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Representation of CLI-Game and the Expected Contract Value
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Abstract

We consider Cooperative Games with Incomplete Information ("CII-Game"), that is, a version of the NTU-characteristic or coalitional function in the presence of incomplete information about players' types. Decisions are parameters entering the utility functions of players, which otherwise depend on randomly chosen types. First of all we study the "canonical choice" of utility functions (e.g., with prevailing linearity in the side payment case). Based on this a "bargaining solution" or "value" is described that results from the expectation of all fair or post (NASH) bargaining solutions of the various realizations of types.
SECTION 1
Clf-Games and Fee Games; Motivation and Notations

A unanimous NTU-Game (or NASH-bargaining situation) for n players is a pair (\( y, V \)) such that \( V \) is a subset of \( \mathbb{R}^n \) ("the feasible utility vectors") and \( y \in V \) (the "status quo" or "threat point").

Within this presentation, we shall always assume that \( V \) is a closed, convex, and comprehensive set such that

\[
V_+ := \{ u \in V \mid u \geq 0 \}
\]

is compact. Also, we shall only consider \( y = 0 \) and require that 0 is within the interior of \( V \) — thus there is \( u \in V_+, u > 0 \).

The elements of \( V \) represent collective decisions that are available to the players if they choose to agree upon one of them, that is if they cooperate. In this context it is of course assumed that all players are aware of the consequences of such a decision for each of them; that is, since the elements of \( V \) are "utility vectors" it is assumed that it is common knowledge that a decision for \( u \in V \) implies the utility \( u_i \) for player \( i \in I := \{1, \ldots, n\} \).

In this paper we want to consider cooperative games with incomplete information; this theory starts out with the work of HARSANYI-SELTEN [9]. In this context it is assumed that there is a fixed set of decisions available to the players by collective and unanimous agreement, but the consequences or utilities resulting from such a decision may vary and depend in addition on random influence.

Most authors assume that the set of decisions in a cooperative game with incomplete information is finite. However, we shall choose

\[
\mathbb{X} = \{ x \in \mathbb{R}_+ \mid e' x \leq 1 \}
\]

as the set of collective decisions or "parameters"; here \( e = (1, \ldots, 1) \in \mathbb{R}_n \). If players fail to reach an agreement we want \( \bar{x} = 0 \in \mathbb{X} \) to represent the "status quo" or "threat" parameter. Also \( \delta \mathbb{X} = \{ x \in \mathbb{X} \mid e' x = 1 \} \) shall represent the "Pareto efficient" boundary of \( \mathbb{X} \).
I = \{1, \ldots, n\} represents the set of players. For i \in I let T^i denote the possible types of i;
T := T^1 \times \cdots \times T^n is the set of possible types of all players. The notion of types as used
in game theory goes back to HARSANYI [4], other authors prefer to speak about the
"state of nature" see e.g. ALLEN [1, 2].

The distribution of types is represented by a probability p on T.

Next, assume that for each i \in I a function

U^i : T^i \times \mathbb{X} \to \mathbb{R}

represents player i's utility depending on the actual types and the collective decision
agreed upon. Then the six-tuple

\[ (I, (T, p; \mathbb{X}, \mathbb{U}) \] (3)

is a (unanimous) cooperative game with incomplete information or for short, a
CII-game.

There are two stories related to the question as to how this game is being played. The
first, naive, or basic interpretation is the obvious one: consider some abstract probability
space (\mathbb{U}, \mathbb{P}, \mathbb{F}) and let \tau : \mathbb{U} \to T be a mapping with distribution p.
Nature performs a chance experiment represented by (\mathbb{U}, \mathbb{P}, \mathbb{F}), which results in some
\omega \in \mathbb{U}. Each player observes his "true type" \tau_i(\omega) and thereafter the players may agree
upon some "collective decision" or parameter x \in \mathbb{X}. If so, each of them receives the
utility U_i(\omega; x). This means that, while bargaining, players have knowledge about
their own type and may infer about the types of the other players from the conditional probability
resulting from \tau given that \tau_i takes a certain value.

The problem is that in such a situation the utilities of the resulting "bargaining solution" depend on the true types and hence, there is an incentive for players to possibly pretend that they are in a false type, thus increasing their true utility.

This leads to the slightly more refined story of the bayesian incentive compatible mechanism (in the sense of HURWICZ [7]).

According to this story a bayesian incentive compatible or BIC-mechanism is a
mapping \( \mu : T \to \mathbb{X} \) such that the following inequalities are satisfied.

\[ E[U|\mu'|z_1 = t_1] \geq E[U|\mu'|z_1 = t_1\ldots z_i = t_i] \quad (i \in I, t_i \in T^i, \omega \in T^n). \] (4)

The mechanism is interpreted as an agreement among the players (to be recorded and enforced by some outside agency or court). Accordingly everyone will make an announcement once the types are generally observed and the collective decision will depend on these announcements. A mechanism is BIC if making the true announcements about one's type maximizes each players utilities provided the others stick to this kind of policy. More precisely, in the resulting non-cooperative game induced by the mechanism \( \mu \), the strategy to report one's true type is a Nash equilibrium.

For fixed \( t \in T^n \), let

\[ V^t := \{ U^i(x) \mid x \in \mathbb{X} \} \]

\[ U^t := U^t(\omega) \] (5)

Then, given some suitable conditions to be imposed on \( U(\cdot) \), \( \{t, V^t\} = \{0, V^t\} \) is a
unanimous NTU-game; clearly it is the one generated by \( t \in T \) "with complete information". Thus, \( U_t : \mathbb{X} \to \mathbb{R} \) can be regarded as a parameterization. For various \( t \in T \)
(i.e. various \( \omega \in \Omega \), \( V^t \) represents the different consequences resulting from decisions \( x \in \mathbb{X} \)).

Fig. 1.1.
The appropriate form of the mappings \( U(\cdot) \) is certainly not without importance. We shall first and for some considerable time deal with these mappings. What influence does the parameterization have on possible solution concepts? Can we, at least for some class of games, axiomatically define something like a *canonical representation* or *parametrization* of CII-games?

The first requirement to every \( U(\cdot) \) is that we want it to be bijective. For, otherwise, the introduction of certain equivalence classes of parameters, which result in equal utility as far as certain players are concerned, would further complicate the situation. Indeed, if our parameterizations are bijective, then we actually attempt the identification of certain points of \( V^t \ (t \in T) \), namely those resulting from the same parameter or collective decision \( x \in \mathcal{X} \). Hence a CII induces an *equivalence relation* between various (unanimous) NTU-games, i.e., between the \((0, V^t) \), \( t \in T \).

A second requirement we attempt to impose on the parameterizations is monotonicity. It certainly clarifies the situation for the players involved if, when bargaining about some \( x \in \mathcal{X} \), they can behave like when bargaining about utility vectors: by this we mean that player \( i \) wants to maximize his coordinate \( x_i \); and hence we should require that every coordinate \( U_i(\cdot) \) of \( U(\cdot) \) is strictly monotone in \( x_i \).

Furthermore, it seems reasonable to require that players are aware when they are dealing with Pareto efficient utility vectors. Hence it seems reasonable to construct \( U \) in such a way that \( U(x) \) is Pareto efficient in \( V^t \) whenever \( x \in \partial \mathcal{X} \).

The above requirements exhibit the tendency to represent the situation as simple as possible and to include as many of the familiar features of a bargaining situation or NTU-Game as possible in the situation with incomplete information ("semia non sunt multiplicanda praeter necessitatem"). Still, further requirements are certainly necessary (e.g., convexity and continuity).

In order to exhibit a "canonical parametrization" let us start out with the simplest case, that is, the side-payment game. As parametrization can be studied without referring to types and the distribution of types, we shall tentatively omit the type vector \( t \) and start out with some (unanimous) NTU-game, say \( (y, V) \) (with \( y = 0 \)). The side-payment character is stressed by putting, for \( c > 0 \) and \( e = (1, \ldots, 1) \),

\[
V = V^{e/c} := \{ u \in \mathbb{R}^n \mid e_f u \leq 1 \} = \{ u \in \mathbb{R}^n \mid c u \leq c \}
\]

Now, as \( (0, V^{e/c}) \) represents a side-payment situation, we want \( U(\cdot) \) to reflect this character as much as possible. This means, that in every coalition \( S \subseteq I \) the result of a redistribution of parameter units ("money") does not change total utility. Clearly, linearity is the least we should ask for in this context.

Having this in mind, we come up with

**Theorem 1.1** Let \( c > 0 \) and let

\[
U : \mathcal{X} \rightarrow V^{e/c}
\]

be a mapping satisfying the following conditions.

1. \( U \) is linear.
2. \( U_t \) is strictly monotone in \( x \).
3. For any \( y \in \mathbb{R}^n \) and any \( S \subseteq I \) such that \( e_y = 0 \) and \( e_t = 0 \) (\( i \not\in S \)) it follows that, for all \( x \in \mathcal{X} \),

\[
\sum_{i \in S} U_i(x + y) = \sum_{i \in S} U_i(x)
\]

4. \( U \) maps \( \partial \mathcal{X} \) on \( \partial V^{e/c} \).

Then there is \( C > 0 \) and \( b^0 \in \mathbb{R}^n \) with \( e b^0 < 1 \) such that

\[
U(x) = C(x - (ex)b^0) \quad (x \in \mathcal{X})
\]
Proof:

\[ e^T x = 1 \]

\[ e^T x = c \]

Fig. 1.2.

As \( U \) is linear, there is an \( n \times n \)-matrix \( A \) such that

\[ U(x) = Ax \quad (x \in \Sigma). \]

Consider some \( S \subseteq I, S \neq \emptyset \).

Clearly, it follows from condition 3 that

\[ \sum_{i \in S} A_i \cdot y = 0 \]

for any \( y \in \mathbb{R}^n \) (except \( y \) such that \( y_l = 0 \) for some \( i \notin S \)). Consequently, the \( S \)-coordinates of \( \sum_{i \in S} A_i \cdot y \) have to be equal, say

\[ \left( \sum_{i \in S} A_i \right)_S = \lambda_S e_S \quad (9) \]

for suitable \( \lambda_S \in \mathbb{R} \). Replace \( S \) by \( S + \{i_0\} \) for some \( i_0 \notin S \) and subtract equations (9) corresponding to \( S \) and \( S + \{i_0\} \); it turns out immediately that all \( a_{i,j} \) corresponding to \( j \notin S \) must be equal. By varying \( S \) we find that all \( a_{i,j} \) outside the diagonal have to be equal. Thus,

\[ A_{i,j} = c_j (e_l - b_l) \]

with \( C_j \in \mathbb{R} \) and \( b_l \in \mathbb{R}^n \).

Now, condition 2 requires \( a_{i,j} > 0 \), thus \( C_j > 0 \) \((j \in I)\).

Next, by condition 4, the image of \( e^T \) has to be Pareto-efficient, that is, we obtain

\[ c = c^A e^A - e^T A_{i,j} = C_j (1 - a_{i,j}) \]

Hence we conclude that \( C_j \) equals \( c^A \), i.e.,

\[ A x = c^A (x - (e^T) b^i), C = \frac{c}{1 - a_{i,j}} \]

as \( c > 0 \).

\[ \text{q.e.d.} \]

If \( C = 1 \) then \( U \) clearly is the parametrization used in the context of "fee games" (see [13,14]). That is, we may interpret \( X \) to be the possible distributions of a unit of money among the players. If players agree on \( x \in X \) than each one of them has to pay a fee (say to the court or referee) which is proportional to the total amount of money agreed upon. The share of this fee is specific to each player and random in the case of a CII-game (a fee game). In this context, types of players are being distinguished only by the fact that they have to pay different fees according to the chance moves and each of them is aware only of the fee he has to pay for himself.

The constant \( C \) amounts to a contraction or extension of the Pareto efficient interval or to a common rescaling of utility for all players; thus \( C \) has normalizing character. In the context of Section 2 we shall find that \( C = \frac{1}{2} \) is more suitable for our present purpose of defining the canonical representation.
SECTION 2

Canonical Representation of Hyperplane-Games

As previously we write \( c = (1, \ldots, 1) \in \mathbb{R}^n \), \( c^t = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n \) is the i'th unit vector.

Next, if \( \mathbb{R}^n \alpha \neq 0 \) and \( x \in \mathbb{R}^n \), we write

\[
\frac{1}{\alpha} = \left( \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n} \right)
\]

and

\[
\alpha \otimes x = (\alpha_1 x_1, \ldots, \alpha_n x_n)
\]

(the tensor product – or rather an affine transformation of utility, "a.t.u"); consequently

\[
\frac{1}{\alpha} \otimes x = \frac{1}{\alpha_1} x_1, \ldots, \frac{1}{\alpha_n} x_n).
\]

Note that

\[
\alpha \left( \frac{1}{\alpha} \otimes x \right) = c \cdot x = \sum_{i=1}^{n} x_i.
\]

Next \( \bar{X} = \{ x \in \mathbb{R}^n \mid c \cdot x \leq 1 \} \) and \( \partial \bar{X} = \{ x \in \mathbb{X} \mid c \cdot x = 1 \} \).

Similarly

\( V^n = \{ x \in \mathbb{R}^n \mid c \cdot x \leq 1 \} \)

and

\( \partial V^n = \{ x \in V^n \mid c \cdot x = 1 \} \).

For fixed \( b^* \in \partial \bar{X} \), define

\[
U((\alpha, \cdot)) : \bar{X} \rightarrow V^n
\]

\[
U((\alpha, x)) = \frac{\alpha x - (c x - (c x - (c x - (c x - \frac{1}{\alpha} \otimes b^*)))}{\alpha}
\]

(1)

Of course, \( U((\alpha, \cdot)) \) maps \( \partial \bar{X} \) onto \( \partial V^n \) (bijectively) because of

\[
\frac{\alpha U((\alpha, x))}{\alpha} = \frac{(\alpha x) - (c x - (c x - (c x - \frac{1}{\alpha} \otimes b^*)))}{\alpha}
\]

\( = \frac{c x}{\alpha} \)

(2)

Definition 2.1: \( U((\alpha, \cdot)) \) is the canonical parametrization of \( V^n \).

\( U((\alpha, \cdot)) \) is linear and "monotone in each coordinate" (cf. SECTION 1). In fact, the connection to "free-games" as discussed shortly in SECTION 1 (and generally in [ ] ) is evident:

Remark 2.2: We may regard \( V^n \) as a transformed free-game as follows:

Write

\[
U((\alpha, x)) = \frac{c x}{\alpha} \otimes (c x - (c x - (c x - \frac{1}{\alpha} \otimes b^*)))
\]

and put

\[
\alpha^n := \frac{c x - 1}{c x}, \quad \lambda = \frac{\alpha}{c x},
\]

then

\[
U((\alpha, x)) = \frac{1}{\lambda} \otimes (c x - (c x - \alpha^n \otimes b^*))
\]

\( = \frac{1}{\lambda} \otimes (c x - (c x - \frac{1}{\alpha} \otimes b^*))
\]

(5)

E.g., for \( \alpha = \left( \frac{1}{n^2}, \ldots, \frac{1}{n^2} \right) \) with \( c x = 1 \) and \( \lambda = \frac{1}{\alpha^n}, \alpha^n = \frac{1}{n^2} \), we come up with

\[
U((\alpha, x)) = \frac{1}{n} \otimes (c x - (c x - \frac{1}{\alpha} \otimes b^*))
\]

(6)

This is the utility as used in free games – up to the factor \( \frac{1}{n} \) (The factor \( \frac{1}{n} \) could be avoided by introducing "\( Y^n = \{ u \in \mathbb{R}^n \mid c u \leq n \} \)", the advantage would be invariance of length measurement as discusses in SECTION 1 – the disadvantage occurs in some notational scramble.)
Remark 2.3: A geometrical interpretation of the mapping $U(\alpha_x)$ is given as follows. Note that

$$U(\alpha, b^*) = \frac{1}{\alpha} \ast b^* = b$$

(7)

(the "rescaled fee vector...".) Therefore, if we define

$$a^i = a^{i0} := \frac{1}{\alpha_a} e^i + \left(1 - \frac{1}{\alpha_a}\right) b^* \quad (i \in I)$$

(8)

then

$$U(\alpha, a^i) = \frac{1}{\alpha} \ast e^i = \frac{a^i}{\alpha}.$$  

Now, consider the simplex

$$I^0 = [a^1, ... , a^p].$$

(9)

Observe that $U(\alpha_x)$ acts in a way such that $a^i$ is thrown into $\frac{e^i}{\alpha}$, $b^*$ is distorted to $\frac{a^i}{\alpha}$, and the points of $I^0$ are exactly mapped into the Pareto efficient and individually rational utility-vectors of $\nu \in \nu^0$.

In particular, if $\alpha = t \left(\frac{1}{\alpha_1}, ... , \frac{1}{\alpha_p}\right)$ (cf. Remark 2.2), then $\alpha_a = t$ and thus

$$a^i = \frac{1}{\alpha} e^i + \frac{t-1}{t} b^*.$$  

Thus, distance measurement on $X$ is distorted by a factor $\frac{t}{\alpha}.$

![Diagram](image)

Fig. 2.2.

Lemma 2.4: The inverse mapping to $U(\alpha_x)$ is given by

$$G(\alpha_x) : \nu^0 \rightarrow \mathbb{R}$$

$$G(\alpha, a) = \alpha_a a + \frac{\alpha}{\alpha_a} (\alpha_a) b^*$$

(10)

Proof: $G(\alpha_x)$ is a linear mapping such that $G(\alpha, 0) = 0$ and $G(a, e^i) = a^i$, for the role of $a^i$ cf. Remark 2.3 and (7) in particular.

The following monotonicity property of $G$ will be important for our presentation in SECTION 3. In this context, let $\bar{u} \in \mathbb{R}^n$, and let
Thus we have

\[ \mathcal{Y}^0 = \{ a \in \mathbb{R}^3 \mid a \cdot a = 1 \} \]  

(11)

Lemma 2.5:

Let \( n = 2 \). Consider \( G_i : \mathcal{Y}^0 \to \mathbb{R} \). Then \( G_i(a, b) \) is strictly monotone in \( a_i \) for all \( a \) with \( e \cdot a \geq 0 \). More precisely, if \( a' \) is close to \( a \) and \( a_i > a'_i \) then

\[ G_i(a, b) > G_i(a', b). \]  

(14)

Proof:

Our claim amounts to showing that the directional derivative of

\[ G_i(a, b) = G_i(a', b) \]

for \( \left( \frac{1}{u_1^*} \right), \left( \frac{1}{u_2^*} \right) \) points in direction of increasing \( a \). We shall do this for \( a > 0 \) only and leave the details for the other cases as an exercise.

To this end, observe that

\[ \frac{\partial}{\partial a_i} G_i(a) = \frac{u_i}{\epsilon} \frac{a_i + b_i}{(e \cdot a)^2} \]

and

\[ \frac{\partial}{\partial a_i} G_i(a) = -\frac{u_i}{\epsilon} \frac{a_i + b_i}{(e \cdot a)^2} \]

such that

\[ \left( \frac{1}{u_1^*} \right), \left( \frac{1}{u_2^*} \right) \]

\[ = \left( \frac{1}{u_1^*} \right) \frac{1}{u_2^*} \frac{a_i + b_i}{(e \cdot a)^2} \]

\[ = \left( \frac{1}{u_2^*} \right) \frac{1}{u_1^*} \frac{a_i + b_i}{(e \cdot a)^2} \]

\[ = \left( \frac{1}{u_1^*} \right) \frac{1}{u_2^*} (1 - \frac{1}{u_1^*} \frac{1}{u_2^*}) \]

\[ > 0 \quad \text{as } b_i \leq \frac{1}{u_1^*} \]

q.e.d.

Now we turn shortly to the role of \( \alpha, u, s \) in connection with the canonical representation.

Definition 2.6:

For \( \alpha > 0 \) define

\[ S_0 : 1^a \to 1^u \]

\[ S_0(x) = \frac{1}{x} + (1 - \frac{1}{x}) \cdot b_s \]  

(11)

The inverse mapping \( T_0 : 1^u \to 1^a \) is given by

\[ R_0(y) = ty + (1-t) \cdot b_s = S_0^{-1}(y) = S_0(y) \]  

(12)

(or course \( S \) and \( R \) can be defined as mappings \( \Xi \to \mathcal{X} \))

Lemma 2.7:

For \( \alpha > 0 \), \( a \in \mathcal{Y}^0 \), and \( z \in \mathbb{R}^3 \)

\[ U(\alpha, z) = U(\alpha, z) - (e \cdot z) \frac{1}{\alpha} \cdot b_s^* \]  

(18)

with

\[ b_s = \frac{1}{\alpha} \cdot b_s^* . \]  

(17)

Proof:

By definition, i.e. (17), \( U(\alpha, b_s^* ) = 0 \), hence

\[ U(\alpha, b_s^* ) = \frac{1}{\alpha} \cdot b_s = \frac{1}{\alpha} \cdot b_s \]  

(18)

and hence

\[ U(\alpha, b_s^* ) - U(\alpha, b_s^* ) = (1 - \frac{1}{\alpha}) \cdot b_s \]  

(19)

Consequently,

\[ U(\alpha, z) - U(\alpha, z) = -(e \cdot z) \frac{1}{\alpha} \cdot b_s^* \]  

\[ = -(e \cdot z) \frac{1}{\alpha} \cdot b_s^* . \]
Lemma 2.8: For \( t > 0 \) and \( \alpha > 0 \), \( x \in \partial X \)

1. \( \frac{1}{t} V^0 = V^{t \alpha} \)

2. \( \frac{1}{t} U(\alpha, x) = U(\alpha, S_{\alpha}(x)) \)

3. Thus, if \( u \in \partial V^0 \) is parametrized by \( x \in I^\alpha \), then \( \frac{1}{t} u \in \frac{1}{t} V^0 = \partial V^{t \alpha} \) is parametrized by \( \frac{1}{t} x + (1 - \frac{1}{t}) b^* \).

Proof: The first statement follows immediately from the definition of \( V^0 \).

As for the second, we rewrite Lemma 2.7 for \( x \in \partial X \), we have

\[
U(\alpha, x) - \frac{1}{t} b = U(\alpha, x) - b; \quad (20)
\]

also

\[
U(\alpha, b^*) = \frac{1}{t} b = \frac{1}{t} u(\alpha, b^*). \quad (21)
\]

Hence, for \( x \in \partial X \)

\[
U(\alpha, \frac{1}{t} x + (1 - \frac{1}{t}) b^*) = \frac{1}{t} U(\alpha, x) + (1 - \frac{1}{t}) U(\alpha, b^*) \quad (22)
\]

\[
= \frac{1}{t} (U(\alpha, x) - U(\alpha, b^*)) + U(\alpha, b^*)
\]

\[
= \frac{1}{t} (U(\alpha, x) - U(\alpha, b^*)) + \frac{1}{t} U(\alpha, b^*)
\]

(in view of (22) and (21))

\[
= \frac{1}{t} U(\alpha, x).
\]

The following sketch relates the a.t.a. induced by \( 1/t \) and the action of \( S_1 \) (for \( t > 1 \)).

**Fig. 2.4.**

\( S_t \) shrinks the Pareto efficient interval \( I^a \) to \( I^t \).

**Theorem 2.9:** Let \( \alpha, \beta \) be l.u.b and let \( x \in \partial X \). Then

1. \( \frac{1}{\beta} \ast V^0 = V^{t \beta \alpha} \)

2. \( \frac{1}{\beta} \ast U(\alpha, x) = U(\beta \ast \alpha, S_{\beta \alpha}(x)) \)

3. That is, if \( u \in V^0 \) is parametrized by \( \alpha \), then \( \frac{1}{\beta} \ast u \in V^{t \beta \alpha} \) is parametrized by \( S_{\beta \alpha}(x) \).
Proof: 1st STEP:

If \( \beta \alpha = \infty \), then

\[
\frac{1}{\beta} \cdot U(\alpha, x) = U(\beta \cdot \alpha, x)
\]

(22)

follows immediately from the definition of \( U \) by taking into account that \( \epsilon(\beta \cdot \alpha) = \beta \alpha \).

Hence it is seen that in the general case

\[
\frac{1}{\beta} \cdot U(\alpha, x) = \beta \alpha \cdot U(\frac{\alpha}{\beta \alpha}, x).
\]

Taking \( \tau := \frac{\beta \alpha}{\infty} \), this means

\[
\frac{1}{\beta} \cdot U(\alpha, x) = \frac{1}{\tau} \cdot U(\frac{1}{\beta} \beta \cdot \alpha, x)
\]

which by 2. of Lemma 2.7 is continued by

\[
= U(\frac{1}{\tau} \cdot \beta \cdot \alpha, S_{\frac{1}{\beta \alpha}}(x))
= U(\beta \cdot \alpha, S_{\frac{1}{\beta \alpha}}(x))
\]

q.e.d.

\[\text{SECTION 3}
\]

\[\text{Parametrization of the General NTU-Game}\]

Consider an NTU-game \((0, V)\) and some utility vector \( \bar{u} \) that is Pareto efficient, i.e., \( \bar{u} \in \partial V \).

For the normal vectors \( \alpha > 0 \) at \( \partial V \) in \( \bar{u} \) we choose a normalization such that

\( \alpha \bar{u} = 1 \)

such that the tangential hyperplane

\( \{ u \mid \alpha u = 1 \} = \partial V^\alpha \)

cuts through axis \( i \) in \( \frac{1}{\alpha} \cdot \alpha \).

\[\text{Fig. 3.1.}\]

\( V^\alpha \) constitutes the "side payment" alternatives available if, at \( \bar{u} \in V \), side payments are permitted "at rate \( \alpha \)." Clearly, \( V^\alpha \) is "canonically" represented / parametrized by \( U(\alpha, \cdot)\).
Denote the normals by
\[ N(u) = \{ \alpha : \text{a normal in } u, \alpha u = 1 \} \]
for \( u \in \partial V \). Moreover
\[ N(V) = \bigcup_{u \in \partial V} N(u) = \text{convex hull} \{ \{ N(u) | u \in \partial V \} \}. \]
(Clearly a convex set.) Next, define the (negative) dual cone of \( V \) to be
\[ K(V) = \{ u \in \mathbb{R}^n | \alpha u \leq 0 \quad (\alpha \in N(V)) \} \]

Note that for any \( u \in V \), \( u \not\in \partial V \), \( u \notin K(V) \) we have \( \lambda u \in \partial V \) for some \( \lambda > 0 \).

The dual cone is also defined for \( X \); clearly \( K(X) = \{ x \in X | ex \leq 0 \} \).

Fig. 3.2.

and, again for \( x \in X \) and \( ex > 0 \) it follows that \( \frac{x}{ex} \in \partial X \).

Our parametrization of \( V \) by \( X \) should desirably transfer halflines \( (\lambda x | 0 \leq \lambda \leq \frac{1}{ex}) \) in halflines \( (\lambda u | 0 \leq \lambda \leq \frac{1}{ex}) \) so that players know simultaneously the impact of a proportional reduction of parameter with respect to utility. (At least, this would mean linearity on lines connecting the origin and points in \( \partial X \).)

Monotonicity is a further requirement, thus player \( i \) is interested in increasing \( x_i \) — in other words, we want to save as much as possible of the properties of the parametrization \( U(\alpha, \cdot) \). And possibly, the \( \alpha \) should be the one (anyone) suggested by \( u \in \partial V \), i.e., we should consider \( \alpha \in N(u) \).

Definition 3.1:
1. Let \((O, V)\) be a standard NTU-game and let \( A : \partial X \rightarrow \mathbb{R}^n \) be a mapping. Consider the mapping...
(4) \[ U^A : \{ x \in \mathbb{R}^n \mid \text{ex} > 0 \} \rightarrow \mathbb{R}^n \]
(5) \[ U^A(x) = U(A(x), x) \quad (x \in \mathbb{R}^n, \text{ex} > 0) \]

2. Suppose \( A \) satisfies the following conditions:

\( A \) is continuous.

(6) \[ A(x) \in N(U^A(x)) \quad (x \in \partial \mathbb{R}^n). \]

(7) For all \( u \in \partial V \), \( A^{-1}(N(u)) \neq \emptyset \).

Then \( A \) is called the (the) canonical normal mapping of \( V \) (or just "\( A \) is normal to \( V \))."

Any parametrization

\[ U : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

such that \( U = U^A \) on \( \{ x \in \mathbb{R}^n \mid \text{ex} > 0 \} \) is called "canonical" as well.

Remark 3.2: Conceivably, the behavior of a mapping \( U \) outside the set \( \{ x \in \mathbb{R}^n \mid \text{ex} > 0 \} \) is of no interest. Once we have established existence and uniqueness of \( A \) we shall therefore speak of "the" canonical representation of \( V \) as well and mention \( U^A \) only.

Indeed, uniqueness of \( A \) does not pose a serious problem. For, given \( \hat{u} \in \partial V \) with a unique \( \hat{x} \in N(\hat{u}) \), there is a unique \( \tilde{x} \in \mathbb{R}^n \) such that \( U(\hat{a}, \tilde{x}) = \hat{u} \) (i.e., \( \tilde{x} = G(\hat{a}, \hat{u}) \)) and clearly \( A \) should satisfy \( A(\tilde{x}) = \hat{a} \). And if \( N(\hat{u}) \) consists of more than one point, then (again providing the argument for \( n = 2 \) only) it is an interval. Within this interval we must have

\[ G(A(x), \hat{u}) = \tilde{x} \]

i.e.,

\[ A(x) = G(\cdot, \hat{u})^{-1}(x) \]

provided \( G(\cdot, \hat{u}) \) is invertible. This property we constitute in passing by pointing out that Lemma 2.5 induces the monotonicity of \( G(\cdot, \hat{u}) \) in the first coordinate.

Perhaps it should be repeated at this instant that \( U(a, \cdot) \) is defined with reference to some fixed \( b' \). Of course the same holds true with respect to the "canonical" representation \( U^A \) for some \( V \).

Example 3.3: Let \( \alpha' \in \mathbb{R}^n \) and \( \alpha'' \in \mathbb{R}^n \) be as sketched in Fig. 3.4.

\[ \partial \mathbb{R}^n \text{ is decomposed into } 3 \text{ "intervals" which is in a self-explaining notation represented by} \]

\[ \partial \mathbb{R}^n = \{ -\alpha, \alpha' \} \cup \{ \alpha', \alpha'' \} \cup \{ \alpha'', \alpha \}. \]

Define \( A : \partial \mathbb{R}^n \rightarrow \mathbb{R}^n \) by

\[ A(x) = \begin{cases} \alpha' & x \in (-\alpha, \alpha') \\ \alpha & x \in (\alpha', \alpha'') \\ \alpha'' & x \in (\alpha'', \alpha) \end{cases} \]

Now, as \( eA(x) = 1 \ (x \in \partial \mathbb{R}^n) \), we come up with

\[ U^A(x) = \frac{1}{A(x)} \cdot x \quad x \in \partial \mathbb{R}^n \]

i.e.,

\[ U^A(x) = \begin{cases} \frac{1}{\alpha'} \cdot x & x \in (-\alpha, \alpha') \\ (1,1) & x \in (\alpha', \alpha'') \\ \frac{1}{\alpha''} \cdot x & x \in (\alpha'', \alpha) \end{cases} \]

Next, \( U(a', \cdot) = \frac{1}{\alpha'} \cdot \) parametrizes \( \mathbb{R}^n \) and, in addition \( U(a', a) = (1,1) \).

From this it is seen at once that \( U^A \) parametrizes \( \mathbb{R}^n \). Since \( \alpha', \alpha'' \in \partial \mathbb{R}^n \), it follows that \( (1,1) \in \partial \mathbb{R}^n \). Thus the parametrization is such that \( (-\alpha, \alpha') \) is mapped bijectively on \( (-\alpha, (1,1)) \) and similarly \( \partial \mathbb{R}^n \) on \((1,1), \alpha) \). All \( x \in (\alpha', \alpha'') \) are mapped into \((1,1) \) etc.
A(x) yields always the normal in \( U(A(x), x) = U^A(x) \) and while \( x \) is running through \([\alpha', \alpha'']\), \( A(x) \) describes the normal cone at \((1, 1)\).

\[ \text{Fig. 3.5.} \]

Note that in this example \( \beta^* \) does not occur as \( eA(x) = 1 \ (x \in X) \).

Remark 3.4: In view of Theorem 2.0 we may attempt to compute the action of \( L \) on "normal functions''.

If \( A \) is normal to \( V \) and, for some \( \tilde{u} \in \partial V \), it happens that

\[ \hat{u} = U(A(\tilde{x}), \tilde{x}) \]

for suitable \( \tilde{x} \in \partial X \), then it follows by virtue of Theorem 2.9 that

\[ \frac{1}{\beta} \cdot \hat{u} = U(\beta \cdot A(\tilde{x}), S_{\partial A}(\tilde{x})), \]

\[ eA(\tilde{x}) \]

If one can show that

\[ x \rightarrow S_{\partial A}(\tilde{x})(x) \]

is invertible (we omit this discussion...) with some function \( Q \), say, then clearly (11) reads

\[ \frac{1}{\beta} \cdot \hat{u} = U(\beta \cdot A \circ Q(\tilde{y}), \tilde{y}), \]

that is \( \beta \cdot A \circ Q \) is normal to \( \frac{1}{\beta} V \).

Example 3.5: (cf. Example 3.3). For \( \alpha', \alpha'' \in \partial X \) we know the function \( A = A^V \) which is normal to \( V = V_{\alpha'} \cap V_{\alpha''} \). This is (cf. (9))

\[ A(x) = \begin{cases} \alpha' & x \in (-\infty, \alpha') \\ \kappa & x \in [\alpha', \alpha''] \\ \alpha'' & x \in (\alpha'', \infty) \end{cases} \]

Now, consider \( \frac{1}{\beta} \cdot (V_{\alpha'} \cap V_{\alpha''}) = \frac{1}{\beta} \cdot V \).

\[ \text{Fig. 2.6.} \]

Now, as \( eA(x) = 1 \) for \( x \in \partial X \) we have

\[ S_{\partial A}(\tilde{x})(x) = S_{\partial A}(\tilde{x})(x), \]

hence, if \( \hat{u} = U(A(\tilde{x}), \tilde{x}) \), then

\[ \frac{1}{\beta} \cdot \hat{u} = U(\beta \cdot A(\tilde{x}), S_{\partial A}(\tilde{x})(\tilde{x})) \]

\[ = U(\beta \cdot A \circ R_{\partial A}(\tilde{y}), \tilde{y}) \]

with \( \tilde{y} = S_{\partial A}(\tilde{x})(\tilde{x}) \). Clearly, for \( x \in (-\infty, \alpha'] \) it follows that \( y = S_{\partial A}(\tilde{x})(x) \in (-\infty, S_{\partial A}(\alpha') \) and

\[ \frac{1}{\beta} \cdot V \]

Thus the canonical normal mapping corresponding to \( \frac{1}{\beta} V \) is given by \( A = A^V \) via
A direct computation is even more revealing since it suggests the unique procedure defining the normal function "at a corner of \( V \)."

To this end consider again

\[
V = V' \cap V''
\]

as indicated in Example 3.5; we assume that \( \gamma = \frac{1}{\beta} \) is the corner point, thus

\[
\gamma' \cdot \frac{1}{\beta} = 1 = \gamma'' \cdot \frac{1}{\beta}
\]

Fig. 3.7.

Clearly, we must have

\[
A(x) = \begin{cases} 
    \gamma' \times \epsilon \left( -\epsilon, G(\gamma', \frac{1}{\beta}) \right) \\
    \gamma'' \times \epsilon \left[ G(\gamma'', \frac{1}{\beta}), \sigma \right] 
\end{cases}
\]  

(14)

In addition we must have

\[
U(A(x), x) = \frac{1}{\beta} \times \epsilon \left[ G(\gamma', \frac{1}{\beta}), G(\gamma'', \frac{1}{\beta}) \right]
\]  

(15)

Indeed, (15) determines \( A \) uniquely within the interval in question for it implies for

\[
A(x) = \alpha:
\]

\[
\frac{\alpha}{\alpha} \times (x - \frac{\alpha-1}{\alpha} b^*) = b^*
\]

\[
\alpha = \epsilon(0a) \beta \times (x - \frac{\alpha-1}{\alpha} b^*)
\]

\[
= \beta \times ((0a) x + (1 - \epsilon(0a)) b^*)
\]

\[
= \beta \times R_{(0a)}(x) = A(x);
\]

here \( t(x) \) is computed as

\[
t(x) = \epsilon n = \beta(0a) x + (1 - t(x)) b^*;
\]

i.e.

\[
t(x) = \frac{\beta b^*}{1 - \beta + \beta b^*}
\]

Thus

\[
A(x) = \beta \times R_{\frac{\beta b^*}{1 - \beta + \beta b^*}} \times \epsilon \left[ G(\gamma', \frac{1}{\beta}), G(\gamma'', \frac{1}{\beta}) \right]
\]  

(18)

and (14) as well as (18) describe \( A \) uniquely.

Fig. 3.8.

Remark 3.6: Clearly, the construction for the canonical representation, as indicated by the second part of Example 3.5, makes sense only if we can be sure that
\[ G(\gamma^*, \frac{1}{\beta}) > G(\gamma, \frac{1}{\beta}) \] (10)

whenever \( \gamma^*_1 > \gamma^*_1 \). However, as \( \gamma^* \frac{1}{\beta} = \gamma^* \frac{1}{\beta} = 1 \), this is a consequence of Lemma 2.5, that is, of the monotonicity of \( G(\cdot, \beta) \) in the first coordinate.

**Theorem 3.7:** For every \( V \) the canonical mapping \( A \) is uniquely defined.

**Proof:**

1st STEP: We shall restrict our exposition on the case that \( V \) is smooth, i.e., there is a unique supporting hyperplane at any \( u \in \partial V \). (For the case of a kink consider Remark 3.2 and Example 3.6.) For any such \( u \in \partial V \) denote by \( \mathcal{A}(u) \) the unique element in \( N(u) \).

Clearly, \( G(\mathcal{A}(u), u) \in X \) is the element of \( X \) that should be thrown into \( \mathcal{A}(u) \) such that \( U(\mathcal{A}(u), G(\mathcal{A}(u), u)) = u \). Therefore, it is conceivably sufficient to show that \( G^0(u) := G(\mathcal{A}(u), u) \) induces a bijective mapping \( G^0 \).

For indeed, assume that \( G^0 \) is bijective and let \( G^{0^{-1}} \) denote the inverse function

\[ G^{0^{-1}} : \partial X \to \partial V. \]

Define

\[ A(x) := \mathcal{A} \circ G^{0^{-1}}(x). \] (20)

That, for any \( u \in \partial V \) and \( x = G^0(u) \), we have to show that

\[ U(A(x), x) = u. \] (21)

However,

\[ x = G^0(u) = G(\mathcal{A} \circ G^{0^{-1}}(x), u) \]
\[ = G(\mathcal{A} \circ G^{0^{-1}}(x), u) \]
\[ = G(A(x), u), \]

and hence

\[ U(A(x), x) = U(A(x), G(A(x), u)) = u; \] (22)

this concludes our first step, i.e., it is indeed sufficient to show that \( G^0 \) is bijective.

**2nd STEP:** We shall restrict our argument to the case that \( n = 2 \) and \( V \) is strictly concave, i.e., no line segments appear in \( \partial V \) — otherwise some details will have to be supplied which does not necessarily enlighten the argument.

First of all consider \( \alpha \) and \( \alpha' \) (\( \in \mathbb{R}^n \)) such that

\[ \alpha > \alpha'. \]

Let \( \bar{u} \in \partial V^0 \cap \partial V^{10} \), i.e., \( \alpha \bar{u} = \alpha' \bar{u} = 1 \).

\[ \alpha \]
\[ \alpha' \]

\[ \text{Fig. 3.6.} \]

In view of Lemma 2.5 we know that

\[ G(\alpha, \bar{u}) > G(\alpha', \bar{u}). \]

In other words, given \( \bar{u} \), we know that \( G_1 \) is strictly monotone in \( \alpha \).
Next, consider the situation where \( \alpha' \) as above, but assume in addition that \( \alpha \in N(u) \) and \( \alpha' \in N(u') \) with (necessarily) \( u_1 > u'_1 \). This time, let \( \bar{\alpha} \) denote the intersection of \( \partial V^o \) and \( \partial V^o' \).

Clearly,
\[
G(\alpha', u') < G(\alpha, \bar{\alpha}) < G(\alpha, \bar{\alpha}) < G(\alpha, u).
\]

This means of course that
\[
G(\alpha, u) \neq G(\alpha, u)
\]
for \( u \neq u' \), i.e., \( G^o \) is 1-1.

Moreover, as \( u_1 \to -\infty \) (using the facts that \( V \) is compact and \( V \) is comprehensive), we observe that \( G(\alpha, u) \to -\infty \). Similarly, as \( u_2 \to -\infty \), \( G(\alpha, u) \to -\infty \) thus by continuity we conclude that \( G^o \) indeed maps \( \partial V \) uniquely onto \( \partial X \), q.e.d.

**Remark 3.8:** If \( A \) is the (canonical) normal mapping of some feasible set \( V \) which results from a standard NTU-game, then there is a simplex \( I^A \subseteq X \) such that \( U^A \) maps \( I^A \) bijectively onto \( \partial V \), i.e., on the Pareto efficient and individually rational points of \( V \). If \( A \) is constant, hence \( V = V^o \), then \( I^A = I^o \) as discussed in Remark 2.3.

**Example 3.9:** Clearly \( A(x) = t = (t > 0, x \in \partial X) \) yields a free game and \( A(x) = x \) \( (x \in \partial X) \) yields \( V = \{ (x, 1) \mid x \leq 1 \} \) (not included in our formal framework). This suggests that these two cases in some sense are "extreme".

Consider
\[
A(x) = \frac{6}{2} + x \quad (x \in \partial X),
\]
here \( eA(x) = 2 \) and hence for \( x \in \partial X \)
\[
U(A(x), x) = \left[ \frac{2x_1 - b_1^i}{x_1 + 1/2}, \frac{2x_2 - b_2^i}{x_2 + 1/2} \right]
\]
As \( u_i = \frac{2x_i - b_i^i}{x_i + 1/2} \) \((i = 1, 2)\) we have
\[
x_1 = \frac{u_1 + 2b_1^i}{4 - 2u_1},
\]
Thus
\[
1 = x_1 + x_2 = \frac{u_1 + 2b_1^i}{4 - 2u_1} + \frac{u_2 + 2b_2^i}{4 - 2u_2}.
\]
is a description of the Pareto efficient boundary of \( V \). \( \partial V \) is a certain hyperbola asymptotically approaching the lines \( u_1 = 2 \) and \( u_2 = 2 \).

If \( u_1 = 0 \) then \( x_1 = \frac{b_1^i}{2} \) and
\[
u_{3,4} = \frac{2}{3 - b_1^i}.
SECTION 4
The Expected Contract Value

By a "value" or "bargaining solution" we mean a mapping which is defined on CH-games (0-tupels as in SEC.1 (3), that is and attaining mechanisms, i.e. a mapping

\[ \chi : \{t\} | t is a CH-game \rightarrow \mathbb{R}^T \]

such that

\[ \chi(t) \in \mathcal{M}(t) = \{\mu | \mu is BIC and "in modis" individually rational\} \]

The term BIC has been explained in SEC.1 (see formula (4)); as for "i.r. in modis" ("IR") the term is almost self explaining, of course this means

\[ E(U_i \circ \mu \circ | r_i = t_i) \geq 0 \quad (i \in I, t_i \in T_i). \]

Note that T is considered fix in (1).

Thus, a value assigns a BIC and IR mechanism to every CH-game. Clearly, the definition of such a value should be governed by a set of axioms. However, in the spirit of the NASH-bargaining solution (NASH [12]), we shall exhibit, as a first approach, a value on "side payment games". It is then a second step to produce an extension on general CH-games using an appropriate version of the IIA-requirement. This extension would then be based on an axiomatic approach.

In the case of complete information, a solution for the side payment game is considered obvious ("equal share of net gains") or justified by the "obvious axioms" (symmetry, Pareto efficiency, translation covariance). With incomplete information at hand, this is by no means obvious. Let us first discuss the situation within the "side payment" territory.

Definition 4.1: A CHI game

\[ \Gamma = (1, T; p, \mathcal{X}, 0, U) \]

is said to be a jex-game ("in the wide sense") if, for any \( t \in T \), there is \( c^t \in \mathbb{R}^T \) and \( b^t \in \mathcal{X} \) such that

\[ U(t, x) = U(c^t, x) = \frac{c^t}{c^t} \circ x - (c^t) \frac{c^t-1}{c^t} \circ b^t \]

holds true.

Fig. 3.11.

This way the "interval" \( I^A = \left[ \left( \frac{b^t}{2}, 1 - \frac{b^t}{2} \right) \cup \left( 1 - \frac{b^t}{2}, \frac{b^t}{2} \right) \right] \) is uniquely mapped onto the Pareto efficient and individually rational boundary of \( V \).
Remark 4.2:
1. The term "fee-game" is explained at length in ROSENmüLLER [14]. If we return to Remark 2.2 and formula (5) in Sect 2, then we may write

\[ U(x) = \frac{1}{n} \cdot (x - (e \cdot x) \cdot c_{\lambda}) \]

\[ := \frac{1}{n} \cdot (x - (e \cdot b)) \]

with \( c_{\lambda} = \frac{\epsilon \cdot \lambda}{e \cdot \lambda} \) and \( b = x \cdot c_{\lambda} \).

If \( \lambda := \frac{a}{e - \lambda} \) happens to be \( \left( \frac{1}{n^{\lambda - 1}}, \frac{1}{n} \right) = \frac{1}{n} \), then \( \lambda \) is a fee-game in the narrow sense. For, in this case \( U \) coincides with the version in [14] (up to the factor \( 1/n \)).

2. Clearly, if \( \lambda \) is a fee game, then \( V^1 = V^n \), thus \( (0, V^1) \) has side-payment character up to a constant rescaling of utility.

Definition 4.3:
Let \( \lambda \) be a fee-game. For any \( t \in T \) define \( u^t \in \bar{V}^1 \) by

\[ g^t = \frac{1}{n} \cdot \frac{1}{\lambda} \]

and \( x^t \in \bar{X}^t \) by

\[ g^t = \frac{1}{n} \sum_{i=1}^{n} a^i \cdot \frac{1}{\lambda} \cdot b + \frac{1}{e} = \frac{1}{\lambda} \cdot b + \frac{1}{e} \]

\[ = c_{\lambda} \cdot b + \frac{1}{\lambda} \cdot b + \frac{1}{e} = b + \frac{1}{e} \]

where \( \lambda = \frac{a}{e - \lambda} \), \( b = c_{\lambda} \cdot b + \frac{1}{e} \) in accordance with Remark 2.2.. Therefore

\[ y^X = \{ EU(a^t, x) \mid x \in \bar{X}^t \} = V^\lambda \]

constitutes an NTU-game \((0, V^X)\), here \( \lambda \) is computed via

\[ \lambda = -\frac{1}{E(P)} \quad \bar{a} = \lambda - \frac{1}{e} \lambda \]

\[ \bar{a} \]

(mutatis mutandis for \( e = 1 \)). As (9) provides the parametrization for \( V^\lambda \), we observe that analogously to (7)

\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x^i = b + \frac{1}{e} \sum_{i=1}^{n} \frac{1}{e} \]

parametrizes the midpoint of \( V^\lambda \). That is, in a situation without information, it seems likely, that players would register \( \bar{x} \), thus ensuring the ex ante NASH-value. (Any symmetric, efficient, and translation invariant "value" leads to the midpoint of \( V^\lambda \).)

Note that, on the other hand, we have of course that \( u := E \cdot u^t \) is the midpoint of \( V^\lambda \). Thus, in a world of truth-speaking individuals, players would register the mechanism
\[ \mu^* = \mathbb{X}(\mathbb{E} \in \mathbb{T}), \] the expectation of which (ex-ante) is \( \mathbb{E}(\mathbb{U}(\mu^*, \mathbb{X})) = \overline{a}. \] Thus, agreeing on \( \mathbb{X} \) yields the same ex-ante expectation as playing the NASH-value ex post in the ideal world.

Definition 4.4: If \( \Gamma \) is a fee game, then the expected contract is given by

\[
\mathbb{X}(\mathbb{E} \mathbb{X}) = \mathbb{E}(\mathbb{X} | \mathbb{E}) = \mathbb{X}^\mathbb{E} = \mathbb{X}
\]

\[
\mathbb{X}(\mathbb{E} \mathbb{X}) = \overline{a} + \frac{1}{e} \frac{e}{e} \mathbb{X}
\]

\[
\mathbb{X}(\mathbb{E} \mathbb{X}) = \frac{e}{e} \mathbb{X}
\]

here \( a = \frac{e}{e} \mathbb{X} \).

It is important to note that \( \mathbb{X} \) and \( \mathbb{E} \mathbb{X} \) are different quantities— they coincide, however, in the case of a fee game in the narrow sense. But, of course \( \mathbb{E}(\mathbb{X}) = \mathbb{E}(\mathbb{X}) = \overline{a} \) holds always true.

Of course, the constant mechanism \( \mu^* \) defined by \( \mu^* = \mathbb{X}(\mathbb{E} \mathbb{X}) \) is always BIC. However, it is not necessarily individually rational, i.e., \( \mu \in \mathbb{E} \mathbb{X} \) is not guaranteed. In [14] it is argued that for fee games (in the narrow sense) a value should always be equal to \( \mu \) whenever \( \mu \in \mathbb{E} \mathbb{X} \). We shall pursue and extend this argument in order to justify what we shall consider the "natural" value on fee-games (and the one to be justified axiomatically for the general case later on).

In order to avoid the term "generalized NASH-value"— which it is— we use the term "expected contract value" or EC-value. Of course, it generalizes the NASH-value— like the ones proposed by HARSANYI-SELTEN [5] and MYERSON [9] (see also WEIDNER [10]).

Within the following, \( \mathbb{E}(\mathbb{X}, \mathbb{X}) \) denotes the NASH-value of a standard NTU-game \( (0, \mathbb{X}) \). Also we use

\[
\mathbb{V} := \{ \mu | \mathbb{E}(\mathbb{X}, \mathbb{X}) = \mu \}
\]

Definition 4.5: \( \mathbb{X} : \{T \} \xrightarrow{\mu^*} \mathbb{X}^\mathbb{X} \) is called a (the) EC-value if

\[
\mathbb{E}(\mathbb{X}, \mathbb{X}) = \mu^* \in \mathbb{V}
\]

holds true.

Let us now observe that there is a nice and useable class of fee-games allowing for a unique EC-value as characterized by 4.4. This is the case of two players, one of them having full information about his two types (in reality), and the other one resting in one type only. Conveniently we write

\[
\mathbb{X} = \sum_{i \in \mathbb{I}} \mathbb{X}_i, \quad \mathbb{E} = \mathbb{E}_i
\]

and refer to the situation as described by (10) as the case of incomplete information on one side (the "II1-CASE"). For simplicity, we assume always \( \mathbb{X}_i < \mathbb{X}_i \) in this situation, see (7) and Theorem 3.4 of [14]. This will ensure that \( \mu^* \) is player i's "worse situation", as \( \mathbb{U}_i(x) < \mathbb{U}_i(x) \) whenever \( \mathbb{E}^i(x) \).

Theorem 4.5: In the II1-case, the EC-value is uniquely defined.

Proof. As for fee-games in the narrow sense, this is more or less the content of Corollary 3.6 and Lemma 4.2 of [14]. The proof can be extended to fee-games in the wide sense as follows.

Let \( \Gamma \) be a fee-game (in the wide sense) for any system

\[
\mathbb{B} = (\mathbb{B}_i, \mathbb{E}), \quad \mathbb{B} \in \mathbb{B}^\mathbb{X}((\mathbb{E} \in \mathbb{E}))
\]

define the mapping

\[
\mathbb{B} : \Gamma \rightarrow \Gamma^* = \mathbb{B}^\mathbb{X}
\]

via the "formal rescaling of utilities", i.e., by fixing the utility functions of \( \mu \) to be
Indeed, as \( E(U^1 \circ \mu^1 | r_i = 1) = U^1_1(\mu^1) \) for \( r_i = \alpha^0 \), it is clear that the inequalities (4) of Sec.1 and hence the BIC-property are invariant into transformation with \( B \). Similarly, individual rationality (in medius) for player \( i \) is preserved.

Also, recall that we have assumed that \( \beta = \text{"player } 1\text{'s worse situation" i.e., that} \end{equation}
\( b^0_1 > b^0_i > 1 \). This is at once transformed into the corresponding property of [14].

Finally, although the I.R. property for player 2 is not preserved \( (E(U^1 \circ \mu^1) \text{ is the uninformd players expectation ex ante and in medius, the relevant proofs in [14] do only require that a non constant BIC mechanism } \mu \text{ satisfies } U^1(\mu_2) > 0 \text{ and is preserved by the formal rescaling } B \).

Thus, Corollary 3.6 and Lemma 4.2 of [14] do also apply in our present III-case, q.e.d.

The problem with the formal rescaling \( B \) is that various types of one player change their utility scale in a different manner. However, this is not compatible with our present philosophy: it is only a player who could apply a change of his utility scale (simultaneously for all his types). This is reflected by the fact that \( \mathcal{A} \) will be invariant only under such restricted type of rescaling.

Nevertheless, in the III-case we may carry over more results from [14], using our above arguments.

Corollary 4.6: The \( \Gamma \) be a fee game (in the wide sense) with III-property. Also, let \( \mu \in \mathcal{A} \) be a Pareto-efficient ex ante mechanism. Assume that \( \mu \) is non-constant, say \( \mu = (\mu^0, \mu^0) \).

Assume, on the other hand, that \( \chi^1 \) is constant, i.e., \( \chi^0(\Gamma) = \chi^1(\Gamma) = 1 \). Then
\[ U^N(\mu^0) < U^N(\bar{x}), \quad U^N(0) = 0 \]
\[ \mathbb{E}[U^N(\tau^0)] < \mathbb{E}[U^N(\bar{x})] \]  

(27)

The proof follows from Lemma 4.3 of [14]. Note that our notation for the III-case ([16] that is) implies that
\[ U^N(\bar{x}) = \mathbb{E}[U^N(x) | \tau^0 = \mu^0] \]  

(28)
e.g. holds true (27) implies that player 1's expectation in medias res (which is deterministic, as he has full information in medias res) is worse at any non constant mechanism - compared to the one at \( \bar{x} \) - and the same holds true ex ante.

Corollary 4.7: Let \( \Gamma \) be a fee game (in the narrow sense) with III-property. Then
\[ \rho^0 \psi^0 \in V_{\kappa}. \]

This we prove only for a fee game in the narrow sense. We may directly refer to Theorem 3.4 of [14] (see also Fig.6 of SEC.3 of [14], which characterizes the non constant mechanisms).

From this it follows that the BiC-mechanisms yielding ex ante expectations \( u^L, u^M, u^R \), and \( u \) are easily identified (see Fig.4.1). We find
\[ (L) \quad \mu = (a^0,1,0), \quad u = BU^0(\mu^0) = \rho^0(a^0,1) - b^0 \]  

and
\[ (M) \quad \mu = (a^0,0,0) \]

with
\[ (M) \quad u = BU^0(\mu^0) = \rho^0(a^0,0) + \rho^0 U^0(a^0) = a^0 - E b^0 \]  

(29)

while \( u^R \) results from the fact that \( V_{\kappa} \in V^X_{\bar{X}} \) and \( V^X_{\bar{X}} \) has side payment character.

Now, as the simplex \( p^0 \psi^0 \) has vertices \( p^0(a^0,1 - b^0) = u^L \) and \( p^0(a^0,1 - b^0) \), we observe that the situation has to be as depicted in Fig.4.1, i.e., \( p^0 \psi^0 \in V_{\kappa} \in V^X_{\bar{X}}, \) q.e.d.

We are now going to collect some arguments in order to support the EC-value in the III-CASE. Some of this will resemble expositions presented in [14].

Player 2, the uninformed one, will start his analysis by considering (cf. (10))
\[ V^X_{\bar{X}} = \{ EU^0(x) | x \in \Sigma \} \cap \mathbb{R}_{+}^2 \]
However, in general the elements of $V_X^N$ may not be acceptable to player 1. For, if he observes his type in $\text{median}$, it might turn out that, for some $x \in \underline{x}$, $E(U'(x)) < 0$. This is of course what leads the players to discuss mechanisms, i.e., agreements depending on the type player 1 observes and announces.

If so, player 2's interest requires to accept RIC mechanism only, thus $\mathcal{M}$ is convex, compact polyhedron contained in $V_X$.

$$V_X^{\mathcal{M}} \subseteq V_X^+$$  \hspace{1cm} (31)

From the viewpoint of player 2, the inclusion (31) is not too exciting as it decreases the feasible alternatives. For player 1 this may also be considered a drawback – but for him it constitutes an insurance against finding himself in a non IR (in median) situation.

Given these general observations, let us now focus to the discussion of a "value" or "bargaining solution".

We are going to distinguish two cases.

1st CASE: Assume that the EC-value is constant. Clearly this means

$$\chi(1) = \chi(1) = x = \bar{x}(1)$$  \hspace{1cm} (32)

or, in other words, the constant mechanism $\bar{x}$ is IR.

In this case the argument is straightforward as far as the uninformed player (player 2) is concerned. For, in view of (31), he was not very happy anyway with mechanisms $\mu \in \mathcal{M}$. Now, he observes that

$$E(U'(\chi(1))) = E(\bar{x}) = \bar{u} = u(0, V^{\mathcal{M}})$$  \hspace{1cm} (33)

holds true. As $V_X^{\mathcal{M}} \subseteq V^+_X$ player 2, an ardent supporter of the IIA-axioms due to NASH, greatly relieved accepts $\chi(1)$ as a mechanism that assures what he wanted from the beginning anyway: the NASH-value of $V_X$.

What could player 1 object to this kind of reasoning? On one hand he could propose another constant mechanism, say $\bar{x}$. There seems to be no sense in his considering none IR (in median) mechanism, thus we expect him to choose $\bar{x}$ such that $\bar{x} = E(U'(\bar{x})) \in V^{\mathcal{M}}$. Since he waves the opportunity to make any use of his in median information, the argument boils down to proposing some utility $\bar{u} \in V^{\mathcal{M}}$ which differs from the midpoint $\bar{u}$ of $V_X$ satisfying $\bar{u} \in V^{\mathcal{M}}$. It is difficult to imagine any believer in the NASH-value bringing forward such a proposition.

On the other hand, player 1 could bring up a non constant mechanism, say $\mu = (\mu_1, \mu_2)$. Since we are still discussing the case in which the constant mechanism $\bar{x} = x$ satisfies $\bar{u} \in V^{\mathcal{M}}$, we have $U(\bar{u}) = 0$. Now, (37) shows that ex ante as well as in median, player 1 is worse off at $\mu$ than at $\chi(1)$ ($= \bar{x}$) – so why should he forward non constant mechanisms at all?

Combining our arguments for the 1st Case, we find it hard to believe that (given the NASH-value as the basis for the discussion in the traditional non random situation) anyone could reject the constant mechanism $\mu = x(1) = \bar{x}$, if $\bar{u} \in \mathcal{M}$, since $E(U'(\chi(1))) \in V^{\mathcal{M}} \subseteq V_X$. Of course, this is what is called "the expected contract axiom" in [14].

Now we are concerned with the case that $\chi(1)$ is non constant, $x = (x^0, x^1)$ (we omit the argument $I$ for the moment).
Again we analyze the situation according to whether player 1 argues for alternative mechanisms that are constant or not.

On one hand there could be constant mechanisms that are IR (in medias), say \( \tilde{\mu} = \tilde{x} \). Now, in our present situation, since \( \tilde{\mu} \) is not IR (in medias). This implies

\[
U_B^p(x) < 0 = U_B^{p,0} \leq U_B^p(\tilde{x})
\]

(cf. Corollary 4.6 and Formula (27)). (Recall that "\( p,0 \) is player 1's worse situation!).

As \( U_B^p \) is strictly monotone in \( x \) (see Theorem 1.1), it follows that

\[
x < x
\]

and

\[
U_B^p(x) < U_B^p(\tilde{x})
\]

as well as

\[
EU_B^p(x) < EU_B^p(\tilde{x})
\]

Since \( x \in \partial X \) it is seen at once, that the inequalities are the other way around for player 2.

Thus, by proposing \( \tilde{\mu} \), player 1 is asking not only for more compared to \( \tilde{x} \) but even for more compared to \( \tilde{x} \). He is asking this by using a constant mechanism, thus without referring to his information in medias, meaning that he argues more or less in view of the ex ante situation. Indeed, some direct analysis verifies the situation (ex ante) as described in Fig. 4.2: \( \tilde{\mu} \) is the midpoint of \( VX \) (and not available in \( V\mu \)), \( E(U_B^p(x)) \) is the midpoint of \( [u_B, u_B] \) (and hence the NASH-value of \( V\mu \)), and \( E(U_B^p(x)) \) is an element of \( \gamma VX \cap V\mu \).

So the first coordinate of \( E(U_B^p(x)) \) exceed the expectation of \( x \) in medias and ex ante. As player 1 is essentially arguing from ex ante, it seems to be rather impertinent to ask for that much and there is no way to justify such demand by the usual reasons of equity or fairness as presented in the context of the NASH-value.

Finally, let us, on the other hand, deal with the situation that player 1 is proposing a non constant mechanism say \( \mu \) instead of \( x \).

Because of Corollary 4.6 we have \( U_B^p(x) = U_B^p(x^0) = 0 \), that is, player 1 always gets 0 in his "worse situation". Now, this could lead him to shift the weight of his argument to situation \( a \) ("if \( a \) occurs I get 0 utility whatsoever, so I want to talk about \( \tilde{b} \), let's treat the \( a \) medias situation as only"). Conceivably in this case he could attempt to ask for \( a^0 \) and utility \( U_B^p(x^0) = 0 \) (the mid-point of \( V\mu \)).

This would mean that he is asking for a mechanism \( \mu = (x^0, \tilde{x}) \), where \( \tilde{x} \) is suitably chosen (by Theorem 3.4 and Corollary 3.6 of [14]) yields 0 utility for player 2. That is, the utility is

\[
EU_B^p(x^0) = p^B \cdot U_B^p(x^0)
\]

Now, let us return to Fig. 4.1. Clearly \( p^B \cdot U_B^p(x) \) is the first coordinate of the NASH-value of \( p^B V\mu \). Hence, \( x^0 = E(U_B^{p,0}) \) is obtained at the corresponding point on the Pareto boundary \( V\mu \).

And clearly, in this situation, the NASH-value of \( V\mu \) (i.e., \( E(U_B^p(x)) \) is to the right of \( x^0 \). (We do not refer to the missing monotonicity of the NASH-value -- the situation is much simpler.)

Thus we conclude: by asking for discussion only "in medias" and claiming \( p^B(V\mu) \) in situation \( a \), player 1 is worse off compared to \( x \) and again, this is so ex ante and in medias (since the corresponding mechanism yields always 0 to player 1 in situation \( p \)). So eventually we end up seeing player 1 resolving himself to go back and discuss not "situation a only". His argument will be that since he is so badly off in situation \( p \), getting 0 at any no constant mechanism
he wants compensation in situation \( x \). But this can only mean that he wants to evaluate the situation \( x \) ante and ask for fair \( x \) ante considerations. In this case there is (for NASH believers) nothing but to stick to just \( X \), because it yields the NASH-value of \( V^M \).

\[ \varepsilon (U^T(x)) \]

\[ V^M \]

Fig 4.2

At this stage we shall end our present discussion. We believe to have supported the expected contract value by quite some arguments; hopefully one could consider \( x \) to be the "natural" value for side payment situations (i.e., for lce-games).

The next task will be to establish \( x \) by an axiomatic approach. If this is possible at all, then the canonical representation will play an important role. For, if something like an IIA-axiom will be suitable then the original game and the one used for comparison, ("a foo-game yielding the irrelevant alternatives") would have to be parameterized simultaneously. We feel that the canonical representation is the only "sensible" (that is canonical) way to attempt such a simultaneous representation by reasons that should be clear at this stage of the development.

**Literatur**


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