kabanov and Stricker (2001) there exist a sequence $\{\tilde{H}_1^k\}$ such that $\{\tilde{H}_1^k(\omega)\}$ is a convergent subsequence of $\{H_1^k(\omega)\}$ for every $\omega \in \Omega_1$. Let $H_1 := \liminf H_1^n \chi_{\Omega_1}$ and $\ell := H_1 \cdot \Delta S_1$.

Note now that $Z^n \leq_\Omega 0$, hence, if $\liminf |Z_n| = \infty$ we have $\liminf Z_n = -\infty$. We show that we can choose $\tilde{Z}^n \in \mathcal{Z}^-$, $\tilde{K}^n \geq_\Omega 0$ such that $\tilde{W}^n := \ell^n - \tilde{K}^n - \tilde{Z}^n \rightarrow W$ and $\liminf \tilde{Z}^n$ is finite on Ω_1 . On $\{\ell^n \geq_\Omega W\}$ set $\tilde{Z}^n = 0$ and $\tilde{K}^n = \ell^n - W$. On $\{\ell^n <_\Omega W\}$ set

$$\tilde{Z}^n = Z^n \vee (\ell^n - W), \quad \tilde{K}^n = K^n \chi_{\{Z^n = \tilde{Z}^n\}}.$$

It is clear that $Z^n \leq_\Omega \tilde{Z}^n \leq_\Omega 0$. From Lemma F.1 we have $\tilde{Z}^n \in \mathcal{Z}$. Moreover, it is easily checked that $\tilde{W}^n := \ell^n - \tilde{K}^n - \tilde{Z}^n \rightarrow W$. Nevertheless, from the convergence of ℓ^n on Ω_1 and $\tilde{Z}^n \geq_\Omega -(W - \ell^n)^+$, we obtain $\{\omega \in \Omega_1 \mid \liminf \tilde{Z}^n > -\infty\} = \Omega_1$. As a consequence also $\liminf \tilde{K}^n$ is finite on Ω_1 , otherwise we could not have that $\tilde{W}^n \rightarrow W$. Thus, by setting $\tilde{Z} := \liminf \tilde{Z}^n$ and $\tilde{K} := \liminf \tilde{K}^n$, we have $W = \ell - \tilde{K} - \tilde{Z} \in \mathcal{C}$.

On Ω_1^C we may take $G_1^n := H_1^n / |H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^C}$. Define, $\ell_G := G_1 \cdot \Delta S_1$. We now observe that,

$$\{\omega \in \Omega_1^C \mid \ell_G(\omega) \leq 0\} \subseteq \{\omega \in \Omega_1^C \mid \liminf Z_n(\omega) = -\infty\}.$$

Indeed, if $\omega \in \Omega_1^C$ is such that $\liminf Z^n(\omega) > -\infty$, applying again Lemma 2 in Kabanov and Stricker (2001), we have that

$$\liminf_{n \rightarrow \infty} \frac{X(\omega) + Z^n(\omega)}{|H_1^n(\omega)|} = 0,$$

implying $\ell_G(\omega)$ is non-negative. Set now

$$\tilde{Z}^n := Z^n \vee -(\ell_G)^-.$$

From $Z^n \leq_\Omega \tilde{Z}^n \leq_\Omega 0$, again by Lemma F.1, $\tilde{Z}^n \in \mathcal{Z}$. By taking the limit for $n \rightarrow \infty$ we obtain $(\ell_G)^- \in \mathcal{Z}$ and thus, $\ell_G \in \mathcal{P}$. Since Θ has no arbitrages $G_1 \cdot \Delta S_1 = Z \in \mathcal{Z}$ and hence one asset is redundant. Consider a partition Ω_2^i of Ω_1^C on which $G_1^i \neq 0$. Since \mathcal{Z} is stable under multiplication (Lemma F.2), for any $\ell^* \in \mathcal{I}$, there exists $Z^* \in \mathcal{Z}$ and $H^* \in \mathcal{L}^0(\Omega_2^i, \mathcal{F}_0)$ with $(H^*)^i = 0$, such that $\ell^* = H^* \cdot \Delta S_1 + Z^*$ on Ω_2^i . Therefore, the term ℓ^n in (D.1) is composed of trading strategies involving only $d - 1$ assets. Iterating the procedure up to d -steps we have the conclusion.

Assuming now that D.1 holds for markets with $T - 1$ periods, with the same argument we show that we can extend to markets with T periods. Set again $\Omega_1 := \{\omega \in \Omega \mid \liminf |H_1^n| < \infty\}$. Since on Ω_1 we have that,

$$W_n - H_1^n \cdot \Delta S_1 = \sum_{t=2}^T H_t^n \cdot \Delta S_t - K^n - Z^n \rightarrow W - H_1 \cdot \Delta S_1.$$

The induction hypothesis allows to conclude that $W - H_1 \cdot S_1 \in \mathcal{C}$ and therefore $W \in \mathcal{C}$. On $\Omega_1^{\mathcal{C}}$ we may take $G_1^n := H_1^n / |H_1^n|$ and let $G_1 := \liminf G_1^n \chi_{\Omega_1^{\mathcal{C}}}$. Note that $W^n / |H_1^n| \rightarrow 0$ and hence

$$\sum_{t=2}^T \frac{H_t^n}{|H_1^n|} \cdot \Delta S_t - \frac{K^n}{|H_1^n|} - \frac{Z^n}{|H_1^n|} \rightarrow -G_1 \cdot \Delta S_1.$$

Since \mathcal{Z} is stable under multiplication $\frac{Z^n}{|H_1^n|} \in \mathcal{Z}$ and hence, by inductive hypothesis, there exists \tilde{H}_t for $t = 2, \dots, T$ and $\tilde{Z} \in \mathcal{Z}$ such that

$$\tilde{\ell} := G_1 \cdot \Delta S_1 + \sum_{t=2}^T \tilde{H}_t \cdot \Delta S_t \geq_{\Omega} \tilde{Z} \in \mathcal{Z}.$$

The No Arbitrage condition implies that $\tilde{\ell} \in \mathcal{Z}$. Once again, this means that one asset is redundant and, by considering a partition Ω_2^i of $\Omega_1^{\mathcal{C}}$ on which $G_1^i \neq 0$, we can rewrite the term ℓ^n in (D.1) with $d - 1$ assets. Iterating the procedure up to d -steps we have the conclusion. \square

The above result is consistent with the fact that in classical “probabilistic” model for finite discrete-time markets only the no-arbitrage condition and not the no-free lunch condition has been utilized.

E Countably Additive Measures

In this section, we show that in general finite discrete time markets, it is possible to characterize viability through countably additive functionals. We prove this result by combining the results of the previous subsection, Theorem C.7 and some results from Burzoni, Frittelli, Hou, Maggis, and Obłój (2017) which we collect in Appendix F.2. In order to use the results of Burzoni, Frittelli, Hou, Maggis, and Obłój (2017) we only require, in addition to the previous setting, that Ω is a Polish space and that the filtration \mathbb{F} contains analytic sets.¹⁴

We let $\mathcal{Q}^{ca}(\Theta)$ be the set of countably additive positive probability measures \mathbb{Q} such that S is a \mathbb{Q} -martingale and $\mathcal{Z}^- := \{-Z^- \mid Z \in \mathcal{Z}\}$. For $X \in \mathcal{H}$, set

$$\mathcal{Z}(X) := \{Z \in \mathcal{Z}^- : \exists \ell \in \mathcal{I} \text{ such that } \mathcal{D}(X) + \ell \geq_{\Omega} X + Z\}.$$

By the lattice property of \mathcal{Z} , if $\mathcal{D}(X) + \ell \geq_{\Omega} X + Z$ the same is true if we take $Z = Z^-$. From Theorem D.6 we know that, under no arbitrage, the attainment property holds and, hence, $\mathcal{Z}(X)$ is non-empty for every $X \in \mathcal{H}$. For $A \in \mathcal{F}$, we define

$$\begin{aligned} \mathcal{D}_A(X) &:= \inf \{c \in \mathbb{R} : \exists \ell \in \mathcal{I} \text{ such that } c + \ell(\omega) \geq X(\omega), \forall \omega \in A\} \\ \mathcal{Q}_A^{ca}(\Theta) &:= \{\mathbb{Q} \in \mathcal{Q}^{ca}(\Theta) : \mathbb{Q}(A) = 1\}. \end{aligned}$$

We need the following technical result in the proof of the main Theorem.

¹⁴Note that this technical aspect is always considered when a reference probability \mathbb{P} is fixed. Analytic sets are indeed contained in the \mathbb{P} -completion of \mathbb{F} .

Proposition E.1. *Suppose Assumption D.3 holds and (Θ, \mathcal{P}^+) has no arbitrages. Then, for every $X \in \mathcal{H}$ and $Z \in \mathcal{Z}(X)$, there exists $A_{X,Z}$ such that*

$$A_{X,Z} \subset \{ \omega \in \Omega : Z(\omega) = 0 \}, \quad (\text{E.1})$$

and

$$\mathcal{D}(X) = \mathcal{D}_{A_{X,Z}}(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_{A_{X,Z}}^{ca}(\Theta)} \mathbb{E}_{\mathbb{Q}}[X].$$

Before proving this result, we state the main result of this section.

Theorem E.2. *Suppose Assumption D.3 holds. Then, (Θ, \mathcal{P}^+) has no arbitrages if and only if for every $(Z, R) \in \mathcal{Z}^- \times \mathcal{P}^+$ there exists $\mathbb{Q}_{Z,R} \in \mathcal{Q}^{ca}(\Theta)$ satisfying*

$$\mathbb{E}_{\mathbb{Q}_{Z,R}}[R] > 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}_{Z,R}}[Z] = 0. \quad (\text{E.2})$$

Proof. Suppose that (Θ, \mathcal{P}^+) has no arbitrages. Fix $(Z, R) \in \mathcal{Z}^- \times \mathcal{P}^+$ and $Z_R \in \mathcal{Z}(R)$. Set $Z^* := Z_R + Z \in \mathcal{Z}(R)$. By Proposition E.1, there exists $A_* := A_{R,Z^*}$ satisfying the properties listed there. In particular,

$$0 < \mathcal{D}(R) = \sup_{\mathbb{Q} \in \mathcal{Q}_{A_*}^{ca}(\Theta)} \mathbb{E}_{\mathbb{Q}}[R].$$

Hence, there is $\mathbb{Q}^* \in \mathcal{Q}_{A_*}^{ca}(\Theta)$ so that $\mathbb{E}_{\mathbb{Q}^*}[R] > 0$. Moreover, since $Z_R, Z \in \mathcal{Z}^-$,

$$A_* \subset \{Z^* = 0\} = \{Z_R = 0\} \cap \{Z = 0\}.$$

In particular, $\mathbb{E}_{\mathbb{Q}^*}[Z] = 0$.

To prove the opposite implication, suppose that there exists $R \in \mathcal{P}^+$, $\ell \in \mathcal{I}$ and $Z \in \mathcal{Z}$ such that $\ell \geq_{\Omega} R + Z$. Then, it is clear that $\ell \geq_{\Omega} R - Z^-$. Let $\mathbb{Q}^* := \mathbb{Q}_{-Z^-, R} \in \mathcal{Q}^{ca}(\Theta)$ satisfying (E.2). By integrating both sides against \mathbb{Q}^* , we obtain

$$0 = \mathbb{E}_{\mathbb{Q}^*}[\ell] \geq \mathbb{E}_{\mathbb{Q}^*}[R - Z^-] = \mathbb{E}_{\mathbb{Q}^*}[R] > 0.$$

which is a contradiction. Thus, there are no arbitrages. \square

We continue with the proof of Proposition E.1.

proof of Proposition E.1. Since there are no arbitrages, by Theorem D.6 we have the attainment property. Hence, for a given $X \in \mathcal{H}$, the set $\mathcal{Z}(X)$ is non-empty.

Step 1. We show that, for any $Z \in \mathcal{Z}(X)$, $\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}}(X)$. Note that, since $\mathcal{D}(X) + \ell \geq_{\Omega} X + Z$, for some $\ell \in \mathcal{I}$, the inequality $\mathcal{D}_{\{Z=0\}}(X) \leq \mathcal{D}(X)$ is always true. Towards a contradiction, suppose that the inequality is strict, namely, there exist $c < \mathcal{D}(X)$ and $\tilde{\ell} \in \mathcal{I}$ such that $c + \tilde{\ell}(\omega) \geq X(\omega)$ for any $\omega \in \{Z = 0\}$. We show that

$$\tilde{Z} := (c + \tilde{\ell} - X)^- \chi_{\{Z < 0\}} \in \mathcal{Z}.$$

This together with $c + \tilde{\ell} \geq_{\Omega} X + \tilde{Z}$ yields a contradiction. Recall that \mathcal{Z} is a linear space so that $nZ \in \mathcal{Z}$ for any $n \in \mathbb{N}$. From $nZ \leq_{\Omega} \tilde{Z} \vee (nZ) \leq_{\Omega} 0$, we also have $\tilde{Z}_n := \tilde{Z} \vee (nZ) \in \mathcal{Z}$, by Lemma F.1. By noting that $\{\tilde{Z} < 0\} \subset \{Z < 0\}$ we have that $\tilde{Z}_n(\omega) \rightarrow \tilde{Z}(\omega)$ for every $\omega \in \Omega$. From the closure of \mathcal{Z} under pointwise convergence, we conclude that $\tilde{Z} \in \mathcal{Z}$.

Step 2. For a given set $A \in \mathcal{F}_T$, we let $A^* \subset A$ be the set of scenarios visited by martingale measures (see (F.2) in the Appendix for more details). We show that, for any $Z \in \mathcal{Z}(X)$, $\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}^*}(X)$.

Suppose that $\{Z = 0\}^*$ is a proper subset of $\{Z = 0\}$ otherwise, from Step 1, there is nothing to show. From Lemma F.6 there is a strategy $\tilde{\ell} \in \mathcal{I}$ such that $\tilde{\ell} \geq 0$ on $\{Z = 0\}$ ¹⁵. Lemma F.5 (and in particular (F.4)) yield a finite number of strategies $\ell_1^t, \dots, \ell_{\beta_t}^t$ with $t = 1, \dots, T$, such that

$$\{\hat{Z} = 0\} = \{Z = 0\}^* \quad \text{where} \quad \hat{Z} := Z - \sum_{t=1}^T \sum_{i=1}^{\beta_t} \chi_{\{Z=0\}}(\ell_i^t)^+ . \quad (\text{E.3})$$

Moreover, for any $\omega \in \{Z = 0\} \setminus \{Z = 0\}^*$, there exists (i, t) such that $\ell_i^t(\omega) > 0$. We are going to show that, under the no arbitrage hypothesis, $\ell_i^t \in \mathcal{Z}$ for any $i = 1, \dots, \beta_t$, $t = 1, \dots, T$. In particular, from the lattice property of the linear space \mathcal{Z} , we have $\hat{Z} \in \mathcal{Z}$.

We illustrate the reason for $t = T$, by repeating the same argument up to $t = 1$ we have the thesis. We proceed by induction on i . Start with $i = 1$. From Lemma F.5 we have that $\ell_1^T \geq 0$ on $\{Z = 0\}$ and, therefore, $\{\ell_1^T < 0\} \subseteq \{Z < 0\}$. Define $\tilde{Z} := -(\ell_1^T)^- \leq_{\Omega} 0$. By using the same argument as in Step 1, we observe that $n\tilde{Z} \leq_{\Omega} \tilde{Z} \vee (n\tilde{Z}) \leq_{\Omega} 0$ with $n\tilde{Z} \in \mathcal{Z}$ for any $n \in \mathbb{N}$. From $\{\ell_1^T < 0\} \subseteq \{Z < 0\}$ and the closure of \mathcal{Z} under pointwise convergence, we conclude that $\tilde{Z} \in \mathcal{Z}$. From $NA(\Theta)$, we must have $\ell_1^T \in \mathcal{Z}$.

Suppose now that $\ell_j^T \in \mathcal{Z}$ for every $1 \leq j \leq i - 1$. From Lemma F.5, we have that $\ell_i^T \geq 0$ on $\{Z - \sum_{j=1}^{i-1} \ell_j^T = 0\}$ and, therefore, $\{\ell_i^T < 0\} \subseteq \{Z - \sum_{j=1}^{i-1} \ell_j^T < 0\}$. The argument of Step 1 allows to conclude that $\ell_i^T \in \mathcal{Z}$.

We are now able to show the claim. The inequality $\mathcal{D}_{\{Z=0\}^*}(X) \leq \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}(X)$ is always true. Towards a contradiction, suppose that the inequality is strict, namely, there exist $c < \mathcal{D}(X)$ and $\tilde{\ell} \in \mathcal{I}$ such that $c + \tilde{\ell}(\omega) \geq X(\omega)$ for any $\omega \in \{Z = 0\}^*$. We show that

$$\tilde{Z} := (c + \tilde{\ell} - X)^- \chi_{\Omega \setminus \{Z=0\}^*} \in \mathcal{Z}.$$

This together with $c + \tilde{\ell} \geq_{\Omega} X + \tilde{Z}$, yields a contradiction. To see this recall that, from the above argument, $\hat{Z} \in \mathcal{Z}$ with \hat{Z} as in (E.3). Moreover, again by (E.3), we have

¹⁵Note that restricted to $\{Z = 0\}$ this strategy yields no risk and possibly positive gains, in other words, this is a good candidate for being an arbitrage.

$\{\tilde{Z} < 0\} \subset \{\hat{Z} < 0\}$. The argument of Step 1 allows to conclude that $\tilde{Z} \in \mathcal{Z}$.

Step 3. We are now able to conclude the proof. Fix $Z \in \mathcal{Z}(X)$ and set $A_{X,Z} := \{Z = 0\}^*$. Then,

$$\mathcal{D}(X) = \mathcal{D}_{\{Z=0\}}(X) = \mathcal{D}_{(A_{X,Z})^*}(X) = \sup_{Q \in \mathcal{Q}_{A_{X,Z}}^{ca}(\Theta)} \mathbb{E}_Q[X],$$

where the first two equalities follow from Step 1 and Step 2 and the last equality follows from Proposition F.7. \square

F Some technical tools

F.1 Preferences

We start with a simple but a useful condition for negligibility.

Lemma F.1. *Consider two negligible contracts $\hat{Z}, \tilde{Z} \in \mathcal{Z}$. Then, any contract $Z \in \mathcal{H}$ satisfying $\hat{Z} \leq Z \leq \tilde{Z}$ is negligible as well.*

Proof. By definitions, we have,

$$X \leq X + \hat{Z} \leq X + Z \leq X + \tilde{Z} \leq X \Rightarrow X \sim X + Z.$$

Thus, $Z \in \mathcal{Z}$. \square

Lemma F.2. *Suppose that \mathcal{Z} is closed under pointwise convergence and Assumption 4.1 is in force. Then, \mathcal{Z} is stable under multiplication, i.e., $ZH \in \mathcal{Z}$ for any $H \in \mathcal{H}$.*

Proof. Note first that $Z^n := Z((H \wedge n) \vee -n) \in \mathcal{Z}$. This follows from by Lemma F.1 and the fact that \mathcal{Z} is a cone. By taking the limit for $n \rightarrow \infty$, the result follows. \square

We next prove that $\mathcal{E}(Z) = 0$ for every $Z \in \mathcal{Z}$.

Lemma F.3. *Let \mathcal{E} be a coherent sublinear expectation. Then,*

$$\begin{aligned} \mathcal{E}(c + \lambda[X + Y]) &= c + \mathcal{E}(\lambda[X + Y]) = c + \lambda\mathcal{E}(X + Y) \\ &\leq c + \lambda[-(-\mathcal{E}(X) - \mathcal{E}(Y))], \end{aligned} \tag{F.1}$$

for every $c \in \mathbb{R}$, $\lambda \geq 0$, $X, Y \in \mathcal{H}$. In particular,

$$\mathcal{E}(Z) = 0, \quad \forall Z \in \mathcal{Z}.$$

Proof. Let $X, Y \in \mathcal{H}$. The sub-additivity of $U_{\mathcal{E}}$ implies that

$$U_{\mathcal{E}}(X') + U_{\mathcal{E}}(Y') \leq U_{\mathcal{E}}(X' + Y'), \quad \forall X', Y' \in \mathcal{H},$$

even when they take values $\pm\infty$. The definition of $U_{\mathcal{E}}$ now yields,

$$\mathcal{E}(X + Y) = -U_{\mathcal{E}}(-X - Y) \leq -[U_{\mathcal{E}}(-X) + U_{\mathcal{E}}(-Y)] = -(-\mathcal{E}(X) - \mathcal{E}(Y)).$$

Then, (F.1) follows directly from the definitions.

Let $Z \in \mathcal{Z}$. Then, $-Z, Z \in \mathcal{P}$ and $\mathcal{E}(Z), \mathcal{E}(-Z) \geq 0$. Since $-Z \in \mathcal{P}$, the monotonicity of \mathcal{E} implies that $\mathcal{E}(X - Z) \geq \mathcal{E}(X)$ for any $X \in \mathcal{H}$. Choose $X = Z$ to arrive at

$$0 = \mathcal{E}(\mathbf{0}) = \mathcal{E}(Z - Z) \geq \mathcal{E}(Z) \geq 0.$$

Hence, $\mathcal{E}(Z)$ is equal to zero. □

F.2 Finite Time Markets

We here recall some results from Burzoni, Frittelli, Hou, Maggis, and Oblój (2017). We first need some notation. For a given sigma-algebra \mathcal{G} , we denote by $\mathcal{G}^{\mathcal{A}}$ the sigma-algebra generated by the analytic sets of \mathcal{G} . Let $(\mathcal{F}_t)_{t=0, \dots, T}$ be the natural filtration of the process S and \mathcal{F} the Borel sigma-algebra. Fix a set $A \in \mathcal{F}^{\mathcal{A}}$. Denote by \mathcal{Q}_A the set of martingale measure Q for S such that $Q(A) = 1$. With \mathcal{Q}_A^f we denote those with finite support. We define the set of scenarios charged by martingale measures as

$$A^* := \left\{ \omega \in \Omega \mid \exists Q \in \mathcal{Q}_A^f \text{ s.t. } Q(\omega) > 0 \right\} = \bigcup_{Q \in \mathcal{Q}_A^f} \text{supp}(Q). \quad (\text{F.2})$$

Definition F.4. We say that $\ell \in \mathcal{I}$ is a one-step strategy if $\ell = H_t \cdot (S_t - S_{t-1})$ with $H_t \in \mathcal{L}(X, \mathcal{F}_{t-1}^{\mathcal{A}})$ for some $t \in \{1, \dots, T\}$. We say that $a \in \mathcal{I}$ is a one-point Arbitrage on A iff $a(\omega) \geq 0 \forall \omega \in A$ and $a(\omega) > 0$ for some $\omega \in A$.

The following Lemma is crucial for the characterization of the set A^* in terms of arbitrage considerations.

Lemma F.5. Fix any $t \in \{1, \dots, T\}$ and $\Gamma \in \mathcal{F}^{\mathcal{A}}$. There exist an index $\beta \in \{0, \dots, d\}$, one-step strategies $\ell^1, \dots, \ell^\beta \in \mathcal{I}$ and B^0, \dots, B^β , a partition of Γ , satisfying:

1. if $\beta = 0$ then $B^0 = \Gamma$ and there are No one-point Arbitrages, i.e.,

$$\ell(\omega) \geq 0 \forall \omega \in B^0 \Rightarrow \ell(\omega) = 0 \forall \omega \in B^0.$$

2. if $\beta > 0$ and $i = 1, \dots, \beta$ then:

- ▷ $B^i \neq \emptyset$;
- ▷ $\ell^i(\omega) > 0$ for all $\omega \in B^i$
- ▷ $\ell^i(\omega) \geq 0$ for all $\omega \in \cup_{j=i}^{\beta} B^j \cup B^0$.

We are now using the previous result, which is for some fixed t , to identify A^* . Define

$$\begin{aligned} A_T &:= A \\ A_{t-1} &:= A_t \setminus \bigcup_{i=1}^{\beta_t} B_t^i, \quad t \in \{1, \dots, T\}, \end{aligned} \quad (\text{F.3})$$

where $B_t^i := B_t^{i,\Gamma}$, $\beta_t := \beta_t^\Gamma$ are the sets and index constructed in Lemma F.5 with $\Gamma = A_t$, for $1 \leq t \leq T$. Note that, for the corresponding strategies ℓ_i^t we have

$$A_0 = \bigcap_{t=1}^T \bigcap_{i=1}^{\beta_t} \{\ell_i^t = 0\}. \quad (\text{F.4})$$

Lemma F.6. A_0 as constructed in (F.3) satisfies $A_0 = A^*$. Moreover, No one-point Arbitrage on $A \Leftrightarrow A^* = A$.

Proposition F.7. Let $A \in \mathcal{F}^A$. We have that for any \mathcal{F}^A -measurable random variable g ,

$$\pi_{A^*}(g) = \sup_{Q \in \mathcal{Q}_A} \mathbb{E}_Q[g]. \quad (\text{F.5})$$

with $\pi_{A^*}(g) = \inf \{x \in \mathbb{R} \mid \exists a \in \mathcal{I} \text{ such that } x + a_T(\omega) \geq g(\omega) \ \forall \omega \in A^*\}$. In particular, the left hand side of (F.5) is attained by some strategy $a \in \mathcal{I}$.

F.3 Properties of $\mathcal{L}^1(\Omega, \varphi)$.

Here we collect some elementary properties of integrals with respect to a bounded additive measure. The only minor difficulty arises from the fact that this integral may not be additive when the integrals are extended real valued.

Lemma F.8. Let $\varphi \in (\mathcal{B}_I)'_+$. φ is additive on $\mathcal{L}^1(\Omega, \varphi)$.

Proof. First we show that for $X \in \mathcal{L}^1(\Omega, \varphi)$ we have $\varphi(-X) = -\varphi(X)$. Note that, for $X \in \mathcal{L}^1(\Omega, \varphi)$, $\varphi(X) = \lim_{K \rightarrow \infty} \varphi((X \wedge K) \vee -K)$. Thus, since $(X \wedge K) \vee -K$ is bounded and $\varphi \in ba$, then

$$\varphi((-X \wedge K) \vee -K) = \varphi(-((X \wedge K) \vee -K)) = -\varphi((X \wedge K) \vee -K).$$

By taking the limit in both sides the result follows. Now, take $X, Y \in \mathcal{L}^1(\Omega, \varphi)$. Let $\alpha, \beta > 0$ and denote by $X^a := X \wedge \alpha$ and $Y^b := Y \wedge \beta$ observe that

$$((X + Y) \wedge K) \vee -K \geq ((X^a + Y^b) \wedge K) \vee -K.$$

For $K > \alpha + \beta$, we have

$$\begin{aligned} ((X^a + Y^b) \wedge K) \vee -K &= (X^a + Y^b) \vee -K \\ &\geq (X^a \vee -K) + (Y^b \vee -K). \end{aligned}$$

From these we obtain,

$$\varphi(((X + Y) \wedge K) \vee -K) \geq \varphi(X^a \vee -K) + \varphi(Y^b \vee -K).$$

Since $X, Y \in \mathcal{L}^1(\Omega, \varphi)$, by taking the limit for $K \rightarrow \infty$, we obtain $\varphi(X + Y) \geq \varphi(X^a) + \varphi(Y^b)$. By taking now the limit for $\alpha, \beta \rightarrow \infty$ we get

$$\varphi(X + Y) \geq \varphi(X) + \varphi(Y).$$

Since this holds for arbitrary $X, Y \in \mathcal{L}^1(\Omega, \varphi)$ and since $\varphi(-Y) = -\varphi(Y)$, we might replace X with $X + Y$ and Y with $-Y$ to obtain the converse inequality. \square

Lemma F.9. *Let $\varphi \in (\mathcal{B}_l)'_+$. For any $X \in \mathcal{H}$ and $Y \in \mathcal{L}^1(\Omega, \varphi)$,*

$$\varphi(X + Y) = \varphi(X) + \varphi(Y). \tag{F.6}$$

Proof. Since $Y \in \mathcal{L}^1(\Omega, \varphi)$, both $\varphi(Y^+)$ and $\varphi(Y^-)$ are finite. Since $\mathcal{L}^1(\Omega, \varphi)$ is a vector space, if X is also integrable (F.6) holds. Also,

$$[X^+ - X^-] + [Y^+ - Y^-] = X + Y = (X + Y)^+ - (X + Y)^-$$

Hence,

$$(X + Y)^+ + X^- + Y^- = (X + Y)^- + X^+ + Y^+.$$

Since $x^+x^- = 0$ for any real number, the above implies that

$$0 \leq X^- \leq (X + Y)^- + Y^+, \quad \text{and} \quad 0 \leq (X + Y)^- \leq X^- + Y^-.$$

Since Y is integrable, this implies that $\varphi((X + Y)^-)$ is finite if and only if $\varphi(X^-)$ is finite. Same argument also implies that $\varphi((X + Y)^+)$ is finite if and only if $\varphi(X^+)$ is finite. So if $\varphi(X^-) = \infty$, then $\varphi((X + Y)^-) = \infty$ and both sides of (F.6) are equal to minus infinity. Suppose that both $\varphi((X + Y)^-)$ and $\varphi(X^+)$ are finite. If $\varphi(X^+)$ is finite, then (F.6) holds and both sides are finite. If $\varphi(X^+) = \infty$, the both sides (F.6) are equal to infinity. \square

We conclude with a limit theorem for integrals. Let

$$L^* := \mathbf{1} + \mathbf{c}^* + \hat{\ell},$$

be as in Assumption C.1.

Lemma F.10. *Let $\varphi \in (\mathcal{B}_l)'_+$. Suppose $X \in \mathcal{H}$ satisfies $X \geq_\Omega -\alpha L^*$ for some $\alpha \in \mathbb{R}^+$. Then,*

$$\varphi(X) = \lim_{K \uparrow \infty} \varphi(X \wedge K).$$

Proof. Since $\varphi \in (\mathcal{B}_t)'$, $\alpha L^* \in \mathcal{L}^1(\Omega, \varphi)$. Set $Y = X + \alpha L^*$. Then, $Y \geq_{\Omega} 0$ and by definition,

$$\varphi(Y) = \lim_{K \uparrow \infty} \varphi(Y \wedge K).$$

Also, by the previous lemma, and the fact that $\alpha L^* > 0$,

$$\begin{aligned} \varphi(X) &= \varphi(Y) - \varphi(\alpha L^*) = \lim_{K \uparrow \infty} \varphi(Y \wedge K) - \varphi(\alpha L^*) \\ &= \lim_{K \uparrow \infty} \varphi([Y \wedge K] - \alpha L^*) \leq \lim_{K \uparrow \infty} \varphi([Y - \alpha L^*] \wedge K) \\ &= \lim_{K \uparrow \infty} \varphi(X \wedge K) = \varphi(X). \end{aligned}$$

Therefore, they are all equalities. □

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