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Characterizations of Two Extended Walras Solutions for Open Economies

by

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Abstract

Various papers from the intersection of Game Theory and General Equilibrium Theory have shown that the property of consistency, which is quite prominent in the framework of cooperative solution concepts, can be used to characterize solution concepts for economies, as well. Papers like Dagan [1] and van den Nouweland, Peleg, Tijs [9] show that using the notion of open or generalized economies is a very fruitful approach to this idea. Here, we will give characterizations for two interesting extensions of the Walras solution to these open economies, the proportional and the equal sharing equilibrium. To do so, it is used that both concepts are minimal non-empty consistent extensions of the empty solution as proved in Korthues [5]. Moreover, Pareto optimality and variants of converse consistency are employed to get that result.

1 Introduction

It was recently shown, that the idea of consistency can be appropiatly transferred from a game theoretic setting to economies. In the case of pure exchange economies it is useful to consider so-called generalized or open economies as was done in Dagan [1] and in van den Nouweland, Peleg, Tijs [9]. Their approaches use a form of consistency without recontracting possibilities. On the other hand, Serrano and Volij [10]. developed – inspired by Dagan [3] – notions of consistency with recontracting possibilities, which are very close to game theoretic approaches to consistency. The authors also show. using the notion of production possibility sets of subgroups of agents, that production can be included. However, their approach does not lead to an axiomatic characterization of the behaviour of firms, since production is introduced without mentioning firms, shares or shareholders.

We will give characterizations of extended Walras solutions by means of consistency without recontracting possibilities. Considerations of minimal consistency, which was introduced by Thomson in a general setting [15] and for economies without private endowments [14] and later on transferred to open economies (including private endowments) by Korthues [5], play an important role. Korthues showed that two Walrasian concepts, the proportional and the equal sharing one, are even minimal non-empty consistent on reasonable classes of open (or generalized) economies. Using this fact, we will be able to give characterizations of these two concepts.

Apart from the "top-down" argument of consistency we will use a variant of converse consistency called symmetrized converse consistency. A solution concept is called conversely consistent, if it suggests an efficient allocation x for an economy E, whenever it suggests the projections x^S of x for the reduced economy $E^{S,x}$ for **all** subgroups S of agents. That is, converse consistency requires the consideration of **all** reduced economies. In some cases, especially if an economy is the "union" of a number of identical or similar disjoint economies, this treatment is much too restrictive. The notion of symmetrized converse consistency does justice to this fact.

Moreover, we will use a property of invariance with respect to rescaling in the commodity space. Proportionality (resp. equal sharing) enters the scene only via one axiom which provides a rule of division in "comparable situations". These are economies the relevant commodity vectors of which lie on the diagonal of the commodity space.

2 Generalized Economies and Solution Concepts

A generalized or open economy E is a tuple $((\omega_i)_{i\in N}, (\succeq_i)_{i\in N}, T)$. Here, $N = \{1, \ldots, n\}$ is the set of the agents of the economy, who are represented by their initial endowments $\omega_i \in \mathbf{R}^l_{++}$ and their preferences $\succeq_i \subset \mathbf{R}^l_+ \times \mathbf{R}^l_+$. In addition, $T \in \mathbf{R}^l$ with $\sum_{j=1}^n \omega_j + T \in \mathbf{R}^l_+$ represents the net trade vector of this economy. Its components can be positive (indicating imports of the commodity in question) as well as negative (indicating exports of the commodity in question). Imports can be distributed among the economy's agents; exports have to be brought up by them. In this wider context a usual economy can be seen as a generalized economy E' with net trade vector $T' = 0$. An **allocation** of a generalized economy (for short: economy) $E = ((\omega_i)_{i\in N}, (\succeq_i)_{i\in N}, T)$ is a vector $z = (z_1, \ldots, z_n) \in (\mathbf{R}^l_+)^n$ such that $\sum_{j\in N} z_j = \sum_{j=1}^n \omega_j + T$. The set of all allocations $\{\zeta \mid \sum_{j=1}^n \zeta_j = \sum_{j=1}^n \omega_j + T\}$ is denoted by $\mathcal{A}(E)$. A price system is a vector P in the $(l-1)$-dimensional unit simplex Δ^l. Often boundary prices will be excluded from consideration. Then we will use the notation $P \in \overset{\circ}{\Delta}{}^l$, where $\overset{\circ}{\Delta}{}^l$ is the interior of the price simplex Δ^l.

As is known from cooperative game theory, varying the notion of reduced games has a great impact on what solutions turn out to be consistent. The specific way of definition of reduced economies seems to be important, too. The easiest and most adhoc way of doing it is described in

Definition 2.1 *For every economy E, every subset of its agents $S \subset N$ and every allocation $x \in \mathcal{A}(E)$ the* **reduced economy** *$E^{S,x}$ is given by*

$$E^{S,x} := ((\omega_i)_{i\in S}, (\succeq_i)_{i\in S}, T^{S,x})$$

with $T^{S,x} := T + \sum_{j\in N\setminus S}(\omega_j - x_j)$.

The definition follows the idea that every agents who leaves the economy is paid in commodities according to the outcome x and leaves his initial endowments in the economy. This automatically leads to the definition of the net trade vector $T^{S,x}$ of the reduced economy. Initial endowments of the remaining agents are kept fixed to give them the same starting position for the discussion of redistribution in the reduced economy.

Throughout the paper, preferences are assumed to be reflexive, transitive, complete, continuous, monotonic and strictly convex. Sometimes consumers' tastes will be described by means of utility functions representing their preferences. Furthermore, generalized economies will be denoted by E or E_T.

Economies with same initial endowments and preferences but with net trade vector 0 will be called **corresponding usual economies** and will be denoted by E_0.

The characterizing aspects of the Walras equilibrium are the market clearing condition and the preference maximization of the agents as regards their budget constraints. The following definition is made to emphasize these aspects, based on which several generalizations of the Walras correspondence can be obtained by varying only the amount of the budget constraints.

Definition 2.2 *$(z, P) \in \mathcal{A}(E) \times \Delta^l$ is called an* **equilibrium** *of E* **relative to the budget constraints** *$v_i(P)$, if and only if*

1. $\sum_{j=1}^{n} z_j = \sum_{j=1}^{n} \omega_j + T$ *(market clearing condition)*
2. $z_i \in B_i(P) := \{x \in \mathbf{R}_+^l | \langle P, x\rangle \leq v_i(P)\} \quad \forall i$
3. $\forall x_i \in B_i(P): \quad z_i \succeq_i x_i \quad \forall i.$

That is, agent i chooses his consumption bundle z_i within his budget set $B_i(P) := \{x \in \mathbf{R}_+^l | \langle P, x\rangle \leq v_i(P)\}$ such that his preferences are maximized. Given monotonicity of preferences $\langle P, z_i\rangle = v_i(P)$ is satisfied for all $i \in N$.[1] Thus

$$\begin{aligned} \sum_{j=1}^{n} v_j(P) &= \langle P, \sum_{j=1}^{n} \omega_j + T\rangle = \langle P, \sum_{j=1}^{n} \omega_j\rangle + \langle P, T\rangle \\ &= \sum_{j=1}^{n} \langle P, \omega_j\rangle + \langle P, T\rangle = \sum_{j=1}^{n} w_j(P) + \langle P, T\rangle \qquad (1) \end{aligned}$$

where $w_j(P) := \langle P, \omega_j\rangle$ are the budget constraints of the corresponding usual economy. Since the value $\langle P, T\rangle$ of the net trade vector does not have to be zero – think for example of $T \in \mathbf{R}_{++}^l$ –, one cannot expect that v_j and w_j are always equal. To give a starting point for our discussion we state

Definition 2.3 *(z, P) is called a* **simple equilibrium**, *if it is an equilibrium relative to the budget constraints $v_i(P) := w_i(P)$.*

In the case of $T = 0$ this coincides with the original Walras equilibrium. Obviously, for almost every net trade vector there will be no simple equilibrium.

[1] Independent of monotonicity of preferences this equality has to be fulfilled in equilibrium, because if anyone does not choose z_i in the boundary of his budget set, someone else has to exceed his budget set, which is not allowed in equilibrium.

2.1 The Proportional Equilibrium

We are now looking for new concepts which generalize the concept of Walras equilibrium. As it will, in general, not be possible to choose budget constraints $v_i = w_i$ for all $i \in N$, one has to think about how to deviate from equality without causing too much damage (and without violating equation (1), of course). One way to solve the problem is to do it proportionally, i.e. to choose budget constraints v_i such that v_i/w_i is independent of i. This ensures equality $v_i(P) = w_i(P)$ in the case that $\langle P,T\rangle = 0$, which leads to the concept of simple equilibrium.

Definition 2.4 *(z,P) is called a* **proportional equilibrium**, *if it is an equilibrium relative to the budget constraints $v_i(P) := \lambda_i\langle P, \sum_{j=1}^n \omega_j + T\rangle$ with $\lambda_i := \langle P,\omega_i\rangle / \langle P, \sum_{j=1}^n \omega_j\rangle$.*

Since monotonicity of preferences is assumed, j's share of the value of total endowments is the same in $E_T := ((\omega_i)_{i\in N}, (\succeq_i)_{i\in N}, T)$ and $E_0 := ((\omega_i)_{i\in N}, (\succeq_i)_{i\in N}, 0)$, i.e.

$$\frac{v_i(P)}{\sum_{j=1}^n v_i(P)} = \frac{v_i(P)}{\langle P, \sum_{j=1}^n \omega_j + T\rangle} = \lambda_i = \frac{\langle P,\omega_i\rangle}{\langle P, \sum_{j=1}^n \omega_j\rangle} = \frac{w_i(P)}{\sum_{j=1}^n w_j(P)}.$$

The foregoing concept is the same as the one defined by the budget constraints

$$\overline{v_i}(P) := \langle P,\omega_i\rangle + \lambda_i\langle P,T\rangle \text{ with } \lambda_i := \frac{\langle P,\omega_i\rangle}{\langle P, \sum_{j=1}^n \omega_j\rangle} \quad .$$

Here, agents get their budget constraints w_i plus their shares λ_i of the value of the net trade vector, where λ_i is proportional to w_i.[2] Both ways lead to the same concept because the budget constraints are equal as can be seen from the following chain of equations.

$$\begin{aligned}
\frac{\overline{v_i}(P)}{\langle P, \sum_{j=1}^n \omega_j + T\rangle} &= \frac{\langle P,\omega_i\rangle + \lambda_i\langle P,T\rangle}{\langle P, \sum_{j=1}^n \omega_j + T\rangle} \\
&= \frac{\langle P,\omega_i\rangle}{\langle P, \sum_{j=1}^n \omega_j\rangle}\frac{\langle P, \sum_{j=1}^n \omega_j\rangle + \langle P,T\rangle}{\langle P, \sum_{j=1}^n \omega_j + T\rangle} \\
&= \lambda_i = \frac{v_i(P)}{\langle P, \sum_{j=1}^n \omega_j + T\rangle}
\end{aligned}$$

2.2 The Equal Sharing Equilibrium

Another idea of sharing $\langle P,T\rangle$ is, of course, to distribute it equally among the agents. This may sometimes lead to problems since not every agent is able to

[2]This concept is due to Thomson, see [13] and [15].

bear the nth part of the net trade vector and has to declare bankruptcy, i.e. is assigned the commodity bundle 0. But in this context, we will also speak of an equilibrium, if for one agent i the bundle $\omega_i + T/n$ is not in the strictly positive orthant but equilibrium trades lead him to a strictly positive commodity bundle. Since for agent i the necessity to declare bankruptcy heavily depends on prices, definition of equal sharing equilibrium is necessarily a bit blown up.

Definition 2.5 *(z, P) is called* **equal sharing equilibrium**, *if a permutation*

$$\Pi := \Pi_P : N \to N$$

exists with $0 \leq \langle P, \omega_{\Pi(1)} \rangle \leq \ldots \leq \langle P, \omega_{\Pi(n)} \rangle$ *and*

$$m := m(P) := \min\{\ i \ | \langle P, \omega_{\Pi(i)} \rangle \geq -\frac{1}{n-i+1} \langle P, T + \sum_{j=1}^{i-1} \omega_{\Pi(j)} \rangle \} \quad ,$$

such that (z, P) is an equilibrium relative to the budget constraints

$$v_i(P) := \begin{cases} 0 & \text{if} \quad \Pi(i) < m(P) \\ \langle P, \omega_i \rangle + \frac{1}{n-m+1} \langle P, T + \sum_{j=1}^{m-1} \omega_{\Pi(j)} \rangle \} & \text{if} \quad \Pi(i) \geq m(P) \end{cases} .$$

The easiest way to understand the definition is to assume first that the agents are without loss of generality ordered by increasing value of initial endowments evaluated at prices P. Then, the definition just means that agents beginning with the poorest declare bancruptcy until all of the remaining agents can pay their part of $|\langle P, T \rangle|$ diminished by the value of the private endowments of the bancruptcy declaring agents.

2.3 Solution Concepts

We will get more insight into the nature of the equilibrium notions we defined up to now, if we compare them in different economies. This will be done by considering solution concepts. A **solution concept** (or just **solution**) Φ on $\mathcal{F}$ is a correspondence that assigns to each economy $E \in \mathcal{F}$ a (possibly empty) set of allocations $\Phi(E) \subset \mathcal{A}(E)$. For the largest possible class of generalized economies, $\mathcal{E}$, we are now able to introduce the following solution concepts.

Definition 2.6

- **Empty solution** $\emptyset$ *with* $\emptyset(E) := \emptyset$,
- **Solution of all allocations** $\mathcal{A}$ *with $\mathcal{A}(E)$ consisting of all allocations of economy E,*

- **Pareto optimal solution** *PO with PO(E) consisting of all Pareto optimal allocations* $x \in \mathcal{A}(E)$,
- **Proportional solution** W_P *with* $W_P(E)$ *consisting of all proportional equilibrium allocations of E,*
- **Equal sharing solution** W_E *with* $W_E(E)$ *consisting of all equal sharing equilibrium allocations of E.*

We will characterize W_P and W_E and will use $\emptyset$, $\mathcal{A}$ and PO only for the sake of comparison.

3 The Axioms

For our characterizations we will be especially interested in top-down and bottom-up arguments, such as consistency or variants of converse consistency.

3.1 Consistency

Consistency, i.e. consistent treatment of economic situations by solution concepts, is, of course, a question of the considered class of economic situations, too. To start with, we shall first define some classes of generalized economies we will discuss later on.

Definition 3.1 *The class of all generalized economies is denoted by* $\mathcal{E}$. *The class of all economies with all agents having strictly convex and monotone preferences representable by n times continuously differentiable utility functions is denoted by* $\mathcal{E}^n$.

To avoid emptyness of W_E for some economies of the considered domain we introduce

Definition 3.2 *By* $\mathcal{E}_E$ *we denote the class of all generalized economies E which satisfy*

$$\forall i \in N(E): \quad \omega_i + T(E)/n \in \mathbf{R}^l_+$$

$\mathcal{E}^n_E$ *is defined analogously.*

Then, consistency can be defined as follows:

Definition 3.3 *(Consistency) A solution concept* Φ *is called* **consistent** *on* $\mathcal{F}$ *if for all* $E \in \mathcal{F}$ *and for all* $x \in \Phi(E)$ *we get*

$$\forall S \neq \emptyset, S \subset N: \quad x^S \in \Phi(E^{S,x}) \text{ whenever } E^{S,x} \in \mathcal{F} \quad .$$

If one considered $E^{S,x} \in \mathcal{F}$ as a consequence rather than as a condition, the consistency notion would be much stronger. It would then imply closedness of $\mathcal{F}$ with respect to formation of reduced economies. Since we will consider consistency only on closed classes of economies, this would not change anything. The following two propositions can be found, for example, in Korthues [5]. Proofs will therefore be omitted.

Proposition 3.4 *The solution concepts $\emptyset$, $\mathcal{A}$, PO, W_E and W_P are consistent on the class $\mathcal{E}$ of generalized economies.*

Proposition 3.5 *Intersections and unions of consistent solution concepts are consistent as well.*

The last proposition enables us to consider minimal consistent extensions (MCE) of solution concepts as proposed in a general setting by Thomson [15]. Furthermore, we can define a minimal non-empty consistent extension (MNCE) of some solution concept Φ, being a minimal extension among the non-empty and consistent extensions of Φ. A solution concept may possibly have several MNCEs, whereas it has a unique MCE

$$\overline{\Phi} := \bigcap_{\Psi \supset \Phi,\, \Psi \text{cons.}} \Psi \quad .$$

Then, we get

Theorem 3.6 (Korthues [5], Theorems 3.9 and 3.11) *W_P is a MNCE (minimal non-empty consistent extension) of $\emptyset$ on $\mathcal{E}^2$ and W_E is a MNCE of $\emptyset$ on $\mathcal{E}^2_E$.*

3.2 Converse Consistency

One can also consider consistency the other way round from the bottom to the top: Let x be an allocation of some generalized economy E, such that all reduced economies $E^{S,x}$ with $S \subset N, S \neq N$ agree upon x^S as one (but not necessarily the only) reasonable outcome. Why then not choosing x as an outcome of economy E? Formally we have

Definition 3.7 *A solution concept Φ is called* **conversely consistent** *(CO-CONS) on $\mathcal{F}$, if for every economy $E \in \mathcal{F}$ with at least 3 agents and $x \in \mathcal{A}(E)$ we get*

$$\left[\forall S \subset N, |S| \leq 2 : E^{S,x} \in \mathcal{F} \text{ and } x^S \in \Phi(E^{S,x}) \Rightarrow x \in \Phi(E)\right] .$$

The restriction to $|E| \geq 3$ is quite important. Dropping this assumption and considering all proper subsets S of the set of agents N leads to the fact that only $\mathcal{A}$ would be conversely consistent **and** everywhere non-empty.[3]

Van den Nouweland, Peleg and Tijs [9] use a weaker version of converse consistency, since their definition only requires considering allocations $x \in PO(E)$ rather than general allocations.[4] They characterize the simple Walras solution by means of converse consistency and are thus interested in a version of converse consistency which is as weak as possible. Their definition makes PO conversely consistent by definition. The following three propositions apear in Korthues [5] not will therefore not be proved.

Proposition 3.8 *The solution concepts $\emptyset$ and $\mathcal{A}$ are conversely consistent on $\mathcal{E}$. Moreover, PO, W_0, W_E and W_P are conversely consistent on the class $\mathcal{E}^1$.*

Proposition 3.9 *Intersections of conversely consistent solution concepts are conversely consistent as well.*

Proposition 3.10 *Unions of conversely consistent solution concepts are not necessarily conversely consistent.*

3.3 A Variant of Converse Consistency

Converse consistency means that starting from a special allocation in the original economy one considers all proper reduced economies. If the solution concept proposes the projection of the allocation onto the set of agents S for every proper subset of agents S, then it should also propose the original allocation as an outcome of the original economy whenever this allocation is Pareto optimal. Consider the case, where $x^S \in \Phi(E^{S,x})$ not for all proper subset of agents S, but only for those S in a proper subset $\mathcal{S}$ of the power set $\mathcal{P}(N)$? Are we still able to say something about the outcome $\Phi(E)$ of the original economy? In general, the minimal requirements we need to make statements on $\Phi(E)$ are, that

- every agent is at least in one of the considered coalitions $S \in \mathcal{S}$, i.e. $\mathcal{S}$ is a covering of N, and

[3] Let $(x_1, x_2) \in \mathcal{A}(E)$ be an allocation of a two-agent economy. If Φ is an everywhere non-empty concept we get for the one-agent reduced economies $\Phi(E^{i,x}) = \{x_i\}$. If converse consistency had been defined without the restriction, converse consistency of Φ would imply $(x_1, x_2) \in \Phi(E)$ and thus $\mathcal{A}(E) \subset \Phi(E)$ for all E.

[4] In their case dropping the restriction $|E| \geq 3$ would lead to the fact that only extensions of PO could be conversely consistent **and** everywhere non-empty.

- agents are somehow linked together by coalitions of $\mathcal{S}$, i.e. for every $i, j \in N$ there exists $S_1, \ldots, S_{m(i,j)} \in \mathcal{S}$ such that $i \in S_1$, $j \in S_{m(i,j)}$ and for all $\mu = 1, \ldots, m(i,j) - 1$ we get $S_\mu \cap S_{\mu+1} \neq \emptyset$.

However, in special symmetric cases it may be reasonable to neglect the second item. One of the symmetric cases we will consider here, is the symmetrization E_Π of some economy $E = ((\omega_i)_{i\in N}, (\succeq_i)_{i\in N}, T)$ which is defined as

$$E_\Pi := ((\omega_{\pi,i})_{\pi\in\Pi, i\in N}, (\succeq_{\pi,i})_{\pi\in\Pi, i\in N}, T_\Pi) \quad ,$$

where Π is the group of permutations (of names of commodities) on $\{1, \ldots, l\}$. For $\pi \in \Pi$ we define

- $x_\pi := (x_{\pi(\lambda)})_{\lambda=1,\ldots,l}$ for all $x \in \mathbf{R}^l_{++}$,
- $\omega_{\pi,i} := (\omega_i)_\pi$,
- $\succeq_\pi$ defined by $x_\pi \succeq_\pi y_\pi :\Leftrightarrow x \succeq y$ (or in terms of utility functions: If $\succeq$ can be represented by u then $\succeq_\pi$ can be represented by u_π given by $u_\pi(x_\pi) := u(x)$) and
- $T_\Pi := \sum_{\pi\in\Pi} T_\pi$.

The set N_Π of agents of the economy E_Π is given by the cross product $\Pi \times N$ of the set of permutations Π and the set of agents N of the original economy E. E_Π is built together by economies of the type $E_\pi := ((\omega_{\pi,i})_{i\in N}, (\succeq_{\pi,i})_{i\in N}, T_\pi)$ being almost cloned versions of E. The difference between E and E_π is that agents' endowments of and preferences on commodity λ in E are equal to agents' endowments of and preferences on commodity $\pi(\lambda)$ in E_π. If there exists an agent in economy E who likes apples and dislikes pears having a high initial endowment of bananas, there exists a permutation π such that the corresponding agent in E_π likes pears and dislikes bananas having a high endowment of apples (or likes pears and dislikes apples having high endowments of bananas, etc.).

E_Π wipes out the effect extremists can have on the economy: If there is an agent who wants to consume almost only apples, then there will also be agents in E_Π who want to consume almost only pears resp. almost only bananas. E_Π is called symmetrization of E because endowments of the agents of all groups $\Pi \times \{i\}$ (and thus also total endowments) sum up to a positive multiple of $\mathbb{I} = \frac{1}{l}(1, \ldots, 1)$. Remind that E_Π is different from $\bigcup_{\pi\in\Pi} E_\pi$ since T_Π is taken to be the sum of all T_π rather than the vector consisting of the components T_π.

Definition 3.11 *(Symmetrized Converse Consistency (SYMCONS)) A solution concept Φ on $\mathcal{F}$ exhibits* **symmetrized converse consistency**, *if for every $E \in \mathcal{F}$ with $E_\Pi \in \mathcal{F}$,*

$$x \in \Phi(E) \text{ and } x_\Pi \in PO(E_\Pi) \text{ imply } x_\Pi \in \Phi(E_\Pi) \quad .$$

That is, if Φ suggests x for an economy E and the symmetrization x_Π is Pareto optimal, then x_Π should be a reasonable outcome for the symmetrization E_Π, as well.

Remark 3.12 *Symmetrized converse consistency is really a variant of converse consistency.*

This can be seen as follows: Suppose $x_\Pi \in PO(E_\Pi)$. For $\pi \in \Pi$ and $S := S_\pi := \{\pi\} \times N \subset N_\Pi$ we get

$$\begin{aligned} (x_\Pi)^S &= x_\pi \qquad \text{and} \\ E_\Pi^{S,x_\Pi} &= E_\pi \quad . \end{aligned}$$

The identity for the reduced economies holds because

$$\begin{aligned} (T_\Pi)^{S_\pi,x_\Pi} &= T_\Pi + \sum_{\pi' \neq \pi} \sum_{j \in N} (\omega_{\pi',j} - x_{\pi',j}) \\ &= T_\Pi - \sum_{\pi' \neq \pi} T_{\pi'} \\ &= T_\pi \quad . \end{aligned}$$

Thus, we do not make assumptions on every proper subset S of the set of agents N_Π, but only on those S of the shape $S = S_\pi$ where $\pi \in \Pi$.

3.4 The Other Axioms

It does not make sense to define proportionality of budget constraints to be an axiom, since that would imply that we a priori restrict ourselves to considering solution concepts which make use of the notion of a budget set. But we can speak of proportionality if all the available data, i.e. initial endowments, net trade vector and the commodity bundles assigned to every agent by an allocation, are comparable in the sense that they all lie on the same line through the origin in $\mathbf{R}^l$. Then, every part of the data is a scalar multiple of some other part of the data and vice versa unless one of the two parts happens to be zero. Such situations could be called "comparable". However, we will only refer to comparable situations if all relevant data lies on the diagonal of the commodity space, which will make the axioms using comparable situations more general. Requiring that all data lie on the diagonal will sometimes be too much, so that we will also talk of a comparable situation if the data lies on the diagonal after an aggregation on suitable sets of agents.

Definition 3.13 *Let E be an economy, $x \in \mathcal{A}(E)$ be an allocation of E and $\tau := \{\tau_1, \tau_2\}$ be a non-trivial partition*[5] *of the set $N(E)$ of agents of the economy E. Then (E, τ, x) is called a* **comparable situation** *if*

$$\sum_{j\in\tau_\kappa} \omega_j, \sum_{j\in\tau_\kappa} x_j \in \mathbf{R}_+\mathbf{I} \qquad \text{for } \kappa = 1, 2 \quad .$$

As a consequence, the net trade vector T of an economy E belonging to a comparable situation (E, τ, x) has to lie on the diagonal, since

$$T := \sum_{j\in N} x_j - \sum_{j\in N} \omega_j = \sum_{j\in\tau_1} x_j - \sum_{j\in\tau_1} \omega_j + \sum_{j\in\tau_2} x_j - \sum_{j\in\tau_2} \omega_j \in \mathbf{R}\mathbf{I} \quad .$$

We shall say that vectors of $\mathbf{R}^l$ are comparable if they lie on the diagonal of $\mathbf{R}^l$.

Definition 3.14 *(Proportionality in comparable situations (PROCS)) A solution concept Φ on $\mathcal{F}$ exhibits* **proportionality in comparable situations**, *if for every $E \in \mathcal{F}$, for every partition $\tau := \{\tau_1, \tau_2\}$ of the set N of agents of E, and for every $x \in \Phi(E)$ such that (E, τ, x) is a comparable situation, there exists an $\alpha_E := \alpha_{E,\tau} \in \mathbf{R}_{++}$, so that*

$$\sum_{j\in\tau_\kappa} x_j = \alpha_E \sum_{j\in\tau_\kappa} \omega_j \qquad \text{for } \kappa = 1, 2 \quad .$$

The α_E given by this definition is the solution β of $\sum_{j\in N} x_j = \beta \sum_{j=1}^{n} \omega_j$ and is thus independent of the chosen partition (if there is more than one partition satisfying the conditions of the definition).

A similar axiom can be stated for the equal sharing case.

Definition 3.15 *(Equal sharing in comparable situations (EQSCS)) A solution concept Φ on $\mathcal{F}$ exhibits* **equal sharing in comparable situations**, *if for every $E \in \mathcal{F}$, for every partition $\tau := \{\tau_1, \tau_2\}$ of the set N of agents of E, and for every $x \in \Phi(E)$ such that (E, τ, x) is a comparable situation,*

$$\frac{1}{|\tau_1|} \sum_{j\in\tau_1} (x_j - \omega_j) = \frac{1}{|\tau_2|} \sum_{j\in\tau_2} (x_j - \omega_j) \quad .$$

[5] A partition $\tau := \{\tau_1, \tau_2\}$ is called non-trivial, if $\tau_\kappa \neq \emptyset$ for $\kappa = 1, 2$.

Then, the averaged excesses for τ_1 and τ_2 are equal to $\gamma l\mathbf{I} = \gamma(1,\ldots,1)$ for some real number $\gamma = \gamma_{E,\tau}$, which is independent of the partition τ since

$$|N|\gamma l\mathbf{I} = |\tau_1|\gamma l\mathbf{I} + |\tau_2|\gamma l\mathbf{I} = \sum_{j\in\tau_1}(x_j - \omega_j) + \sum_{j\in\tau_2}(x_j - \omega_j) = \sum_{j\in N}(x_j - \omega_j) = T.$$

The axiom can be described as follows: If the basic data of the economy and what the concept proposes as commodity bundles for the agents appropiatly aggregated according to the partition τ lie on the diagonal, then aggregated excesses should be proportional to the size of τ_1 resp. τ_2. Especially, if τ_1 and τ_2 both contain only one agent, the excesses $x_1 - \omega_1$ and $x_2 - \omega_2$ should be both equal to $\gamma l\mathbf{I}$.

The axioms PROCS and EQSCS are basically of the same shape. Both give reasonable splittings of excesses in comparable situations, if the initial data and the outcome proposed by the solution concept fit into some symmetrical framework. However, they only provide insight for one-dimensional problems, for which the problem of dividing excesses can be easily treated. In addition, they do not require solution concepts to propose outcomes on the diagonal if the initial data are on the diagonal. They just propose a division of excesses if the initial data as well as the outcome proposed by the solution concept lie on the diagonal. For some solution concepts PROCS and EQSCS could very well be empty assumptions, if, for example, these solution concepts never propose tuples of commodity bundles which lie on the diagonal (and not even in an aggregated form).

Solution concepts satisfying PROCS resp. EQSCS will in general propose different outcomes in comparable situations unless $T = 0$ or the size of τ_i relative to the size of N is as big as the fraction it owns of total endowments.[6]

[6]If τ_i's initial endowments sum up to $w_i\mathbf{I}$ and

$$\frac{w_i}{w_1 + w_2} = \frac{|\tau_i|}{|N|}$$

then a concept satisfying PROCS and a concept satisfying EQSCS yield the same outcome. To see this let $\xi_i^P\mathbf{I}$ resp. $\xi_i^E\mathbf{I}$ be the aggregated consumption bundle of τ_i proposed by a concept satisfying PROCS resp. by a concept satisfying EQSCS. Furthermore, let $T = t\mathbf{I}$. Then, we get for the proportional factor $\alpha = \alpha_E = \frac{w_1+w_2+t}{w_1+w_2}$ and for the equal sharing constant $\gamma = \gamma_E = \frac{t}{|N|}$. That yields

$$\begin{aligned}
\xi_i^P &= \alpha w_i = \alpha\frac{|\tau_i|}{|N|}(w_1 + w_2) \\
&= \frac{|\tau_i|}{|N|}(w_1 + w_2 + t) = \frac{|\tau_i|}{|N|}(w_1 + w_2) + |\tau_i|\gamma \\
&= w_i + |\tau_i|\gamma = \xi_i^E \quad .
\end{aligned}$$

Up to now in this section, we have restricted ourselves to comparable situations, i.e. to cases on the diagonal. Often, the data lie on a line different from the diagonal. These cases are very similar to comparable situations and they can be mapped to comparable situations by only rescaling commodities. It makes sense to assume that rescaling commodities does not change the outcome proposed by a solution concept, since the input data are basically the same, apart from the normalization of the axes in the commodity space. Redefining units of commodities should not do any harm to the solution proposed by a reasonable solution concept. Let $\alpha \in \mathbb{R}^l_{++}$. Then rescaling commodities in economy $E = ((\omega_i)_{i\in N}, (u_i)_{i\in N}, T)$ by α leeds to the new economy $\alpha * E = ((\omega_i')_{i\in N}, (\succeq_i')_{i\in N}, T')$ defined by

- $\omega_i' := \alpha * \omega_i$,
- $\succeq_i'$ defined by $x \succeq_i' y :\Leftrightarrow (\alpha^{-1} * x) \succeq_i (\alpha^{-1} * y)$
(or in terms of utility functions: If $\succeq_i$ can be represented by u_i then $\succeq_i'$ can be represented by u_i' defined via $u_i'(x) := u_i(\alpha^{-1} * x)$.) and
- $T' := \alpha * T$.

Here, $\alpha * x := (\alpha_\lambda x_\lambda)_{\lambda=1,\ldots,l}$ is defined to be the componentwise product. $\alpha^{-1} := (\alpha_\lambda^{-1})_{\lambda=1,\ldots,l}$ is defined to be the componentwise inverse of α. For $y = (y_1, \ldots, y_n) \in (\mathbb{R}^l)^n$ we will write $\alpha * y := (\alpha * y_1, \ldots, \alpha * y_n)$ using the * operator also for tuples of vectors with l components.

Definition 3.16 *(Invariance with respect to rescaling commodities (IRC)) A solution concept Φ on $\mathcal{F}$ exhibits* **invariance with respect to rescaling commodities**, *if for every $E \in \mathcal{F}$ and for every $\alpha \in \mathbb{R}^l_{++}$ such that $\alpha * E \in \mathcal{F}$, we get that*

$$\Phi(\alpha * E) = \alpha * \Phi(E) \quad .$$

4 The Characterizations

The characterization results we will provide here will essentially make use of the property of minimal non-empty consistency, which both W_P and W_E exhibit. Thus, the axioms consistency and non-emptyness will be of special interest. In addition to the axioms we introduced above we will need Pareto optimality.

Theorem 4.1 *Every solution concept Φ on $\mathcal{E}^2$ that satisfies Non-emptyness, CONS, SYMCONS, PO, PROCS and IRC is equal to W_P.*

Proof: Note that by minimal non-empty consistency of W_P all we have to show is that $\Phi \subset W_P$.

Let $E \in \mathcal{E}^2$ and $x \in \Phi(E) \subset PO(E)$. Then there exists a unique price system $P \in \Delta^l$ that supports every x_i by choice of the considered class of economies $\mathcal{E}^2$. Choose $\alpha := P$ and look at the economy $\alpha * E$. Then $\mathbb{I}$ is the unique price system that supports $\alpha * x_i$ in $\alpha * E$. Since $\mathbf{I}$ is permutation-invariant, we get that $\mathbf{I}$ is the unique price system that supports $(\alpha * x)_{\pi,i}$ in $(\alpha * E)_\pi$. Therefore, $\mathbf{I}$ is the unique price system that supports $(\alpha * x)_{\pi,i}$ for all i and π in $(\alpha * E)_\Pi$.[7] Hence, $(\alpha * x)_\Pi = ((\alpha * x)_{\pi,i})_{\pi\in\Pi, i\in N} \in PO(E_\Pi)$[8]and therefore,

$$(\alpha * x)_\Pi \in \Phi(E_\Pi)$$

by SYMCONS.

Now, define a partition $\tau^i := \{\tau_1^i, \tau_2^i\}$ for every agent $i \in N$ by $\tau_1^i := \{i\} \times \Pi$ and $\tau_2^i := N \setminus \{i\} \times \Pi$. Check that

$$\begin{aligned}
\sum_{j\in\tau_1^i} (\alpha * \omega)_j &= \sum_{\pi\in\Pi} (\alpha * \omega_i)_\pi \\
&= \sum_{\pi\in\Pi} \left((\alpha * \omega_i)_{\pi(\lambda)}\right)_{\lambda=1,\ldots,l} \\
&= \frac{1}{(l-1)!} \sum_{\lambda=1}^{l} (\alpha * \omega_i)_\lambda \, \mathbf{I}, \\
\sum_{j\in\tau_1^i} (\alpha * x)_j &= \frac{1}{(l-1)!} \sum_{\lambda=1}^{l} (\alpha * x_i)_\lambda \, \mathbb{I}
\end{aligned}$$

[7]More precisely we have

$$x_\Pi \in PO(E_\Pi) \Leftrightarrow \forall \pi \in \Pi \forall i \in N \operatorname{grad} u_{\pi,i}(x_{\pi,i}) \in \mathbb{R}_{++}\mathbf{I} \quad .$$

For the gradient of a function u the following chain of equations holds under a slight abuse of notation:

$$\begin{aligned}
\operatorname{grad} u_\pi(x) &= \operatorname{grad} u \circ \pi^{-1}(x) = \pi^{-1} \operatorname{grad} u(\pi^{-1}(x)) \\
&= (\operatorname{grad} u(x_{\pi^{-1}}))_{\pi^{-1}} \\
\Rightarrow \operatorname{grad} u_\pi(x_\pi) &= (\operatorname{grad} u(x))_{\pi^{-1}} \quad .
\end{aligned}$$

That implies, that for all π and for all i we get $\operatorname{grad} u_i(x_i) \in \mathbb{R}_{++} v_\pi$, so that v has to be permutation-invariant. Without loss of generality, we can assume $v = \mathbf{I}$.

[8]The proceeding notation is consistent with the notation chosen before:

$$\begin{aligned}
(\alpha * x)_\Pi &= ((\alpha * (x_i)_{i\in N})_\pi)_{\pi\in\Pi} = ((\alpha * x_i)_{i\in N})_{\pi\in\Pi} \\
&= ((\alpha * x_i)_\pi)_{\pi\in\Pi, i\in N} = ((\alpha * x)_{\pi,i})_{\pi\in\Pi, i\in N} \quad .
\end{aligned}$$

and analogously

$$\begin{aligned}\sum_{j\in\tau_2^i}(\alpha*\omega)_j &= \sum_{\pi\in\Pi}\sum_{k\neq i}(\alpha*\omega_k)_\pi \\ &= \frac{1}{(l-1)!}\sum_{k\neq i}\sum_{\lambda=1}^{l}(\alpha*\omega_k)_\lambda\,\mathbb{I}, \\ \sum_{j\in\tau_2^i}(\alpha*x)_j &= \frac{1}{(l-1)!}\sum_{k\neq i}\sum_{\lambda=1}^{l}(\alpha*x_i)_\lambda\,\mathbb{I}\end{aligned}$$

What we have proved up to now is that $((\alpha*E)_\Pi, \tau^i, (\alpha*x)_\Pi)$ is a comparable situation. From PROCS we then know that there exists a $\beta > 0$ such that

$$\sum_{j\in\tau_\kappa^i}(\alpha*x)_j = \beta\sum_{j\in\tau_\kappa^i}(\alpha*\omega)_j \text{ for all } \kappa = 1,2 \quad .$$

The summation of all components of a vector is invariant with respect to permutations of the components. That is why we get

$$\begin{aligned}\langle\mathbb{I}, \alpha*x_i\rangle &= \frac{1}{l!}\langle\mathbb{I}, \sum_{j\in\tau_1^i}(\alpha*x)_j\rangle \\ &= \beta\langle\mathbb{I}, \frac{1}{l!}\sum_{j\in\tau_1^i}(\alpha*\omega)_j\rangle \\ &= \beta\langle\mathbb{I}, \alpha*\omega_i\rangle \quad .\end{aligned}$$

$\Rightarrow$ $(\mathbb{I}, \alpha*x)$ is a proportional equilibrium of $\alpha*E$.
$\Rightarrow$ $\alpha*x \in W_P(\alpha*E) = \alpha*W_P(E)$ by IRC.
Finally, we get that $x \in W_P(E)$. □

Theorem 4.2 *Every solution concept Φ on $\mathcal{E}_E^2$ that satisfies Non-emptyness, CONS, SYMCONS, PO, EQSCS and IRC is equal to W_E.*

Proof: The proof is similar to the foregoing one. Define the same α and the same partitions τ^i of N_Π. Showing that $((\alpha*E)_\Pi, \tau^i, (\alpha*x)_\Pi)$ is a comparable situation is exactly the same than in the foregoing proof. From EQSCS we then know that

$$\frac{1}{|\tau_1^i|}\sum_{j\in\tau_1^i}((\alpha*x)_j-(\alpha*\omega)_j) = \frac{1}{|\tau_2^i|}\sum_{j\in\tau_2^i}((\alpha*x)_j-(\alpha*\omega)_j) = \gamma l\mathbb{I}$$

independent of i for some $\gamma \in \mathbf{R}_{++}$. The summation of all components of a vector is invariant with respect to permutations of the components. That is why we get

$$\begin{aligned}\langle \mathbf{I}, \alpha * x_i - \alpha * \omega_i \rangle &= \frac{1}{l!}\langle \mathbf{I}, \sum_{j \in \tau_1^i} (\alpha * x)_j - (\alpha * \omega)_j \rangle \\ &= \frac{\gamma |\tau_1^i|}{l!} \langle \mathbf{I}, l\mathbf{I} \rangle = \gamma \quad .\end{aligned}$$

$\Rightarrow$ $(\mathbf{I}, \alpha * x)$ is an equal sharing equilibrium of $\alpha * E$.

$\Rightarrow$ $\alpha * x \in W_E(\alpha * E) = \alpha * W_E(E)$ by IRC.

We hence get $x \in W_E(E)$. □

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