An Algorithm for the Construction of Homogeneous Games
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Abstract

Suppose, a weighted majority simple n-person game is to be specified by allotting weight $g_0 = 0$ to $k_0$ players, weight $g_1$ to the next $k_1$ player, ..., weight $g_r$ to the last $k_r$ players; here $g_i \in \mathbb{N}$ is increasing and $k_i \in \mathbb{N}$. ($i = 0, \ldots, r$). A coalition is winning if the total weight of its members is at least $\lambda \in \mathbb{N}$. An algorithm is provided such that, given $(g_0, g_1, \ldots, g_r)$ and $(k_0, k_1, \ldots, k_r)$, every $\lambda$ is produced which renders the resulting simple game to be homogeneous.
SEC. 1 The matrix of homogeneity

Let \( k = (k_0, \ldots, k_r) \in \mathbb{N}_0^{r+1} \) satisfy

\[
(1) \quad k_0 \geq 0, \quad k_1, \ldots, k_r \geq 1;
\]

a vector \( s = (s_0, \ldots, s_r) \in \mathbb{N}_0^{r+1} \) is a feasible profile (for \( k \)) if \( s \leq k \). Next, let \( g = (g_0, g_1, \ldots, g_r) \in \mathbb{N}_0^{r+1} \) satisfy

\[
(2) \quad 0 = g_0 \leq g_1 \leq g_2 \ldots \leq g_r \neq 0 .
\]

\( g \) induces the function

\[
g : \{s \leq k\} \to \mathbb{N}_0
\]

\[
(3) \quad g(s) = \sum_{i=0}^{r} s_i g_i .
\]

The function \( g \) as well as the pair \( M = (g, k) \) is called a measure. A measure and a constant \( \lambda \in \mathbb{N} \) such that \( g(k) \geq \lambda \) generate a characteristic function \( v = v^M_\lambda : \{s \leq k\} \to \{0, 1\} \) on the profiles of \( k \) via

\[
(4) \quad v(s) = \begin{cases} 
1 & g(s) \geq \lambda \\
0 & g(s) < \lambda 
\end{cases} \quad (s \leq k)
\]

The familiar framework of n-person cooperative game theory is easily obtained; put \( n = \sum_{i=0}^{r} k_i \) and \( \Omega = 1, \ldots, n \). Decompose

\( \Omega = K_0 + K_1 + \ldots + K_r \) (\( + = \) "disjoint union") such that \( |K_i| = k_i \). \( \Omega \) is the "set of players" and any coalition \( S \subseteq \Omega \) has a profile \( s = (|S \cap K_0|, \ldots, |S \cap K_r|) \). Then \( (M, \lambda) = (g, k; \lambda) \) induce a cf. (in the familiar sense) say, by

\[
w(S) = v^M_\lambda (|S \cap K_0|, \ldots, |S \cap K_r|) = v(s).
\]
Thus, players in $K_i$ have the same weight $g_i$ and there are $k_i$ players with this property. Therefore, $i \in \{0, \ldots, r\}$ (or $K_i$) is called a **fellowship**. $\lambda$ is the "majority level".

A profile $s \preceq k$ is winning if $v(s) = 1$ (losing otherwise) and minimal winning if any winning profile $t \preceq s$ satisfies $t = s$.

Profiles will be ordered **lexicographically** (from right to left, i.e., $s$ precedes $s'$ if $s_\rho > s'_\rho$ and $s_i = s'_i$ ($i > \rho$)). The **lex-max** profile is the lexicographically first min-win profile (containing the largest fellows).

A pair $M = (g,k)$ is said to be **homogeneous** w.r.t. $\lambda \in \mathbb{N}$ if $g(k) > \lambda$ and

\[
\text{For any } s \preceq k \text{, } g(s) > \lambda \text{ there is } t \preceq s \text{ such that } g(t) = \lambda
\]

We use the notation $M \text{ hom } \lambda$ in order to indicate homogeneity; $M \text{ hom } 0 \lambda$ means that either $M \text{ hom } \lambda$ or $g(k) < \lambda$.

Assume that $M \text{ hom } \lambda$ and let us construct the **lex-max** profile, say $s_0^\lambda$, by collecting first the weights of the largest fellowship, then those of the second largest fellowship etc. until we have the total mass $\lambda$ is combined. By homogeneity, the majority level is indeed exactly hit by this procedure, i.e., there is $i_0 \in \{1, \ldots, r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$ such that

\[
\lambda = cg_{i_0} + \sum_{i=i_0+1}^{r} k_i g_i
\]

and

\[
s_0^\lambda = (0, \ldots, 0, c, k_{i_0+1}, \ldots, k_r)
\]
The remaining players (fellowships) constitute a smaller measure which is a projection of \( M = (g,k) \); this measure is (for \( c < k_{i_0} \))

\[
M^{C}_{i_0} := (g_0, g_1, \ldots, g_{i_0}; k_0, k_1, \ldots, k_{i_0} - c)
\]

or (if \( c = k_{i_0} \))

\[
k_{i_0} = M^{0}_{i_0} := (g_0, g_1, \ldots, g_{i_1}; k_0, k_1, \ldots, k_{i_1 - 1})
\]

Now the remaining players may try to replace a member of the larger fellowships (those already engaged in the lex-max profile) in order to enter a min-win coalition (profile); more precisely, the measure \( M^{C}_{i_0} \) and the weight \( g_j \) \( (j > i_0) \) constitute a weighted majority game which, as it turns out, is a homogeneous one. Indeed we have

**Lemma 1.1.** (The BASIC LEMMA, see [8])

Let \( M = (g,k) \) satisfy (1) and (2) and let \( \lambda \in \mathbb{N}, g(k) \geq \lambda \). Then \( M \) hom \( \lambda \) if and only if there is \( i_0 \in \{1, \ldots, r\} \) and \( c \in \mathbb{N}, 1 \leq c \leq k_{i_0} \) such that (6) is satisfied and

\[
M^{C}_{i_0} \text{ hom } g_j \quad (i_0 \leq j \leq r)
\]

holds true.

Note that \( M^{C}_{i_0} \text{ hom } g_{i_0} \) is equivalent to \( M^{0}_{i_0} \text{ hom } g_{i_0} \).

Let us introduce the following notations

\[
\mathcal{M}^r := \{(g,k) \in \mathbb{N}_0^{2(r+1)} \mid k \text{ satisfies (1)} \quad \text{and } g \text{ satisfies (2)}\}
\]
\( (11) \)

\[
\mathcal{M}^0 := \{(0, k_0) \mid k_0 \in \mathbb{N}_0 \}
\]

\[
\mathcal{M} = \bigcup_{r=0}^{\infty} \mathcal{M}^r
\]

For any \( M \in \mathcal{M}^r \), \( m \) denotes "total mass", i.e.

\[
m = \sum_{i=1}^{r} k_i g_i
\]

and indices are carried accordingly, i.e.

\[
i_0 - 1
\]

\[
m_{i_0}^c = \sum_{i=1}^{i_0 - 1} k_i g_i + (k_{i_0} - c) g_{i_0}
\]

etc.

**Definition 1.2.** Let \( M = (g, k) \in \mathcal{M}^s \) and for \( 1 \leq r \leq s \) consider \( M_r = (g_0, g_1, \ldots, g_r; k_0 k_1, \ldots, k_r) \).

For \( 1 \leq i_0 \leq r \) and \( 1 \leq c \leq k_{i_0} \) consider

\[
\lambda_{i_0}^c = \lambda_{i_0}^c (g, k) = c g_{i_0} + \sum_{i=i_0+1}^{r} k_i g_i
\]

and

\[
c_{i_0}^r = c_{i_0}^r (g, k) = \min \{ c \in \mathbb{N} \mid 1 \leq c \leq k_{i_0}, M_r \text{ hom } \lambda_{i_0}^c \}
\]

(where \( \min \emptyset = \infty \)). Then \( C = (c_{i_0}^r)_{1 \leq i \leq s} \)

\[
C_{i_0}^r = 0 \quad \text{for} \quad r < i
\]

is called the **matrix of homogeneity** of \( M = (g, k) \).

**Lemma 1.3.** Let \( M \in \mathcal{M}^s \). For \( 1 \leq i_0 \leq r \leq s \) and \( 1 \leq c \leq k_{i_0} \),

\[
M_r \text{ hom } \lambda_{i_0}^c
\]

if and only if \( c \geq c_{i_0}^r \).
Proof: This follows from the BASIC LEMMA since \( M^C_{i_0} \) hom \( g_j \) for 
\[ j > i_0 \] implies \( M^C_{i_0} \) hom \( g_j \); see Lemma 2.1. of [8]

Thus, we have now slightly changed our view point: Fix \( M \in \mathfrak{M}^S \) and 
consider any "projection" \( M_r \in \mathfrak{M}^R \) \((r \leq s)\). The numbers \( \lambda = \lambda^C_{i_0} \) are 
the only candidates such that \( (M_r, \lambda^C_{i_0}) \) generates a "homogeneous" cf. 
\( \lambda^C_{i_0} \), i.e., such that \( M_r \) hom \( \lambda^C_{i_0} \). Knowledge of the matrix \( C \) is 
sufficient in order to decide whether \( M \) hom \( \lambda^C_{i_0} \) holds true. Now, 
Lemma 1.1. suggests that homogeneity is a property which is acquired 
or disturbed by a recursive procedure.

As has been shown in [8], \( C \) also allows for a recursive computation.

Our present purpose is to exhibit regularity properties of the matrix \( C \). 
This will provide faster algorithms for the actual computation of \( C \). 
Besides, as has been elaborated in [5], [8], [9] the recursive structure 
allows for the definition of certain characters of players (as well as 
fellowships and types) in a homogeneous game. These types are the 
(familiar) dummy, the sum and the step. It will turn out that some 
properties of \( C \) also reflect the "strength" of certain players (fellow- 
ships) in the corresponding games in a way such that the character of a 
fellowship may be decided upon by inspection of \( C \).

Thus, the matrix \( C \) at once yields all homogeneous games that may result 
from any projection \( M_r \) of \( M \in \mathfrak{M}^S \) and shows something about the 
strength of the players in these games. Fast algorithms in order to 
compute \( C \) are therefore considered to be desirable.
SEC. 2  Properties of C

Lemma 1.3 suggests that, given $M = (g, k) \in \mathcal{K}^S$, it suffices to know the matrix of homogeneity $C$ in order to specify all "majority levels", $\lambda \in \mathbb{N}$ such that $M \hom C \lambda$. Let us, therefore, exhibit some properties of this matrix that will turn out to be useful for a recursive computation. Parts of these arguments we shall quote from [8], however, we want to use the fact that "many" entries of the matrix $C$ are not finite, thus providing eventually a more effective algorithm.

Lemma 2.1.  (cf. 2.3. of [8])

Let $M \in \mathcal{K}^S$. Then, for $1 \leq i_0 \leq r \leq s$, we have

$$c_{i_0}^r < \infty \text{ if and only if } M_{i_0-1} \hom g_i \quad (i = i_0, \ldots, r).$$

Proof: $c_{i_0}^r < \infty$ is equivalent to $M_r \hom \lambda_i^{C_{i_0}}$ for some $c$, $1 \leq c \leq k_{i_0}$ (Definition 1.2.) and by Lemma 1.3. this is equivalent

$$M_r \hom \lambda_i^{C_{i_0}} = \sum_{i = i_0}^{k_{i_0}} k_i g_i$$

Finally, by Lemma 1.1., this is equivalent to

$$M_{i_0-1} \hom g_i \quad (i = i_0, \ldots, r)$$

Lemma 2.2.  (cf. 2.3. of [8])

Let $M \in \mathcal{K}^S$. Then

$$c_1^1 = 1$$

and, for $r \geq 2$

$$c_r^r = \begin{cases} 1 & \text{if } M_{r-1} \hom g_r \\ \infty & \text{otherwise.} \end{cases}$$
Proof: \( c_{i_0}^1 = 1 \) follows from the definition of \( C \). In order to check (3), observe that Lemma 2.1. implies that
\[ c_r^r < \infty \text{ if and only if } M_{r-1} \text{ hom}_o g_r. \text{ But } M_{r-1} \text{ hom}_o g_r \text{ implies } M^C_r \text{ hom}_o g_r \text{ for all } c \in \mathbb{N}, 1 \leq c \leq k_r, \text{ q.e.d.} \]

Next, let us define the quantity

\[ (4) \quad \gamma_{i_0}^r := \min \{ c \mid 1 \leq c \leq k_{i_0}, M^C_{i_0} \text{ hom}_o g_r \}, \]

then, using \( \alpha \lor \beta \) in order to denote the maximum of reals \( \alpha \) and \( \beta \), we have

**Lemma 2.3.** (cf. [8])

Let \( \mathcal{M} \in M^S \) and \( 2 \leq r \leq s \). For \( i_0 < r \) it follows that

\[ (5) \quad c_{i_0}^r = c_{i_0}^{r-1} \lor \gamma_{i_0}^r \]

**Proof:** Obvious, in view of Lemma 1.1. we have

\[ c_{i_0}^r = \min \{ c \in \mathbb{N} \mid 1 \leq c \leq k_{i_0}, M^C_{i_0} \text{ hom}_o g_i \text{ } (i=i_0, \ldots, r) \} \]

\[ = \min \{ c \in \mathbb{N} \mid 1 \leq c \leq k_{i_0}, M^C_{i_0} \text{ hom}_o g_i \text{ } (i=i_0, \ldots, r-1) \} \]

\[ \lor \min \{ c \in \mathbb{N} \mid 1 \leq c \leq k_{i_0}, M^C_{i_0} \text{ hom}_o g_r \} \]

\[ = c_{i_0}^{r-1} \lor \gamma_{i_0}^r. \]

Denote by \( [\alpha] \) the largest integer less than or equal to \( \alpha \) and put, for \( 1 \leq i \leq s \)

\[ (7) \quad l_i^i := \begin{cases} 
1 & \text{if } g_{i_1} \mid g_i \text{ or } k_{i_1} g_{i_1} \leq g_i \\
& \text{otherwise.}
\end{cases} \]
Lemma 2.4. (cf. [8])

Let \( M \in \mathcal{M}^S \) and \( 1 < r \leq s \). Then

\[
  c_{1}^{r} = c_{1}^{r-1} \vee l_{1}^{r}.
\]  

(8)

Proof: This follows from 2.3., since

\[
  \gamma_{1}^{r} = \min \{ c \mid 1 \leq c \leq k_{1}, M_{1}^{c} \text{ hom}_{o} g_{r} \}
  = \min \{ c \mid 1 \leq c \leq k_{1}, g_{1} \mid g_{r} \text{ or } k_{1}g_{1} < g_{r} \}
  = l_{1}^{r}
\]

Corollary 2.5. Let \( M = (g,k) \in \mathcal{M}^S \) and let

\[
  C = C(M) = (c_{i}^{r})_{1 \leq i \leq r \leq s}
\]

denote the matrix of homogeneity of \( M \).

Each column of \( C \) is monotone increasing. If there are finite entries in a column at all, then the first entry equals 1.

This is obvious since monotonicity follows from (8) and the shape of the first entry (the diagonal element of \( C \)) is specified by (3).

Corollary 2.6. Let \( M \in \mathcal{M}^S \) and \( 2 < r \leq s \). Suppose that

\[
  c_{1}^{r} > c_{1}^{r-1}.
\]

Then

\[
  c_{2}^{r} = \ldots = c_{r}^{r} = \infty
\]

(9)

(and, by monotonicity, \( c_{i}^{j} = \infty \) for \( j \geq r \) and \( 2 \leq i \leq r \)).
Proof: If \( c_i^r > c_i^{r-1} \), then \( \gamma_1^r = \gamma_1^{r-1} > 1 \); i.e., in view of (7)

\[ g_1 \neq g_r \quad \text{and} \quad k_1 g_1 > g_r \]

It follows that, for any \( i_0, 2 \leq i_0 \leq r \),

\[ M_{i_0-1} h \in \mathbb{L}_{i_0} g_r \]

That is, \( c_{i_0}^r = \infty \) by Lemma 2.1., q.e.d.

Consider the matrix \( C = C(M) \). Given \( r, i_0 \), we may say that the entries \( c_{j}^{i}, j \geq r, i_0+1 \leq i \leq r \) constitute the "south-east-stripe" of \( c_{i_0}^r \). Thus, whenever there occurs a jump in the first column, then the entries in the south-east-stripe are all \( \infty \); this suggests vaguely the following form of \( C \):

```
1
... ...
1 ...
1 \infty \infty \infty
```

Now, it turns out, that the principle is a general one: whenever jumps occur in any column, then the south-east-stripe is rendered \( \infty \).

**Theorem 2.7.** Let \( M \in \mathcal{R}^s \) and \( C \) the matrix of homogeneity. Whenever, for some \( 2 \leq i_0 \leq r-1 \leq r \leq s \),

\[ c_{i_0}^r > c_{i_0}^{r-1} \]

then
(10) \[ c_{i_0}^r = \ldots = c_r^r = \infty \]

(and, by monotonicity \( c_j^j = \infty \) for \( j \geq r \), \( i_0 + 1 \leq i \leq r \)).

**Proof:** Let \( c = c_{i_0}^{r-1} \). Then we have

\[ M_{i_0}^C \hom \rightarrow g_j \quad (i_0 \leq j \leq r - 1) \]

(see e.g. formula (6)). On the other hand, we have \( \gamma_{i_0}^r > c \), thus

\[ M_{i_0}^C \hom \rightarrow g_r \]

Now, \( M_{i_0}^C \) is a projection of \( M_{i_0}^C \) and hence we have a fortiori

\[ M_{i_0} \hom \rightarrow g_r \]

which by Lemma 2.1. implies \( c_{i_0}^{r-1} = \infty \). Similarly, for any \( i \), \( i_0 \leq i \leq r \), \( M_{i_0}^C \) is a projection of \( M_i \) and thus \( M_i \hom \rightarrow g_r \), implying \( c_i^r = \infty \), q.e.d.

**Remark 2.8.** The computational procedure for obtaining the matrix \( C \)

is greatly simplified by the fact that frequently \( c_{i_0}^r = \infty \)

is implied by the "south-east-stripe" rule indicated via Theorem 2.7.

However, in some cases we actually need to compute \( c_{i_0}^r \) given that the

entries \( c_1^{r'} \) are already known for \( r' = r \), \( i < i_0 \) and for \( r' < r \), \( i \leq r' \). To this end we proceed as follows.

2.8. A. If \( i_0 = r \), then we know that \( c_r^r = 1 \) if and only if

\[ M_{r-1} \hom \rightarrow g_r \]. The matrix of homogeneity w.r.t. \( M_{r-1} \) is

(recursively) known; this is \((c_i^{r'})_{i \leq i < r' \leq r-1} \).
2.8.B. Consider the case that $1 < i_0 < r$. We may assume $c_{i_0}^i = 1$ and $c_{i_0}^r < \infty (i_0 \leq r < r-1)$, for otherwise we have $c_{i_0}^r = \infty$ by monotonicity. Thus we have

\[(11) \quad M_{i_0}^{i-1} \hom g_{i_0}\]

(by Theorem 2.1.). Given this hypothesis, we have to compute

\[(12) \quad \gamma_{i_0}^r = \min \{c \in \mathbb{N} \mid 1 \leq c \leq k_{i_0}, M_{i_0}^c \hom g_r\} .\]

To this end we first apply a test in order to check whether

\[(13) \quad M_{i_0}^{i-1} \hom g_r\]

holds true. (Again, the C-matrix for $M_{i_0}^{i-1}$ is already known)

2.8.B.a. If the answer is \textbf{no} and $M_{i_0}^{i-1} \hom g_r$ then

\[\gamma_{i_0}^r = \infty\]

as $M_{i_0}^{i-1} = M_{i_0}^c$.

2.8.B.b. If the answer is \textbf{yes} and $M_{i_0}^{i-1} \hom g_r$, then, by the same reasoning we have $\gamma_{i_0}^r \leq k_{i_0}$ (i.e. $c = k_{i_0}$ is admitted in (12); thus, in particular the "min" operation in (12) is not taken w.r.t. the empty set). Our next test consists of a check whether

\[(14) \quad g_{i_0} \mid g_r\]

holds true.
2.8.B.b.a. If "yes", and \( g_{i_0} \mid g_r \), then (13) implies \( M_{i_0}^1 \hom g_{i_0} \) and hence
\[
\gamma_{i_0}^r = 1.
\]
Indeed, if \( M_{i_0}^{i_0-1} \hom g_{i_0} \) then \( M_{i_0}^1 \hom g_r \) is trivial and if \( m_{i_0}^{i_0-1} < g_{i_0} \) then \( M_{i_0}^1 \hom g_r \) is a simple exercise.

2.8.B.b. If "no", and \( g_{i_0} \not\mid g_r \), then we call upon Lemma 3.4. in [8] which tells us that, given the present conditions
\[
M_{i_0}^c \hom g_r
\]
is equivalent to
\[
M_{i_0}^{i_0-1} \hom (g_r - (k_{i_0} - c) g_{i_0})
\]
if \( (k_{i_0} - c) g_{i_0} < g_r \), that is, if \( c g_{i_0} > k_{i_0} g_{i_0} - g_r \).

Now, for \( c = k_{i_0} \) this is satisfied, indeed, we know already that
\[
\gamma_{i_0}^r < k_{i_0}.
\]
Hence, for \( c = k_{i_0} - 1, k_{i_0} - 2, \ldots, 1 \) let us check whether
\[
(k_{i_0} - c) g_{i_0} < g_r
\]
and
\[
M_{i_0}^{i_0-1} \hom (g_r - (k_{i_0} - c) g_{i_0})
\]
are simultaneously satisfied. Again the \( c \)-matrix for \( M_{i_0}^{i_0-1} \) is known.

Once for some \( c \) the answer is "no" we put \( \gamma_{i_0}^r = c + 1 \). If the answer is "yes" for all \( c \), we put \( \gamma_{i_0}^r = 1 \).
Presumably, it is preferable to check for \( t = 1, 2, \ldots, k_{i_0} - 1 \) (i.e. \( t = k_{i_0} - c \)) whether

\[
(17) \quad t g_{i_0} < g_r
\]

\[
(18) \quad M_{i_0-1} \hom_0(g_r - t g_{i_0})
\]

If, for some \( t \) the answer is "no" put

\[
(19) \quad \gamma^r_{i_0} = k_{i_0} - t + 1
\]

and if the answer is "yes" always, put

\[
\gamma^r_{i_0} = 1.
\]

**Remark 2.9.** Any test for homogeneity, given that the matrix \( C \) is known, takes place according to Lemma 1.3. That is, given \( M_r \) and \( \lambda \), check first whether there is \( i_0, 1 \leq i_0 \leq r \) and \( c, 1 \leq c \leq k_{i_0} \) such that \( \lambda = \lambda_{i_0}^c \). Then check whether \( c \geq c_{i_0}^r \).
SEC. 3  The algorithm

Collecting the pieces we now want to describe an algorithm for the matrix of homogeneity of a given measure \( M = (g,k) \).

Now, for the sake of a consistent representation it is useful to carry a fellowship with players having weight 0; i.e. to consider \( k \) and \( g \) as specified by (1) and (2) of SEC. 1, where \( g_0 = 0 \). However, for the present algorithm this is not necessary; thus we deal with \( g = (g_1, \ldots, g_r) \in \mathbb{N}^r \) and \( k = (k_0, \ldots, k_r) \in \mathbb{N}^r \).

The algorithm is described in terms of "functions" defined on vectors of integers. The essential one is the last function, called \( CE \), which yields the matrix \( C \). However, the functions defined by I, II, III are necessary because of the recursive nature of our procedure.

I. Function IOC \((g; k; \lambda)\).

Entries: \( g = (g_1, \ldots, g_r) \in \mathbb{N}^r \); \( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \); \( \lambda \in \mathbb{N} \).

Output: \( (i_0, c) \in \mathbb{N}_0 \times \mathbb{N}_0 \).

Task: Determines \( i_0, c \) such that \( \lambda = \lambda_{i_0}^c \) or otherwise reports failure if no such quantities exist.

Procedure:

1. Choose \( i_0 \in \mathbb{N} \) such that

\[
\sum_{i=i_0}^{i+1} k_i g_i < \lambda \leq \sum_{i=i_0}^{i+1} k_i g_i
\]

2. Put \( \Delta := \lambda - \sum_{i=i_0}^{i+1} k_i g_i \)
3. If \( g_{i_0} \mid A \), then \( \Rightarrow 4 \); otherwise \( \Rightarrow 5 \)

4. Put \( c := \frac{A}{g_{i_0}} \); put IOC \((g; k; \lambda) = (i_0, c) \Rightarrow 6 \)

5. Put \( \text{IOC} (g; k; \lambda) = (0, 0) \Rightarrow 6 \)

6. END.

II. Function HOMN \((g; k; \lambda; c, )\)

Entries: \( g = (g_1, \ldots, g_r) \in \mathbb{N}^r \)
\( k = (k_1, \ldots, k_r) \in \mathbb{N}^r \)
\( \lambda \in \mathbb{N} \)
\( c. = (c_1, \ldots, c_r) \in \mathbb{N}^r \)

Output: \( + \) or \( - \) (HOMN is a "Boolean function")

Task: Given (the last row of the matrix C) \( c. \), HOMN decides whether \( M = (g, k) \text{ hom}_0 \lambda \) or not.

Procedure:
1. If \( \sum_{i=1}^{r} k_i g_i \leq \lambda \), then put \( \text{HOMN} (\ast, \ast, \ast, \ast) = + \) and \( \Rightarrow 4 \). Otherwise \( \Rightarrow 2 \).

2. If \( \text{IOC} (g; k; \lambda) = (0, 0) \), then put \( \text{HOMN} (\ast, \ast, \ast, \ast) = - \) and \( \Rightarrow 4 \).
   Otherwise put \( (i_0, c) := \text{IOC} (g; k; \lambda) \) and \( \Rightarrow 3 \).
3. If \( c \geq c_{i_0} \), then put \( \text{HOMN} (\cdot, \cdot, \cdot, \cdot, \cdot) \)
\[ = + \] and \( \Rightarrow 4 \). Otherwise, put \( \text{HOMN} (\cdot, \cdot, \cdot, \cdot, \cdot) \)
\[ = - \] and \( \Rightarrow 4 \).

4. END.

III. Function \( \text{GAM} (g; k; g_{i_0}; g_r; c_r) \)

Entries:
\[
g = (g_1, \ldots, g_{i_0-1}) \in \mathbb{N}_{i_0-1}^i
\]
\[
k = (k_1, \ldots, k_{i_0-1}) \in \mathbb{N}_{i_0-1}^i
\]
\[
g_{i_0}, g_r \in \mathbb{N}
\]
\[
c_r = (c_1, \ldots, c_{i_0-1}) \in \mathbb{N}_{i_0-1}^i
\]

Output:
\[
\gamma_{i_0}^r \in \mathbb{N} \cup \{\infty\}
\]

Task: Computes \( \gamma_{i_0}^r \) given row \( i_0 \) of \( C \).

Procedure:
1. If \( \text{HOMN} (g; k; g_r; c_r) = - \), then put \( \text{GAM} (\cdot, \cdot, \cdot, \cdot, \cdot) = \infty \)
and \( \Rightarrow 9 \). Otherwise \( \Rightarrow 2 \).

2. If \( g_{i_0} \mid g_r \), then put \( \text{GAM} (\cdot, \cdot, \cdot, \cdot, \cdot) = 1 \) and \( \Rightarrow 9 \).
Otherwise \( \Rightarrow 3 \).

3. Let \( t = 1 \) and \( \Rightarrow 4 \).

4. If \( t g_{i_0} < g_r \) and \( \text{HOMN} (g; k; g_r - tg_{i_0}; c_r) = + \),
then \( \Rightarrow 5 \). Otherwise \( \Rightarrow 7 \).

5. \( t \rightarrow t + 1 \); \( \Rightarrow 6 \).
6. If \( t \leq k_{i_0} - 1 \) then \( \Rightarrow 4 \). Otherwise \( \Rightarrow 7 \).

7. Put \( \text{GAM} (\ast, \ast, \ast, \ast, \ast) = 1 \) and \( \Rightarrow 9 \).

8. Put \( \text{GAM} (\ast, \ast, \ast, \ast, \ast) = k_{i_0} - t + 1 \) and \( \Rightarrow 9 \).

9. END.

IV. Function CE \((g, k)\)

Entries: \( g = (g_1, \ldots, g_s) \in \mathbb{N}^s \)
\( k = (k_1, \ldots, k_s) \in \mathbb{N}^s \)

Output: Matrix \( C = (c_{i,r})_{i,r=1,\ldots,s} \in \mathbb{N}^{s \times s} \)
\( c_{i,r} = 0 \ (i > r) \) (or \( C \) triangular and \( c_{i,r} \) not defined for \( i > r \))

Procedure:
1. Put \( c_{i}^r = 0 \) for all \( i \) and \( r \).
2. Put \( c_{1}^1 = 1 \)
3. For \( j = 1, \ldots, s \) put \( 1_{j}^1 = 1 \) if \( k_1 g_1 < g_j \) or \( g_1 \mid g_j \).
   Otherwise, put \( 1_{j}^1 = k_1 - \frac{g_j}{g_1} \).
4. If \( k_1 g_1 < g_2 \) or \( g_1 \mid g_2 \), then put \( c_{1}^2 = 1 \) and \( c_{2}^2 = 1 \).
   Otherwise put \( c_{1}^2 = 1^2 \) and \( c_{2}^2 = m \).
5. Put \( r = 3 \) and \( i_0 = 1 \).
6. If \( c_{i_0}^r = m \), then \( \Rightarrow 12 \). Otherwise \( \Rightarrow 7 \).
REFERENCES


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