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A Note on Existence of Equilibria in Generalized Economies

by

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Abstract

If the budget constraints of a generalized economy are continuous and balanced, then there exists a generalized Walras equilibrium.
A Note on Existence of Equilibria in Generalized Economies

Let \( N = \{1, \ldots, n\} \) be a set of traders and let \( R^l_+ \) be the commodity space. A generalized economy is a \((2n + 1)\)-tuple \( E = < w^1, \ldots, w^n; u^1, \ldots, u^n; t > \) where \( w^i \in R^l_+ \) is the initial endowment of trader \( i \in N; u^i : R^l_+ \rightarrow R \) is the utility function of \( i \in N; \) and \( t \in R^l \) is the net trade vector of \( E \) (with the outside world). We shall assume in the sequel that

\[
\sum_{i=1}^n w^i + t \in R^l_+ (R^l_+ = \{ x \in R^l | x_j > 0 \text{ for } j = 1, \ldots, l \}) \tag{1.1}
\]

(1.2)

\( u^i \) is continuous, quasi-concave, and strictly monotonic for all \( i \in N \)

Generalized economies were introduced in Thomson (1992). Let \( E = < w^1, \ldots, w^n; u^1, \ldots, u^n; t > \) be a generalized economy.

Definition 1.1 A system of budget constraints for \( E \) is an \( n \)-tuple of function \( v^i : \Delta^l \rightarrow R_+, i \in N, \) that satisfy

\[
v^i \text{ is continuous for all } i \in N; \tag{1.3}
\]

\[
\sum_{i=1}^n v^i(p) = \sum_{i=1}^n p \cdot w^i + p \cdot t \text{ for all } p \in \Delta^l. \tag{1.4}
\]

(Here \( \Delta^l \) is the \((l-1)\)-dimensional standard simplex.)

Example 1.2 Let \( w = \sum_{i=1}^n w^i \in R^l_+ \) and define

\[
v^i(p) = p \cdot w^i + \frac{p \cdot w^i}{p \cdot w} p \cdot t \text{ for all } i \in N \text{ and } p \in \Delta^l. \tag{1.5}
\]

Thus, we obtain Thomson's proportional rule. It satisfies (1.3) and (1.4).

Definition 1.3 A competitive equilibrium of \( E \) relative to the budget constraints \( v^1, \ldots, v^n \) is an \((n+1)\)-tuple \( < x^1, \ldots, x^n; p > \) that satisfies

\[
x^i \in R^l_+ \text{ for } i \in N \text{ and } \sum_{i=1}^n x^i = w + t \text{ (here } w = \sum_{i=1}^n w^i). \tag{1.6}
\]

\[
p \in \Delta^l. \tag{1.7}
\]

\[
p \cdot x^i \leq v^i(p) \text{ for } i \in N. \tag{1.8}
\]

\[
[y \in R^l_+, i \in N, \text{ and } u^i(y) > u^i(x^i)] \Rightarrow p \cdot y > v^i(p). \tag{1.9}
\]
Theorem 1.4 Let \( E = \langle w^1, ..., w^n; u^1, ..., u^n; t \rangle \) be a generalized economy that satisfies (1.1) and (1.2). If \( v^1, ..., v^n \) is a system of budget constraints for \( E \) that satisfies (1.3) and (1.4), then there exists a competitive equilibrium of \( E \) relative to \( v^1, ..., v^n \).

Korthues [1995] proved the existence of proportional equilibrium, that is, equilibrium relative to (1.5).

Proof: Step 1: Let \( q = 2[\sum_{i=1}^n w^i + t] \) and let \( Q = \{ x \in R^{n+1}_+ \mid x \leq q \} \). We claim that if the \((n+1)\)-tuple \( < x^1, ..., x^n, p > \) satisfies (1.6), (1.7), (1.8) and

\[
[y \in Q, i \in N, \text{ and } u^i(y) > u^i(x^i)] \Rightarrow p \cdot y > v^i(p),
\]

then it also satisfies (1.9), that is, it is an equilibrium. Assume, on the contrary, that there exist \( y \in R^{n+1}_+ \) and \( i \in N \) such that \( u^i(y) > u^i(x^i) \) and \( p \cdot y \leq v^i(p) \). Clearly, \( y \neq 0 \).

Hence, there is \( z \in R^{n+1}_+ \), \( z \leq y, z \neq y \), such that \( u^i(z) > u^i(x^i) \). For every \( 0 < \lambda < 1 \), \( u^i(\lambda z + (1 - \lambda)x^i) \geq u^i(x^i) \). Thus, by strict monotonicity, \( u^i(\lambda y + (1 - \lambda)x^i) > u^i(x^i) \).

However, \( x^i \leq w + t \leq q \). Hence, for \( \lambda > 0 \) sufficiently small, \( \lambda y + (1 - \lambda)x^i \leq q \) contradicting (1.10).

Step 2: For \( i \in N \) and \( p \in \Delta^i \) we denote

\[
\hat{B}^i(p) = \{ x \in Q \mid p \cdot x \leq v^i(p) \}
\]

Then \( \hat{B}^i(\cdot) \) is upper hemicontinuous. Furthermore, if \( v^i(p) > 0 \) then \( \hat{B}^i(\cdot) \) is lower hemicontinuous at \( p \). Indeed, upper hemicontinuity of \( \hat{B}^i(\cdot) \) follows from the continuity of \( v^i \).

Assume now that \( v^i(p) > 0, x \in \hat{B}^i(p), p(k) \in \Delta^i, k = 1, 2, ..., \) and \( p(k) \to p \). Let \( y(k) \) be defined by

\[
[0, y(k)] = \hat{B}^i(p(k)) \cap [0, x]
\]

Clearly, if \( p \cdot x < v^i(p) \) then \( y(k) = x \) for \( k \) sufficiently large. Thus, assume \( p \cdot x = v^i(p) \). We claim that \( y(k) \to x \). Assume, on the contrary, that there exists a subsequence \( y(k_j) \to y \) and \( y \leq x \). Then \( p \cdot y < p \cdot x \) (because \( p \cdot x = v^i(p) > 0 \)). Hence \( p(k_j) \cdot y(k_j) \to p \cdot y < p \cdot x \).

Thus, for \( j \) sufficiently large, \( p(k_j) : y(k_j) = v^i(p(k_j)) \), and, therefore, \( v^i(p(k_j)) \to p \cdot y \) while \( p \cdot y < v^i(p) \), which is impossible because \( v^i(\cdot) \) is continuous.

Step 3: For \( i \in N \) and \( p \in \Delta^i \) define

\[
\hat{D}^i(p) = \{ x \in \hat{B}^i(p) \mid u^i(x) \geq u^i(y) \text{ for all } y \in \hat{B}^i(p) \}.
\]

\( \hat{D}^i(p) \) is \( i \)'s demand correspondence for the bounded economy. Further, let

\[
F^i(p) = \begin{cases} 
\hat{D}^i(p), & \text{if } v^i(p) > 0 \\
\hat{B}^i(p), & \text{if } v^i(p) = 0
\end{cases}
\]
Then $F^i(p)$ is upper hemicontinuous. The proof is straightforward. We also notice that if $x \in F^i(p)$ then $p \cdot x = v^i(p)$.

**STEP 4:** Let $\tilde{e}(p) = \sum_{i=1}^{n} F^i(p) - w - t$ and $y \in \tilde{e}(p)$. Then $y = \sum_{i=1}^{n} x^i - w - t$ where $x^i \in F^i(p)$, $i \in N$. Thus

$$p \cdot y = \sum_{i=1}^{n} p \cdot x^i - p \cdot w - p \cdot t = \sum_{i=1}^{n} v^i(p) - p \cdot w - p \cdot t = 0 \quad (1.14)$$

Therefore, by Lemma AIV.1 of Hildenbrand and Kirman (1988) there exists $\bar{p} \in \Delta^l$ such that $\tilde{e}(\bar{p}) \cap R^I_+ \neq \emptyset$. Thus, there exists a feasible allocation $\bar{x} = < \bar{x}^1, ..., \bar{x}^n >$ such that $\bar{x}^i \in F^i(\bar{p})$, $i \in N$.

**STEP 5:** $< \bar{x}^1, ..., \bar{x}^n, \bar{p} >$ is a competitive equilibrium of the bounded economy. By strict monotonicity of $u^1, ..., u^n$, $\bar{p}_j > 0$, $j = 1, ..., l$. Hence, by (1.13), $\bar{x}^i \in \hat{D}^i(\bar{p})$ for $i \in N$. Q.E.D.

**References**

