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Evolutionary Stability in Extensive 2-Person Games
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Evolutionary Stability in Extensive 2-Person Games

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10. Decomposition

The analysis of complex extensive game models can often be facilitated by the application of a decomposition principle. In perturbed games with subgames it is possible to solve the subgames first and then to replace them by their solution payoff vectors in order to obtain a "truncated game" whose solution together with the solutions of the subgames yields the solution of the whole game. The smallest units obtained by the application of this decomposition principle to the game as a whole and to its subgames are called "elementary games".

The decomposition principle has been applied successfully in other game theoretical contexts (e.g., see Selten 1973). However, it is by no means obvious that this principle can be used for the determination of evolutionarily stable strategies in extensive games. As we have seen in the last section a direct ESS of a perturbed game cannot be locally characterized as an LSS.

The results of this section will be based on the assumption that the natural symmetry is subgame preserving. In view of theorem 1 this is a very mild restriction. Eventually, it will be shown that in perturbed games with subgames the analysis of the whole game can be reduced to the successive analysis of its elementary games. A number of auxiliary results must be obtained before this goal can be achieved.

It will be important to distinguish between symmetric and asymmetric subgames. A symmetric subgame coincides with its image under the natural symmetry; an asymmetric subgame is mapped to a different subgame. It will be shown that all information sets of an asymmetric subgame are image detached. This will lead to the conclusion that the restriction of a direct ESS of a perturbed game to one of its asymmetric subgames must be a strong equilibrium point.
An analogous distinction will be made between symmetric and asymmetric elementary games. It will be shown that a direct ESS induces a direct ESS on every symmetric elementary game and that a direct ESS together with its symmetric image induces a strong equilibrium point on every asymmetric elementary game. Moreover, the induced strategies must be such that origins of subsequent subgames are reached with positive probability. These conditions are not only necessary but sufficient for a direct ESS.

Finally, theorem 10 will summarize necessary conditions for a limit ESS which can be obtained as a consequence of decomposition results for perturbed games.

10.1 Symmetric and asymmetric subgames: Let \((\Gamma, f)\) be a symmetric extensive 2-person game whose natural symmetry is subgame preserving.

Let \(y\) be a decomposition point and let \(\Gamma_y\) be the subgame at \(y\). If \(\Gamma_y\) is the symmetric image of itself, i.e., if we have \(\Gamma_y = f(\Gamma_y)\) then the restriction \(f_y\) of \(f\) to the choice set \(\mathcal{O}_y\) of \(\Gamma_y\) is called the natural symmetry of \(\Gamma_y\) in \((\Gamma, f)\).

Assume that we have \(\Gamma_y = f(\Gamma_y)\). It can be seen easily that the natural symmetry \(f_y\) of \(\Gamma_y\) has all the properties of a symmetry of \(\Gamma_y\). The pair \((\Gamma_y, f_y)\) is a symmetric game in the sense of definition 5.5. We call \((\Gamma_y, f_y)\) the symmetric subgame of \((\Gamma, f)\) at \(y\).

Now assume that we have \(\Gamma_y \neq f(\Gamma_y)\). In this case we call \(\Gamma_y\) an asymmetric subgame of \((\Gamma, f)\). Note that symmetric subgames and asymmetric subgames are different kinds of mathematical objects. Unlike a symmetric subgame and asymmetric subgame is not endowed with a natural symmetry.

Consider an arbitrary subgame \(\Gamma_y\) of \(\Gamma\). Let \(b\) be a behavior strategy of player 1 in \(\Gamma\) and let \(b_y\) be that be-
behavior strategy of player 1 in $\Gamma_y$ which assigns the same local strategy as $b$ to every information set $v$ in $\Gamma_y$. We call this strategy $b_y$ the strategy induced by $b$ on $\Gamma_y$. For behavior strategies $b'$ of player 2 in $\Gamma$ the strategy $b'_y$ induced by $b'$ on $\Gamma_y$ is defined analogously.

Let $\eta$ be a perturbation of $(\Gamma,f)$. For every subgame $\Gamma_y$ of $(\Gamma,f)$ the restriction $\eta_y$ of $\eta$ to the choice set $C_y$ of $\Gamma_y$ is called the perturbation induced by $\eta$ on $\Gamma_y$. If $(\Gamma_y,f_y)$ is a symmetric subgame of $(\Gamma,f)$ then $\Gamma_y,f_y$ and the perturbation $\eta_y$ induced by $\eta$ on $\Gamma$ form a perturbed game

$\tilde{\Gamma}_y = (\Gamma_y,f_y,\eta_y)$ of $(\Gamma_y,f_y)$. This perturbed game $\tilde{\Gamma}_y$ is called the symmetric subgame of $\tilde{\Gamma} = (\Gamma,f,\eta)$ at $y$.

If $\Gamma_y$ is an asymmetric subgame and $\eta_y$ is induced by $\eta$ on $\Gamma_y$ then the pair $\tilde{\Gamma}_y = (\Gamma_y,\eta_y)$ is called an asymmetric subgame of $\tilde{\Gamma} = (\Gamma,f,\eta)$. In this case the perturbation $\eta_y$ induced by $\eta$ on $\Gamma_y$ has all the properties of a perturbation of $\Gamma_y$ as defined in 7.1 except (40) which refers to the natural symmetry $f$. Perturbations of this kind are called asymmetric perturbances. An extensive game $\Gamma$ together with an asymmetric perturbance $\eta$ forms an asymmetric perturbed game $(\Gamma,\eta)$ of $\Gamma$. In this sense an asymmetric subgame $(\Gamma_y,\eta_y)$ of $\tilde{\Gamma} = (\Gamma,f,\eta)$ is an asymmetric perturbed game of $\Gamma_y$.

Local strategies, behavior strategies, best replies, strong best replies, equilibrium points and strong equilibrium points of asymmetric perturbed games are defined in the same way as for perturbed games of symmetric games. The definitions of "dispersed", "permeable" and "pervasive" are also transferred in the obvious way to asymmetric perturbed games.

10.2 Remark: Let $(\Gamma,f)$ be a symmetric extensive 2-person game and let $\Gamma_y$ be a subgame at a decomposition point $y$ of $\Gamma$. Let $b$ and $b'$ be two behavior strategies for players 1 and 2, respectively in $\Gamma$ and let $b_y$ and $b'_y$ be the strategies induced on $\Gamma_y$ by $b$ and $b'$, respectively. We use the symbol $E_y$ for player 1's expected payoff in $\Gamma_y$ and the symbol $E_yu$
for local payoffs in $\Gamma_Y$. Let $u$ be an information set in $\Gamma_Y$ which in $\Gamma$ is not blocked by $b'$. Obviously, $u$ is not blocked by $b'_y$ in $\Gamma_Y$ either. With the help of (51) it can be seen immediately that local payoffs in $\Gamma$ and in $\Gamma_Y$ agree in the following sense:

$$E_{yu}(s_u, b_y, b'_y) = E_u(s_u, b, b')$$

for every local strategy $s_u$ at $u$.

10.3 Lemma 5 (image detachment of asymmetric subgames): Let $(\Gamma, f)$ be a symmetric extensive 2-person game with a subgame preserving natural symmetry $f$ and let $\Gamma_Y$ be an asymmetric subgame of $(\Gamma, f)$. Then every information set of $\Gamma_Y$ is image detached.

**Proof:** Let $y$ be the origin of $\Gamma_Y$ and let $\tilde{y}$ be the origin of $\Gamma_{\tilde{y}} = f(\Gamma_Y)$. Obviously, we have $\tilde{y} \neq y$. Moreover, as we shall see $\tilde{y}$ does not belong to $\Gamma_{\tilde{y}}$. This can be seen as follows. $y$ is the only element of an information set $v = \{y\}$. If $\tilde{y}$ comes after $y$ then $\Gamma_{\tilde{y}}$ is a subgame of $\Gamma_Y$ which contains fewer information sets than $\Gamma_Y$. Two different information sets must have different symmetric images, since otherwise $f(f(u)) = u$ could not hold for both of them (see (21) in 5.6). Therefore, $f(\Gamma_Y)$ must have the same number of information sets as $\Gamma_{\tilde{y}}$. This shows that $\tilde{y}$ cannot come after $y$. The same argument applied to $\Gamma_{\tilde{y}}$ instead of $\Gamma_Y$ shows that $y$ cannot come after $\tilde{y}$. Therefore, $y$ and $\tilde{y}$ cannot be on the same play. If $u$ is an information set in $\Gamma_Y$ then $y$ is on every play which intersects $u$. Moreover, $f(u)$ belongs to $\Gamma_{\tilde{y}}$ and $\tilde{y}$ is on every play which intersects $f(u)$. This shows that there cannot be any play which intersects both $u$ and $f(u)$. All information sets in $\Gamma_Y$ are image detached.

10.4 Lemma 6 (local characterization of strong equilibrium points): Let $\tilde{\Gamma} = (\Gamma, n)$ be a asymmetric perturbed game of an extensive 2-person game $\Gamma$ and let $r$ and $r'$ be behavior strategies of players 1 and 2, respectively in $\tilde{\Gamma}$. 


The pair \((r, r')\) is a strong equilibrium point of \(\tilde{\pi}\), if and only if both of the following conditions are satisfied:

(i) For every essential information set \(u\) in \(\tilde{\pi}\) the realization probability \(\gamma(u, r, r')\) is positive.

(ii) For every information set \(u\) of player 1 or 2 in \(\tilde{\pi}\) the local strategy \(r_u\) or \(r'_u\) assigned to \(u\) by \(r\) or \(r'\), respectively, is a strong local best reply to \(r\) and \(r'\) in \(\tilde{\pi}\).

Proof: Assume that \((r, r')\) is a strong equilibrium point of \(\tilde{\pi}\). Suppose that (i) is not satisfied. Let \(u\) be an essential information set of player 1 or 2 with \(\gamma(u, r, r') = 0\). Let \(s_u\) be a local strategy at \(u\) in \(\tilde{\pi}\) which is different from the local strategy assigned to \(u\) by \(r\) or \(r'\), respectively. If \(u\) is an information set of player 1 then \(s = r/s_u\) is an alternative best reply to \(r'\) and if \(u\) is an information set of player 2, then \(s' = r'/s_u\) is an alternative best reply to \(r\). In both cases expected payoffs are not influenced by the local strategy change. Therefore, (i) holds if \((r, r')\) is a strong equilibrium point of \(\tilde{\pi}\).

Suppose that (i) holds and that (ii) is violated for at least one information set \(u\) of player 1 or 2. Since the assertion of the lemma is completely symmetric with respect to players 1 and 2, it is sufficient to examine the case that \(u\) belongs to player 1. Let \(u\) be an information set of player 1, such that the local strategy \(r_u\) assigned by \(r\) to \(u\) is not a strong local best reply to \(r'\) in \(\tilde{\pi}\). Let \(s_u\) be a local best reply to \(r'\) at \(u\) in \(\tilde{\pi}\) with \(s_u \neq r_u\). The local payoff at \(u\) remains unchanged by a local strategy change to \(s = r/s_u\). It follows by (b) in 8.4 that player 1's global payoff remains unchanged if he uses \(s\) instead of \(r\) against \(r'\). This shows that \(s\) is a best reply to \(r'\) in \(\tilde{\pi}\). In view of \(s \neq r\) the equilibrium point \((r, r')\) fails to be strong, contrary to the assumption made above. Therefore, (ii) holds.
Now assume that (i) and (ii) are satisfied. Since the assertion of the lemma is symmetric with respect to players 1 and 2, it is sufficient to examine the case that player 1 has an alternative best reply. Let $S$ be the set of all best replies $s$ with $s \neq r$ to $r'$ in $\tilde{r}$. For any $s \in S$ let $s_u$ be the local strategy assigned to an information set of player 1 by $s$ and let $k(s)$ be the number of information sets of player 1 with $s_u \neq r_u$. Let $s \in S$ be a strategy such that for no $t \in S$ we have $k(t) < k(s)$. We shall proceed in a similar fashion as in the proof of lemma 3.

Let $u$ be an information set of player 1 with $s_u \neq r_u$ such that $u$ does not precede any information set $v$ of player 1 with $s_v \neq r_v$. In view of (51) we must have:

\[(81) \quad E_v(t_v, s, r') = E_v(t_v, r, r')\]

for every local strategy $t_v$ at $v$. Since in view of (ii) the local strategy $s_v$ cannot be a local best reply to $r$ and $r'$ in $\tilde{r}$, it cannot be a local best reply to $s$ and $r'$ in $\tilde{r}$, either. This yields:

\[(82) \quad E_v(s_v, s, r') < E_v(r_v, s, r')\]

Consider the strategy $t = s/r_v$. If $\gamma(v, s, r')$ were positive then $t$ would yield a higher global payoff than $s$ against $r'$ (see (d) in 8.4). Therefore, we must have $\gamma(v, s, r') = 0$. However, in this case a local change at $v$ has no influence on global payoffs. Consequently, $t$ is in $S$. Moreover, $k(t) = k(s) - 1$ contrary to the assumption that $s$ is minimal with respect to $k(s)$. Therefore, $r$ must be a strong best reply to $r'$ in $\tilde{r}$.

We can conclude that $(r, r')$ is a strong equilibrium point of $\tilde{r}$ if (i) and (ii) are satisfied.

10.5 Theorem 6 (strong equilibrium points in asymmetric subgames): Let $\tilde{r} = (r, f, n)$ be a perturbed game of a symmetric extensive 2-person game $(r, f)$ with a subgame preserving natural symmetry $f$ and let $\tilde{r}_y = (r_y, n)$ be an asymmetric subgame of $\tilde{r}$. Let $b$ be a direct ESS of $\tilde{r}$. Let $b_y$ and $b'_y$ be the strategies induced
on $\Gamma_y$ by $b$ and $b' = f(b)$, respectively. Then $(b_y, b'_y)$ is a strong equilibrium point of $\hat{\Gamma}_y$.

Proof: Let $b_u$ and $b'_u$ be the local strategies assigned by $b$ and $b'$, respectively to information sets of players 1 or 2. In view of lemma 6 it is sufficient to show that the following statements are true.

(i) For every essential information set $u$ in $\Gamma_y$ the realization probability $\gamma_y(u, b_y, b'_y)$ of $u$ in $\Gamma_y$ is positive.

(ii) For every information set $u$ of player 1 or 2 in $\Gamma_y$ the local strategy $b_u$ or $b'_u$ assigned to $u$ by $b_y$ or $b'_y$ is a strong local best reply to $b_y$ and $b'_y$ in $\hat{\Gamma}_y$.

Since $b$ is a direct ESS of $\hat{\Gamma}$ it follows by theorem 2 that $b$ is pervasive. In view of lemma 2 we can conclude that the realization probabilities $\gamma(u, b, b')$ of essential information sets in $\Gamma$ are positive. Consequently, (i) holds.

Lemma 5 shows that every information set of $\Gamma_y$ is image detached. It follows by (a) in theorem 4 and by (a') in 9.3 that for every information set $u$ of player 1 or 2 in $\Gamma$ the local strategy $b_u$ or $b'_u$ is a strong local best reply to $b$ and $b'$ in $\Gamma$. Since local payoffs in $\Gamma$ and $\Gamma_y$ agree in the sense of (80) in 10.2, we can conclude that (ii) holds.

10.6 Theorem 7 (induced ESS on symmetric subgames): Let $\hat{\Gamma} = (\Gamma, f, n)$ be a perturbed game of a symmetric extensive 2-person game $(\Gamma, f)$ with a subgame preserving natural symmetry $f$. Let $\hat{\Gamma}_y = (\Gamma_y, f_y, n_y)$ be a symmetric subgame of $\hat{\Gamma}$. Let $b$ be a direct ESS of $\hat{\Gamma}$. Then the strategy $b_y$ induced by $b$ on $\hat{\Gamma}_y$ is a direct ESS of $\hat{\Gamma}_y$.

Proof: Theorem 2 shows that $b$ is pervasive. In view of lemma 2 all realization probabilities $\gamma(u, b, f(b))$ of essential information sets are positive. Consequently, the
same is true for all realization probabilities $\gamma_{y}(u,b_{y},f_{y}(b_{y}))$ of essential information sets $u$ in $\Gamma_{y}$. In view of lemma 2 we can conclude that $b_{y}$ is pervasive in $(\Gamma_{y},f_{y})$.

Since local payoffs in $\Gamma$ and $\Gamma_{y}$ agree in the sense of (80) in 10.2 it is clear that $b_{u}$ is a local best reply to $b_{y}$ and $f_{y}(b_{y})$ in $\Gamma_{y}$ if and only if $b_{u}$ is a local best reply to $b$ and $f(b)$ in $\Gamma$. Since $b$ is a pervasive symmetric equilibrium strategy, it follows by theorem 3 that for every information set $u$ of player 1 the local strategy $b_{u}$ assigned to $u$ by $b$ is a local best reply to $b$ and $f(b)$ in $\Gamma$. We can conclude that for every information set $u$ of player 1 the local strategy $b_{u}$ assigned to $u$ by $b_{y}$ is a local best reply to $b_{y}$ and $f_{y}(b_{y})$ in $\Gamma_{y}$. Since $b_{y}$ is pervasive it follows by theorem 3 that $b_{y}$ is a symmetric equilibrium strategy for $(\Gamma_{y},f_{y})$.

It remains to be shown that $b_{y}$ satisfies the second condition in the definition of a direct ESS for a perturbed game (see 9.6). Assume that $r_{y}$ with $r_{y} \neq b_{y}$ is a best reply to $f_{y}(b_{y})$ in $\Gamma_{y}$ and that in addition to this we have:

\[ E_{y}(b_{y},f_{y}(b_{y})) \leq E_{y}(r_{y},f_{y}(b_{y})) \]

Let $r$ be that behavior strategy of player 1 in $\Gamma$ which agrees with $r_{y}$ at information sets in $\Gamma_{y}$ and with $b$ at information sets outside $\Gamma_{y}$. Local strategy changes at information sets in $\Gamma_{y}$ do not influence realization probabilities of endpoints outside of $\Gamma_{y}$. This together with the fact that $r_{y}$ is a best reply to $f_{y}(r_{y})$ in $\Gamma_{y}$ permits the conclusion that $r$ is a best reply to $f(b)$ in $\Gamma$. Moreover, we can conclude that the following is true:

\[ E(b,f(r)) \leq E(r,f(r)) \]

In view of the second condition in the definition of a direct ESS for a perturbed game (84) contradicts the assumption that $b$ is a direct ESS of $\Gamma$. Consequently, an alternative best reply $r_{y}$ to $f_{y}(b_{y})$ with (84) cannot be
10.7 Truncations and elementary games: Let \( \Gamma \) be an extensive 2-person game and let \( b \) and \( b' \) be behavior strategies of players 1 and 2 for \( \Gamma \). A multizubame \( M \) of \( \Gamma \) is a non-empty set of subgames of \( \Gamma \) which are pairwise non-intersecting in the sense that two subgames in \( M \) do not have any vertex in common. Let \( M \) be a multizubame of \( \Gamma \).

We shall define a game \( \tilde{\Gamma} = (\tilde{R}, \tilde{P}, \tilde{V}, \tilde{C}, \tilde{p}, \tilde{h}, \tilde{h}') \) which will be called the \( (b, b') \)-truncation of \( \Gamma \) with respect to \( M \). The components of \( \tilde{\Gamma} \) are as follows:

(a) the origin \( o \) of \( \tilde{R} \) is the origin of \( K \). The decision points of \( \tilde{R} \) are those decision points of \( K \) which do not belong to subgames in \( M \). The endpoints of \( \tilde{R} \) are either endpoints of \( \Gamma \) which do not belong to subgames in \( M \) or origins of subgames of \( M \). The origins of subgames in \( M \) are also called decompos-ition endpoints of \( \tilde{R} \),

(b) \( P, V, C \) and \( \tilde{P} \) are the restrictions of \( P, U, C \) and \( p \), respectively to \( \tilde{R} \).

(c) Let \( z \) be an endpoint of \( \tilde{\Gamma} \) which also belongs to \( \Gamma \). Then we have:

\[
\tilde{h}(z) = h(z)
\]

and

\[
\tilde{h}'(z) = h'(z)
\]

Let \( y \) be the origin of a subgame \( \Gamma_y \in M \) and let \( \tilde{b}_y \) and \( \tilde{b}'_y \) be the strategies induced by \( b \) and \( b' \), respectively on \( \Gamma_y \). then we have:

\[
\tilde{h}(y) = E_y(b_y, b'_y)
\]

\[
\tilde{h}'(y) = E'_y(b_y, b'_y)
\]

where \( E_y \) and \( E'_y \) denote the expected payoff functions of players 1 and 2 in \( \Gamma_y \).

Let \( \eta \) be a perturbation for \( \Gamma \) and let \( \tilde{\eta} \) be the restriction of \( \eta \) to the union \( \mathcal{C}_1 \cup \mathcal{C}_2 \) of the choice sets of players 1
and 2 in \( \tilde{r} \). Then \( \tilde{\gamma} = (\tilde{r}, \tilde{\eta}) \) is called the \((b, b')\)-truncation of \( \tilde{r} = (r, \eta) \). Note that \( b \) and \( b' \) need not belong to \( \tilde{\mathcal{B}} \) and \( \tilde{\mathcal{B}}' \), respectively; \((b, b')\)-truncations of \( \tilde{r} = (r, \eta) \) can be formed, too, if \( b \) and \( b' \) do belong to \( \tilde{r} \) but not to \( \tilde{r} \).

A subgame \( \gamma_y \) of \( \Gamma \) is called **maximal** if \( \gamma_y \) is not a subgame of another subgame of \( \Gamma \). It is clear that two maximal subgames cannot have any vertices in common. Therefore, the set of all maximal subgames of \( \Gamma \) is a multisubgame. The \((b, b')\)-truncation of \( \Gamma \) with respect to this multisubgame is called the **main** \((b, b')\)-truncation of \( \Gamma \); in the case that \( \Gamma \) has no subgames, this name is applied to \( \Gamma \) itself. The main \((b, b')\)-truncation \( \tilde{\gamma} = (\tilde{r}, \tilde{\eta}) \) of a perturbed game \((r, \eta)\) is formed by the main \((b, b')\)-truncation \( \tilde{\Gamma} \) of \( \Gamma \) together with the restriction \( \tilde{\eta} \) of \( \eta \) to the choices in \( \tilde{r} \).

Let \( f \) be a subgame preserving symmetry of \( \Gamma \). A multisubgame \( M \) of \( \Gamma \) is called **symmetric** multisubgame of \((\Gamma, f)\) if for every \( \gamma_y \in M \) the symmetric image \( f(\gamma_y) \) belongs to \( M \), too.

It can be seen easily that the symmetric image \( f(\gamma_y) \) of a maximal subgame \( \gamma_y \) is maximal; if \( f(\gamma_y) \) were a subgame of a subgame \( \gamma'_y \) then \( f(\gamma'_y) \) would contain \( \gamma_y \) as a subgame and \( \gamma_y \) could not be maximal. Therefore, the set of all maximal subgames of \( \Gamma \) is symmetric with respect to \( f \).

Assume \( b' = f(b) \) and let \( M \) be a symmetric multisubgame of \((\Gamma, f)\) where \( f \) is subgame preserving. Consider the \((b, b')\)-truncation \( \tilde{r} \) of \( \Gamma \) with respect to \( M \). Since \( f \) is subgame preserving, the symmetric image \( f(u) \) of an information set \( u \) in \( \tilde{r} \) is an information set of \( \tilde{r} \). With the help of (85) to (87) it can be seen that the restriction \( f \) of \( \tilde{f} \) to \( f \) is a symmetry of \( \tilde{r} \). We call \((\tilde{r}, \tilde{f})\) the \( b \)-truncation of \((\Gamma, f)\) with respect to \( M \). In view of \( b' = f(b) \) it is convenient to use a name which does not explicitly mention \( b' \). The \( b \)-truncation of \((\Gamma, f)\) with respect to the set of all maximal subgames of \( \Gamma \) is called the **main** \( b \)-truncation of \((\Gamma, f)\).
Let \( \tilde{\Gamma} = (\tilde{\Gamma}, \tilde{f}, \tilde{\eta}) \) be a perturbed game of \((\Gamma, f)\) and let \( \tilde{\eta} \) be the restriction of \( \eta \) to the choices in the b-truncation \((\tilde{\Gamma}, \tilde{f})\) with respect to a symmetric multisubgame \( M \). Then the perturbed game \( \tilde{\Gamma} = (\tilde{\Gamma}, \tilde{f}, \tilde{\eta}) \) is called the b-truncation of \( \tilde{\Gamma} \) with respect to \( M \). If \( M \) is the set of all maximal subgames of \( \Gamma \), then \( \tilde{\Gamma} \) is the main b-truncation of \( \Gamma \).

An elementary game of \( \tilde{\Gamma} = (\tilde{\Gamma}, \tilde{f}, \tilde{\eta}) \) or shortly a b-element of \( \tilde{\Gamma} \) is a game \( \tilde{\gamma} \) which fits one of the following descriptions (i), (ii) or (iii):

(i) \( \tilde{\gamma} = (\tilde{\Gamma}, \tilde{f}, \tilde{\eta}) \) is the main b-truncation of \( \tilde{\Gamma} \).

(ii) \( \tilde{\gamma} = (\tilde{\Gamma}_y, \tilde{f}_y, \tilde{\eta}_y) \) is the main \( b_y \)-truncation of a symmetric subgame \((\Gamma_y, f_y, \eta_y)\) of \( \Gamma \) where \( b_y \) is the strategy induced by \( b \) on \( \Gamma_y \).

(iii) \( \tilde{\gamma} = (\tilde{\Gamma}_y, \tilde{\eta}_y) \) is the main \((b_y, b'_y)\)-truncation of an asymmetric subgame \( \Gamma'_y = (\Gamma_y, \eta_y) \) of \( \Gamma \) where \( b_y \) and \( b'_y \) are the strategies induced by \( b \) and \( f(b) \), respectively on \( \Gamma_y \).

A behavior strategy for a b-element is called induced by a behavior strategy for the whole game if both strategies assign the same local strategies to the information sets in the b-element. The term "induced" will also be used analogously for truncations in general.

A b-element is called symmetric, if it fits one of the descriptions (i) and (ii) and asymmetric in the case of (iii). From what has been said before, it is clear that symmetric b-elements are symmetric games.

Since we do not distinguish between \((\Gamma, f)\) and the special case of a perturbed game of \((\Gamma, f)\) where all minimum probabilities are zero, the definition of b-elements also covers the case of an unperturbed symmetric extensive two-person game.
A maximal subgame of a perturbed game \( \tilde{\Gamma} = (\Gamma, f, \eta) \) either is a symmetric subgame \( (\Gamma_y, f_y, \eta_y) \) or an asymmetric subgame \( (\Gamma_y, \eta_y) \) of \( \tilde{\Gamma} \) where \( \Gamma_y \) is a maximal subgame of \( \Gamma \).

We say that a subgame is blocked by a behavior strategy if the information set containing the origin of \( \Gamma_y \) is blocked by this strategy. A subgame or more generally an extensive game is called essential, if it contains at least one essential information set (see 8.6 for the definition of "essential").

10.8 Comment: The strategies induced by a direct ESS of a perturbed game on subgames are characterized by theorems 6 and 7. It is necessary to derive a similar result for truncations with respect to symmetric multigames. This will be the content of lemma 8. On the basis of lemma 8, theorem 8 will characterize a direct ESS of a perturbed game in terms of the strategies induced on the subgames in a symmetric multigame and the truncation with respect to this subgame. Theorem 9 will give a similar characterization in terms of strategies induced on elementary games.

Even if all the results of this section may seem to be fairly obvious at first glance, careful proofs require much more detail than one might think.

It will be necessary to derive a result which is not directly connected to subgames and truncations. If one wants to check whether a symmetric equilibrium strategy is an ESS, one can restrict ones attention to alternative best replies which do not differ from the ESS on information sets blocked by the alternative best reply. This will be the content of lemma 7. One can expect that this lemma is useful not only as a step towards further results, but also as a tool for the analysis of specific game models.

10.9 Lemma 7 (on alternative best replies): Let \( \tilde{\Gamma} = (\Gamma, f, \eta) \) be a perturbed game of a symmetric extensive 2-person
game $(\hat{\Gamma}, f)$ and let $b$ be a symmetric equilibrium strategy for $\hat{\Gamma}$. If $b$ is not a direct ESS of $\hat{\Gamma}$, then a best reply $r$ to $f(b)$ in $\hat{\Gamma}$ with $r \not= b$ and

\[(89) \quad E(b, f(r)) \leq E(r, f(r))\]

can be found which has the following additional property: If an information set $u$ of player 1 is blocked by $r$, then $b$ and $r$ assign the same local strategy to $u$.

Proof: Let $R$ be the set of all best replies $r$ to $f(b)$ in $\hat{\Gamma}$ with $r \not= b$ and $(89)$. Assume that $b$ is not a direct ESS. It follows by $(89)$ together with condition (b) in definition 7.5 of a direct ESS for a perturbed game that $R$ is not empty. For every $r \in R$ let $k(r)$ be the number of information sets of player 1, blocked by $r$, where $b$ and $r$ prescribe different local strategies. Let $r \in R$ be a strategy with $k(r) \leq k(t)$ for every $t \in R$.

We have to show $k(r) = 0$. Assume $k(r) > 0$ and let $u$ be an information set of player 1, blocked by $r$, where the local strategies $b_u$ and $r_u$ assigned to $u$ by $b$ and $r$, respectively, are different from each other.

Consider the strategy $s = r/b_u$. An information set $v$ is blocked by $s$, if and only if it is blocked by $r$. It does not matter that different local strategies are prescribed at an information set which is not reached anyhow. Therefore, an information set $v$ is blocked by $f(s)$, if and only if it is blocked by $f(r)$. Consequently, $(89)$ continues to hold, if $f(r)$ is replaced by $f(s)$ on both sides. Moreover, the payoff on the right hand side remains unchanged if in addition to this $r$ is replaced by $s$, since $u$ is blocked by $r$ and by $s$. This shows that we have:

\[(90) \quad E(b, f(s)) \leq E(s, f(s))\]

Moreover, it is clear that $s$ is a best reply to $f(b)$ in $\hat{\Gamma}$ since in view of the fact that $u$ is blocked by $r$ and $s$ the payoffs $E(r, f(b))$ and $E(s, f(b))$ must be equal. Therefore
s belongs to R. However k(s) < k(r) contrary to the assumption that r is minimal in R with respect to k(r). Therefore, the assertion of the lemma is true.

10.10 Lemma 8 (truncation lemma): Let \( \hat{\Gamma} = (\Gamma, f, n) \) be a perturbed game of a symmetric extensive 2-person game \((\Gamma, f)\) with a subgame preserving natural symmetry f. Let M be a symmetric multisubgame of \((\Gamma, f)\) and let b be a direct ESS for \( \hat{\Gamma} \). Then the strategy \( \hat{b} \) induced by b on the b-truncation \( \hat{\Sigma} = (\hat{\Gamma}, \hat{f}, \hat{n}) \) of \( \hat{\Gamma} \) with respect to M is a direct ESS of this b-truncation \( \hat{\Sigma} \).

Proof: We use the symbol \( E_u \) in order to denote local payoffs in \( \hat{\Gamma} \) at an information set u of \( \hat{\Gamma} \). It can be seen immediately that local payoffs in \( \Gamma \) and \( \hat{\Gamma} \) satisfy the following condition:

\[
(91) \quad E_u(s_u, \hat{b}, \hat{f}(\hat{b})) = E_u(s_u, b, f(b))
\]

for every local strategy \( s_u \) at u. Since b is pervasive by theorem 2, it is clear that \( \hat{b} \) is pervasive in \( \hat{\Gamma} \). Local payoffs are defined everywhere. It follows by theorem 3 that b satisfies local conditions which again by theorem 3 together with (91) permit the conclusion that \( \hat{b} \) is a pervasive symmetric equilibrium strategy for \( \hat{\Sigma} \). Assume that nevertheless \( \hat{b} \) is not a direct ESS of \( \hat{\Sigma} \). Then we can find an alternative best reply \( \hat{F} \) to \( \hat{f}(\hat{b}) \) in \( \hat{\Sigma} \) with \( \hat{F} + \hat{b} \) and with the following property:

\[
(92) \quad E(\hat{b}, \hat{F}(\hat{F})) \leq E(\hat{F}, \hat{f}(\hat{F}))
\]

where \( E \) denotes the expected payoff function for \( \hat{\Gamma} \). This follows by the second condition (b) in definition 7.5 of a perturbed game direct ESS. Let r be that behavior strategy of player 1 for \( \hat{\Gamma} \) which agrees with \( \hat{F} \) on \( \hat{\Gamma} \) and with b on all subgames in M. It is clear that we have:

\[
(93) \quad E(r, f(b)) = E(\hat{F}, f(\hat{b})) = E(\hat{b}, f(\hat{b})) = E(b, f(b))
\]

This shows that r is an alternative best reply to \( f(b) \) in
Moreover, we have:

\[ E(\hat{\mathcal{B}}, \hat{r}(\hat{r})) = E(b, f(r)) \]

and

\[ E(\hat{r}, f(\hat{r})) = E(r, f(r)) \]

Inequality (92) together with (94) and (95) shows that \( r \) violates the second condition (b) in definition 7.5 of a perturbed game direct ESS. Therefore, no alternative best reply \( \hat{r} \) with (92) can be found. \( \hat{\mathcal{B}} \) is a direct ESS of \( \hat{r} \).

10.11 Theorem 8 (truncation theorem): Let \( \hat{r} = (r, f) \) be a perturbed game of a symmetric extensive 2-person game \((r, f)\) with a subgame preserving symmetry \( f \). Let \( M \) be a symmetric multistrategy of \((r, f)\) and let \( b \) be a behavior strategy of player 1 for \( \hat{r} \). Then \( b \) is a direct ESS of \( \hat{r} \), if and only if the following conditions are satisfied for all \( \Gamma_y \in M \) and for the \( b \)-truncation \( \hat{r} = (\hat{r}, \hat{f}, \hat{\eta}) \) of \( \hat{r} \) with respect to \( M \):

(i) The strategy \( \hat{\mathcal{B}} \) induced by \( b \) on \( \hat{r} \) is a direct ESS of \( \hat{r} \). Moreover if \( \Gamma_y \in M \) is essential, then the origin \( y \) of \( \Gamma_y \) has a positive realization probability \( \hat{\gamma}(y, \hat{\mathcal{B}}, \hat{f}(\hat{\mathcal{B}})) \) under \( \hat{\mathcal{B}} \) and \( f(\hat{\mathcal{B}}) \) in \( \hat{r} \).

(ii) If \( \hat{r}_y = (r_y, \eta_y) \) is an essential asymmetric subgame of \( \hat{r} \), then the pair \( (b_y, b'_y) \) of strategies induced by \( b \) and \( f(b) \), respectively on \( \hat{r}_y \) is a strong equilibrium point of \( \hat{r}_y \).

(iii) If \( \hat{r}_y = (r_y, f_y, \eta_y) \) is an essential symmetric subgame of \( \hat{r} \), then the strategy \( b_y \) induced by \( b \) on \( \Gamma_y \) is a direct ESS of \( \hat{r}_y \).

Proof: It follows by lemma 8 that \( \hat{\mathcal{B}} \) is a direct ESS of \( \hat{r} \) if \( b \) is a direct ESS of \( \hat{r} \). Moreover, \( b \) could not be pervasive if \( \hat{\gamma}(y, \hat{\mathcal{B}}, \hat{f}(\hat{\mathcal{B}})) \) would not be positive for the origin \( y \) of every essential subgame in \( M \). If \( b \) is a direct
ESS of \( \tilde{T} \), then (ii) and (iii) follow by theorems 6 and 7.

It remains to show that (i), (ii) and (iii) together imply that \( b \) is a direct ESS of \( \tilde{T} \). It will first be shown that \( b \) is pervasive, then that \( b \) is a symmetric equilibrium strategy and finally that \( b \) is a direct ESS.

In view of theorem 2, it is clear that the strategies \( \tilde{b} \) and \( by \) in (i) and (iii) are pervasive. Lemma 6 shows that the realization probabilities \( \gamma(y, \tilde{b}, by) \) of essential information sets \( u \) in asymmetric subgames \( \Gamma_y \in \mathcal{M} \) are positive. Since \( \gamma(y, \tilde{b}, f(\tilde{b})) \) is positive for origins \( y \) of essential subgames in \( \mathcal{M} \) we can conclude that \( b \) is pervasive. Note that without the assumption on \( \gamma(y, \tilde{b}, f(\tilde{b})) \) the pervasiveness of \( b \) would not follow.

In order to prove that \( b \) is a symmetric equilibrium strategy of \( \tilde{T} \) we show that the local conditions required by theorem 3 are satisfied. In this respect, it is important to remember that local payoffs in a subgame agree with local payoffs in the whole game in the sense of (80) in 10.2. Moreover, we can make use of (91) in the proof of lemma 8. Therefore, the local optimality conditions implied by theorem 3 and lemma 6 applied to the induced strategies in (i), (ii) and (iii) are nothing else than the local optimality conditions required for \( b \) by theorem 3. This shows that \( b \) is a pervasive symmetric equilibrium strategy of \( \tilde{T} \).

It remains to show that the second condition in definition 7.5 of a perturbed game direct ESS is satisfied for \( b \). We can restrict our attention to best replies to \( f(b) \) in \( \tilde{T} \) with \( r \neq b \) and the additional property from lemma 7. Let \( r \) be a strategy of this kind. For every subgame \( \Gamma_y \in \mathcal{M} \) let \( r_y \) be the strategy induced by \( r \) on \( \Gamma_y \). Let \( \tilde{r} \) be the strategy induced by \( r \) on \( \tilde{T} \).

If a subgame \( \Gamma_y \in \mathcal{M} \) is blocked by \( r \), then for this subgame \( r_y \) agrees with \( by \) in view of the additional property from
lemma 7.

Consider an asymmetric subgame $\Gamma_y \in \mathcal{M}$ not blocked by $r$. Then it follows by (iii) that $b_y$ is the only best reply in $\tilde{\Gamma}_y$ to the strategy $b'_y$ induced by $f(b)$ on $\tilde{\Gamma}_y$. It follows by lemma 6 applied to $b$ and by lemma 3 applied to $r$ that in this case $r_y$ agrees with $b_y$.

Suppose that for a subgame $\Gamma_y \in \mathcal{M}$ we have $r_y \neq b_y$. Then $r_y$ is not blocked by $r$ and $r_y$ belongs to a symmetric subgame $(\Gamma_y, f_y)$ of $(\Gamma, f)$. Moreover, by the additional property of lemma 7 the local strategies assigned to information sets blocked by $r_y$ in $\Gamma_y$ are the same for $b_y$ and $r_y$. It follows by lemma 3 that the sufficient local conditions required by lemma 4 for a best reply to $f_y(b_y)$ in $\tilde{\Gamma}_y = (\Gamma_y, f_y, \eta_y)$ are satisfied by $r_y$. Therefore $r_y$ is a best reply to $f_y(b_y)$ in $\tilde{\Gamma}_y$.

Consider the $(r, f(b))$-truncation $\tilde{\Gamma}_x$ of $\tilde{\Gamma}$ with respect to $M$. Since $r$ induces best replies to the strategies induced by $f(b)$ in the subgames $\tilde{\Gamma}_y$ of $\tilde{\Gamma}$ mentioned in (ii) and (iii), the game $\tilde{\Gamma}_x$ differs from $\tilde{\Gamma}$ only with respect to player 2's payoff. Therefore $\tilde{\Gamma}$ must be a best reply to $b$ in $\tilde{\Gamma}$.

For every subgame $\Gamma_y \in \mathcal{M}$ let $r'_y$ be the strategy induced by $f(r)$ on $\Gamma_y$. It is clear that for asymmetric subgames $(r_y, r'_y)$ agrees with $(b_y, b'_y)$. Wherever $(r_y, r'_y)$ does not agree with $(b_y, b'_y)$ the game $\Gamma_y = (\Gamma_y, f_y, \eta_y)$ is a symmetric subgame of $\tilde{\Gamma}$ and $r_y$ is an alternative best reply to $f_y(b_y)$ which by the second condition in the direct ESS definition 7.5 satisfies the following inequality:

\[(96) \quad E_y(b_y, f_y(r_y)) - E_y(r_y, f_y(r_y)) > 0\]

Assume that for at least one subgame $\Gamma_y \in \mathcal{M}$ we have $r_y \neq b_y$. As we shall see under this assumption (96) has the following consequence:
(97) \[ E(b, f(r)) - E(r, f(r)) > E(b, \tilde{f}(\tilde{r})) - E(\tilde{r}, \tilde{f}(\tilde{r})) \]

where \( E \) denotes payoffs in \( \tilde{r} \). On the right hand side of (97) we find the payoff difference between \( b \) and \( \tilde{r} \) against \( f(\tilde{r}) \), if player 1 behaves according to \( b \) in all subgames \( \Gamma_y \in M \). Inequality (96) shows that this difference is increased if player 1 changes his strategy from \( b_y \) to \( r_y \) in a subgame \( \Gamma_y \in M \) with \( b_y \neq r_y \). This yields (97).

\( \tilde{b} \) is a direct ESS of \( \tilde{r} \). If \( \tilde{r} = \tilde{b} \), then the right hand side of (97) is zero. If \( \tilde{r} \neq \tilde{b} \), then it follows by the second condition in 7.5 that the right hand side of (97) is positive. Therefore, the left hand side of (97) is positive. This shows that \( r \) does not violate the second condition in 7.5 as an alternative best reply to \( f(b) \) in \( \tilde{r} \).

Now assume that we have \( b_y = r_y \) for all subgames \( \Gamma_y \in M \). In this case we have:

(98) \[ E(b, f(r)) - E(r, f(r)) = E(b, \tilde{f}(\tilde{r})) - E(\tilde{r}, \tilde{f}(\tilde{r})) \]

Moreover, \( \tilde{r} \) and \( \tilde{b} \) are different from each other since otherwise \( r \) could not be different from \( b \). Since \( \tilde{b} \) is a direct ESS for \( \tilde{r} \) it follows by the second condition in 7.5 that the right hand side of (98) is positive. Consequently, in this case, too, \( r \) does not violate the second condition in 7.5 applied to \( b \). We have shown that \( b \) is a direct ESS of \( \tilde{r} \).

10.12 Theorem 9 (decomposition theorem): Let \( \tilde{r} = (r, f, n) \) be a perturbed game of a symmetric extensive game \((x, f)\) with a subgame preserving symmetry \( f \). Let \( b \) be a behavior strategy of player 1 for \( x \). For every \( b \)-element \( \tilde{x} \) of \( \tilde{r} \) let \( \tilde{b} \) and \( \tilde{b}' \) be the strategies induced by \( b \) and \( f(b) \), respectively on \( \tilde{x} \). Then \( b \) is a direct ESS of \( \tilde{r} \), if and only if the following conditions (a), (b) and (c) are satisfied for all \( b \)-elements \( \tilde{x} \) of \( \tilde{r} \):

(a) If \( \tilde{x} = (\tilde{r}, \tilde{n}) \) is an asymmetric \( b \)-element of \( \tilde{r} \), then \((\tilde{b}, \tilde{b}')\) is a strong equilibrium point of \( \tilde{r} \).
(b) If $\bar{y} = (\bar{r}, \bar{f}, \bar{u})$ is a symmetric b-element of $\bar{\tau}$, then $b$ is a direct ESS of $\bar{\tau}$.

(c) If a decomposition endpoint $y$ of $\bar{\tau}$ is the origin of an essential subgame $\Gamma_y$ of $\tau$, then the realization probability $\gamma(y, b, b')$ of $y$ in $\bar{\tau}$ is positive.

Proof: The decomposition rank of a game $\tau$ is defined recursively by the following two conditions (i) and (ii):

(i) If $\tau$ has no subgame then the decomposition rank of $\tau$ is 1.

(ii) If $\tau$ has at least one maximal subgame of decomposition rank $k$ and all maximal subgames have decomposition ranks of at most $k$, then the decomposition rank of $\tau$ is $k+1$.

The theorem will be proved by induction on the decomposition rank of $\tau$. It is trivially true if $\tau$ has decomposition rank 1. Suppose that the assertion holds if $\tau$ has a decomposition rank of at most $k$. Then the theorem holds for the maximal subgames of $\tau$. For every maximal subgame $\Gamma_y$ of $\tau$ let $b_y$ be the strategy induced by $b$ on $\Gamma_y$. The $b_y$-elements of a maximal subgame $\Gamma_y$ of $\bar{\tau}$ are b-elements of $\Gamma_y$. The main $b$-truncation of $\bar{\tau}$ is the only b-element of $\bar{\tau}$ which is not a $b_y$-element of a maximal subgame $\Gamma_y$ of $\bar{\tau}$. The decomposition rank connected to the main b-truncation is 1.

Assume that $\tau$ has decomposition rank $k+1$. If $b$ is a direct ESS of $\bar{\tau}$, then it follows by theorem 8 and the induction hypothesis applied to the maximal subgames of $\bar{\tau}$ that (a), (b) and (c) hold. If (a), (b) and (c) hold for $\bar{\tau}$, then we can conclude that (ii) and (iii) in theorem 8 hold for the maximal subgame of $\bar{\tau}$. Theorem 8 together with (b) and (c) applied to the main b-truncation of $\bar{\tau}$ permits the conclusion that $b$ is a direct ESS of $\bar{\tau}$. Consequently, the assertion of the theorem is true.
10.13 Comment: Theorem 9 characterizes a direct ESS of a perturbed game in terms of the strategies induced on the elementary games generated by the direct ESS. It would be desirable to obtain a similar characterization of a limit ESS. Unfortunately, a result analogous to theorem 9 cannot be derived since the strong inequalities in the definition of a strong equilibrium point and in the second ESS condition need not be preserved by a transition to the limit. A limit ESS cannot be expected to satisfy these strong inequalities but only weak inequalities of the same kind. These weak inequalities are not sufficient for a limit ESS but since they are necessary, they may still serve to exclude many symmetric equilibrium strategies as possible candidates for a limit ESS. In order to be able to express these necessary conditions in a convenient way we shall introduce the notion of a "semi-stable strategy".

10.14 Semistable strategies: A behavior strategy \( b \) for player 1 in a symmetric extensive 2-person game \((\Gamma,f)\) is called a semistable strategy of \((\Gamma,f)\) if it is a symmetric equilibrium strategy which satisfies the following additional condition: For every best reply \( r \) to \( f(b) \) in \((\Gamma,f)\) we have:

\[
E(b,f(r)) \geq E(r,f(r))
\]

Analogously a semistable strategy \( q \) of a bimatrix game \( G = (I,E) \) is a symmetric equilibrium strategy for \( G \) which satisfies the following condition:

\[
E(q,r) \geq E(r,r)
\]

for every best reply \( r \) to \( q \).

10.15 Theorem 10 (necessary conditions for a limit ESS): Let \((\Gamma,f)\) be a symmetric extensive 2-person game with a subgame preserving natural symmetry \( f \). Let \( b \) be a limit ESS of \((\Gamma,f)\)
and for every subgame \( \Gamma_y \) of \( \Gamma \) let \( b_y \) and \( b'_y \) be the strategies induced by \( b \) and \( f(b) \), respectively on \( \Gamma_y \). Then the following conditions (a) to (e) are satisfied:

(a) If \( \Gamma_y \) is an asymmetric subgame of \( \Gamma, f \) then \( (b_y, b'_y) \) is an equilibrium point in pure strategies of \( \Gamma_y \).

(b) If \( \bar{\Gamma} \) is a \( (b_y, b'_y) \)-truncation of an asymmetric subgame \( \Gamma_y \) of \( \Gamma, f \) then the strategies \( b \) and \( b' \) induced on \( \bar{\Gamma} \) by \( b \) and \( f(b) \), respectively form an equilibrium point \( (b, b') \) in pure strategies of \( \bar{\Gamma} \).

(c) If \( (\Gamma_y, f_y) \) is a symmetric subgame of \( \Gamma, f \) then \( b_y \) is a semistable strategy of \( (\Gamma_y, f_y) \).

(d) If \( (\bar{\Gamma}, f) \) is a \( b \)-truncation of \( \Gamma, f \) with respect to a symmetric multisubgame, then the strategy \( b \) induced on \( \bar{\Gamma} \) by \( b \) is a semistable strategy of \( (\bar{\Gamma}, f) \).

(e) If \( (\bar{\Gamma}, f) \) is a \( b \)-truncation of a symmetric subgame \( (\Gamma_y, f_y) \) of \( \Gamma, f \) with respect to a symmetric multisubgame of \( (\Gamma_y, f_y) \), then the strategy \( b \) induced on \( \bar{\Gamma} \) by \( b \) is a semistable strategy of \( (\bar{\Gamma}, f) \).

Proof: We can find a test sequence \( \bar{\Gamma}^1, \bar{\Gamma}^2, ... \) with \( \bar{\Gamma} = (\Gamma, f, \eta) \) such that \( b \) is a limit ESS of this test sequence, i.e. the limit of a sequence \( b^1, b^2, ... \) of direct ESS's \( b^k \) for the corresponding perturbed games \( \bar{\Gamma}^k \). Theorem 6 can be applied to each of the \( b^k \). Let \( b^k_y \) and \( b'^k_y \) be the strategies induced on \( \Gamma_y \) by \( b^k \) and \( f(b^k) \), respectively and let \( \eta^k_y \) be the perturbation induced on \( \Gamma_y \) by \( \eta^k \). Obviously \( \bar{\Gamma}^k = (\Gamma_y, \eta^k_y) \) is an asymmetric subgame of \( \bar{\Gamma}^k \). It follows by theorem 6 that \( (b^k_y, b'^k_y) \) is a strong equilibrium point of \( \bar{\Gamma}^k \). Therefore, the local strategies prescribed by \( b^k_y \) and \( b'^k_y \) must be extreme in \( \bar{\Gamma}^k_y \). Consequently, the limit \( (b_y, b'_y) \) of the sequence of the pairs \( (b^k_y, b'^k_y) \) must be a pair of pure strategies. Moreover, in view of the continuity of the payoff \( E_y \) of \( \Gamma_y \), it can be shown that \( (b_y, b'_y) \) is an equilibrium point of \( \bar{\Gamma}_y \) (compare remark 7.7). Consequently (a) holds.
We now turn our attention to (b). For \( k = 1, 2, \ldots \) let \( \tilde{\pi}^k = (\tilde{r}^k, \tilde{y}^k) \) be the \( (b^k, b_y^k) \)-truncation of \( \pi_y^k \) with respect to a fixed multisubgame \( M \) of \( \Gamma_y \); moreover, let \( b^k \) and \( \delta^k \) be the strategies induced by \( b^k \) and \( f(b^k) \), respectively on \( \tilde{\pi}^k \). Local payoffs in \( \Gamma_y \) and \( \tilde{\pi}^k \) agree in the following sense:

\[
E_{yu}(r_u, b^k, b_y^k) = \tilde{E}_u(r_u, \delta^k, \delta_y^k)
\]

where \( E_{yu} \) and \( \tilde{E}_u^k \) are the local payoff functions at \( u \) in \( \Gamma_y \) and \( \tilde{\pi}^k \), respectively. It follows by lemma 6 that \( (\delta^k, \delta_y^k) \) is a strong equilibrium point of \( \tilde{\pi}^k \). An argument analogous to that used at the end of the proof for statement (a) shows that the limit \( (\delta, \delta_y^k) \) of the sequence of the \( (\delta^k, \delta_y^k) \) for \( k \rightarrow \infty \) is an equilibrium point in pure strategies of the \( (b^k, b_y^k) \)-truncation \( \tilde{\pi} \) of \( \Gamma_y \) with respect to \( M \). Statement (b) holds.

We now prove (c). For \( k = 1, 2, \ldots \) let \( \tilde{\pi}_y^k = (\tilde{r}_y^k, \tilde{r}_y^k, \tilde{n}_y^k) \) be the subgame of \( \tilde{\pi}^k \) at \( y \) and let \( b_y^k \) be the strategy induced by \( b^k \) on \( \Gamma_y \). It follows by theorem 7 that \( b_y^k \) is a direct ESS of \( \tilde{\pi}_y^k \). Obviously, the strategy \( b_y^k \) induced on \( \Gamma_y \) by \( b^k \) is the limit of the \( b_y^k \) for \( k \rightarrow \infty \). This shows that \( b_y^k \) is a limit ESS of the test sequence \( \tilde{\pi}_y^1, \tilde{\pi}_y^2, \ldots \). Therefore, \( b_y^k \) is a symmetric equilibrium strategy of \( \Gamma_y \) (compare remark 7.12). Moreover, in view of the continuity of \( E_y \), the second ESS condition in 7.5 applied to \( b^k \) secures the semistability of \( b_y^k \). Statement (c) holds.

The next statement to be proved is (d). For \( k = 1, 2, \ldots \) let \( \hat{\pi}^k = (\hat{r}^k, \hat{f}^k, \hat{r}^k) \) be the \( b^k \)-truncation of \( \pi^k \) with respect to a fixed symmetric subgame \( M \); moreover, let \( \delta^k \) be the strategy induced by \( b^k \) on \( \hat{\pi}^k \). Let \( (\tilde{\pi}, \tilde{f}) \) be the \( b \)-truncation of \( (\pi, f) \) with respect to \( M \). It follows by (i) in theorem 8 that \( \delta^k \) is a direct ESS of \( \hat{\pi}^k \). In view of the continuity of \( E_y \), the strategy \( \delta \) induced on \( \tilde{\pi} \) by \( b \) is the limit of the sequence of \( \delta^k \) for \( k \rightarrow \infty \). The inequalities satisfied by a symmetric equilibrium point remain valid in the transition to the limit. The second ESS
condition in 7.5 applied to $b^k$ secures the semistability of $\tilde{b}$ in $(\tilde{r}, \tilde{f})$. Statement (d) holds.

It remains to show (e). The proof of (c) has shown that $b_y$ is a limit ESS of the test sequence $\tilde{r}_y^1, \tilde{r}_y^2, \ldots$. Therefore $(\tilde{r}_y, f_y)$ instead of $(\tilde{r}, f)$ and $b_y$ instead of $b$ satisfy the assumptions of the theorem. We can apply statement (d) to $(\tilde{r}_y, f_y)$ and $b_y$. Thereby we receive statement (e).

10.16 Remark: The proof of (c) has shown that a limit ESS of a symmetric extensive 2-person game $(\tilde{r}, f)$ induces a limit ESS of $(\tilde{r}_y, f_y)$ on every symmetric subgame of $(\tilde{r}, f)$. This property of the notion of a limit ESS will be referred to as subgame consistency (compare Selten 1973).

It is not true in general that a limit ESS $b$ of a symmetric extensive 2-person game $(\tilde{r}, f)$ induces a limit ESS of $(\tilde{r}, \tilde{f})$ on every $b$-truncation $(\tilde{r}, \tilde{f})$ of $(\tilde{r}, f)$. Truncation consistency in this sense is not a property of the notion of a limit ESS. A counterexample is provided by the male desertion game of figure 9 in 6.5 for the case $a+s=1$. Consider a perturbed game with minimum probabilities $\epsilon > 0$ for $L$ at $v$ and $v'$ and with zero minimum probabilities everywhere else. A perturbed game of this kind has exactly one direct ESS which prescribes $R$ at $u$ and $v$. This shows that the behavior strategy $b$ of player 1 which prescribes $R$ at $u$ and $v$ is a limit ESS of the game of figure 9 with $a+s=1$. However, the $b$-truncation with respect to the symmetric multisubgame containing the subgame at $x_2$ and its symmetric image does not have any limit ESS.
11. Simultaneity games

Theorem 9 shows that one can find the direct ESS's of a perturbed game by a successive analysis of elementary games. One begins with the smallest subgames near the end. It may, of course, happen that these subgames have several direct ESS's. Truncations with respect to the smallest subgames must be formed for each possible case. Then the process is continued by the analysis of the smallest subgames of the truncations, etc. At least, in principle all direct ESS's of the whole game can be found in this way.

The procedure outlined above can be expected to simplify the task of analyzing a game model, in cases where a big complex game can be decomposed into many simple elementary games. In this section we shall turn our attention to a class of games which exhibit a very high degree of decomposability. Such games arise from models with a period structure where the two players have the opportunity to make simultaneous decisions in each of a finite number of periods. In each period the players know everything which happened in previous periods, but they do not know the random choices and the choices of the opponent in the same period. In the literature such games have been referred to as simultaneity games (Selten 1973).

In this paper a simultaneity game will be formally introduced as a symmetric extensive 2-person game whose elementary games contain at most one information set for each of both players. It is clear that in this way one obtains a class of games which includes the period structure models described above.

It will be shown that for perturbed simultaneity games there is no difference between an LSS in the sense of 9.9 and a direct ESS.

The notion of a regular ESS for a symmetric bimatrix game will be introduced in order to derive sufficient local conditions for a limit ESS of a simultaneity game. A behavior strategy \( \mathbf{b} \) of player 1 for a simultaneity game is
a limit ESS, if strong equilibrium points are induced by b and its symmetric image on asymmetric b-elements and regular ESS's are induced on symmetric b-elements. This will be the final result.

11.1 Simultaneity games: A simultaneous game is a symmetric extensive 2-person game \((\Gamma, f)\) with a subgame preserving natural symmetry \(f\) and with the additional property that every b-element of \((\Gamma, f)\) has at most one information set for each of both players 1 and 2. This definition does not really depend on b. With the exception of the payoff functions the components of the b-elements are not influenced by b.

Consider a symmetric b-element \((\tilde{\Gamma}, \tilde{f})\) of a simultaneity game \((\Gamma, f)\). Assume that \((\tilde{\Gamma}, \tilde{f})\) is essential or, in other words, that \(\tilde{\Gamma}\) has at least one essential information set. Then \(\tilde{\Gamma}\) has exactly one information set for each of both players 1 and 2. Let \(u\) be player 1's information set in \(\tilde{\Gamma}\). Then \(f(u)\) is player 2's information set in \(\tilde{\Gamma}\). Consider the local game \(G_{ub} = (C_u, E_{ub})\) of \((\Gamma, f)\) at \(u\) under b (see 9.8). Obviously, \(G_{ub}\) is nothing else than the symmetric normal form of \((\tilde{\Gamma}, \tilde{f})\). (For the definition of the symmetric normal form, see 5.8). This fact is the basis of the following theorem.

11.2 Theorem 11 (on local and global stability): Let \(\tilde{\Gamma} = (\Gamma, f, n)\) be a perturbed game of a simultaneity game \((\Gamma, f)\) and let b be a behavior strategy of player 1 for \(\tilde{\Gamma}\). Then b is a direct ESS of \(\tilde{\Gamma}\), if and only if b is an LSS of \(\tilde{\Gamma}\).

Proof: As has been pointed out in 9.9 a direct ESS of \(\tilde{\Gamma}\) always is an LSS of \(\tilde{\Gamma}\). It remains to show that in view of the simultaneity game property of \((\Gamma, f)\), an LSS of \(\tilde{\Gamma}\) is a direct ESS of \(\tilde{\Gamma}\). Let b be an LSS of \(\tilde{\Gamma}\). Then b is pervasive in view of (i) in 9.9. Therefore, condition (c) of theorem 9 is satisfied. Condition (a) of theorem 3 follows by (ii) in 9.9. Condition (b) of theorem 3 follows by (iii) in view of the agreement between local games.
and symmetric normal forms of symmetric b-elements discussed in 11.1. Theorem 3 permits the conclusion that b is a direct ESS of \( \pi \).

11.3 Essential ESS: Let \( G = (\Pi, E) \) and \( G_+ = (\Pi, E_+) \) be two symmetric bimatrix games which differ only with respect to their payoff functions \( E \) and \( E_+ \). Define

\[
|E - E_+| = \max_{\gamma, \phi \in \Pi} |E(\gamma, \phi) - E_+(\gamma, \phi)|
\]

We call \( |E - E_+| \) the distance between \( G \) and \( G_+ \). Let \( q \) and \( r \) be two mixed strategies for \( G = (\Pi, E) \). Define

\[
|q - r| = \max_{\pi \in \Pi} |q(\pi) - r(\pi)|
\]

We call \( |q - r| \) the distance between \( q \) and \( r \). A mixed strategy \( q \) for \( G = (\Pi, E) \) is an essential ESS of \( G = (\Pi, E) \) if \( q \) is an ESS of \( G \) and in addition to this satisfies the following condition: For every \( \varepsilon > 0 \) we can find a \( \delta > 0 \) such that every symmetric bimatrix game \( G_+ = (\Pi, E_+) \) whose distance from \( G \) is smaller than \( \delta \), has an ESS \( q_+ \) whose distance from \( q \) is smaller than \( \varepsilon \).

Wu Wen-tsün and Jiang Jia-he have introduced the notion of an essential equilibrium point (Wu-Wen-tsün and Jiang Jia-he 1962). The definition of an essential ESS is analogous to their definition.

11.4 Regularity: A symmetric equilibrium strategy \( q \) of a symmetric bimatrix game \( G = (\Pi, E) \) is called regular if it assigns a positive probability \( q(\pi) \) to every pure best reply \( \pi \) to \( q \) in \( G \). An ESS of \( G \) is called regular if it is a regular symmetric equilibrium strategy of \( G \).

In the literature the term "regular" applied to equilibrium points has a different sense (Harsanyi 1973, van Damme 1983). Our use of the word "regular" corresponds to van Damme's use of the word "quasi-strong". It has been shown that an equilibrium point of a bimatrix game is regular in the sense of Harsanyi and van Damme, if and only if it is essential.
and quasi-strong (Jansen 1981, van Damme 1983, theorem 3.4.5). It will be shown in lemma 9 that a regular ESS (in our sense) is always essential. The equilibrium point connected to a regular ESS is also regular in the sense of Harsanyi and van Damme. This justifies our language use.

11.5 Haigh's criterion: Let \( G = (\Pi, E) \) be a symmetric bimatrix game. The **carrier** of a mixed strategy \( r \) for \( G \) is the set of all pure strategies \( \pi \in \Pi \) with \( r(\pi) > 0 \). Let \( q \) be a regular symmetric equilibrium strategy for \( G \). The fact that \( q \) is regular can be expressed by saying that all pure best replies to \( q \) are in the carrier of \( q \). Let \( \pi_1, \ldots, \pi_n \) be the pure strategies in the carrier of \( q \). Define

\[
    a_{ij} = E(\pi_i, \pi_j)
\]

for \( i,j = 1, \ldots, n \). The payoffs \( a_{ij} \) form an \( nxn \)-matrix

\[
    A = (a_{ij})_{nxn}
\]

We call this matrix the **carrier matrix** of \( q \). The carrier matrix is not determined by \( q \) alone but also by the numbering of the pure strategies in the carrier. The definition must be understood relative to a fixed numbering.

Let \( R \) be the set of all mixed strategies for \( G \) which assign positive probabilities to the pure strategies in the carrier of \( q \) only. Obviously, \( R \) can be described as the set of all best replies to \( q \) in \( G \). For any \( r \in R \) let \( r_i \) be the probability \( r(\pi_i) \) assigned to \( \pi_i \) by \( r \). We can think of \( r \) as a column vector.

\[
    r = \begin{pmatrix}
        r_1 \\
        \vdots \\
        r_n
    \end{pmatrix}
\]
The same symbol \( r \) will be used for the strategy \( r \) and for this column vector. No confusion can arise from this notational convention. In the same way \( q \) also represents the \( n \)-dimensional column vector whose components \( q_i \) are the probabilities assigned to \( \pi_i \) by \( q \).

We use the upper index \( T \) to indicate transposition. For any two strategies \( r \) and \( s \) in \( R \) the expected payoff in \( G \) can be written as follows:

\[
(107) \quad E(r,s) = r^TAs
\]

We now examine the conditions under which \( q \) is an ESS of \( G \). The second condition (b) in 2.11 is satisfied for \( q \), if and only if for every \( r \in R \) with \( r \neq q \) we have:

\[
(108) \quad q^TAr > r^TAr
\]

If there is only one pure strategy in the carrier of \( q \), then this is trivially true since in this case \( q \) is the only best reply to \( q \) in \( G \). Therefore, in the following we shall assume \( n \geq 1 \). The fact that \( r \) is a best reply to \( q \) in \( G \) can be expressed as follows:

\[
(109) \quad q^TAr = r^TAr
\]

Subtraction of (106) from (105) yields:

\[
(110) \quad q^TAr - q^TAr > r^TAr - r^TAr
\]

Therefore, the second condition (b) in 2.11 is satisfied for \( q \), if and only if for every \( r \in R \) with \( r \neq q \) we have:

\[
(111) \quad (r-q)^TAr(r-q) < 0
\]

The components of \( r-q \) sum up to zero. Let \( S \) be the set of all vectors

\[
(112) \quad s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}
\]
with

\[ \sum_{i=1}^{n} s_i = 0 \]

which are different from the \( n \)-dimensional zero vector. Since \( q_i \) is positive for \( i=1, \ldots, n \) for every \( s \in S \) we can find an \( r \in \mathbb{R} \) and a number \( \lambda > 0 \) such that

\[ r - q = \lambda s \]

Therefore, the second condition (b) in 2.11 is satisfied for \( q \), if and only if for every \( s \in S \) we have:

\[ s^T A s < 0 \]

Haigh's criterion is a condition on \( A \) which guarantees that (115) holds for every \( s \in S \) (Haigh 1975). In order to derive this criterion, it is useful to look at the left hand side of (115) as a quadratic form in the first \( n-1 \) components of \( r - q \). For every \( s \in S \), let \( \tilde{s} \) be the \( (n-1) \)-dimensional vector:

\[ \tilde{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_{n-1} \end{bmatrix} \]

Let \( D \) be the following \( n \times (n-1) \)-matrix:

\[ D = (d_{ij})_{n \times (n-1)} = \begin{bmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \\ -1 & \ldots & -1 \end{bmatrix} \]

with

\[ d_{ij} = \begin{cases} 1 & \text{for } i=j < n \\ -1 & \text{for } i=n \\ 0 & \text{else} \end{cases} \]

Obviously we have

\[ s = D \tilde{s} \]
Therefore the left hand side of (115) can be rewritten as follows:

\[(120) \quad s^T A s = s^T D^T A s\]

We say that **Haigh's criterion** is satisfied for \( q \) if the \((n-1) \times (n-1)\)-matrix \( D^T A D \) is negative quasi-definite. (120) shows that the second condition (b) holds for \( q \), if and only if Haigh's condition is satisfied. In order to extend this statement to the trivial case of only one pure strategy in the carrier of \( q \), we shall count **Haigh's criterion** as always satisfied in this case.

Whether Haigh's criterion is satisfied or not does not depend on the numbering of the pure strategies in the carrier. Therefore, it is not necessary to refer to a particular numbering if we speak of Haigh's criterion.

The result obtained above is expressed by the first part of the following lemma.

**11.6 Lemma 9 (on Haigh's criterion):** A regular symmetric equilibrium strategy \( q \) of a symmetric bimatrix game \( G = (\Pi, E) \) is an ESS of \( G \), if and only if it satisfies Haigh's criterion. Moreover, a regular ESS of \( G \) is an essential ESS of \( G \).

**Proof:** The first part of the lemma has been shown in 11.5. It remains to prove the second part. Let \( q \) be a regular ESS of \( G \).

If the carrier of \( q \) has only one element, then \((q,q)\) is a strong equilibrium point of \( G \) and also a strong equilibrium point of every game \( G_+ = (\Pi, E_+) \) sufficiently near to \( G \). Therefore, in this case the assertion is true. In the following we shall assume that there are at least two elements in the carrier of \( q \). We shall use the notation of 11.5. Define:
\[(121) \quad y = q^\top A q\]

\(y\) is the equilibrium payoff connected to \(q\). The components \(q_1, \ldots, q_n\) of \(q\) together with \(y\) can be looked upon as \(n+1\) unknowns in a system of \(n+1\) linear equations:

\[(122) \quad a_iq_1 + \cdots + a_nq_n - y = 0 \quad \text{for } i = 1, \ldots, n\]

\[(123) \quad q_1 + \cdots + q_n = 1\]

We shall first show that this system has only one solution. Suppose that \(q_1, \ldots, q_n, y\) is not the only solution. The set of all solution vectors of a linear equation system forms a linear subspace in the space of the unknowns. Therefore, a different solution \(\tilde{q}_1, \ldots, \tilde{q}_n, \tilde{y}\) can be found arbitrarily near to \(q_1, \ldots, q_n, y\). If \(\pi \in \Pi\) is not in the carrier of \(q\), then \(E(\pi, q) < E(q, q)\). In view of the continuity of \(E\) we can find an \(\varepsilon > 0\) such that \(E(\pi, q) < E(\tilde{q}, q)\) holds for every \(\tilde{q}\) with \(|\tilde{q}_i - q_i| < \varepsilon\) and every \(\pi \in \Pi\) outside the carrier of \(q\). Let \(\varepsilon\) be a number of this kind and let \(\tilde{q}_1, \ldots, \tilde{q}_n\) be a solution of the system such that \(\tilde{q}_i\) is positive for \(i = 1, \ldots, n\) and \(|\tilde{q}_i - q_i| < \varepsilon\) holds for \(i = 1, \ldots, n\). In view of (120) the \(\tilde{q}_i\) form a strategy \(\tilde{q}\) in the carrier of \(q\). Moreover, the choice of \(\varepsilon\) guarantees that all best replies to \(\tilde{q}\) in \(G\) are in the carrier of \(q\). In view of (122) all pure strategies in the common carrier of \(q\) and \(\tilde{q}\) yield the same payoff \(\tilde{y}\) against \(\tilde{q}\). Therefore, all these pure strategies are best replies to \(\tilde{q}\). It follows that \(\tilde{q}\) is a symmetric equilibrium strategy of \(G\). Since \(q\) and \(\tilde{q}\) have the same carrier, both \(q\) and \(\tilde{q}\) are best replies to \(q\):

\[(124) \quad E(q, q) = E(\tilde{q}, q)\]

This shows that \(\tilde{q}\) is an alternative best reply to \(q\), which violates the second condition (b) in 2.11.
we know by the first part of the lemma that \( q \) is an ESS
we can conclude that \( q_1, \ldots, q_n, y \) is the only solution
of the system formed by (122) and (123).

The fact that \( q_1, \ldots, q_n, y \) is the only solution of the
system has the consequence that the matrix of the system
is non-singular. If \( G_+ = (\Pi, E_+) \) is sufficiently near to
\( G = (\Pi, E) \), then the matrix \( A_+ = (a_+_{ij}) \) with

\[
(125) \quad a_+_{ij} = E_+ (\pi_i^+, \pi_j^+)
\]

will still be non-singular and the system formed by (122)
and (123) with coefficients \( a_+_{ij} \) instead of \( a_{ij} \) will still
have a unique solution \( q_1^+, \ldots, q_n^+, y^+ \). Moreover, this
unique solution depends continuously on the payoffs in \( E_+ \).
Therefore, if \( G_+ \) is sufficiently near to \( G \), then
\( E_+ (\pi, q_+) \leq E_+ (q_+, q_+) \) holds for all pure strategies \( \pi \) not in
the carrier of \( q \). In a similar fashion as for \( \bar{q} \) in the
first part of the proof we can conclude that \( q_+ \) is a symme-
tric equilibrium strategy of \( G_+ \).

An \((n \times n)\)-matrix \( M \) is negative quasidefinite, if and only if
the so-called north-west subdeterminants \( \Delta_1, \ldots, \Delta_n \) of
\( M + M^T \) have alternating signs, beginning with a negative
sign for \( \Delta_1 \) (see for example Beckmann and Künzi 1973).
This shows that Haigh's criterion is satisfied for the
symmetric equilibrium strategy \( q_+ \) of \( G_+ \) described above,
if \( G_+ \) is sufficiently near to \( G \); it follows that \( q_+ \) is
an ESS of \( G_+ \). We can conclude that \( q \) is an essential ESS
of \( G \).

11.7 Remark: If \( q \) is a regular ESS of a symmetric bimatrix
game \( G = (\Pi, E) \), then for every \( \varepsilon > 0 \) a \( \delta > 0 \) can be found
such that a game \( G_+ = (\Pi, E_+) \) whose distance from \( G \) is
smaller than \( \delta \), has a regular ESS \( q_+ \) with \( lq_+ - q_0 < \varepsilon \). This
statement is a consequence of the proof of lemma 9. The
ESS \( q_+ \) of a game \( G_+ \) sufficiently near to \( G \) which has been
constructed there, satisfies Haigh's criterion. Therefore,
the second part of the lemma can be applied to \( q_+ \). It is
also clear that the trivial case of only one pure strategy in the carrier of \( q \) and \( q_+ \) poses no difficulties.

11.8 Example of an inessential ESS: Figure 15 shows a class of symmetric bimatrix games \( G_\epsilon \). The pure strategy \( \pi_1 \) is an ESS of the game \( G_0 \) with \( \epsilon = 0 \). It can be seen immediately that for \( \epsilon > 0 \) a mixed strategy \( q \) is a symmetric equilibrium strategy of \( G_\epsilon \), if and only if it satisfies the following condition:

\[
q(\pi_1) = \frac{1}{1+\epsilon}
\]

The complementary probability \( \epsilon/(1+\epsilon) \) can be distributed in any way on the two remaining pure strategies \( \pi_2 \) and \( \pi_3 \). Let \( q \) and \( \tilde{q} \) be two different mixed strategies which assign \( 1/(1+\epsilon) \) to \( \pi_1 \). Obviously, both \( q \) and \( \tilde{q} \) yield the same payoff 1 against \( \tilde{q} \) in \( G_\epsilon \). Therefore \( \tilde{q} \) violates the second condition as an alternative best reply to \( q \). We can conclude that none of the games \( G_\epsilon \) with \( \epsilon > 0 \) have an ESS. It follows that \( \pi_1 \) fails to be an essential ESS of \( G_0 \).

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<td>( \pi_2 )</td>
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<td>( \pi_3 )</td>
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Figure 15: A class of symmetric bimatrix games \( G_\epsilon \). The pure strategy \( \pi_1 \) is an ESS of \( G_0 \); however, this ESS is not essential since for \( \epsilon > 0 \) the game \( G_\epsilon \) does not have an ESS.
Lemma 9 cannot be applied to $\pi_1$ in $G_0$ since $\pi_1$ is not regular. According to the language use introduced in 11.4 Haigh's criterion is trivially satisfied for $\pi_1$ in $G_0$. However, in cases where a symmetric equilibrium strategy $q$ is not regular one should maybe look at the set of all pure best replies to $q$ in $G$ instead of the carrier of $q$. We call the set of all pure best replies to $q$ in $G$ the extended carrier of $q$. A matrix of the extended carrier of $q$ can be defined analogously to the carrier matrix in 11.5. In the case of $G_0$ in figure 15 the matrix $\bar{A}$ of the extended carrier does not satisfy Haigh's criterion.

One obtains:

\[(127) \quad D^TA_0D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\]

This matrix is not negative quasi-definite.

It may be true that an ESS of a symmetric bimatrix game is essential if and only if Haigh's criterion applied to the extended carrier is satisfied. This question will not be pursued here.

11.9 Lemma 10 (composition of strategy and payoff perturbation):
Let $r$ be a regular ESS of a symmetric bimatrix game $G=(\Pi, E)$. Then for every $\epsilon > 0$ a number $\delta > 0$ can be found such that every perturbed game $\hat{G}_+^+ = (\Pi, E_+^+, r_+)$ with $|\ln l| < \delta$ and $|E_+^+ - E_0| < \delta$ has an ESS $\hat{r}_+^+$ with $|\hat{r}_+^+ - r| < \epsilon$.

Proof: Let $\hat{G}_+ = (\Pi, E_+, r)$ be a perturbed game of $G_+ = (\Pi, E_+)$. For every pure strategy $\pi \in \Pi$ let $q^\pi$ be that mixed strategy for $\hat{G}_+$ which concentrates as much probability on $\pi$ as possible. We call $q^\pi$ the extreme strategy of $G$ with the intended choice $\pi$ (see 8.5). We construct a symmetric bimatrix game $G_+^+ = (\Pi, E_+^+)$ which will be called the equivalent unperturbed game of $\hat{G}_+$. The payoff function $E_+^+$ of $G_+^+$ is defined as follows:

\[(128) \quad E_+^+(\pi, \gamma) = E_+(q^\pi, q^\gamma)\]
for every $\pi \in \Pi$ and every $\gamma \in \Pi$. For every mixed stragey $s$ in $G_{++}$, we define a mixed strategy $\tilde{s}$ for $\tilde{G}_+$ which corresponds to $s$ in $\tilde{G}_+$:

$$(129) \quad \tilde{s}(\pi) = \eta_\pi + (1 - \Sigma_{\varphi \in \Pi} \eta_\varphi)s(\pi)$$

for every $\pi \in \Pi$. Suppose that a player in $G_+$ plays each of the extreme strategies $q^\pi$ with its probability $s(\pi)$. Obviously, this is the same as playing each $\pi \in \Pi$ with probability $\tilde{s}(\pi)$. Therefore, the following relationship between $E_+$ and $E_{++}$ holds, whenever $\tilde{s}$ and $\tilde{t}$ correspond to $s$ and $t$, respectively in $\tilde{G}_+$:

$$(130) \quad E_+ (\tilde{s}, \tilde{t}) = E_{++}(s,t)$$

A one-to-one mapping of the mixed strategies in $G_+$ onto the mixed strategies in $\tilde{G}$ is defined by (129). Moreover, this mapping conserves payoffs in the sense of (130). This permits the conclusion that a strategy $\tilde{r}_+$ which in $\tilde{G}_+$ corresponds to an ESS $r_{++}$ of $G_{++}$ is an ESS of $\tilde{G}_+$.

Since $r$ is a regular ESS of $G$, it follows by remark 11.7 that for every $\varepsilon > 0$ a number $\delta > 0$ can be found such that a game $G_+ = (\Pi, E_+)$ with $|E_+ - E| < \delta$ has a regular ESS $r_+$ with $|r_+ - r| < \varepsilon/3$. If $\delta$ is chosen sufficiently small then for $|n| < \delta$ the equivalent unperturbed game $G_{++} = (\Pi, E_+, n)$ of $\tilde{G}_+ = (\Pi, E_+, n)$ has an ESS $r_{++}$ with $|r_{++} - r_+| < \varepsilon/3$; this follows by lemma 9 and the regularity of $r_+$. Moreover $\delta$ can be chosen sufficiently small to secure that $|\tilde{s} - s| < \varepsilon/3$ holds if $\tilde{s}$ corresponds to $s$ in $\tilde{G}_+$. Let $\tilde{r}_+$ be the ESS of $\tilde{G}_+$ which corresponds to an ESS $r_{++}$ of $G_{++}$ in $\tilde{G}_+$ with $|r_{++} - r_+| < \varepsilon/3$. Then we have $|r - \tilde{r}_+| < \varepsilon$, if $\delta$ is chosen sufficiently small. This shows that the assertion of the lemma is true.

11.10 Theorem 12 (sufficient local conditions for a limit ESS): Let $(r,f)$ be a simultaneity game and let $b$ be a behavior strategy of player 1 for $r$. Then $b$ is a limit ESS of $(r,f)$, if the following conditions (a) and (b) are
satisfied for the b-elements of \((\mathfrak{E}, f)\).

(a) If \(\mathfrak{T}\) is an essential asymmetric b-element of \((\mathfrak{E}, f)\)
then the strategies \(\mathfrak{S}\) and \(\mathfrak{S}'\) induced by \(b\) and \(f(b)\),
respectively on \(\mathfrak{T}\) form a strong equilibrium point
\((\mathfrak{S}, \mathfrak{S}')\) of \(\mathfrak{T}\).

(b) If \((\mathfrak{T}, \mathfrak{F})\) is an essential symmetric b-element of \((\mathfrak{E}, f)\),
then the strategy \(\mathfrak{S}\) induced by \(b\) on \(\mathfrak{T}\) is a regular ESS
of the symmetric normal form of \((\mathfrak{T}, \mathfrak{F})\).

Corrolary: Under the assumptions of the theorem an \(\varepsilon > 0\)
can be found for every \(b\) with (a) and (b) such that \(b\) is
a limit equilibrium point of every test sequence \(\mathfrak{T}^1, \mathfrak{T}^2, \ldots\)
for \((\mathfrak{T}, f)\) with \(\mathfrak{T}^k = (\mathfrak{T}, f, n^k)\) which has the following pro-
erties:

(i) \(\ln^k| < \varepsilon\) for \(k = 1, 2, \ldots\)

(ii) For every \(k = 1, 2, \ldots\) the minimum probabilities \(n^k_c\)
assigned by \(n^k\) to choices \(c\) of players 1 or 2 are
always positive.

Proof: We shall prove the corrolary since the corrolary
implies the theorem. We shall use induction on the number
of essential b-elements of \((\mathfrak{T}, f)\). If the game has no es-
tential b-element, then player 1 has only one behavior
strategy and the assertion is trivially true. Assume that
the assertion holds for symmetric extensive 2-person games
with up to \(n\) essential b-elements. Let \(b\) be a behavior
strategy of player 1 which satisfies (a) and (b). Further
assume that \((\mathfrak{T}, f)\) has \(n + 1\) essential b-elements.

It is clear that the main b-truncation of \((\mathfrak{T}, f)\) is symmetric.
Let \((\mathfrak{P}, \mathfrak{F})\) be the main b-truncation of \((\mathfrak{T}, f)\). We shall first
consider the case that \((\mathfrak{P}, \mathfrak{F})\) is essential. The result will
be used in the proof of the assertion for the case that
\((\mathfrak{P}, \mathfrak{F})\) is not essential.

Assume that the main b-truncation \((\mathfrak{P}, \mathfrak{F})\) of \((\mathfrak{T}, f)\) is essential.
Obviously, in this case fewer than \(n+1\) essential b-elements
belong to a maximal subgame of \((\Gamma, f)\). The corollary can be applied to the maximal subgames of \((\Gamma, f)\).

Let \(\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots\) be a test sequence with \(\Gamma^k = (\Gamma, f, \eta^k)\) such that all minimum probabilities prescribed by the \(\eta^k\) are positive. For every maximal subgame \(\Gamma_y\) of \(\hat{\Gamma}\) let \(\hat{\eta}^k\) be the perturbation induced on \(\Gamma_y\) by \(\eta^k\) and let \(b^y\) and \(b^{y'}\) be the strategies induced by \(b^y\) and \(f(b^y)\), respectively on \(\Gamma_y\).

Consider an asymmetric subgame \(\Gamma_y\) of \((\Gamma, f)\) and the sequence \(\hat{\Gamma}_y^1, \hat{\Gamma}_y^2, \ldots\) of perturbed games of \(\Gamma_y\) with \(\hat{\Gamma}^k = (\Gamma_y, \hat{\eta}^k)\). For every \(k = 1, 2, \ldots\) let \(b^y_k\) and \(b^{y'}_k\) be those strategies of players 1 and 2 for \(\hat{\Gamma}^k_y\) which prescribe the extreme local strategies with the intended choice prescribed by \(b^y\) and \(b^{y'}\), respectively. If \(\varepsilon\) is sufficiently small and \(\ln^k | < \varepsilon\) holds for \(k = 1, 2, \ldots\) then \((b^y_k, b^{y'}_k)\) is a strong equilibrium point of \(\hat{\Gamma}^k_y\), this can be seen with the help of (a) and Lemma 6.

Now consider a symmetric subgame \((\Gamma_y, f_y)\) of \((\Gamma, f)\). The corollary can be applied to \((\Gamma_y, f_y)\). If \(\varepsilon\) is sufficiently small and \(\ln^k | < \varepsilon\) holds for \(k = 1, 2, \ldots\), then \(\hat{\Gamma}^1_y, \hat{\Gamma}^2_y, \ldots\) with \(\hat{\Gamma}^k_y = (\Gamma_y, f_y, \hat{\eta}^k_y)\) is a test sequence for \((\Gamma_y, f_y)\) which has \(b^y\) as a limit ESS. For every \(\hat{\Gamma}^k_y\) in the sequence we can find an ESS \(b^y_k\) such that \(b^y_k\) converges to \(b^y\) for \(k \to \omega\). For \(k = 1, 2, \ldots\) define \(b^{y'}_k = f_y(b^y_k)\).

For \(k = 1, 2, \ldots\) let \((\hat{\xi}^k, \hat{\xi})\) be defined as follows: with the exception of the payoffs at decomposition endpoints \(\hat{\xi}^k\) agrees with \(\hat{\xi}\); at the origin \(y\) of a maximal subgame the payoffs \(\hat{\xi}(y)\) and \(\hat{\xi}'(y)\) of players 1 and 2 are the expected payoffs \(E_y(b^y_k, b^{y'}_k)\) and \(E'_y(b^y_k, b^{y'}_k)\) of players 1 and 2, respectively in \(\Gamma_y\) for the strategy pair \((b^y_k, b^{y'}_k)\) defined above.

Let \(\hat{G} = (_, \hat{E})\) be the symmetric normal form of \((\hat{\xi}, \hat{\xi})\) and for \(k = 1, 2, \ldots\) let \(\hat{G}^k = (\hat{\xi}, \hat{E}^k)\) be the symmetric normal form of \((\hat{\xi}^k, \hat{\xi})\). In view of (b) the strategy \(B\) induced
by $b$ on $(\bar{r}, \bar{f})$ is a regular ESS of $\mathcal{G}$. Let $\bar{\eta}^k$ be the perturbation induced by $\eta^k$. It follows by lemma 10 that for sufficiently small $\varepsilon$ and for $|\eta^k| < \varepsilon$ for $k = 1, 2, \ldots$ we can find an ESS $\bar{b}^k$ for each of the perturbed games $(\bar{r}^k, \bar{r}, \bar{\eta}^k)$ such that the sequence $\bar{b}^1, \bar{b}^2, \ldots$ converges to $\bar{b}$.

Assume $|\eta^k| < \varepsilon$ for $k = 1, 2, \ldots$ with an $\varepsilon$ sufficiently small to permit the construction of all the sequences $b^1, b^2, \ldots$ for symmetric and asymmetric maximal subgames and of $\bar{b}^1, \bar{b}^2, \ldots$. Since $\Gamma$ is finite, an $\varepsilon$ of this kind can be found. For $k = 1, 2, \ldots$ let $b^k$ be that strategy of player 1 which agrees with the pertinent $b_y^k$ on every maximal subgame and with $\bar{b}^k$ on the main truncation. The construction guarantees that the conditions (i), (ii) and (iii) in definition 9.9 of an LSS are satisfied for $b^k$ with respect to $\bar{r}^k$. It follows by theorem 10 that $b^k$ is an ESS of $\bar{r}^k$. Therefore, $b$ is a limit ESS of the test sequence $\bar{r}^1, \bar{r}^2, \ldots$. This completes the induction step as far as games $(\Gamma, f)$ with an essential main b-truncation are concerned.

We continue to assume that $(\Gamma, f)$ has $n+1$ essential b-elements but we now do not exclude the case of a main b-truncation $(\bar{\mathcal{G}}, \bar{f})$ which fails to be essential. Let $m$ be the number of inessential b-elements of $(\Gamma, f)$. We shall use induction on $m$. We already know that the assertion holds for $m = 0$, since in this case $(\bar{\mathcal{G}}, \bar{f})$ is essential. Assume the assertion holds for up to $m-1$ inessential b-elements, where $m$ is positive. We have to show that the assertion holds for $m$ inessential b-elements in $(\Gamma, f)$ in the case that the main b-truncation $(\bar{\mathcal{G}}, \bar{f})$ is inessential. In this case the maximal subgames of $(\Gamma, f)$ have at most $m-1$ inessential b-elements and at most $n+1$ essential b-elements. The assertion of the corollary can be applied to the maximal subgames. A construction analogous to that used in the case of an essential b-main truncation can be employed here, too, in order to show that the assertion holds for $(\Gamma, f)$. The construction is even easier than there, since $(\bar{\mathcal{G}}, \bar{f})$ is inessential. It is not necessary to describe the construction in detail. It is now clear
that the theorem and the corrolary hold.

11.11 Comment: The corrolary shows that the sufficient conditions (a) and (b) of theorem 12 guarantee an additional robustness property. An analogous property for equilibrium points has been introduced by Okada under the name of "strict perfectness" (Okada 1981). The robustness property permits us to say that it does not matter which slight mistakes are how much more probable than others provided that all mistake probabilities are sufficiently small and positive.

It may be true that the regularity of the induced ESS is not really needed in condition (b). The theorem may still be true with "essential" instead of "regular" in condition (b). This conjecture is connected to the conjecture expressed at the end of 11.8. However, a sharper version of theorem 12 would probably require much longer proofs.
12. A many period model with ritual fights and escalated conflicts

The results obtained in sections 10 and 11 can be used as tools for the analysis of special models which take the form of a simultaneity game. Even if the game structure is quite complicated one may be able to describe the properties of a limit ESS in sufficient detail to answer theoretically important questions. This will be demonstrated for a many period model with ritual fights and escalated conflicts.

In the model two animals are in conflict over a resource like a territory, a female, etc. They may engage in ritual fights or serious fights. Ritual fight is modelled as a random mechanism which permits the conventional determination of a "winner" and a "looser", and does not directly influence payoffs.

A contestant who wants to engage in a ritual fight, but finds himself seriously attacked, may either defend himself or flee. In the latter case the attacker gets the resource. If there is a serious fight, both face a risk of receiving a serious wound like in the hawk-dove game explained in 2.10. However, the probability of being wounded may be small in one period of serious fight; after each period of serious fight, both players have to decide whether they want to continue the fight or to flee. In this way a serious fight may end without serious consequences.

It will be shown that a limit ESS must be of one of two types. In one case referred to by the name "attacker advantage", the asymmetry created by a unilateral attack leads to a subgame equilibrium which favors the attacker; he gets the resource and the other animal flees. In the second case, named "defender advantage", the attacked animal gets the resource if he defends himself and the fight ends in a draw. However, this does not necessarily mean that a unilaterally attacked animal actually will
defend himself. Therefore, two subcases must be distinguished. In the first subcase a unilaterally attacked animal flees, since the defender advantage of getting the resource after a draw is not great enough in comparison to the risk of being wounded. We may refer to this subcase as "ineffective defender advantage". In the second subcase the defender advantage is strong enough to make it worthwhile for a unilaterally attacked animal to defend himself. This subcase can be called "peaceful" since it completely excludes serious fights; unilateral attacks are effectively deterred by the willingness to defend. It is interesting that this kind of deterrence cannot work, unless the risk of being wounded within one round of serious fight is not too high.

12.1 Explanation of the model: The model takes the form of a simultaneity game. The extensive game can be thought of as composed of many copies of two building blocks \( \Lambda_1 \) and \( \Lambda_2 \) shown in figures 16 and 17. The game is played over \( T \) periods \( 1, \ldots, T \) with \( T \geq 2 \). The game structure for period 1 is that of \( \Lambda_1 \). First, both players have to decide independently whether they want to choose A (serious attack) or R (ritual fight). If both choose A, a round of serious fighting results whose outcome is decided by the random choice at \( x_4 \). With probability \( \alpha \) player 2 receives a serious wound and suffers a fitness loss of \( W \) and player 1 gets the resource whose fitness value is normed to 1. With the same probability \( \alpha \) the opposite result occurs. With probability \( 1-2\alpha \) the round of fight results in a draw and the second period begins at \( y_1 \). This vertex \( y_1 \) is a connecting point.

Connecting points are graphically indicated by little enclosing squares. The name of the building block used for continuation is shown above this little square. Pictorially speaking, the origin of a copy of this building block is glued to the connecting point. A new period begins after a connecting point, as long as period \( T \) has not yet been reached. Of course, the game is not continued
Symbols:

- A: serious attack
- R: ritual fight
- D: defend
- F: flee
- W: fitness loss of wound
- \( \alpha \): probability being wounded

Assumptions on the parameters

- \( W > 1 \)
- \( 0 < \alpha < 1 \)

Figure 16: The building block \( A_1 \)
Figure 17: The building block $A_2$. For the meaning of symbols see figure 16.

beyond period $T$. At the end of period $T$ connecting points become endpoints with zero payoffs.

If player 1 selects A and player 2 chooses R, then at $x_5$ in $A_1$ player 2 has to decide between D (defend) and F (flee). In the case of D a serious fight takes place with the same array of consequences as after a double choice of A. In the case of F player 1 receives the resource whose value is 1 and player 2 receives zero payoffs.

The same situation with reversed roles results from a choice of R by player 1 and of A by player 2.
If both players choose R, then a round of ritual fighting takes place. Ritual fighting is not modelled in detail but simply as a random mechanism whose outcome clearly distinguishes both players. This is expressed by the distinguishing random choise at x7. From there two connecting points y4 and y5 are reached with equal probabilities. Who is regarded as the "winner" or "looser" of a round of ritual fight is not determined by the rules of the game but by the conventions embodied in an ESS.

At y4 and y5 the game continues with copies of A1. We do not exclude the possibility of several rounds of ritual fight. However, once serious fighting has begun, it cannot stop unless one of the animals decides to flee or receives a serious wound; in the latter case the wounded animal has no choice and must flee. These assumptions underly the structure of building block A2 where the players have to decide between A (serious attack) or F (flee). If both choose A a new round of serious fight takes place with the same array of consequences as in A1. If one player attacks and the other flees, the attacker receives the resource. We do not exclude the possibility that both of them flee and none of them receives the resource.

It is now clear how in principle a graphical representation of the full extensive game can be constructed. One begins with a copy of A1 for period 1. One continues with copies of A1 at y4 and y5 and with copies of A2 at y1, y2 and y3; thereby one builds up period 2. As long as the end of period T has not yet been reached, new copies of building blocks are glued to the connecting points at the end of period t in order to build up period t+1. Finally, at the end of period T the connecting points become endpoints with zero payoffs for both players.

The natural symmetry of the game is based on the convention that choices are mapped to choices with the same name. The symmetric images of information sets are indicated by double arrows in figure 16 and 17. However, these arrows describe
the natural symmetry only for those copies of building blocks which are not preceded by an asymmetric pair of choices like \((A,R)\) or \((R,A)\) in a copy of \(A_1\) or by a distinguishing random choice at a copy of \(x_7\). Such copies are called symmetric; other copies are asymmetric.

The copy of \(A_1\) for the first period is the only symmetric copy of \(A_1\). The only symmetric copies of \(A_2\) are those which follow an unbroken sequence of pairs \((A,A)\) of choices in previous periods.

The naming convention for choices together with the double arrows in figures 16 and 17, understood in the way described above, completely determine the natural symmetry of the game. It can be seen immediately that the origin of a symmetric building block copy is the origin of a symmetric subgame and that the origin of an asymmetric building block copy is the origin of an asymmetric subgame.

A parameter triple \((\alpha, W, T)\) will be called admissible if \(T\) is an integer with \(T \geq 2\) and if in addition to this

\[(131) \quad 0 < \alpha < 1/2\]

and

\[(132) \quad W > 1\]

hold for \(\alpha\) and \(W\). For every admissible parameter triple \((\alpha, W, T)\) the model generates a simultaneity game \((\Gamma,f)\). The structure of this game is sufficiently clear from the explanations given above. It seems to be unnecessary to add a description by a more precise set theoretical formalism. It can be seen easily without such a formalism that the game \((\Gamma,f)\) generated by the model for an admissible parameter combination \((\alpha, W, T)\) is a simultaneity game in the sense of definition 11.1.
12.2 Preview of the analysis: We shall first rely on necessary conditions which must be satisfied by a limit ESS according to theorem 10. It will be shown that every limit ESS must be of one of two types named "attacker advantage" and "defender advantage" which have been mentioned already in the introduction to this section. There, it has been explained already that the defender advantage may be ineffective in the sense that it does not provide sufficient incentive to choose D in period 1. Whether this is the case or not depends on the following constant g, called defense gain:

\[(133) \quad g = 1 - \alpha - \alpha W\]

In a defender advantage limit ESS a player who selects D in period 1 can expect to win the resource with probability \(\alpha\) by inflicting a serious wound and with probability \(1 - 2\alpha\) after a draw, with probability \(\alpha\) he himself will be wounded. The defense gain g is nothing else than the local payoff for D in period 1 under a defender advantage limit ESS.

For \(g > 0\) a defender advantage limit ESS is peaceful in the sense explained in the introduction of this section. It will turn out that for \(g < 0\) a defender advantage ESS differs from an attacker advantage ESS only in unreached parts of the game.

Some questions concerning the borderline case \(g = 0\) will be left unanswered.

In the course of the analysis the necessary conditions of theorem 10 will be examined for various subgames and truncations. A \(\Lambda_1\)-subgame is a subgame which starts with a copy of \(\Lambda_1\) and a \(\Lambda_2\)-subgame is a subgame which starts with a copy of \(\Lambda_2\). First we shall look at the last period. The next step will be the analysis of the symmetric \(\Lambda_2\)-subgames. It will be shown that payoffs of these subgames are zero, if a limit ESS is played. Then we shall prove
that the necessary conditions for a limit ESS exclude serious fighting in asymmetric $A_2$-subgames. Already at the beginning of such subgames one player must attack and the other must flee.

The analysis of the asymmetric $A_1$-subgames will reveal that in these subgames, too, the necessary conditions of theorem 10 exclude serious fighting; moreover, one of both players must receive the resource in the end.

On the basis of the results on $A_1$-subgames and $A_2$-subgames, it will be possible to investigate the first period. It will be shown that the necessary conditions for a limit ESS permit only two patterns of first period behavior. One of these patterns is produced by a defender advantage limit ESS with $g > 0$ and the other one is characteristic for all other cases with $g \neq 0$.

After the exploitation of the necessary conditions of theorem 10, the existence of an attacker advantage limit ESS and a defender advantage limit ESS will be shown for every admissible parameter triple with $g > 0$ by the construction of specific strategies of both types. It will not be difficult to check the sufficient conditions of theorem 12.

In sections 12.3 and 12.8 we shall always assume that $b$ is a limit ESS of the game $(I,f)$ generated by the model for an admissible parameter triple $(\alpha, W, T)$.

12.3 The last period: A subgame which starts with the beginning of the last period will be called a *last period subgame*. Obviously, a last period subgame is either a copy of $A_1$ or a copy of $A_2$ with zero payoffs for both players at the connecting points.

It is convenient to introduce the following constant $a$:

(134) \[ a = \alpha W - \alpha \]

In a last period subgame the expected payoff of each of both players at vertices corresponding to $x_4$, $x_7$, $x_8$ and $x_9$ is $-a$. The constant $a$ can be interpreted as the expected loss connected to a fight with zero payoffs after a draw. In view of $W > 1$ the constant $a$ is always positive.

Consider a last period subgame which is a $A_1$-subgame. In view of $a > 0$ it is clear that $F$ is the only best reply.
Figure 18: The common structure of all main b-truncations of last period subgames.

Figure 19: Normal form of the game of figure 18.
at the vertices corresponding to $x_5$ and $x_6$. It follows by (a) in theorem 10 that $b$ prescribes $F$ at these vertices. It follows that the main $b$-truncation of a last period subgame of this kind looks like the game of figure 18.

It can be seen immediately that the main $b$-truncation of a last period subgame derived from a copy of $A_2$ also looks like the game of figure 18.

With the help of figure 19 it can be seen easily that the following is true: If a last period subgame is symmetric, then its main $b$-truncation has exactly one symmetric equilibrium strategy which assigns the following probability $q_A$ to choice $A$:

\[
q_A = \frac{1}{1+a}
\]  

(135)

In view of (d) in theorem 10 the local strategies prescribed by $b$ in a symmetric last period subgame select $A$ with probability $q_A$ at information sets corresponding to $u_1$ and $u_3$ in figures 16 and 17. If in figure 19 both players play the symmetric equilibrium strategy, then both of them receive zero payoffs.

In view of (a) in theorem 10 a pure strategy equilibrium point is induced by $(b,f(b))$ on an asymmetric last period subgame. One of the players must choose $A$ and the other has to choose $R$ in the case of a $A_2$-subgame. This results in payoffs of 1 for the player who chooses $A$ and in zero payoffs for the other player.

The results obtained for the last period are summarized by the table in figure 20.

12.4 Symmetric $A_2$-subgames: Consider a symmetric $A_2$-subgame beginning in period $T-n$. We shall show by induction on $n$ that the main $b$-truncation of a subgame of this kind has the structure of the game of figure 18. As we have seen
<table>
<thead>
<tr>
<th>Type of last period subgame</th>
<th>main decision 1)</th>
<th>second decision in $A_1$ 2)</th>
<th>subgame payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>symmetric $A_2$-subgame</td>
<td>$q_A = \frac{1}{1+a}$ 3)</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>asymmetric $A_1$-subgame</td>
<td>A or R</td>
<td>F</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>asymmetric $A_2$-subgame</td>
<td>A or F</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>F</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 20: Strategies induced by a limit ESS and its symmetric image on last period subgames and corresponding payoffs. (Last period subgames cannot be symmetric $A_1$-subgames.)

1) Decisions at information sets corresponding to $u_1$ and $u'_1$ in $A_1$ and to $u_3$ and $u'_3$ in $A_2$

2) Decisions at information set corresponding to $u_2$ and $u'_2$ in $A_1$

3) Probability for $A$
in 11.3 this is true for \( n = 0 \). Suppose that the assertion holds for \( n \). Then in a symmetric \( \Lambda_2 \)-subgame beginning in period \( T-n \) the expected payoffs for both players are zero if \( b \) and \( f(b) \) are played. Therefore, the expected payoffs for \((F,F)\) in the main \( b \)-truncation of a symmetric \( \Lambda_2 \)-subgame beginning in period \( T-n-1 \) are zero for both players. This shows that the assertion is true. It follows by (d) in theorem 10 that \( b \) and \( f(b) \) induce local strategies on a main \( b \)-truncation of a symmetric \( \Lambda_2 \)-subgame which select \( A \) with probability \( q_A \). The results are summarized by table 21.

<table>
<thead>
<tr>
<th>decisions</th>
<th>payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1</td>
<td>player 2</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
q_A = \frac{1}{1+a}
\]

**Figure 21:** Strategies induced on main \( b \)-truncations of symmetric \( \Lambda_2 \)-subgames by a limit ESS \( b \) and its symmetric image \( f(b) \) and corresponding payoffs.
12.5 Asymmetric $\Lambda_2$-subgames: Consider an asymmetric $\Lambda_2$-subgame $\Gamma_y$. In view of (a) a pure strategy equilibrium point $(b_y, b'_y)$ is induced on $\Gamma_y$ by $(b, f(b))$. In the last period one of the players must choose $F$ in $(b_y, b'_y)$. This follows by the table in figure 20. If in some earlier period one of the players chooses $F$ in $(b_y, b'_y)$, then the other one must choose $A$ in this period. Consider that player who is the first to choose $F$ in $(b_y, b'_y)$. If this choice of $F$ did not occur already in the first period of $\Gamma_y$, then he could improve his payoff by choosing $F$ in the first period of $\Gamma_y$. In this way, he would avoid unnecessary losses incurred by serious fighting. It follows that $(b_y, b'_y)$ must prescribe $A$ to one player and $F$ to the other player already at the beginning of $\Gamma_y$. This results in subgame payoffs of 1 for the player who chooses $A$ at the beginning of $\Gamma_y$; the other one receives zero. The results are summarized by the table in figure 22.

<table>
<thead>
<tr>
<th>decisions</th>
<th>payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1</td>
<td>player 2</td>
</tr>
<tr>
<td>A</td>
<td>F</td>
</tr>
<tr>
<td>or</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>A</td>
</tr>
</tbody>
</table>

Figure 22: Strategies induced on main $(b, f(b))$-truncations of asymmetric $\Lambda_2$-subgames by a limit ESS $b$ and its symmetric image $f(b)$ and corresponding payoffs.
12.6 Asymmetric $\Lambda_1$-subgames: Asymmetric $\Lambda_1$-subgames always start at a vertex corresponding to a connecting point $y_4$ in a copy of $\Lambda_1$. Let $\Gamma_y$ be an asymmetric $\Lambda_1$-subgame which starts in a period before the last period. We shall use the notation $\bar{x}_k$ for the vertex which corresponds to $x_k$ in the starting period of $\Gamma_y$. Similarly $\bar{z}_k, \bar{u}_k, \bar{u}_k'$ denote that endpoint or information set of $\Gamma_y$ which corresponds to $z_k, u_k, u_k'$, respectively, in the starting period of $\Gamma_y$.

In order to have a convenient way of speaking we shall introduce the "starting period truncation" of $\Gamma_y$. Let $M$ be the multisubgame of $\Gamma_y$ containing the subgames at $\bar{x}_4, \bar{x}_8, \bar{x}_9$ and $\bar{x}_7$. Moreover, let $b_y$ and $b'_y$ be the strategies induced by $b$ and $f(b)$ on $\Gamma_y$. The starting period truncation $\bar{\Gamma}_y$ of $\Gamma_y$ is the $(b_y, b'_y)$-truncation of $\Gamma_y$ with respect to $M$.

Let $\bar{\Gamma}_4, \bar{\Gamma}_8, \bar{\Gamma}_9$ and $\bar{\Gamma}_7$ be the subgames at $\bar{x}_4, \bar{x}_8, \bar{x}_9$ and $\bar{x}_7$ of $\Gamma_y$. In view of the table in figure 22 it is clear that in the asymmetric $\Lambda_2$-subgames of $\bar{\Gamma}_4, \bar{\Gamma}_8, \bar{\Gamma}_9$ and $\bar{\Gamma}_7$ one of both players receives 1 and the other one receives 0, if $b$ and $f(b)$ are played. This has the consequence that in the starting period truncation $\bar{\Gamma}_y$ at each of the endpoints $\bar{x}_4, \bar{x}_8$ and $\bar{x}_9$ one of the players receives $-a$ and the other receives $g$.

$a$ can be interpreted as the expected loss for a serious fight if 0 is the payoff after a draw and $g$ as the expected gain for a serious fight if 1 is the payoff after a draw.

Figure 23 shows all possible structures of a starting period truncation $\bar{\Gamma}_y$ of an asymmetric $\Lambda_1$-subgame $\Gamma_y$. The payoffs $\beta_1$ and $\beta_2$ remain unspecified.

It follows by (b) in theorem 10 that a pure strategy equilibrium point $(b_y, b'_y)$ is induced on $\bar{\Gamma}_y$ by $(b, f(b))$. 
The equilibrium payoffs connected to \((\bar{b}_y, \bar{b}_y')\) cannot be negative since each of both players can enforce zero by choosing \(R\) and \(F\). Therefore, the equilibrium play generated by \((\bar{b}_y, \bar{b}_y')\) cannot end in one of the endpoints \(\bar{x}_4, \bar{x}_8\) or \(\bar{x}_9\) since there, one of both players receives a payoff of \(-1\). Consequently, the play generated by \((\bar{b}_y, \bar{b}_y')\) leads to \(\bar{z}_5, \bar{z}_6\) or \(\bar{x}_7\).

We shall continue the investigation of asymmetric \(A_1\)-subgames in 12.7. The results obtained up to now are summarized by figure 24.

12.7 The payoffs \(\beta_1\) and \(\beta_2\): The payoffs \(\beta_1\) and \(\beta_2\) in figure 23 are averages of equilibrium payoffs for two asym-

![Figure 23: Possible structures of a starting period truncation of an asymmetric \(A_1\)-subgame. Both of the possible payoff vectors are shown above \(\bar{x}_4, \bar{x}_8\) and \(\bar{x}_9\).](image-url)
metric $\Lambda_1$-subgames. In these subgames each of both players can enforce zero, simply by always choosing $R$ and $F$. Therefore $\beta_1$ and $\beta_2$ are nonnegative:

\begin{equation}
\beta_1 \geq 0
\end{equation}

\begin{equation}
\beta_2 \geq 0
\end{equation}

<table>
<thead>
<tr>
<th></th>
<th>player 1</th>
<th>player 2</th>
<th>player 1</th>
<th>player 2</th>
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<td>1</td>
<td>$A$</td>
<td>$R$</td>
<td>$F$</td>
<td></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>or $R$</td>
<td>or $A$</td>
<td></td>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>or $R$</td>
<td></td>
<td></td>
<td></td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
</tr>
</tbody>
</table>

Figure 24: Possible equilibrium plays induced on starting period truncations of asymmetric $\Lambda_1$-subgames by a limit ESS and its symmetric image and corresponding payoffs.

1) decisions at information sets corresponding to $u_1^1$ and $u_1^1$
2) decisions at information sets corresponding to $u_2^2$ and $u_2^2$
We shall show that in addition to this we have:

\[(138) \quad \beta_1 + \beta_2 = 1\]

Consider the strategy pair \((b_y, b'_y)\) induced on an asymmetric \(A_1\)-subgame \(Y_y\) by \((b, f(b))\). Let \(z\) be an endpoint of \(Y_y\) whose realization probability \(\gamma_y(z, b_y, b'_y)\) in \(Y_y\) under \(b_y\) and \(b'_y\) is positive. We shall show that the payoffs at \(z\) are either \(h(z) = 1\) and \(h'(z) = 0\) or \(h(z) = 0\) and \(h'(z) = 1\). In view of figure 20 this is the case, if \(Y_y\) is a last period subgame. If \(Y_y\) starts earlier, then it follows by figure 24 that \(z\) has this property if it belongs to the starting period; otherwise it belongs to a shorter \(A_1\)-subgame. A simple induction argument shows that also in this case \(z\) has the asserted property. \(138\) is an immediate consequence of this.

### 12.8 The first period:

For the sake of simplicity the same names of vertices and information sets as in \(A_1\) are used for the copy of \(A_1\) which constitutes the first period of \((\Gamma, f)\).

Let \((\Gamma_4, f_4), \Gamma_8, \Gamma_9, \) and \((\Gamma_7, f_7)\) be the subgames of \((\Gamma, f)\) at \(x_4, x_8, x_9\) and \(x_7\), respectively. Let \(M\) be the multigame containing these four subgames. Since the two asymmetric subgames \(\Gamma_8\) and \(\Gamma_9\) in \(M\) are symmetric images of each other, \(M\) is symmetric. The first period truncation \((\tilde{\Gamma}, \tilde{f})\) is the \(b\)-truncation of \((\Gamma, f)\) with respect to \(M\).

As we shall see the first period truncation either has the form of figure 25 or that of figure 26. In view of figure 21 the payoffs for the strategies induced by \(b\) and \(f(b)\) in the symmetric \(A_2\)-subgame at the connecting point \(y_1\) immediately following \(x_4\) are zero for both players. Therefore, in figures 25 and 26 both players receive \(-a\) at \(x_4\).

In the same way as for the starting period truncation in figure 23 it can be seen that at \(x_8\) and \(x_9\) one player receives \(g\) and the other receives \(-a\); moreover, in view of
Figure 25: Structure of the first period truncation in the case of an attacker advantage limit ESS.

Figure 26: Structure of the first period truncation in the case of a defender advantage limit ESS.
\( \Gamma_9 = f(\Gamma_8) \) one player's payoff at \( x_8 \) is the other player's payoff at \( x_9 \). There are only two possibilities with respect to the payoffs at \( x_8 \) and \( x_9 \) which are shown by figure 25 and 26.

The subgame \((\Gamma_7, f_7)\) begins with a random choice leading with equal probabilities to two asymmetric \( \Lambda_1 \)-subgames \( \Gamma_4 \) and \( \Gamma_5 \) which are symmetric images of each other; this has the consequence that both players have the same payoffs at \( x_7 \). Moreover, it is clear from figure 24 and (138) that the payoffs of both players must sum up to 1. Therefore, each of both players receives 1/2 at \( x_7 \).

We call \( b \) an attacker advantage limit ESS, if the first period truncation has the form of figure 25 and a defender advantage limit ESS, if it has the form of figure 26. The names are suggested by the interpretation of the model. The difference concerns the convention in the two subgames reached by a unilateral attack in period 1 followed by a defense and a draw in the resulting fight. There, in the attacker advantage case the attacker gets the resource and the defender flees and in the defender advantage case the defender gets the resource and the attacker flees.

Statement (b) in theorem 10 can be applied to the subgames of the first period truncation at \( x_5 \) and \( x_6 \). Equilibrium points in pure strategies must be induced there by \((b, f(b))\). This has the consequence that \( b \) and \( f(b) \) must prescribe \( F \) at \( u_2 \) and \( u_2' \) in figure 25.

It is necessary to make a case distinction between \( g < 0 \) and \( g > 0 \) in the case of figure 26. We shall omit the treatment of the borderline case \( g = 0 \). For \( g < 0 \) the situation is essentially the same as in figure 25. The choice \( F \) must be prescribed by \( b \) and \( f(b) \). For \( g > 0 \) the choice \( D \) must be prescribed by \( b \) and \( f(b) \) at \( u_2 \) and \( u_2' \) in figure 26.
Figure 27: Symmetric normal form of the main b-truncation in the case of an attacker advantage and in the case of a defender advantage with $g < 0$.

Figure 28: Symmetric normal form of the main b-truncation in the case of a defender advantage with $g > 0$. 
<table>
<thead>
<tr>
<th>type of limit ESS</th>
<th>main decision 1)</th>
<th>second decision 2)</th>
<th>individual payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>attacker advantage</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q(A) = \frac{1}{1+2a}</td>
<td>F</td>
<td>\frac{1}{1+2a}</td>
<td></td>
</tr>
<tr>
<td>defender advantage for g &lt; 0</td>
<td></td>
<td>3)</td>
<td></td>
</tr>
<tr>
<td>defender advantage for g &gt; 0</td>
<td>R</td>
<td>D</td>
<td>\frac{1}{2}</td>
</tr>
</tbody>
</table>

Figure 29: Strategies induced on the first period truncation by a limit ESS and corresponding payoff.

1) decision at \( u_1 \)
2) decision at \( u_2 \)
3) probability of A

The main b-truncation of \((\Gamma, f)\) is nothing else than the main 5-truncation of the first period truncation \((\Gamma, \Gamma)\) where 5 is the strategy induced by b on \( \Gamma \). From what has been said on the decisions at \( u_2 \) and \( u_2' \) in figures 25 and 26, it can be seen immediately how the main b-truncation of \((\Gamma, f)\) looks like. The two cases which can arise with respect to the symmetric normal form of the main b-truncation are shown by figures 27 and 28.

It follows by (d) in theorem 10 that a symmetric equilibrium strategy is induced by b on the main b-truncation. The induced equilibrium strategies can be examined in the symmetric normal form of the main b-truncation.

The game of figure 27 has exactly one symmetric equilibrium strategy \( q \) which assigns the following probability \( q(A) \) to A.

\[
q(A) = \frac{1}{1 + 2a}
\]
Obviously, A and R fail to be symmetric equilibrium strategies of the game of figure 27. It can be checked easily that (139) describes the only mixed symmetric equilibrium strategy.

The game of figure 28 also has exactly one symmetric equilibrium strategy, namely R. In view of \( g > 0 \) it is clear that R is not only a strong best reply to R but to all mixed strategies.

The results obtained for the first period are summarized by figure 29.

12.9 Discussion: The application of the necessary conditions of theorem 10 to the model has covered all admissible parameter triples with the exception of the borderline case \( g = 0 \). In the following we shall always assume \( g \neq 0 \).

It is interesting to know whether serious fights occur with positive probability or not, if a limit ESS and its symmetric image are played. This is maybe the most important question to be asked about a limit ESS. The answer to this question depends on the distinction between an attacker advantage limit ESS and a defender advantage limit ESS. It has been shown that no serious fights occur for \( g > 0 \) in the case of a defender advantage limit ESS and that at least one round of serious fighting occurs with probability \( \frac{1}{(1+2a)^2} \) in all other cases.

The necessary conditions exclude serious fights after a round of ritual fight in the first period. Ritual fight always leads to a peaceful settlement where one contestant flees and the other receives the resource. However, a great variety of conventions can be used in order to determine the final owner of the resource.
The first round of ritual fighting may be decisive or many rounds may be necessary before a settlement can be reached. Obviously, there is not much interest in a complete classification of all possibilities which may arise in this respect. The probability distribution of final results does not depend on the details of the conventions surrounding settlement by ritual fight. After a ritual fight in the first period serious fights do not occur and both players have the same chance of $1/2$ to receive the resource.

One may be interested to know what sequences of interactions involving no ritual fights can occur with positive probability if a limit ESS and its symmetric image are played. On the basis of the distinction between an attacker advantage limit ESS and a defender advantage limit ESS the conclusions obtained from the necessary conditions permit a complete answer to this question.

For $g > 0$ a defender advantage limit ESS excludes a choice of A in period 1. Consider an attacker advantage limit ESS or a defender advantage limit ESS with $g < 0$. Figure 21 shows what happens if a symmetric $A_2$-subgame is reached after a choice of A by both players in the first period. After a unilateral attack in the first period, the attacked animal flees and the attacker receives the resource.

A choice of D cannot be observed if a limit ESS and its symmetric image are played. Nevertheless, it is important that in the case of a defender advantage ESS with $g > 0$ a unilaterally attacked animal would respond with D. The contingent threat of defense deters an attack.

The discussion has shown that the salient features of a limit ESS are captured by the distinction between the two cases which can arise with respect to the behavior in the first period. In order to have a convenient way
of speaking we shall refer to the case of an attacker advantage limit ESS or a defender advantage limit ESS with \( g < 0 \) as partially ritualized and to the case of a defender ESS with \( g > 0 \) as completed ritualized.

Complete ritualization requires a positive value of the defense gain \( g \). Since \( g \) decreases with \( \alpha \) and \( W \) this leads to the conclusion that complete ritualization is favored by a relatively low risk of being wounded in one round of serious fighting.

If it is easy to disengage from a serious fight, a round of serious fighting must be thought of as short. \( \alpha \) can be expected to be small, if a round of serious fighting is short. This suggests that easiness of disengagement from serious fight favors complete ritualization.

It has not yet been shown that an attacker advantage limit ESS and a defender advantage limit ESS exist for every \( g \neq 0 \). This will be done in 12.10. A limit ESS in each of both classes will be constructed for every admissible parameter triple with \( g \neq 0 \). The conventions surrounding settlements by ritual fight will be fixed in the simplest possible way which is maybe not the most plausible one. The same is true for other conventions concerning unreached parts of the game which also need to be specified by the complete description of a limit ESS.

12.10 Construction of specific examples of both types of a limit ESS: For \( i = 1, \ldots, 5 \) let \( \tilde{y}_i \) be that vertex of the copy of \( A_1 \) for the first period which corresponds to \( y_1 \) in figure 16. Let \( (r^1, r^1) \) and \( r^2, \ldots, r^5 \) be the subgames at \( \tilde{y}_1, \ldots, \tilde{y}_5 \), respectively. The conventions surrounding settlement by ritual fight and other conventions concerning unreached parts of the game will be fixed in such a way that the player who is favored by the settlement is always the same in the same asymmetric subgame \( r^i \) with \( i = 2, \ldots, 5 \). This permits a relatively simple description of the construction.
### Decisions in Copies of Λ₁

<table>
<thead>
<tr>
<th>after first period connecting point</th>
<th>main decision</th>
<th>second decision</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>attacker advantage</td>
<td>defender advantage</td>
</tr>
<tr>
<td></td>
<td>g &lt; 0</td>
<td>g &gt; 0</td>
</tr>
<tr>
<td>symmetric t = 1</td>
<td>1</td>
<td>1 + 2a</td>
</tr>
<tr>
<td>asymmetric t = 2, ..., T - 1</td>
<td>y₄</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>y₅</td>
<td>R</td>
</tr>
<tr>
<td>asymmetric t = T</td>
<td>y₄</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>y₅</td>
<td>R</td>
</tr>
</tbody>
</table>

### Decisions in Copies of Λ₂

<table>
<thead>
<tr>
<th>after first period connecting point</th>
<th>decisions in copies of Λ₂</th>
<th>1) decisions at information sets corresponding to u₁ in figure 16</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>attacker advantage</td>
<td>defender advantage</td>
</tr>
<tr>
<td></td>
<td>g &lt; 0</td>
<td>g &gt; 0</td>
</tr>
<tr>
<td>symmetric t = 2, ..., T</td>
<td>y₁</td>
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<tr>
<td>asymmetric t = 2, ..., T</td>
<td>y₂</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>y₃</td>
<td>F</td>
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<tr>
<td></td>
<td>y₄</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>y₅</td>
<td>F</td>
</tr>
</tbody>
</table>

Figure 30: Special attacker advantage limit ESS and special defender advantage limit ESS for g + 0. 
Figure 30 shows a specific attacker advantage limit ESS and a specific defender advantage ESS for every \( g \neq 0 \). The figure contains two separate tables for decisions in copies of \( A_1 \) and \( A_2 \). The first column indicates whether a symmetric or an asymmetric subgame starts with the copy under consideration in one of the periods listed there. The second column shows after which of the vertices \( \bar{y}_1, \ldots, \bar{y}_5 \) the decision has to be made.

No case distinction between \( g < 0 \) and \( g > 0 \) needs to be made with respect to the decisions in copies of \( A_2 \). However, the distinction is important for copies of \( A_1 \).

It can be seen without difficulty that the strategies shown in figure 30 satisfy the necessary conditions summarized by figures 20, 21, 22, 24 and 29. In the following we shall show that in all cases the sufficient conditions of theorem 12 are satisfied, too.

We begin with the main b-truncation of \((\Gamma, f)\). It can be verified easily that the symmetric equilibrium strategy for the game of figure 28 satisfies Haigh's criterion (see 11.5). The same is true for the symmetric equilibrium strategy of the game of figure 29. Therefore, condition (a) in theorem 12 is satisfied for the main b-truncation in both cases.

Now consider the main b-truncations of the asymmetric subgames at the vertices \( x_5 \) and \( x_6 \) of the first period copy of \( A_1 \) (see figures 25 and 26). It can be seen that a strong equilibrium point is induced there in both cases. Condition (b) of theorem 12 is satisfied for the second decisions in period 1.

The conventions surrounding settlement by ritual fight are fixed in favor of player 1 after \( \bar{y}_4 \) and in favor of player 2 after \( \bar{y}_5 \). Consider an asymmetric \( A_1 \)-subgame which starts in one of the periods \( 2, \ldots, T-1 \).
The starting period truncation looks like figure 23 with the upper payoff vectors at $\bar{x}_4$, $\bar{x}_8$ and $\bar{x}_9$ for a subgame after $\bar{y}_4$ and with the lower payoff vectors at $\bar{x}_4$, $\bar{x}_8$ and $\bar{x}_9$ for a subgame after $\bar{y}_5$. It can be seen that in all cases strong equilibrium points are induced on the $b$-elements which belong to starting period truncations of $A_1$-subgames beginning before the last period. It is also clear that strong equilibrium points are induced on $b$-elements belonging to last period asymmetric $A_1$-subgames.

Haigh's criterion is satisfied for the symmetric equilibrium strategy of the game of figure 19. Therefore, condition (a) of theorem 12 is satisfied for the main $b$-truncations of symmetric $A_2$-subgames. In view of $g < 1$ it can be seen immediately that strong equilibrium points are induced on all main $b$-truncations of asymmetric $A_2$-subgames.

It is now clear that the sufficient conditions of theorem 12 are satisfied for the strategies described by figure 30. We can conclude that these strategies have the properties of a limit ESS.

12.11 Concluding remark: The analysis of the model has shown how the necessary conditions of theorem 10 and the sufficient conditions of theorem 12 can be used in order to analyse simultaneity games of considerable complexity. The results obtained are not without interest from the substantial point of view. New light is thrown on the old problem of stable conventions involving a choice between ritual fight and serious attack. The hawk-dove-game is an ingenuously simple way of posing the problem. The model examined here is necessarily much more complex since it explicitly considers a great variety of possible interactions extended over many periods.
The model examined here partially confirms the intuitions underlying the simplifying assumptions embodied in the hawk-dove-game. In fact, the main b-truncation in figure 28 has the form of a hawk-dove-game. Figure 29, however, shows a different picture. The total absence of serious fights can also be supported by stable conventions if the risk of being wounded within one period of serious fighting is sufficiently low in the sense of a positive defense gain.
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