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Construction of
Homogeneous Zero-Sum Games
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Introduction

This paper presents two procedures to construct all equivalence-classes of homogeneous n-person zero-sum games without dummies, in form of certain canonical representatives, recursively on the number of persons. The concept of homogeneous games was introduced by VON NEUMANN und MORGENSTERN in [7]. In both methods only the weights of these canonical representatives of the equivalence-classes are of importance.

The first method, as described in the second paragraph, produces some unwanted objects, which only can be eliminated by testing whether they are homogeneous. This method, however, shows that the number of equivalence-classes of the n-person games mentioned above, grows more slowly than $3^{n-k}$, for each natural number k. Up to now no smaller upper bound than $(n-1)!$ was known ([2]).

The second method first divides the canonical representatives with length at least three - with the length being the number of members of the lex-max coalition (see [5,6]) - into disjoint sets, namely into the set of decomposable and of reducible games. For the definition of these sets I refer to § 3. From the first method the construction of the representatives with length two is kept, being bijective. The resulting recursive construction generates all equivalence-classes of homogeneous zero-sum games without dummies in an injective way, so that no further tests are required.

In the last paragraph a set of representatives of n-person games with length two is produced, whose cardinality grows faster than each $2^{n+k}$, with k being a natural number. ISBELL ([1]) showed that $2^{n-4}$ is a lower bound, but no larger lower bound was known up to now.
§ 1  Some Notations and Definitions

A simple n-person game is a pair \((\Omega, v)\) with \(\Omega = \{1, \ldots, n\}\), which is called the set of players, and \(v : \mathcal{P}(\Omega) \to \{0,1\}\), \(v(\emptyset) = 0\). An element of \(\mathcal{P}(\Omega)\), i.e. a subset of \(\Omega\), is a coalition. A coalition \(S\) is winning if \(v(S) = 1\) and losing if \(v(S) = 0\). If each proper sub-coalition of a winning coalition is losing, this winning coalition is called minimal.

The expression "n-person" is often deleted.

A nonnegative vector \(m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n\) together with a natural number \(\lambda, 0 < \lambda < m(\Omega)\) - where \(m(S) = \sum_{i \in S} m_i\) is the weight of coalition \(S\) - defines a simple game \(v\) by

\[
v(S) = \begin{cases} 1, & \text{if } m(S) \geq \lambda \\ 0, & \text{if } m(S) < \lambda, \end{cases}
\]

or shortly by \(v = 1_{[\lambda, m(\Omega)]}^\circ m\), \(1_T\) is the indicator function of \(T\).

The pair \((m, \lambda)\) is a representation of \((\Omega, v)\).

A weighted majority game is a simple game which has a representation.

If a weighted majority game has a representation \((m, \lambda)\) such that all minimal winning coalitions have exactly weight \(\lambda\), then both the simple game and the representation are called homogeneous.

**Definition 1.1:** A vector \(m = (m_1, \ldots, m_r) \in \mathbb{N}^r\) is said to be homogeneous with respect to \(\lambda \in \mathbb{N}\) - written \(m\ hom \lambda\) - iff

1: \(1 \leq \lambda \leq \sum_{i=1}^{r} m_i\) and

2: For \(S \subseteq \{1, \ldots, r\}\), \(m(S) > \lambda\), there is \(T \subseteq S\) such that \(m(T) = \lambda\).

For this definition see also [4].
Thus a representation \((m, \lambda)\) of a simple game is homogeneous iff 
\(m\) hom \(\lambda\).
A simple game has zero-sum, iff 
\(v(S) + v(\Omega - S) = 1\) for all \(S \in \mathcal{P}(\Omega)\).
A simple game \((\Omega, v)\) is equivalent to a simple game \((\Omega, v')\), if 
\(v = v' \circ \pi\) for some permutation \(\pi\) of \(\Omega\).

The representations \((m, \lambda)\) of a weighted majority game are ordered by 
the total mass \(m(\Omega)\). The minimal elements with respect to the total 
mass are called minimal representations. A member \(i\) of \(\Omega\) is called 
dummy of \((\Omega, v)\), if for each subset \(S\) of \(\Omega\) containing \(i\) with 
\(v(S) = 1\) the equation \(v(S \setminus \{i\}) = 1\) is valid.

OSTMANN ([3]) has shown that there is a unique minimal representation 
\((m, \lambda)\) of a homogeneous zero-sum game \((\Omega, v)\) without dummies, which is 
automatically homogeneous and satisfies \(m(\Omega) = 2\lambda - 1\). The term 
"homogeneous zero-sum game without dummies" is abbreviated by 
"h. o-s. game w.d."

Thus the representation \((m, \lambda)\) can be identified with \(m\), because 
\[\lambda = \frac{m(\Omega) + 1}{2}\] In this case we write \(\lambda = \lambda(m)\) and call it the level of \(m\).
The equivalence-class - for short class - of \((\Omega, v)\) is identified with 
the ordered representative, i. e. with the representation 
\[m' = (m_{\pi(1)}, \ldots, m_{\pi(n)})\] such that
\[m_{\pi(1)} \geq m_{\pi(2)} \geq \cdots \geq m_{\pi(n)}\]

The vector \(m'\) is called ordered minimal representation of \((\Omega, v)\).
If \(m'(\{1, \ldots, k\}) = \lambda(m)\), then \(k\) is called the length of \(m\) - 
written \(l(m)\). In the sense of [5,6] the set \(\{1, \ldots, l(m)\}\) is the 
lex-max coalition of \(m'\). In the following the class of a h. o-s. 
game w.d. is identified with its ordered minimal representation.

**Definition 1.2:** A \((n\text{-person})\) court \(m'\) is a vector of \(\mathbb{N}^n\) such 
that there is an ordered minimal representation 
and a \(j, 1 \leq j \leq l(m)\), with 
\[m' = (m_j, m_1, \ldots, m_{j-1}, m_{j+1}, \ldots, m_n)\], \(m'\) is a 
court of \(m\).
Without the restriction \( j \leq 1(m) \) this definition of a court is equal to that of [2].

The ordered minimal representation \( m \) is called associated with \( m' \).
Each ordered minimal representation has at least one court, namely itself. The number of courts of \( m \) is the number of types in the lex-max coalition. Two players are of the same type, if their weights are equal ([5,6]).

Let \( S_n \) resp. \( T_n \) be the set of all classes of homogeneous \( n \)-person zero-sum games without dummies resp. \( n \)-person courts and \( S_n^k \) resp. \( T_n^k \) the subset of classes resp. courts with length \( k \). The length of an ordered minimal representation and of a court, which particularly are representations of simple games, was defined above.

The number of elements of sets \( S_n^k, T_n^k, S_n \) and \( T_n \) are defined as \( s_n^k, t_n^k, s_n \) and \( t_n \).

A member of \( S_n^k \) resp. \( S_n \) sometimes is called game in \( S_n^k \) resp. \( S_n \).

From OSTMANN ([3]) and the "Basic Lemma" of ROSENMOELLER ([4,5]) and with the help of the definition of \( S_n^k \) we know the following

**Theorem 1.3:**

\[ m \in S_n^k \iff \]

(i) \( m_1 \geq m_2 \geq \ldots \geq m_n, \ m_i \in \mathbb{N}, \)

(ii) \( m_n = 1, \)

(iii) \( (m_{k+1}, \ldots, m_n) \) hom \( m_j, \ j \leq k, \)

(iv) \( m \{(1, \ldots, k)\} = m \{(k+1, \ldots, n)\} + 1 \)

(v) If \( n > j > k \) then \( m_j = \sum_{i=j+1}^{n} m_i. \)

This important theorem will be used several times in the sequel and has the following consequences:

For \( x \in \mathbb{R} \) define \( [x] = \max \{ z \in \mathbb{Z} | z \leq x \}. \)

If \( k > [\frac{n+2}{2}] \), then \( S_n^k = \emptyset \) because of (i) and (iv).

\( S_n^{2k-1} = \{(1, \ldots, 1)\} \) because of (i), (ii), (iv).

(2k-1) times
§ 2  A Construction of the Classes of the Homogeneous n-Person Zero-Sum Games without Dummies from Certain Courts with Less Persons Providing an Upper Bound for the Number of these Classes.

In the sequel we presume $n \geq 3$. Indeed there is only one 1-person zero-sum game and none with two persons.

Lemma 2.1: Let $m$ be in $S_n^k$ and $S$ be a proper nonempty subset of $K = \{1, \ldots, k\}$. Then there is a $r$, $k < r < n$, such that

$$\sum_{i \in S} m_i = \sum_{j=k+1}^{r} m_j.$$ 

Proof: For each $i$, $n \geq i \geq k$, define

$$S_i = S \cup \{k+1, \ldots, i\}.$$ 

Thus we have the following increasing chain:

$$S_k \subset \subset S_{k+1} \subset \subset \cdots \subset \subset S_n.$$ 

$S_n$ is a winning coalition because $\sum_{j=k+1}^{n} m_j = \lambda(m)-1$ and $S$ is nonempty. $S_k$ is losing because $m(S) < m(K) = \lambda(m)$. If $S_j$ is winning but not minimal winning, then $S_{j-1}$ is winning because $m_j$ is the smallest weight in $S_j$ and the representation is homogeneous.

We conclude that there is a $j$ with $m(S_j) = \lambda(m)$.

Thus

$$\sum_{j \in S} m_j = \sum_{i=k+1}^{j} m_i.$$ 

q.e.d.

If $n \geq 4$ define for each $m \in T_n$

$$H(m) = \begin{cases} (m_2 - m_1, m_3, \ldots, m_n), & \text{if } m_1 \neq m_2, \\ (m_3, \ldots, m_n), & \text{if } m_1 = m_2. \end{cases}$$

For example $H(T_4) = \{(1,1,1)\}$ because $T_4 = \{(2,1,1,1), (1,2,1,1)\}$ (see [2]).
Theorem 2.2: Let \( m \) be in \( T_k^n \), \( n \geq 5 \). Then there is a natural
number \( r, k < r \leq n \), such that \( \min \{ m_1, m_2 \} = \sum_{j=k+1}^{r} m_j \)
and \( H(m) \) is either a court in \( T_{n-1}^{r-1} \) or a
h. o-s. game w.d. in \( S_{n-2}^{r-2} \).

Further the two mappings

\[
\begin{align*}
H & \quad \text{and} \quad H \bigg|_{S_k^n} \\
& \quad \text{and} \quad H \bigg|_{T_k^n \setminus S_k^n}
\end{align*}
\]

are injective.

Proof: Lemma 2.1 guarantees the existence of \( r \). Let \( m_1 \) be different
from \( m_2 \). (The other case can be treated analogously.)

In order to finish the first part of the proof we show that \( H(m) \) has
the properties (iii) and (iv) of Theorem 1.3 and that the inequality
\[
|m_1 - m_2| \geq m_{r+1}
\]
is valid.
The fact
\[
|m_1 - m_2| = \max \{ m_1, m_2 \} - \min \{ m_1, m_2 \}
\]
implies
\[
|m_1 - m_2| + m_3 + \ldots + m_r = \max \{ m_1, m_2 \} + \lambda(m) - m_1 - m_2
\]
\[
= \lambda(m) - \min \{ m_1, m_2 \} = m \{ (r+1, \ldots, n) \} + 1
\]
and
\[
|m_1 - m_2| \geq m_{r+1}
\]
Define \( R = \{ m_{r+1}, \ldots, m_n \} \). In this first part it remains to show
(a) \( R \) hom \( |m_1 - m_2| \) and (b) \( R \) hom \( m_j \), \( k+1 \leq j \leq r \),
because the assertion
\( R \) hom \( m_j \), \( 3 \leq j \leq k \)
is obvious.
ad (b): Let \( S \) be a subset of \( \{r+1, \ldots, n\} \) with \( m(S) > m_j \) for some \( j, \ k+1 \leq j \leq r \). Then there is a subset \( T \) of \( S \), such that \( m_j < m(T) < m_j + m_r \) because the weights \( m_t \), \( t \in S \), are at most \( m_r \).

Assume \( m(T) > m_j \), then

\[
m((\{k+1, \ldots, r\} \setminus \{j\}) \cup T) > \min \{m_1, m_2\}.
\]

By the homogeneity of \( m \) this superset of \( T \) has a subset \( S' \) with \( m(S') = \min \{m_1, m_2\} \). Because of the fact \( m(T) < m_j + m_r \), the form of \( S' \) must be

\[
S' = (\{k+1, \ldots, r\} \setminus \{j\}) \cup T'
\]

for some \( T' \subseteq S \).

This implies \( m(T') = m_j \), thus \( R \) hom \( m_j \).

The case (a) can be treated analogously.

The injectivity remains to be shown.

Let \( m \) be in \( S_n^k \), \( H(m) \) in \( T_{n-1}^j \). Define

\[
(m_1', \ldots, m_{n-1}'): = H(m).
\]

Then we conclude

\[
m = (m_1' + m_{k+1}^j + \ldots + m_j^j, m_{k+1}^j + \ldots + m_j^j, m_2, \ldots, m_{n-1}').
\]

If \( H(m) \) is in \( S_{n-2}^j \), define

\[
(m_1', \ldots, m_{n-2}^j) = H(m)
\]

and conclude

\[
m = (\Sigma m_i^j, \Sigma m_i^j, m_1^j, \ldots, m_{n-2}^j)
\]

The other function can be treated analogously, thus both mappings are injective.

q.e.d.

With the help of the function \( H \) two other mappings will be defined which allow us to construct inductively all \( S_n^k, T_n^k \) and which deliver upper bounds for the cardinalities of \( S_n^k \) and \( T_n^k \).

Formally, let \( k \geq 3, n \geq 2k \). Then define for all \( m \in S_n^k \):
\[ \sigma(m) = \begin{cases} 
H(m), & \text{if } m_{k+1} \neq m_2 \\
The ordered minimal representation & \text{associated with } H(m), \text{ if } m_{k+1} = m_2 \neq m_1 \\
\frac{[k+1]+1}{2}, & \text{if } m_1 = m_{k+1} > m_{k+1}, \\
H(m), & 1 \leq 1 \leq k-2 
\end{cases} \]

As for the expression "ordered minimal representation associated with \(H(m)\)" the passage after Definition 1.2. is referred to.

Under the prior assumptions \(k \geq 3\) and \(n \geq 2k\) define for all \(m \in T_n^k:\)

\[ \tau(m) = \begin{cases} (0, \sigma(m)), & \text{if } m \in S_n^k \\
(1, H(m)), & \text{if } m \in T_n^k \setminus S_n^k 
\end{cases} \]

First it can be noticed that both mappings are well defined because for each \(m\) with \(m_1 = m_{k+1} > m_{k+1} \) for some \(1 \leq 1 \leq k-2, \)

\[ \frac{[k+1]+1}{2} \]

the term "\(H(m)\)" is defined. In the definition of \(\sigma\) no \(m \in S_n^k\) was "forgotten" because in view of Theorem 1.3 the case \(m_1 = m_{2k-1}\) can only occur if \(m_{2k-1} = 1\) and therefore \(n = 2k-1,\)

which contradicts the assumptions.

The "artificial" definition of \(\tau\) guarantees that certain restrictions of \(\tau\) are injective, which is very important for the enumeration.

**Lemma 2.3:** Let \(n \geq 2k, \ k \geq 3.\) Then

\[ \sigma(S_n^k) \subseteq S_{n-1}^k \cup \bigcup_{j=k+1}^{\frac{n}{2}} T_{n-1}^j \cup \bigcup_{j=k}^{\frac{n-1}{2}} S_{n-2}^j \cup \bigcup_{1=1}^{k-2} S_{n-k-1}^j \]
\[ \tau(\tau^k_n) \subseteq \{\{0\} \times \sigma(S^k_n)\} \cup \{\{1\} \times \bigcup_{j=k}^{n-1} \tau^j_{n-1}\} \]

and the mappings

\[ \sigma, \tau \]

\[ |S^k_n| |\tau^k_n| \]

both are injective.

**Proof:** Let \( m \) be an ordered minimal representation in \( S^k_n \).

Distinguish the following four cases.

1. \( m_1 \not= m_2 = m_{k+1} \)
   
   The ordered minimal representation associated with \( H(m) \) is of the form
   
   \[ (m_1, m_2, \ldots, m_2, m_{k+2}, \ldots, m_n) \]
   
   \( (k-1) \) times
   
   or
   
   \[ (m_2, \ldots, m_2, m'_1, m_{k+2}, \ldots, m_n) \]
   
   for some weight \( m'_1 \).
   
   Notice that \( m'_1 \) equals \( |m_1 - m_2| \).
   
   As the total weight of \( H(m) \) is \( 2\lambda(m) - 1 - 2m_2 \), the length of \( H(m) \) is \( k \), thus \( S^k_{n-1} \) contains \( \sigma(m) \). The injectivity of \( \sigma \) restricted to such elements is an obvious consequence of Theorem 2.2.

2. \( m_1 = m_2 \not= m_{k+1} \)
   
   Pick \( t \in \mathbb{N} \) such that \( m_{k+1} + \ldots + m_t = m_1 \) (see Lemma 2.1).
   
   Then it follows from Theorem 2.2 that
   
   \[ \sigma(m) \in S^2_{n-2} \]
   
   Moreover it is seen that the restriction of \( \sigma \) to such minimal representations is injective.
3. \( m_{k+1} \neq m_2 \neq m_1 \).  

From Theorem 2.2 it is clear that \( \sigma(m) \in T_{n-1}^j \) for some \( j \) and the restriction of \( \sigma \) to such representation is injective. It remains to consider the case

4. \( m_i = m_{k+1} > m_{k+1+1} \) for some \( 1 \leq i \leq k-2 \).

Assume that \( k-1 \) is even (the case "\( k-1 \) is odd" can be treated analogously).

By definition of \( H \) it is clear that 
\[
H(m) = (m_1, \ldots, m_1, m_{k+1+1}, \ldots, m_n). \\
\]
\((k+1-2)\) times

Successively applying \( H \) we yield the following equation:

\[
1 + \frac{k-1}{2} \mathcal{H}(m) = (m_1, \ldots , m_n) = \sigma(m').
\]

This new representation \( m' \) equals \( \sigma(m) \) because 
\[
\left[ \frac{k+1+1}{2} \right] = 1 + \frac{k-1}{2} \] by the assumption.

The fact that \( m' \) is an element of \( S_{n-k-1} \) is obvious.

The level of \( m' \) is \( \left( \frac{k-1}{2} \right) \cdot m_1 \), thus \( \langle m' \rangle \geq k-1 \) because

\[
m_1 \geq m_{k+1} + m_{k+1+2}, \ m_1 \geq m_{k+1+2} + m_{k+1+4} \ldots
\]

The easy proof that \( \sigma \) restricted to these representations is injective is left to the reader.

As the images of \( \sigma \) restricted to these four cases are disjoint, the injectivity of \( S_n^k \) is shown.

It remains to consider the mapping \( \tau \).

If \( m \) is a court in \( T_n^k \setminus S_n^k \), then \( H(m) \) is in \( T_{n-1}^k \) because \( m_1 \) and \( m_2 \) do not coincide.

The assertions concerning \( \tau \) are now direct consequences of the definition of \( \tau \), of Theorem 2.2, and the preceding proof.

q.e.d.

We proceed by defining the "inverse" of \( \sigma \), \( k \geq 3 \), in order to construct \( S_n^k \) from \( S_n^k, T_n^k, n' < n \).
Formally define
\[
\tilde{\sigma}(m) = \begin{cases}
(m_1, m_2, \ldots, m_k, m_{k+1}, \ldots, m_{n-1}) & \text{if } m = (m_1, m_2, \ldots, m_k, m_{k+1}, \ldots, m_{n-1}) \text{ } k \text{-times}
\frac{(k-1)\text{times}}{
\text{or } m = (m_2, \ldots, m_k, m_1, m_{k+1}, \ldots, m_{n-1}) \text{ and } m \in S_{n-1}^k}
\end{cases}
\]
\[
\tilde{S}_{n-1}^k = \{ m \in S_{n-1}^k | m \text{ is of the form } (m_1, m_2, \ldots, m_k, m_{k+1}, \ldots, m_{n-1}) \\
\text{or } (m_1, \ldots, m_1, m_{k+1}, \ldots, m_{n-1}) \}
\]

Consequently the image of \( \tilde{\sigma} \) is a subset of the domain of \( \sigma \).

A simple computation shows that \( \tilde{\sigma} \) is the left-inverse of \( \sigma \), i.e.
\[
\tilde{\sigma}(\sigma(m)) = m
\]
for all \( m \in S_n^k \).

Thus \( \tilde{\sigma} \) allows to construct the elements of \( S_{n-1}^k \) from those of
\[
\tilde{S}_{n-1}^k \cup \bigcup_{j \geq k+1} T^j_{n-1} \cup \bigcup_{j > k} S_{n-2}^j \cup \bigcup_{l=1}^{k-2} S_{n-k-1}^j \cup S_{n-k-1}^j.
\]

The fact that not all elements of the image of \( \tilde{\sigma} \) are actually ordered minimal representations in \( S_n^k \) is the main disadvantage of this mapping \( \tilde{\sigma} \). In order to construct all minimal ordered representations inductively by applying \( \tilde{\sigma} \), those elements which are not representations have to be eliminated, a very difficult task.
The advantage of considering \( \sigma \) is that this mapping together with the mapping \( H \) yields an upper bound of \(|S_n^k|\). The next lemma shows that at least \( S_n^k \) can be constructed in a satisfying way.

**Lemma 2.4:** The function

\[
H: S_n^k \rightarrow \bigcup_{j \geq 2} t_{n-1}^j \cup \bigcup_{j \geq 2} s_{n-2}^j
\]

is bijective, if \( n \geq 5 \).

**Proof:** By Theorem 2.2 this function is injective.

An inverse of this function remains to be defined.

Let \((m_1, \ldots, m_{n-1}) \in t_{n-1}^j\), then \( \overline{m} = (m_1^1 + \ldots + m_j^1 + m_{n-1}^2, m_1^2 + \ldots + m_{n-1}^2) \) is an element of \( S_n \) and \( H(\overline{m}) = m \).

Analogously we have for \((m_1, \ldots, m_{n-1}) \in s_{n-2}^j\):

\[
\overline{m} = (\sum_{r=1}^j m_r, \sum_{r=1}^j m_r, m_1, \ldots, m_{n-2}) \in S_n^2 \quad \text{and} \quad H(\overline{m}) = m.
\]

q.e.d.

Up to the end of this chapter the maps \( \sigma, \tau, H \) are used to give an upper bound of \(|S_n^k|\) resp. \(|S_n^k|\).

**Corollary 2.5:**

(i) \( s_{2k-1}^k = t_{2k-1}^k = 1 \), if \( k \geq 2 \).

(ii) \( s_{n+2}^k \leq s_{n+1}^k + \sum_{j=k+2}^{n+1} t_{n+1}^j \)

\[+ \sum_{j=k+1}^{n+1} s_{n+1}^j \]

\[+ \sum_{j=1}^{k-1} \sum_{j=1}^{k-1} s_{n-k+1+1}^j \]

and

\( t_{n+2}^k \leq t_{n+1}^k + 2 \sum_{j=k+2}^{n+1} t_{n+1}^j \)

\[+ \sum_{j=k+1}^{n+1} s_{n+1}^j \]

\[+ \sum_{j=1}^{k-1} \sum_{j=1}^{k-1} s_{n-k+1+1}^j \]

if \( k \geq 2, n \geq 3 \).

(iii) \( s_{n+2}^2 = \sum_{j \geq 2} t_{n+1}^j + \sum_{j \geq 2} s_{n}^j \),

\( t_{n+2}^2 = 2 \sum_{j \geq 2} t_{n+1}^j + \sum_{j \geq 2} s_{n}^j \),

if \( n \geq 3 \).
Proof: (i) is obvious from the remark following Theorem 1.3.
(ii) is a direct consequence of Lemma 2.3.
(iii) follows from Lemma 2.4 because

\[
(H_{n+1}^{-1}(m) \text{ has exactly } \begin{cases} 
\text{two courts, if } m \in \bigcup_{j \geq 2} T_{n+1}^j \\
\text{one court, if } m \in \bigcup_{j \geq 2} S_{n+1}^j.
\end{cases}
\]

q.e.d.

In order to define an upper bound of \( S_n^k \) resp. \( T_n^k \) we need the following

Lemma 2.6: If \( n \geq 5 \) and \( n \leq 3k-3 \), then

\[
S_n^k \leq S_{n-2}^{k-1} \quad \text{and} \quad T_n^k \leq T_{n-2}^{k-1}.
\]

Later it will be shown that these inequalities in fact are equations (at the end of § 3).

Proof: Let \( m \) be an element of \( S_n^k \) and define

\[
\tilde{m} = (m_1, \ldots, m_{k-1}, m_{k+2}, \ldots, m_n).
\]

If \( m_{k-1} = m_k = m_{k+1} \), then \( \tilde{m} \) is an element of \( S_{n-2}^{k-1} \)
and the number of courts of \( \tilde{m} \) is still the number of courts of \( m \). If all elements of \( S_n^k \) have this property the proof is
finished because the map \( m \mapsto \tilde{m} \) is injective and preserves
the number of courts.

It remains to be shown that \( m_{k-1} = m_{k+1} \) for all \( m \in S_n^k \).
Assume \( m_{k-1} \neq m_{k+1} \), then we have:

\[
m_{k-1} \geq m_{k+1} + m_{k+2}, \quad m_{k-2} \geq m_{k+3} + m_{k+4}, \ldots, \quad m_1 \geq m_{k+2(k-2)} + 1 + m_{3k-2}.
\]

This is a consequence of the expression for the level of \( m \), i.e.

\[
\lambda(m) = \sum_{j=1}^k m_j = \sum_{j=k+1}^n m_j + 1.
\]

The last inequality shows that \( n \geq 3k-2 \), a contradiction.

q.e.d.

Now new sequences which majorize \( S_n^k, T_n^k \) can be defined in the
following inductive way:
\[ s_3^2 = s_4^2 = \tau_3^2 = 1, \tau_4^2 = 2. \]

\[
\hspace{1cm} \hat{s}_n^k = \hat{s}_{n-1}^k + \frac{2}{n} \sum_{j=k+1}^{n} \tau_j^{k-1} + \frac{2}{n-2} \sum_{j=n-2}^{k+1} \tau_j^2 + \frac{2}{n-1} \sum_{j=n-1}^{k+1} \tau_j^2 + \tau_j^{n-k-1} \text{ and} \]

\[
\hat{t}_n^k = \hat{t}_{n-1}^k + \frac{2}{n} \sum_{j=k}^{n} \tau_j^{k-1}, \text{ if } n \geq 3k-2, k \geq 3. \]

\[
\hat{s}_n^k = \hat{s}_{n-2}^{k-1}, \tau_{n}^k = \hat{t}_{n-2}^{k-1}, \text{ if } k \geq 3, n \leq 3k-3, n \geq 2k-1. \]

\[
\hat{s}_n^2 = \frac{2}{n} \sum_{j=2}^{n} \tau_j^{n-1} + \frac{2}{n-2} \sum_{j=2}^{n-2} \tau_j^2, \tau_n^2 = \frac{2}{n} \sum_{j=2}^{n} \tau_j^{n-1} + \frac{2}{n-2} \sum_{j=2}^{n} \tau_j^2, n \geq 5. \]

A direct consequence of Lemma 2.6 and Corollary 2.5 is

**Corollary 2.7:**

The sequence \( \hat{s}_n^k \) resp. \( \tau_n^k \) majorizes \( s_n^k \) resp. \( t_n^k \), i.e.

\[
s_n^k \leq \hat{s}_n^k, \quad t_n^k \leq \tau_n^k, \quad \text{if } n \geq 3 \text{ and } 2 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor. \]

Some of the \( \hat{s}_n^k, \tau_n^k \) are tabulated in the supplement.

**Lemma 2.8:**

\[
\hat{s}_n^3 < \hat{s}_n^4 < \ldots < \hat{s}_{n+2k-6}^k < \ldots < \hat{s}_{3n-14}^{k-n-4}, \]

\[
\tau_n^3 < \tau_n^4 < \ldots < \tau_{n+2k-6}^k < \ldots < \tau_{3n-14}^{k-n-4}, \quad \text{if } n \geq 7. \]

**Proof (per induction on n):**

If \( n = 7 \), nothing has to be shown because in this case \( 3n - 14 = n \).

Assume the assertion is valid for some \( n \geq 7 \), we are going to verify it for \( n + 1 \). Observe the following inequalities which follow from the inductive hypothesis and the definition:
\[ \tilde{s}_{n+2k-5} = \sum_{j \geq k+1} \tilde{s}_{n+2k-6} + \sum_{j \geq k} \tilde{s}_{n+2k-7} + \sum_{j \geq k-1} \tilde{s}_{n+k-1-5} \]
\[ \geq \tilde{s}_{n+2k-8} + \sum_{j \geq k} \tilde{s}_{n+2k-9} + \sum_{j \geq k-1} \tilde{s}_{n+k-1-6} + \sum_{j \geq k-1} \tilde{s}_{n+k-6} \]
\[ = \tilde{s}_{n+2k-7} + \sum_{j \geq k-1} \tilde{s}_{n+k-6} \]
\[ > \tilde{s}_{n+2k-7} \text{ for each } 4 \leq k \leq n-3. \]

The last inequality is strict because
\[ \sum_{j \geq k-1} \tilde{s}_{n+k-6} > 0. \]

The inequality \( \tilde{s}_{n+2k-5} > \tilde{s}_{n+2k-7} \) can be shown analogously.

q.e.d.

Define
\[ a_3^2 = a_4^2 = b_3^2 = 1, \quad b_4^2 = 2, \]
\[ a_n = \begin{cases} \frac{n}{2} & \text{if } n \equiv 1 \pmod{2} \\ \frac{n-1}{2} & \text{if } n \equiv 3 \pmod{2} \end{cases} \]
\[ b_n = b_{n-1}^2 + \sum_{j=2}^{n-2} a_j^2, \quad n \geq 5; \]
\[ b_n = 2a_n^2 - \sum_{j=2}^{n-2} a_j^2, \quad n > 5; \]
\[ a_n^k = a_{n-2}, \quad b_n^k = b_{n-2}^k, \text{ if } k \geq 3, n \geq 2k-1. \]

Theorem 2.9:
The sequence \( a_n^k \) resp. \( b_n^k \) majorizes \( \tilde{s}_n^k \) resp. \( \tilde{t}_n \)
- i.e. \( \tilde{s}_n^k \leq a_n^k \) resp. \( \tilde{t}_n^k \leq b_n^k \) - if \( 3 \leq n, 2 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor \).

In the sequel this theorem allows us to give simple upper bounds of \( s_n^k, t_n^k, s_n, \) and \( t_n \).

If \( \tilde{s}_{n-1}^j \leq a_{n-1}^j, \tilde{t}_{n}^j \leq b_{n}^j \) then \( \tilde{s}_{n+1}^2 \leq a_{n+1}^2, \tilde{t}_{n+1}^2 \leq b_{n+1}^2 \),
which is a consequence of the definitions of these sequences.
Lemma 2.8 suggests the following procedure: By the definitions of \( \xi_n, \eta_n, a_n^j, b_n^j \) in the case \( n > 3j-3, j \geq 3 \), it is realized that Theorem 2.9 is proved once the inequalities
\[
\xi_{3k-2}^k \leq a_{3k-2}^k \quad \text{and} \quad \eta_{3k-2}^k \leq b_{3k-2}^k
\]
are established. In view of the definitions of \( \xi_n^k, \eta_n^k \) it is sufficient to show the following

**Lemma 2.10:**

\[
a_{3k-2}^k \geq a_{3k-3}^k + \sum_{j \geq k+1} b_{3k-3}^j + \sum_{j \geq k} a_{3k-4}^j + \sum_{j \geq k-1} a_{2k-1}^j,
\]

\[
b_{3k-2}^k \geq b_{3k-3}^k + a_{3k-3}^k + 2 \sum_{j \geq k+1} b_{3k-3}^j + \sum_{j \geq k} a_{3k-4}^j + \sum_{j \geq k-1} a_{2k-1}^j,
\]

if \( k \geq 3 \).

**Preliminary Remarks:**

From the definition it can be seen directly that

1. \( a_n^k = a_{n-2(k-2)} \),
2. \( b_n^k = b_{n-2(k-2)} \) and
3. \( a_{n+1} = 2a_n + b_n \),
4. \( b_{n+1} = b_n + 2b_n + a_{n-1} \), if \( n \geq 5 \).

From (3) and (4) it is easy to see the following equation:

1. \( a_{n+1} = 2b_{n+1} - 3b_n - 2b_{n-1} \),

thus

2. \( b_n = 2b_{n-1} + 3b_{n-2} - 3b_{n-3} - 2b_{n-4} \), if \( n \geq 8 \).

**Proof of Lemma 2.10:**

Put \( x_k = \sum_{j=1}^{k-2} a_{3j-2}^j \)\(^{-2} \). \( a_{k+2} \)

\[
a_{k+2}^k \geq a_{k+1}^k + \sum_{j=1}^{k-2} b_{k-2j+1}^j + \sum_{j=1}^{k-1} a_{k-2j+2}^j + x_k
\]

\[
def. \quad a_{k+2}^k \geq a_{k+1}^k + a_{k+1}^k + x_k
\]

resp.

\[
b_{k+2}^k \geq b_{k+2}^k - b_{k+1}^k + a_{k+1}^k + x_k \quad (k \geq 3).
\]
Consequently it is sufficient to show that
\[ b_{k+1} - a_{k+1} - x_k \geq 0. \]
From the preliminary remarks (1), (3) and the definition of \( b_n \) we get for each \( n \geq 6 \):
\[ a_n = 2a_{n-2} + 2a_{n-1} - \sum_{j=1}^{\lfloor \frac{n-4}{2} \rfloor} a_{n-2j-1}, \]
thus
\[ (III) \quad x_n = \sum_{l=1}^{n-2} (2(a_{l+4} + a_{l+3}) - a_{l+5}) \]
\[ = 2a_4 + 4a_5 + 3 \sum_{l=6}^{n-2} a_l + a_{n+2} - a_{n+3} \]
The case \( k \leq 12 \) is checked easily with the help of the list of the \( a_j^m, b_j^m \) \( (n \leq 16) \) in the supplement.
Let \( k \geq 12 \). Then we have the following equation:
\[ z_k = b_{k+1} - a_{k+1} - x_k \]
\[ = b_{k+1} - 4a_{k+1} + b_{k+3} - a_{k+2} - 3 \sum_{l=6}^{k-2} a_l - 4a_5 - 2a_4 \]
\[ = b_{k+1} - 4b_{k+1} + 12b_k + 8b_{k-1} + 2b_{k+3} - 3b_{k+2} - 2b_{k+1} - 2b_{k-2} \]
\[ + 3b_{k+1} + 2b_k - 3 \sum_{l=1}^{k-2} (2b_l - 3b_{l-1} - 2b_{l-1}) - 14 \]
\[ = 2b_{k+3} - 5b_{k+2} - 6b_{k+1} + 14b_k + 8b_{k-1} - 6b_k + 3b_{k-1} - \sum_{l=6}^{k-2} 9b_l \]
\[ + 15b_5 + 6b_4 - 14 \]
\[ = 2b_{k+3} - 5b_{k+2} - 6b_{k+1} + 8b_k + 11b_{k-1} - \sum_{l=6}^{k-10} 9b_l + 15b_5 + 6b_4 - 14 \]
\[ = \begin{cases} 10b_{k-6} + 53b_{k-7} + 50b_{k-8} + 19b_{k-9} + \sum_{l=6}^{k-10} 9b_l + 15b_4 + 6b_4 - 14, & \text{if } k \geq 15 \\ 10b_8 + 53b_7 + 50b_6 + 25b_5 + 6b_4 - 14, & \text{if } k = 14 \\ 10b_7 + 56b_6 + 56b_5 + 16b_5 - 14, & \text{if } k = 13, \end{cases} \]
thus \( z_k \geq 0 \).
In this last equality the equation (II) is successively applied nine times.
The general case \( k \geq 16 \) is illustrated in the following diagram.
Lemma 2.12:

(i) \[ 2b_n < b_{n+1}, \quad n \geq 4 \]
(ii) \[ 2a_{n+1} > b_{n+1}, \quad n \geq 4 \]
(iii) \[ 2a_n < a_{n+1}, \quad n \geq 4 \]
(iv) \[ b_{n+1} < 3b_n, \quad n \geq 3 \]
(v) \[ a_{n+1} < 3a_n, \quad n \geq 5 \]

Proof:

(i) \[ b_{n+1} = b_{n-1} + 2b_n + a_{n-1} > 2, \quad \text{if } n \geq 5 \]  
   (the case \( n = 4 \) is trivial).

(ii) \[ b_{n+1} = 2a_n - \sum_{j=2}^{n-2} a_j \leq 2a_n \]

(iii) For \( n \leq 5 \) the assertion is valid.
If $2a_{n-1} < a_n$, then by (i) we have the following inequality:

$$2a_{n+1} = 2(2a_{n-1} + b_n) < 2a_n + b_{n+1} = a_{n+2}$$

(3)

Thus the proof is finished by induction.

(iv) The list in the supplement shows the assertion in case $n \leq 7$. If the inequality is valid for $j < n \geq 7$ then

$$b_{n+1} < 3b_n + b_{n-1} - 3b_{n-2} - 2b_{n-3}$$

(II)

$$< 3b_n - 2b_{n-3}$$

$$< 3b_n$$

and the proof is finished by induction.

(v) $a_{n+1} = 2a_{n-1} + b_n < 2a_{n-1} + 2a_n$

(3)

(ii) $< 3a_n$

(iii) $q.e.d.$

Lemma 2.13.

(i) $b_{2n} \leq 2 \cdot 7^{n-2}$ and

$$b_{2n+1} \leq 5 \cdot 7^{n-2}$$

if $n \geq 2$.

(ii) $a_{2n} \leq 7^{n-2}$ and

$$a_{2n+1} \leq 3 \cdot 7^{n-2}$$

if $n \geq 2$.

Proof:

(i) For $n \leq 4$ the assertion follows from the list in the supplement. For $n \geq 5$ the inequalities are implied by induction on $n$ with the help of Lemma 2.12. (iv):

$$b_1 = 2b_{1-1} + 3b_{1-2} - 3b_{1-3} - 2b_{1-4}$$

$$= 7b_{1-2} + 3b_{1-3} - 8b_{1-4} - 4b_{1-5}$$

$$< 7b_{1-2}$$

because (iv) guarantees

$$8b_{1-4} + 4b_{1-5} > 9b_{1-4} > 3b_{1-3},$$

where $1 = 2n$ resp. $1 = 2n+1$.  


(ii) Again for \( n \leq 4 \) the assertion follows from the list. For \( n \geq 5 \) the inequalities are implied by induction with the help of (i) and the preliminary remark (3):

\[
\begin{align*}
a_{2n+2} &= 2a_{2n} + b_{2n+1} \leq 2 \cdot 7^{n-2} + 5 \cdot 7^{n-2} = 7^{n-1}, \\
\text{resp.} \\
a_{2n+3} &= 2a_{2n+1} + b_{2n+2} \leq 3 \cdot 7^{n-2} + 2 \cdot 7^{n-1} = 17 \cdot 7^{n-2} \leq 3 \cdot 7^{n-1}.
\end{align*}
\]

q.e.d.

A consequence of the last lemma is

**Lemma 2.14:** Let \( k \geq 4, \, n \geq 3 \).

(i) If \( n \geq 9k-25 \), then \( b_n \leq 3^{n-k} \).

(ii) If \( n \geq 9k-29 \), then \( a_n \leq 3^{n-k} \).

Proof of (i) if \( n \) is even (the other cases can be treated analogously):

Put \( n = 21 \geq 9k-25 \), then \( 1 \geq \frac{k \cdot \ln 3 - \ln 49 + \ln 2}{\ln 9 - \ln 7} \), which

is equivalent to \( 2 \cdot 7^{n-2} \leq 3^{21-k} \), thus \( b_{21} \leq 3^{21-k} \) by the last lemma.

q.e.d.

Now the aim to construct simple upper bounds is reached and can be expressed in form of the following

**Proposition 2.15:**

Assume \( k \geq 4, \, n \geq 6 \).

If \( n \geq 9k-27 \), then \( s_{n-3} \leq 3^{n-k} \),

\[
s_{n-3} \leq t_{n-3} \leq 3^{n-k+1}.
\]

Proof: With the help of the preliminary remarks (1),(3) and the last lemma the following inequalities are implied:

\[
\begin{align*}
s_{n-3} &\leq \sum_{j=1}^{\frac{n-4}{2}} a_{n-2j-1} = 2a_{n-1} + 2a_{n-2} - a_n \\
&\leq 2a_{n-1} + 2a_{n-2} \leq 2 \cdot 3^{n-k-1} + 2 \cdot 3^{n-k-2} \\
&= 8 \cdot 3^{n-k-2} < 3^{n-k}.
\end{align*}
\]
The other inequality follows from the fact that
\[ b_n \leq 2a_n \leq 3a_n. \]

q.e.d.

We have shown that the number of equivalence-classes of the homogeneous n-person zero-sum games without dummies is less than \( 3^{n-k} \), for \( n \) sufficiently large.
§ 3  A Procedure to Construct all Classes of
Homogeneous n-Person Zero-Sum Games Without
Dummies From Those With Less Persons.

Unless otherwise specified, assume \( n \geq 5 \).
The games in \( S^2_n \) can easily be constructed from those of \( S_{n-1} \) and \( S_{n-2} \) by means of
\[
H|_{S^2_n}^{-1} : T_{n-1} \cup S_{n-2} \rightarrow S^2_n.
\]
Indeed, Lemma 2.4 guarantees the bijectivity of this function.

Definition 3.1:  For \( m \in S^k_n \) define
\[
R(m) = (m_{k+1}, \ldots, m_n).
\]
A representation \( m' \) of \( S_j \) is called subgame of \( m \), if the following conditions are fulfilled:

(i) \( m_{k+r+i} = m'_i \), for some not negative integer \( r \) and \( i \in \{1, \ldots, j\} \).

(ii) \( m_i \equiv 0 \mod \lambda(m') \), if \( 1 \leq i \leq k + r \).

If \( i < k \) and \( k \geq 3 \), we call \( m_i \) small in \( m \), if
\[
n \geq m_i + k, \quad m_{n-m_i+1} = m_{n-m_i+2} = \ldots = m_n = 1
\]
and
\[
(m_{i-1}, m_{i+1}, \ldots, m_{n-m_i}) \in S^{k-1}_{n-m_i}.
\]
For example the representation \( m = (3,3,3,3,2,1,1,1) \) has the subgame \( (2,1,1,1) \) but \( m_i \) \( i = 1, 2, 3 \) is not small in \( m \) although \( m_{n-m_i+1} = 1 \) and \( n \geq m_i + k \).

If \( m' = (m_1', \ldots, m_s') \) is a subgame of the game \( m \in S^n_k \), then
\[
n - s = c \cdot \lambda(m') \quad \text{for some } c \in \mathbb{N}.
\]
In the following certain representations are "decomposed".

This decomposition is based on the next important

Theorem 3.2:  Let \( m \) be a game in \( S^n_k \) for some \( k \geq 3 \).
If \( m \) has no subgame, then \( m_2 \) is small in \( m \).
Proof: The cases $n = 3, 4$ are trivial because then $S^k_3 = \emptyset = S^k_4$. The cases $n = 5, 6$ can be seen easily because all classes of homogeneous 5-resp. 6-person zero-sum games without dummies are well-known (compare with the supplement).

The proof proceeds by induction on $n$.

Assume the assertion for $j \leq n_0 \geq 6$ and put $n = n_0 + 1$.

The following six cases are distinguished:

1. There is a natural number $i$, $3 \leq i \leq k$, with $m_{i-1} = m_i > m_{i+1}$:
   A game $m' \in S^k_{n-1}$ is constructed in order to apply the inductive hypothesis:
   If $i < k$, then there is a number $j$ with $i < j \leq k$ such that
   \[ m' = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_j, m_{i} - m_{k+1}, m_{j+1}, \ldots, m_k, m_{k+2}, \ldots, m_n) \]
   is a member of $S^k_{n-1}$. If $i = k$, consider
   \[ m' = (m_1, \ldots, m_{k-1}, m_k, m_{k+1}, m_{k+2}, \ldots, m_n), \]
   which is an element of $S^k_{n-1}$, too.
   If this newly formed game $m'$ had a subgame $m''$, so would the old game $m$ because with
   \[ m_{i-1} \equiv 0 \mod \lambda(m''), \ m_i - m_{i+1} \equiv 0 \mod \lambda(m'') \]
   it is also valid that
   \[ m_i \equiv 0 \mod \lambda(m''), \ m_{i+1} \equiv 0 \mod \lambda(m'') \] as $m_{i-1} = m_i$.
   By hypothesis $m_2$ is small in $m'$ and therefore in $m$ as well.

2. $k \geq 4$ and $m_1 = m_2$:
   Then
   \[ m' = (m_1, m_k, m_3, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n) \]
   is a game in $S^k_{n-1}$ and has no subgame either, thus $m_1$ is small in this game as it is in $m$.

3. $m_{k-1} = m_k = m_{k+1}$:
   If $k \geq 4$, then
   \[ m' = (m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n) \]
   is in $S^k_{n-2}$. This game $m'$ cannot have a subgame, otherwise $m$ would have it as well. With the help of the inductive hypothesis it follows that $m_2$ is small even in $m$. 
Now consider the case \( k = 3 \) and \( m_k > 1 \), the case \( m_k = 1 \) being obvious. It is \( m_1 \neq m_2 \), otherwise

\((m_5, \ldots m_n)\)

would be a subgame of \( m \).

In view of \( \S \) 2 it is clear that \( \sigma(m) \) is a game in \( S_{n-1}^3 \) and \( \sigma(m) \) has no subgame. As \( \sigma(m)_2 = m_2 \), we have shown that \( m_2 \) is small in \( \sigma(m) \) by the hypothesis, thus \( m_2 \) is small in \( m \).

4. \( m_3 > m_4 \):

Take \( j \geq k+1 \) such that \( m_3 = m_k + \ldots + m_j \) and put \( \widetilde{m}_3 = m_3 - m_j \).

If the game

\[ m' = (m_1, m_2, \widetilde{m}_3, m_4, \ldots, m_{j-1}, m_{j+1}, \ldots, m_n) \]

which is in \( S_{n-1}^k \), has no subgame, nothing further has to be shown.

Assume \( m' \) has a subgame

\[ m'' = (m_1, \ldots, m_s) \in S_{s-1}^t \text{ resp. } S_{s-i}^t \]

If \( i < j-1 \), then \( t+s < j \) because of the fact

\( m_3 \equiv 0 \mod \lambda(m'') \).

This implies

\[ m_{s+1} = 1 \text{ and } \sum_{l=s+1}^{n} m_l \geq m_1 + m_2 + \sum_{l=4}^{k} m_l - \lambda(m'') \]

\[ \geq m_1 + m_2 - \lambda(m'') \],

thus \( m_2 \) is small in \( m \).

If \( i > j-1 \), then because of \( m_j \neq 0 \mod \lambda(m'') \) we have

\( m_j = m_{j+1} + \ldots + m_x \) for some \( x \geq i \). This and the fact that

\( m_1 \equiv 0 \mod \lambda(m'') \) has the consequence

\[ m_1 = m_{k+1} + \ldots + m_{j-1} + m_{j+1} + \ldots + m_{i+t-1} + c \cdot \lambda(m'') \]

for some not negative integer \( c \), implying the following equation:

\[ \sum_{j=s+1}^{n} m_j - m_2 + \sum_{j=i}^{k} m_j \]

\[ = m''_3 + \sum_{j=4}^{k} m_j + (c-1) \cdot \lambda(m'') + \lambda(m'') = c \cdot \lambda(m'') + \widetilde{m}_3 + \sum_{j=4}^{k} m_j \]

By the inductive hypothesis the proof is finished in this case.
5. $m_1 > m_2 > m_3 = m_4$, $k = 3$.

Put $\tilde{m}_1 = m_1 - m_r$, where $m_1 = m_4 + \ldots + m_r$, and

$$m^\prime = (\tilde{m}_1, m_2, \ldots, m_{r-1}, m_r, \ldots, m_n),$$

which is a game of $S_n^{k+1}$. The case $m^\prime$ having no subgame is trivial and the case $m^\prime$ having a

subgame $m^\prime = (m_i, \ldots, m_s)$ with $i \leq r-1$ can be treated analogously to

4. and is skipped here.

Assume $i > r+1$, which yields $m_3 = 2(m_r + \ldots + m_i - 1) + k \cdot \lambda(m_i)$ for some $k \geq 1$, thus $m((s+1, \ldots, n)) + 2\lambda(m_i) - 1 - m_2 = m_{r+1} + \ldots + m_{i-1} + k \cdot \lambda(m_i) - 1 \geq m_{r+1}$

and $m_{r+1} + \ldots + m_n - m_2 = m_3 - 1$. It remains to be shown that $m_3 - m_2 + 1 = 1$.

If $i > r+1$ or $i = r+1$ and $m_3 \geq 2\lambda(m_i)$ the proof is finished "directly.

Otherwise $m^\prime = (\tilde{m}_1, m_2, m_3, \ldots, m_{r-1}, m_i, \ldots, m_n)$. Observe that $m_3 - m_2$ is

small in the game $m = (m_1 - m_3, m_2 - m_3, m_3, \ldots, m_{r-1}, m_i, \ldots, m_n)$, as it has no

subgame, thus $m_2$ is small in $m$.

6. $m_1 = m_2 > m_3 = m_4$, $k = 3$.

Put $m^\prime = (2m_1 - m_3, m_3, m_4, \ldots, m_n)$ and consider that $m^\prime$ is a game

in $S_n^3$. The following two cases may occur:

(A) $m^\prime$ has a subgame.
(B) $m^\prime$ has no subgame.

ad (A): Let $m^\prime = (m_i, \ldots, m_s)$, $m^\prime \in S_n^{k-1}$, be a subgame of $m$.

Take $r \in \mathbb{N}$ such that $m_1 = m_4 + \ldots + m_r$. The fact that

$i + \lambda - 1 \leq r$ means that $m_2$ is small in $m$.

Now, assume $i + \lambda - 1 > r$. It is easy to see that $i \leq r$,

otherwise $m^\prime$ would be a subgame of $m$.

The fact $2m_1 \equiv 0 \mod \lambda(m^\prime)$ implies

$$m_1 + \ldots + m_r = \frac{\lambda(m^\prime)}{2}.$$ 

Put $m_3 = c \cdot \lambda(m^\prime)$ and observe the following equality:

$$m_3 + \ldots + m_n = m_1 + (c - \frac{3}{2}) \lambda(m^\prime).$$

Thus $m_2$ is small in $m$ if $c \geq 2$.

If $c = 1$, then $m_3 = \ldots = m_{i-1} = \lambda(m^\prime)$. Concerning the

case $i-1 > 3$ look at

$$(m_1 - m_3, m_1 - m_3, m_3, \ldots, m_3, m_i, \ldots, m_n - \lambda(m^\prime)), (i-4)\text{times}$$.
which is a game in $S_{n-m_3-1}^3$ and has no subgame (otherwise m would have the same).
The inductive hypothesis guarantees that $m_1 - m_3$ is small in this newly formed game, thus $m_1$ is small in m. The case i-1 equals 3 can be treated analogously by looking at the game

$$(m_3, m_1-m_3, m_1-m_3, m_5, \ldots, m_{n-m_3}),$$

which is in $S_{n-m_3-1}^3$ again.

ad (B): In this case (3) guarantees that $m_3$ is small in $m'$, thus

$$m'' = (m_1, m_1, m_3, m_5, \ldots, m_{n-m_3})$$

is a game in $S_{n-m_3-1}^2$.

Put

$$\tilde{m} = \begin{cases} 
(m_1-m_3, m_1-m_3, m_3, m_5, \ldots, m_{n-m_3}), & \text{if } m_1 - m_3 \geq m_3. \\
(m_3, m_1-m_3, m_1-m_3, m_5, \ldots, m_{n-m_3}), & \text{otherwise.}
\end{cases}$$

Consequently $m$ is the ordered minimal representation of a $(n-m_3-1)$-person game of length 3 and there is no subgame of $\tilde{m}$, otherwise $m$ would have the same subgame. We conclude that $(m_1-m_3)$ must be small in $\tilde{m}$, and therefore $m_1$ is small in $m$.

Each game in $S_n^k$, $k \geq 3$, has at least one of the previous properties (1)-(6), so that the proof is finished.

In this proof Theorem 1.3 has been used several times without being mentioned explicitly.

q.e.d.

**Definition 3.3:**

Let $m \in S_n^k$, $k \geq 3$.

Then $m$ is called reducible, if

1. $n \geq \sum_{j=2}^{k-1} m_j + k + 1$ and $m_{n-k-1} = 1$. 

(ii) \((m_1, m_k, m_{k+1}, \ldots, m_{n-k+1}, m_j, \ldots, m_j, m_{n-k+2}^k) \in S^{n-k+1}_{n-k+2} \)

\(m\) is called irreducible, if it is not reducible.

**Lemma 3.4:** Let \(m \in S^n_k\), which has a subgame.

If \(i\) resp. \(s\) is the minimal resp. maximal number such that there is a subgame of the form 
\((m_i, \ldots, m_s)\) resp. \((m_i, \ldots, m_s)\)

then

\((m_i, \ldots, m_s)\)

is a subgame, too.

**Definition 3.5:** With the notations of Lemma 3.4 the subgame 
\((m_i, \ldots, m_s)\) is called maximal subgame of \(m\).

**Proof of Lemma 3.4:**

Put \(m^1 = (m_i, \ldots, m_{s'})\), \(m^2 = (m_i, \ldots, m_{s'})\).

The fact \(m_{i'-1}^1 \equiv 0 \mod \lambda(m^2)\) and the existence of at least two persons with weight one in \(m^1\) imply \(s' > i'\).

If \(i'\) is not a member of the lex-max coalition of \(m^1\), i.e. \(i + 1(m^1) \leq i'\), then \(m_i + \ldots + m_s = c \lambda(m^2) - 1\) for some natural number \(c\). If \(c > 2\), then \(s' > s\), a contradiction.

If \(c = 1\), then \(m_i + \ldots + m_{i'} = s\), which is a contradiction, too (see Theorem 1.3(v)).

Thus, we have \(c = 2\) and \(s' = s\).

If \(i + 1(m^1) > i'\), we conclude \(i + 1(m^1) < i' + 1(m^2)\), as \(s' \leq s\), thus \(m_j \equiv 0 \mod \lambda\), \(1 \leq j < i\), where \(\lambda\) is the least common multiple of \(\lambda(m^1)\) and \(\lambda(m^2)\). The facts \(m_{i-1} \leq m_i + \ldots + m_s\) and \(\lambda \geq 2\lambda(m^1)\) imply the existence of some \(p > s'\) such that

\(m_i + \ldots + m_p = \lambda\), thus \((m_i, \ldots, m_{p+\lambda-1})\)

is a subgame of \(m\), which is a contradiction to the maximality of \(s\).

q. e. d.
Lemma 3.6 Let $m \in S_n^k$, $k \geq 3$.

Then $m$ is reducible, if and only if $m$ has no subgame or the maximal subgame

$m' = (m_i, \ldots, m_s)$

of $m$ has the property

$m_1 + m_k > m_{k+1} + \ldots + m_s$.

Proof:

1st STEP: Suppose that $m$ is reducible.

Assume the contrary, i.e. $m$ has a subgame and the maximal subgame has the property $m_1 + m_k \leq m_{k+1} + \ldots + m_s$.

This last inequality is actually proper because of the congruence

$m_j \equiv 0 \mod \lambda(m')$, $1 \leq j \leq i-1$.

As the total mass of $m$ is $2\lambda(m)-1$, we have the following inequality:

$m((s+1), \ldots, n)) \leq m_2 + \ldots + m_{k-1} - \lambda(m')$,

thus

$m((i, \ldots, n)) - m_2 - \ldots - m_{k-1} \leq \lambda(m') - 1$.

The reducibility together with Theorem 1.3 shows that

$m_{i-1} \leq m((i, \ldots, n)) - m_2 - \ldots - m_{k-1}$,

but

$m_{i-1} \equiv 0 \mod \lambda(m')$,

which contradicts the assumption.

2nd STEP: Suppose that $m$ has no subgame.

The assertion follows by induction on $k$.

For $k = 3$ the assertion is valid by the last theorem.

Assume the assertion for $k_0 \leq k-1 > 3$. From Theorem 3.2 we know that

$\tilde{m} = (m_1, m_3, \ldots, m_{n-m_2})$

is a game in $S_{n-m_2-1}^{k-1}$.

The case $m$ having no subgame is clear by the inductive hypothesis.
Now consider the case \( m \) having a subgame \( m' = (m_1', \ldots, m_s') \). Pick a nonnegative integer \( r \) such that \( m'_r = m_{k+1}' + \ldots + m_{r'}' \), then \( j \leq r \), otherwise \( m' \) would be a subgame of \( m \).

We conclude that there is a natural number \( c \) such that

\[
m_1 = m_{k+1} + \ldots + m_{j-1} + \lambda(m') \cdot c,
\]

thus

\[
m_{s+1}' + \ldots + m_n' = m_2' + \ldots + m_k'(c-2) \cdot \lambda(m'),
\]

which implies the inequality

\[
m_1' + \ldots + m_n' = m_{k-1}' \geq m_k,
\]

thus

\[
\sum_{j=2}^{k-1} \sum_{j=2}^{n} m_j \geq s + 1
\]

and the proof is finished.

---

**3rd STEP:**

Suppose that \( m \) has a subgame and the maximal subgame \( m' \) of \( m \) has the property

\[
m_1 + m_k > m_{k+1} + \ldots + m_s.
\]

We proceed by induction on \( n \).

If \( n \) is less than six the assertion is certainly valid, since all such games are explicitly known.

Assume the assertion for \( r \leq n-1 \geq 5 \).

Let \( m \) be a game in \( S_n^k \).

If \( m_1 \geq m_{k+1} + \ldots + m_i \), the proof can be finished directly.

Otherwise look at

\[
\tilde{m} = \left( \frac{m_1}{\lambda(m')}, \ldots, \frac{m_i-1}{\lambda(m')}, 1, \ldots, 1 \right)_{(n-s) \lambda(m') + 1 \text{ times}}
\]

which is in \( S_i^k \)

\[
i + \frac{n-s}{\lambda(m')}
\]

If \( \tilde{m} \) has no subgame resp. only subgames \( (m_1', \ldots, m_s') \) with \( s' \leq i-1 \), then \( m \) is reducible by the 2nd step resp. the inductive hypothesis, consequently the assertion is true.

If \( \tilde{m} \) had a subgame \( \tilde{m}' = (m_1', \ldots, m_s') \) with \( s' > i \), then
\[(m_1, \ldots, m_s, 1, \ldots, 1)\]
\[(\lambda(m')) \cdot (s' - 1) \text{ times}\]

would be a subgame of \(m\) with level \(\lambda(m') \lambda(\tilde{m}')\).
This would be a contradiction to the fact that \(m'\) is maximal because \(i' < i\) or \(s' > s\).

\[q.e.d.\]

Now "the decomposition" of irreducible games in \(S_n^k\) can be defined.
Let \(m\) be irreducible and \(m' = (m_1, \ldots, m_s)\) be the maximal subgame
(m' exists because of Lemma 3.6).
Then
\[
\tilde{m}' = (\frac{m_1}{\lambda(m')}, \ldots, \frac{m_{i-1}}{\lambda(m')}, 1, \ldots, 1), \quad \text{where}
\]
\[(c+1) \text{ times}\]
\[n - s = c \cdot \lambda(m'), \text{ is in } S_{i+c}^k.\]

**Definition 3.7:** With the previous notations \(m\) is called
**decomposable** (with respect to \(i\)), written
\[m = \tilde{m}' \otimes_i m'.\]

In order to "compose" two representations, first define for each
\(\tilde{m} \in S_n^k, k \geq 3\), the following:
\[p(\tilde{m}) = \max \{j \in \{k+1, \ldots, n+1\} | \tilde{m}_1 \tilde{m}_k = \sum_{i=k+1}^{j} \tilde{m}_i \text{ or} \]
there is a subgame \((\tilde{m}_1, \ldots, \tilde{m}_{j-1})\) of \(\tilde{m}\) for some \(i\).

In view of the last lemma and Theorem 1.3 we know that
\[\tilde{m} p(m) - 1 = \tilde{m} p(m) - 2 = 1.\]

If \(p(\tilde{m}) < n\), a natural number \(i\), \(p(\tilde{m}) \leq i \leq n\), and an arbitrary game
\(m' \in S_n\), define the **composition** of \(\tilde{m}\) and \(m'\) resp. \(i\) by
\[m = (\lambda(m') \cdot \tilde{m}_1, \ldots, \lambda(m') \cdot \tilde{m}_{i-1}, \tilde{m}_1, \ldots, m'_n, 1, \ldots, 1).\]
\[(n - i) \cdot \lambda(m') \text{ times}\]

**Lemma 3.8:** With the previous notations \(m'\) is the maximal subgame
of \(m\) and \(m\) is decomposable with respect to \(i\),
in addition
\[ m = \tilde{m} \otimes \ldots \otimes m'. \]

**Proof:** From the definition of \( p(\tilde{m}) \) and the last lemma it is easy to see that \( m \) is irreducible, so it suffices to show that \( m' \) is the maximal subgame of \( m \).

The fact \( m' \) being a subgame of \( m \) is a direct consequence of Definition 3.1.

Let \( m'' = (m_1, \ldots, m_s) \) be the maximal subgame.

Assume \( i' < i \), then

\[ (\tilde{m}_1, \ldots, \tilde{m}_{i-1}, 1, \ldots, 1) \]

\( (s'-s+1) \) times

\( \lambda(m') \)

is a subgame of \( \tilde{m} \), thus \( p(\tilde{m}) > i+1 \), a contradiction.

Moreover \( \tilde{m} p(\tilde{m})-1 \) equals one, that means

\[ m p(\tilde{m})-1 = \lambda(m'), \]

thus \( s' = s \).

q.e.d.

Notice that \( p(m) = i+n' \).

**Corollary 3.9.** Let \( m \in S_n^k \), \( k \geq 3 \), irreducible. Then \( m \) is decomposable and the decomposition is unique, i.e. if

\[ m = \tilde{m}^1 \otimes i_1 \quad m'' = \tilde{m}^2 \otimes i_2 \quad m', \]

then \( \tilde{m}^1 = \tilde{m}^2 \), \( i_1 = i_2 \) and \( m'' = m' \).

Furthermore \( i_1 \geq p(\tilde{m}^1) \).

The last results can be summarized in the following important

**Theorem 3.10:** Let \( m \in S_n^k \), \( k \geq 3 \).

Then \( m \) is irreducible, iff there is exactly one "complete decomposition" of \( m \), that means:

\[ m = (\ldots (\underbrace{m^1 \otimes i_1 m^2} \otimes i_2 m^3) \otimes i_3 \ldots) \otimes i_{m} \]

\( (r-1) \) brackets
with a reducible game \( m^1 \) in \( S^k_{n_1} \) and games \( m^t \in S^t_{n_t} \) and
\[
i_1 > p(m^1), \quad i_t > i_{t-1} + n_t.
\]
In addition the parameters \( r, i_t, m^t \) are uniquely determined by \( m \) and vice versa.
This theorem yields a procedure to construct all games in \( S^k_{n} \), \( k \geq 3 \),
which are irreducible from the sets \( S^m_{m}, \ m \leq n-2 \).
It remains to look at the reducible games of \( S^k_{n} \) which may be constructed in
another way, shown in the sequel.

Remark 3.11: If \( m \) is a reducible element of \( S^k_{n} \) for some \( k \geq 3 \),
then as well
\[
f(m) = (m_1, m_2, \ldots, m_{n-m})
\]
is in \( S^{k-1}_{n-m-1} \) as
\[
g(m) = (m_1, m_2, m_3, \ldots, m_{n-m-3})
\]
is in \( S^{k-1}_{n-m-3-1} \) and in addition \( f(m) \) and \( g(m) \) are reducible, if \( k \geq 4 \).
The other way round, if
\( m^1 \) is in \( S^{k-1}_{n_1} \) and \( m^2 \) is in \( S^{k-1}_{n_2} \), both are reducible,
if \( k > 4 \), such that
\[
m^2 = m^1, m^2 \geq m^1, m^2 = m^1, \ldots, m^2 = m^1 (especially \ n_2 > n_1)
\]
then
\[
m = (m^1_1, m^2_2, m^1_2, m^2_3, \ldots, m^1_{n_2}, 1, \ldots, 1)
\]
is a reducible game in \( S^{k}_{n_2+m^2_1} \) and in addition
\[
f(m) = m^1, \ g(m) = m^2.
\]
So we have proved the following

Theorem 3.12: Let \( m \in S^k_{n} \), \( k \geq 3 \).
\( m \) is reducible, iff there are two - in case \( k > 4 \)
reducible- games \( m^1 \in S^{k-1}_{n_2} \) with \( n_2 > n_1 \) and \( m^1_i = m^2_i \)
for all \( i \in \{1, 3, \ldots, n_1\} \) such that
\[ m = (m_1^1, m_2^1, m_2^2, m_3^2, \ldots, m_{n_2}^2, 1, \ldots, 1). \]

Moreover these games \( m^1 \) and \( m^2 \) are uniquely determined by \( m \) and vice versa. We say that \( m \) can be reduced to \( m^1 \) and \( m^2 \).

The last two theorems together with the bijections \( (H_{n_2}^2)^{-1} \) yield a procedure to construct all classes of homogeneous zero-sum games without dummies successively.

The last section of this chapter refers to Lemma 2.6 in order to show that the inequalities in fact are equations, applying the previous results. More precisely it must be shown that \( s_n^k \) and \( s_{n-2}^{k-1} \) resp. \( t_n^k \) and \( t_{n-2}^{k-1} \) coincide, if \( 3k-3 \geq n \geq 5 \). In view of the proof of Lemma 2.6 it suffices to show that \( m_i \) equals \( m_{k-1}^k \cdot c_i \) for all \( 1 \leq i \leq k-2 \) (for some natural numbers \( c_i \)) and all \( m \in S_{n-2}^{k-1} \) - because then

\[
(m_{k-1}^k, \ldots, m_n^k) \text{ hom } m_i, 1 \leq i \leq k-1,
\]

thus

\[
(m_1^1, \ldots, m_{k-1}^k, m_{k-1}^k, m_{k-1}^k, m_k^1, \ldots, m_n^k)
\]

is a game in \( S_n^k \), implying the other inequalities.

This can be done by induction on \( n \):

Assume the assertion is true for all \( 1 \leq n-1 \leq 5 \) as it is for \( l = 5 \).

Let \( m \) be a game in \( S_{n-2}^{k-1} \).

1st STEP: Propose \( m \) is decomposable, let us say that

\[
m = m^1 \boxplus j m^2,
\]

\( m^1 \in S_{l-1}^{k-1} \) for some \( 1 \leq n-3 \), by induction hypothesis

\[
m_i^1 = c_i \cdot m_{k-1}^1, 1 \leq k-1 \text{ because } k \geq \frac{l+3}{3}. \]

The fact \( m_i = \lambda(m^2) \cdot m_i^1 \) for \( i \leq k-1 \) finishes the proof in this case.

2nd STEP: Propose \( m \) is reducible or \( k = 3 \). Assume there is 

\( i \leq k-2 \) such that \( m_i \not\equiv 0 \mod m_{k-1}^1 \), then \( m_i \geq m_k^1 + m_{k+1}^1 \), thus \( m_1^1 \geq m_k^1 + m_{k+1}^1 \) and \( m_{k-1}^1 \geq 1 \).
Consequently we have
\[ \sum_{i=2}^{k-2} m_i \geq 2(k-3), \]
thus
\[ n-2 \geq 2(k-3) + 2 + k = 3k-4 \]
because
\[ \left( m_1, m_{k-1}, m_k, \ldots, m_{k-2} \right) \]
is an ordered minimal representation by the fact that \( m \) is reducible resp. \( k = 3 \). This shows that \( n \) must be greater than \( 3k-3 \), a contradiction.

As all cases were treated the proof is finished.
§ 4  A Lower Bound for the Number of Classes
of Games with Length Two

At first the definition of a set of $n$-person games of length two, introduced by ISBELL ([1]), is recalled, from which he derived the well-known lower bound $2^{n-4}$, i.e. the cardinality of this set.

Let
\[ \hat{m} = (m_1, m_2, \ldots, m_2, m_3, \ldots, m_3, \ldots, m_j, \ldots, m_j) \]

be a vector in $\mathbb{N}^n$, such that

1. $k_2 + \ldots + k_j = n-2$, $k_1 \geq 1$, for $2 \leq l \leq j$;

2. $m_1 = \sum_{l=2}^{j-1} k_l m_1 - m_2 + 1$;

3. $m_1 = \begin{cases} \frac{j-1-2}{2} & \sum_{r=1}^{j-1-1} k_{1+2r-1} m_{1+2r-1} + 1, \text{ if } j-1 \equiv 0 \text{ mod } 2. \\ \frac{j-1-2}{2} & \sum_{r=1}^{j-1-1} k_{1+2r-1} m_{1+2r-1}, \text{ if } j-1 \equiv 1 \text{ mod } 2. \end{cases}$

Then $m$ is a game in $S_n^2$.

Exactly these games possess a "minimal" incidence-matrix, i.e. one with $n$ rows. For this property and the definition of the incidence-matrix I refer to ISBELL ([1,2]) and OSTMANN ([3]).

Define
\[ C_n = \{ m \in S_n^2 | m \text{ has a minimal incidence-matrix} \}. \]

It is known from the paper mentioned above [1], that $C_3 = \{(1,1,1)\}$ and $|C_n| = 2^{n-4}$ in case $n \geq 4$.

The rest of this chapter is used to enlarge the sets $C_n$.

Continue the composition $\otimes_k$ of the last paragraph formally on $C_n$, which yields a mapping
\[ \varphi_n : \bigcup_{k+1=n+1} C_k \otimes S_1 \to S_n^2 \\
(m, m') \mapsto m \otimes_k m', \]

where
\[ m \otimes_k m' = (m_1 \lambda(m'), \ldots, m_{k-1} \lambda(m'), m_1', \ldots, m_1'). \]
Lemma 4.1: A game in $C_n$ has no subgame.

Proof: Assume the contrary and $n \geq 4$, the case $n = 3$ being trivial. Let $m' = (m_j, \ldots, m_s)$ be a subgame. From the fact that $m_2 = m_3$ we know that $j$ must be larger than three.

Put

$$c = \frac{m_{j-1}}{\lambda(m')} = \frac{m_{s+1}^{'} + \cdots + m_n}{\lambda(m')}$$

It follows from Theorem 1.3.(v) that $c$ is a positive and $c'$ a nonnegative integer, such that $c' > c - 1$.

The game $m$ has the property (3) of the beginning of this paragraph. Applying (3) twice it follows that

$$m_{j-1} = \sum_{r=j+1}^{n} m_r + 1,$$

thus

$$m_{j-1} = 2 \cdot \lambda(m') + c' \lambda(m') - m_j.$$

On the other hand $m_{j-1} = c \lambda(m')$, which leads to

$$(c' + 2 - c) \lambda(m') = m_j,$$

thus

$$m_j > \lambda(m'),$$

which contradicts the fact that $m'$ is a game in some $S_t$.

q.e.d.

Corollary 4.2: The mapping $\varphi_n$ is injective and the image of $\varphi_n$ does not contain any game with a minimal incidence-matrix, formally

$$\text{im } \varphi_n \cap C_n = \emptyset.$$

Proof: The second assertion is implied by the last lemma because of the fact that $m'$ is a subgame of $\varphi_n(m, m')$. The first assertion follows directly by considering $m'$ as the maximal subgame of $\varphi_n(m, m')$, otherwise $m$ would have a subgame, too.

q.e.d.
Consequently the set $S_n^2$ contains two disjoint subsets $C_n$ and $\varphi_n\left( \bigcup_{k=3}^{n-2} C_k \times S_{n-j+1} \right)$.

Now we define subsets of $S_n^2$, which contain $C_n$; inductively by

\[ \tilde{C}_3^i = C_3, \quad \tilde{C}_4^i = C_4, \quad \tilde{C}_{n+1}^i = C_{n+1} \cup \varphi_{n+1} \left( \bigcup_{k=3}^{n-1} \tilde{C}_k \times \tilde{C}_{n+2-k} \right). \]

Let $C_n$ denote the cardinality of $\tilde{C}_n$.

**Remark 4.3:** In the case $n \geq 7$ the set $C_n$ does not contain the image of $\varphi_n$ because the game

(1,1,1,1,1)

is in $S_5$ but not in $C_5$.

In order to get a lower bound of the cardinality of $S_n^2$, the cardinality of $\tilde{C}_n$ is determined in the next

**Lemma 4.4:**

(i) $c_3 = c_4 = 1$, $c_5 = 3$.

(ii) If $n \geq 5$, then $c_{n+1} = 2c_n + c_{n-1} - c_{n-2}$.

(iii) $c_{n+1} \geq 2c_n$, if $n \geq 4$.

**Proof:** ad (i) The equation $c_3 = c_4 = 1$ is obvious.

The fact $c_5 = 3$ is a direct consequence of the last corollary.

ad (ii) The last corollary implies for all $n \geq 5$ that

\[ c_n = 2^{n-4} + \sum_{k=4}^{n-2} 2^{k-4} c_{n-k+1} + c_{n-2}. \]

A simple induction on $n$ shows the assertion, applying the last formula.

(iii) is a direct consequence of (ii), since the increase of $c_n$ with respect to $n$ is obvious.

* q.e.d.

Furthermore in case $n \geq 5$ we have - by applying Lemma 4.4(ii) twice:

\[ c_{n+2} = 5c_n + c_{n-1} - 2c_{n-2}. \]
This equality yields the following inequality by Lemma 4.4(iii):
\[ c_{n+2} \geq 5c_n, \quad n \geq 6. \]
From the list in the supplement it can be seen that \( c_6 = 6 \) and \( c_7 = 14 \), thus
\[ c_{6+2^j} \geq 6 \cdot 5^j, \quad j \geq 0. \]
\[ c_{7+2^j} \geq 14 \cdot 5^j, \quad j \geq 0. \]

With the help of the preceding results we can give a real number \( n \), such that \( 2^{l+k} \) is at most the cardinality of \( C_l \) for all \( l \geq n \). Indeed, the next theorem yields this natural number.

**Theorem 4.5:** If
\[ n \geq 2 \frac{(k+4)\ln 2 - \ln 3}{\ln 5 - \ln 4} + 7, \quad k \geq 1 \text{ fixed,} \]
then
\[ c_n \geq 2^{n+k}. \]

The proof can be treated analogously to the proof of Lemma 2.14 and is therefore omitted.
Some lists:
In List 1 the entries in the rectangles represent $\frac{k^k}{n^k}$.
In List 2 they represent $\frac{a^n b^k}{p^n}$.

**List 1:**

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In the marked area of List 1 there are only entries with respect to \( k \) and \( n \) with \( \tau_n^k = \tau_{n-2}^{k-1} \) and \( \tau_n^k = \tau_{n-2}^{k-1} \) because there the inequality \( n \leq 3k-3 \) is valid.

**List 2:**

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Next the first eleven $c_n$ are tabulated (see §4):

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As an example all games in $S_9$ with length at least three are listed with the help of the results of §3, the sets $S_3, \ldots, S_7$ being explicitly known from ISBELL ([2]).

The reducible games in $S_9^k$, $k \leq 3$:

$(3,3,1,1,1,1,1,1,1) \in S_9^3$ can be reduced to $(3,1,1,1,1,1)$ and $(3,1,1,1,1,1)$.  

$(4,2,1,1,1,1,1,1,1) \in \quad \quad \quad \quad (4,1,1,1,1,1,1,1)$ 

$(4,2,2,1,1,1,1,1,1) \in \quad (4,2,2,1,1,1,1,1)$ 

$(5,1,1,1,1,1,1,1,1,1) \in \quad (5,1,1,1,1,1,1,1)$ 

$(3,3,2,1,1,1,1,1,1,1) \in \quad (3,2,2,1,1,1,1,1)$ 

$(3,2;2,1,1,1,1,1,1,1) \in \quad (3,2,1,1,1,1,1,1)$ 

$(5,2,2,2,1,1,1,1,1,1) \in \quad (5,2,2,2,1,1,1,1)$ 

$(2,2,1,1,1,1,1,1,1,1) \in S_9^4 \quad \quad \quad \quad (2,1,1,1,1,1,1)$ 

$(3,1,1,1,1,1,1,1,1,1) \in \quad \quad (3,1,1,1,1,1,1,1)$ 

$(1,1,1,1,1,1,1,1,1,1) \in S_9^5 \quad \quad \quad \quad (1,1,1,1,1,1,1,1)$

The decomposable games in $S_9$:

$(1,1,1,1,1,1) \otimes_6 (1,1,1,1,1) = (3,3,3,3,1,1,1,1,1)$

$(1,1,1,1,1,1) \otimes_5 (3,1,1,1,1,1) = (4,4,4,4,3,1,1,1,1)$

$(1,1,1,1,1,1) \otimes_5 (2,2,1,1,1) = (4,4,4,4,2,1,1,1,1)$

$(1,1,1,1,1,1) \otimes_5 (3,2,2,1,1) = (5,5,5,5,3,2,1,1,1)$

$(2,1,1,1,1,1,1) \otimes_5 (2,1,1,1,1) = (6,3,3,3,3,2,1,1,1)$

$(3,1,1,1,1,1,1,1) \otimes_7 (1,1,1) = (6,2,2,2,2,2,1,1,1)$

$(2,2,1,1,1,1,1,1,1) \otimes_7 (1,1,1) = (4,4,2,2,2,2,1,1,1)$

$(1,1,1,1,1,1,1,1,1,1) \otimes_7 (1,1,1) = (2,2,2,2,2,2,2,1,1,1)$
As last example the elements of $C_9$ are written down:

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