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A NOTE ON "AN IMPOSSIBILITY RESULT CONCERNING N-PERSON BARGAINING GAMES"

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In a recent paper published in this journal A. E. ROTH [1979] demonstrated that the axiomatization of E. KALAI and M. SMORODINSKY [1975] of solutions for two person bargaining games cannot be generalized to more than two persons. This result should not pass without due qualification. We are able to provide a straightforward generalization of the KALAI-SMORODINSKY axiomatization if only the system of feasible bargaining games is appropriately restricted. The difference between the general case and the two person case thus seems to be less "fundamental" (ROTH [1979, p. 132]) than ROTH's counterexample suggests.
1. A POSSIBILITY RESULT

We employ ROTH's [1979] definitions and notation. A \( n \)-person bargaining game \((S,d) \in B\) is then given whenever \( S \subseteq \mathbb{R}^n\) is compact and convex and if some \( x \in S \) exists with \( x > d \in S\). In contrast with this general definition we shall restrict attention to some subset \( B_0 \subseteq B\). However, we first need some further notational conventions.

Let \( \bar{x} = \bar{x}(S,d) \) denote the ideal point. Put \( A = A^{S,d} : \mathbb{R}^n \to \mathbb{R}^n\), \( A_i(x) = (x_i - d_i) / (\bar{x}_i - d_i) \) for \( i \in \mathbb{N}\). Then \( A \) denotes the affine transformation of utilities normalizing threat- and ideal point: \( A(d) = 0 \) and \( A(\bar{x}) = e = (1, \ldots, 1) \). Note that \((S,d) \in B\) implies \( \bar{x} > d \).

Let \((S,d)\) be an element of \( B_0 \) by definition iff

\[
d \leq y \leq x \in S \implies y \in S \tag{1}\]

and

whenever \( x \in S \) can be written as \( x = \lambda \bar{x} + (1 - \lambda)d \) for some \( \lambda \in [0,1] \) and there exists \( y \in S \) such that \( y \geq x \), \( y \neq x \) then there also exists \( z \in S \) such that \( z > x \).
(2) is less restrictive than to consider only such (S,d) for which the sets of strong and weak PARETO-optimal points coincide. Note that the application of any affine transformation of utilities $\bar{A} : \mathbb{R}^n \to \mathbb{R}^n$ with $\bar{A}_i(x) = \alpha_i x_i + \beta_i$; $\alpha_i, \beta_i \in \mathbb{R}$, and $\alpha_i > 0$ does not lead out of $B_0$.

Now we define the solution $G : B_0 \to \mathbb{R}^n$ by

$$G(S,d) := \arg(\max_{x \in S} \min_{i \in \mathbb{N}} A_i(x))$$

Hence $G(S,d) \in S$ maximizes $\min_{i \in \mathbb{N}} A_i(x)$. Let $g = G(S,d)$ for fixed $(S,d) \in B_0$. We claim:

There is some $\gamma \in [0,1]$ s.t.

$$A_i(g) = \gamma \quad \text{for all } i \in \mathbb{N}.$$  \hspace{1cm} (4)

This is the more conventional definition of KALAI-SMORODINSKY’s monotone solution. ((4) might also be written as $g = \gamma \bar{x} + (1 - \gamma)d$.)

Suppose that in contradiction to (4)

$$0 < A_j(g) = \min_{i \in \mathbb{N}} A_k(g) < A_i(g) \leq 1.$$  

Then define $y := A_j(g)e$. Note first that $y \in S$ by (1). Secondly, $y$ can be written as

$$y = A_j(g)e + (1 - A_j(g))0.$$  

So we may apply property (2) which yields some $x \in S$ with $x > y = A_j(g)e$, contradicting the definition of $g$.  

Note that by (4) it is clear that there is a unique maximizer of $\min_{i \in \mathbb{N}} A_i(x)$. Thus $G(S,d)$ is a well defined mapping. Let us now state the

**THEOREM:** $G : B_0 \to \mathbb{R}^n$ satisfies PARETO-optimality, symmetry, independence of affine transformations of utilities, and restricted monotonicity. Furthermore $G$ is uniquely determined by these properties (1,2,3,5; see ROTH [1979], p. 130 - 131).

**PROOF:** Let us only show PARETO-optimality and restricted monotonicity for $G$.

Assume that there exists some $x \in S$ with $x \geq g$, $x \neq g$. (4) implies that the assumptions of (2) are fulfilled so that there is some $z \in S$ with $z > g$, contradicting (3).

To show the restricted monotonicity let us assume $(T,d), (S,d) \in B_0$, $T \subseteq S$, $\bar{x} = \bar{x}(T,d) = \bar{x}(S,d)$. Then

$$\max_{x \in T} \min_{i \in \mathbb{N}} A_i(x) \leq \max_{x \in S} \min_{i \in \mathbb{N}} A_i(x)$$

By (4) we obtain $A_i(G(T,d)) \leq A_i(G(S,d))$ for all $i \in \mathbb{N}$ which in turn implies $G_i(T,d) \leq G_i(S,d)$. 


To show that $G$ is uniquely determined we essentially reproduce KALAI-SMORODINSKY's argument [1975, p. 517]. Let $f : B_0 \rightarrow \mathbb{R}^n$ meet all noted axioms. Fix $(S, d) \in B_0$ where $x(S, d) = e$, $d = 0$, w.l.o.g.

We have to show $f(S, 0) = G(S, 0)$.

By (4) $G(S, 0) = \gamma e$ with $\gamma \in [0, 1]$. Define $T \subset S$ to be the convex hull of $0, e^1 = (1, 0, \ldots, 0), \ldots, e^n, \gamma e$. Clearly, $(T, 0) \in B_0$. The symmetry of $T$ implies $f(T, 0) = \gamma e = G(S, 0)$.

The restricted monotonicity requires $f(S, 0) \succeq f(T, 0) = G(S, 0)$, but as $G(S, 0)$ is PARETO-optimal in $S$ we obtain $f(S, 0) = G(S, 0)$, which was to be shown.

2. A HISTORICAL NOTE

ROTH quotes several papers where the solution $G$ has been discussed before. We would like to add to this list some prominent nineteenth century's economists like F. Y. EDGECROWTH [1897] and A. J. COHEN-STUART [1889]. The solution $G$ is no invention of Game Theory but of Public Finance. In the latter field this solution is better known as equal proportional sacrifice.
To see this relationship we have to say a word on the problem of normatively justifying progressive taxation. Consider \( n \) tax payers with incomes \( y_i > 0 \) before, \( y_i - t_i \) after taxes. \( i \in N \) derives utility from money income according to some utility function \( u_i : \mathbb{R}_+ \rightarrow \mathbb{R} \) with neoclassical properties \( (u_i' > 0, u_i'' \leq 0) \). Denote \( y = (y_1, \ldots, y_n) \), \( t = (t_1, \ldots, t_n) \), \( u = (u_1, \ldots, u_n) \). The set of all feasible tax distributions is assumed to be 
\[ t := \{ t \in \mathbb{R}^n \mid \sum_{i=1}^n t_i = y, \quad 0 \leq t \leq y \}. \]
The tax revenue \( y \) is exogenously fixed.

The classical question is: Which properties characterize a just and equitable distribution of taxes \( t^* \in t \)? The classics more specifically tried hard to justify and designate some tax function \( t^* : ((u, y)) \rightarrow t \) holding (at least generically)
\[ 1 > at_i^* / ay_i > t_i^* / y_i \quad \text{for all} \quad i \in N \quad (5) \]

independently of the utility functions under consideration. The right hand inequality of (5) defines progressivity of \( t_i^* \) with respect to \( y_i \) in a local sense.

It is enough to say that neither the classics nor their followers solved the problem. They confined
themselves to a discussion of several specific choice functions ("sacrifice concepts") among which

\[
(u_i(y_i) - u_i(y_i - t^*)) / u_i(y_i) = \text{const.}
\]

(4')
is known to be the equal proportional sacrifice. Setting \( d := u(0) \),

\[
S := \{ x \in \mathbb{R}^n \mid \exists t \in T : d \leq x \leq (u_i(y_i - t_i))_{i \in N} \}
\]

and assuming \( u(0) = 0 \) (by transformation invariance) we realize that (4') is equivalent to the application of solution \( G \) to \((S, 0)\). By the way, the tax problem yields a nice convincing interpretation of the ideal-point \( \bar{x}(S, 0) \) as the utility of income before taxes: \( \bar{x}_i = u_i(y_i) \).

Obviously, \((S, 0) \in B_0\): Convexity of \( S \) follows by concavity of \( u_i(\cdot) \). \( S \) is compact as \( t \) shares this property and as the \( u_i(\cdot) \) are continuous. Finally \((S, 0)\) holds (2) because of \( u_i' > 0 \). Consequently all bargaining games derived in this way from the task of distributing tax shares are in \( B_0 \).

It might be of general interest that the Dutch COHEN-STUART [1889] managed to construct some "nice" utility function such that the progressivity condition (5) is violated under solution \( G \) ((4')). The determination and axiomatic characterization of choice functions \( t^* \) holding (5) is relegated to two forthcoming papers (RICHTER [1980 a,b]).
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