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by
Reinhard Selten

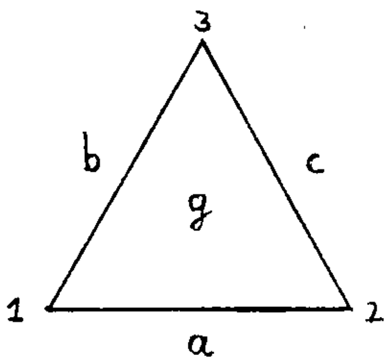
A new descriptive theory for 3-person characteristic function games will be presented in this paper. This theory of "equal division payoff bounds" is based on the idea that players form aspiration levels which are lower bounds for the payoffs they are willing to accept in a coalition of two or three players. These aspiration levels are derived from the structure of the game. They are based on equal shares of coalition values and of increments of coalition values. The theory takes the form of a hypothetical reasoning process which looks at the players in the order of their strength. Considerations based on equal shares determine lower payoff bounds for one player after the other.

Equal shares of coalition values and the order of strength are basic concepts which have been introduced in the framework of equal share analysis (Selten 1972). In this respect, the new theory is related to equal share analysis.

In the paper by Wilhelm Krischker and the author included in this volume a method has been developed which permits the comparison of different theories for characteristic function game experiments relative to a body of data. This method will be applied to a sample of 571 runs of three person games reported in the literature (Maschler 1978 , Kahan and Rapoport 1974, Rapoport and Kahan 1975). The results of the paper by Wilhelm Krischker and the author are confirmed inasmuch as the equal division core shows a better overall performance than the bargaining set, even if power transformations are taken into account. However, the comparison also reveals certain weaknesses of the equal division core which are avoided by the new theory.

1. Normed 3-person games

In a 3 person characteristic function game three players 1, 2, 3 can form one of several coalitions in order to divide a certain



$a \geq b \geq c \geq 0$
$g \geq 0$
Figure 1: Graphical representation of 3-person characteristic function games.

payoff amount which depends on the coalition. In this paper we shall only look at 3-person games where a player who does not succeed to enter the final coalition receives zero. Such games are called normed. We use the symbol ij in order to denote the two person coaltion of i and j and the symbol 123 for the grand coalition of all three players. We restrict our attention to two types of normed 3-person games: in games without the grand coalition only the two person coalitions 12, 13 and 23 are permitted; in games with the grand coalition 12, 13, 23 and 123 can be formed. As indicated in the graphical representation of figure 1, the amounts available to 12, 13, 23 and 123 are denoted by a, b, c and g, respectively. Without loss of generality we can assume $a \geq b \geq c$. If necessary, this condition can be achieved by a renumbering of the players. We assume $c \geq 0$ and $g \geq 0$.

In the following the word game will always refer to a normed 3-person characteristic function game of one of both types introduced above. A superadditive game is a game with the grand coalition which satisfies the additional condition $g \geq a$.

The games used in experiments are scaled in the sense that there is a smallest indivisible payoff unit which also serves as the unit of measurement for the payoff scale.

Sometimes it will be convenient to use the notation v(12), v(13), v(23) and v(123) for a, b, c and g, respectively.

2. Equal division payoff bounds

In an intuitively obvious sense player 1 is stronger than player 2 for $b>c$. Similarily, player 2 is stronger than player 3 for $a>b$. We use the symbols $\succ$ and $\sim$ in order to express the relationships "stronger" and "equally strong", respectively.

Our conventions of numbering the players permit the following orders of strength:

$1 \succ 2 \succ 3$	for $a > b > c$
$1 \sim 2 \succ 3$	for $a > b = c$
$1 \succ 2 \sim 3$	for $a = b > c$
$1 \sim 2 \sim 3$	for $a = b = c$

A general definition of the order of strength for arbitrary characteristic function games can be found elsewhere (Selten 1972).

The order of strength has an important role in the theory of equal division payoff bounds. The payoff bounds determined in this theory have the character of aspiration levels. In the tradition of limited rationality theory going back to H.A. Simon, aspiration levels are lower bounds on goal variables (Simon 1957, Sauermann and Selten 1962). The reasoning process postulated by the theory of equal division bounds determines such aspiration levels for one player after the other following the order of strength. The aspiration levels are called equal division payoff bounds or shortly payoff bounds.

The payoff bounds of players 1, 2 and 3 will be denoted by u_1, u_2 and u_3, respectively. We shall now introduce some auxiliary concepts and some assumptions on the u_i connected to them.

<u>Coalition shares</u>: Coalition shares are equal shares of coalition values. The <u>coalition shares</u> of 12, 13, 23 and 123 are a/2, b/2, c/2 and g/3, respectively.

<u>Assumption 1</u>: Consider a coalition C where i is one of the strongest members, i.e. C contains no member stronger than i. Then u_i is at least as high as the greatest integer which does not exceed the coalition share of C.

<u>Substitution shares</u>: Let i, j, k be the three players 1, 2, 3, not necessarily in that order. We have $v(ik) > v(jk)$ if and only if i is stronger than j. For $i \succ j$ the substitution of j by i in jk yields a positive increment $v(ik) - v(jk)$. This

increment is controlled by i and k. Therefore, we divide by 2 in order to define an equal share:

$$e_{ij} = \frac{v(ik) - v(jk)}{2} \tag{1}$$

we call e_{ij} player i's <u>substitution share</u> with respect to j.

<u>Assumption 2</u>: For $i \neq j$ player i's payoff bound u_i is at least as high as the greatest integer which does not exceed his substitution share with respect to j.

<u>Completion share</u>: As above, let i, j, k be the three players 1, 2, 3, not necessarily in this order. Assume $g > v(jk)$. If 123 is formed instead of jk the joint payoff of all players is increased by the increment $g - v(jk)$. This increment is controlled by all three players. Therefore, we divide by 3 in order to form the equal share $(g-v(jk))/3$. This number is called player i's <u>completion share</u>.

<u>Assumption 3</u>: Player i's payoff bound u_i is at least as high as the greatest integer which does not exceed his completion share.

<u>Remark</u>: Obviously, assumption 3 applies to games with the grand coalition only. If such a game is not superadditive, then $(g-v(jk))/3$ may be negative. This does not matter since we are going to assume that u_i will always be at least 1. In this respect, it is important to remember that the games considered here are scaled. 1 is the smallest feasible positive payoff.

<u>Assumption 4</u>: Player i's payoff bound u_i is at least 1.

<u>Remark</u>: Our assumptions have the character of arguments which can be put forward in order to justify lower bounds on a player's payoff. Later we shall introduce a general principle to the effect that the highest bound which can be justified by one of the arguments is player i's payoff bound u_i. The payoff bounds u_1 and u_2 of players 1 and 2 are already determined by assumptions 1 to 4 together with this general principle. The only argument which will be added to the four as-

sumptions above concerns player 3 alone.

Player 3's competitive bound: In order to motivate this concept it is useful to focus attention on the case of a game without the grand coalition which satisfies the "triangular equation":

(2) $$b + c \geq a$$

Moreover, assume that a, b, c are positive and divisible by 2. By assumption 1 we must have $u_1 \geq a/2$ and $u_2 \geq c/2$. The substitution shares $(a-c)/2$ and $(b-c)/2$ of player 1 with respect to 2 and 3 cannot be greater than $a/2$. The triangular equation (2) has the consequence that player 2's substitution share $(a-b)/2$ with respect to 3 is not greater than $c/2$. The general principle informally introduced above yields $u_1 = a/2$ and $u_2 = c/2$. If 12 is formed player 1 will receive at least u_1 and player 2 will receive at least u_2. Consequently, player 1 will receive at most $a-u_2$ and player 2 will receive at most $a-u_1 = a/2$. After these preparations we now introduce the intuitive idea which leads to the definition of player 3's competitive bound. In order to break the natural tendency of 1 and 2 to form 12 player 3 has to be prepared to offer each of them the maximum he can get in 12. In order to match player 1's maximum he has to be satisfied with $b-(a-u_2)$. In order to match player 2's maximum he has to be satisfied with $c-(a-u_1)$. The lower of both numbers is his competitive bound w:

(3) $$w = \min\,[b-a+\frac{c}{2},\ c-\frac{a}{2}]$$

Of course, w may be negative. In this case, player 3 has no chance to compete with maximum offers in 12.

The argument given above is also relevant for games with the grand coalition provided we have $u_1 = a/2$ and $u_2 = c/2$. If this is the case coalition 12 appears to be attractive since it has the highest equal share and it is not unreasonable to look at $a-u_1$ and $a-u_2$ as tentative upper bounds for reasonable payoff expectations.

It can also be seen that it is not reasonable to apply the same argument in modified form to cases with $u_1 > a/2$ or $u_2 > c/2$. As we have seen assumption 2 has no relevance for player 1. His payoff bound is determined either by $a/2$ or by $g/3$. In the latter case, coalition 12 is not very attractive and cannot serve to yield tentative upper payoff bounds. If player 2's substitution share $(a-b)/2$ is greater than $c/2$, the competitive bound w is negative anyhow, even if it is computed with $u_2 = (a-b)/2$ instead of $u_2 = c/2$. If player 2's completion share determines his payoff bound, then 12 is not attractive to him and cannot serve to yield tentative upper payoff bounds.

<u>Assumption 5</u>: If we have $u_1 = a/2$ and $u_2 = c/2$ then u_3 is at least as high as the greatest integer which does not exceed the competitive payoff bound w defined by (3).

<u>Assumption 6:</u> Player i's payoff bound is the highest lower bound determined by one of the assumptions 1 to 5.

<u>Formulas</u>: It is now possible to describe u_1, u_2 and u_3 by closed formulas. For any real number λ let int λ denote the greatest integer which does not exceed λ.

(4) $$u_1 = \text{int}\max\left[\frac{a}{2}, \frac{g}{3}, 1\right]$$

(5) $$u_2 = u_1 \quad \text{for } b = c$$

(6) $$u_2 = \text{int}\max\left[\frac{c}{2}, \frac{a-b}{2}, \frac{g-b}{3}, 1\right] \quad \text{for } b > c$$

(7) $$u_3 = u_2 \quad \text{for } a = b$$

(8) $$u_3 = \text{int}\max\left[w, \frac{g-a}{3}, 1\right]$$
$$\text{for } u_1 = \text{int}\,\frac{a}{2} \text{ and } u_2 = \text{int}\,\frac{c}{2} \text{ and } a > b$$
$$\text{with } w = \min\left[b-a+\frac{c}{2}, \; c-\frac{a}{2}\right]$$

(9) $$u_3 = \text{int}\max\left[\frac{g-a}{3}, 1\right] \quad \text{for } a > b$$
$$\text{together with } u_1 > \text{int}\,\frac{a}{2} \text{ or } u_2 > \text{int}\,\frac{c}{2}$$

The formulas also apply to games without the grand coalition, if zero is inserted for g.

It is not difficult to see that (4) to (9) follow by assumptions 1 to 6 and that the numbers u_i computed in this way satisfy assumptions 1 to 6.

Rounded payoff bounds: Subjects in experimental games must be expected to form their aspiration levels at round numbers. If for example c/2 determines player 2's payoff bound and we have c = 55, then we should not be surprised to observe that he accepts a payoff of 25 in the final coalition. For most of the experimental games in the sample considered here numbers divisible by 5 can be regarded as sufficiently round. Therefore, we define rounded payoff bounds r_i as follows: If $u_i \geq 5$ then r_i is the greatest number divisible by 5 which does not exceed u_i; if $u_i < 5$ then $r_i = 1$. This is expressed by (10):

(10) $$r_i = \max\left[1,\ 5 \text{ int } \frac{u_i}{5}\right]$$
for i = 1, 2, 3.

Prediction: The theory of rounded equal payoff bounds makes the following predictions:

(A) If there is at least one 2- or 3-person coalition C with

(11) $$\sum_{i \in C} r_i \leq v(C)$$
then a coalition of this kind will be formed.

(B) If a 2- or 3-person coalition C is formed then the final payoffs x_i will obey the following condition:

(12) $x_i \geq r_i$ for every $i \in C$

The final result of a game has the form of a configuration $(C_1,\ldots,C_m;\ x_1,\ x_2,\ x_3)$ where $C_1,\ldots,C_m$ is a partition of the player set into non-empty coalitions and x_1, x_2 and x_3 are integer payoffs with

(13) $$\sum_{i \in C_j} x_i = v(C_j) \quad \text{for } j = 1,\ldots,m$$

and

(14) $$x_i \geq 0 \quad \text{for } i = 1, 2, 3$$

In all practical cases prediction (A) excludes the coalition structure where each player forms a coalition where he is the only member. This is the formal interpretation of a result where none of the 2- or 3-person coalitions is formed. This coalition structure is called the null structure.

Since r_i always is at least 1, prediction (A) also excludes the formation of 2- or 3-person coalitions C with $v(C) = 0$.

Prediction (B) excludes coalition structures with 2- or 3-person coalitions C which do not satisfy (11).

Limited rationality aspects: The theory of equal division payoff bounds has some interesting aspects of limited rationality. First of all it portrays the players as satisficing rather than maximizing. Their behavior is guided by aspiration levels.

Second, players are not supposed to perform complicated computations. They do not have to solve any systems of simultaneous equations. They add and subtract and divide by 2 or 3.

Equations (4) to (10) may convey the impression of complexity. However, the arguments which lead to these equations are extremely simple. The apparent complexity arises from the fact that different heuristic principles of aspiration level formation are decisive in different cases. Actually, in every single case the application of the theory is very easy.

Experimental findings suggest that human decision behavior is casuistic in the sense that it is based on complicated case distinctions and simple rules for every single case (Selten 1979, Selten and Tietz 1980).

Finally, it is worth pointing out that the theory of equal division payoff bounds does not involve the usual game theoretical circularity. One payoff bound can be determined after the other following the order of strength. This kind of linearity may be a typical feature of boundedly rational reasoning processes.

3. Comparisons of predictive success

The method developed by Wilhelm Krischker and the author will be applied to several theories for characteristic function games including the theory of equal division payoff bounds.

Based on a body of data the method computes a gross rate of success, the number of correct predictions divided by the number of cases. A measure of the relative size of the predicted area is subtracted from the gross rate of success in order to obtain the net rate of success. The measure of relative size weighs coalition structures equally and weighs configurations equally within each coalition structure. The precise definitions cannot be repeated here.

We shall concentrate our attention on four theories. For the sake of shortness we shall use combinations of two capital letters as abbreviations:

BS: The bargaining set without null structure and with deviations up to 5 (described in the paper by Krischker and the author).

UB: United bargaining sets without null structure and with deviations up to 5 (described below).

EC: Equal division core (described in the paper by Krischker and the author).

EB: Rounded equal division payoff bounds.

United bargaining sets: Maschler has argued that in some cases the bargaining set should be applied to certain transformations of the original characteristic function called power functions. He considered two power functions v_1 and v_2

which in the case of games with the grand coalition can be described as follows:

(15) $v_1(i) = [g-v(jk)]/2$

(16) $v_1(jk) = v(jk) + [g-v(jk)]/2$

(17) $v_1(123) = g$

(18) $v_2(i) = [g-v(jk)]/3$

(19) $v_2(jk) = v(jk) + 2[g-v(jk)]/3$

(20) $v_2(123) = g$

where i, j, k are the player 1, 2, 3 in any order.

It is not completely clear what predictions should be associated with the bargaining sets of v_1 and v_2. It seems to be appropriate to resolve this ambiguity in the following way:

(a) such predictions are made for superadditive games only,

(b) the prediction excludes two person coalitions ij with $v(ij) < g$.

Let B be the bargaining set without null structure for the original game. Let B_1 and B_2 be the bargaining sets of v_1 and v_2 without the null structure and without the structures excluded by (b). Define

(21) $U = B \cup B_1 \cup B_2$

Theory UB predicts the set of all configurations $\alpha = (C_1,\ldots,C_m;\ x_1,\ x_2,\ x_3)$ such that a configuration $\beta = (C_1,\ldots,C_m;\ y_1,\ y_2,y_3) \in U$ can be found which satisfies $|x_i-y_i| \leq 5$ for i = 1, 2, 3.

The united bargaining set obtained in this way performs better than its individual components.

4. Results of the comparison

The results are summarized by the tables at the end of the paper. Table 1 evaluates the 27 cases of superadditive games reported by Maschler. The united bargaining sets UB have a much higher net rate of success than the ordinary bargaining set; the gross rate is much greater and the area is only slightly greater. The power bargaining sets are very small since they exclude many coalition structures and have small areas for other coalition structures.

The difference between UB and BS is much less pronounced for the experiments of Rapoport and Kahan shown in table 4.

The united bargaining sets UB perform a little better than the equal division core EC in table 1. In table 4, however, the net rate of success for UB is considerably smaller than that for EC.

The equal division core does quite well in all of the tables 1 to 4 but it is inferior to the theory of rounded equal division bounds PB. It is worth pointing out that PB has the smallest area in tables 1 and 2 whereas in tables 3 and 4 the area of PB is greater than that of the other theories.

Table 5 shows the games I to V used by Kahan and Rapoport. In games I, II and III the values of the 2-person coalitions are relatively near to each other whereas in games V and VI they are farther apart.

For $a \geq b > c$ the equal division core excludes coalition 23. In games I, II, III without the grand coalition this coalition occurs sufficiently often even if it tends to be less frequent than the other 2-person coalitions. As Kahan and Rapoport pointed out this is probably due to the relatively small differences between the values of the two-person coalitions.

Table 6 shows a poor performance of EC in the games I, II and III without the grand coalition. This is due to the exclusion of coalition 23 by EC. For V, VI without the grand coalition EC does quite well. The same is true for all five games with the grand coalition. This can be seen in table 7. If the grand coalition is available the 2-person coalitions become less importortant

The new theory PB is quite successfull everywhere in tables 6 and 7. It achieves considerably higher net rates of success than BS and UB for all five games with and without the grand coalition.

Further investigations are needed in order to confirm the theory of equal division payoff bounds for a wider range of data.

Table 1: Maschlers 27 plays of superadditive 3-person games (Maschler 1978)

	BS	UB	EC	PB
gross rate	.59	.89	.85	.89
area	.19	.20	.19	.13
net rate	.40	.69	.66	.76

Table 2: Maschlers 51 plays of non-superadditive 3-person games (Maschler 1978)

	BS	EC	PB
gross rate	.45	.78	.92
area	.32	.20	.20
success	.13	.58	.72

Table 3: Games I to V without the grand coalition. 240 plays (Kahan and Rapoport 1974)

	BS	EC	PB
gross rate	.59	.58	.95
area	.10	.07	.22
net rate	.49	.51	.73

Table 4: Games I to V with the grand coalition. 160 plays (Rapoport and Kahan 1975)

	BS	UB	EC	PB
gross rate	.51	.55	.78	.94
area	.08	.08	.11	.19
net rate	.43	.47	.67	.75

Table 5: Coalition values for the games I to V of Kahan and Rapoport

Coalition

	12	13	23	123
I	95	90	65	120
II	115	90	85	140
III	95	88	81	127
IV	106	86	66	124
V	118	84	50	121

Table 6: Net rates of success for the games I to V without the grand coalition

	BS	EC	PB
I	.57	.44	.74
II	.66	.46	.79
III	.64	.32	.77
IV	.35	.61	.68
V	.22	.72	.71

Table 7: Net rates of success for the games I to V with the grand coalition

	BS	UB	EC	PB
I	.44	.45	.59	.75
II	.52	.65	.66	.84
III	.45	.49	.63	.71
IV	.42	.51	.73	.74
V	.29	.28	.77	.74

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