Information Uncertainty in Auction Theory

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Except where otherwise indicated, this thesis is my own original work.

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In memory of my father Mirko
To my mother Ljiljana and my brother Novak

Summary

For a long time in the literature, the usual way of modeling an auction participant's beliefs has been exogenous - the distribution of private value is simply given, fixed, commonly known and out of any doubt for the involved participants. Growing awareness on information uncertainty and its strong influence on decision choices and social outcomes is the reason why we focus on introducing information uncertainty in auction theory. In particular, we allow involved subjects to question underlying information structure. We propose three different ways to design the endogenous belief formation process and explore its influence on state of the art in auction theory. This research approach results in three different setups, each examined in depth in a separate chapter.

Chapter 2 shows the importance of our modeling approach. The genericity of an optimal auction format - in the literature known as the Crémer-McLean auction - is challenged and its robustness against a change of information structure is examined. We show that participation constraints in the auction format fail to hold once the seller and bidders have slightly different beliefs over the joint distribution of private values. We propose the definition of a *belief neighborhood* in order to capture this slight discrepancy between seller's and bidders' beliefs. Moreover, we also give a quantitative assessment of how often the failure occurs in the case of a common prior assumption as well as once the common prior assumption among bidders is relaxed. In particular, participation constraints fail in at least one half of any plausible belief's neighborhood. Once the common prior assumption among bidders is relaxed, the failure occurs everywhere except in $1/2^{2m}$ of a belief's neighborhood, where m is the number of possible private values for each bidder.

The *origin* of private values in auction models is examined in Chapter 3. In contrast with the standard literature, we let participants question the information structure involved in auctions. We focus on the most prevalent forms of sealed-bid auctions on the market: the first-price and second-price auctions. Instead of being exogenously given, we permit the observed distribution of private value to take the form of mixture probability distribution, which in turn allows participants to speculate on the true distribution of the values. The speculations are done in accordance with Bayes' rule - the bidders use their private value to infer the distribution, whereas the seller bases his belief on an observed mixture form of the values' distribution. We explore the consequences of allowing this particular belief formation on bidders' and seller's behavior. First of all, despite the independently and identically distributed values, we show that the belief formation leads to the revenue equivalence failure in such a way that the *seller* prefers the *second-price auction*. Moreover, in our setup, truthfully bidding one's own private value continues to be optimal in the second-price auction.

However, the *bidding behavior* in the first-price auction is influenced by the belief formation in the following way. Comparison of bidding strategies in the standard framework and our model leads to the statement that these two strategies cross at most once. In other words, we discover that there is either *dominance* ordering or a *single-crossing property* between the compared strategies.

The final chapter presents a model with a new kind of information uncertainty - the uncertainty about *seller's type*. Motivated by recent security and fraud issues in auctions, we develop a model where a manipulative seller has an opportunity to send an agent to bid secretly on the seller's behalf. Out of the possible auction formats, we choose the second-price auction and the all-pay auction. We look at the all-pay auction because of its design: everyone pays her own bid, irrespective of whether she wins the auction. Consequently, it seems that the manipulative seller can make the best use of his *manipulation* in the all-pay auction. In the all-pay auction, the seller may keep the object and collect all bids, which is not the case with either a second-price or first-price auction. In this way we change the standard information structure of an auction by proposing that the seller also holds private information, whether or not he is a manipulative (with an agent) or an honest seller (without an agent). Even though a priori there is a common prior over the possibility of facing a cheating seller, bidders in our model perceive the choice of auction format as a signal about the seller's intentions. Thus, our setup extends the standard auction framework to a form of signaling game. To this end, we explore the pure weak perfect Bayesian Nash equilibria of this game. We show that the only robust weak Bayesian Nash equilibrium is *pooling on the second-price*: that is, it is beneficial for both type of sellers to choose the second-price auction. In addition, there is a special non-generic equilibrium scenario in which the honest seller chooses the all-pay auction and the cheating seller chooses the second-price auction. However, this equilibrium is very unstable and not robust against the smallest change of bidders' belief on seller's honesty. Thus, it turns out that the signaling effect is stronger than the effect of manipulation. Unlike the discussion in related literature on the disadvantage of the application of the second-price auction in a setup with independently and identically distributed private values and a similar seller's manipulation (with different timing), our model favors the second-price auction.

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Contents

Chapter 1

Introduction

Auctions, or more precisely information involved in auction models is the topic covered in this thesis. We are tackling the problem regarding the source of knowledge involved in auction models. There is a gap in the literature both in explaining the thought formation process and on the origin of information possessed by participants (here also called players). On one side, it has been long assumed that knowledge is simply given and that participants always hold perfectly correct (private or not) information. On the other side, the recent literature has been going in the opposite direction: any kind of information must be taken into account and assumed as relevant for a model and the players might be uncertain about anything in the model. Our goal here is to point out and fill the gap between these two extremes and come up with models which would be, in a certain sense, closer to the reality where some pieces of information possessed by players will be correct and well-justifiable whereas some other pieces of information will be missing from players' knowledge and thus treated as a variable of the model.

1.1 Research Focus

In economics, auctions, as one of the most popular and prominent selling mechanisms have for a long time been modeled as a well-known game-theoretic concept of a certain class of games. These games involve players with private information. The economic model representing an auction is indeed nothing more than Harsanyi's game of incomplete information introduced in Harsanyi (1967). Roughly speaking, an auction might be seen as a game of incomplete information with two sorts of players: "seller" and "bidder". We focus on single-object auctions where there is typically one seller, selling a non-divisible object to two or more potential buyers or "bidders". The information structure of Harsanyi's model is represented by type spaces of players. In an auction framework, the Harsanyi's type spaces are identified by valuation spaces, where a valuation is the private information of each bidder and denotes the value which they attach to the object. All other aspects of Harsanyi's model, such as strategy space, equilibrium definition and so on, are naturally translated into the auction model.

Furthermore, the auction model has also inherited the underlying assumptions of Harsanyi's concept. One of these assumptions is the common prior assumption (CPA). It supposes that players hold identical beliefs (named common prior) about the probability of each single profile of a bidder's values. Furthermore, based on the common prior and their own private information - the valuation - a bidder acts in the model and influences the outcome of the game. Beside this assumption, the common knowledge assumption (CKA) has also been long kept as a matter of course in auction theory. The CKA is nothing more than assuming that the CPA is known by all players in the following sense: each player is aware of the common prior's existence, they know that the other players know the prior and know that the other knows that they know it and so on ad infinitum. The combination of CPA and CKA has been an inevitable element of any model describing an auction format in the literature. Usually the CPA was explicitly assumed, whereas the CKA was implicitly assumed. We question and address these assumptions in our research.

These assumptions might be a justified and appropriate description of certain real life situations, whereas they might be too restrictive and unrealistic in some other situations (for a good overview and examples, see Morris (1995)). Where does the common prior come from? Aren't the participants in the real world exposed to different kinds of information source which might cause not only different private values, but also different priors over the set of all possible value profiles? These are the questions which motivate and determine our research road map.

Related to our work, there is already a vast literature criticizing the CKA and CPA assumptions. The origin of the critique dates back to Wilson (1987) and is commonly known as "Wilson's Doctrine". The article has brought to light the problem of the CKA assumption and strong dependence of auction models that use this assumption on assumed players' knowledge and private information. The work has initiated a long series of research papers ever since and still represents one of the most popular research topics. The literature following Wilson's critique heavily employs the universal type space formally introduced by Mertens and Zamir (1985). The type space is a full mathematical description of all possible types of player, where a type is determined by an infinite consistent hierarchy of beliefs. The scope of this universal type spaces is as large as one can imagine and also incorporates types of player who do not satisfy CPA. In addition, the literature following this research agenda gives rise to the term "detail-free mechanism".[1] Loosely speaking, the detail-free is an auction which does not depend on the assumptions of players' knowledge at all. For example, an auction with an ex post dominant-strategy is an obvious instance of the detail-free mechanism.

Therefore, on one side, by employing a detail-free mechanism one overcomes the problem of assuming a specific information structure imposed on a player's knowledge and in particular avoids difficulties caused by CPA or CKA. However, on the other side, use of such a mechanism may completely ignore the eventual benefit from information which is certainly available to players of the auction. It is unclear why, for example, a seller would like

[1] Some of the pioneer works in this research area are: d'Aspremont and Gerard-Varet (1979), Dasgupta and Maskin (2000), Perry and Reny (2002), Bergemann and Morris (2005) and Chung and Ely (2007).

to employ a detail-free mechanism and thus ignore some "certain" or "almost certain" information, whose use could lead to higher expected revenue than the one obtained by the detail-free mechanism.

Overall, as we can see, there exist two extreme approaches to modeling the information structure in auctions. The earliest auction models have heavily and essentially depended on assumptions of player's perfectly correct beliefs and did not leave any space to possibility of having any slightly inaccurate information. That is, this models cannot address any form of information uncertainty. On the other side, the current literature on "robust mechanism design" completely rules out the role of private information by allowing that any potential information structure is always possible and must be taken into consideration by designing a detail-free auction. In other words, absolute information uncertainty turns out to be an inevitable building block of the models. This is the issue we want to address. We make a compromise between these two approaches by introducing auction models which involve information uncertainty with limited scope: in our setups, the knowledge of the players is uncertain, but this uncertainty is limited and partial.

1.2 Roadmap of This Thesis

Our goal is to introduce information uncertainty into the auction models and model the participants' beliefs as an endogenous intellectual process rather than an exogenously given and non-questionable component of the model. There are several ways we approach our research goal: adding a possibility of erroneous seller belief, allowing bidders to update their private values or allowing the seller to manipulate auctions. With the different methods we use in order to address the research question, we have defined three different research projects. Each are separately covered in great detail in the form of a chapter of this thesis. The following subsections give short descriptions of each project separately, and as such introduce the content of this thesis.

1.2.1 Non-Robustness of the Crémer-McLean Auction

Our departing point is a chapter whose goal is to formally (in terms of modeling) represent the relevance of this research focus. Namely, in this chapter we test the robustness of an optimal auction format: the Crémer-McLean Auction. In spirit of our focus, the robustness test is designed by introducing a change in information structure. More precisely, we introduce uncertainty in the accuracy of the involved beliefs. That is, instead of assuming complete certainty about the distribution of private values, we consider a framework where beliefs are estimations and analyze the case in which these estimates are partially incorrect. To this end, the CPA is relaxed in such a way that the bidders may or may not share the same prior, but the seller certainly does not have to have an identical prior as the bidders. As the seller and bidders belong to opposite sides of the trade, it seems plausible to model a situation where the seller and bidders might possess different knowledge. Finally, the core of the chapter is the analysis of the consequences of this relaxation on bidders' behavior and the selling mechanism. In conclusion, you will discover within the lines of the chapter that even the smallest fluctuation of belief certainty leads to a dramatic influence on the model's outcome. The novelty of the chapter is our quantitative qualification of the consequences on bidders' participation that the information uncertainty caused.

1.2.2 Bayesian Updating on Origin of Private Values

Here we grasp the problem of the origin of knowledge about the distributions of values in standard auction formats, in particular, within first-price and second-price auctions. As in the standard models, we assume that private values are independently identically distributed (i.i.d.). However, the new component of modeling is that we assume that the exact distribution of a value is unknown and comes from a finite set of distributions, with the property that each element of the set is equally likely to be the true distribution. Depending on the manner of how players resolve the introduced uncertainty, we create two scenarios. The benchmark scenario is pretty much in line with the standard scenario in the literature - the players treat the true distribution to be equal to the average distribution over the given set and behave accordingly. In the second scenario, we allow bidders to update their beliefs (in a Bayesian manner) once they become aware of their own private value. Meanwhile, the seller considers the convex combination of all joint distributions as the one defining the joint distribution of values. Therefore, in this scenario we emphasize the role of private information on the belief formation procedure. Furthermore, we compare the bidding behaviors and expected revenue across these two scenarios. Under some imposed structure on the set of possible marginal distributions (namely, we assumed that the set is ordered in the terms of likelihood ratio dominance), we have shown the failure of the revenue equivalence principle within the second scenario - in particular, that the seller favors a second-price auction for the sale. Moreover, once the bidding strategies in the first-price auction between scenarios are opposed to each other, we show that the bidding strategies can be either ordered in terms of dominance or exhibit a single-crossing property.

1.2.3 Endogenous Auction Choice under Possible Seller's Manipulation: All-Pay and Second-Price Auction

We end our research journey in this thesis with a model that differs in several ways to the ones developed in the earlier described projects. The set of bidders' beliefs is enriched by introducing two types of the seller: an honest seller and a manipulative seller. To the manipulative type of seller we give the possibility of sending a secret agent in the second-price or all-pay auction. The secret selfless agent bids on the behalf of the manipulative seller and takes role similar to a double account in online auctions. Meanwhile, the honest seller may choose only regular forms (without reserve price or reserve payments) of these two auction formats for sale. Therefore, the existence of two seller's types gives rise to a new kind of bidder belief - belief about the honesty of the seller. It is intuitively clear, that this new belief impacts bidding strategy - the stronger the belief that a bidder faces a cheating seller, the more probable it is to expect, at least in the all-pay auction, that the bidder will bid more modestly and shade her bids. The motivation to give the seller an opportunity to place a "false bid" might be found in recent literature development on fraudulent seller behavior as well as in various online auction security measures (for example, Ebay's double account policies). In the spirit of our research focus, we allow bidders to endogenously form their beliefs about honesty and base their judgment on the seller's choice of the auction format. Namely, in game-theoretical terms, we add a signaling process analogous to the one introduced by In-Koo Cho (1987). Once the bidders perceive the choice of auction, they are asked to place a bid. Bidders can then use this information to directly shape their beliefs about the seller's type which ultimately influences their bidding behavior. The exact strength of the signaling effect on bidders' strategies as well as its influence on the seller's auction choice at pure weak Bayesian Nash equilibria are explored and characterized in the main findings of the chapter. Unlike previous studies on similar set-ups with identically and independent distributed private values and a possible seller's fraudulent behavior (see Rothkopf and Harstad (1995) and Porter and Shoham (2005)), once the bidders' belief formation incorporates the signaling effect, we obtain one more argument in favor of the wide use of the second-price auction, especially to avoid seller's abuse via a secret double account.

Chapter 2

Non-Robustness of the Crémer-McLean Auction

In this chapter we examine the consequences of introducing information uncertainty in a well-defined optimal auction format. The importance of this approach lies in the fact that it helps in identifying elements of the standard auction model that are vulnerable to a change of information structure. In this particular case, we show the failure of participation constraints. That is, once the seller has even a slightly inaccurate belief about the distribution of joint values, the bidders will probably refuse to participate to the auction.

We are looking at the informational robustness of an optimal auction format in the case of a finite and fixed-size valuation space and a single-unit object offered for sale. Crémer and McLean (1988) have defined the optimal auction as one which guarantees the full extraction of the surplus. Given that the distribution of bidders' valuations is common knowledge among all participants and the cardinality of value space is fixed, they have shown that the full surplus extraction (FSE) is generically possible. Once the assumptions on common prior and fixed size of valuation space are relaxed, Heifetz and Neeman (2006) have shown the opposite result - the generic impossibility of FSE. In particular, they have considered a setting where the seller is uncertain both about the common prior shared by bidders and the number of possible valuations. Furthermore, Barelli (2009) has shown the impossibility of FSE in the the case of a common prior without a fixed-size valuation space. We combine the last two approaches and consider the set-up with a fixed-sized valuation space and the seller possibly holding an inaccurate belief about the joint distribution of bidders' private values. As a consequence, the seller then has a different prior to the one the bidders consider. Namely, the true distribution of values is close to the seller's belief. Given this set-up, we show that participation constraints failure occurs in a significant fraction of any "reasonable" neighborhood of the seller's belief. Finally, we also consider the case where bidders themselves do not share a prior, but may have different estimates on the origin of private values. We show that the failure of participation constraints occurs even more often in this case.

This chapter is organized as follows. The next section introduces the model. Sections 2.2 and 2.3 present the main results with, or without the common prior assumption respectively. We conclude this chapter with some final comments in Section 2.4. All proofs may be found in Appendix 2.A. Appendix 2.B provides a deep analysis of the participation constraints for the specific case of two private values with a common prior assumption.

2.1 The Set-Up

We consider the standard single-unit auction setting. There is a seller who sells his indivisible good to N potential buyers (bidders). The bidders compete for the good by placing bids. Without loss of generality, we assume that there are only two bidders. As usual, each bidder has her own value, which she attaches to the good. We usually refer to these values as the bidder's "private values" and it is her private information. To simplify the notation, we assume that the seller attaches the value 0 to the good and this fact is known to the bidders. Regarding the space of private values, the setting identical to the one used in Crémer and McLean (1988) is considered - that is, the space is finite. In particular, let bidder i's private value be denoted by the random variable X_i, which may take one of the following m values: 0 ,$\Delta, \ldots, (m-1)\Delta$, with $\Delta > 0$ and $m \geq 2$. In order to keep the argument as simple as possible, we choose equidistant values and an identical set of values among bidders. The result may easily be generalized for the case of any finite (potentially different) sets of values.

In line with the literature, the economic model for an auction corresponds to a game of incomplete information introduced by Harsanyi (1967). That is, an auction is a game played by the seller and bidders with the feature that the seller does not know the private valuations of the bidders. Indeed, if the seller knew the private values of the bidders, he would sell his good to the bidder with the highest valuation. Hence, there would not be room for strategic decision making. Thus, the key secret to the model are the private values.

Consequently, participants hold beliefs about the distribution of values and act accordingly. In addition, participants are assumed to have a risk-neutral attitude. The next subsections are devoted to full descriptions of the beliefs and information structure involved in the game and of the game itself.

2.1.1 Formation of Beliefs

Each participant forms beliefs conditional on all information they possess. For example, a piece of information available to a bidder is her private value. The only other piece of information available to both bidders and the seller is the joint probability distribution of the bidders' values.

Crémer and McLean (1988), in their model, assume that this distribution is common knowledge, that is, each participant knows the distribution, knows that all other participants know it, knows that the others know that they know it and so on, and that their knowledge is correct. Similarly, in our set-up we assume that each participant believes that the distribution is common knowledge and has no doubt about this belief. However, unlike the above mentioned authors, we explore what happens once the seller holds a "slightly" inaccurate belief about the joint distribution of values and is not aware of his possible error. Note that the probability distribution of joint values is the only piece of information given to the seller who derives all his beliefs based on this distribution.

Let us now express the above described beliefs through an appropriate mathematical apparatus. We firstly depict the seller's mind; i.e. his two types of beliefs: belief about the joint distribution and his belief about bidders' beliefs.

2.1.2 The Seller's Perspective

The seller's point of view, including his role and participation in the model, is the subject of this subsection. The seller's task is to set the auction game which will be used as the mechanism to sell his object. In other words, the seller sets the rules of the game which will be played by both himself and bidders. His obligations in the game are definition of the auction game and commitment to complete execution of the promised game.

The seller defines the auction game in such a way that his expected revenue is maximized. In order to be able to calculate the expected revenue, the seller needs to hold a belief about the joint distribution of the private values of bidders. In addition, he has to hold a belief about the bidding strategy played by any single type of each bidder. For this purpose, the seller forms his belief about each bidder's belief about the distribution of other bidders' valuations. Indeed, whenever a rational bidder places a bid, she has to take into account the probability distribution of other bidders' values in order to predict others' bidding behavior and, thus, optimize her own behavior.

Let us now express the above described beliefs by means of a suitable mathematical tools. The seller believes that the joint distribution of private values (X_1, X_2) is given by the following matrix $\bar{P}$:

$$\bar{P} = \begin{bmatrix} \bar{p}_{00} & \bar{p}_{01} & \cdots & \bar{p}_{0\,m-2} & \bar{p}_{0\,m-1} \\ \bar{p}_{10} & \bar{p}_{11} & \cdots & \bar{p}_{1\,m-2} & \bar{p}_{1\,m-1} \\ & & \vdots & & \\ \bar{p}_{m-2\,0} & \bar{p}_{m-2\,1} & \cdots & \bar{p}_{m-2\,m-2} & \bar{p}_{m-2\,m-1} \\ \bar{p}_{m-1\,0} & \bar{p}_{m-1\,1} & \cdots & \bar{p}_{m-1\,m-2} & \bar{p}_{m-1\,m-1} \end{bmatrix},$$

where $\bar{p}_{ij}$ denotes the probability of the event that bidder 1's private value is $i\Delta$ and bidder 2's value is $j\Delta$, for any $i, j \in \{0, 1, \ldots, m-1\}$. Table 2.1 visualizes this belief for the case of two bidders.

Bidder 2

Bidder 1 Values	0	Δ	$\cdots$	$(m-1)\,\Delta$
0	$\bar{p}_{00}$	$\bar{p}_{01}$	$\cdots$	$\bar{p}_{0\,m-1}$
Δ	$\bar{p}_{10}$	$\bar{p}_{11}$	$\cdots$	$\bar{p}_{1\,m-1}$
$\vdots$			$\ddots$	
$(m-1)\,\Delta$	$\bar{p}_{(m-1)0}$	$\bar{p}_{(m-1)\,1}$	$\cdots$	$\bar{p}_{m-1\,m-1}$

Table 2.1: *The seller's belief about joint distribution of private values.*

Thus, the marginal distributions of X_1 and X_2 are given by :

$$X_1 : \begin{array}{cccc} 0 & \Delta & \cdots & (m-1)\Delta \\ \bar{p}_0 & \bar{p}_1 & \cdots & \bar{p}_{m-1} \end{array} \text{ and } X_2 : \begin{array}{cccc} 0 & \Delta & \cdots & (m-1)\Delta \\ \bar{p}^0 & \bar{p}^1 & \cdots & \bar{p}^{m-1} \end{array},$$

with $\bar{p}_i = \sum\limits_{j=0}^{m-1} \bar{p}_{ij}$ and $\bar{p}^j = \sum\limits_{i=0}^{m-1} \bar{p}_{ij}$.

Let $\bar{P}_i$ be beliefs of bidder $i \in \{1,2\}$ over the distribution of the other bidder's value, given her own value, i.e. let

$$\bar{P}_i = \begin{bmatrix} \bar{P}(X_j = 0|X_i = 0) & \bar{P}(X_j = \Delta|X_i = 0) & \cdots & \bar{P}(X_j = (m-1)\Delta|X_i = 0) \\ \bar{P}(X_j = 0|X_i = \Delta) & \bar{P}(X_j = \Delta|X_i = \Delta) & \cdots & \bar{P}(X_j = (m-1)\Delta|X_i = \Delta) \\ & & \vdots & \\ \bar{P}(X_j = 0|X_i = (m-1)\Delta) & \bar{P}(X_j = \Delta|X_i = (m-1)\Delta) & \cdots & \bar{P}(X_j = (m-1)\Delta|X_i = (m-1)\Delta) \end{bmatrix},$$

for any $i \in \{1,2\}$ and $j \in \{1,2\} / \{i\}$. Therefore, the k^{th} row of the matrix $\bar{P}_i$ denotes the so called first-order belief held by bidder i with private value equal to $(k-1)\Delta$ for $k \in \{1,2,\ldots,m\}$. The first-order belief of a bidder is simply her belief about the distribution of the other bidders' values, given their own value. We stress that this is the way the seller thinks and solely represents his point of view, not necessarily the bidders' true beliefs.

The Optimal Auction Game: the Crémer-McLean Auction

Once we have defined beliefs, we are able to calculate the expected revenue obtained from any auction and its associated equilibrium strategy profile. We shall consider the following two auction formats: the second-price auction and the Crémer-McLean auction. The later auction is introduced in Crémer and McLean (1988) and presents the optimal auction format, that is it leads to the highest expected revenue for the seller among all possible selling mechanisms given his personal beliefs. Thus, it will be the auction format chosen by the seller.

However, we start our analysis by introducing the second-price auction, mainly because of the following two reasons. Firstly, the auction represents the essential building block for the optimal auction and other models explained in this thesis. It is therefore, of crucial importance to understand its rules before the rules for the auction format, actually played by participants are introduced. The second reason for consideration of the second-price auction is its property of being one of the so-called "detailed-free" auction formats. In other words, the second-price auction has its equilibrium in ex post (weakly) dominant strategies (which are truth-telling strategies) and therefore is robust against fluctuations of participants' beliefs.

In order to define an auction game it is sufficient and necessary to define the probability of getting the object and the ex post (expected) payment made to the seller for each bidder and each possible combination of bids. Thus, the second-price auction is identified with a pair of mappings $\left(Q^{II}, M^{II}\right)$, where:

1. $Q^{II} : \{0, \Delta, \ldots, (m-1)\Delta\}^2 \to [0,1]^2$ with coordinate functions[1] defined as follows:
$$Q_i^{II}(b_i, b_j) = \begin{cases} 1, \, b_i > b_j \\ 0, \, b_i < b_j \\ \frac{1}{2}, \, b_i = b_j \end{cases} \text{for all } i, j \in \{1, 2\} \text{ and } j \neq i.$$
$Q_i^{II}(b_i, b_j)$ is the probability that bidder i gets the object when she bids b_i and her opponent bids b_j in the second-price auction. Therefore, in the auction the good is sold to the bidder with the highest bid. This allocation rule is also known as the **standard auction** rule.

2. $M^{II} : \{0, \Delta, \ldots, (m-1)\Delta\}^2 \to \mathbb{R}^2$
$$M_i^{II}(b_i, b_j) = \begin{cases} b_j, \, b_i > b_j \\ 0, \, b_i < b_j \\ \frac{b_i}{2}, \, b_i = b_j \end{cases},$$
$M_i^{II}(b_i, b_j)$ is the ex post (expected) payment that bidder i pays to the seller when she bids b_i and her opponent bids b_j in the second-price auction. From the definition, one may notice in the auction's name, that the winner of the auction (the bidder whom the good is sold to) pays the second-highest bid as the price for the good.

Given this definition, one can easily show that truth telling is the ex post weakly dominant strategy[2] and therefore, also the Bayesian Nash equilibrium of this game. Thus, under the implicit rationality assumption, we shall assume that bidders use the truth-telling strategy once they are called upon to participate in the second-price auction.

For the purpose of further analysis, it is useful to determine the ex post payoff and interim expected payoff for each type of bidder at the equilibrium, which will be denoted as Π^{II} and

[1]For any vector-valued function $f : A \to C$ with $C \subseteq \mathbb{R}^k$ its i^{th} coordinate function, $1 \leq i \leq k$, denoted by f_i, is the mapping from set A to set C_i, with $C_i = \Pr_i(C) = \{c \in \mathbb{R} \,|\, (\exists a \in A)\,(f(a) = (c_1, c_2, \ldots, c_k) \text{ and } c_i = c)\}$, which maps any $a \in A$ into real number $\Pr_i(f(a))$, where $\Pr_i(\mathbf{x})$ is the i^{th} coordinate of the vector $\mathbf{x} \in \mathbb{R}^k$.

[2]For example, please check Vickrey (1961).

$\bar{u}_i^{II}$, respectively. More precisely, we define $\Pi^{II} : \{0, \Delta, \ldots, (m-1)\Delta\}^2 \to \mathbb{R}^2$ as $\Pi_i^{II}(x_i, x_j) = Q_i^{II}(x_i, x_j)x_i - M_i^{II}(x_i, x_j)$, that is, $\Pi_i^{II}(x_i, x_j)$ is the payoff of bidder i when her value is x_i and the opponent's value is x_j at the truth-telling equilibrium. However, we might sometimes order the arguments of the function Π_i^{II}, i.e. the bids, by bidder's names and, in order to avoid ambiguity, we shall use a right arrow whenever we mean ordered bids. For instance, $\Pi_2^{II}(k\Delta, \ell\Delta)$ is the payoff of bidder 2 when her value and bid are $k\Delta$ and the opponent's value and bid are $\ell\Delta$; whereas, $\Pi_2^{II}(\overrightarrow{k\Delta, \ell\Delta})$ denotes the payoff of bidder 2 when bidder 1's value and bid are $k\Delta$ and bidder $2's$ value and bid are $\ell\Delta$.

Using this notation, the expected (interim) payoffs for each type of bidder i in the second-price auction, from seller's point of view, is given by:

$$\bar{u}_i^{II} = \begin{bmatrix} \bar{u}_i^{II}(0) \\ \bar{u}_i^{II}(\Delta) \\ \vdots \\ \bar{u}_i^{II}((m-1)\Delta) \end{bmatrix}, \tag{2.1}$$

with

$$\begin{aligned} \bar{u}_i^{II}(k\Delta) &= \sum_{j=0}^{m-1} \bar{P}(X_j = j\Delta | X_i = k\Delta)\Pi_i^{II}(k\Delta, j\Delta), \; k \in \{0, 1, \ldots, m-1\} \\ &= \sum_{j<i} \bar{P}(X_j = j\Delta | X_i = k\Delta)(i-j)\Delta, \; k \in \{0, 1, \ldots, m-1\}, \end{aligned}$$

for any $i \in \{1, 2\}$. Thus, $\bar{u}_i^{II}(k\Delta)$ denotes the expected (interim) payoff of bidder i at the truth-telling equilibrium when her own value is equal to $k\Delta$. Note that $\bar{u}_i^{II}(k\Delta)$ depends on $\bar{P}$. This is the reason for overlining the notation of the expected (interim) payoffs. In this text, we shall stick to this notional rule - each variable that depends on $\bar{P}$ will be overlined. This is important in order to get a clear insight into the part of the seller's derivation, which depends heavily on his belief about the joint distribution of private values.

Let us now introduce the auction game which is optimal from the seller's point of view, that therefore will be actually played: the so-called Crémer-McLean (CM) auction. The allocation rule $Q^{CM} : \{0, \Delta, \ldots, (m-1)\Delta\}^2 \to [0, 1]^2$ and payment rule $\bar{M}^{CM} : \{0, \Delta, \ldots, (m-1)\Delta\} \to \mathbb{R}^2$ of the auction are defined as follows:

1. $Q^{CM} = Q^{II}$, that is, the CM auction uses the same allocation rule as the second-price auction: the good is allocated to the bidder with the highest bid.

2. $\bar{M}^{CM} : \{0, \Delta, \ldots, (m-1)\Delta\}^2 \to \mathbb{R}^2$ with coordinate functions defined by

$$\bar{M}_i^{CM}(b_i, b_j) = M_i^{II}(b_i, b_j) + \bar{c}_i(b_j), \tag{2.2}$$

where vectors

$$\bar{c}_i = \begin{bmatrix} \bar{c}_i(0) \\ \bar{c}_i(\Delta) \\ \vdots \\ \bar{c}_i((m-1)\Delta) \end{bmatrix}$$

are such that they satisfy:

$$\bar{P}_i \bar{c}_i = \bar{u}_i^{II}, \tag{2.3}$$

for any $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$.

Let us have a closer look at the definition of the bidder's payment rule and its building blocks. From equation (2.2) we see that bidder i's payment is made of two terms: $M_i^{II}(b_i, b_j)$ and $\bar{c}_i(b_j)$. The former term is the payment that the bidder would make if she participated in a second-price auction. The later term depends solely on the bid placed by the opponent. Thus, we may interpret the CM auction as a second-price auction with a lottery over opponents' bids. Namely, each bidder, by participating in a CM auction, accepts to (i) compete over a good being sold under second-price auction rules and (ii) bet on opponents' bids. More precisely, by definition of the payment rule, for each possible bid b_j of the opponent, bidder i will ex post pay $c_i(b_j)$ to the seller (if $c_i(b_j) > 0$) or will be paid $-c_i(b_j)$ (if $c_i(b_j) < 0$). Note that the payments from the lottery are ex post executed by all bidders and not exclusively by the winner. Therefore, each potential buyer, once invited to participate in the CM auction, is actually invited to play in a second-price auction and to bet on each opponent's bid, whose payments may be illustrated by Table 2.2.

Opponent's bid	0	Δ	$\cdots$	$(m-1)\Delta$
Ex post payment from bidder i	$\bar{c}_i(0)$	$\bar{c}_i(\Delta)$	$\cdots$	$\bar{c}_i((m-1)\Delta)$

Table 2.2: *The lottery in CM auction offered to a bidder.*

The key to the optimality of the auction is the proper choice of the vectors $\bar{c}_i$, which is made by solving the system of equations (2.3). First of all, note that in order to guarantee existence of a solution to the system we must impose some further assumptions on matrices $\bar{P}_i$. The sufficient and necessary condition for existence and uniqueness of vectors $\bar{c}_i$ is that the matrices $\bar{P}_i$ are non-singular, i.e. that $\det \bar{P}_i \neq 0$ for any i. Therefore, the following assumption is imposed:

Assumption 2.1. (Existence and Uniqueness of the CM Auction) *$\bar{P}_i$ is a non-singular matrix for each $i \in \{1, 2\}$ or equivalently, the system of linear equations* (2.3) *has a unique solution, for any i.*

Remark 2.1. Note that, due to the continuity of the det mapping,[3] once we limit our attention to the space of all $m \times m$ matrices, Assumption 2.1 is generically satisfied.[4] This is one of the most often stated argument in favor of the CM auction.

In order to understand the definition of $\bar{c}_i$, we shall first have a look at the bidding strategy of the auction. For a moment, we shall take vectors $\bar{c}_i$ as given and take into account only the fact that $\bar{c}_i(b_j)$ depends only on the bid of i's opponents and not on her own bid b_i. Having this fact in mind, we can immediately see that the bidding behavior of a bidder involved in a CM auction will coincide with her behavior in the second-price auction - the truth-telling strategy is also an equilibrium. More precisely, by (2.2), the difference of ex post (expected) payoffs in the CM auction between any two bids $b_i^{'}$ and b_i coincides with the analogous ex post difference in the second-price auction. Thus, the same analysis to the second-price auction case shows that truth-telling is ex post (weakly) dominant in a CM auction and that this bidding strategy will be played by a rational bidder in the CM auction as well.

Now that we have established that the truth-telling is the equilibrium played by bidders, we may go back to the interpretation of the choice of vectors $\bar{c}_i$. The $(k+1)^{th}$ equation of the (2.3) is as follows:

$$\sum_{j=0}^{m-1} \bar{P}(X_j = j\Delta | X_i = k\Delta)\bar{c}_i(j\Delta) = \bar{u}_i^{II}(k\Delta),$$

for any $i \in \{1,2\}$ and any $k \in \{0,1,\ldots m-1\}$. Thus, the vector $\bar{c}_i$ is chosen in a such a way that, at the truth-telling weakly dominant equilibrium, the expected (interim) payoff of the offered lottery (illustrated by Table 2.2) is equal to the expected (interim) payoffs in the second-price auction for each type of bidder i. Loosely speaking, the CM auction is basically a second-price auction with an additional (mandatory) bet over the opponents' bids, such that the expected payoff solely from the bet coincides with expected payoffs from the second-price auction.

Finally, we show that the CM auction is the optimal selling mechanism for the seller. Moreover, we show that the full extraction of surplus from bidders is guaranteed at the equilibrium. Analogously to the second-price auction, we derive the ex post and interim expected payoff for each type of bidder at the equilibrium, denoted as $\bar{\Pi}^{CM}$ and $\bar{u}_i^{CM}$, respectively. The ex post (expected) payoff in a CM auction is given by:

$$\begin{aligned} \bar{\Pi}_i^{CM}(x_i, x_j) &= Q_i^{CM}(x_i, x_j)x_i - \bar{M}_i^{CM}(x_i, x_j) \\ &= \Pi_i^{II}(x_i, x_j) - \bar{c}_i(x_j), \end{aligned} \tag{2.4}$$

[3] Under the det mapping we understand the function which assigns to each square matrix the value of its determinant.

[4] This is because Assumption 2.1 rules out only set of singular matrices in the space of all $m \times m$ matrices. The set of singular matrices is negligible compared to the whole space.

for any pair of private values $(x_i, x_j) \in \{0, \Delta, \ldots, (m-1)\Delta\}^2$.

The interim expected payoffs for bidders in the CM auction, from a seller's point of view, is therefore given by:

$$\bar{u}_i^{CM}(x_i) = \sum_{j=0}^{m-1} \bar{P}(X_j = j\Delta | X_i = x_i) \bar{\Pi}_i^{CM}(x_i, j\Delta),$$

for any $i \in \{1, 2\}$ and $x_i \in \{0, \Delta, \ldots, (m-1)\Delta\}$.

Plugging the expression for $\bar{\Pi}_i^{CM}(x_i, x_j)$, from (2.4), into the last equation, we obtain:

$$\bar{u}_i^{CM}(x_i) \quad = \quad \bar{u}_i^{II}(x_i) - \sum_{j=0}^{m-1} \bar{P}(X_j = j\Delta | X_i = x_i) \bar{c}_i(j\Delta).$$

In vector notation, we have just obtained that

$$\bar{u}_i^{CM} = \bar{u}_i^{II} - \bar{P}_i \cdot \bar{c}_i. \tag{2.5}$$

From the definition of vectors $\bar{c}_i$ given in (2.3), it follows:

$$\bar{u}_i^{CM} = \bar{u}_i^{II} - \bar{P}_i \cdot \bar{c}_i = \bar{u}_i^{II} - \bar{u}_i^{II} = 0. \tag{2.6}$$

That is, the expected (interim) payoffs for bidders in the CM auction is equal to zero, i.e. we have just derived the full surplus extraction property of the CM auction. Theoretically, given Assumption 2.1, which is not restrictive, the full surplus extraction is generically possible. However, the CM auction is rarely used in practice. There are many reasons not to use the CM auction, some of which were stressed in the original paper Crémer and McLean (1988). Among the listed reasons were potentially too high absolute bet payment values $\bar{c}_i$ or the assumption on risk-neutrality of the bidders, which might be far away from the "real world". The authors also mentioned the assumption of common knowledge as a potential danger of the auction's usage in practice. This is the direction our research path leads.

The focus of this chapter is on the vulnerability of the underlying information structure in the optimal auction. Namely, the importance imposed on the correctness of the seller's belief is stressed. More precisely, all the calculations in this section depict the seller's point of view as in Crémer and McLean (1988) and there it is assumed that miscalculations by the participants do not occur. However, in the next section we describe a typical bidder's mind and allow them to hold beliefs that are arbitrarily close to the seller's belief. The consequences of this allowance will be assessed in subsequent sections.

2.1.3 The Bidders' Perspective

Once the seller has set the rules of the auction game, which are described as the CM auction in the previous section, we now have a look at the auction game from the perspective of a bidder. Each bidder competes against other bidders for the good being sold by posting a bid. The level of the bid is derived by the bidder given her private value. Therefore, the bidder's belief over possible opponents' private values plays a key role in determining her bidding strategy.

In line with the original work, Crémer and McLean (1988), all bidders are assumed to have a common prior over the joint distribution of private values. More precisely, let us denote the common prior by a $m \times m$ matrix $P = \left[p_{ij}\right]_{i,j=0}^{m-1}$, where p_{ij} denotes the probability of the event that bidder 1's private value is $i\Delta$ and bidder 2's value is $j\Delta$, for any $i, j \in \{0, 1, \ldots, m-1\}$ from bidders' point of view. The bidders' belief about the joint distribution of private values is represented in Table 2.3.

Bidder 2 (columns); *Bidder 1* (rows)

Values	0	Δ	$\cdots$	$(m-1)\,\Delta$
0	p_{00}	p_{01}	$\cdots$	$p_{0\,m-1}$
Δ	p_{10}	p_{11}	$\cdots$	$p_{1\,m-1}$
$\vdots$			$\ddots$	
$(m-1)\,\Delta$	$p_{m-1\,0}$	$p_{m-1\,1}$	$\cdots$	$p_{m-1\,m-1}$

Table 2.3: *The bidders' belief about the joint distribution of private values.*

Let P_i be the true beliefs of bidder i derived from P given her own private value. That is, let

$$P_i = \begin{bmatrix} P(X_j = 0|X_i = 0) & P(X_j = \Delta|X_i = 0) & \cdots & P(X_j = (m-1)\Delta|X_i = 0) \\ P(X_j = 0|X_i = \Delta) & P(X_j = \Delta|X_i = \Delta) & \cdots & P(X_j = (m-1)\Delta|X_i = \Delta) \\ & & \vdots & \\ P(X_j = 0|X_i = (m-1)\Delta) & P(X_j = \Delta|X_i = (m-1)\Delta) & \cdots & P(X_j = (m-1)\Delta|X_i = (m-1)\Delta) \end{bmatrix} \tag{2.7}$$

be such that P_i is a row stochastic matrix[5] and $P(X_j = \ell\Delta|X_i = k\Delta)$ denotes the probability that bidder i, whose private value is $k\triangle$, attaches to the event that bidder j's private value is $\ell\Delta$.

In general, we allow that $P \neq \bar{P}$. In particular, we are interested in the case where P is very "close" to $\bar{P}$, where the term of "closeness" will be defined later. This is the crucial difference

[5]A matrix $M = \left[m_{ij}\right]_{i=\overline{1,n}}^{j=\overline{1,k}}$ is a row stochastic matrix if and only if all its entries are non-negative and for any $i \in \{1, 2, \ldots, n\}$ we have $\sum_{j=1}^{k} m_{ij} = 1$.

to the original paper. Namely, Crémer and McLean (1988) assume implicitly that $P = \bar{P}$. Thus, we depart from this assumption by allowing that the seller holds slightly different belief to bidders. This is the place where we attack the common prior assumption. Doing so requires us to depart from the usual modeling approach.

Let us now continue with the analysis of the situation from the bidders' perspective. Given the auction game $\left(Q^{CM}, \bar{M}^{CM}\right)$, truth-telling remains to be an equilibrium in a weakly dominant strategy even if P eventually differs from $\bar{P}$. That is, even though the expected payoff changes, the truth-telling stays ex post weakly dominant and therefore it is still an optimal interim bidding choice, irrespective of bidders' belief. This becomes straightforward once one considers the fact that the ex post dominance concept does not depend on the interim beliefs, which are in this case given by P_i.

The bidders' true interim expected payoffs (at the truth-telling equilibrium) in the Crémer-McLean mechanism are given by:

$$\bar{u}_i^{CM_P}(x_i) = \sum_{j=0}^{m-1} P(X_j = j\Delta | X_i = x_i)\bar{\Pi}_i^{CM}(x_i, j\Delta).$$

Plugging in the expression for $\bar{\Pi}_i^{CM}(x_i, x_j)$ from (2.4), we obtain:

$$\begin{aligned} \bar{u}_i^{CM_P}(x_i) &= \sum_{j=0}^{m-1} P(X_j = j\Delta | X_i = x_i) \left[\Pi_i^{II}(x_i, j\Delta) - \bar{c}_i(j\Delta)\right] && (2.8) \\ &= u_i^{II_P}(x_i) - \sum_{j=0}^{m-1} P(X_j = j\Delta | X_i = x_i)\bar{c}_i(j\Delta), && (2.9) \end{aligned}$$

where $u_i^{II_P}(x_i)$ is the expected payoff of bidder i in the second-price auction from her point of view with the belief P_i. Note that the expected payoff also depends on the seller's belief about the joint distribution of private values $\bar{P}$, through the vector $\bar{c}_i$.

In matrix notation, we have obtained that the vector of bidder i's expected payoffs is given by the following formula:

$$\bar{u}_i^{CM_P} = u_i^{II_P} - P_i \cdot \bar{c}_i. \qquad (2.10)$$

With the explicit derivation of the expected payoff in hand, we can now asses whether participation constraints still hold: that is, whether the bidder will accept to play in the offered auction game. Participation constraints are defined as the requirement that the interim expected payoff of any bidder with any private value must be non-negative from the bidder's point of view. If the opposite is true, a bidder will refuse to participate in the auction game - explaining the requirement's name.

The justification for addressing the participation constraints might be seen from a comparison between equations (2.5) and (2.10). Namely, once we consider the vectors $\bar{c}_i$ as exogenously given, we might obtain (2.10) from (2.5) by replacing $\bar{P}_i$ with P_i. Therefore, it is natural to suspect that the equality (2.6) does not hold anymore once we make the replacement. Possible changes of signs of the expected payoffs, as well as determining whether these changes are negligible or significant will be explored in subsequent sections. Moreover, we are interested to quantify, in terms of geometry, the frequency that eventual participation constraint failure occurs.

2.2 Non-Robustness with Common Prior Assumption

As stated in the previous section, we allow here that the seller's belief over distribution of private values may differ from the analogous belief of bidders; i.e. we allow $P \neq \bar{P}$. Hereby, we have implicitly assumed that the bidders share a common prior assumption P. Namely, by the definition of bidders' belief, given by (2.7), we have assumed that all matrices P_i are derived from one common belief over joint distribution, denoted by P. This assumption will be fixed throughout this section. Therefore, here the primitives are given by belief over the vector of private values, which is in line with Crémer and McLean (1988). In the next section we drop this constraint.

All in all, bidders and the seller might hold different beliefs, but bidders among themselves do not differ in their belief of the joint distribution of private values. These two beliefs, P and $\bar{P}$, will be "very close": we consider the case where P is in a neighborhood of $\bar{P}$. The following definitions should clarify what is meant by term "neighborhood of a belief".

Definition 2.1. An **ε-neighborhood of a belief matrix** $\bar{P} = \left[\bar{p}_{ij}\right]_{i,j=0}^{m-1}$, $\varepsilon > 0$, is the set of matrices $P = \left[p_{ij}\right]_{i,j=0}^{m-1}$ such that $\|P - \bar{P}\|_2 \leq \varepsilon$ and P is also a belief matrix (that is, $p_{ij} \in [0,1]$ and $\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} p_{ij} = 1$), where $\|\cdot\|_2$ is the standard Euclidean norm.[6]

Once we consider the set of all $m \times m$ belief matrices as the topological subspace of the topological space $\left(\mathbb{R}^{m^2}, \tau_2\right)$ where the topology τ_2 denotes topology induced by the standard (Euclidean) norm, then an ε-neighborhood of a belief matrix $\bar{P} = \left[\bar{p}_{ij}\right]_{i=\overline{0,m-1}}^{j=\overline{0,m-1}}$ is nothing more than a closed ball with its center at $\bar{P}$ and radius ε in the subspace of belief matrices.

[6]The standard (Euclidean) norm on $\mathbb{R}^k$ is defined by $\|x\|_2 = \sqrt{\sum_{i=1}^{k} x_i^2}$ for any $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$.

Definition 2.2. An ε-neighborhood of a belief matrix $\bar{P}$ is said to be a **bidirectional ε-neighborhood** if it contains two matrices $P = \left[p_{ij}\right]_{i=\overline{0,m-1}}^{j=\overline{0,m-1}}$ and $P' = \left[p'_{ij}\right]_{i=\overline{0,m-1}}^{j=\overline{0,m-1}}$ such that $\left(p_{ij} - \bar{p}_{ij}\right) \cdot \left(p'_{ij} - \bar{p}_{ij}\right) < 0$ for all possible pairs of i and j.

Remark. There exists a bidirectional ε-neighborhood of a belief matrix $\bar{P}$, if and only if all entries of $\bar{P}$ are strictly positive.

Definition 2.3. An ε-neighborhood of a belief matrix $\bar{P}$ is said to be a **fair ε-neighborhood** if it is bidirectional ε-neighborhood; and if it contains a belief matrix P then it must also contain the belief matrix $P' = \bar{P} - D$, with D being the $m \times m$ matrix given by $D = P - \bar{P}$.

Remark. A bidirectional ε-neighborhood of a belief matrix $\bar{P}$ is fair ε-neighborhood if and only if $0 < \varepsilon \leq min\left(\left\{\bar{p}_{ij} \,|i,j \in \{0,1,\ldots,m-1\}\right\}\right)$.

Finally, we define that P is very "close" to $\bar{P}$ if there is an ε-neighborhood of $\bar{P}$ that contains P with ε being arbitrarily small. Thus, our aim is to analyze expected payoffs of bidders when their belief of the joint distribution of private values lies in an ε-neighborhood of $\bar{P}$. For some results, we might also require the fairness property of the considered neighborhood.

2.2.1 Participation Constraints Failure

Now that our measure of the closeness is defined, we can address the analysis of the main question, that is to characterize the set of all possible values that the expected payoffs of bidders may take. The first step is to collect all expected payoffs in the form of a function whose argument is the common prior belief of bidders. The image of this function is the associated vector of expected payoffs for each bidder and each possible private value - firstly ordered by bidder and then by private values. The function will be denoted by $\bar{U}$ and is formally defined as follows.

Definition 2.4. Let $\bar{U} : \mathbb{R}^{m^2} \to \mathbb{R}^{2m}$ be the following mapping:

$$\bar{U}(\bar{P}) = \begin{bmatrix} \bar{u}_1^{CM_P}(0) \\ \bar{u}_1^{CM_P}(\Delta) \\ \vdots \\ \bar{u}_1^{CM_P}((m-1)\Delta) \\ \bar{u}_2^{CM_P}(0) \\ \bar{u}_2^{CM_P}(\Delta) \\ \vdots \\ \bar{u}_2^{CM_P}((m-1)\Delta) \end{bmatrix}, \tag{2.11}$$

where $\tilde{P} = (\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_{m^2-1}) \in \mathbb{R}^{m^2}$ and $\bar{u}_i^{CM_P}(x_i)$ is the expected payoff in the Crémer-McLean auction from Section 2.1, whose explicit form is given by (2.8). The matrix P is derived from $\tilde{P}$ as follows:

$$\tilde{P} = (\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_{m^2-1}) \in \mathbb{R}^{m^2} \Rightarrow P = \begin{bmatrix} \tilde{p}_0 & \tilde{p}_1 & \cdots & \tilde{p}_{m-1} \\ \tilde{p}_m & \tilde{p}_{m+1} & \cdots & \tilde{p}_{2m-1} \\ & & \vdots & \\ \tilde{p}_{(m-1)\cdot m} & \tilde{p}_{(m-1)\cdot m+1} & \cdots & \tilde{p}_{m^2-1} \end{bmatrix}.$$

Without loss of formality, we shall sometimes write P instead of $\tilde{P}$. Note that $\bar{U}$ crucially depends on the chosen seller's belief $\bar{P}$.

Remark 2.2. Note that $\bar{U}(\bar{P}) = [0]_{2m\times 1}$, since

$$\bar{U}(\bar{P}) \overset{\text{by definition of } \bar{U}}{=} \left[\ \bar{u}_i^{CM}(j\Delta)\ \right]_{i=1,2;\, j=\overline{0,m-1}} \overset{(2.6)}{=} [0]_{2m\times 1}.$$

This is the case where both seller and bidders agree on the common prior being matrix $\bar{P}$. That is, it represents the scenario analyzed in Crémer and McLean (1988), where all beliefs are aligned and correct.

Since we are essentially interested in values that mapping $\bar{U}$ takes in an Euclidean neighborhood of matrix $\bar{P}$, the natural tool to approximate these values is the Taylor expansion. Therefore, the first degree Taylor polynomial is used to approximate the value of $\bar{U}$ near point $\bar{P}$. The following lemma and proposition are devoted to obtaining the explicit formula of the polynomial.

Lemma 2.1. *The Jacobian of the mapping $\bar{U}$ defined in* (2.11) *for a matrix*

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0\,m-1} \\ p_{10} & p_{11} & \cdots & p_{1\,m-1} \\ & & \vdots & \\ p_{m-1\,0} & p_{m-1\,1} & \cdots & p_{m-1\,m-1} \end{bmatrix},$$

is given by following formula

$$J\bar{U}(P) = \begin{bmatrix} J\bar{U}_1(P) \\ J\bar{U}_2(P) \end{bmatrix}_{2m\times m^2}.$$

$J\bar{U}_1(P)$ is a matrix of format $m \times m^2$ whose element in $(i+1)^{th}$ row and $(j+1)^{th}$ column, denoted by $J\bar{U}_1(P)_{ij}$, is given by

$$J\bar{U}_1(P)_{ij} = \begin{cases} 0, & j \notin \{im, im+1, \ldots, (i+1)m-1\} \\ \frac{\bar{\Pi}_1^{CM}(\overrightarrow{i\Delta,(j-im)\Delta}) - \bar{u}_1^{CM_P}(i\Delta)}{p_i}, & \textit{otherwise} \end{cases},$$

for $i = \overline{0, m-1}$ *and* $j = \overline{0, m^2-1}$.

Similarly, $J\bar{U}_2(P)$ *is a matrix of format* $m \times m^2$ *whose element in* $(i+1)^{th}$ *row and* $(j+1)^{th}$ *column is of the following form* [7]

$$J\bar{U}_2(P)_{ij} = \begin{cases} 0, & j \bmod m \neq i \\ \frac{\bar{\Pi}_2^{CM}(\overrightarrow{((j\%m)\Delta, i\Delta)}) - \bar{u}_2^{CM_P}(i\Delta)}{p^i}, & \textit{otherwise} \end{cases},$$

for $i = \overline{0, m-1}$ *and* $j = \overline{0, m^2-1}$.

Here p_i *denotes the sum of the elements in the* $(i+1)^{th}$ *row of the matrix* P *whereas* p^i *denotes the sum of the elements belonging to its* $(i+1)^{th}$ *column.*

$J\bar{U}_k(P)$ is the matrix of partial derivatives where each i^{th} row is the gradient of $\bar{u}_k^{CM_P}(i\Delta)$ once it is considered as a mapping of belief matrix P. In other words, the element of $J\bar{U}_k(P)$ in $(i+1)^{th}$ and $(j+1)^{th}$ column is $\frac{\partial \bar{u}_k^{CM_P}(i\Delta)}{\partial \tilde{p}_j}$, that is, the element reflects how the expected payoff of bidder k with private value $i\Delta$ changes with a change of the element $\tilde{p}_j$.

Matrices $J\bar{U}_k(P)$ are sparse, because each particular $\bar{u}_k^{CM_P}(i\Delta)$ depends solely on the $(i+1)^{th}$ row (when $k = 1$) or $(i+1)^{th}$ column (when $k = 2$) of P, and is independent of all other elements of P (see expression (2.8)). In order to make this claim clear, let us visualize these two matrices where we denote by $*$ any potentially non-zero entry. Thus, the matrices take the forms as illustrated by Figure 2.1.

$$\begin{bmatrix} \overbrace{\begin{matrix} * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \end{matrix}}^{\text{first } m \text{ entries}} & \overbrace{\begin{matrix} 0 & 0 & \cdots & 0 \\ & * & \cdots & * \\ & & & \\ 0 & 0 & \cdots & 0 \end{matrix}}^{\text{second } m \text{ entries}} & \ddots & \overbrace{\begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \end{matrix}}^{\text{third } m \text{ entries}} & \cdots & \overbrace{\begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ & * & \cdots & * \end{matrix}}^{\text{last } m \text{ entries}} \end{bmatrix}_{m \times m^2}$$

$$\begin{bmatrix} \overbrace{\begin{matrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & * \end{matrix}}^{\text{first } m \text{ entries}} & \overbrace{\begin{matrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & * \end{matrix}}^{\text{second } m \text{ entries}} & \cdots & \overbrace{\begin{matrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & * \end{matrix}}^{\text{third } m \text{ entries}} & \cdots & \overbrace{\begin{matrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & * \end{matrix}}^{\text{last } m \text{ entries}} \end{bmatrix}_{m \times m^2}$$

Figure 2.1: *An illustration of Jacobian submatrices.* $J\bar{U}_1(P)$ (*upper*) and $J\bar{U}_2(P)$ (*lower*).

[7] The operation mod is the modulo operation, i.e. it finds remainder of division. For example, 7 mod 3 = 1. The operation % is floored division, e.g. 7%3 = 2.

The previous lemma gives expression for the Taylor expansion at any belief matrix P. Since the bidders' belief is in a neighborhood of the seller's belief $\bar{P}$, the following proposition gives the explicit expression of the Taylor polynomial at exactly the matrix $\bar{P}$.

Proposition 2.1. (The Taylor Expansion Matrix) *The Taylor expansion of the mapping $\bar{U}$ defined in* (2.11) *in the neighborhood of the matrix $\bar{P}$ is given by following formula:*

$$\bar{U}(P) = J\bar{U}(\bar{P})\,(P-\bar{P}) + o\left(\|P-\bar{P}\|^2\right),$$

where $J\bar{U}(\bar{P})$ is the Jacobian of the mapping $\bar{U}$ at the point $\bar{P}$. Its explicit form is

$$J\bar{U}(\bar{P}) = \begin{bmatrix} J\bar{U}_1(\bar{P}) \\ J\bar{U}_2(\bar{P}) \end{bmatrix}_{2m\times m^2}. \tag{2.12}$$

$J\bar{U}_1(\bar{P})$ is a matrix of format $m \times m^2$ whose element in $(i+1)^{th}$ row and $(j+1)^{th}$ column, denoted by $J\bar{U}_1(\overline{P})_{ij}$, is defined as follows:

$$J\bar{U}_1(\bar{P})_{ij} = \begin{cases} 0, & j \notin \{im+1, im+2, \dots (i+1)m\} \\ \frac{\bar{\Pi}_1^{CM}(\overrightarrow{i\Delta,(j-im)\Delta})}{\bar{p}_i}, & \textit{otherwise} \end{cases},$$

for any $i, j \in \{0, 1, \dots, m-1\}$.

Similarly, $J\bar{U}_2(\bar{P})$ is a matrix of format $m \times m^2$ whose element in $(i+1)^{th}$ row and $(j+1)^{th}$ column, denoted by $J\bar{U}_2(\overline{P})_{ij}$, is given by the following formula:

$$J\bar{U}_2(\bar{P})_{ij} = \begin{cases} 0, & j \bmod m \neq i \\ \frac{\bar{\Pi}_2^{CM}(\overrightarrow{(j\%m)\Delta,i\Delta})}{\bar{p}^i}, & \textit{otherwise} \end{cases},$$

for any $i, j \in \{0, 1, \dots, m-1\}$.

Proof. By the general Taylor expansion theorem, we have that

$$\bar{U}(P) = \bar{U}(\bar{P}) + J\bar{U}(\bar{P})\,(P-\bar{P}) + o\left(\|P-\bar{P}\|^2\right),$$

for all matrices P in some neighborhood $\mathcal{O}(\bar{P})$ of matrix $\bar{P}$. By definition of $\bar{U}$, we have that $\bar{U}(\bar{P}) = [0]_{2m\times 1}$ (see Remark 2.2 after the definition of the mapping $\bar{U}$). Furthermore, using the general formula for the Jacobian matrix $J\bar{U}(P)$, derived in Lemma 2.1, by setting $P = \bar{P}$ and using the fact that $\bar{u}_i^{CM}(k\Delta) = 0$ for any bidder i and any private value $k\Delta$, we obtained the desired result. □

More precisely, for any $i \in \{0, 1, \ldots, m-1\}$, the $(i+1)^{th}$ row of the matrix $J\bar{U}_1(\bar{P})$ takes the following form:

$$\underbrace{0 \quad 0 \quad \cdots \quad 0}_{i \cdot m \text{ times}} \quad \underbrace{\frac{\bar{\Pi}_1^{CM}(\overrightarrow{i\Delta, 0})}{\bar{p}_i} \quad \frac{\bar{\Pi}_1^{CM}(\overrightarrow{i\Delta, \Delta})}{\bar{p}_i} \quad \cdots \quad \frac{\bar{\Pi}_1^{CM}(\overrightarrow{i\Delta, (m-1)\Delta})}{\bar{p}_i}}_{m \text{ places}} \quad \underbrace{0 \quad 0 \quad \cdots \quad 0}_{(m-i-1) \cdot m \text{ times}}.$$

Similarly, the $(i+1)^{th}$ row of the matrix $J\bar{U}_2(\bar{P})$ has the following form:

$$\underbrace{\overbrace{0 \quad \cdots \quad 0}^{i \text{ times}} \quad \frac{\bar{\Pi}_2^{CM}(\overrightarrow{0, i\Delta})}{\bar{p}^i} \quad 0 \quad \cdots \quad 0}_{\text{First } m \text{ places}} \quad \underbrace{\overbrace{0 \quad \cdots \quad 0}^{i \text{ times}} \frac{\bar{\Pi}_2^{CM}(\overrightarrow{\Delta, i\Delta})}{\bar{p}^i} \quad 0 \quad \cdots \quad 0}_{\text{Second } m \text{ places}} \quad \cdots$$

$$\cdots \quad \underbrace{\overbrace{0 \quad \cdots \quad 0}^{i \text{ times}} \frac{\bar{\Pi}_2^{CM}(\overrightarrow{(m-1)\Delta, i\Delta})}{\bar{p}^i} \quad 0 \quad \cdots \quad 0}_{\text{Last } m \text{ places}}.$$

For the case where $m = 2$, the Jacobian submatrices $J\bar{U}_1(\bar{P})$ and $J\bar{U}_2(\bar{P})$ are given in Figure 2.2.

$$\begin{bmatrix} \frac{\bar{\Pi}_1^{CM}(\overrightarrow{0,0})}{\bar{p}_0} & \frac{\bar{\Pi}_1^{CM}(\overrightarrow{0,\Delta})}{\bar{p}_0} & 0 & 0 \\ 0 & 0 & \frac{\bar{\Pi}_1^{CM}(\overrightarrow{\Delta,0})}{\bar{p}_1} & \frac{\bar{\Pi}_1^{CM}(\overrightarrow{\Delta,\Delta})}{\bar{p}_1} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\bar{\Pi}_2^{CM}(\overrightarrow{0,0})}{\bar{p}^0} & 0 & \frac{\bar{\Pi}_2^{CM}(\overrightarrow{\Delta,0})}{\bar{p}^0} & 0 \\ 0 & \frac{\bar{\Pi}_2^{CM}(\overrightarrow{0,\Delta})}{\bar{p}^1} & 0 & \frac{\bar{\Pi}_2^{CM}(\overrightarrow{\Delta,\Delta})}{\bar{p}^1} \end{bmatrix}$$

Figure 2.2: *Jacobian submatrices for case $m = 2$. $J\bar{U}_1(\bar{P})$ (upper)* and *$J\bar{U}_2(\bar{P})$(lower)*.

Overall, we have obtained a good linear approximation of the set of all expected payoff values for all bidders, $J\bar{U}(\bar{P})\,(P - \bar{P})$, which is uniquely defined by matrix $J\bar{U}(\bar{P})$. Thus, the set of all possible values that an expected payoff may take is approximated by the image space of the linear mapping. Moreover, our research question may be now refined in the following two subquestions:

1. How large is the image space?
2. How often the value of the expected payoff will be different to zero, or in the worst case scenario less than zero?

The following results give answers to these subquestions and represent our main results.

Proposition 2.2. (Rank of the Taylor Expansion Matrix) *For any $m \times m$ belief matrix $\bar{P}$ with all non-zero entries and for which a Crémer-McLean auction exists, the rank of the associated matrix $J\bar{U}(\bar{P})$, defined by* (2.12)*, is at least* $2m-1$.

The proof is derived by elementary transformations of the Jacobian matrix such that it will be turned into a matrix of a row echelon form. Here is an example which shows that the full rank does not hold in general:

Example 2.1. Let us consider the CM auction in the case of two bidders with two possible private values: 0 or 1 (thus, $m=2$ and $\Delta=1$). Let us assume that the seller's belief is given by matrix

$$\bar{P} = \begin{bmatrix} 1/4 & 1/6 \\ 1/3 & 1/4 \end{bmatrix}.$$

In this case we obtain that $\bar{c}_1^T = \begin{bmatrix} -8 & 12 \end{bmatrix}$ and $\bar{c}_2^T = \begin{bmatrix} -8 & 6 \end{bmatrix}$.[8] The associated Jacobian matrix at $\bar{P}$ turns out to take the following values:

$$\begin{bmatrix} \frac{12}{5}\cdot 8 & \frac{12}{5}\cdot(-12) & 0 & 0 \\ 0 & 0 & \frac{12}{7}\cdot 9 & \frac{12}{7}(-12) \\ \frac{12}{7}\cdot 8 & 0 & \frac{12}{7}\cdot(-6) & 0 \\ 0 & \frac{12}{5}\cdot 9 & 0 & \frac{12}{5}\cdot(-6) \end{bmatrix} = \begin{bmatrix} \frac{96}{5} & -\frac{144}{5} & 0 & 0 \\ 0 & 0 & \frac{108}{7} & -\frac{144}{7} \\ \frac{96}{7} & 0 & -\frac{72}{7} & 0 \\ 0 & \frac{108}{5} & 0 & -\frac{72}{5} \end{bmatrix}.$$

One can easily see that the last three rows of the last matrix are linearly independent, therefore, the rank is at least 3. However, the first row is the linear combination of the last three rows (with linear coefficients equal to $14/15$, $7/5$ and $-4/3$, respectively). Therefore, the rank is exactly equal to 3.

The previous result tells us that the set of all possible expected payoff values is at least $(2m-1)-$ dimensional subset in the $2m-$dimensional space $\mathbb{R}^{2m}$. This makes a huge difference compared to the standard setting of the original Crémer-McLean auction where this set was singleton $\{0\}$. Namely, we have just shown the non-robustness of the auction once the seller might hold a slightly different belief to the bidders. In particular, the full surplus extraction is marginalized.

However, the question of the sign of the possible expected payoffs has not been addressed so far. Are the bidders, most of the time, better off to participate in this new modified

[8] v^T is the transpose of a vector v.

set compared to the original surplus extracting auction (i.e. do they get positive expected payoffs most of the time) or would they not want to participate in the auction at all? To address this question we focus on the participation constraints of the new set-up and analyze whether they still hold. The following result shows the participation constraints failure in the modified setting.

Corollary 2.1. *For any seller's belief $\bar{P}$ about the joint distribution of bidders' values, for which the Crémer-McLean auction exists and whose entries are positive, there exists another belief P in any neighborhood of $\bar{P}$, such that the Crémer-McLean auction related to $\bar{P}$ and with bidders holding P as their beliefs, will result in a negative expected payoff for at least one value of some bidder.*

Therefore, we have just shown that the participation constraints do not hold anymore in the new set-up. Thus, sometimes a bidder might expect a negative payoff and decide not to participate, leading to the break down of the mechanism. We are interested in assessing what fraction of any fair ε-neighborhood of a seller's belief leads to the failure of the participation constraints. This answer is given by the following proposition:

Proposition 2.3. (Participation Constraints Failure with Common Prior) *For any seller's belief $\bar{P}$ about the joint distribution of bidders' values, for which the Crémer-McLean mechanism exists and whose entries are positive, the set of matrices P that ensures the non-negative expected payoffs for each possible value of any bidder, occupies up to half of any fair neighborhood of $\bar{P}$.*

The argument of the previous proposition is quite simple. Namely, one should firstly note that a fair neighborhood of a belief is a $(m^2 - 1)$ -dimensional Euclidean ball that entirely lies in the simplex of the same dimension. Let us, for a moment fix a bidder with an arbitrarily chosen fixed value. Furthermore, let us concentrate on the hyperplane in $\mathbb{R}^{m^2}$ containing the simplex and refer to this subspace as "simplex space". According to the approximation from Proposition 2.1, the set of the neighborhood that guarantees the bidder a zero expected payoff is an intersection of a hyperplane in $\mathbb{R}^{m^2}$ and the neighborhood (and thus it is the intersection of the hyperplane and the "simplex space"). As an intersection of non-parallel hyperplanes has one dimension less than the hyperplanes, the set with zero-payoff property for the chosen bidder with the chosen value is contained in a hyperplane of the "simplex space" considered as initial space - the hyperplane will from now on be named "zero-payoff hyperplane". Thus, this proves that the set of matrices with full surplus extraction is a negligible part of a neighborhood, as its dimension is less that the dimension of the neighborhood. Note that $\bar{P}$ belongs to the "zero-payoff hyperplane", as the original CM auction has the full surplus extraction property. Furthermore, since the neighborhood is an Euclidean ball whose center $\bar{P}$ lies on the "zero-payoff hyperplane", it must be the case that the hyperplane separates the ball into two equal parts. As the set where the expected payoff is non-negative is characterized by belonging to exactly one of these two parts, the claim of the last proposition follows.

2.2.2 A Suggestion to Overcome the Participation Constraints Failure

Here, a way to overcome to the participation constraints failure will be proposed. Suppose that the seller is aware that his belief might be wrong and that the true bidders' belief matrices are somewhere in an ε-neighborhood of $\bar{P}$, which we shall denote by $\varepsilon(\bar{P})$. According to the considerations from the previous section, the seller is aware of the generic participation failure and therefore should adjust the selling mechanism in order to make the mechanism attractive to all bidders. The seller might for instance define rules for the situation when a bidder does not appear or change the payment from the lotteries (the $\bar{c}_i s$ vectors). We are interested into the later approach, that is, we assess how the seller should adjust vectors $\bar{c}_i s$ in order to guarantee the participation constraints within the whole neighborhood.

Firstly, we shall suppose that ε is small enough, such that for any bidder's belief matrix $P \in \varepsilon(\bar{P})$ we have $\det P_i \neq 0$, i.e. we may construct the associated Crémer-McLean auction. The continuity of the determinant as a function, with matrices entries as arguments, guarantees the existence of such ε. Therefore, we consider $\varepsilon(\bar{P})$ where $\varepsilon(\bar{P}) = \left\{P \in \mathbb{R}^{m^2} \,\middle|\, P \text{ is a belief matrix and } \left|p_{kj} - \bar{p}_{kj}\right| \leq \varepsilon\right\}$ and $P \in \varepsilon(\bar{P}) \Rightarrow \det P_i \neq 0$.

Let us define **the minimal vectors for second price auction**

$$\bar{u}_i^{II_{min}} = \begin{bmatrix} u_i^{II_{min}}(0) \\ u_i^{II_{min}}(\Delta) \\ \vdots \\ u_i^{II_{min}}((m-1)\Delta) \end{bmatrix}$$

for $i \in \{1,2\}$, such that

$$\bar{u}_i^{II_{min}}(k\Delta) = \min_{P\in\varepsilon(\bar{P})} u_i^{II_P}(k\Delta), \; k \in \{0,1,\ldots,m-1\}, i \in \{1,2\}. \tag{2.13}$$

In other words, $\bar{u}_i^{II_{min}}(k\Delta)$ is the minimal expected payoff in the second-price auction that bidder i, with value $k\Delta$, may obtain given that her belief matrix is an element in $\varepsilon(\bar{P})$. Note that $\bar{u}_i^{II_{min}}$ are well-defined by the Weierstrass extreme value theorem.

Similarly, let us define **the minimal $\bar{c}-$vectors**

$$\bar{c}_i^{min} = \begin{bmatrix} \bar{c}_i^{min}(0) & \bar{c}_i^{min}(\Delta) & \cdots & \bar{c}_i^{min}((m-1)\Delta) \end{bmatrix}^T$$

for any $i \in \{1,2\}$ as follows. Firstly, for any $P \in \varepsilon(\bar{P})$ and any bidder i, let $c_i^P = \begin{bmatrix} c_i^P(0) & c_i^P(\Delta) & \cdots & c_i^P((m-1)\Delta) \end{bmatrix}^T$ be the following vector

$$c_i^P = (P_i)^{-1} \cdot \bar{u}_i^{II_{min}}.$$

Further,[9] let

$$\bar{c}_i^{min}(k\Delta) = \min_{P\in\varepsilon(\bar{P})} c_i^P(k\Delta),\ k \in \{0, 1, \ldots, m-1\}, i \in \{1, 2\}. \tag{2.14}$$

Finally, in order to avoid participation constraints failure, the seller could set an analogy to the Crémer-McLean auction with lottery payment vectors $\bar{c}_i^{min}$; that is, he sets the mechanism $\left(Q^{CM_\varepsilon}, \bar{M}^{CM_\varepsilon}\right)$ with:

- $Q^{CM_\varepsilon} = Q^{II}$ (the allocation rule remains the same - it is the standard auction rule where the bidder with the highest bid wins)
- $\bar{M}_i^{CM_\varepsilon}\left(b_i, b_j\right) = M_i^{II}\left(b_i, b_j\right) + \bar{c}_i^{min}\left(b_j\right)$

In this new auction mechanism,[10] the truth-telling is still an equilibrium and participation constraints for this particular mechanism are satisfied everywhere in $\varepsilon(\bar{P})$. Analogously to the derivation of (2.4), the vector of the expected payoffs for bidder i with belief matrix P is

$$\bar{u}_i^{CM_\varepsilon}(P) = u_i^{II_P} - P_i \cdot \bar{c}_i^{min}.$$

If $P \in \varepsilon(\bar{P})$ then using definitions given in expressions (2.13) and (2.14) we have that $u_i^{II_P} \geq \bar{u}_i^{II_{min}}$ and $P_i \cdot \bar{c}_i^{min} \leq P_i \cdot c_i^P = \bar{u}_i^{II_{min}}$ and thus

$$\bar{u}_i^{CM_\varepsilon}(P) \geq \bar{u}_i^{II_{min}} - \bar{u}_i^{II_{min}} = 0.$$

Overall, we suggest a mechanism which has the same construction as the optimal mechanism but with the payments in lotteries being adjusted in order to guarantee participation constraints in a neighborhood of the seller's belief.

The new proposed mechanism is a solution to the problem stated in last subsection but lacks a lot of sound properties. First of all, it depends on the chosen neighborhood of the seller belief (through selection of $\bar{c}_i^{min}(k\Delta)$, which directly depends on ε). Secondly, how restrictive the chosen $\bar{c}_i^{min}$ is (for example, the height of the bet compared to the value of the good), might be an interesting question to tackle. Lastly, the expected payoffs are generically positive and thus the full surplus extraction is not guaranteed.

[9] Note that $c_i^{P_i}$ is not the $c-$vector associated with Crémer-McLean mechanism with belief matrix P_i, because $\bar{u}_i^{II_{min}} \neq \bar{u}_i^{II_P}$.

[10] Note that the payment rule is still defined as earlier introduced in "The Optimal Auction Game: The Crémer-McLean Auction" of Subsection 2.1.2, but with replacement of $\bar{c}_i$ by $\bar{c}_i^{min}$.

2.3 Common Prior Assumption's Relaxation

Here the common prior assumption is relaxed; that is, the bidders do not necessarily share the same belief about the joint distribution of private values. Note that this does not essentially change the analysis of Section 2.1. Indeed, all derivations in that section, where perspectives of different participant types were considered and explored, stay valid. The crucial assumption, which guarantees the existence of the Crémer - McLean auction, is a premise on non-singularity of matrices P_i, and, as such, does not require that these matrices are derived from the same prior, over the set of all combination of private values. Thus, all introduced concepts in 2.1 also hold in non-common prior set-up.

In terms of modeling tools, in this section we suppose that the primitives of the setting are the belief matrices P_is and $\bar{P}_i$s themselves, rather than the joint distribution matrices P and $\bar{P}$. In other words, let us denote a participant's (in particular, the seller's or a bidder's) belief over the joint distribution of private values as his/her zero-order belief. Furthermore, we define as a first-order belief a participant's belief regarding what bidders believe about the distribution of opponents' private values conditional on their own value. Using this new terminology, in the previous sections matrices $\bar{P}$ and P were the seller's and the bidders' zero-order beliefs, whereas $\bar{P}_i$s and P_is were their first-order beliefs, respectively. Moreover, in the previous text the main building blocks were zero-order beliefs. The first-order bidders' beliefs were assumed to be directly derived from the zero-order belief conditioning on their own private values. However, here the building blocks will be the first-order beliefs themselves. These matrices have identical interpretation to previous sections. Thus, $\bar{P}_i$ collects bidder i 's belief over the opponent's private value distribution,[11] conditional on their own private value, from the seller's point of view. Furthermore, we assume that bidder $i's$ true belief matrix is given by P_i. The only difference in the current section is that we do not require that the first-order beliefs are derived from the same zero-order belief, that is, we abandon common prior assumption over the joint distribution of the private values.

Further, note that the seller's and bidders' perspectives stay the same as in Section 2.1. More precisely, the seller sets the same Crémer - McLean auction; that is, the auction with standard allocation rules and payments given by (2.2) and (2.3). The truth-telling continues to be an equilibrium in weakly dominant strategies and the full extraction property still holds true, from the seller's point of view. Therefore, the optimal auction derivation may be extended from the standard set-up to the set-up without the common prior among bidders.

Once we consider the model without common prior, we will examine the robustness under these new conditions. Namely, what happens here when the seller holds slightly inaccurate beliefs about first-order beliefs of bidders? This is the topic tackled in the following subsection.

[11] Note that also here, without loss of generality, we assume there are only two bidders.

2.3.1 Failure of the Participation Constraints

In the spirit of the previous robustness check in Section 2.2, we analyze the set of possible expected payoffs once the seller's beliefs differ from those of the bidders, but "beliefs that are close enough". Because of the relaxation of the common prior assumption, it is clear that this section needs a new definition of the term "being close". Thus, we are going to specify the term "neighborhood" by the following few definitions which are analogies of the neighborhood terms in set-up with common prior.

Definition 2.5. An **ε-neighborhood of a row stochastic non-negative matrix** $\bar{Q} = \left[\bar{q}_{ij}\right]_{i,j=0}^{m-1}$ with $\varepsilon > 0$, is the set of matrices $Q = \left[q_{ij}\right]_{i,j=0}^{j=\overline{0,m-1}}$ such that:

(i) $\left|q_{ij} - \bar{q}_{ij}\right| \leq \varepsilon$ for all $i, j \in \{0, 1, \ldots, m-1\}$ and

(ii) Q is also a row stochastic non-negative matrix.

The neighborhood will be usually denoted by $\varepsilon\left(\bar{Q}\right)$.

Remark. Note that an ε-neighborhood of a matrix $\bar{Q}$ may be equivalently defined as the set of all matrices Q such that $Q = \bar{Q} + D$ where D is a matrix whose entries, denoted by δ_{ij}:

(i) belong to the interval $[-\varepsilon, \varepsilon]$,

(ii) sum up to 0 in each row and

(iii) $q_{ij} = \bar{q}_{ij} + \delta_{ij} \in [0, 1]$ for all i, $j \in \{0, 1, \ldots, m-1\}$

Definition 2.6. An ε-neighborhood of a row stochastic non-negative matrix $\bar{Q} = \left[\bar{q}_{ij}\right]_{i,j=0}^{m-1}$ is said to be **symmetric** if ε satisfies inequality $0 < \varepsilon \leq \min\limits_{i,j:\bar{q}_{ij} \neq 0} \bar{q}_{ij}$

Definition 2.7. An ε-neighborhood of a row stochastic non-negative matrix $\bar{Q} = \left[\bar{q}_{ij}\right]_{i,j=0}^{m-1}$ is said to be **bidirectional** if it contains two matrices Q and Q' such that $\left(q_{ij} - \bar{q}_{ij}\right) \cdot \left(q'_{ij} - \bar{q}_{ij}\right) < 0$ for all possible pairs of i and j.

Remark. There is a bidirectional ε-neighborhood of a row stochastic non-negative matrix $\bar{Q}$ if and only if all entries of the $\bar{Q}$ are strictly positive (and, hence, less than 1).

Definition 2.8. An ε-neighborhood of a row stochastic non-negative matrix $\bar{Q}$ is said to be **fair** if it is bidirectional and if it contains matrix $Q = \bar{Q} + D$ then it contains also matrix $Q' = \bar{Q} - D$.

Fairness is different to symmetry. For example, if $\bar{Q} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$ then its $\frac{1}{2}$-neighborhood is symmetric. However, $Q = \bar{Q} + \begin{bmatrix} \delta & -\delta \\ 1/4 & -1/4 \end{bmatrix}$ belongs to it, but $Q' = \bar{Q} - \begin{bmatrix} \delta & -\delta \\ 1/4 & -1/4 \end{bmatrix}$ does not, for $|\delta| \leq \frac{1}{2}$.

Remark. An ε-neighborhood of a row stochastic non-negative matrix $\bar{Q}$ is fair if and only if all entries of the $\bar{Q}$ are strictly positive and $0 < \varepsilon \leq \min_{i,j \in \{0,1,\ldots,m-1\}} \bar{q_{ij}}$, i.e. if and only if it is a bidirectional and symmetric neighborhood.

Definition 2.9. An (fair) ε-neighborhood of a pair of matrices $(\bar{P}_1, \bar{P}_2)$, denoted as $\varepsilon(\bar{P}_1, \bar{P}_2)$, is the set of all pairs of matrices (P_1, P_2) such that P_i is an element of the (fair) ε-neighborhood of of the matrix $\bar{P}_i$ for $i \in \{1,2\}$.

Mathematically, $\varepsilon(\bar{P}_1, \bar{P}_2)$ is defined as the Cartesian product of $\varepsilon(\bar{P}_1)$ and $\varepsilon(\bar{P}_2)$, that is $\varepsilon(\bar{P}_1, \bar{P}_2) = \varepsilon(\bar{P}_1) \times \varepsilon(\bar{P}_2)$. This emphasizes the fact that the common prior assumption is abandoned as the belief matrices in a neighborhood are chosen independently.

Finally, from now on, we assume that the seller believes that bidders' belief matrices are $(\bar{P}_1, \bar{P}_2)$ and that true bidders' belief matrices, denoted by (P_1, P_2) lie in some fair ε-neighborhood of $(\bar{P}_1, \bar{P}_2)$. We shall show that in any fair ε-neighborhood of seller's belief matrices $(\bar{P}_1, \bar{P}_2)$, satisfying some mild assumptions, there is a substantial part of the neighborhood where participation constraints fail.

2.3.2 Special Case of Two Possible Private Values

Let us first consider a special case in which there are only two possible private values: 0 and Δ. Keeping notation from Section 2.1, once more, we emphasize that, despite the identical notation, the belief matrices $\bar{P}_i$ may not be derived from the same common prior. In other words, $\bar{P}_i$ are independently and arbitrarily chosen from the set of all non-negative row stochastic matrices. Under the usual assumption, that is, the non-singularity of matrices $\bar{P}_i$, the seller is able to uniquely define the Crémer-McLean auction. In addition, we assume that all entries of $\bar{P}_i$ are positive for both bidders in order to be able to consider fair neighborhoods. Given that there are only two private values, from (2.3) we can derive explicit formulas for vectors $\bar{c}_i$. Namely, simple calculations led to the following expressions:

$$\bar{c}_i = \begin{bmatrix} -\Delta \bar{P}(X_{-i} = \Delta \,|X_i = 0)\bar{P}(X_{-i} = 0\,|X_i = \Delta)/\det \bar{P}_i \\ \Delta \bar{P}(X_{-i} = 0\,|X_i = 0)\bar{P}(X_{-i} = 0\,|X_i = \Delta)/\det \bar{P}_i \end{bmatrix}$$

with $\bar{P}(X_{-i} = j\Delta \,|X_i = k\Delta)$ being the element of matrix $\bar{P}_i$ in $(k+1)^{th}$ row and $(j+1)^{th}$ column for any $i \in \{1,2\}$ and $j \in \{0,1\}$.

Furthermore, the bidders' true belief matrices are P_i, where P_i belongs to some fair ε-neighborhood of $\bar{P}_i$. Equivalently, $P_i = \bar{P}_i + D_i$, where $D_i = \begin{bmatrix} \delta^i_{00} & -\delta^i_{00} \\ \delta^i_{\Delta 0} & -\delta^i_{\Delta 0} \end{bmatrix}$ is such that $|\delta^i_{00}| \leq \varepsilon$, $|\delta^i_{\Delta 0}| \leq \varepsilon$ with ε not being greater than any entry of both matrices $\bar{P}_i$.

Applying the last notation in equations (2.10), we obtain expected payoffs for bidders in terms of elements of D_i. Precisely, we find that

$$\begin{aligned} \bar{u}_i^{CM_{P_1,P_2}} &= \begin{bmatrix} \delta^i_{00}\left(\bar{c}_i(\Delta)-\bar{c}_i(0)\right) \\ \delta^i_{\Delta 0}\left(\Delta+\bar{c}_i(\Delta)-\bar{c}_i(0)\right) \end{bmatrix} \qquad (2.15) \\ &= \begin{bmatrix} \Delta\delta^i_{00}\bar{P}(X_{-i}=0\,|X_i=\Delta\,)/\det\bar{P}_i \\ \Delta\delta^i_{\Delta 0}\left(1+\bar{P}(X_{-i}=0\,|X_i=\Delta\,)\right)/\det\bar{P}_i \end{bmatrix}. \end{aligned}$$

The choice of the pair matrices $(D_1,\, D_2)$ which leads to non-negative payoffs defined by (2.15) will be analyzed in the following lines.

For this purpose, note that $\operatorname{sgn}(\det\bar{P}_i) = \operatorname{sgn}(1+\bar{P}(X_{-i}=0\,|X_i=\Delta\,)/\det\bar{P}_i)$.[12] Indeed, it is clearly seen that, when $\det\bar{P}_i$ is positive, the expression $1+\bar{P}(X_{-i}=0\,|X_i=\Delta\,)/\det\bar{P}_i$ is also positive. Now supposing that $\det\bar{P}_i$ is negative, then

$$\begin{aligned} 1+\bar{P}(X_{-i}=0\,|X_i=\Delta\,)/\det\bar{P}_i &< 0 \\ &\Leftrightarrow \\ \bar{P}(X_{-i}=0\,|X_i=\Delta\,) &> -\det\bar{P}_i. \end{aligned}$$

Finally, the last inequality might be written as

$$\bar{P}\left(X_{-i}=0\,|X_i=0\,\right)\bar{P}\left(X_{-i}=\Delta\,|X_i=\Delta\,\right) > \bar{P}\left(X_{-i}=0\,|X_i=\Delta\,\right)\left[P\left(X_{-i}=\Delta\,|X_i=0\,\right)-1\right],$$

where the equality between $\det\bar{P}_i$ and the expression $\bar{P}(X_{-i}=0\,|X_i=0\,)\bar{P}(X_{-i}=\Delta\,|X_i=\Delta\,)$ $-\bar{P}(X_{-i}=\Delta\,|X_i=0\,)\bar{P}(X_{-i}=0\,|X_i=\Delta\,)$ is used. Since all entries of $\bar{P}_i$ lie in the open interval $(0,\,1)$, the right-hand side of the last inequality is always negative whereas the left-hand side of the inequality is always positive. Therefore, the initial inequality holds true for any matrix $\bar{P}_i$ with negative determinant. Finally, we have shown the equality between $\operatorname{sgn}(\det\bar{P}_i)$ and $\operatorname{sgn}(1+\bar{P}(X_{-i}=0\,|X_i=\Delta\,)/\det\bar{P}_i)$.

From this feature, it follows immediately that any bidder with any value will get a non-negative payoff if and only if $\operatorname{sgn}\left(\delta^i_{00}\right)$ and $\operatorname{sgn}\left(\delta^i_{\Delta 0}\right)$ are elements of the set $\{0,\, \operatorname{sgn}(\det\bar{P}_i)\}$, for each i. This sufficient and necessary condition for participation constraints is quite strong. In terms of geometry, this means that only 1/16 of the fair ε-neighborhood satisfies participation constraints.

Let us illustrate this claim. Firstly, note that D_i uniquely defines the pair $\left(\delta^i_{00},\ \delta^i_{\Delta 0}\right)$ in $[-\varepsilon,\varepsilon]^2$ and vice versa. More precisely, by fairness of the neighborhood, any pair of two numbers belonging to $[-\varepsilon,\varepsilon]^2$ uniquely defines a matrix D_i such that matrix P_i with $P_i=\bar{P}_i+D_i$ being an element of the ε-neighborhood of the matrix $\bar{P}_i$. Consequently, the ε-neighborhood of the

[12] For any real number x, $\operatorname{sgn}(x) = \begin{cases} 1, & x>0 \\ -1, & x<0. \\ 0, & x=0 \end{cases}$

matrix $\bar{P}_i$ may be illustrated by a square in the Cartesian coordinate system, whose center is at the origin of the system and an edge equal to 2ε. Note that the origin represents the matrix $\bar{P}_i$. Since, the matrices $P_i s$ are also allowed to be chosen independently and within the ε-neighborhood of the matrices $\bar{P}_i s$, the set of all feasible pair matrices (P_1, P_2) may be represented as pair of squares in $\mathbb{R}^2$ with center at the origin and edges equal to 2ε. Figure 2.3 illustrates this point, i.e. it depicts a fair neighborhood $\varepsilon\left(\bar{P}_1, \bar{P}_2\right)$.

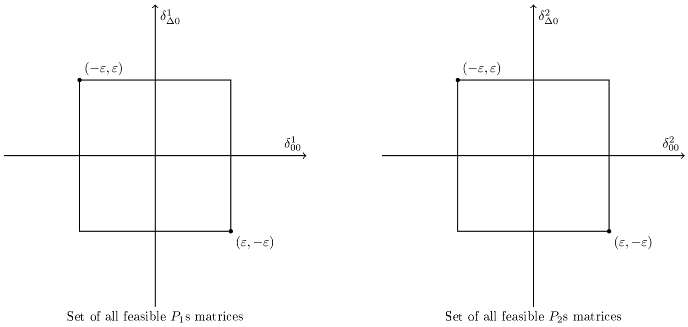

Figure 2.3: *A fair neighborhood* $\varepsilon\left(\bar{P}_1, \bar{P}_2\right)$*- the set of all feasible matrices* $P_i s$*.*

Further, by the sufficient and necessary condition for non-negative payoffs, $\operatorname{sgn}\left(\delta^i_{00}\right)$, $\operatorname{sgn}\left(\delta^i_{\Delta 0}\right) \in \{0, \operatorname{sgn}(\det \bar{P}_i)\}$ for each i, we obtain that all D_is that guarantee non-negative payoff for each value of bidder i lie in one quadrant of the square $[-\varepsilon, \varepsilon]^2$. This is either the first or the third quadrant, depending on $\operatorname{sgn}(\det \bar{P}_i)$. Thus, only $1/4$ of each square satisfies participation constraints. Given that D_1 and D_2 are chosen independently, it follows that only $1/4 \cdot 1/4 = 1/16$ of all feasible pairs (P_1, P_2) guarantees a non-negative payoff for any type of each bidder. Figure 2.4 shows the region where participation constraints hold true.

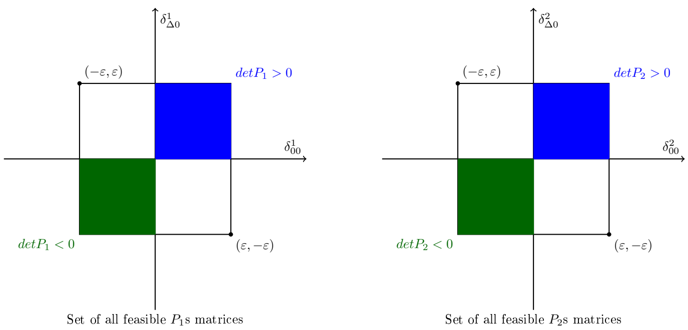

Figure 2.4: *The fraction of feasible* $P_i s$ *that satisfies participation constraints.* In each square, the blue (green) area denote the quadrant where the expected payoff for the corresponding bidder is non-negative given that $\det \bar{P}_i > 0$ ($\det \bar{P}_i < 0$). Hence, for any combination of signs of the determinants, the overall fraction of all feasible pairs of (P_1, P_2), where the payoffs are non-negative, is $1/4 \cdot 1/4 = 1/16$ of the whole neighborhood.

2.3.3 General Case with m Private Values

Here a generalization of the result from the previous subsection to the setting with an arbitrarily finite set of equidistant private values and the lowest value being 0 is derived.

We express the expected payoff as a function of matrices which are analogies of D_i matrices from the special case. This is summarized in the following lemma.

Lemma 2.2. *Suppose that the seller's belief about bidders' belief matrices is $(\bar{P}_1, \bar{P}_2)$ and they are two non-singular matrices. Further, let the bidders' true belief matrices be (P_1, P_2). Then the expected payoff of bidder i with private value $k\Delta$ in the associated Crémer-McLean auction is given by:*

$$\bar{u}_i^{CM_{P_i}}(k\Delta) = \left\langle \delta_i^k, \bar{\Pi}_i^{CM}(k\Delta) \right\rangle, \tag{2.16}$$

where

- *$\langle \cdot, \cdot \rangle$ is the standard scalar product[13] of vectors in $\mathbb{R}^m$*
- *δ_i^k is the vector which denotes $(k+1)^{th}$ row of the matrix $D_i = P_i - \bar{P}_i$*
- *$\bar{\Pi}_i^{CM}(k\Delta)$ is the vector of all possible actual payoffs of a bidder with value $k\Delta$ in the associated Crémer-McLean auction; i.e. its ℓ^{th} entry is equal to the actual payoff of a bidder in the auction when the bid is $k\Delta$ and the opponent's bid is equal to $(\ell-1)\Delta$.*

Proof. The proof follows immediately from equations (2.6) and (2.8). Indeed,

$$\begin{aligned}
\bar{u}_i^{CM_{P_i}}(k\Delta) &= \sum_{j=0}^{m-1} P(X_j = j\Delta | X_i = k\Delta)\bar{\Pi}_i^{CM}(k\Delta, j\Delta) \\
&= \sum_{j=0}^{m-1} \left[\bar{P}(X_j = j\Delta | X_i = k\Delta) + \delta_{ij}\right] \bar{\Pi}_i^{CM}(k\Delta, j\Delta) \\
&= \sum_{j=0}^{m-1} \bar{P}(X_j = j\Delta | X_i = k\Delta)\bar{\Pi}_i^{CM}(k\Delta, j\Delta) + \sum_{j=0}^{m-1} \delta_{ij}\bar{\Pi}_i^{CM}(k\Delta, j\Delta) \\
&= \bar{u}_i^{CM}(k\Delta) + \left\langle \delta_i^k, \bar{\Pi}_i^{CM}(k\Delta) \right\rangle \\
&\overset{(2.6)}{=} 0 + \left\langle \delta_i^k, \bar{\Pi}_i^{CM}(k\Delta) \right\rangle = \left\langle \delta_i^k, \bar{\Pi}_i^{CM}(k\Delta) \right\rangle,
\end{aligned}$$

where $D_i = P_i - \bar{P}_i$ with a typical entry denoted by δ_{ij} and its $(k+1)^{th}$ row vector is denoted by δ_i^k, for $k \in \{0, 1, ..., m-1\}$. □

[13]The standard scalar product of two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ in $\mathbb{R}^m$ is defined as $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{m} x_i y_i$.

In the following proposition, the assessment of the fraction of a fair neighborhood, where participation constraints fail to hold, is given.

Proposition 2.4. (Participation Constraints Failure without Common Prior) *Suppose that the seller's belief about bidders' belief matrices is $(\bar{P}_1, \bar{P}_2)$ where each entry of any of these matrices are positive and the matrices are non-singular. If we consider any fair ε-neighborhood of the pair $(\bar{P}_1, \bar{P}_2)$, then the set of pairs of bidders' true belief matrices (P_1, P_2) in the neighborhood that guarantees participation constraints to be satisfied, occupies maximally $1/2^{2m}$ of the ε-neighborhood, where m is the number of possible values for each bidder.*

The proof is based on the following two properties. Firstly, as Lemma 2.1 also shows, the expected payoff of any bidder and any value is controlled by a single row of her belief matrix (as is the case in the set-up with a common prior). Secondly, the common prior relaxation leads to the fact that the neighborhood of the belief matrix is a Cartesian product of neighborhoods of its rows (which is not the case in the set-up with the common prior). Moreover, exactly half of each of these $2m$ fair row-neighborhoods lead to a non-negative expected payoff to the associated bidder with the associated value. Lastly, the independence in the choice of rows in the belief matrices, together with the previous considerations lead to the result.

Overall, we have shown that in this modified setting without the common prior assumption, the participation constraints fail to hold even more often than in the standard set-up.

Finally, we conclude this section on common prior relaxation by giving a suggestion on overcoming participation constraints. Namely, a solution which is analogous to the solution proposed in Subsection 2.2.2, is considered. We introduce vectors $\bar{u}_i^{II_{min}}, \bar{c}_i^{min}$ in an identical way to before, with the difference that each appearance of $\varepsilon(\bar{P})$ is literally replaced by $\varepsilon(\bar{P}_i)$. Without any other change, all conclusions from the subsection apply also in this new environment. In particular, the proposed mechanism satisfies the participation constraints.

2.4 Concluding Remarks

In this chapter we examined the robustness of an optimal auction format once uncertainty over the distribution of private values is imposed. Even though one can estimate the distribution of private values from data (obtained from empirical records on auctions) by means of statistical inference, there is no reason not to believe that the distribution might be slightly different than the estimate. Once we allow this type of uncertainty, the selling mechanism will break down, due to participation constraints failure. This particular result is not novel in the literature. However, the level of the introduced uncertainty lies, in a way, between existing models (see Chapter 1-"Introduction" for more details).

The main contribution to the existing models is the quantitative description of the participation failure - a quantitative assessment of how often the participation constraints will fail is given. While other authors concentrated on the feature of full surplus extraction and on showing that it is non-generic, we have in addition, assessed the frequency of participation constraint failure. Furthermore, we have extended the analysis to a set-up without the common prior assumption among bidders and the same results turn out to hold true, even with a stronger violation of the constraints.

Let us comment on a few of the imposed assumptions and methods used in current chapter. Firstly, the results may be easily extended to more than two bidders as well as to a non equal-distanced set of private values. Secondly, in Section 2.2, the Euclidean norm was used as measure of "being closed", whereas in Section 2.3, we used the Chebyshev norm[14] for the same purpose. We believe that the set-up without the common prior assumption from Section 2.3 would lead to the exact the same results once we used the Euclidean norm. The results there mostly rely on the fact that a neighborhood is a Cartesian product, and it generalizes to the case where we use the Euclidean norm. However, one can not claim the same for using the Chebyshev norm in Section 2.2. The results from this section are mostly based on the special geometric feature of an Euclidean ball. Namely, we heavily used the fact that the intersection of an Euclidean ball and a hyperplane containing its center that is also an Euclidean ball, but with one dimension less. The Chebyshev ball does not have this property and, therefore, the same argument could not be applied.

Lastly, let us comment on the result of Proposition 2.3. Namely, we originally believed that the participation constraints fail more than half of the time (as the proposition states). The argument for this belief lies in the fact that the region where the participation constraints for any bidder and any value hold true lies in an orthant whose boundaries are $2m$ hyperplanes in $\mathbb{R}^{m^2}$, where each hyperplane comes from the surplus extraction condition of a certain value of a bidder. The normal vectors of these hyperplanes are given by rows of the Jacobian matrix and therefore, at least m of them are mutually orthogonal to each other. Let us consider the first m rows of the matrix and concentrate on the orthant determined by these m hyperplanes where the participation constraint for bidder 1 of any value hold true. Let us name the orthant as "bidder 1's orthant". On the other side, let us call the set in a fair neighborhood of matrix $\bar{P}_1$, where the participation constraint for bidder 1 always holds true (independent of her value), as "bidder 1's region". Clearly, "bidder 1's region" is the superset of the region where participation constraints for any bidder and any value hold true. "Bidder 1's orthant" is an orthant between hyperplanes which are mutually orthogonal. If we would, theoretically, for the moment , allow matrix P_1 to come from an m^2-dimensional "fair" Euclidean ball with center at $\bar{P}_1$, then "bidder 1's region" would be the intersection of "bidder 1's orthant" and the ball. Furthermore, "bidder 1's region" would occupy $1/2^m$ of the ball, as all hyperplanes which form the region pass through its center $\bar{P}_1$, are orthogonal, and the neighborhood is a symmetric and fair Euclidean ball. Let us name "bidder 1's region" from this theoretical exercise as "theoretical bidder 1's region". Thus, in this theoretical case, the participation constraints would hold true only in $1/2^m$ of any Euclidean ball. However, the result can not be immediately applied to our setting. Namely, matrices P_1 must be belief matrices and

[14]The Chebyshev norm on $\mathbb{R}^k$ is defined by $\|x\|_\infty = \max\limits_{1\le i\le k} |x_i|$ for any $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$.

therefore, matrix P_1 can not be any matrix in an Euclidean ball around a belief matrix. In particular, P_1 is an element of the intersection of the m^2-dimensional "fair" Euclidean ball with center at $\bar{P}_1$ and the simplex. Thus, "bidder 1's region" is actually the intersection of the simplex and "theoretical bidder 1's region". As the simplex is not mutually orthogonal with the hyperplanes forming the "bidder 1's orthant", we can not be sure that the participation constraints hold true in at most $1/2^m$ of any fair neighborhood. However, the case with two private values was analyzed (for details, please see Appendix 2.B) and it shows that participation constraints hold true in up to 25% of any fair neighborhood. For the general case, our best approximation remains to be given by Proposition 2.3.

2.A Proofs

2.A.1 Proof of Lemma 2.1

From the definition of $\bar{U}$, it is clear that all rows of its Jacobian matrix are equal to the gradients of $\bar{u}_i^{CM_P}(x_i)$. $\bar{u}_i^{CM_P}(x_i)$ is here considered as a function from $\mathbb{R}^{m^2}$ to $\mathbb{R}$, which assigns to any matrix P a value equal to the expected payoff $\bar{u}_i^{CM_P}(x_i)$, $i \in \{1,2\}$ and $x_i \in \{0, \Delta, \dots, (m-1)\Delta\}$. Moreover, we shall denote these coordinate functions of $\bar{U}$ by $\{\bar{u}_1^{CM}(x_1)\}_{x_1=0}^{(m-1)\Delta}$ and $\{\bar{u}_2^{CM}(x_2)\}_{x_2=0}^{(m-1)\Delta}$, whereas the value assigned to a matrix P by a typical coordinate function will be written as $\bar{u}_i^{CM}(x_i)(P)$.

Let us obtain the explicit formula for the gradient $\nabla\bar{u}_i^{CM}(\ell\Delta)(P)$, for any private value $\ell\Delta$ and matrix P. For this purpose, a matrix $P \in \mathbb{R}^{m^2}$ will be interpreted as a $m \times m$ matrix and denoted by

$$\begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0\,m-1} \\ p_{10} & p_{11} & \cdots & p_{1\,m-1} \\ & & \vdots & \\ p_{m-1\,0} & p_{m-1\,1} & \cdots & p_{m-1\,m-1} \end{bmatrix}.$$

That is, P is understood to be a matrix whose typical element p_{ij} denotes probability that $X_1 = i\Delta$ and $X_2 = j\Delta$. In future, we will denote by p_ℓ the sum of the elements in the $(\ell+1)^{th}$ row of the matrix P and by p^ℓ the sum of the elements belonging to its $(\ell+1)^{th}$ column, for $\ell = \overline{0, m-1}$.

The explicit form of $\bar{u}_i^{CM}(\ell\Delta)(P)$ is obtained earlier and is equal to:

$$\bar{u}_i^{CM}(\ell\Delta)(P) = \sum_{k=0}^{m-1} P(X_j = k\Delta | X_i = \ell\Delta)\bar{\Pi}_i^{CM}(\ell\Delta, k\Delta).$$

Note that terms $\bar{\Pi}_i^{CM}(\ell\Delta, k\Delta)$s do not depend on the argument P. The terms of the right-hand side in the last equation, which depend on the argument P, are:

$$P(X_j = k\Delta | X_i = \ell\Delta) = \frac{P(X_i = \ell\Delta,\, X_j = k\Delta)}{\sum\limits_{r=0}^{m-1} P\left(X_i = \ell\Delta, X_j = r\Delta\right)},\ k = \overline{0, m-1}.$$

Further, depending whether $i = 1$ or $i = 2$, $P(X_j = k\Delta | X_i = \ell\Delta)$ depends only on the ℓ^{th} row or the ℓ^{th} column of P, respectively. More precisely,

$$P(X_j = k\Delta | X_i = \ell\Delta) = \begin{cases} \frac{p_{\ell k}}{\sum\limits_{r=0}^{m-1} p_{\ell r}}, & i = 1 \\ \frac{p_{k\ell}}{\sum\limits_{r=0}^{m-1} p_{r\ell}}, & i = 2 \end{cases}.$$

Firstly, suppose $i = 1$. Then, the partial derivative of $\bar{u}_1^{CM}(\ell\Delta)(P)$ with respect to any variable, which does not belong to the $(\ell+1)^{th}$ row of P, is equal to zero. This is because the expression for $\bar{u}_1^{CM}(\ell\Delta)(P)$ depends only on the $(\ell+1)^{th}$ row, thus, $\frac{\partial \bar{u}_1^{CM}(\ell\Delta)(P)}{\partial p_{rk}} = 0$ for any $r \neq \ell$. Moreover, from the last formula we get

$$\begin{aligned}
\frac{\partial \bar{u}_1^{CM}(\ell\Delta)(P)}{\partial p_{\ell k}} &= \frac{\sum\limits_{r=0}^{m-1} p_{\ell r} - p_{\ell k}}{\left(\sum\limits_{r=0}^{m-1} p_{\ell r}\right)^2} \bar{\Pi}_1^{CM}(\ell\Delta, k\Delta) - \sum_{\substack{r=0 \\ r\neq k}}^{m-1} \frac{p_{\ell r}}{\left(\sum\limits_{q=0}^{m-1} p_{\ell q}\right)^2} \bar{\Pi}_1^{CM}(\ell\Delta, r\Delta) \\
&= \frac{\sum\limits_{r=0}^{m-1} p_{\ell r}}{\left(\sum\limits_{r=0}^{m-1} p_{\ell r}\right)^2} \bar{\Pi}_1^{CM}(\ell\Delta, k\Delta) - \sum_{r=0}^{m-1} \frac{p_{\ell r}}{\left(\sum\limits_{q=0}^{m-1} p_{\ell q}\right)^2} \bar{\Pi}_1^{CM}(\ell\Delta, r\Delta) \\
&= \frac{1}{\sum\limits_{r=0}^{m-1} p_{\ell r}} \bar{\Pi}_1^{CM}(\ell\Delta, k\Delta) \\
&\quad - \frac{1}{\sum\limits_{q=0}^{m-1} p_{\ell q}} \sum_{r=0}^{m-1} P(X_2 = r\Delta | X_1 = \ell\Delta) \bar{\Pi}_1^{CM}(\ell\Delta, r\Delta) \\
&= \frac{1}{p_\ell} \bar{\Pi}_1^{CM}(\ell\Delta, k\Delta) - \frac{1}{p_\ell} \bar{u}_1^{CM_P}(\ell\Delta).
\end{aligned}$$

Overall, we obtained that the gradient $\nabla \bar{u}_1^{CM}(\ell\Delta)(P) = \left(\frac{\partial \bar{u}_1^{CM}(\ell\triangle)(P)}{\partial p_{rk}}\right)_{r,k=0}^{m-1}$ takes the following form:

$$\frac{\partial \bar{u}_1^{CM}(\ell\Delta)(P)}{\partial p_{rk}} = \begin{cases} 0, & r \neq \ell \\ \frac{1}{p_\ell}\bar{\Pi}_1^{CM}(\ell\Delta, k\Delta) - \frac{1}{p_\ell}\bar{u}_1^{CM_P}(\ell\Delta), & r = \ell \end{cases}. \tag{2.17}$$

Analogously, for $i = 2$ we can obtain that the gradient $\nabla \bar{u}_2^{CM}(\ell\Delta)(P) = \left(\frac{\partial \bar{u}_2^{CM}(\ell\Delta)(P)}{\partial p_{rk}}\right)_{r,k=0}^{m-1}$ has the form:

$$\frac{\partial \bar{u}_2^{CM}(\ell\Delta)(P)}{\partial p_{rk}} = \begin{cases} 0, & k \neq \ell \\ \frac{1}{p^\ell}\bar{\Pi}_2^{CM}(k\Delta, \ell\Delta) - \frac{1}{p^\ell}\bar{u}_2^{CM_P}(\ell\Delta), & k = \ell \end{cases}. \tag{2.18}$$

Thus, we have that the Jacobian of the mapping $\bar{U}$ at the matrix P, $J\bar{U}(P)$, may be denoted as

$$J\bar{U}(P) = \begin{bmatrix} J\bar{U}_1(P) \\ J\bar{U}_2(P) \end{bmatrix}_{2m\times m^2},$$

where

$$J\bar{U}_i(P) = \begin{bmatrix} \nabla \bar{u}_i^{CM}(0)(P) \\ \nabla \bar{u}_i^{CM}(\Delta)(P) \\ \vdots \\ \nabla \bar{u}_i^{CM}((m-1)\Delta)(P) \end{bmatrix}_{m\times m^2}, i = \overline{1,2}.$$

In the last equality the gradient $\nabla \bar{u}_i^{CM}(\ell\Delta)(P)$ was considered as a row vector, whose elements, $\frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{rk}}$, are sorted firstly by index r and then index k. That is, $\nabla \bar{u}_i^{CM}(\ell\Delta)(P)$ takes the following form:

$$\Bigg[\frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{00}} \quad \frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{01}} \quad \dots \quad \frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{0\,m-1}}$$

$$\frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{10}} \quad \frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{11}} \quad \dots \quad \frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{1\,m-1}} \quad \dots$$

$$\frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{m-1\,0}} \quad \frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{m-1\,1}} \quad \dots \quad \frac{\partial \bar{u}_i^{CM}(\ell\Delta)(P)}{\partial p_{m-1\,m-1}}\Bigg] \quad .$$

Finally, plugging expressions from (2.17) and (2.18) into the Jacobian matrix, we obtain the claim of the lemma.

2.A.2 Proof of Proposition 2.2

As a reference, which supports the mathematical tools used here, we kindly recommend Meyer (2000). In order to provide the proof, we establish the following lemma.

Lemma. *Given assumptions in the Proposition 2.2, it follows that:*

For any bidder $i \in \{1,2\}$, there exists a value $j\Delta$ of the other bidder such that the payoff $\bar{\Pi}_i^{CM}(k\Delta,\ j\Delta)$ is non-zero, irrespective of the value $k\Delta$ of bidder i, i.e.

$(\forall i \in \{1,2\})\,(\exists\, j \in \{0,1,\ldots,m-1\})\,(\forall k \in \{0,1,\ldots,m-1\})\,\big(\bar{\Pi}_i^{CM}(k\Delta,\ j\Delta) \neq 0\big)$.

<u>*Proof of the Lemma:*</u>

Suppose the opposite, that is

$$(\exists i \in \{1,2\})\,(\forall j \in \{0,1,\ldots,m-1\})\,(\exists k \in \{0,1,\ldots,m-1\})\,\Big(\bar{\Pi}_i^{CM}(k\Delta, j\Delta) = 0\Big). \tag{2.19}$$

Without loss of generality, we take $i = 1$. We will look at the matrix $\bar{\Pi}_1^{CM}$ whose entry in the $(i+1)^{th}$ row and the $(j+1)^{th}$ column is equal to $\bar{\Pi}_1^{CM}(\overrightarrow{i\Delta,\ j\Delta})$, that is, the matrix is equal to:

$$\begin{bmatrix} -\bar{c}_1\,(0) & -\bar{c}_1\,(\Delta) & \cdots & -\bar{c}_1\,((m-2)\Delta) & -\bar{c}_1\,((m-1)\Delta) \\ \Delta-\bar{c}_1\,(0) & -\bar{c}_1\,(\Delta) & \cdots & -\bar{c}_1\,((m-2)\Delta) & -\bar{c}_1\,((m-1)\Delta) \\ & & \vdots & & \\ (m-2)\Delta-\bar{c}_1\,(0) & (m-3)\Delta-\bar{c}_1\,(\Delta) & \cdots & -\bar{c}_1\,((m-2)\Delta) & -\bar{c}_1\,((m-1)\Delta) \\ (m-1)\Delta-\bar{c}_1\,(0) & (m-2)\Delta-\bar{c}_1\,(\Delta) & \cdots & \Delta-\bar{c}_1\,((m-2)\Delta) & -\bar{c}_1\,((m-1)\Delta) \end{bmatrix}.$$

Therefore, the assumption (2.19) is equivalent to say that a row of the last matrix has all elements equal to zero. Consequently, the claim (2.19) is equivalent to assuming the following set of conditions:

$(a_0)\ \bar{c}_1\,(0) \in \{0,\Delta,\ldots,(m-1)\Delta\}$,

$(a_1)\ \bar{c}_1\,(\Delta) \in \{0,\Delta,\ldots,(m-2)\Delta\}$,

$(a_2)\ \bar{c}_1\,(2\Delta) \in \{0,\Delta,\ldots,(m-3)\Delta\}$,

$\vdots$

$(a_k)\ \bar{c}_1\,(k\Delta) \in \{0,\Delta,\ldots,(m-k-1)\Delta\}$,

$\vdots$

$(a_{nm-2})\,\bar{c}_1\left((m-2)\Delta\right) \in \{0, \Delta\}$ and

$(a_{nm-2})\,\bar{c}_1\left((m-1)\Delta\right) = 0.$

This set of conditions implies that all components of the vector $\bar{c}_1$ are non-negative. In addition, according to the first equation of the system $\bar{P}_1\bar{c}_1 = \bar{u}_1^{II}$ and the assumption on the non-vanishing entries of $\bar{P}$, it must be that $\bar{c}_1\,(k\Delta) = 0$ for all k. However, this is impossible, because of the second equation in the system $\bar{P}_1\bar{c}_1 = \bar{u}_1^{II}$ and the assumption that all entries of $\bar{P}$ are non-zero (which also implies that each entry of $\bar{P}_1$ are positive and $\bar{u}_1^{II}(\Delta) > 0$). Hence, we have obtained a contradiction. $\diamondsuit$

Proof of the Proposition:

By the previous lemma, we may suppose that $i_0 \in \{0, 1, \ldots, m-1\}$ is such that $\bar{\Pi}_2^{CM}(\overrightarrow{i_0\Delta,\, j\Delta}) \neq 0$ for all js.

Further, we perform the following elementary transformations on the matrix $J\bar{U}(\bar{P})$:

1. Switch positions of the first m rows with the next m rows, i.e. exchange the position of the submatrix $J\bar{U}_1(\bar{P})_{m\times m^2}$ with the position of the submatrix $J\bar{U}_2(\bar{P})_{m\times m^2}$.
2. If $i_0 = 0$, then skip this step. Otherwise, in case of $i_0 \neq 0$:
   (a) switch the place of the first m columns of the matrix $J\bar{U}(\bar{P})$ with the place of the m consecutive columns, starting from the $(i_0+1)^{th}$ column and
   (b) exchange the position of the $(m+1)^{st}$ row with the position of the $(m+i_0+1)^{th}$ row (this means, in the transformed submatrix $J\bar{U}_1(\bar{P})_{m\times m^2}$, replace the position of the first and the $(i_0+1)^{th}$ row).
3. Reorder the $(m+1)^{th}$,$(m+2)^{th}$,..., $(2m)^{th}$ rows in the following way:
   (a) put $(m+1)^{st}$ row at the position of the $(2m)^{th}$ row and
   (b) shift other rows for one place up, i.e. put the k^{th} row at the position of the $(k-1)^{th}$ row, for all $k \in \{m+2, m+3, ...2m\}$.

   In other words, shift circularly to the left the set made of the $(m+1)^{st}$,$(m+2)^{th}$,... and the $(2m)^{th}$ rows.

In this way, we have transformed the matrix $J\bar{U}(\bar{P})$ into the following matrix:

$$\begin{bmatrix} J\bar{U}_2^{i_0}(\bar{P})_{m\times m^2} \\ J_{-i_0}\bar{U}_1(\bar{P})_{(m-1)\times m^2} \\ \left(u_1^{i_0}\right)_{1\times m^2} \end{bmatrix}_{2m\times m^2}.$$

The submatrix

$$\begin{bmatrix} J\bar{U}_2^{i_0}(\bar{P})_{m\times m^2} \\ J_{-i_0}\bar{U}_1(\bar{P})_{(m-1)\times m^2} \end{bmatrix}_{(2m\text{-}1)\times m^2}$$

has the following form:

$$\begin{bmatrix}
* & 0 & \cdots & 0| & * & 0 & \cdots & 0| & \cdots & \cdots & \cdots & |* & 0 & \dots & 0 \\
0 & * & \dots & 0| & 0 & * & \dots & 0| & \cdots & \cdots & & |0 & * & \dots & 0 \\
 & & \ddots & & & & \ddots & & & & & & & \ddots & \\
0 & 0 & \dots & *| & 0 & 0 & \cdots & *| & \cdots & \cdots & \cdots & |0 & 0 & \dots & * \\
0 & 0 & \dots & 0| & * & * & \dots & *| & 0 & \dots & 0 & |0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0| & 0 & 0 & \dots & 0| & * & \dots & * & |0 & 0 & \dots & 0 \\
 & & & & & & & & & & & \ddots & & & \\
0 & 0 & \dots & 0| & 0 & 0 & \dots & 0| & 0 & \dots & 0 & |* & * & \dots & *
\end{bmatrix}.$$

According to the definition of the submatrix and the last lemma, the submatrix has a row echelon form. Therefore, the rank of the submatrix is $2m-1$. Consequently, the rank of the matrix $J\bar{U}(\bar{P})$ is at least $2m-1$.

2.A.3 Proof of Corollary 2.1

Let $\bar{P}$ be a matrix of the joint distribution of the private values believed by the seller, which satisfies the condition of the corollary. Let $\varepsilon(\bar{P})$ be a neighborhood of $\bar{P}$ in $\mathbb{R}^{m^2}$ as in Definition 2.1. By the Taylor expansion in Proposition 2.1, for any the bidders' true belief matrix $P \in \varepsilon(\bar{P})$ that is close enough to the $\bar{P}$, the vector of their expected payoffs can be well-approximated by the vector $J\bar{U}(\bar{P})(P-\bar{P})$.

Let us consider the restriction of the linear mapping $J\bar{U}(\bar{P})$ to a hyperplane $\mathcal{D} = \left\{ D \in \mathbb{R}^{m^2} \,\middle|\, \sum_i \sum_j d_{ij} = 0 \right\}$. Note that $\mathcal{D}$ is a subspace in the space $\mathbb{R}^{m^2}$. The kernel of the restriction mapping is a subspace of the kernel of the original linear mapping. Thus, by that fact, Proposition 2.2 and rank-nullity theorem for a linear mapping, we obtain that the dimension of the kernel of the restriction is at most equal to m^2-2m-1, which is strictly less than m^2. Hence, there exists some $D \in \mathcal{D}$ such that $J\bar{U}(\bar{P})D \neq 0$. If all components of the vector $J\bar{U}(\bar{P})D$ would be positive we redefine the matrix D by assigning it the value of its complement: $-D$. At the end, we have found a matrix $D \in \mathcal{D}$ such that the vector $J\bar{U}(\bar{P})D$ has at least one negative component. Let $P = \bar{P} + \delta D$, where $\delta > 0$ is small enough to satisfy the following two conditions:

1. P is a belief matrix, i.e. its entries are non-negative and less than 1 and
2. P is close enough to the $\bar{P}$, i.e. $P \in \varepsilon(\bar{P})$.

The parameter $\delta > 0$ is possible to choose, because all entries of the $\bar{P}$ belong to the open interval $(0, 1)$.

Thus, vector $\bar{U}(P)$ may be approximated by $J\bar{U}(\bar{P})\,(P - \bar{P})$, i.e.

$$\begin{aligned} \bar{U}(P) &\approx J\bar{U}(\bar{P})\,(P - \bar{P}) \\ \bar{U}(P) &\approx \delta \cdot J\bar{U}(\bar{P})D. \end{aligned}$$

Since δ is positive and there exists a negative entry of vector $J\bar{U}(\bar{P})D$, $\bar{U}(P)$ is such vector whose at least one components is negative. Overall, we have found a well-defined belief matrix P close enough to $\bar{P}$ such that it guarantees a negative payoff for at least one value of some bidder, given that the true joint distribution of private values is according to P, whereas the seller believes that the distribution is given by $\bar{P}$.

2.A.4 Proof of Proposition 2.3

Let $\bar{P}$ be a matrix satisfying the conditions of the proposition and let $\varepsilon(\bar{P})$ be a fair neighborhood of $\bar{P}$ for some $\varepsilon > 0$. Furthermore, let $B_2(\bar{P}, \varepsilon)$ be an Euclidean ball with center at $\bar{P}$ and a radius $\varepsilon > 0$ in $\mathbb{R}^{m^2}$. Let $\mathcal{H}_\mathbf{1}$ denote a hyperplane in $\mathbb{R}^{m^2}$ whose normal vector is equal to the vector $\mathbf{1} = (1, 1, \ldots, 1)$. That is, $\mathcal{H}_\mathbf{1}$ is the hyperplane which contains the $(m^2 - 1)-$dimensional simplex. Thus, by definition of a neighborhood, we have that $\varepsilon(\bar{P}) = B_2(\bar{P}, \varepsilon) \cap \mathcal{H}_\mathbf{1}$, i.e the neighborhood is an intersection of the Euclidean ball and the hyperplane. Since the hyperplane contains the center of the ball, $\varepsilon(\bar{P})$ is also an Euclidean ball whose dimension is $m^2 - 1$. Let us look at a subset of the neighborhood, which guarantees bidders non-negative payoff.

We approximate the vector of the expected payoffs for the bidders by $J\bar{U}(\bar{P})\,(P - \bar{P})$, where $P \in \varepsilon(\bar{P})$ is the true joint distribution of private values. We want to prove that set X, defined as

$$X = \{\, P \in \varepsilon(\bar{P}) |\ \text{All entries of the vector } \bar{U}(P) \text{ are non-negative} \},$$

occupies maximally 1/2 of $\varepsilon(\bar{P})$. According to the Taylor approximation, we approximate the set X by a set $\tilde{X}$, where

$$\tilde{X} = \{\, P \in \varepsilon(\bar{P}) |\ \text{All entries of the vector } J\bar{U}(\bar{P})\,(P - \bar{P}) \ \text{ are non-negative} \}.$$

Let the i^{th} row of $J\bar{U}(\bar{P})$ be denoted by $v^1_{i-1} \in \mathbb{R}^{m^2}$ for $i \le m$ and $v^2_{i-m-1} \in \mathbb{R}^{m^2}$ for $i > m$.

Using this notation, we obtain that

$$\tilde{X} = \left\{ P \in \varepsilon(\bar{P}) |\ \left\langle v^i_j, P - \bar{P} \right\rangle \ge 0 \text{ for all } i \in \{1, 2\} \text{ and } j \in \{0, 1, \ldots, m-1\} \right\},$$

with $\langle \cdot, \cdot \rangle$ being Euclidean scalar product of vectors.

For all $i \in \{1, 2\}$ and $j \in \{0, 1, \ldots, m-1\}$ let us introduce the following sets:

$$\begin{aligned} B_{ij}^{0} &= \left\{ P \in \varepsilon(\bar{P}) | \left\langle v_j^i, P - \bar{P} \right\rangle = 0 \right\}, \\ B_{ij}^{+} &= \left\{ P \in \varepsilon(\bar{P}) | \left\langle v_j^i, P - \bar{P} \right\rangle > 0 \right\}, \\ B_{ij}^{-} &= \left\{ P \in \varepsilon(\bar{P}) | \left\langle v_j^i, P - \bar{P} \right\rangle < 0 \right\} \text{ and} \\ B_{ij}^{\geq 0} &= \left\{ P \in \varepsilon(\bar{P}) | \left\langle v_j^i, P - \bar{P} \right\rangle \geq 0 \right\}. \end{aligned}$$

Therefore , we have that

$$\tilde{X} = \bigcap_{i \in \{1,2\}, j \in \{0,1,\ldots,m-1\}} B_{ij}^{\geq 0}.$$

Hence, the fraction that $\tilde{X}$ occupies in the neighborhood $\varepsilon(\bar{P})$ is less or equal to the fraction that $B_{ij}^{\geq 0}$ occupies in the $\varepsilon(\bar{P})$, for arbitrarily chosen i and j. Let us show that $B_{ij}^{\geq 0}$ occupies maximally a half of the neighborhood. This follows trivially from the following property: For any $j \in \{0, 1, \ldots, m-1\}$, B_{ij}^{0} is an $\left(m^2 - 2\right)$ −dimensional subset in $\varepsilon(\bar{P})$ that divides the neighborhood into two congruent sets:[15] B_{ij}^{+} and B_{ij}^{-}. Indeed, B_{ij}^{0} is a subset of an intersection of two hyperplanes: $\mathcal{H}_{\mathbf{1}}$ and $\mathcal{H}_{\mathbf{v_j^i}}$, where $\mathcal{H}_{v_j^i}$ is the hyperplane whose normal vector is v_j^i. Since v_j^i and $\mathbf{1}$ are not collinear vectors, we have that $\mathcal{H}_{\mathbf{1}} \cap H_{v_j^i}$ is an $\left(m^2 - 2\right)$ −dimensional set. Thus, B_{ij}^{0} is also an $\left(m^2 - 2\right)$ −dimensional subset in $\varepsilon(\bar{P})$, and, as such, it is negligible (because its dimensionality is strictly less than dimensionality of the neighborhood). Furthermore, $\mathcal{H}_{\mathbf{1}} \cap H_{v_j^i}$ is a hyperplane in $\mathcal{H}_{\mathbf{1}}$, which contains $\bar{P}$, therefore, $\mathcal{H}_{\mathbf{1}} \cap H_{v_j^i}$ splits $\mathcal{H}_{\mathbf{1}}$ into two congruent halfspaces. Moreover, B_{ij}^{0} splits $\varepsilon(\bar{P})$ into two congruent sets: B_{ij}^{+} and B_{ij}^{-}. Indeed, in the space $\mathcal{H}_{\mathbf{1}}$, $\varepsilon(\bar{P})$ is a fair Euclidean ball with center at $\bar{P}$. Furthermore, in that space B_{ij}^{0} is such that it belongs to a hyperplane containing the center of the ball. Therefore, each B_{ij}^{+} and B_{ij}^{-} occupies a half of the $\varepsilon(\bar{P})$. Since B_{ij}^{0} is a negligible part of $\varepsilon(\bar{P})$, we may say that $B_{ij}^{\geq 0} = B_{ij}^{+} \cup B_{ij}^{0}$ occupies a half of $\varepsilon(\bar{P})$.

Finally, as $\tilde{X} \subseteq B_{ij}^{\geq 0}$, the set $\tilde{X}$ occupies maximally a half of the neighborhood, $P \in \varepsilon(\bar{P})$, which was to be demonstrated.

[15] Two sets of points are called congruent if and only if one can be transformed into the other by an isometry. In particular, translations, rotations or reflections are isometries.

2.A.5 Proof of Proposition 2.4

Let each of two bidders have m possible private values: $0, \Delta, 2\Delta, \ldots, (m-1)\Delta$. Suppose that the seller's belief about the belief matrices is $(\bar{P}_1, \bar{P}_2)$. Moreover, $\bar{P}_1$ and $\bar{P}_2$ are assumed to be non-singular matrices and each entry of any of these matrices is positive. Let us consider an arbitrary fair ε-neighborhood of the belief matrices $(\bar{P}_1, \bar{P}_2)$. We wish to assess the fraction of the neighborhood, that contains any pair of the bidders' belief matrices, (P_1, P_2), which guarantees a non-negative payoff for any bidder with any possible private value in the CM auction associated to the pair $(\bar{P}_1, \bar{P}_2)$.

Let (P_1, P_2) be a pair of the bidder's belief matrices in the neighborhood. Let $D_i = P_i - \bar{P}_i$ and $\delta_i^k \in \mathbb{R}^m$ be the $(k+1)^{th}$ row of the matrix D_i, for any $i \in \{1,2\}$ and $k \in \{0, 1, \ldots, m-1\}$.

Firstly, we will consider bidder 1 with an arbitrarily chosen value $k\triangle$. According to the definition of the CM auction, we have that

$$\bar{u}_1^{CM_{P_1}}(k\Delta) = \left\langle p_1^k, \bar{\Pi}_i^{CM}(k\Delta) \right\rangle, \tag{2.20}$$

where p_1^k is the $(k+1)^{th}$ row of the matrix P_1.

For purpose of this section, we shall firstly establish the following properties.

Property 1.

(i) The vector $\bar{\Pi}_1^{CM}(k\Delta) = \left(\bar{\Pi}_1^{CM}(k\Delta, j\Delta)\right)_{j=0}^{m-1}$ *is not a zero-vector, i.e.*

$\bar{\Pi}_1^{CM}(k\Delta) \neq (0, 0, \ldots, 0).$

(ii) The vector $\bar{\Pi}_1^{CM}(k\Delta) = \left(\bar{\Pi}_1^{CM}(k\Delta, j\Delta)\right)_{j=0}^{m-1}$ *has both negative and positive entries.*

Proof for claims in Property 1:

(i) Suppose the opposite, there exists $k_0 \in \{0, 1, \ldots, m-1\}$ such that $\bar{\Pi}_1^{CM}(k_0\Delta) = (0, 0, \ldots, 0)$. By definition of the actual payoff in CM auction (see equation (2.4)) this is equivalent to saying that there exists $k_0 \in \{0, 1, \ldots, m-1\}$ such that $\bar{c}_1 = \Pi_1^{II}(k_0\Delta)$, where $\Pi_1^{II}(k_0\Delta)$ is the vector of all possible actual payoffs for bidder 1 with value $k_0\Delta$ in a second-price auction. Mathematically, $\Pi_1^{II}(k_0\Delta) = \left[\Pi_1^{II}(k_0\Delta, \ell\Delta)\right]_{\ell=0}^{m-1}$ with a typical element

$$\Pi_1^{II}(k_0\Delta, \ell\Delta) = \begin{cases} (k_0 - \ell)\Delta, & \ell < k_0 \\ 0, & \text{otherwise} \end{cases},$$

for any $\ell \in \{0, 1, \ldots, m-1\}$.

Furthermore, we have defined the vector $\bar{c}_1$ as a unique solution of the following system of linear equations:

$$\bar{P}_1 \cdot \bar{c}_1 = u_1^{II}.$$

Or, equivalently, in terms of the scalar product, it becomes:

$$\left\langle \bar{p}_1^{\ell}, \bar{c_1} \right\rangle = \bar{u}_1^{II}\left(\ell\Delta\right), \ell = 0, \ldots, m-1,$$

where $\bar{p}_1^{\ell}$ denotes the $(\ell+1)^{th}$ row of the matrix $\bar{P}_1$. By the similar reasoning to the representation of the expected payoff in Crémer-McLean mechanism and by definition of the actual payoff in the second-price auction , Π^{II}, we obtain that the last system of equations may be rewritten as:

$$\begin{aligned} \left\langle \bar{p}_1^{\ell}, \bar{c}_1 \right\rangle &= \left\langle \bar{p}_1^{\ell}, \Pi_1^{II}\left(\ell\Delta\right) \right\rangle \quad , \ell = 0, \ldots, m-1 \\ &\Updownarrow \\ \left\langle \bar{p}_1^{\ell}, \bar{c}_1 - \Pi_1^{II}\left(\ell\Delta\right) \right\rangle &= 0 \quad , \ell = 0, \ldots, m-1. \end{aligned}$$

By the assumption on $\Pi_1^{II}(k_0\Delta)$, we have that $\bar{c}_1 = \Pi_1^{II}(k_0\Delta)$ and, hence, we have obtained that the following system must be satisfied:

$$\left\langle \bar{p}_1^{\ell}, \Pi_1^{II}(k_0\Delta) - \Pi_1^{II}\left(\ell\Delta\right) \right\rangle = 0,\ \ell = 0, \ldots, m-1.$$

However, if we take a closer look to the explicit form of the vector $\Pi_1^{II}(k_0\Delta) - \Pi_1^{II}\left(\ell\Delta\right)$, we obtain that the above system can not be satisfied. Indeed, by definition of the actual payoff in the second-price auction, we obtained that, if $k_0 > \ell$, the vector $\Pi_1^{II}(k_0\Delta) - \Pi_1^{II}\left(\ell\Delta\right)$ is equal to

$$\left(\overbrace{\left(k_0-\ell\right)\Delta, \left(k_0-\ell\right)\Delta \ldots, \left(k_0-\ell\right)\Delta,}^{\ell \text{ times}} \right.$$

$$\overbrace{\left(k_0-\ell\right)\Delta, \left(k_0-\ell-1\right)\Delta, \ldots, \Delta,}^{(k_0-\ell) \text{ places}}$$

$$\left. \overbrace{0, 0, \ldots 0}^{(m-k_0) \text{ times}} \right).$$

In the case where $k_0 < \ell$, we have that

$$\Pi_1^{II}(k_0\Delta) - \Pi_1^{II}(\ell\Delta) = -\left(\Pi_1^{II}(\ell\Delta) - \Pi_1^{II}(k_0\Delta)\right) .$$

Therefore, the vector $\Pi_1^{II}(k_0\Delta) - \Pi_1^{II}(\ell\Delta)$ has either all non-negative entries and at least one positive entry (if $k_0 > \ell$) or it has all non-positive entries and at least one negative entry (if $k_0 > \ell$).[16] Together with the assumption that all entries of the matrix P_1 are positive, we obtained that $\left\langle \bar{p}_1^{\ell}, \Pi_1^{II}(k_0\Delta) - \Pi_1^{II}(\ell\Delta) \right\rangle \gtrless 0$ (if $\ell \lesseqgtr k_0$) for all $\ell \neq k_0$. We have obtained a contradiction. Therefore, it must be that $\bar{\Pi}_1^{CM}(k\Delta) \neq (0, 0, \ldots 0)$ for any k.

(ii) This claim follows from the fact that the expected payoffs under Crémer-McLean mechanism where $(P_1, P_2) = (\bar{P}_1, \bar{P}_2)$ (i.e. the seller has correct beliefs) are equal to 0 and the assumption that $\bar{P}_i$ has positive entries. Indeed, by the definition of the mechanism and the expected payoff we obtain that

$$\begin{aligned} \bar{u}_1^{CM}(k\Delta) &= \sum_{k=0}^{m-1} \bar{P}_1(X_2 = \ell\Delta | X_1 = k\Delta) \bar{\Pi}_1^{CM}(k\Delta, \ell\Delta) \\ &= \left\langle \bar{p}_1^k, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle = 0, \end{aligned}$$

where $\bar{p}_1^k$ is the $(k+1)^{th}$ row of the matrix $\bar{P}_1$. Given that the vector has only positive entries it must be that either $\bar{\Pi}_1^{CM}(k\Delta)$ is a zero-vector (the vector with all entries equal to zero) or it has both negative and positive entries (and perhaps contains some zeros). By (i) of this property, the latter must be true, which proves the claim (ii). ♢

Property 2. *In a fair ε-neighborhood of $\bar{P}_1$, the set of matrices P_1 that guarantees non-negative payoff to bidder 1 with value $k\Delta$, occupies at most a half of the neighborhood.*

Proof of Property 2:

For this purpose, we introduce the following notation. Let the fair ε-neighborhood of $\bar{P}_1$ be denoted by $\varepsilon(\bar{P}_1)$. Since, in the current set-up without common prior assumption, we are allowed to perturb each row of $\bar{P}_1$ independently, we may write $\varepsilon(\bar{P}_1)$ as a Cartesian product of neighborhoods of its row vectors. That is, we have that

$$\varepsilon(\bar{P}_1) = \mathop{\times}_{j=0}^{m-1} \varepsilon\left(\bar{p}_1^j\right),$$

where

$$\varepsilon\left(\bar{p}_1{}^j\right) = \left\{ p = (p_0, p_1, \ldots, p_{m-1}) \in \mathbb{R}^m | \sum_{k=0}^{m-1} p_k = 1 \text{ and } \left| p_k - \bar{p}_{jk}^1 \right| \leq \varepsilon \text{ for any } k \right\}$$

($\bar{p}_{jk}^1$ is the element in the $(j+1)^{th}$ row and the $(k+1)^{th}$ column of the matrix $\bar{P}_1$).

The neighborhood $\varepsilon(\bar{P}_1)$ may be represented as a disjoint union of the following three sets:

[16] Note that, since $m \geq 2$, there exists $\ell \neq k_0$.

- $\varepsilon_k^+(\bar{P}_1) = \left\{ P_1 \in \varepsilon(\bar{P}_1) \,\middle|\, \bar{u}_1^{CM_{P_1}}(k\Delta) = \left\langle p_1^k, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle > 0 \right\}$,
- $\varepsilon_k^-(\bar{P}_1) = \left\{ P_1 \in \varepsilon(\bar{P}_1) \,\middle|\, \bar{u}_1^{CM_{P_1}}(k\Delta) = \left\langle p_1^k, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle < 0 \right\}$ and
- $\varepsilon_k^0(\bar{P}_1) = \left\{ P_1 \in \varepsilon(\bar{P}_1) \,\middle|\, \bar{u}_1^{CM_{P_1}}(k\Delta) = \left\langle p_1^k, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle = 0 \right\}$.

We want to assess the fraction that the set $\varepsilon_k^+(\bar{P}_1) \cup \varepsilon_k^0(\bar{P}_1)$ occupies in $\varepsilon(\bar{P}_1)$ and show that it can not be greater than a half of the neighborhood.

Further, since the expected payoff for bidder 1 with value $k\triangle$ depends solely on the $(k+1)^{th}$ row of $P_1 \in \varepsilon(\bar{P}_1)$, denoted as p_1^k, we get:[17]

- $\varepsilon_k^+(\bar{P}_1) = \bigtimes_{j=0}^{k-1} \varepsilon\left(\bar{p_1}^j\right) \times \varepsilon^+\left(\bar{p_1}^k\right) \bigtimes_{j=k+1}^{m-1} \varepsilon\left(\bar{p_1}^j\right)$,
- $\varepsilon_k^-(\bar{P}_1) = \bigtimes_{j=0}^{k-1} \varepsilon\left(\bar{p_1}^j\right) \times \varepsilon^-\left(\bar{p_1}^k\right) \bigtimes_{j=k+1}^{m-1} \varepsilon\left(\bar{p_1}^j\right)$ and
- $\varepsilon_k^0(\bar{P}_1) = \bigtimes_{j=0}^{k-1} \varepsilon\left(\bar{p_1}^j\right) \times \varepsilon^0\left(\bar{p_1}^k\right) \bigtimes_{j=k+1}^{m-1} \varepsilon\left(\bar{p_1}^j\right)$,

where

- $\varepsilon^+\left(\bar{p_1}^k\right) = \left\{ p = (p_0, p_1, \ldots, p_{m-1}) \in \mathbb{R}^m \,\middle|\, p \in \varepsilon\left(\bar{p_1}^k\right) \text{ and } \left\langle p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle > 0 \right\}$,
- $\varepsilon^0\left(\bar{p_1}^k\right) = \left\{ p = (p_0, p_1, \ldots, p_{m-1}) \in \mathbb{R}^m \,\middle|\, p \in \varepsilon\left(\bar{p_1}^k\right) \text{ and } \left\langle p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle = 0 \right\}$ and
- $\varepsilon^+\left(\bar{p_1}^k\right) = \left\{ p = (p_0, p_1, \ldots, p_{m-1}) \in \mathbb{R}^m \,\middle|\, p \in \varepsilon\left(\bar{p_1}^k\right) \text{ and } \left\langle p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle < 0 \right\}$.

Consequently, the fraction that $\varepsilon_k^s(\bar{P}_1)$ occupies in $\varepsilon(\bar{P}_1)$ is identical to the fraction that $\varepsilon^s\left(\bar{p_1}^k\right)$ occupies in the $\varepsilon\left(\bar{p_1}^k\right)$ for any $s \in \{+,-,0\}$. Therefore, we concentrate on the fraction that the set $\varepsilon^+\left(\bar{p_1}^k\right) \cup \varepsilon^0\left(\bar{p_1}^k\right)$ occupies in $\varepsilon\left(\bar{p_1}^k\right)$.

First of all, note that we may leave out the set $\varepsilon^0\left(\bar{p_1}^k\right)$ from consideration and neglect it. Indeed,

$$\varepsilon^0\left(\bar{p_1}^k\right) = \left\{ p = (p_0, p_1, \ldots, p_{m-1}) \in \mathbb{R}^m \,\middle|\, p \in \varepsilon\left(\bar{p_1}^k\right) \text{ and } \left\langle p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle = 0 \right\}$$

Therefore, $p = (p_0, p_1, \ldots, p_{m-1}) \in \varepsilon^0\left(\bar{p_1}^k\right)$ is an $m-$dimensional vector in the neighborhood of $\bar{p_1}^k$ which satisfies the following system of two linear equations:

[17]If $k = 0$ ($k = m-1$), the set $\bigtimes_{j=0}^{k-1} \varepsilon\left(\bar{p_1}^j\right)$($\bigtimes_{j=k+1}^{m-1} \varepsilon\left(\bar{p_1}^j\right)$) should be ignored from the equation.

$$\begin{aligned} \left\langle p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle &= 0 \\ \langle p, \mathbf{1} \rangle &= 1, \end{aligned}$$

where $\mathbf{1} = \left[\begin{array}{cccc} 1 & 1 & \ldots & 1 \end{array} \right]_{1 \times m}$. In other words, $\varepsilon^0 \left(\bar{p_1}^k \right)$ is an intersection of two hyperplanes (with normal vectors $\bar{\Pi}_1^{CM}(k\Delta)$ and $\mathbf{1}$) and the $\varepsilon \left(\bar{p_1}^k \right)$ in the vector space $\mathbb{R}^m$. Since the normal vectors, $\bar{\Pi}_1^{CM}(k\Delta)$ and $\mathbf{1}$, are not collinear (from Property 1 (ii)), the intersection of these two hyperplanes is an $(m-2)$-dimensional set, therefore, the dimension of $\varepsilon^0 \left(\bar{p_1}^k \right)$ is also $m-2$. This means that $\varepsilon^0 \left(\bar{p_1}^k \right)$ has one dimension less[18] than $\varepsilon \left(\bar{p_1}^k \right)$, consequently, it is a negligible part of $\varepsilon \left(\bar{p_1}^k \right)$.

Now, we will show that sets $\varepsilon^+ \left(\bar{p_1}^k \right)$ and $\varepsilon^- \left(\bar{p_1}^k \right)$ are congruent sets. Indeed, let σ : $\varepsilon^+ \left(\bar{p_1}^k \right) \to \varepsilon^- \left(\bar{p_1}^k \right)$ be defined as follows:

$$\sigma(p) = \bar{p_1}^k - \delta,$$

where $\delta = p - \bar{p_1}^k$, for any vector $p \in \varepsilon^+ \left(\bar{p_1}^k \right)$. From definition of σ, it is clear that the mapping σ is a translation. It is left to show that σ is well-defined, that is, for each $p \in \varepsilon^+ \left(\bar{p_1}^k \right)$ the vector $\sigma(p) \in \varepsilon^- \left(\bar{p_1}^k \right)$.

Let $p \in \varepsilon^+ \left(\bar{p_1}^k \right)$ be arbitrarily chosen. Since $\varepsilon \left(\bar{p_1}^k \right)$ is also a fair neighborhood (see Definition 2.8), the vector $\sigma(p) = \bar{p_1}^k - \delta$ belongs to the $\varepsilon \left(\bar{p_1}^k \right)$. Moreover, we want to show that $\sigma(p)$ is an element in $\varepsilon^- \left(\bar{p_1}^k \right)$. Note that, $\sigma(p) = \bar{p_1}^k - \delta = \bar{p_1}^k - (p - \bar{p_1}^k) = 2\bar{p_1}^k - p$.

Therefore,

$$\begin{aligned} \left\langle \sigma(p), \bar{\Pi}_1^{CM}(k\Delta) \right\rangle &= \left\langle 2\bar{p_1}^k - p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle \\ &= 2 \left\langle \bar{p_1}^k, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle - \left\langle p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle. \end{aligned}$$

Since $\left\langle \bar{p_1}^k, \bar{\Pi}_i^{CM}(k\Delta) \right\rangle = \bar{u}_1^{CM}(k\Delta) = 0$ and $p \in \varepsilon^+ \left(\bar{p_1}^k \right) \Rightarrow \langle p, \bar{\Pi}_1^{CM}(k\Delta) \rangle > 0$ we get:

$$\left\langle \sigma(p), \bar{\Pi}_1^{CM}(k\Delta) \right\rangle = - \left\langle p, \bar{\Pi}_1^{CM}(k\Delta) \right\rangle < 0.$$

[18] $\varepsilon \left(\bar{p_1}^k \right)$ is an intersection of the simplex in $\mathbb{R}^m$ and a Chebyshev ball with center belonging to the simplex, therefore, its dimension is $m-1$.

Overall, we have obtained that $\sigma(p) \in \varepsilon^{-}\left(\bar{p_1}^k\right)$.

Hence, we have proved that there is a translation between the disjoint sets $\varepsilon^{+}\left(\bar{p_1}^k\right)$ and $\varepsilon^{-}\left(\bar{p_1}^k\right)$, which means that these sets are two congruent sets. Consequently, they occupy identical fractions within the set $\varepsilon\left(\bar{p_1}^k\right)$. Furthermore, since $\varepsilon\left(\bar{p_1}^k\right) \approx \varepsilon^{+}\left(\bar{p_1}^k\right) \cup \varepsilon^{-}\left(\bar{p_1}^k\right)$, we obtain that $\varepsilon^{+}\left(\bar{p_1}^k\right)$ occupies exactly a half of $\varepsilon\left(\bar{p_1}^k\right)$, which proves Property 2. ♢

Proof of the Proposition:

From the Property 2 and using notations from its proof, we obtained that the set of matrices P_1 that guarantees non-negative payoffs to bidder 1, irrespective of her value, is the set $\left(\bigcap_{k=0}^{m-1} \varepsilon_k^{+}(\bar{P}_1)\right) \cup \left(\bigcap_{k=0}^{m-1} \varepsilon_k^{0}(\bar{P}_1)\right)$. Given the reasoning above, we may neglect the set $\bigcap_{k=0}^{m-1} \varepsilon_k^{0}(\bar{P}_1)$ and concentrate to the set $\bigcap_{k=0}^{m-1} \varepsilon_k^{+}(\bar{P}_1)$. Since $\varepsilon_k^{+}(\bar{P}_1) = \bigtimes_{j=0}^{k-1} \varepsilon\left(\bar{p_1}^j\right) \times \varepsilon^{+}\left(\bar{p_1}^k\right) \bigtimes_{j=k+1}^{m-1} \varepsilon\left(\bar{p_1}^j\right)$, we obtain:

$$\begin{aligned}
\bigcap_{k=0}^{m-1} \varepsilon_k^{+}(\bar{P}_1) &= \bigcap_{k=0}^{m-1}\left[\bigtimes_{j=0}^{k-1} \varepsilon\left(\bar{p_1}^j\right) \times \varepsilon^{+}\left(\bar{p_1}^k\right) \bigtimes_{j=k+1}^{m-1} \varepsilon\left(\bar{p_1}^j\right)\right] \\
&= \bigtimes_{k=0}^{m-1} \varepsilon^{+}\left(\bar{p_1}^k\right).
\end{aligned}$$

We have shown that each set $\varepsilon^{+}\left(\bar{p_1}^k\right)$ occupies approximately 1/2 of the neighborhood $\varepsilon\left(\bar{p_1}^k\right)$, hence, $\bigcap_{k=0}^{m-1} \varepsilon_k^{+}(\bar{P}_1)$ occupies maximally $\overbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2}}^{m \text{ times}} = \frac{1}{2^m}$ of the neighborhood $\varepsilon(\bar{P}_1)$.

Analogously, we may obtain that the set of matrices P_2 in the ε-neighborhood of the matrix $\bar{P}_2$ occupy maximally $\frac{1}{2^m}$ of the neighborhood.

Overall, the set of all feasible pairs of matrices (P_1, P_2) belonging to ε-neighborhood of the pair $(\bar{P}_1, \bar{P}_2)$, which guarantee the non-negative payoffs to any possible type of any bidder, occupies maximally $\frac{1}{2^m} \cdot \frac{1}{2^m} = \frac{1}{2^{2m}}$ of the neighborhood. Q.E.D.

2.B Special Case: Two Private Values and Common Prior

In the following lines we provide a stronger result than one stated in Proposition 2.3, for a special case of two private values. Namely, it will be shown that the participation constraints hold true in maximally 25% of any fair neighborhood of a seller's belief matrix.

The apparatus used here comes from linear algebra and vector space literature. Furthermore, the further derivations rely heavily on the following two properties of our model. Firstly, the belief matrices belong to a hyperplane that contains the simplex (in this particular case of $m = 2$, the hyperplane is 3-dimensional); let us name the hyperplane as "simplex hyperplane". Secondly, the set where full surplus extraction for a bidder with certain value holds true, is well-approximated by an intersection of the previously mentioned hyperplane and a hyperplane, whose normal vector is a corresponding row of the associated Jacobian matrix defined in Lemma 2.1.

Furthermore, as these hyperplanes are not parallel or identical, it is clear that the intersection must be a hyperplane in "simplex hyperplane", once "simplex hyperplane" is considered as a space on its own. Let us name the intersections as "zero-expected-payoff hyperplanes". Therefore, for each bidder and each possible value, the associated "zero-expected-payoff hyperplane" is a set that contains all belief matrices leading to the bidder's expected payoff equal to exactly zero. For each bidder and each possible value, we shall explicitly calculate the normal vectors of "zero-payoff hyperplanes". The normal vectors will be expressed in terms of an orthonormal basis in "simplex hyperplane" space. Furthermore, let us fix a fair neighborhood of a seller's belief matrix. Note that the set, where participation constraints hold true, is a subset of the neighborhood that belongs to a hyperorthant, whose borders are "zero-expected-payoff hyperplanes". By the fairness property of the neighborhood,the angles between these hyperplanes completely reflect the share that the participation constraints occupy in the neighborhood. Once we have calculated the normal vectors, we are able to assess the angles between each of two of these hyperplanes (we have a hyperplane per a pair (bidder, value)). It turned out that at least one of the angles is acute, and the result follows.

Let us formally derive the result. For the sake of simplicity, we leave out details of the calculations and just give the final results of each derivation step. I am willing to make details of the derivations available upon a request.

In general, we shall denote by $\mathcal{H}_v$ a hyperplane in $\mathbb{R}^4$ whose normal vector is given by $v = (v_1, v_2, v_3, v_4)$. In addition, we shall reserve notation $\mathcal{H}_s$ for a hyperplane which contains the simplex ("simplex hyperplane"); that is, $\mathcal{H}_s$ is the hyperplane whose normal vector is $s = \mathbf{1} = (1, 1, 1, 1)$.

2.B.1 The Intersections of the Hyperplanes with The "Simplex Hyperplane" in $\mathbb{R}^4$

Let us fix a seller's belief matrix $\bar{P}$, such that all its entries are positive. We are interested in determining sets $\mathcal{H}_s \cap \mathcal{H}_v$ where v is a row of the Jacobian matrix determined by (2.12). Therefore, $\mathcal{H}_v$ is the hyperplane in $\mathbb{R}^4$ which contains "the point" $\overline{P}$ and whose normal vector is some row of the Jacobian matrix. This set shall contain all belief matrices which give zero expected payoff to the associated pair of the bidder and her value.

More generally, exploiting the special form of the row vectors of the Jacobian, we find $\mathcal{H}_s \cap \mathcal{H}_v$ are such that v takes one of the following four general forms:

$$\begin{aligned} v_1 &= (a, b, 0, 0)\,, \\ v_2 &= (0, 0, c, d)\,, \\ v_3 &= (e, 0, f, 0) \text{ and} \\ v_4 &= (0, g, 0, h)\,, \end{aligned}$$

assuming that $aceg \neq 0$.

Solving four linear systems, each made of two equations with four unknown variables, we obtained that

$$\mathcal{H}_s \cap \mathcal{H}_{v_1} = \mathcal{L}\left(\left(-\frac{b}{a}, 1, -\frac{a-b}{a}, 0\right), (0, 0, -1, 1)\right) + \bar{P},$$

$$\mathcal{H}_s \cap \mathcal{H}_{v_2} = \mathcal{L}\left((-1, 1, 0, 0), \left(-\frac{c-d}{c}, 0, -\frac{d}{c}, 1\right)\right) + \bar{P},$$

$$\mathcal{H}_s \cap \mathcal{H}_{v_3} = \mathcal{L}\left((0, 1, 0, -1), \left(-\frac{f}{e}, 0, 1, -\frac{e-f}{e}\right)\right) + \bar{P} \text{ and}$$

$$\mathcal{H}_s \cap \mathcal{H}_{v_4} = \mathcal{L}\left((1, 0, -1, 0), \left(0, -\frac{h}{g}, -\frac{g-h}{g}, 1\right)\right) + \bar{P},$$

where $\mathcal{L}(w_1, w_2, \ldots, w_k)$ is a subspace in $\mathbb{R}^4$ spanned by the vectors w_ℓs (i.e. a set of all possible linear combinations of the vectors w_1, w_2, … , w_k). Note that $\mathcal{H}_s \cap \mathcal{H}_v$ are indeed two-dimensional sets and, hence, they are hyperplanes in space $\mathcal{H}_s$. All vector coordinates here are expressed in terms of the standard Euclidean basis. In the following section we switch coordinate notation in terms of a basis in the subspace $\mathcal{H}_s$.

2.B.2 Defining an Orthonormal Basis in $\mathcal{H}_s$

Here we shall find an orthonormal basis in the "simplex hyperplane". As the departing point, we choose two vectors of the basis using the previous calculations. Namely, we choose the following vectors:

$$\begin{aligned} \mathbf{i} &= \frac{1}{\sqrt{2}}(0, 0, -1, 1) \text{ and} \\ \mathbf{j} &= \frac{1}{\sqrt{2}}(-1, 1, 0, 0)\,. \end{aligned}$$

The choice of vectors **i** and **j** is due to the last equalities in the previous subsection. Namely, these vectors belong to some of the intersections in which we are interested in. After few mathematical steps, we extend the set made of these two vectors with a vector in such way, that the obtained triple form an orthonormal basis in the space $\mathcal{H}_s$. Thus, we obtained an orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where

$$\mathbf{k} = \frac{1}{2}\left(1, 1, -1, -1\right).$$

The vector k was found as a solution of the following equations:

1. $\langle \mathbf{k}, \mathbf{1} \rangle = 0$, (a condition that $k \in \mathcal{H}_s$)
2. $\langle \mathbf{k}, \boldsymbol{i} \rangle = 0$,
3. $\langle \mathbf{k}, \boldsymbol{j} \rangle = 0$ and
4. $\langle \mathbf{k}, \mathbf{k} \rangle = 1$ (**k** must be an unit vector).

2.B.3 The Intersections as Hyperplanes in the "Simplex Hyperplane" Space

We shall here express each of the sets $\mathcal{H}_s \cap \mathcal{H}_v$ in terms of the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Moreover, we shall find their normal vectors (also expressed in the basis from the previous subsection), because $\mathcal{H}_s \cap \mathcal{H}_v$ is a hyperplane in $\mathcal{H}_s$. Therefore, from now on, we consider the set $\mathcal{H}_s$ as our working vector space and the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ will be the default basis. The results are, as follows:

$$\begin{aligned}
\mathcal{H}_s \cap \mathcal{H}_{v_1} &= \mathcal{L}\left(\left(\frac{a-b}{\sqrt{2}a}, \frac{a+b}{\sqrt{2}a}, \frac{a-b}{a}\right), (1,0,0)\right) + \bar{P}, \\
\mathcal{H}_s \cap \mathcal{H}_{v_2} &= \mathcal{L}\left(\left(\frac{c+d}{\sqrt{2}c}, \frac{c-d}{\sqrt{2}c}, -\frac{c-d}{c}\right), (0,1,0)\right) + \bar{P}, \\
\mathcal{H}_s \cap \mathcal{H}_{v_3} &= \mathcal{L}\left(\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right), \left(-\sqrt{2} + \frac{f}{\sqrt{2}e}, \frac{f}{\sqrt{2}e}, -\frac{f}{e}\right)\right) + \bar{P}, \\
\mathcal{H}_s \cap \mathcal{H}_{v_4} &= \mathcal{L}\left(\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1\right), \left(\sqrt{2} - \frac{h}{\sqrt{2}g}, -\frac{h}{\sqrt{2}g}, -\frac{h}{g}\right)\right) + \bar{P},
\end{aligned}$$

and the normal vectors are, respectively, given by:

$$\mathbf{n_1} = \frac{1}{\sqrt{1+\frac{1}{2}\left(\frac{a+b}{a-b}\right)^2}}\left(0,1,-\frac{a+b}{\sqrt{2}\,(a-b)}\right),$$

$$\mathbf{n_2} = \frac{1}{\sqrt{1+\frac{1}{2}\left(\frac{c+d}{c-d}\right)^2}}\left(1,0,\frac{c+d}{\sqrt{2}\,(c-d)}\right),$$

$$\mathbf{n_3} = \frac{1}{\sqrt{1+\frac{1}{2}\left(1-\frac{f}{e}\right)^2+\frac{f^2}{e^2}}}\left(\frac{f}{e},1,-\frac{e-f}{\sqrt{2}e}\right) \text{ and}$$

$$\mathbf{n_4} = \frac{1}{\sqrt{1+\frac{1}{2}\left(1-\frac{h}{g}\right)^2+\frac{h^2}{g^2}}}\left(\frac{h}{g},1,\frac{g-h}{\sqrt{2}g}\right).$$

2.B.4 The Angles between the Intersections

Here we would like to determine the angles between $\mathcal{H}_s \cap \mathcal{H}_{v_i}$ and $\mathcal{H}_s \cap \mathcal{H}_{v_j}$ for any $i \neq j$. The angle is actually the supplementary angle to the angle between $\mathbf{n}_i$ and $\mathbf{n}_j$, therefore,

$$\begin{aligned}\cos\sphericalangle\left(\mathcal{H}_s \cap \mathcal{H}_{v_i}, \mathcal{H}_s \cap \mathcal{H}_{v_j}\right) &= -\frac{\langle \mathbf{n}_i, \mathbf{n}_j\rangle}{\sqrt{\langle \mathbf{n}_i, \mathbf{n}_i\rangle \langle \mathbf{n}_j, \mathbf{n}_j\rangle}} \\ &= -\langle \mathbf{n}_i, \mathbf{n}_j\rangle,\end{aligned}$$

where $\cos\sphericalangle\left(\mathcal{H}_s \cap \mathcal{H}_{v_i}, \mathcal{H}_s \cap \mathcal{H}_{v_j}\right)$ is the cosine of the angle between involved hyperplanes.

Let us fix a fair neighborhood of a seller's belief matrix $\bar{P}$. The neighborhood is, actually (in this special case), a three-dimensional ball within the simplex and with center at $\bar{P}$, thus, it is a ball in $\mathcal{H}_s$. The subset of the neighborhood, where participation constraints hold true, is approximated by a hyperorthant, whose boundary is determined by the four hyperplanes,that we calculated in the previous subsection. Let us call this subset of the neighborhood as "participation constraints region". Note that $\bar{P}$ belongs to each of these four hyperplanes. Thus, the share that "participation constraints region" occupies in the neighborhood, is completely determined by the angles among the hyperplanes. For example, if the angle between $\mathcal{H}_s \cap \mathcal{H}_{v_1}$ and $\mathcal{H}_s \cap \mathcal{H}_{v_2}$ is $60°$, the participation constraints can not hold true in more than $60° : 360° = 1/6$ of the whole neighborhood (ball).

Let us find these angles. After performing simple calculations from linear algebra, we obtain:

1. $\cos\sphericalangle\left(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_2}\right) = \frac{\lambda_1}{\sqrt{1+(\lambda_1)^2}} \cdot \frac{\lambda_2}{\sqrt{1+(\lambda_2)^2}}$,

2. $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_3}, \mathcal{H}_s \cap \mathcal{H}_{v_4}) = -\frac{2-\sqrt{2}(\lambda_3+\lambda_4)+\lambda_3\lambda_4}{\sqrt{(2-2\sqrt{2}\lambda_3+3(\lambda_3)^2)(2-2\sqrt{2}\lambda_4+3(\lambda_4)^2)}}$,
3. $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_3}) = -\frac{1+\lambda_1\lambda_3}{\sqrt{(2-2\sqrt{2}\lambda_3+3(\lambda_3)^2)(1+(\lambda_1)^2)}}$,
4. $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_4}) = -\frac{1-\lambda_1\lambda_4}{\sqrt{(2-2\sqrt{2}\lambda_4+3(\lambda_4)^2)(1+(\lambda_1)^2)}}$,
5. $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_2}, \mathcal{H}_s \cap \mathcal{H}_{v_3}) = -\frac{1-\sqrt{2}\lambda_3-\lambda_2\lambda_3}{\sqrt{(2-2\sqrt{2}\lambda_3+3(\lambda_3)^2)(1+(\lambda_2)^2)}}$,
6. $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_2}, \mathcal{H}_s \cap \mathcal{H}_{v_4}) = -\frac{1-\sqrt{2}\lambda_4+\lambda_2\lambda_4}{\sqrt{(2-2\sqrt{2}\lambda_4+3(\lambda_4)^2)(1+(\lambda_2)^2)}}$,

with $\lambda_1 = \frac{a+b}{\sqrt{2}(a-b)}$, $\lambda_2 = \frac{c+d}{\sqrt{2}(c-d)}$, $\lambda_3 = \frac{e-f}{\sqrt{2}e}$, $\lambda_4 = \frac{g-h}{\sqrt{2}g}$.

If we add restrictions that $a \neq b$ and $c \neq d$ then we have that $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \left\{-\frac{1}{\sqrt{2}}\right\}$, because

$$\lambda_1 = \frac{2a - x}{\sqrt{2}x}$$

with $x = a - b$, $a \in \mathbb{R} \setminus \{0\}$.

Note that the following statements:

1. $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_2}) > 0 \Leftrightarrow \operatorname{sgn} \lambda_1 = \operatorname{sgn} \lambda_2$,
2. $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_2}) = 0 \Leftrightarrow (a = -b)$ or $(c = -d)$,
3. If $\lambda_i \to \lambda_j, \lambda_j \to \pm\infty$ then $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_2}) \to 1$ and
4. If $\lambda_i \to -\lambda_j, \lambda_j \to \pm\infty$ then $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_2}) \to -1$

hold true.

2.B.5 The Possible Range of the Angles Using the Special Forms of the Vectors v_is

Here we characterize what values $\cos\sphericalangle\left(\mathcal{H}_s \cap \mathcal{H}_{v_i}, \mathcal{H}_s \cap \mathcal{H}_{v_j}\right)$ may take, once we replace the general numbers a, b, c, d, e, f by the corresponding values obtained from the rows of the Jacobian matrix. In particular, we have that

$$\begin{aligned} a &= -\tfrac{\bar{c}_1(0)}{\bar{p}_0}; & e &= -\tfrac{\bar{c}_2(0)}{\bar{p}^0} \\ b &= -\tfrac{\bar{c}_1(\Delta)}{\bar{p}_0}; & f &= -\tfrac{\bar{c}_1(\Delta)}{\bar{p}^0} \\ c &= \tfrac{\Delta-\bar{c}_1(0)}{\bar{p}_\Delta}; & g &= \tfrac{\Delta-\bar{c}_1(0)}{\bar{p}^\Delta} \\ d &= -\tfrac{\bar{c}_1(\Delta)}{\bar{p}_\Delta}; & h &= -\tfrac{\bar{c}_2(\Delta)}{\bar{p}^\Delta}. \end{aligned}$$

If we use the following notation:

$$\begin{aligned}\bar{p} &= \bar{P}(X_2 = 0 \,|\, X_1 = 0), \\ \bar{q} &= \bar{P}(X_2 = \Delta \,|\, X_1 = \Delta), \\ p' &= \bar{P}(X_1 = 0 \,|\, X_2 = 0) \text{ and} \\ q' &= \bar{P}(X_1 = \Delta \,|\, X_2 = \Delta),\end{aligned}$$

then we obtain the following equalities:

- $a = \frac{\Delta(1-\bar{p})(1-\bar{q})}{\bar{p}_0(\bar{p}+\bar{q}-1)}$, $b = -\frac{\Delta\bar{p}(1-\bar{q})}{\bar{p}_0(\bar{p}+\bar{q}-1)}$, $c = \frac{\Delta\bar{p}\bar{q}}{\bar{p}_\Delta(\bar{p}+\bar{q}-1)}$, $d = -\frac{\Delta\bar{p}(1-\bar{q})}{\bar{p}_\Delta(\bar{p}+\bar{q}-1)}$;
- $e = \frac{\Delta(1-p')(1-q')}{\bar{p}^0(p'+q'-1)}$, $f = -\frac{\Delta p'(1-q')}{\bar{p}^o(p'+q'-1)}$, $g = \frac{\Delta p'q'}{\bar{p}^\Delta(p'+q'-1)}$, $h = -\frac{\Delta p'(1-q')}{\bar{p}^\Delta(p'+q'-1)}$;
- $\lambda_1 = \frac{1-2\bar{p}}{\sqrt{2}}$, $\lambda_2 = \frac{2\bar{q}-1}{\sqrt{2}}$, $\lambda_3 = \frac{1}{\sqrt{2}(1-p')}$, $\lambda_4 = \frac{1}{\sqrt{2}q'}$.

Overall, we get that the following statements:

- $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_2}) > 0 \Leftrightarrow \left(\frac{1}{2} - \bar{p}\right)\left(\bar{q} - \frac{1}{2}\right) > 0$,
- $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_3}, \mathcal{H}_s \cap \mathcal{H}_{v_4}) > 0 \Leftrightarrow p' + q' - 2p'q' < \frac{1}{2}$,
- $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_3}) > 0 \Leftrightarrow \bar{p} + p' > \frac{3}{2}$,
- $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_1}, \mathcal{H}_s \cap \mathcal{H}_{v_4}) > 0 \Leftrightarrow \bar{p} + q' < \frac{1}{2}$,
- $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_2}, \mathcal{H}_s \cap \mathcal{H}_{v_3}) > 0 \Leftrightarrow p' + \bar{q} > \frac{1}{2}$ and
- $\cos\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_2}, \mathcal{H}_s \cap \mathcal{H}_{v_4}) > 0 \Leftrightarrow \bar{q} + q' < \frac{3}{2}$

hold true.

Finally, we can derive the following claim: For any matrix $\bar{P}$ and its fair neighborhood, the "participation constraint region" does not occupy more than 25% of the neighborhood.

The claim follows from the fact that, for any matrix $\bar{P}$, we have that $\bar{q} > 1/2$ or $\bar{q} \leq 1/2$. In any of these two cases at least one of the above listed angles is acute (for example, please look at $\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_2}, \mathcal{H}_s \cap \mathcal{H}_{v_3})$ and $\sphericalangle(\mathcal{H}_s \cap \mathcal{H}_{v_2}, \mathcal{H}_s \cap \mathcal{H}_{v_4})$, respectively). Consequently, the "participation constraint region" occupies maximally $1/4$ of the neighborhood.

Chapter 3

Bayesian Updating on Origin of Private Values

The starting point of modeling in this chapter is the origin of the distribution of the private values. We assume that there is enough data collected (e.g. reports on winning prices from Ebay or second-price auctions),[1] so that one can observe the distribution of the values. However, in contrast to the practice in established literature, the observed distribution will be assumed to take a special form: a convex combination of distributions. This particular form of distribution is known as a mixture distribution. There are several reasons why we consider this special form of private value distribution. First of all, it allows us to give participants the opportunity to speculate on the the true distribution of values, as they assume that one component of the observed mixture is the true source of the values. Thus, in this way we introduce the information uncertainty and model the endogenous formation of participants' beliefs about the involved distributions. Secondly, the chosen family of the mixture distribution is not too restrictive because approximations of a probability distribution by a mixture distribution is widely used and well justified in the literature.[2]

Using this, we stay in the framework of independently and identically distributed (i.i.d.) private values as well as assume common knowledge on a participant's beliefs. However, the special shape of the observed marginal distribution will give rise to several ways of modeling the auction participant's behavior. One way to deal with the situation is the classical approach used in the literature (this will be referred to as the "classical model"), where participants do not question the observed distribution and their belief are given and fixed.[3]

We introduce another model as an answer to the ambiguous origin of the marginal distribution - a model where bidders will update their knowledge about the origin of values conditioning on their own value. We will call this the "Bayesian updaters model". In this

[1]For example, one possible way to obtain a selling price from the past is to use the "Advanced Search" function from Ebay's website: http://www.ebay.com/sch/ebayadvsearch.

[2]For more details on the approximations and mixture distributions see Diaconis and Freedman (1980), Escobar and West (1995) or Everitt and Hand (1981).

[3]Under classical models we understand models as the ones reviewed in Vickrey (1961) or Chapters 2 and 3 in Krishna (2010).

set up, we allow participants to question the origin of, and endogenously form their beliefs about auction information structure - a step on from the standard literature. Namely, given the special mixture form of the observed distribution of private vales, the participants question which distribution involved in the mixture generates the private values and form their beliefs accordingly. For the belief formation process they use both common and private information. They are the observed data pattern and private value. We are interested in the performance of first-price and second-price auctions as well as comparison of the Bayesian updaters model's components to the associated features of various relevant models.

It turns out that the revenue equivalence principle does not hold in the "Bayesian updaters model" in such a way that the seller favors the second-price auction over the first-price auction. Comparison between the "classical model" and the "Bayesian updaters model" results in no difference in bidding behavior in the second-price auction (truth telling remains the optimal choice). However, there is either dominance order or a single-crossing property of respective bidding strategy functions in the first-price auction. Mathematically, the result of the revenue equivalence failure is not as surprising as it turns out that in terms of statistics, the model is analogous to the model of interdependent private values introduced in Milgrom and Weber (1982). However, the crucial difference lies in the semantics of a participant's belief structure, as Bayesian updaters assume that the values are i.i.d. and the belief formation process makes it only mathematically similar to their model. Moreover, some assumptions, which have been previously considered as technical assumptions, obtain a meaningful interpretation in terms of beliefs (for example, refer to Lemma 3.1).

The chapter is organized as follows. Sections 3.1 - 3.3 are devoted to our new model with Bayesian updaters providing insights into the model's components. Section 3.1 explains the information structure of the model. It presents the common premises for both the classical and the new model as well as the new way of thinking about the origin of private values in the new "Bayesian updater" model. The section explains relationships between the model's assumptions and involved participants' beliefs. We then continue with Section 3.2 that gives an answer to the question of how Bayesian updaters bid in the first-price and second-price auction. Once we find the bidding strategies, we continue by taking the seller's point of view. Section 3.3 compares the seller's expected revenue between first-price and second-price auctions with the Bayesian updaters. Comparison of bidding behavior, as well as the expected revenues of our model to other benchmark models and the classical one is provided in Section 3.4. We sum up the results and give a short summary of this chapter in Section 3.5. Some proofs are omitted in the main body, but you can find them in Appendix 3.A.

3.1 The Bayesian Updaters Model

Before going into details on the core model, we shall here firstly describe the common building blocks for the two models, which are the focus of this chapter - previously defined as the "classical model" and the "Bayesian updaters model". At the same time, we introduce notation necessary to understand the later parts of current chapter.

3.1.1 The Common Ground of the Classical and Bayesian Updaters Models

There is a seller who wants to sell an object to $N \geq 2$ potential buyers - so-called bidders. We denote by $\mathcal{N} = \{1, 2, \ldots, N\}$ the set of bidders. Each bidder receives her private signal x_i which is a realization of a random variable X_i. As in the previous chapter, the private signal, known also as private value, is the maximal willingness to pay; that is the value that the corresponding bidder attaches to the object. For technical reasons, we suppose that the seller's value attached to the object is zero. The seller chooses an auction format $A \in \mathcal{A} = \{I, II\}$, where if

- $A = I$: A first-price auction is chosen for the sale, and
- $A = II$: A second-price auction is chosen as the selling mechanism.

We assume that the X_i are identically distributed, which makes the bidders symmetric and anonymous. Based on observations of outcomes of previous auctions (i.e. using statistics from past), the values appear to be distributed according to a cumulative distribution function (c.d.f.) $\widetilde{F}$ over an interval $[0, \omega]$ with $\omega \leq +\infty$. Furthermore, the distribution $\widetilde{F}$ is known both by bidders and the seller. Let $\mathcal{F} = \left\{F_j \,|\, j \in \mathcal{J}\right\}$ be a given family of strictly increasing cumulative distribution functions over the interval $[0, \omega]$. Let us denote by f_j the density function (d.f.) associated to F_j. Note that f_j takes strictly positive values everywhere. Specifically, we assume that $\widetilde{F}$ is a convex combination of the marginal distributions from set $\mathcal{F}$. In probability theory, $\widetilde{F}$ is known as a mixture distribution. Our attention is restricted to the case where $\mathcal{J} = \{1, 2\}$ and $\widetilde{F} = \frac{1}{2}F_1 + \frac{1}{2}F_2$.

Now we concentrate on the seller's belief and bidders' beliefs about the origin of the marginal distribution of the values. We consider two possible types of participants: the "classical model" and "Bayesian updaters model".

In the "classical model" all participants, including the seller, believe that that all X_is are identically identically distributed (i.i.d.) according to $\widetilde{F}$; that is, we are in a standard private value single-object auction set-up. As this is a well-known model, we shall when needed, only state the known facts about the model.

The core of the chapter is devoted to the "Bayesian updaters model". In this model, each participant uses the fact that the observed distribution $\widetilde{F}$ is a mixture distribution. More precisely, each participant, including the seller, believes that private values are i.i.d. according to either F_1 or F_2, with (a priori) equal probability of being either of these two options. As the seller has no further information, his belief about the joint distribution remains unchanged: that is, he believes that the joint distribution is either $\prod_{i \in \mathcal{N}} F_1(x_i)$ or $\prod_{i \in \mathcal{N}} F_2(x_i)$, with equal chances. On the other side, at the interim, the bidders have their private information - the private values. Accordingly, the bidders are assumed to update their belief about the distribution of the signals; upon the receiving private signal $x \in [0, \omega]$, each bidder updates her belief about the distribution of others' private values according to the Bayes rule. This model, as well as its features and components is the subject of our research that follows in the next sections.

3.1.2 Bayesian Updaters

Here, we have a closer look at the components of our model. As mentioned earlier, in this model all participants believe that the observed distribution $\widetilde{F}$ means that the private values are i.i.d. realization of either F_1 and F_2, with equal chance.

The observed distribution is the only piece of information that the seller accesses. Therefore, it is natural to assume that the seller does not change his initial belief. As the main interest of the seller is to achieve the best price for the good, firstly, the seller must hold a belief about the joint distribution of private values in order to calculate the expected revenue from an auction. In order to be in line with the a priori belief and set-up, the seller believes that the joint distribution of the signals is $\tilde{\Phi}(x_1, x_2, \ldots, x_N) = \frac{1}{2}\prod_{i\in\mathcal{N}} F_1(x_i) + \frac{1}{2}\prod_{i\in\mathcal{N}} F_2(x_i)$, which gives rise to the observed marginal $\widetilde{F}$. Moreover, in order to optimize their bidding behavior, all involved participants hold a belief about the bidders' bidding strategies. This belief formation will be explained in the following paragraph.

Bidders in the model, in the course of time, depart from their initial common belief. Namely, upon "receiving" her private signal - the private value - a bidder updates her belief accordingly. Namely, if her private value is x, then the probability that "the Bayesian updater" attaches to the event that private values are coming from F_1 is $\alpha(x)$, where $\alpha : [0, \omega] \to [0, 1]$ is a function defined as:

$$\alpha(x) = \frac{f_1(x)}{f_1(x) + f_2(x)},\ x \in [0, \omega]$$

This function will be known as **the belief function**. Based on the value of the function α, a bidder forms a belief about the bidding behavior of the opponents. An estimate of the distribution of the highest private value of the other bidders plays an important role in the belief formation process. This is because the bidders participate in the standard auction formats - auctions where the highest bid wins. The distribution of the highest value of the other bidders will be denoted as Y_1. Please note that this distribution does not depend on the identity of the excluded bidder, as bidders are supposed to be symmetric (due to the "anonymity" assumptions of the bidders). Furthermore, in order to optimally infer the opponents' bidding strategy, a bidder i will base her estimation on the distribution of the highest value among opponents, conditional on her own private value, denoted by $Y_1 | X_i = x$. From the belief function, it naturally comes out that this conditional random variable has the following c.d.f.:

$$G_x(y) = \alpha(x)\left(F_1(y)\right)^{N-1} + (1 - \alpha(x))\left(F_2(y)\right)^{N-1}$$

We summarize the mathematical apparatus of the model and, at the same time, its notation in Table 3.1:

Random variable	**C.d.f. and d.f.**	**Participant(s)**
X_i Private signal of bidder i	$\widetilde{F}(x_i) = \frac{1}{2}F_1(x_i) + \frac{1}{2}F_2(x_i),\ x_i \in [0,\omega]$ $\widetilde{f}(x_i) = \frac{1}{2}f_1(x_i) + \frac{1}{2}f_2(x_i), x_i \in [0,\omega]$	Seller A priori: Bidders
$(X_1, X_2, \ldots, X_N)$ Joint random vector of signals	$\tilde{\Phi}(x_1, x_2, \ldots, x_N) = \frac{1}{2}\prod_{i\in\mathcal{N}} F_1(x_i) + \frac{1}{2}\prod_{i\in\mathcal{N}} F_2(x_i)$ $\tilde{\phi}(x_1, x_2, \ldots, x_N) = \frac{1}{2}\prod_{i\in\mathcal{N}} f_1(x_i) + \frac{1}{2}\prod_{i\in\mathcal{N}} f_2(x_i)$	Seller
$Y_1 \mid X_i = x$ where $Y_1 = \max_{j\neq i} X_j$	$G_x(y) = \alpha(x)(F_1(y))^{N-1} + (1-\alpha(x))(F_2(y))^{N-1}$ $g_x(y) = (N-1)\left[\alpha(x)(F_1(y))^{N-2} f_1(y)\right] + (N-1)\left[(1-\alpha(x))(F_2(y))^{N-2} f_2(y)\right]$	Interim: Bidder

Table 3.1: *The beliefs in the Bayesian updaters model.*

Note that the right hand column of the table associates participant(s) whose belief is depicted in the respective row. However, all beliefs are common knowledge; that is, the whole table is known to all participants in the auction and they know that the others know it and so on ad infinitum.

Now we look at the bidding behavior of the Bayesian updaters in the first-price and second-price auctions. In order to be able to explicitly infer pattern of bids, we imposed some additional assumptions such as: domination in terms of likelihood ratio[4] or affiliation.[5] These assumptions are known in the auction literature and were already imposed as "technical" assumptions. They were firstly imposed in Milgrom and Weber (1982) to make it convenient to calculate optimal bidding strategies (see the appendices of Krishna (2010) for a brief introduction of these assumptions). Indeed, the mathematical representation of our model given by Table 3.1 resembles the model developed by Milgrom and Weber (1982). However, the semantic interpretation and distribution forms are different. The uniqueness of our model is that some of these assumptions also get a behavioral interpretation. Therefore, in the following lines of this section we explore the link between these "technical" assumptions and the participants' behavior and beliefs.

3.1.3 Dominance in Terms of the Likelihood Ratio, Affiliation and Beliefs

The next lemma shows the link between dominance in terms of the likelihood ratio among F_1 and F_2 and the belief function α. Namely, it turns out that the existence of dominance in

[4]A distribution F dominates some distribution G in terms of the likelihood ratio if for all $t < x$,

$$\frac{f(t)}{g(t)} \leq \frac{f(x)}{g(x)}.$$

[5]A joint signal distribution ϕ is affiliated if for all x', x'' $\phi(x' \vee x'')\phi(x' \wedge x'') \geq \phi(x')\phi(x'')$, where $x' \vee x''$ is the component-wise maximum of x' and x'' and $x' \wedge x''$ the component-wise minimum of x' and x''.

terms of the likelihood ratio between marginal distributions is equivalent to the statement that the belief function is monotonic. Loosely speaking, for example, the property that higher values are more likely to be generated by F_1 is shown to be equivalent to the property of the belief function, where the higher value means the higher belief on the event that the true marginal distribution function is F_1. More precisely,

Lemma 3.1. *The following statements are equivalent:*

(a) F_1 and F_2 can be ordered by dominance in terms of likelihood ratio, i.e. F_1 dominates F_2 in terms of likelihood ratio or vice versa.

(b) Function α is a monotonic function, i.e. α is non-decreasing (if F_1 dominates F_2) or non-increasing (if F_2 dominates F_1).

(c) For any $x' < x''$ we have that $G_{x''}$ first-order[6] stochastically dominates $G_{x'}$.

Proof. Firstly we will prove that (a) is equivalent to (b). Suppose that α is non-decreasing, which is identical to stating the following:

$$\begin{array}{lc}
(\forall t < x) & (\alpha(t) \leq \alpha(x)) \\
& \Updownarrow \\
(\forall t < x) & \left(\frac{f_1(t)}{f_1(t)+f_2(t)} \leq \frac{f_1(x)}{f_1(x)+f_2(x)}\right) \\
& \Updownarrow \\
(\forall t < x) & (f_1(t)f_2(x) \leq f_1(x)f_2(t)) \\
& \Updownarrow \\
(\forall t < x) & \left(\frac{f_1(t)}{f_2(t)} \leq \frac{f_1(x)}{f_2(x)}\right).
\end{array}$$

The last expression is equivalent to the statement that F_1 dominates F_2 in terms of likelihood ratio.

Similarly, one may prove that α is non-increasing, if and only if F_2 dominates F_1 in terms of likelihood ratio. Thus, we have just shown the equivalence between expressions (a) and (b).

We continue proving that (c) is equivalent to (b). By definition of the first-order stochastic dominance we derive that statement (c) is equivalent to the following:

[6] A distribution F first-order stochastically dominates some distribution G on $[0, \omega]$ if and only if $F(z) \leq G(z)$ for any $z \in [0, \omega]$.

$$\left(G_{x''}(y) \leq G_{x'}(y)\right)$$
$$\Updownarrow$$
$$\alpha(x'')(F_1(y))^{N-1} + (1-\alpha(x''))(F_2(y))^{N-1} \leq$$
$$\alpha(x')(F_1(y))^{N-1} + (1-\alpha(x'))(F_2(y))^{N-1}$$
$$\Updownarrow$$
$$(\alpha(x'') - \alpha(x'))\left((F_1(y))^{N-1} - (F_2(y))^{N-1}\right) \leq 0.$$

Consequently, we obtain that statement (c) is equivalent to saying that either α is non-decreasing and F_1 first-order stochastically dominates F_2 or the reverse - α is non-increasing and F_2 first-order dominates F_1. Furthermore, it is known that dominance in terms of likelihood ratio implies first-order stochastic dominance (for example, see Appendix B in Krishna (2010)). Thus, as we have just proved that (b) implies the dominance in the terms of the likelihood ratio, we may conclude that (b) is a necessary and sufficient condition for (c). Q.E.D. □

The implication "$(a) \Rightarrow (b)$" from the last lemma explicitly shows the usual interpretation of the monotone likelihood ratio property: if a bidder has a higher value, then she believes that the private values comes from the distribution which dominates the other. However, in this particular set-up the opposite is also true: if the bidder believes that higher private values always come from a particular distribution, then the distribution must be a dominant one in terms of likelihood ratio. Moreover, according to part (c), the same positive correlation between private value and estimated distribution of the highest opponent's value given her own private value holds true.

Given the equivalence of the conditions, in our particular model we may introduce the following (re)definitions:

Definition 3.1. We say that F_1 **dominates (strictly dominates)** F_2 **in terms of likelihood ratio** if the belief function α is non-decreasing (strictly increasing). Equivalently, we say that F_2 **dominates (strictly dominates)** F_1 **in terms of likelihood ratio** if the belief function α is non-increasing (strictly decreasing).

The following lemma shows that likelihood ratio dominance is a sufficient condition for so-called affiliated private values. The term affiliation was introduced in Milgrom and Weber (1982). Affiliation stands for a form of strong positive dependence. If private values are affiliated, then a high value of one's private value implies high probability that others' private values are also high.

Lemma 3.2. *Suppose that F_1 and F_2 can be ordered by domination in terms of likelihood ratio, i.e. F_1 dominates F_2 in terms of likelihood ratio or vice versa. $\tilde{\phi}(x_1, x_2, \ldots, x_N)$ is then affiliated.*

Proof. The joint signal distribution $\tilde{\phi}$ is affiliated if, for all vectors $\mathbf{x}', \mathbf{x}'' \in [0, \omega]^N$,

$$\tilde{\phi}(\mathbf{x}' \vee \mathbf{x}'')\tilde{\phi}(\mathbf{x}' \wedge \mathbf{x}'') \geq \tilde{\phi}(\mathbf{x}')\tilde{\phi}(\mathbf{x}''), \tag{3.1}$$

where $\mathbf{x}' \vee \mathbf{x}''$ is the component-wise maximum and $\mathbf{x}' \wedge \mathbf{x}''$ is the component-wise minimum of $\mathbf{x}'$ and $\mathbf{x}''$.

Let $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_N) \in [0, \omega]^N$ and $\mathbf{x}'' = (x''_1, x''_2, \cdots, x''_N) \in [0, \omega]^N$ be arbitrary. Let us denote

$$\check{x}_i = \max\left\{x'_i, x''_i\right\}, \ \hat{x}_i = \min\left\{x'_i, x''_i\right\},$$

for all $i \in \mathcal{N}$.

Thus, inequality (3.1) is equivalent to

$$\begin{aligned} &\tilde{\phi}\left(\check{x}_1, \check{x}_2, \ldots, \check{x}_N\right) \cdot \tilde{\phi}\left(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N\right) \geq \\ &\quad \tilde{\phi}(x'_1, x'_2, \ldots, x'_N) \cdot \tilde{\phi}(x''_1, x''_2, \ldots, x''_N). \end{aligned}$$

Plugging the definition of $\tilde{\phi}$ (see Table 3.1) into the last inequality, we obtain that

$$\begin{aligned} &\left[\prod_{i\in\mathcal{N}} f_1\left(\check{x}_i\right) + \prod_{i\in\mathcal{N}} f_2\left(\check{x}_i\right)\right] \cdot \left[\prod_{i\in\mathcal{N}} f_1\left(\hat{x}_i\right) + \prod_{i\in\mathcal{N}} f_2\left(\hat{x}_i\right)\right] \geq \\ &\left[\prod_{i\in\mathcal{N}} f_1\left(x'_i\right) + \prod_{i\in\mathcal{N}} f_2\left(x'_i\right)\right] \cdot \left[\prod_{i\in\mathcal{N}} f_1\left(x''_i\right) + \prod_{i\in\mathcal{N}} f_2\left(x''_i\right)\right]. \end{aligned}$$

Equivalently,

$$\begin{aligned} &\prod_{i\in\mathcal{N}} f_1\left(\check{x}_i\right) f_1\left(\hat{x}_i\right) + \prod_{i\in\mathcal{N}} f_1\left(\check{x}_i\right) f_2\left(\hat{x}_i\right) + \prod_{i\in\mathcal{N}} f_2\left(\check{x}_i\right) f_1\left(\hat{x}_i\right) + \prod_{i\in\mathcal{N}} f_2\left(\check{x}_i\right) f_2\left(\hat{x}_i\right) \geq \\ &\prod_{i\in\mathcal{N}} f_1\left(x'_i\right) f_1\left(x''_i\right) + \prod_{i\in\mathcal{N}} f_1\left(x'_i\right) f_2\left(x''_i\right) + \prod_{i\in\mathcal{N}} f_2\left(x'_i\right) f_1\left(x''_i\right) + \prod_{i\in\mathcal{N}} f_2\left(x'_i\right) f_2\left(x''_i\right). \end{aligned}$$

Since $f_j\left(\check{x}_i\right) f_j\left(\hat{x}_i\right) = f_j\left(x'_i\right) f_j\left(x''_i\right)$, $j \in \{1, 2\}$, $i \in \mathcal{N}$, a few terms in the last inequality cancel out and we get:

$$\begin{aligned} \prod_{i\in\mathcal{N}} f_1\left(\check{x}_i\right) f_2\left(\hat{x}_i\right) &+ \prod_{i\in\mathcal{N}} f_2\left(\check{x}_i\right) f_1\left(\hat{x}_i\right) \\ &\geq \\ \prod_{i\in\mathcal{N}} f_1\left(x'_i\right) f_2\left(x''_i\right) &+ \prod_{i\in\mathcal{N}} f_2\left(x'_i\right) f_1\left(x''_i\right). \end{aligned} \tag{3.2}$$

Let us define:

$\mathcal{N}' = \left\{i \in \mathcal{N} : \check{x}_i = x'_i\right\}$ and $\mathcal{N}'' = \left\{i \in \mathcal{N} : \check{x}_i = x''_i \text{ and } x''_i \neq x'_i\right\}$.

Hence, $\{\mathcal{N}', \mathcal{N}''\}$ forms a partition of $\mathcal{N}$, and (3.2) becomes:

$$\begin{aligned}\prod_{i\in\mathcal{N}} f_1(\check{x}_i) f_2(\hat{x}_i) + \prod_{i\in\mathcal{N}} f_2(\check{x}_i) f_1(\hat{x}_i) \geq \prod_{i\in\mathcal{N}'} f_1(\check{x}_i) f_2(\hat{x}_i) \prod_{i\in\mathcal{N}''} f_1(\hat{x}_i) f_2(\check{x}_i) \\ + \prod_{i\in\mathcal{N}'} f_1(\hat{x}_i) f_2(\check{x}_i) \prod_{i\in\mathcal{N}''} f_1(\check{x}_i) f_2(\hat{x}_i).\end{aligned}$$

Reordering and factoring the last polynomial expression, we get

$$\left[\prod_{i\in\mathcal{N}'} f_1(\check{x}_i) f_2(\hat{x}_i) - \prod_{i\in\mathcal{N}'} f_1(\hat{x}_i) f_2(\check{x}_i)\right] \cdot \left[\prod_{i\in\mathcal{N}''} f_1(\check{x}_i) f_2(\hat{x}_i) - \prod_{i\in\mathcal{N}''} f_1(\hat{x}_i) f_2(\check{x}_i)\right] \geq 0.$$

This inequality is satisfied if and only if both of its terms are either non-negative or non-positive.

Consequently, in order that the inequality is satisfied, it suffices that

$$\frac{f_2(\hat{x}_i)}{f_2(\check{x}_i)} - \frac{f_1(\hat{x}_i)}{f_1(\check{x}_i)} \leq 0 \tag{3.3}$$

or

$$\frac{f_2(\hat{x}_i)}{f_2(\check{x}_i)} - \frac{f_1(\hat{x}_i)}{f_1(\check{x}_i)} \geq 0, \tag{3.4}$$

for all $i \in \mathcal{N}$.

Finally, assuming (3.3) or (3.4), is nothing more than saying that F_1 dominates F_2 or vice versa. Hence, as the assumption of the lemma is equivalent to supposing that (3.3) or (3.4) holds true, we have just shown that it is a sufficient condition for affiliation of the $\tilde{\phi}$, thus proving the lemma. □

Now that we have established the main components of the model and some links between the assumptions and participants' beliefs, in the next section we continue by derivation of the optimal bidders' behavior.

3.2 Bidding Behavior of Bayesian Updaters

In this section, we explore the pattern of optimal bids for the Bayesian updaters. The bidders might be invited to participate in either a first-price or second-price auction. Thus, we consider their behavior separately in each of these two auction formats. Since assumptions include anonymity and symmetry of the bidders, our focus is set on symmetric bidding strategy profiles. Let us denote by $\tilde{\beta}^A : [0, \omega] \to \mathbb{R}$ the optimal bidding strategy of a bidder participating in auction $A \in \{I, II\}$; that is, a Bayesian updater with private value x bids exactly $\tilde{\beta}^A(x)$ in auction A. We proceed by finding the explicit expressions of the strategies in second-price and first-price auctions, respectively.

The mathematical apparatus used here is analogous to one used in the symmetric model with interdependent values and affiliated signals introduced in Milgrom and Weber (1982) and elaborated in Chapter 6 of Krishna (2010).

3.2.1 Second-Price Auction

Once a Bayesian updater is invited to participate in a second-price auction, she will bid truthfully, that is, she bids her private value. Thus, introduction of our special form of uncertainty about the origin of the private values does not influence the bidding behavior of the bidders in the second-price auction: they behave identically as they do in the classical set-up. Namely, the argument for the truthful bidding strategy is identical to the one used in standard literature: truth telling is ex post weakly dominant (see Vickrey (1961)).

Henceforth, we have that

$$\tilde{\beta}^{II}(x) = x. \tag{3.5}$$

This truth telling behavior is not a surprise once one takes into account that the second-price auction is a so-called detail-free mechanism (see literature on robust mechanism design such as Bergemann and Morris (2005)). In a detail-free mechanism, the bidders' behavior is independent of the assumptions imposed on the information structure. Hence, our transformation of the deterministic information structure into an uncertain one does not effect bidders in the second-price auction.

3.2.2 First-Price Auction

Here we derive the optimal behavior of a Bayesian updater participating in a first-price auction. Mathematically, we use the heuristic method in order to obtain the explicit formula of the bidding strategy. We derive the bidding behavior in a standard manner, similar to Milgrom and Weber (1982). Thus, in a mathematical sense, this subsection is a special case of the existing derivation methods. Using the special form of our assumptions on the model in the following lines we present the derivation of the bidding strategy.

Let us first establish the following two lemmas that have technical use in our model. They represent a sufficient argument for explicit derivation of the Bayesian updater's bidding strategy in the first-price auction.

Lemma 3.3. *Suppose that F_1 and F_2 can be ordered by domination in terms of likelihood ratio, i.e. F_1 dominates F_2 in terms of likelihood ratio or vice versa; then, for any $t, x \in (0, \omega]$ such that $t < x$, the following inequality holds true:*

$$\frac{g_t(t)}{G_t(t)} \leq \frac{g_x(t)}{G_x(t)}. \tag{3.6}$$

Proof. Let $t, x \in [0, \omega]$ be such that $t < x$. By definition of conditional distributions (see Table 3.1), we may rewrite (3.6) as follows:

$$\begin{gathered}
g_t(t)\, G_x(t) \leq g_x(t)\, G_t(t) \\
\Updownarrow \\
\alpha_t \alpha_x F_1^{2N-3}(t) f_1(t) + \alpha_t (1-\alpha_x) F_1^{N-2}(t) F_2^{N-1}(t) f_1(t) \\
+ \quad \alpha_x (1-\alpha_t) F_1^{N-1}(t) F_2^{N-2}(t) f_2(t) + (1-\alpha_x)(1-\alpha_t) F_2^{2N-3}(t) f_2(t) \\
\leq \\
\alpha_t \alpha_x F_1^{2N-3}(t) f_1(t) + \alpha_t (1-\alpha_x) F_1^{N-1}(t) F_2^{N-2}(t) f_2(t) \\
\alpha_x (1-\alpha_t) F_1^{N-2}(t) F_2^{N-1}(t) f_1(t) + (1-\alpha_x)(1-\alpha_t) F_2^{2N-3}(t) f_2(t) \\
\Updownarrow \\
\alpha_t (1-\alpha_x) F_1^{N-2}(t) F_2^{N-2}(t) \left(F_2(t) f_1(t) - F_1(t) f_2(t)\right) \\
\leq \\
\alpha_x (1-\alpha_t) F_1^{N-2}(t) F_2^{N-2}(t) \left(F_2(t) f_1(t) - F_1(t) f_2(t)\right) \\
\Updownarrow \\
F_1^{N-2}(t) F_2^{N-2}(t) \left[f_1(t) F_2(t) - F_1(t) f_2(t)\right] \left[\alpha_t - \alpha_x\right] \leq 0,
\end{gathered} \tag{3.7}$$

where $\alpha_t = \alpha(t)$ and $\alpha_x = \alpha(x)$.

Under the assumption of ordering in terms of likelihood ratio dominance, (3.7) holds true, because $[f_1(t)F_2(t) - F_1(t)f_2(t)][\alpha_t - \alpha_x] \leq 0$. Indeed, suppose that, for example, F_1 dominates F_2 in terms of likelihood ratio. Since likelihood ratio dominance implies reverse hazard rate dominance[7] (as stated in Appendix B in Krishna (2010)), then $f_1(t)F_2(t) - F_1(t)f_2(t)$ is non-negative. On the other side, the fact that F_1 dominates F_2 in terms of likelihood ratio is equivalent to saying that function α is a non-decreasing function (See proof of Lemma 3.1) and, hence, $\alpha_t \leq \alpha_x$, which, in turn, implies (3.7). □

Remark 3.1. If α is strictly monotonic and $F_1(x)F_2(x) \neq 0$, then (3.6) holds true with strict inequality.

A sufficient condition for (3.6) is also affiliation of signals. This implies that we may establish a more general statement, which says that the domination in terms of likelihood between F_1 and F_2 is a sufficient condition so that $Y_1 \,|X_i = x$ dominates $Y_1 \,|X_i = t$ in terms of the reverse hazard rate, for $t < x$. Therefore:

Lemma 3.4. *Suppose that F_1 and F_2 can be ordered by domination in terms of likelihood ratio; i.e. F_1 dominates F_2 in terms of likelihood ratio or vice versa. Then, for any $z, x \in [0, \omega]$ such that $z < x$, we have that*

$$\frac{g_z\,(y)}{G_z\,(y)} \leq \frac{g_x\,(y)}{G_x\,(y)}, \tag{3.8}$$

for any $y \in (0, \omega]$.

Proof. Using Lemma 3.2, it follows that signals are affiliated, which immediately implies the claim of the proposition by applying the same argument as in Krishna (2010) on pages 87-88. Alternatively, one can deduce the proof in an analogous manner as in the previous lemma. □

Heuristics

Let us suppose that $\beta : [0, \omega] \to [0, \omega]$ is a symmetric bidding strategy in the first-price auction. For a moment, suppose that all bidders except bidder 1 follow the strategy β, such that:

- $\beta\,(0) = 0$ and
- β is a strictly increasing and differentiable function.

[7] A distribution F dominates some distribution G in terms of the reverse hazard rate if for all $z \in [0, \omega]$,

$$\frac{f(z)}{F(z)} \geq \frac{g(z)}{G(z)}.$$

We look at the optimization problem that bidder 1 has to solve in order to determine her optimal bid. It is clear that the bidder has to optimize her expected payoff, which depends on her private value x and chosen bid b. Let $\Pi^I(b,x)$ be bidder $1's$ expected payoff when her private value is x and the bid amount is b. Thus, the bidder is searching for bid b^*, which is the solution of the optimization problem $\max_{b\geq 0}\Pi^I(b,x)$. First of all, note that once bidder 1 assumes that other bidders bid according to the rule β, her bid b will not exceed $\beta(\omega)$. Hence, bidder 1 is maximizing $\Pi^I(b,x)$ on the set of bids $b\in\left[0,\beta^{-1}(\omega)\right]$.

Let us find explicit formula for $\Pi^I(b,x)$. By definition, we get

$$\Pi^I(b,x)=\int_0^{\omega}\pi(b,x)\,g_x(y)\,dy,$$

where $\pi^I(b,x)$ is the actual payoff and $g_x(y)$ is d.f. of the $Y_1\,|X_i=x$ (see Table 3.1). This means that

$$\pi^I(b,x)=\begin{cases}x-b, & b>\beta(y_1)\\ 0, & b<\beta(y_1)\end{cases},$$

where $y_1=\max_{j\neq 1}x_j$ (the highest private value of the other bidders).

Since β is strictly increasing, we obtain $b>\beta(y_1)\Leftrightarrow y_1<(\beta)^{-1}(b)$ and

$$\Pi^I(b,x)=\int_0^{(\beta)^{-1}(b)}(x-b)\,g_x(y)\,dy=(x-b)\,G_x\left((\beta)^{-1}(b)\right). \tag{3.9}$$

The first order condition for the problem $\max_{b\in\left[0,\beta^{-1}(\omega)\right]}\Pi^I(b,x)$ gives:

$$\begin{aligned}\frac{\partial}{\partial b}\Pi^I(b,x) &= 0\\ (x-b)\,g_x\left((\beta)^{-1}(b)\right)\frac{1}{\beta'\left((\beta)^{-1}(b)\right)}-G_x\left((\beta)^{-1}(b)\right) &= 0. \end{aligned}\tag{3.10}$$

At the symmetric equilibrium, we have that $(\beta)^{-1}(b)=x$; thus, we obtain the following ordinary differential equation:

$$\beta'(x) = (x-\beta(x))\frac{g_x(x)}{G_x(x)}. \tag{3.11}$$

In order to determine the symmetric equilibrium explicitly, we need to solve the last ordinary differential equation (ODE) and show that the solution has the features we assumed at the beginning of the heuristics. Using the usual mathematical apparatus for first-price auction derivation and the solution for the ODE, we get the expression of the bidding strategy. The expression and sufficient conditions are given in the following proposition:

Proposition 3.1. (First-Price Auction Bid) *F_1 and F_2 can be ordered by dominance in terms of likelihood ratio. Symmetric equilibrium strategies*[8] *in the first-price auction with Bayesian updaters are given by*

$$\tilde{\beta}^I(x) = \int_0^x y dL(y|x) = x - \int_0^x L(y|x)\, dy, \tag{3.12}$$

where

$$L(y|x) = e^{-\int_y^x \frac{g_t(t)}{G_t(t)}dt}. \tag{3.13}$$

Proof. Firstly, we examine some properties of the function L defined in (3.13).

Property 1. *For any $x \in [0, \omega]$, $L(\cdot|x)$ is a cumulative distribution function over $[0, x]$.*

Indeed, $L(\cdot|x)$ is a right continuous and non-decreasing function by its definition. Furthermore, $L(x|x) = 1$. In order to show that $\lim_{y\to 0} L(y|x) = 0$, note that from Lemma 3.4 it follows that, for any $t > 0$

$$\frac{g_t(t)}{G_t(t)} \geq \frac{g_0(t)}{G_0(t)}$$

and thus,

$$\begin{aligned} -\int_y^x \frac{g_t(t)}{G_t(t)}dt &\leq -\int_y^x \frac{g_0(t)}{G_0(t)}dt \\ &= -\int_y^x \frac{d}{dt}(\ln G_0(t))\, dt \\ &= \ln G_0(y) - \ln G_0(x). \end{aligned}$$

[8] Expression (3.12) is the solution of ODE (3.11) with initial condition $\beta(0) = 0$.

Letting that y tends to 0, we obtain that $\lim_{y\to 0}[\ln G_0(y) - \ln G_0(x)] = -\infty$ and, hence, $\lim_{y\to 0}\left(-\int_y^x \frac{g_t(t)}{G_t(t)}dt\right) = -\infty$. Consequently,

$$\lim_{y\to 0} L(y|x) = e^{\lim_{y\to 0}\left(-\int_y^x \frac{g_t(t)}{G_t(t)}dt\right)} = 0.$$

<u>Property 2.</u> *Distribution $L(\cdot|x')$ first-order stochastically dominates the distribution $L(\cdot|x)$ on the interval $[0,x]$ for $x' > x$.*

Property 2 follows straightforwardly from the definition of the function $L(\cdot|\cdot)$.

Now, we return to the proof of the proposition. The first step is to show that the function defined in this proposition satisfies the preconditions assumed in the heuristic properties assumed at the beginning of this section. Afterwards, we prove that for each private value x, the function $\tilde{\beta}^I(x)$ indeed maximizes the expected payoff of a bidder. Let us start with showing that the preconditions hold true.

Firstly, note that $\tilde{\beta}^I(0) = 0$ (by its definition given in (3.12)). Secondly, function $\tilde{\beta}^I$ is a strictly increasing function because of Properties 1 and 2.[9] Thirdly, it satisfies the ordinary differential equation (3.11), because it is defined as the solution of the equation. Indeed, using standard calculus rules (such as the general Leibniz rule for differentiation under the integral sign) one can show that the bidding function satisfies the ODE given by (3.11). Overall, we have shown that the function satisfies all preconditions required in the heuristics. Therefore, $\tilde{\beta}^I(x)$ is at least a local extremum of the expected payoff to a bidder, when her private value is equal to x.

In order to prove that the function value is the global maximum of the expected payoff, we look at the first derivative of the expected payoff within a neighborhood of the $\tilde{\beta}^I(x)$. It is sufficient to show that the expected payoff, $\Pi^I(b,x)$, increases once we approach $\tilde{\beta}^I(x)$ from the left side (by showing that the left derivative of $\Pi^I(b,x)$ at point $b = \tilde{\beta}^I(x)$ is positive), and that the payoff decreases once we go away from $\tilde{\beta}^I(x)$ (by showing that the associated right derivative is negative). Thus, let us look at the first derivative of the expected payoff, $\frac{\partial}{\partial b}\Pi^I(b,x)$, when the received signal is x and the bidder submits bid b, where b is in a neighborhood of $\tilde{\beta}^I(x)$. The derivative is given by (3.10) and is equal to:

$$\frac{\partial}{\partial b}\Pi^I(b,x) = (x-b)\, g_x\left(\left(\tilde{\beta}^I\right)^{-1}(b)\right)\frac{1}{\left(\tilde{\beta}^I\right)'\left(\left(\tilde{\beta}^I\right)^{-1}(b)\right)} - G_x\left(\left(\tilde{\beta}^I\right)^{-1}(b)\right).$$

[9] Note that the difference is given by:
$\tilde{\beta}^I(x') - \tilde{\beta}^I(x) = \left[(x'-x) - \int_x^{x'} L(y|x')\,dy\right] + \int_0^x (L(y|x) - L(y|x'))\,dy.$

Let $z = \left(\tilde{\beta}^I\right)^{-1}(b)$. Then,

$$\begin{aligned}\frac{\partial}{\partial b}\Pi^I(b,x) &= \left(x-\tilde{\beta}^I(z)\right)g_x(z)\frac{1}{\left(\tilde{\beta}^I\right)'(z)} - G_x(z)\\ &= \frac{G_x(z)}{\left(\tilde{\beta}^I\right)'(z)}\left[\left(x-\tilde{\beta}^I(z)\right)\frac{g_x(z)}{G_x(z)} - \left(\tilde{\beta}^I\right)'(z)\right].\end{aligned}$$

Since $\tilde{\beta}^I$ is an increasing function, we have $\frac{G_x(z)}{\left(\tilde{\beta}^I\right)'(z)} \geq 0$. Moreover, by Lemma 3.3 it follows:

(a) If $z < x$ then $\frac{g_x(z)}{G_x(z)} \geq \frac{g_z(z)}{G_z(z)}$ and, hence,

$\frac{\partial}{\partial b}\Pi^I(b,x) > \frac{G_x(z)}{\left(\tilde{\beta}^I\right)'(z)}\left[\left(z-\tilde{\beta}^I(z)\right)\frac{g_z(z)}{G_z(z)} - \left(\tilde{\beta}^I\right)'(z)\right] = 0$

(since $\tilde{\beta}^I$ satisfies the ordinary differential equation).

(b) In the case where $z > x$ then $\frac{g_x(z)}{G_x(z)} \leq \frac{g_z(z)}{G_z(z)}$ and thus,

$\frac{\partial}{\partial b}\Pi^I(b,x) < \frac{G_x(z)}{\left(\tilde{\beta}^I\right)'(z)}\left[\left(z-\tilde{\beta}^I(z)\right)\frac{g_z(z)}{G_z(z)} - \left(\tilde{\beta}^I\right)'(z)\right] = 0$

(since $\tilde{\beta}^I$ satisfies the ordinary differential equation).

Overall, we have shown that $\tilde{\beta}^I$ is indeed the symmetric equilibrium in the first-price auction.

□

We have just derived the optimal behavior for the Bayesian updaters, once they are asked to participate in a second-price or first-price auction. The next step is to analyze the choice the seller has to make in order to optimize his expected revenue. For this purpose, in the rest of current chapter we assume that the sufficient condition from Proposition 3.1 is satisfied; that is we assume monotonicity of belief function α.

3.3 The Seller's Choice Facing Bayesian Updaters

Suppose that the seller has chosen auction format $A \in \mathcal{A}$. Remember that here we consider the Bayesian updaters model; that is we assume that the seller believes he is facing Bayesian updaters, as described in Section 3.1. Ex ante, the expected revenue to the seller in the chosen auction A, denoted by $\mathbb{E}\left[\tilde{R}^A\right]$, is given by the following derivation:

$$
\begin{aligned}
\mathbb{E}\left[\tilde{R}^A\right] &= \int\limits_{[0,\omega]^N} \left(\sum_{i\in\mathcal{N}} M_i^A\left(x_1, x_2, \ldots, x_N\right)\right) \tilde{\phi}(x_1, x_2, \ldots, x_N) dx_1 dx_2 \ldots dx_N \\
&= \sum_{i\in\mathcal{N}} \int\limits_{[0,\omega]^N} M_i^A\left(x_1, x_2, \ldots, x_N\right) \tilde{\phi}(x_1, x_2, \ldots, x_N) dx_1 dx_2 \ldots dx_N \\
&= \sum_{i\in\mathcal{N}} \int_0^\omega \left(\int\limits_{[0,\omega]^{N-1}} M_i^A\left(x_1, x_2, \ldots, x_N\right) \tilde{\phi}(x_1, x_2, \ldots, x_N) dx_1 dx_2 \ldots dx_{i-1} dx_{i+1} dx_N \right) dx_i \\
&= \sum_{i\in\mathcal{N}} \int_0^\omega \left(\int\limits_{[0,\omega]^{N-1}} M_i^A\left(x_1, x_2, \ldots, x_N\right) \tilde{\phi}_{-i}(x_{-i}\left|x_i\right.) \widetilde{f}(x_i) dx_{-i} \right) dx_i \\
&= \sum_{i\in\mathcal{N}} \int_0^\omega \left(\int\limits_{[0,\omega]^{N-1}} M_i^A\left(x_1, x_2, \ldots, x_N\right) \tilde{\phi}_{-i}(x_{-i}\left|x_i\right.) dx_{-i} \right) \widetilde{f}(x_i) dx_i \\
&= \sum_{i\in\mathcal{N}} \int_0^\omega m_i^A(x_i) \widetilde{f}(x_i) dx_i = N \int_0^\omega m_i^A(x_i) \widetilde{f}(x_i) dx_i,
\end{aligned} \tag{3.14}
$$

where $M_i^A\left(x_1, x_2, \ldots, x_N\right)$ and $m_i^A(x_i)$ denote the actual and the expected payment of bidder i with private value x_i in auction format A. Thus,

$$
\begin{aligned}
m_i^A(x_i) &= \int\limits_{[0,\omega]^{N-1}} M_i^A\left(x_1, x_2, \ldots, x_N\right) \tilde{\phi}_{-i}(x_{-i}\left|x_i\right.) dx_{-i} \\
&= \int_0^\omega M_i^A\left(x_i, y\right) g_{x_i}\left(y\right) dy.
\end{aligned} \tag{3.15}
$$

In the case of the first-price auction ($A = I$), we obtain

$$
\begin{aligned}
m_i^I(x_i) = \int_0^\omega M_i^I\left(x_i, y\right) g_{x_i}\left(y\right) dy \quad &= \int_0^{x_i} \tilde{\beta}^I(x_i) g_{x_i}\left(y\right) dy \\
= \tilde{\beta}^I(x_i) \int_0^{x_i} g_{x_i}\left(y\right) dy \quad &= \tilde{\beta}^I(x_i) G_{x_i}\left(x_i\right),
\end{aligned}
$$

whereas in the case of the second-price auction $(A = II)$ it is the case that

$$
m_i^{II}(x_i) = \int_0^\omega M_i^{II}\left(x_i, y\right) g_{x_i}\left(y\right) dy = \int_0^{x_i} \tilde{\beta}^{II}(y) g_{x_i}\left(y\right) dy = \int_0^{x_i} y g_{x_i}\left(y\right) dy.
$$

Hence, we are now able to explicitly calculate the expected revenue to the seller and may analyze his preference of auction format. Note that in the classical model (with i.i.d. private values), the revenue equivalence principle, established by Myerson (1981) and Riley and Samuelson (1981), holds true. Under classical assumptions, the seller gets the same expected revenue from the first-price or second-price auction. The following proposition shows that in our model, once we have strict order by dominance in terms of likelihood ratio, the seller is no longer indifferent and prefers the second-price auction over the first-price auction.

Proposition 3.2. (Revenue Equivalence Failure) *If α is strictly monotonic, then*

$$\mathbb{E}\left[\tilde{R}^{II}\right] > \mathbb{E}\left[\tilde{R}^{I}\right]. \tag{3.16}$$

Before proceeding to the proof of the last proposition, we must formulate a lemma needed for the proof.[10]

Lemma. *Let random variables X and Y be distributed according to F and G respectively, where F first-order stochastically dominates G. Now suppose $\gamma : [0,\omega] \to \mathbb{R}$ is a strictly increasing and differentiable function. Then*

$$\mathbb{E}\left[\gamma(X)\right] - \mathbb{E}\left[\gamma(Y)\right] \geq 0.$$

Let us now proceed with the proof of the proposition.

Proof. Since the assumption on the monotonicity of α is the sufficient condition for Proposition 3.1, it follows that $\tilde{\beta}^I$ describes bidding behavior in the first-price auction. Note that in order to prove inequality (3.16), it suffices to show that $m_i^I(x_i) < m_i^{II}(x_i)$, for any i and any $x_i \in (0,\omega]$, as it would imply $\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\tilde{R}^{I}\right] = \sum\limits_{i\in\mathcal{N}} \int\limits_0^\omega \left(m_i^{II}(x_i) - m_i^I(x_i)\right) \widetilde{f}(x_i)dx_i > 0$. Therefore, our next step is to prove that $m_i^I(x_i) < m_i^{II}(x_i)$, or equivalently that

$$\tilde{\beta}^I(x_i) < \frac{1}{G_{x_i}(x_i)} \int\limits_0^{x_i} y g_{x_i}(y)\, dy,\; x_i \in (0,\omega]. \tag{3.17}$$

The right-hand side of the last inequality may be written as $\int\limits_0^{x_i} ydK(y\,|x_i)$ with $K(y\,|x_i) = \frac{G_{x_i}(y)}{G_{x_i}(x_i)}$. On the other side, from (3.12) we have that the left-side of the inequality is equal to $\int\limits_0^{x_i} ydL(y\,|x_i)$ with $L(y\,|x_i) = e^{-\int\limits_y^{x_i} \frac{g_t(t)}{G_t(t)}dt}$. Hence, showing that (3.17) holds true is equivalent to showing that

[10] The proof of lemma is straightforward and may be found, for example, as last claim and its derivation in Krishna (2010) p. 275.

$$\int_0^x yd\left[K\left(y\,|x\right) - L\left(y\,|x\right)\right] > 0,\; x \in (0,\omega]\,. \tag{3.18}$$

Note that both $L\left(\cdot\,|x\right)$ and $K\left(\cdot\,|x\right)$ are cumulative distribution functions over interval $[0,x]$ (see Property 1 stated in the proof of Proposition 3.1).

According to the previous lemma, in order to establish inequality (3.18), it suffices to show that $L\left(y\,|x\right) > K\left(y\,|x\right)$ for any $0 < y < x \leq \omega$. The last claim follows from Lemma 3.3, Remark 3.1 and the assumption of the proposition. Indeed, the lemma and the remark imply that for any $t < x$, $t, x \in (0,\omega]$

$$\frac{g_t\left(t\right)}{G_t\left(t\right)} < \frac{g_x\left(t\right)}{G_x\left(t\right)},$$

and, thus,

$$\begin{aligned} -\int_y^x \frac{g_t\left(t\right)}{G_t\left(t\right)} dt > & -\int_y^x \frac{g_x\left(t\right)}{G_x\left(t\right)} dt \\ = & -\int_y^x \frac{d}{dt}\left(\ln G_x\left(t\right)\right) dt \\ = & \ln G_x\left(y\right) - \ln G_x\left(x\right) \\ = & \ln \frac{G_x\left(y\right)}{G_x\left(x\right)}. \end{aligned}$$

Applying the exponential function to both sides, we obtain that for all $y < x$, $L\left(y\,|x\right) > K\left(y\,|x\right)$ and this completes the proof. □

Remark 3.2. Note that the last proposition is equivalent to proving the failure of the revenue equivalence principle between the first-price and second-price auctions in our model once ordering by dominance in terms of likelihood ratio is present. Therefore, under these circumstances, a seller facing Bayesian updaters will choose the second-price auction rather than the first-price auction.

In the previous two sections we have analyzed both bidders' and seller's behavior under the conditions of our model. In the following section we compare behavioral patterns among our model with respective behavior in other relevant standard models. The comparison gives us a better insight and understanding of our model's features.

3.4 Comparison to Other Models

Here we compare our model to the other well-known models in literature. For sake of short notation, we will denote our model by $\widetilde{\mathcal{M}}$. The necessary conditions from Proposition 3.1 are assumed throughout the rest of the section. In particular, we assume that F_1 and F_2 can be ordered in terms of likelihood ratio dominance (for equivalent formulations of the assumption, please refer to Lemma 3.1).

3.4.1 Relevant Non-Updating Models

The following three models represent classical models in the related literature. The common feature of these models is that the distribution of the private signals, known a priori by all participants in auction, is also used ad interim; i.e. bidders do not update. Namely, we suppose that bidders' private signals are identically distributed according to some cumulative distribution function H and each bidder uses this c.d.f. as given during all her calculations and actions. We set $H = F_1$, F_2 or $H = \tilde{F} = 1/2F_1 + 1/2F_2$, which gives rise to three different models. Technically, these models differ from the Bayesian updaters model by adding the assumption that private signals are stochastically independent and eventually changing the explicit formula for the c.d.f. of private signals. We analyze how these changes effect the behavior of bidders and the seller by comparing both bidding strategies and the expected revenues across different models. For this purpose, let us summarize the general pattern of each of these models in Table 3.2 :

	Seller and Bidders
X_i Private signal of bidder i	$H(x_1)$, $x_i \in [0, \omega]$ $h(x_i)$, $x_i \in [0, \omega]$
$(X_1, X_2, \ldots, X_N)$ Joint random vector of signals	$\Phi(x_1, x_2, \ldots, x_N) = \prod_{i \in \mathcal{N}} H(x_i)$ $\phi(x_1, x_2, \ldots, x_N) = \prod_{i \in \mathcal{N}} h(x_i)$
$Y_1^H \lvert X_i = x$, $Y_1^H = \max_{j \neq i} X_j$	$G_H(y) = (H(y))^{N-1}$ $g_H(y) = (N-1)(H(y))^{N-2} h(y)$

Table 3.2: *"Classical" and benchmark models: The seller's and bidders' perspective.*

As stated above, we assume $H \in \{F_1, F_2, \tilde{F}\}$. When $H = F_i$, $i \in \{1, 2\}$ we shall refer to the associated model as model $\mathcal{M}_i$, whereas $\overline{\mathcal{M}}$ will be the model in the case where $H = \tilde{F}$. Bidding strategies at a symmetric equilibrium of model $\mathcal{M}_i$ in auction format $A \in \mathcal{A}$ is denoted by $\beta_i^A(\cdot)$, whereas the corresponding expected revenue will be denoted by $E\left[R_i^A\right]$. In the model $\overline{\mathcal{M}}$, the bidding strategies at the symmetric equilibrium and the expected revenue in auction format $A \in \mathcal{A}$ will be denoted by $\bar{\beta}^A(\cdot)$ and $\mathbb{E}\left[\bar{R}^A\right]$, respectively. Moreover, we denote by Y_1^i ($\bar{Y}_1$), the random variable $\max_{j\neq i} X_j$ in the model $\mathcal{M}_i$ ($\overline{\mathcal{M}}$) with c.d.f. G_i ($\bar{G}$) and d.f. g_i ($\bar{g}$). Hence, these functions take the following forms:

$$\begin{aligned} G_i(y) &= (F_i(y))^{N\text{-}1}, \quad g_i(y) = (N-1)\,(F_i(y))^{N\text{-}2} f_i(y), \\ G(y) &= \left(\tilde{F}(y)\right)^{N\text{-}1}, \quad \bar{g}(y) = (N-1)\left(\tilde{F}(y)\right)^{N\text{-}2} \tilde{f}(y). \end{aligned}$$

Using this notation, from the literature (see Vickrey (1961) and Riley and Samuelson (1981)) it is known that:

$$\beta_1^{II}(x) = \beta_2^{II}(x) = \bar{\beta}^{II}(x) = x, \tag{3.19}$$

$$\begin{aligned} \beta_i^I(x) &= E\left[Y_1^i \,\middle|\, Y_1^i < x\right] \text{ and} \\ \bar{\beta}^I(x) &= E\left[\bar{Y}_1 \,\middle|\, \bar{Y}_1 < x\right]. \end{aligned}$$

More precisely, the symmetric equilibria in first-price auctions might be rewritten as

$$\begin{aligned} \beta_i^I(x) &= \frac{1}{G_i(x)} \int_0^x y g_i(y) dy \text{ and} \\ \bar{\beta}^I(x) &= \frac{1}{\bar{G}(x)} \int_0^x y \bar{g}(y) dy. \end{aligned} \tag{3.20}$$

Let us examine the relevance of these three models for our set up. The model $\overline{\mathcal{M}}$ represents the "classical" model described in Section 3.1. Namely, that is the model when bidders do not update on underlying information structure of the set-up. In particular, in framework $\overline{\mathcal{M}}$, the seller has the belief that $\Phi(x_1, x_2, \ldots, x_N) = \prod_{i\in\mathcal{N}} \tilde{F}(x_i)$ is the distribution of the joint private signals and that the bidders are classical bidders.

Here, models $\mathcal{M}_i$ serve as benchmark models. Namely, the Bayesian updaters behave in the way that they believe that their private signal is generated either by F_1 or F_2. Ad interim, they use their private signal to update their belief about the origin of their signals (the belief was given by function α). Hence, it seems appealing to compare the behavior of the Bayesian updaters to the case where bidders know the true distribution. That is why $\mathcal{M}_i$ are labeled as "non-updating models", as their bidders hold fixed, constant beliefs that their signals are i.i.d. distributed with marginal distribution F_i.

We proceed by comparing bidders' performance in each of these models. As the "non-updating models" are known and well-studied in the literature, we consider their features and characteristics as given and state a literature reference when needed.

3.4.2 Comparison of Bidders' Behavior in the First-Price Auction

Note that bidders, independent of the model choice, in the second-price auction, always bid truthfully. However, their behavior differs in the first-price auction. This section is actually devoted to the comparison of their behavior in the first-price auction across models that are the subject of our research. For this purpose, we assume the sufficient condition from Proposition 3.1; that is, we assume that F_1 and F_2 can be ordered by dominance in terms of likelihood ratio. Without loss of generality, we will consider the case where F_1 dominates F_2 in terms of likelihood ratio.

The following proposition compares Bayesian updaters and the classical model to the benchmark models.

Proposition 3.3. (First-Price Auction Bids: Comparison to Benchmarks) *Suppose that F_1 dominates F_2 in terms of likelihood ratio. Then*

$$\beta_2^I(x) \leq \tilde{\beta}^I(x) \leq \beta_1^I(x), \tag{3.21}$$

$$\beta_2^I(x) \leq \bar{\beta}^I(x) \leq \beta_1^I(x), \tag{3.22}$$

for all $x \in [0, \omega]$.

Proof. According to the explicit formula for symmetric bidding equilibria, we have

$$\beta_i^I(x) = \frac{1}{G_i(x)} \int_0^x y g_i(y) dy = \int_0^x y \, d\left(\frac{G_i(y)}{G_i(x)}\right)$$

for $i \in \{1, 2\}$. Furthermore, the bidding strategy of the Bayesian updaters in the first-price auction might be written as

$$\tilde{\beta}^I(x) = \int_0^x y \, dL(y \,|x\,), \tag{3.23}$$

with

$$L(y\,|x\,) = e^{-\int_y^x \sigma_L(t)dt}$$

and

$$\sigma_L(t) = \frac{g_t(t)}{G_t(t)} = \frac{(N-1)\left[\alpha(t)(F_1(t))^{N-2} f_1(t) + (1-\alpha(t))(F_2(t))^{N-2} f_2(t)\right]}{\alpha(t)(F_1(t))^{N-1} + (1-\alpha(t))(F_2(t))^{N-1}}.$$

In general, for a random variable with c.d.f. F and d.f. f, the function $\sigma_F(t) = \frac{f(t)}{F(t)}$ is known as the reverse hazard rate. Moreover, as stated earlier in this chapter (see also Appendix B in Krishna (2010)), the dominance of the likelihood ratio implies the dominance in terms of the reverse hazard rate. Hence, by the assumption of the proposition, it follows that

$$\sigma_{F_1}(t) \geq \sigma_{F_2}(t) \Leftrightarrow \frac{f_1(t)}{F_1(t)} \geq \frac{f_2(t)}{F_2(t)}. \tag{3.24}$$

Furthermore, we may rewrite $\beta_i^I(x)$, for any $i \in \{1,2\}$, as follows:

$$\beta_i^I(x) = \int_0^x y\, dL_i(y\,|x\,), \tag{3.25}$$

where

$$L_i(y\,|x\,) = e^{-\int_y^x \sigma_{G_i}(t)dt} = \frac{G_i(y)}{G_i(x)}$$

and

$$\sigma_{G_i}(t) = \frac{g_i(t)}{G_i(t)}.$$

Integration by parts in equations (3.23) and (3.25), leads to the next expressions:

$$\tilde{\beta}^I(x) = \int_0^x y\, dL(y\,|x\,) = x - \int_0^x L(y\,|x\,)dy$$

and

$$\beta_i^I(x) = \int_0^x y\, dL_i(y\,|x\,) = x - \int_0^x L_i(y\,|x\,)dy$$

for $i \in \{1,2\}$. Thus, in order to prove the claim (3.21), it is sufficient[11] to prove that for any $x \in [0,\omega]$ and $y \leq x$,

$$\begin{array}{ccc} L_1(y\,|x\,) & \leq & L(y\,|x\,) \leq L_2(y\,|x\,) \\ & \Updownarrow & \\ \int_y^x \sigma_{G_2}(t)dt & \leq & \int_y^x \sigma_L(t)dt \leq \int_y^x \sigma_{G_1}(t)dt. \end{array}$$

[11]Note that functions L_i and L are non-negative functions.

It turns out that the domination in terms of reverse hazard rate, given by (3.24), is equivalent to $\sigma_{G_2}(t) \leq \sigma_L(t) \leq \sigma_{G_1}(t)$ for any $t \in [0, \omega]$, that implies the last set of inequalities. Indeed,

$$\begin{array}{rcl} \sigma_{G_2}(t) & \leq & \sigma_L(t) \\ & \Updownarrow & \\ \dfrac{(N-1)\left(F_2(t)\right)^{N-2} f_2(t)}{\left(F_2(t)\right)^{N-1}} & \leq & \dfrac{(N-1)\left[\alpha(t)\left(F_1(t)\right)^{N-2} f_1(t) + (1-\alpha(t))\left(F_2(t)\right)^{N-2} f_2(t)\right]}{\alpha(t)\left(F_1(t)\right)^{N-1} + (1-\alpha(t))\left(F_2(t)\right)^{N-1}} \\ & \Updownarrow & \\ \alpha(t) F_1(t) f_2(t) & \leq & \alpha(t) F_2(t) f_1(t) \\ & \Updownarrow & \\ \dfrac{f_2(t)}{F_2(t)} & \leq & \dfrac{f_1(t)}{F_1(t)}. \end{array}$$

Similarly,

$$\begin{array}{rcl} \sigma_{G_1}(t) & \geq & \sigma_L(t) \\ & \Updownarrow & \\ \dfrac{f_1(t)}{F_1(t)} & \geq & \dfrac{\alpha(t)\left(F_1(t)\right)^{N-2} f_1(t) + (1-\alpha(t))\left(F_2(t)\right)^{N-2} f_2(t)}{\alpha(t)\left(F_1(t)\right)^{N-1} + (1-\alpha(t))\left(F_2(t)\right)^{N-1}} \\ & \Updownarrow & \\ (1-\alpha(t))\left(F_2(t)\right)^{N-1} f_1(t) & \geq & (1-\alpha(t))\left(F_2(t)\right)^{N-2} F_1(t) f_2(t) \\ & \Updownarrow & \\ \dfrac{f_1(t)}{F_1(t)} & \geq & \dfrac{f_2(t)}{F_2(t)}. \end{array}$$

Thus, the statement $\sigma_{G_2}(t) \leq \sigma_L(t) \leq \sigma_{G_1}(t)$ for any $t \in [0, 1]$ is equivalent to the assumption that F_1 dominates F_2 in terms of the reverse hazard rate, which turned out to be a sufficient condition for the statement expressed by (3.21).

In an analogous way, we may show that for (3.22) it suffices to show that

$$\sigma_{G_2}(t) \leq \sigma_{\bar{L}}(t) \leq \sigma_{G_1}(t)$$

with

$$\sigma_{\bar{L}}(t) = \frac{\bar{g}(y)}{\bar{G}_i(x)}.$$

Thus, in a similar manner we obtain that

$$\begin{array}{rcl}
\sigma_{G_2}(t) & \leq & \sigma_{\bar{L}}(t) \\
 & \Updownarrow & \\
\dfrac{(N-1)\,(F_2(t))^{N-2}\,f_2(t)}{(F_2(t))^{N-1}} & \leq & \dfrac{(N-1)\,\left(\tilde{F}(y)\right)^{N\text{-}2}\,\tilde{f}(y)}{\left(\tilde{F}(y)\right)^{N\text{-}1}} \\
 & \Updownarrow & \\
\tilde{F}(y)f_2(t) & \leq & F_2(t)\tilde{f}(y) \\
 & \Updownarrow & \\
\frac{1}{2}\,(F_1(t)+F_2(t))\,f_2(t) & \leq & \frac{1}{2}F_2(t)\,(f_1(t)+f_2(t)) \\
 & \Updownarrow & \\
F_1(t)f_2(t) & +F_2(t)f_2(t) \leq & F_2(t)f_1(t)+F_2(t)f_2(t) \\
 & \Updownarrow & \\
F_1(t)f_2(t) & \leq & F_2(t)f_1(t).
\end{array}$$

It turned out, as before, that the last claim is equivalent to (3.24), which is implied by the assumption. The same argument applies for the second inequality in expression (3.22). Q.E.D. □

The last proposition gives a feeling for the difference in the bidding behavior between the benchmark models and the models that are considered as possible scenarios for the information structure proposed in this chapter. The higher belief is that the values are generated by the distribution that is dominant and favors higher values (in the case of the assumption of the lemma, it is the distribution F_1), the bidders will bid higher in the first-price auction.

However, so far, we did not compare the "classical" model to the model with Bayesian updaters. This is the goal of the rest of this section. We compare the bidding behavior of Bayesian updaters and bidders in the classical model, as these two models are the models that are considered as the only possible scenarios in our model. In other words, we aim to analyze relationship between $\tilde{\beta}^I(x)$ and $\bar{\beta}^I(x)$.

Classical Bidder versus Bayesian Updater

Here we look at the difference in bidding behavior between the classical and Bayesian updater types of bidder. In particular, we are interested in the sign of the difference $\bar{\beta}^I - \tilde{\beta}^I$. Our starting point is the fact, that the strategies are given by the next two expressions (see the definitions given in (3.12) and (3.20) and the proof of the last proposition):

$$\tilde{\beta}^I(x) = x - \int_0^x L(y|x)\,dy,$$
$$\bar{\beta}^I(x) = x - \int_0^x \bar{L}(y|x)dy$$

with

$$L(y|x) = e^{-\int_y^x \frac{g_t(t)}{G_t(t)}dt}$$
$$\bar{L}(y|x) = e^{-\int_y^x \frac{\bar{g}(t)}{\bar{G}(t)}dt}.$$

Therefore, we have that

$$\bar{\beta}^I(x) - \tilde{\beta}^I(x) = \int_0^x [L(y|x) - \bar{L}(y|x)]\,dy.$$

Moreover, the ratio of the functions participating in the last integral takes the following form:

$$\frac{L(y|x)}{\bar{L}(y|x)} = e^{\int_y^x \left[\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)}\right]dt}.$$

Given the equations above, we start our analysis looking at the difference $\frac{\bar{g}(x)}{\bar{G}(x)} - \frac{g_x(x)}{G_x(x)}$. The following lemma establishes the decomposition of this difference. The decomposition helps us to identify functions which are crucial in determining the sign of the difference in bidding behavior.

Lemma 3.5. Given notation from this chapter, for any $x \in (0, \omega]$ it holds true:

$$\frac{\bar{g}(x)}{\bar{G}(x)} - \frac{g_x(x)}{G_x(x)} = \frac{(N-1)(f_1(x)F_2(x) - f_2(x)F_1(x))}{2\tilde{F}(x)\left(\alpha(x)F_1^{N-1}(x) + (1-\alpha(x))F_2^{N-1}(x)\right)} \cdot \left[(1-\alpha(x))F_2^{N-2}(x) - \alpha(x)F_1^{N-2}(x)\right].$$

Proof of the lemma can be found in the Appendix 3.A.1.

Our core assumption is that F_1 dominates F_2 in terms of the likelihood ratio or vice versa. In the case of the former dominance, we have that the first part of the right hand-side of the last equality (i.e. the expression between the equality and multiplication signs) is non-negative, whereas in the case of the later dominance, the part is non-positive. In other words, under our assumption on likelihood ratio dominance, the sign of $\frac{\tilde{g}(x)}{\tilde{G}(x)} - \frac{g_x(x)}{G_x(x)}$ is determined by the sign of the term $(1-\alpha(x)) F_2^{N-2}(x) - \alpha(x) F_1^{N-2}(x)$, for any value x. Thus, for the purpose of the current analysis, we shall denote by $\delta : (0,\omega] \to (0,\omega]$ the following function

$$\delta(x) \quad = \quad \frac{F_2^{N-2}(x)}{F_1^{N-2}(x) + F_2^{N-2}(x)}.$$

Using the notation for function δ, the right hand side of the claim in the last lemma might be transformed to:

$$\underbrace{\frac{(N-1)\left(f_1(x) F_2(x) - f_2(x) F_1(x)\right)}{2\tilde{F}(x)\left(\alpha(x) F_1^{N-1}(x) + (1-\alpha(x)) F_2^{N-1}(x)\right)}\left(F_2^{N-2}(x) + F_1^{N-2}(x)\right)}_{\geq 0} \quad \cdot \quad [\delta(x) - \alpha(x)].$$

Henceforth, we may conclude that the sign of $\frac{\tilde{g}(x)}{\tilde{G}(x)} - \frac{g_x(x)}{G_x(x)}$ depends directly on the sign of the difference between functions δ and α (given the likelihood dominance order).

The further steps of our analysis established the claims given by the next lemma. The lemma gives further insight into some properties and interplay between the functions and brings us one step closer to the final result on the difference of the bidding behaviors.

Lemma 3.6. *Let us suppose that F_1 (strictly) dominates F_2 in terms of the likelihood ratio. Then, for any private value $x \in (0,\omega)$, the following statements are true:*

(i) The belief function α is (strictly increasing) non-decreasing, whereas δ is a non-increasing function of x. Thus, the function $\delta - \alpha$ is (strictly decreasing) a non-increasing function on interval $(0,\omega)$and

(ii) $\lim_{x\to 0} (\delta(x) - \alpha(x)) \geq 0.$

The steps of the proof can be found in Appendix 3.A.2.

Remark. If we consider $\delta - \alpha$ as a function, we may conclude that the function takes a zero value nowhere, or if it does so, it takes a zero value only at some connected interval $[x^*, x^{**}] \subseteq [0,\omega]$, with $x^* = x^{**}$, when strict domination in terms of the likelihood ratio is assumed. This indirectly gives rise to the possibility of the single-crossing property between considered bidding strategies. This is stated as the last statement of the next proposition. Moreover, with the help of the last two lemmas we may establish the following claims about the bidding behavior of different bidder types:

Proposition 3.4. (WEAK DOMINANCE OR SINGLE-CROSSING PROPERTY OF THE BIDS) *Let us suppose that F_1 dominates F_2 in terms of the likelihood ratio. Then, for the bidding strategies in the first-price auction, exactly one of the following two scenarios must be true.*

(i) (Weak dominance) A Bayesian bidder always bids weakly lower / weakly higher than the classical bidder. That is,

$$(\forall x \in (0,\omega)) \left(\tilde{\beta}^I(x) \leq \bar{\beta}^I(x)\right) \text{ or } (\forall x \in (0,\omega)) \left(\tilde{\beta}^I(x) \geq \bar{\beta}^I(x)\right).$$

Or,

(ii) (Single-crossing property) There is a unique private value $\hat{x} \in (0,\omega)$, such that for any value smaller than $\hat{x}$, the Bayesian bidder does not bid higher than the classical bidder. For values higher than $\hat{x}$, the Bayesian bidder bids strictly higher. Mathematically,

$$(\exists! \hat{x} \in (0,\omega)) \left(x \leq \hat{x} \Rightarrow \tilde{\beta}^I(x) \leq \bar{\beta}^I(x) \text{ and } x > \hat{x} \Rightarrow \tilde{\beta}^I(x) > \bar{\beta}^I(x)\right).$$

We define the situation described in part (i) as weak dominance between bidding strategies, where the name is self-explanatory. The situation described in part (ii) of the last proposition is known as "single-crossing property". The term "single-crossing" comes from the fact that bidding strategies cross each other only once on the open interval $(0,\omega)$ (and at value $\hat{x}$). In the case of statement (ii), we say that "$\tilde{\beta}^I(x)$ crosses $\bar{\beta}^I(x)$ from below" or "$\bar{\beta}^I(x)$ crosses $\tilde{\beta}^I(x)$ from above".

Remark. If the opposite likelihood ratio dominance is assumed; that is, if F_2 dominates F_1, one may show same results. In this case, Lemma 3.6 holds true with reverted inequalities and opposite monotonicity trends. Regarding the last proposition, both parts remain to hold true.

Finally, we may summarize the comparison of the classical and Bayesian updater in the first-price auction by the following final corollary:

Corollary 3.1. *Let us suppose that F_1 and F_2 can be ordered by dominance in terms of the likelihood ratio. Then, bidding strategies of classical and Bayesian updaters in the first-price auction, $\bar{\beta}^I$ and $\tilde{\beta}^I$, satisfy exactly one of the following three claims :*

(i) A Bayesian updater always bids weakly lower than the classical type of bidder (with identical private values).

(ii) Given a private value, a Bayesian updater always bids weakly higher than the classical bidder.

(iii) The bidding strategies exhibit a "single-crossing" property; that is, the bidding functions cross themselves only once and $\tilde{\beta}^I(x)$ crosses $\bar{\beta}^I(x)$ from below.

We conclude the current subsection with an example of the bidding strategies. A simple example with $N = 2$, $F_1(x) = x$ and $F_2(x) = x^2$ for $x \in [0, 1]$ is considered. Figure 3.1 illustrates the two bidding strategies (see the plot at the left-hand side of the figure) and their difference as a function (illustrated by the right-hand plot in the figure).

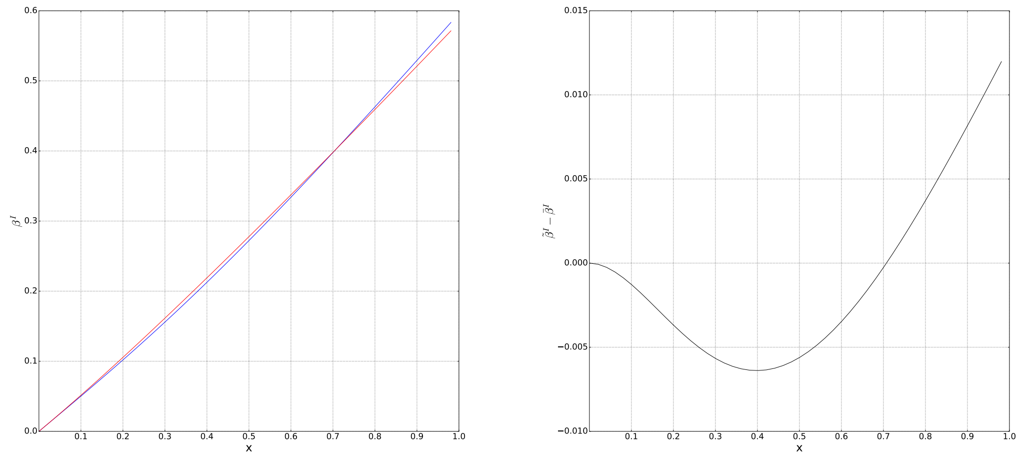

Figure 3.1: *Bidding strategies in the first-price auction - a simple example*. The private value is plotted on the horizontal axis of both plots, whereas the vertical axis represents either bids (*left*) or difference of the bids (*right*). The blue line represents the Bayesian updater's behavior whereas the red line represents the classical bidder's behavior (*left*). The difference between the Bayesian and "classical" bidder is illustrated as a function of the private value (*right*).

3.4.3 The Expected Revenue Puzzle For the Seller

We have compared the performance of the first-price auction and the second-price auction separately in each relevant model, where performance is identified with the expected revenue of the seller. We now make our assumption on the likelihood ratio stricter in the sense that we assume strict dominance in terms of the likelihood ratio. From the obtained revenue comparison in Section 3.3 and the revenue equivalence principle we summarize the comparison in Table 3.3:

Model	First-price auction		Second-price auction
$\mathcal{M}_1$	$\mathbb{E}\left[R_1^I\right]$	$=$	$\mathbb{E}\left[R_1^{II}\right]$
$\mathcal{M}_2$	$\mathbb{E}\left[R_2^I\right]$	$=$	$\mathbb{E}\left[R_2^{II}\right]$
$\overline{\mathcal{M}}$	$\mathbb{E}\left[\bar{R}^I\right]$	$=$	$\mathbb{E}\left[\bar{R}^{II}\right]$
$\widetilde{\mathcal{M}}$	$\mathbb{E}\left[\tilde{R}^I\right]$	$<$	$\mathbb{E}\left[\tilde{R}^{II}\right]$

Table 3.3: *Expected revenue of first-price against second-price auction (under strict dominance in terms of likelihood ratio).*

One may conclude that if the seller is uncertain which of the above listed scenarios occurs and wants to be on the safe side, then he should prefer the second-price auction over the first-price auction. Therefore, let us assume that the seller does so and chooses to run a second-price auction. We shall now focus only on models $\widetilde{\mathcal{M}}$ and $\overline{\mathcal{M}}$, because in our set-up these two models are the only possible ones. In the following lines, we compare the performance of the second-price auction across these two models. More precisely, we are interested in the difference $\mathbb{E}\left[\bar{R}^{II}\right] - \mathbb{E}\left[\tilde{R}^{II}\right]$. Note that even though the bidders' beliefs change across models, their bids are identical in the second-price auction - they bid truthfully. Hence, the difference in the expected revenue then comes from the difference in the seller's beliefs. In the following lines we shall provide our analysis in the form of remarks which might be helpful for the further research of the topic. Proofs of the remarks can be found in appendix.

Firstly, we obtain the general formula for the difference in the expected revenue that second-price auctions deliver in considered models, presented in the following remark.

Remark 3.3. $\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] = N\int\limits_0^{\omega}\left[\int\limits_0^{x} y\left(g_x(y) - \bar{g}(y)\right)dy\right]\tilde{f}\left(x\right)dx.$

Secondly, from the general number of bidders, we restricted our research in the special case of two bidders (i.e. we assumed $N = 2$). This simplification leads to the conclusion that in this special case, the second-price performs better in the case of the Bayesian updaters than in the classical model. Therefore, the seller who interprets the special mixture form of the observed distribution as uncertainty is, in a certain sense, subjectively better off than the "ignorant" seller of the classical model once the second-price auction is chosen for sale. Thus, we conclude this subsection with a remark stating this last fact:

Remark 3.4. In the case of two bidders, $\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] \geq 0$.

Details and derivations behind the previous two remarks can be found in Appendices 3.A.4 and 3.A.5.

3.5 Wrapping Up and Final Comments

We offered an alternative way of modeling the information structure involved in auctions where we departed from the usual assumption that the distribution of private values is simply given. Namely, we have allowed a mild ambiguity regarding the origin of private value. Participants were supposed to deal with ambiguity ad interim in the Bayesian way of thinking - the bidders update their belief about the origin according to Bayes' rule using their private information or private value. The seller, on the other side, can not update (as he has no further private information) but takes into account the mixture form of the observed distribution by his belief formation process.

We analyzed bidders' behavior and the seller's revenue in the first-price and second-price auctions in the set-up and compared the results to the classical set-up. In the second-price auction, bidders' behavior does not change (in comparison to the classical model where the distribution is given and not questioned) - they still bid truthfully. However, in the first-price auction, the behavior of bidders is influenced by updating. It turns out that one can establish either a weak dominance between the classical bidder's and the Bayesian updater's bidding pattern or that the patterns exhibit single-crossing property.

Moreover, questioning on the origin of private values leads to revenue equivalence failure in favor of the second-price auction. Namely, when facing Bayesian updaters, the seller would prefer to run a second-price auction instead of a first-price auction. Furthermore, given that the sellers choose the second-price auction and that there are only two potential buyers, the seller from the Bayesian model is better off than the one from the classical model. This is not due to bidders' behavior but due to the difference in the sellers' beliefs about the joint distribution of private values.

Several assumptions were used in the presented results. Here we discuss how they might be lifted to widen the scope of the presented research. Firstly, we assumed that the observed distribution of the private value is a mixture distribution, i.e. a convex combination of several distributions. This was the approach used to introduce the doubt about the origin of private values, making it possible to question: Which one of the involved distributions is the one which generates the private values? For this purpose, it was enough to consider only two distributions as candidates. Moreover, we assumed a priori, that both distribution were equally likely to be the origin of the values. The equal weights for both candidates were only for the sake of simplicity - the same claims were true if the common prior was an arbitrary choice of convex combination between the two candidates. However, whether or not the same result would hold if more than two distributions are involved is a topic for further research. The introduction of more distributions would definitely make the notation and mathematical apparatus more complicated. Additionally, instead of dominance in terms of the likelihood ratio between two distributions, one would have to consider a finite family of distributions satisfying a monotone likelihood ratio (MLR). An MLR family is a family of distributions, which can be completely ordered by dominance in terms of likelihood ratio. As this type of family generalizes many of the properties used in this chapter (such as first-order dominance, dominance in terms of reverse hazard rate etc.; see Shao (2003)), our conjecture is that the results of the chapter might be extended to MLR families of distributions.

In sum, the introduction of mild ambiguity in standard auction models with the usual probability set-up had led to a failure of the essential revenue equivalence principle. This should be reason enough for a new direction of modeling in auction theory, in such a way that it will allow endogenous belief formation about the distribution of private signals itself.

3.A Proofs

3.A.1 Proof of Lemma 3.5

The equality is shown by replacing the involved functions by expressions from their definitions. As the argument of all functions is identical (that is, the argument is x), for better readability, we will omit the argument, wherever it is unambiguous. Using this notation, hence, we get:

$$
\begin{aligned}
\frac{\bar{g}(x)}{\bar{G}(x)} - \frac{g_x(x)}{G_x(x)} &= \frac{(N-1)\tilde{F}^{N\text{-}2}\tilde{f}}{\tilde{F}^{N\text{-}1}} - \frac{(N-1)\left[\alpha F_1^{N-2} f_1 + (1-\alpha) F_2^{N-2} f_2\right]}{\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}} \\
&= (N-1)\frac{\alpha F_1^{N-1}\tilde{f} + (1-\alpha) F_2^{N-1}\tilde{f} - \alpha F_1^{N-2}\tilde{F} f_1 - (1-\alpha) F_2^{N-2}\tilde{F} f_2}{\tilde{F}\left(\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}\right)} \\
&= (N-1)\frac{\alpha F_1^{N-1}(f_1+f_2) + (1-\alpha) F_2^{N-1}(f_1+f_2) - \alpha F_1^{N-2}(F_1+F_2) f_1 - (1-\alpha) F_2^{N-2}(F_1+F_2) f_2}{2\tilde{F}\left(\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}\right)} \\
&= (N-1)\frac{\alpha F_1^{N-1} f_1 + \alpha F_1^{N-1} f_2 + (1-\alpha) F_2^{N-1} f_1 + (1-\alpha) F_2^{N-1} f_2 - \alpha F_1^{N-1} f_1 - \alpha F_1^{N-2} F_2 f_1}{2\tilde{F}\left(\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}\right)} \\
&\quad -(N-1)\frac{(1-\alpha) F_2^{N-1} f_2 + (1-\alpha) F_2^{N-2} F_1 f_2}{2\tilde{F}\left(\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}\right)} \\
&= (N-1)\frac{\alpha F_1^{N-1} f_2 + (1-\alpha) F_2^{N-1} f_1 - \alpha F_1^{N-2} F_2 f_1 - (1-\alpha) F_2^{N-2} F_1 f_2}{2\tilde{F}\left(\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}\right)} \\
&= (N-1)\frac{(1-\alpha) F_2^{N-2}(F_2 f_1 - F_1 f_2) - \alpha F_1^{N-2}(F_2 f_1 - F_1 f_2)}{2\tilde{F}\left(\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}\right)} \\
&= (N-1)\frac{(1-\alpha) F_2^{N-2}(F_2 f_1 - F_1 f_2) - \alpha F_1^{N-2}(F_2 f_1 - F_1 f_2)}{2\tilde{F}\left(\alpha F_1^{N-1} + (1-\alpha) F_2^{N-1}\right)} \\
&= \frac{(N-1)\left(f_1(x) F_2(x) - f_2(x) F_1(x)\right)}{2\tilde{F}(x)\left(\alpha(x) F_1^{N-1}(x) + (1-\alpha(x)) F_2^{N-1}(x)\right)} \cdot \left[(1-\alpha(x)) F_2^{N-2}(x) - \alpha(x) F_1^{N-2}(x)\right],
\end{aligned}
$$

which was asked to be shown.

3.A.2 Proof of Lemma 3.6

Part (i) From the assumption on likelihood ratio dominance and Lemma 3.1 it follows that the function α is a non-decreasing (strictly increasing) function. It is left to show that δ is a non-increasing function. For this purpose, we calculate the first derivative of the function and show that the derivative is non-positive. Thus,

$$\begin{aligned}
\delta'(x) &= \left(\frac{F_2^{N-2}(x)}{F_1^{N-2}(x)+F_2^{N-2}(x)}\right)' \\
&= (N-2)\frac{F_2^{N-3}(x)\left(F_1^{N-2}(x)+F_2^{N-2}(x)\right)f_2(x)}{\left(F_1^{N-2}(x)+F_2^{N-2}(x)\right)^2} \\
&\quad -(N-2)\frac{F_2^{N-2}(x)\left(F_1^{N-3}(x)f_1(x)+F_2^{N-3}(x)f_2(x)\right)}{\left(F_1^{N-2}(x)+F_2^{N-2}(x)\right)^2} \\
&= (N-2)\frac{F_1^{N-3}(x)F_2^{N-3}(F_1(x)f_2(x)-F_2(x)f_1(x))}{\left(F_1^{N-2}(x)+F_2^{N-2}(x)\right)^2}.
\end{aligned}$$

Given that the reverse hazard ratio dominance is implied by the domination in terms of the likelihood ratio, from the last equality it follows that the derivative is non-positive, which proves the claim.

Part (ii) We calculate the limit value and obtain:

$$\begin{aligned}
\lim_{x\to 0}(\delta(x)-\alpha(x)) &= \lim_{x\to 0}\frac{F_2^{N-2}(x)}{F_1^{N-2}(x)+F_2^{N-2}(x)}-\lim_{x\to 0}\frac{f_1(x)}{f_1(x)+f_2(x)} \\
&= \lim_{x\to 0}\frac{1}{\left(\frac{F_1(x)}{F_2(x)}\right)^{N-2}+1}-\frac{f_1(0)}{f_1(0)+f_2(0)} \\
&\geq \frac{1}{1+1}-\frac{f_1(0)}{f_1(0)+f_2(0)}=\frac{f_2(0)-f_1(0)}{2(f_1(0)+f_2(0))}.
\end{aligned}$$

The last inequality follows from the fact that F_1 first-order stochastically dominates F_2; which is a consequence of the assumption of the lemma. Because of the same reason (that is, the implied first-order stochastic dominance) we have that $f_1(0) \leq f_2(0)$ (as $F_1(0) = F_2(0)$ and $F_1(\varepsilon) \leq F_2(\varepsilon)$ for any $\varepsilon \in (0,\omega)$). Thus, the right-hand side of the last inequality is non-negative, which proves the claim.

3.A.3 Proof of Proposition 3.4

From Lemma 3.6 it follows that there are only three possibilities regarding the sign of the function $\delta - \alpha$ on the interval $(0,\omega)$. Let us consider each of these possibilities separately and analyze implications on the relation between the bidding strategies.

- *Possibility 1:* Assume that $(\delta(t) - \alpha(t)) < 0$ for all $t \in (0, \omega)$.
According to Lemma 3.5 we obtain that $\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)} \leq 0$ for any $t \in (0, \omega)$. Furthermore, this leads to

$$\frac{L(y|x)}{\bar{L}(y|x)} = e^{\int_y^x \left[\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)}\right] dt} \leq 1,$$

or, $L(y|x) \leq \bar{L}(y|x)$, for any $y < x$. Therefore, under this scenario, we obtain that

$$\bar{\beta}^I(x) - \tilde{\beta}^I(x) = \int_0^x [L(y|x) - \bar{L}(y|x)]\, dy \leq 0,$$

for any x.

- *Possibility 2:* Here we suppose that $(\delta(t) - \alpha(t)) > 0$ for all $t \in (0, \omega)$. Analogously to the previous possibility (only reverting all inequalities), one may show that, in this case, we have that

$$\bar{\beta}^I(x) - \tilde{\beta}^I(x) = \int_0^x [L(y|x) - \bar{L}(y|x)]\, dy \geq 0,$$

for any x.

- *Possibility 3:* There is an interval $[x^*, x^{**}] \subseteq (0, \omega)$ such that $\delta(t) - \alpha(t) = 0$ if and only if $t \in [x^*, x^{**}]$ (possibly $x^* = x^{**}$). Because of the last lemma (both part (i) and part (ii)), it follows that $\delta(t) - \alpha(t) > 0$ for $t \in (0, x^*)$ and $\delta(t) - \alpha(t) < 0$ for $t \in (x^{**}, \omega)$. Thus, we may conclude the following:

$$\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)} = \begin{cases} \geq 0, & t \in (0, x^*) \\ = 0, & t \in [x^*, x^{**}] \\ \leq 0, & t \in (x^{**}, \omega) \end{cases}.$$

Consequently, our further analysis leads to the establishment of the following inequalities:

$$\frac{L(y|x)}{\bar{L}(y|x)} = e^{\int_y^x \left[\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)}\right] dt} = \begin{cases} \geq 1, & y < x \leq x^{**} \\ \leq 1 & x^{**} \leq y < x \\ ?, & y < x^* \leq x^{**} < x \end{cases},$$

where "?" expresses uncertainty about the sign of the fraction under consideration. That is, in that specific case we have no certain knowledge about the sign. Equivalently, we may observe

$$L(y|x) - \bar{L}(y|x) = \begin{cases} \geq 0, & y < x \leq x^{**} \\ \leq 0, & x^{**} \leq y < x \\ ?, & y < x^* \leq x^{**} < x \end{cases}.$$

In the same manner, for any $y \leq x_1 < x_2$ we obtain:

$$\begin{aligned}
\frac{L(y|x_2)}{\bar{L}(y|x_2)} \Big/ \frac{L(y|x_1)}{\bar{L}(y|x_1)} &= e^{\int\limits_y^{x_2}\left[\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)}\right]dt - \int\limits_y^{x_1}\left[\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)}\right]dt}, \\
&= e^{\int\limits_{x_1}^{x_2}\left[\frac{\bar{g}(t)}{\bar{G}(t)} - \frac{g_t(t)}{G_t(t)}\right]dt} \\
&= \begin{cases} \geq 1, & x_1 < x_2 \leq x^{**} \\ \leq 1 & x^{**} \leq x_1 < x_2 \\ ?, & x_1 < x^{*} \leq x^{**} < x_2 \end{cases} .
\end{aligned}$$

The last two steps make it possible to partially assess the sign of the difference between the bidding strategies as following:

$$\bar{\beta}^I(x) - \tilde{\beta}^I(x) = \int\limits_0^x \left[L(y|x) - \bar{L}(y|x)\right] dy = \begin{cases} \geq 0, & x \leq x^{**} \\ ?, & x > x^{**} \end{cases} .$$

Additionally, for any $x^{**} < x_1 < x_2$ we have that $\bar{\beta}^I(x_2) - \tilde{\beta}^I(x_2) < \bar{\beta}^I(x_1) - \tilde{\beta}^I(x_1)$, that is, $\bar{\beta}^I - \tilde{\beta}^I$ is decreasing on (x^{**}, ω). Indeed,

$$\begin{aligned}
\left[\bar{\beta}^I(x_2) - \tilde{\beta}^I(x_2)\right] - \left[\bar{\beta}^I(x_1) - \tilde{\beta}^I(x_1)\right] &= \int\limits_0^{x_1} \left\{\left[L(y|x_2) - \bar{L}(y|x_2)\right] - \left[L(y|x_1) - \bar{L}(y|x_1)\right]\right\} dy \\
&+ \underbrace{\int\limits_{x_1}^{x_2} \left[L(y|x_2) - \bar{L}(y|x_2)\right] dy}_{\leq 0} \\
&\leq \int\limits_0^{x_1} \left\{\bar{L}(y|x_2)\left[\frac{L(y|x_2)}{\bar{L}(y|x_2)} - 1\right] - \bar{L}(y|x_1)\left[\frac{L(y|x_1)}{\bar{L}(y|x_1)} - 1\right]\right\} dy \\
&< \int\limits_0^{x_1} \bar{L}(y|x_1)\left\{\left[\frac{L(y|x_2)}{\bar{L}(y|x_2)} - \frac{L(y|x_1)}{\bar{L}(y|x_1)}\right]\right\} dy \\
&\leq 0 \quad .
\end{aligned}$$

The strict inequality in the last expression comes from the fact that $\bar{L}(y|x_2)$ first-order strictly dominates $\bar{L}(y|x_1)$. Indeed, we have that

$$\frac{\bar{L}(y|x_2)}{\bar{L}(y|x_1)} = e^{-\int\limits_{x_1}^{x_2} \frac{\bar{g}(t)}{\bar{G}(t)} dt} < 1.$$

Therefore, finally, we may conclude that in the currently considered scenario, we have either $\bar{\beta}^I(x) - \tilde{\beta}^I(x) = \int\limits_0^x \left[L(y|x) - \bar{L}(y|x)\right] dy \geq 0$ for any x (if $\bar{\beta}^I(\omega) - \tilde{\beta}^I(\omega) \geq 0$) or "single-crossing" where $\tilde{\beta}^I$ crosses $\bar{\beta}^I$ from below at some point $\hat{x} > x^{**}$.

All in all, summarizing results over all possibilities, we have obtained that exactly one of the following three situations must always hold true:

1. $\bar{\beta}^I(x) - \tilde{\beta}^I(x) \leq 0$ for any private value x,
2. $\bar{\beta}^I(x) - \tilde{\beta}^I(x) \geq 0$ for any private value x, or
3. There is some $\hat{x} \in (0, \omega)$ such that $\bar{\beta}^I(x) - \tilde{\beta}^I(x) \geq (\leq) 0$ if and only if $x \leq \hat{x}$ $(x \geq \hat{x})$. In other words, we say that the strategies have "single-crossing" where $\tilde{\beta}^I(x)$ crosses $\bar{\beta}^I(x)$ from below at $\hat{x}$,

which proves the claim of the proposition.

3.A.4 Details behind Remark 3.3

According to general formula for the expected revenue (please check equation (3.14)), we have that

$$\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] = N\int_0^{\omega} \left(\tilde{m}^{II}(x) - \bar{m}^{II}(x)\right) \tilde{f}(x)dx.$$

Let us now calculate $\tilde{m}^{II}(x)$ and $\bar{m}^{II}(x)$. According to their definitions (for example, see (3.15)), we obtain

$$\tilde{m}^{II}(x) = \int_0^{\omega} M^{II}(x,y)\, g_x(y)\, dy$$

and

$$\bar{m}^{II}(x) = \int_0^{\omega} M^{II}(x,y)\, \bar{g}(y)\, dy.$$

Since $M^{II}(x,y) = \begin{cases} 0, & x \leq y \\ y, & x > y \end{cases}$ at the truth telling equilibrium, we obtain

$$\tilde{m}^{II}(x) = \int_0^{x} y g_x(y)\, dy$$

and

$$\bar{m}^{II}(x) = \int_0^{x} y \bar{g}(y)\, dy.$$

Therefore,

$\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] = N\int_0^{\omega} \left[\int_0^{x} y\left(g_x(y) - \bar{g}(y)\right) dy\right] \tilde{f}(x)\, dx.$ Q.E.D.

3.A.5 Details behind Remark 3.4

From Remark 3.3, for $N = 2$, the difference takes the following form:

$$\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] = 2\int_0^{\omega}\left[\int_0^{x} y\left(g_x(y) - \bar{g}(y)\right)dy\right]\tilde{f}\left(x\right)dx.$$

Moreover, if $N = 2$, the relevant density functions are given as following:

$$\begin{aligned} g_x(y) &= \alpha\left(x\right)f_1(y) + \left(1-\alpha\left(x\right)\right)f_2\left(y\right), \\ \bar{g}(y) &= \tilde{f}\left(y\right) = \frac{1}{2}f_1\left(y\right) + \frac{1}{2}f_2\left(y\right), \end{aligned}$$

Plugging these equalities into the above calculated expression for the difference of the expected revenues, we obtain the following equality:

$$\begin{aligned} \mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] &= 2\int_0^{\omega}\left[\int_0^{x} y\left[\alpha\left(x\right)f_1(y) + \left(1-\alpha\left(x\right)\right)f_2\left(y\right) - \frac{1}{2}f_1\left(y\right) - \frac{1}{2}f_2\left(y\right)\right]dy\right]\tilde{f}\left(x\right)dx \\ &= 2\int_0^{\omega}\left[\int_0^{x} y\left[\left(\alpha\left(x\right)-\frac{1}{2}\right)f_1(y) + \left(1-\alpha\left(x\right)-\frac{1}{2}\right)f_2\left(y\right)\right]dy\right]\tilde{f}\left(x\right)dx \\ &= 2\int_0^{\omega}\left[\int_0^{x} y\left(\alpha\left(x\right)-\frac{1}{2}\right)\left(f_1(y) - f_2\left(y\right)\right)dy\right]\frac{1}{2}\left(f_1\left(x\right) + f_2\left(x\right)\right)dx \\ &= \int_0^{\omega}\left[\int_0^{x} y\left(f_1(y) - f_2(y)\right)dy\right]\left(\alpha\left(x\right)-\frac{1}{2}\right)\left(f_1(x) + f_2(x)\right)dx \end{aligned}$$

We further simplify the last expression. For this purpose, note that

$$\begin{aligned} \left(\alpha\left(x\right)-\frac{1}{2}\right)\left(f_1(x) + f_2(x)\right) &= \alpha\left(x\right)\left(f_1(x) + f_2(x)\right) - \frac{1}{2}\left(f_1(x) + f_2(x)\right) \\ &= f_1(x) - \frac{1}{2}f_1(x) - \frac{1}{2}f_2(x) \\ &= \frac{1}{2}\left(f_1(x) - f_2(x)\right) \end{aligned}$$

Hence, overall the difference of the expected revenues is given as:

$$\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] = \frac{1}{2}\int_0^{\omega}\left[\int_0^{x} y\left(f_1(y) - f_2(y)\right)dy\right]\left(f_1(x) - f_2(x)\right)dx.$$

Our further step is to show that the right-hand side of the last expression is non-negative. For this goal, let us denote with $\Theta = \Theta(y)$ the anti derivative of the definite integral $\int_0^x y\,(f_1(y) - f_2(y))\,dy$. With this notation, we have that $\frac{d}{dy}\Theta(y) = y\,(f_1(y) - f_2(y))$ and $\int_0^x y\,(f_1(y) - f_2(y))\,dy = \Theta(x) - \Theta(0)$. Integrating by parts twice, we show that the expression is actually non-negative. Indeed, we may rewrite the difference between the expected revenues as following:

$$
\begin{aligned}
\mathbb{E}\left[\tilde{R}^{II}\right] - \mathbb{E}\left[\bar{R}^{II}\right] &= \frac{1}{2}\int_0^{\omega} \left[\int_0^x y\,(f_1(y) - f_2(y))\,dy\right](f_1(x) - f_2(x))\,dx \\
&= \frac{1}{2}\int_0^{\omega} (\Theta(x) - \Theta(0))\,(f_1(x) - f_2(x))\,dx \\
&= \frac{1}{2}(\Theta(x) - \Theta(x))\,(F_1(x) - F_2(x))\,|_0^{\omega} \\
&\quad -\frac{1}{2}\int_0^{\omega} x\,(f_1(x) - f_2(x))\,(F_1(x) - F_2(x))\,dx \\
&= -\frac{1}{2}\int_0^{\omega} x\,(f_1(x) - f_2(x))\,(F_1(x) - F_2(x))\,dx \\
&= -\frac{1}{2}\left[x \cdot \frac{1}{2}(F_1(x) - F_2(x))^2\,|_0^{\omega} - \frac{1}{2}\int_0^{\omega} (F_1(x) - F_2(x))^2\,dx\right] \\
&= \frac{1}{4}\int_0^{\omega} (F_1(x) - F_2(x))^2\,dx \geq 0. \text{ Q.E.D.}
\end{aligned}
$$

Chapter 4

Endogenous Auction Choice under Possible Seller's Manipulation

All-pay and Second-price Auction

So far we have considered models where bidders were the only participants with private information - their private values. The uniqueness of this chapter (compared to the previous ones) is that we include the possibility of a private piece of information also on the other side of the trade - that is, on the seller's side. This new feature of the model is introduced by giving the seller the possibility of manipulating the auction game. The seller's eventual manipulation of auctions has been a hot topic for some time, and especially popular since Internet auctions became prevalent on the market. There are different kinds of the manipulation that, among other factors, depend on the auction's form. Shill bidding (also known as "phantom bidding" or "false-name bids") is one of the earliest discussed forms of seller's manipulation. Even in early work such as in Vickrey (1961), where the second-price sealed bid was introduced, the possibility of "shill" was considered (see, for example, lines on page 22 of the article). Afterwards, Cassady (1967) was one of the first to formally recognize and consider this fraudulent seller's behavior - though, in the book it is denoted as "trotting" or "running" (for more details, please refer to the page 166 of the book). Shill bidding is a situation where the seller places a bid (or has someone to place a fake bid on seller's behalf) in his own (open English or second-price) auction, in order to push the selling price up. More precisely, in an English (ascending) auction, once there is only one (genuine) bidder left who is interested in buying the good, the seller may make a "phantom bid" in order to make the interested buyer bid higher and continue bidding. In this way, either the seller gets to sell his good for the price equal to (or close to) the bidder's highest private value (once truth-telling is assumed) or the good stays unsold (once the "phantom bid" wins the auction). Under the assumption of truth telling and being able to offer the good for sale again, the seller's shill bidding is always beneficial for him. Recent wide practical employment of the English auction on online platforms (such as Ebay or similar ones) made seller's manipulation even easier (through multiple accounts) and at the same time rose awareness of this type of fraud among potential buyers. This perception of a seller's potential phantom bid causes bidders to counteract: bidders tend not to bid truthfully but choose to shade their bids. The interplay of a seller's shill bidding and a bidder shading bids was explored in numerous theoretical and experimental studies. It led to a long debate on its economical consequences (e.g. social welfare or consumer surplus), adequate normative and security measures against the fraudulent behavior, shill-proof bidding strategies and so on.[1] Relevant to this chapter is the link between shill bidding and the existence of a secret reserve price. These two terms were treated as equivalent in Graham et al. (1990), where shill bidding was nothing more than secretly setting a reserve price (in the paper, a so-called "non-constant reserve price"). The price was revealed by the seller as a regular bid and the revelation comes about just before the "non-manipulated" auction would close. Later, Bag et al. (2000) named the manipulation as "flexible reserve price" as opposite to "fixed reserve price", where latter case refers to the regular public announcement of the reserve price and prior to the start of the bidding process. The "flexible reserve price" is exactly the kind of manipulation we allow in the model we develop in the following sections. Furthermore, instead of an English auction format, we shall consider the second-price sealed-bid auction.

In the second-price auction, shill bidding is the situation when the seller, after learning the winner's bid and the price (that is, after observing the submitted bids), places a "false-bid" which is just a bit below the winner's bid. In this way, the seller maximizes the price of the good without changing the winner of the auction. Note that, unlike the English auction, the "phantom bid" certainly does not win the auction and the seller will never keep the object. Thus, the "flexible reserve price" surely rises the selling price in the second-price sealed bid without changing the winner. The collection of seminal works on shill bidding in sealed-bid auctions, among others, include Rothkopf and Harstad (1995), Porter and Shoham (2005) and Watanabe and Yamato (2008). The common elements of these three works are: (i) the possible selling mechanism is either a second-price or first-price auction without a reserve price, (ii) there are two types of seller: cheating and honest (non-cheating) seller, where the distinction depends on having an opportunity to place a "flexible reserve price" - shill bidding and (iii) the exact timeline of shill bidding is as follows: bidders submit sealed-bids in the second-price auction $\rightarrow$ the seller observes all bids $\rightarrow$ the cheating seller has the opportunity to place a phantom bid just bellow the highest bid (shill bidding) $\rightarrow$ the seller reveals all submitted bids, the sale takes place and the auction closes. In an environment where bidders have identically and independently distributed (i.i.d.) values, irrespective of whether the possibility of there being a cheating seller is endogenous or exogenous (for more details, please refer to Rothkopf and Harstad (1995) and Porter and Shoham (2005)), the studies show that the non-cheating sellers should always favor the first-price auction over the second-price auction.[2] In other words, the studies supported the critique and arguments against use of the second-price auction.[3] On the other side, in the model of interdependent

[1]The collection of such studies include, for example, Graham et al. (1990), Bag et al. (2000), Sinha and Greenleaf (2000) and Chakraborty and Kosmopoulou (2004).

[2]Porter and Shoham (2005) allow cheating also on the bidders' side in the first-price auction by being able to spy and observe the highest competing bid and,thus, optimally shade own bid.

[3]Arguments and critique on applicability of the second-price auction may be found in Rothkopf et al. (1990) or Rothkopf (2007), for instance.

private values developed in Watanabe and Yamato (2008), there exists an equilibrium where both types of seller choose to sell simultaneously via the second-price auction (when the probability of the cheating is relatively small), and the pooling is shown to be robust in a certain sense.

In the model that follows, we consider the case of the i.i.d. private values. There also exists two types of sellers: manipulative (cheating) and honest. The only difference between these two types is that the cheating seller has an opportunity to manipulate the auction in the way previously described under the term "flexible reserve price". More precisely, in this case, the cheating seller has the chance to send a selfless agent who bids a fixed amount on the seller's behalf and the value of agent's bid is defined by the seller. Unlike the literature, the timing of the manipulation in our model is different. In our framework, the seller neither can have early access to the submitted bids nor can take secret action after learning the bids. Similarly, as discussed in Vickrey (1961), we assume that there is a trustful platform or an auction house, independent of seller and bidders that collects submitted bids and reveals them to all participants only at the closing of the auction. Therefore, the manipulative seller's agent may only submit the bid at the same time the other regular bidders do. In this way, the cheating seller will probably not boost the selling price to the bidders' highest value, but according to Myerson (1981), with the optimal choice of the agent's bid, the seller could get the same expected revenue as in the optimal auction, once the range of standard auction formats[4] are available to the seller. Moreover, according to the revenue equivalence theorem, the optimal outcome may be achieved via any standard auction - the seller has to optimally adjust the phantom bid to fit the chosen format. The set of allowed auction formats are only standard ones, in particular only the following two: the second-price auction and the all-pay auction. There are many reasons why we chose the second-price auction as the available format in our model: its theoretical importance, its ex post dominant strategy, the wide practical use of theoretically equivalent auction forms and so on. The all-pay auction is considered as a possible selling mechanism because we believe, at first glance, that the cheating seller may exploit the auction format more than any other standard auction. Namely, in the all-pay auction, all bidders pay their bids (irrespective if they win or not) and the bidder with the highest bid wins the good. Thus, we may imagine that the cheating seller can choose the all-pay auction and send an agent who bids very high, such that the good stays unsold (as the fake bid wins) and the seller collects the bids from all bidders. One of our goals is to find out if the cheating seller is indeed better off with an all-pay auction (where the agent's bid is very high) than with the second-price auction with the optimal (hidden) reserve price. Moreover, it is interesting to see whether the honest seller will still be indifferent between these two standard auctions (as should be the case according to the revenue equivalence principle). Unlike related work, in our framework, the cheating seller can successfully cheat in all available auction formats.[5] The recent literature, as mentioned above, suggested that in the case of i.i.d. private values, the honest seller should prefer the "non-manipulative"

[4]The standard auctions are the auctions where the good is won by the bidder with the highest bid.

[5]The cheating seller in Rothkopf and Harstad (1995), Porter and Shoham (2005) and Watanabe and Yamato (2008) can not influence the outcome of the first-price auction by manipulation at all, thus, the manipulation is effective only in the second-price auction.

first-price auction over the "manipulative second-price auction". With our set-up, we explore the honest seller's choice between an all-pay auction and a second-price auction, both of a "manipulative" nature. Similar to Watanabe and Yamato (2008), the auction game takes the form of a signaling game, where nature chooses the seller's type and the seller sends a signal to bidders by choosing the auction format. Therefore, this signaling effect (for example, the choice of an all-pay auction might be interpreted as a signal sent only by a cheating seller) and its influence on bidders' behavior must be investigated. At first glance, it is reasonable to assume that once the bidders doubt that the seller is a manipulative one and they are asked to play in an all-pay auction, they will probably bid lower than in the usual standard set-up (or maybe, even refuse to bid). Indeed, our analysis of the model confirmed the existence of this bidding shade effect and determined its magnitude. Moreover, the final results show that the bid shading of bidders has a stronger influence on the equilibrium outcomes than the seller's possible manipulation gain. Assuming that the participants use pure strategies, we obtain that it is in the seller's best interest to choose a second-price auction, irrespective of his type. Moreover, there are very rare circumstances when the honest seller may choose the all-pay auction and the manipulative seller runs the second-price auction. However, this equilibrium outcome is very unstable and not robust against the smallest change of bidders' belief on seller's type. Unlike the related frameworks with i.i.d. values and seller's fraudulent action developed in Rothkopf and Harstad (1995) and Porter and Shoham (2005), our model argues and supports the use of the second-price auction and shows its robustness in a case of the eventual seller's shill bidding.

The rest of the chapter is organized as follows. Section 4.1 introduces the model and gives the first insights into its components. In Section 4.2, we look at the signaling game's weak Bayesian Nash equilibria in pure strategies and characterizes them for the case of our model. We show that the only robust equilibrium is pooling on the second-price auction. Our final comments are provided in Section 4.3, whereas all proofs are contained in Appendix 4.A.

4.1 The Model

Similar to the models from previous chapters, participants in our model are: one seller with an indivisible good for sale and N potential buyers, so-called bidders. As before, the bidders have private values which are realizations of random variables X_is. We assume that X_is are independently and identically distributed (i.i.d.) on the final interval $[0,\omega]$ with respect to a continuous cumulative distribution function (c.d.f.) F with density function (d.f.) f. We assume that the hazard rate of F is an increasing function. In addition, F is assumed to be strictly increasing and continuous. These are assumptions also used in Myerson (1981), which ensures the existence of the optimal standard auction (with reserve price). This time, we assume that the seller attaches the value $x_0 \in [0,\omega]$, $\omega \in \mathbb{R}_{++}$ to the object. That is, unlike in the previous chapters, the seller's value x_0 is not necessary equal to zero. There are two types of seller: an honest (non-cheating) and manipulative (cheating) seller. The type of the seller is known only to the seller, though we assume there is some bidders' (a priori) common prior about the seller's type. Namely, with probability $p \in (0,1)$, the seller is believed to be manipulative. What makes the seller manipulative, is the fact that the seller may send an agent to the auction to bid a fixed amount a. The agent himself has no personal interest - we may think of an agent as a seller's double account on some online auction platform.

Independent of type, the seller may choose either a second-price auction or an all-pay auction for the sale, or to give up on the sale before the bidding starts. We chose to include the second-price auction in our current model, because of its theoretical and practical importance. The choice of the an all-pay auction was due to the fact that each bidder has to pay their own bid and, thus, we believe that the agent might bring the most profit to the cheating seller in an all-pay auction compared to other standard auctions. In order to be in line with the assumption that bidders do not know the seller's type or, equivalently, they are not supposed to be able to recognize the agent, we assume anonymity and physical barriers among bidders. In other words, we assume that bidders are not able to observe other participants in the auction (one sufficient condition could also be that N is large enough), so that they can not anticipate with full certainty whether there is an agent or not.

4.1.1 Feasible Bids

Theoretically, we could consider any function of private value as the possible bidding strategy of a bidder. Furthermore, bidders might have different strategies. However, due to anonymity and i.i.d of private values, we assume that all bidders follow the same bidding pattern; that is, we are searching for symmetric equilibria. Based on practical observations we impose some further assumptions on the characteristics of the bidding behavior. For this purpose, let $\sigma_B = \left(\beta^{II}(\cdot), \beta^{All}(\cdot)\right)$ be a typical feasible bidding behavior of a typical bidder, where $\beta^{II}(\cdot)$ is her bid function once she is participating in a second-price auction and $\beta^{All}(\cdot)$ is the bidder's bid function in an all-pay auction. For further analysis we assume that, for any $A \in \mathcal{A} = \{II, All\}$ it holds true:

- $\beta^A(0) = 0$,
- $\beta^A(x) \leq x$ for $x \in [0, \omega]$,
- β^A is non-decreasing and
- β^A is differentiable almost everywhere.

Note that $\beta^A(x) = 0$ for any x is a feasible strategy and it describes the situation when bidders reserve the right not to participate in the auction. According to the existing literature, these assumptions are not too restrictive and in line with our intuition on bidding behavior.

4.1.2 Beliefs

As the type of the seller and the agent's bid are unknown to a bidder and are crucial for the bidder's behavior, it rises naturally that there is a need to model the bidder's beliefs about the honesty of the seller and the agent's bid. We denote $\mu = \left(\left(\mu^{II}, B^{II}\right), \left(\mu^{All}, B^{All}\right)\right)$ as a **belief system of bidders**, where, for any auction format A the following hold true:

- μ^A is the probability that the seller is believed to be manipulative once he chose the auction A; thus, $1 - \mu^A$ is the probability that a bidder attaches to the event that the seller is honest in the auction A, and
- $B^A(a)$ is c.d.f. over $[0, \omega]$, which represents a bidder's belief over the agent's bid a in the auction format A, conditional on the event that the seller is cheating.

As feasible beliefs we take any tuple μ, with $\mu^{II}, \mu^{All} \in [0, 1]$ and B^A being a deterministic distribution. Mathematically, we consider only $B^A(\cdot)$ that are degenerate distributions; that is, the bidders believe that only one certain particular value must be the agent's bid. In the case where $B^A(\cdot)$ puts all probability mass on the agent's bid $a \in [0, \omega]$, we shall introduce a simplified notation $B^A \equiv a$, hopefully without risking any ambiguities in notation. It seems that the deterministic beliefs are the most natural and appealing as we assume that the seller is allowed to choose only pure strategies. The seller's exact feasible choices are contained in the following lines.

4.1.3 Seller's Options

Let us denote by $\sigma_S = (A^H, (A^M, a))$ any seller's pure strategy profile, where A^H denote the honest seller's choice of auction format and (A^M, a) denotes the manipulative seller's choice of auction format, and the agent's bid respectively. So, if $A^S = II$, we assume that the seller chooses the second-price auction, whereas $A^S = All$ means that the seller chooses an all-pay auction format (for any seller's type S). We assume that the seller exclusively uses the pure strategies σ_S with $a \in [0, \omega]$. In addition, we implicitly assume that a seller always has the right to choose not to sell the good, which must be announced prior to bidders being called to place bids.

Lastly, it is clear that our model may be seen in the spirit of the signaling game introduced by In-Koo Cho (1987). The game tree, associated to our model, is illustrated in Figure 4.1.

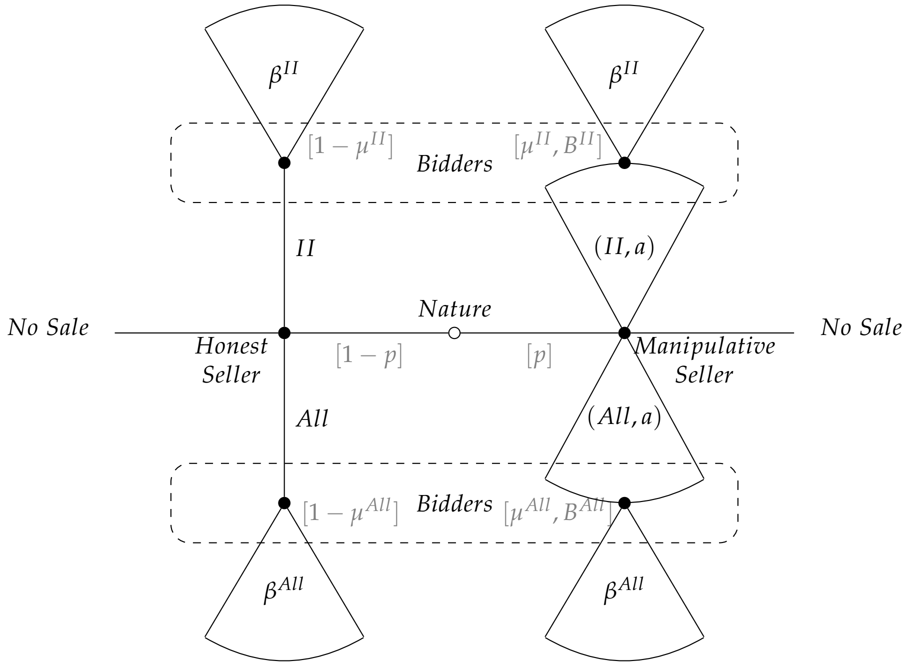

Figure 4.1: *The game tree of the model (with beliefs).*

Note that we have depicted only one bidder as we assume anonymity and symmetry among them. Thus, it suffices to represent one typical bidder. Let us continue with analyzing the optimal behavior and equilibrium outcomes in this particular set-up.

4.2 Equilibrium Behavior

In this section we shall characterize and discuss the equilibrium behavior of the participants. As our model is a kind of signaling game with pure strategy profiles, we consider pure weak Bayesian Nash equilibria. Moreover, we are interested when both types of seller do not choose to exit the auction process, but rather choose to offer the good for sale. As there are only two formats of auction involved, there are four potential candidates for equilibria. It is interesting to discover which type of equilibrium arises: pooling (when both types of sellers choose to run the same auction format) or separating (where different types of seller choose different auctions). Due to the existence of the manipulation, it seems that the cheating seller will go for an all-pay auction, whereas the honest seller might be indifferent between the two auctions. However, we should take the signaling effect into consideration. It is natural to expect that bidders will act more cautiously in the all-pay auction and believe that it is more probable that they face a cheating seller in the auction. Consequently, we should expect that bidders will decrease their bids. Therefore, at first glance, one could speculate that the honest seller will prefer a second-price auction over an all-pay, due to negative effect of signaling on the bids. The equilibrium choice of the manipulative seller remains unclear because of the two opposite effects that the agent's bid and the choice of auction format have. Thus, let us analyze the equilibrium patterns and characterize them.

For this goal, we shall firstly provide the definition of a pure weak Bayesian Nash equilibrium (further denote as PWBNE). We just translate the general defining properties of the concept (given, for example, as Definition 9.C.3 on page 285 in Mas-Colell et al. (1995)) in to the language of our set-up.

Definition 4.1. A pair of feasible seller's and bidders' strategies (σ_S^*, σ_B^*) with feasible belief system $\mu^* = \left(\left(\mu^{II^*}, B^{II^*}\right), \left(\mu^{All^*}, B^{All^*}\right)\right)$ is a **pure weak perfect Bayesian Nash equilibrium (PWBNE)** if:

(i) $\sigma_B^* = \left(\beta^{II^*}, \beta^{All^*}\right)$ are the bidders' sequentially rational responses to σ_S^* with belief μ^*,

(ii) $\sigma_S^* = (A^{H^*}, (A^{M^*}, a^*))$ are the seller's sequentially rational responses to σ_B^* and

(iii) μ^* is derived according to Bayes' rule, whenever possible (that is, along the equilibrium path).

In order to determine the equilibria we shall start by analyzing the consequences of the participants' sequential rationality. Let us start with the bidders' perspective. More precisely, in the following lines we derive the optimal response of the bidders to any given seller's strategy and given any feasible belief system.

4.2.1 Bidders' Sequentially Rational Bids

Here we solve the problem that is being considered by a buyer once asked to place a bid. Thus, we assume that the seller's choice of auction format and the buyer's belief about the honesty of the seller are given. In other words, we determine the best response of a bidder given that all other parameters are fixed. Moreover, we shall only consider symmetric equilibria, due to anonymity and the i.i.d. assumption on bidders and their private values. As there are two possible auction formats (and hence, two information sets of a bidder - see the game tree in Figure 4.1) which might take place, we analyze a buyer's best response at each information set, separately. The following lemma summarizes the results of the analysis.

Before giving the formulation of the lemma, let us introduce some notation. Let G and g denote the c.d.f. and the d.f for the second highest private values among any $N-1-$tuple of private values. Let us denote this random variable as Y_1 , that is, let $Y_1 = \max_{j \neq 1} X_j$. More precisely, since values are i.i.d., we define the functions as follows:

$$\begin{aligned} G(y) &= F^{N-1}(y) \\ g(y) &= (N-1)F^{N-2}(y)f(y), \end{aligned}$$

for any $y \in [0, \omega]$. As we have seen in previous chapters, the second-highest private value among opponents Y_1, plays an important role in shaping a bidder's own bid - also the case within this chapter.

Lemma 4.1. *A bidder's pure best sequentially rational response to any seller's choice* $\sigma_S = \left(A^H, \left(A^M, a\right)\right)$ *with belief system* $\mu = \left(\left(\mu^{II}, B^{II}\right), \left(\mu^{All}, B^{All}\right)\right)$ *is given by* $\sigma_B = \left(\tilde{\beta}^{II}(\cdot), \tilde{\beta}^{All}(\cdot)\right)$*, where* $\tilde{\beta}^{II}(x)$ *is truth-telling and* $\tilde{\beta}^{All}$ *is characterized in the following lines.*

If $B^{All} \equiv a$*; that is, in the case the bidder believes that the manipulative seller's agent certainly bids* a*, then the bidder behaves as follows. If there exists a private value* $\tilde{x} \in [0, \omega]$ *which satisfies the following equality*

$$\left(1 - \mu^{All}\right) \int_0^{\tilde{x}} tg(t)\, dt + \mu^{All} \tilde{x} G(\tilde{x}) = a \tag{4.1}$$

then the optimal bidding strategy in the all-pay auction is given by

$$\tilde{\beta}^{All}(x) = \begin{cases} \left(1 - \mu^{All}\right) \int_0^{x} tg(t)\, dt, & x < \tilde{x} \\ a + \int_{\tilde{x}}^{x} tg(t)\, dt, & x \geq \tilde{x} \end{cases}. \tag{4.2}$$

Otherwise, if there is no such private value $\tilde{x}$*, then the bids take the following pattern:*

$$\tilde{\beta}^{All}(x) = \left(1 - \mu^{All}\right) \int_0^{x} tg(t)\, dt,\ x \in [0, \omega]. \tag{4.3}$$

Remark 4.1. For any fixed $\mu^{All} < 1$, equation (4.1) has no solution in the interval $[0, \omega]$, if and only if $a > \left(1 - \mu^{All}\right) \int_0^{\omega} tg(t)\, dt + \mu^{All} \omega$. Moreover, note that for $\mu^{All} = 0$ and any feasible belief B^{All}, the bidders bid according to $\tilde{\beta}^{All}(x) = \int_0^{x} tg(t)\, dt$, which is the bidding behavior that occurs in the standard all-pay auction.

For the case of being invited to place a bid in the all-pay auction, the proof of the lemma is similar to the heuristic approach we used in Section 3.2, when we derived the bidding strategies in the first-price auction. The only difference is that once calculating the bidder's payoff from a bid, we have to take into account beliefs μ^{All} and B^{All} and replace the rules of the first-price auction by the respective rules of the all-pay auction. All steps of the analysis can be found in Section 4.A.1 of the chapter's appendix.

Note that the analysis resulted in stating that the bidders will bid truthfully in the second-price auction, irrespective of their beliefs. The intuition behind the truth telling is as follows. If the bidder faces an honest seller, then we are back to the standard second-price model where truth-telling is the ex post weakly dominant strategy. In the case of encountering the manipulative seller who is believed to send an agent to bid a, the bidder perceives the situation as the second-price auction with a reserve price, where truth-telling also turns out to be weakly dominant. So for any belief a bidder holds, once asked to participate in the second-price auction, her best choice is to bid her own private value.

However, the beliefs matter once the bidder bids in the all-pay auction. Indeed, in the case where bidders believe that the agent's bid a is very high (meaning that equation (4.1) has no solution in the interval of private values, or $\tilde{x} = \omega$), the bidding pattern takes the form of a scaled version of the bidding strategy that they would use in the scenario with the standard all-pay auction. More precisely, equation (4.3) is nothing more than saying that the bidders bid $\tilde{\beta}^{All}(x) = (1 - \mu^{All})\bar{\beta}^{All}(x)$, where $\bar{\beta}^{All}(x) = \int_0^x tg(t)\,dt$ is the bidding strategy in the standard all-pay auction.[6] Therefore, once the bidders believe that the cheating seller sends an agent who bids high enough, the bidders will behave cautiously and shade their bids. How much they shade their bid, depends on their belief μ^{All} about honesty of the seller they face. For example, in the extreme cases of the belief μ^{All}: either they behave "normally", once they exclude the possibility that the seller is a manipulator ($\mu^{All} = 0$) or they do not participate at all, once they believe that the seller must be a manipulative one ($\mu^{All} = 1$). Therefore, here we capture the negative effect that signaling the seller's type via the all-pay auction has on bids.

Let us consider the other case, when bidders believe that the agent's bid is low enough. The last lemma states that the optimal bidding pattern has a cut-off shape specified by (4.2). Bidders with private values lower than the cutoff value $\tilde{x}$ still behave as in the previous case; that is, they shade the bids they would place in the standard all-pay auction. However, the bidders with higher values bid more aggressively than otherwise and bid at least as high as the believed agent's bid. It is interesting to compare this behavior to the one that occurs at the equilibrium in an all-pay auction with a reserve payment equal to a. In the case of the all-pay auction with the reserve payment equal to a, the bidding strategy is $\bar{\beta_a}^{All}$ where

$$\bar{\beta_a}^{All}(x) = \begin{cases} 0, & x < \bar{x}_a \\ a + \int_{\bar{x}_a}^{x} yg(y)dy, & x \geq \bar{x}_a \end{cases},$$

and $\bar{x}_a$ is the solution of the next equation:

[6] A derivation of the bidding strategy in the standard all-pay auction might be found in Krishna and Morgan (1997).

$$G(x)x = a.$$

At first glance, we can see that the bidders with higher private values behave as if they are taking part in an all-pay auction with reserve payment a. However, the cutoff values differ. A closer look at the defining equations for cutoffs $\tilde{x}$ and $\bar{x}_a$ tells us that the cutoff in our model is greater than the cutoff value in the regular all-pay auction with reserve payment a. Thus, the bidders with values between $\bar{x}_a$ and $\tilde{x}$ are more cautious in our set-up than they would be in the regular model. On the other hand, the bids of the bidders with lower private values seem to be higher in our model. Indeed, whereas in the standard all-pay auction with reserve payment a, the low private value bidders refuse to participate; in our model they eventually do participate, depending on their belief on the honesty of the seller. If there is the smallest hope that the seller is an honest one ($\mu^{All} < 1$), those bidders are going to bid positive amounts in the all-pay auction. Therefore, even though the bidders with "medium values" shade their bids, the bidders with lower values are willing to bid in the all-pay auction once they believe there is a chance to deal with the honest seller. In this way, we have captured both the manipulation's positive and negative effect on the bids from the seller's point of view. Which of these two opposite effects prevails and drives the expected revenue is left to be checked in the following lines.

4.2.2 The Seller's Perspective

Now we consider the seller's perspective, i.e. we determine his best response to bidders' bids. As we assume that the game is common knowledge, we allow the seller to anticipate the claims from Lemma 4.1 and assume that the bidders will use strategy profile $\tilde{\sigma}_B = \left(\tilde{\beta}^{II}(\cdot), \tilde{\beta}^{All}(\cdot)\right)$, as defined in the lemma given by belief system μ. The final goal of this section is to determine situations when a seller prefers a second-price auction over an all-pay or vice versa. As there are two types of seller, we consider each type separately.

The Honest Seller

Here we look at a best response of the honest seller given only feasible (pure) strategies. This seller type has the choice between two auction formats: the second-price or the all-pay auction. Let us denote with $u_H(II\,|\tilde{\sigma}_B)$, his expected revenue from running the second-price auction. If the seller chooses the second-price auction, according to Lemma 4.1, he expects truthful bids. Thus, the seller's expected revenue coincides with the expected revenue in the standard second-price auction, which we denote as $\mathbb{E}R^{II}$. Consequently, it is not hard to conclude that the expected revenue to the honest seller from truth-telling in the second-price auction is equal to the expected value of the second-highest private value whose c.d.f. and d.f. are respectively given by

$$H(y) = F^N(y) + NF^{N-1}(y)\,(1 - F(y)) = NF^{N-1}(y) - (N-1)\,F^N\,(y) \tag{4.4}$$
$$h(y) = N(N-1)F^{N-2}(y)f\,(y)\,(1 - F\,(y))\,. \tag{4.5}$$

for any $y \in [0, \omega]$. So, we have just shown that

$$\begin{aligned} u_H(II\,|\tilde{\sigma}_B\,) &= \mathbb{E}R^{II} \\ &= N\int_0^{\omega}\left[\int_0^{x} yg(y)dy\right] f(x)dx \\ &= \int_0^{\omega} yh\,(y)\,dy. \end{aligned} \tag{4.6}$$

Let us now consider the seller's expected payoff from the all-pay auction, denoted as $u_H(All\,|\tilde{\sigma}_B\,)$. In the case of the non-existence of the cutoff value $\tilde{x}$ on the open interval $(0, \omega)$, for strategy $\tilde{\beta}^{All}\,(\cdot)$ (that is, in the case where $\mu^{All} = 1$, or, the agent is believed to bid $a > (1 - \mu^{All})\int_0^{\omega} tg\,(t)\,dt + \mu^{All}\omega$) we have that $\tilde{\beta}^{All} = \left(1 - \mu^{All}\right)\bar{\beta}^{All}$, where $\bar{\beta}^{All}$ is the bidding pattern from the standard all-pay auction. Using the revenue equivalence principle in this particular case, it follows that the honest seller, by running the all-pay auction, expects to obtain

$$u_H(All\,|\tilde{\sigma}_B\,) = \left(1 - \mu^{All}\right)\mathbb{E}R^{II}\ ,$$

which is definitely less than the expected payoff in the second-price auction.

Finally, let us look at the case where $\tilde{\beta}^{All}$ has a cutoff shape. Then, the best assessment we obtain for the expected revenue to the seller is given by the following expression:

$$u_H(All\,|\tilde{\sigma}_B\,) = N\int_0^{\omega}\tilde{\beta}^{All}\,(x)\,f(x)dx \tag{4.7}$$

with $\tilde{\beta}^{All}$ being defined by (4.2).

In addition, note that the seller might always decide to keep his object and not sell, in which case his gain will be x_0. Since we explored all options and scenarios that the honest seller might be involved in, we may straightforwardly determine the optimal choice of the seller. The findings are given in the following lemma:

Lemma 4.2. *Let us assume that bidders bid according to* $\tilde{\sigma}_B = \left(\tilde{\beta}^{II}, \tilde{\beta}^{All}\right)$*. The following claims hold true.*

(i) If $\tilde{\beta}^{All}$ *is given by* (4.3) *and* $x_0 \leq \mathbb{E}R^{II}$*, then the honest seller chooses to sell via second-price auction and his expected revenue is*

$$u_H(II\,|\tilde{\sigma}_B) = \mathbb{E}R^{II}.$$

(ii) If $\tilde{\beta}^{All}$ *is defined by* (4.3) *and* $x_0 > \mathbb{E}R^{II}$*, then the honest seller decides to keep the object. The object is kept by either not running any auction or, if* $\mu^{All} = 1$*, by choosing the all-pay auction. In this case, his expected revenue is equal to his valuation of the object, that is* x_0*.*

(iii) If $\tilde{\beta}^{All}$ *takes the form as in* (4.2)*, then the honest seller strictly prefers the second-price auction over the all-pay auction if and only if*

$$\int_0^{\omega} \left[\int_0^{x} y g(y) dy - \tilde{\beta}^{All}(x) \right] f(x) dx > 0.$$

If we have $=$ *instead of* $>$ *in the last expression, then the honest seller is indifferent between the second-price and all-pay auctions. His expected revenues from the second-price auction and all-pay auction are given by* (4.6) *and* (4.7)*, respectively. In the case where the seller's value attached to the object* x_0 *is lower than maximum of the expected revenues, he will decide to retain the object and no sale takes place. Otherwise, the seller chooses the (strictly) preferred auction.*

The proof trivially follows from the considerations prior to the lemma. Thus, we have just provided the full characterization of the decision making process that takes part in the honest seller's mind, once he holds certain beliefs about bidder's behavior. Even though the case (iii) still has vague characterization, as we shall see later, deeper insights are not needed for the purpose of analyzing the equilibria from Definition 4.1. Let us now look at the analogous problem of the manipulative seller.

The Manipulative Seller

The best response of the manipulative seller is the topic in this part of the chapter. We consider the case where the seller predicts that bidding strategies are given by $\tilde{\sigma}_B = \left(\tilde{\beta}^{II}(\cdot), \tilde{\beta}^{All}(\cdot)\right)$. Unlike the honest seller, the manipulative seller has one additional action: sending the agent to bid certain amount. Thus, in order to calculate the expected revenue of the cheating seller, we have to determine the optimal agent's bid in each available auction format. By considering the expressions of the expected revenues and solving the raised optimization problems, we obtain the following claims regarding the agent's bid and the expected revenue:

Lemma 4.3. *Let us consider the choice made by the cheating type of the seller.*

(i) If the manipulative seller chooses the second-price auction, then the agent's role is identical to the reserve price in the regular second-price auction. Thus, the agent bids amount $\tilde{a}^{II}$, that is the optimal reserve price in the second-price auction and hence is the solution of the equation

$$\tilde{a}^{II} = x_0 + \frac{1 - F(\tilde{a}^{II})}{f\left(\tilde{a}^{II}\right)}. \tag{4.8}$$

Furthermore, the seller's expected revenue with this choice will be

$$u_M(II, \tilde{a}^{II} \,|\tilde{\sigma}_B) = \mathbb{E}R^{II} + \tilde{\psi}\left(x_0, \tilde{a}^{II}\right)$$

with the function $\tilde{\psi}$ defined as

$$\tilde{\psi}\left(x_0, a\right) = \int_0^a H\left(y\right) dy - F^N\left(a\right)\left(a - x_0\right), \tag{4.9}$$

for any pair (x_0, a).

(ii) In the case where the manipulative seller chooses an all-pay auction, his agent bids $\tilde{a}^{All}$ with

$$\tilde{a}^{All} = \omega.$$

In this case, the expected payoff to the seller is given by

$$u_M(All, \tilde{a}^{All} \,|\tilde{\sigma}_B) = x_0 + N \int_0^\omega \tilde{\beta}^{All}\left(x\right) f\left(x\right) dx.$$

The intuition behind the last lemma is simple. Since we have already inferred that the bidders will bid truthfully in the second-price auction, the manipulation through an agent in this auction has the same impact a regular reserve price has. Therefore, it is clear that the agent will be instructed to bid the optimal reserve price. However, the manipulation in the all-pay auction is different to just setting the reserve payment. This is because in the regular set-up, once a seller publicly announces a reserve payment, the seller will not obtain the bids lower than the reservation payment. However, in our model, the cheating seller gets paid all payments, irrespective of agent's bid. Thus, in our model we have that (i) the seller always gets paid all bids and (ii) he sometimes sells the good, depending on whether the highest bidders' bid outbids the agent's bid. Consequently, it is clear that the best outcome for the manipulative seller in the all-pay auction is to both collect all bids and keep the object. Hence, the seller will send an agent who will definitely outbid all bids. This outcome is almost surely achieved if the agent bids the highest possible private bid - that is, ω. Therefore, if the manipulative seller chooses to run an all-pay auction he will "fake" the sale, in the sense that, no real sale will take place (as his agent will always win the auction) and he will gather all bids. The complete proof of the last lemma can be found in Appendix 4.A.2.

Remark 4.2. Note that, unlike the honest seller, the manipulative seller will never use the outside option - the option not to sell via an auction. Indeed, running the all-pay auction with an agent bidding the value ω makes the seller keep the object and eventually get positive bids. Moreover, the second-price auction with an agent bidding the optimal price $\tilde{a}^{II}$ will give him expected revenue at least as high as x_0. Namely, if it is optimal that the agent bids $\tilde{a}^{II}$, it must be that the seller obtains expected revenue at least as high as in the case where the agent would bid ω in the second-price auction. In the case in which the agent bids ω in the second-price auction, the sale will not take place and the seller will keep the object, meaning his expected revenue is exactly x_0. A close inspection of the proof of the last lemma says that $u_M(II, \tilde{a}^{II} | \tilde{\sigma}_B) > x_0$ for $x_0 < \omega$ and $u_M(II, \tilde{a}^{II} | \tilde{\sigma}_B) = x_0$ for $x_0 = \omega$.

Considering each participant in this model separately, in the previous part of this section we have analyzed their optimal behavior. In particular, we have analyzed the behavior of the participants that satisfies the requirements of the sequential rationality - the requests (i) and (ii) from Definition 4.1. Let us now consider the last requirement of the equilibrium - the consistency of the belief system and therefore, fully determine the set of the equilibria.

4.2.3 Equilibria: No Manipulation via All-Pay Auction

In the following lines we capture the main findings of the analysis in our model. The following proposition characterizes all PWBNE of the model, where the sellers do not go for the outside option but choose to sell via an auction. Rather than using the language of set theory in order to express the set of all equilibria (which, by the way, can be directly derived from the proposition's proof), we provide the full characterization in the terms of economics. Namely, it turns out that the signaling effect (and, hence bid shading) is stronger than the manipulation effect, which leads to no equilibria when the manipulative seller cheats in the all-pay auction. The best outcome the manipulative seller in our model gets is the one obtained by running the second-price auction with the optimal price. The honest seller also (almost surely) prefers the second-price auction over the all-pay auction, due to the bid shading that may occur in the all-pay auction. Here is a summary of the findings:

Proposition 4.1. (Characterization of the Pure Weak Perfect Bayesian Nash Equilibria) *Let us look at the set of all pure weak perfect Bayesian Nash equlibria such that both seller types decide to run an auction format. The following statements give full characterization of the involved seller's actions at the equilibria:*

(i) ***(Non-possibility of manipulation via an all-pay auction)*** *There is no such equilibrium where both seller types run an auction and the manipulative seller chooses an all-pay auction.*

(ii) ***(Preference for trade via a second-price action)*** *If the bidders doubt they might face the manipulative seller in an all-pay auction (that is if the belief* $\mu^{All^*} > 0$*), the honest seller strictly prefers the second-price auction. In the case of no doubt about the honesty of the seller, the honest seller is indifferent between the second-price and all-pay auctions.*

For the goal of proving the above claims, we shall firstly give a lemma that explores the properties of the function $\tilde{\psi}$ defined in (4.9). Remember that the function measures the benefit that the manipulative seller gets from the agent's shill bidding in the second-price auction. The next lemma, in particular, explores how the benefit compares to the good's worth (which from the seller's point of view is equal to x_0), once the agent bids the optimal reserve price. Let us look at the claims:

Lemma 4.4. *For any* $N \geq 2$ *there exists* $\bar{x}_0$ *such that :*

- $x_0 < \bar{x}_0$ *if and only if* $\tilde{\psi}\left(x_0, \tilde{a}^{II}\left(x_0\right)\right) > x_0$*,*
- $x_0 = \bar{x}_0$ *if and only if* $\tilde{\psi}\left(x_0, \tilde{a}^{II}\left(x_0\right)\right) = x_0$*,*
- $x_0 > \bar{x}_0$ *if and only if* $\tilde{\psi}\left(x_0, \tilde{a}^{II}\left(x_0\right)\right) < x_0$ *and*
- $\tilde{\psi}\left(x_0, \tilde{a}^{II}\left(x_0\right)\right) - x_0$ *decreases with* x_0*,*

where $\tilde{\psi}$ *and* $\tilde{a}^{II}$ *are defined in Lemma 4.3.*

The details of the lemma's proof might be found in Appendix 4.A.3. We proceed by proving the statements of the last proposition.

Proof. Let us firstly identify elements of the set of equilibria considered in the proposition. As we consider only those PWBNE where each seller type chooses an auction format, there are four possible candidates: two pooling equilibria and two separating equilibria. In the following lines we check whether each of these candidates can be a part of a PWBNE.

1. Pooling on the all-pay auction

   Here we check if there is a PWBNE such that the seller's strategies are $\sigma_S^* = \left(All, \left(All, a^{All^*}\right)\right)$. For a moment, suppose that such a PWBNE exists. Let $\mu^* = \left(\left(\mu^{II^*}, B^{II^*}\right), \left(\mu^{All^*}, B^{All^*}\right)\right)$ be an arbitrary belief system associated with the equilibrium. According to Lemma 4.3, it must be $a^{All^*} = \omega$. Furthermore, from Bayes' rule, it follows that $\mu^{All^*} = p \in (0,1)$ and $B^{All} \equiv \omega$. Consequently, according to Lemma 4.1, $\sigma_B^* = \left(\beta^{II^*}, \beta^{All^*}\right)$ is such that β^{All^*} takes the form (4.3). However, due to Lemma 4.2, it is not optimal for the honest seller to choose the all-pay auction, contradicting the assumption. Thus, there is no equilibrium with pooling on the all-pay auction.

2. Pooling on the second-price auction

   It is easy to check that for any value $x_0 \leq \mathbb{E}R^{II}$, the strategy profiles $\sigma_S^* = \left(II, \left(II, a^{II^*}\right)\right)$ and $\sigma_B^* = \left(\beta^{II^*}, \beta^{All^*}\right)$ with $a^{II^*} = \tilde{a}^{II}$, $\beta^{All^*}(x) = 0$, $\beta(x) = x$ and any belief system μ^* such that $\mu^{All^*} = 1$ and $B^{All^*} \equiv \omega$ is a PWBNE. Indeed, by deviating to the all-pay auction, both sellers would get an expected revenue of x_0, which is less than or equal to the one they get with the sale via a second-price auction. In particular, Remark 4.2 argues that the manipulative seller has no incentive to change the auction format.

3. Separating candidate: Honest seller runs a second-price auction and manipulative seller chooses an all-pay auction

   Here we check if there is a PWBNE such that the seller's strategies are $\sigma_S^* = (II, \left(All, a^{All^*}\right))$. Let us assume that such a PWBNE exists. As above, we may conclude that $a^{All^*} = \omega$. Let $\mu^* = \left(\left(\mu^{II^*}, B^{II^*}\right), \left(\mu^{All^*}, B^{All^*}\right)\right)$ be the arbitrary belief system associated with the equilibrium. Bayes' rule implies that it must be $\mu^{II^*} = 0$, $\mu^{All^*} = 1$ and $B^{All} \equiv \omega$. According to Lemma 4.1, $\sigma_B^* = \left(\beta^{II^*}, \beta^{All^*}\right)$ is such that $\beta^{All^*}(x) = 0$ and $\beta^{II^*}(x) = x$, for any private value x. However, this is in contradiction to the sequential rationality of the manipulative seller because at this equilibrium the seller gets x_0, whereas deviating to the second-price auction would give at least x_0. Only in the case where $x_0 = \omega$ would the manipulative seller choose an all-pay auction for the sale. However, for $x_0 = \omega$, the honest-seller would keep his object and thus deviate from running the second-price auction to the all-pay auction or to no sale (as $\mathbb{E}R^{II} < \omega$). Hence, there is no such equilibrium where the honest seller runs the second-price and the manipulative seller chooses an all-pay auction.

4. Separating candidate: Manipulative seller chooses a second-price auction and the honest seller runs an all-pay auction

   Let us assume that there is a PWBNE such that $\sigma_S^* = \left(All, \left(II, a^{II^*}\right)\right)$ is a part of it. According to Lemma 4.3, it must be $a^{II^*} = \tilde{a}^{II}$. From Bayes' rule for the belief system, applied along the equilibrium path, it follows that $\mu^{II^*} = 1$ and $\mu^{All^*} = 0$. Accordingly, it follows that the bidders behave as if they are in standard auctions: truth-telling in the second-price and $\beta^{All^*}(x) = \int_0^x tg(t)\,dt$ (see Lemma 4.1 and remarks below it). In this particular case of the belief system, it turns out that the honest seller is indifferent between the two auction formats, and thus, sequential rationality of the seller is satisfied (unless $x_0 > \mathbb{E}R^{II}$, when the seller keeps object). It is left to check the sequential rationality of the manipulative seller. The choice of the second-price auction for the manipulative seller will be sequentially rational if and only if:

$$
\begin{array}{rcl}
u_M(II, a^{II^*} | \sigma_B^*) & \geq & u_M(All, \omega | \sigma_B^*) \\
& \Updownarrow & \\
\mathbb{E}R^{II} + \tilde{\psi}\left(x_0, \tilde{a}^{II}\right) & \geq & \left(1 - \mu^{All^*}\right) \mathbb{E}R^{II} + x_0 \\
& \Updownarrow & \\
\mathbb{E}R^{II} + \tilde{\psi}\left(x_0, \tilde{a}^{II}\right) & \geq & \mathbb{E}R^{II} + x_0 \\
& \Updownarrow & \\
\tilde{\psi}\left(x_0, r^{II}\right) & \geq & x_0.
\end{array}
$$

As Lemma 4.4 shows, the last inequality is satisfied for $x_0 \leq \bar{x}_0$. Consequently, we may conclude that $\tilde{\sigma}_S = \left(All, \left(II, a^{II^*}\right)\right)$ is a PWBNE, if and only if x_0 is low enough, or in particular if $x_0 \leq min\left\{\bar{x}_0, \mathbb{E}R^{II}\right\}$.

Overall, once we consider any PWBNE in the set of such equilibria where both sellers are to sell their object, we conclude that either $\sigma_S^* = \left(II, \left(II, a^{II^*}\right)\right)$ or $\sigma_S^* = \left(All, \left(II, a^{II^*}\right)\right)$. Moreover, at any of these equilibria, the honest seller either strictly prefers the second-price auction or is indifferent between the all-pay and second-price auctions and that indifference happens if and only if $\mu^{All^*} = 0$. These concluding remarks are the exact claims of the proposition. □

According to the statements of the proposition, pooling on the second-price is the most frequent and robust outcome in our model and chosen by all types of seller. There is a possible equilibrium outcome where the honest seller would run an all-pay auction, but as it turns out, the equilibrium is non-robust against the minimal belief fluctuation of the bidders.

4.3 Summary

We tested the second-price auction against the all-pay auction once the seller has a possibility to cheat by sending an agent to place a bid on his behalf. In addition, we analyzed the choice an honest seller has to make once the participants are aware of the possible manipulation. Examining the concept of the pure weak perfect Bayesian Nash equilibrium we obtain two types of equilibria: pooling on the second-price auction or the separating equilibrium where the honest seller will choose the all-pay auction. In any case, at all equilibria, the honest seller at least weakly prefers the second-price auction. Moreover, we have seen that the bidders still bid truthfully in the second-price auction, whereas they shade their bids in the case of all-pay auction. Hence, our model gives one more argument in favor of the second-price auction and supports its growing use in practice. Namely, even in the environment of i.i.d. private values and a special form of manipulation, the second-price auction will be chosen as the selling mechanism by both cheating and non-cheating sellers.

4.A Proofs

4.A.1 Proof of Lemma 4.1

Regarding the optimal bid in a second-price auction, it is trivial to show that the truth telling is ex post weakly dominant and, thus, the optimal choice of the bidders.

Let us now consider the case where a bidder is called to place a bid in an all-pay auction. We assume that the bidders hold the following belief: if there is an agent then the agent bids an amount a, where a is any fixed value between 0 and ω. According to the notation introduce earlier, it means that we assume that $B^{All} \equiv a$. Let us denote by $\tilde{\beta}^{All}$ the optimal bidding strategy of the bidders. We aim to derive $\tilde{\beta}^{All}$.

Furthermore, in the case where $\mu^{All} = 1$ (that is the bidders are sure that the seller is manipulative), by description of the model, it follows that the bidders will behave as if they participate in an all-pay auction with a reserve payment equal to the agent's believed bid a. Consequently, $\tilde{\beta}^{All}$ takes the following form:

$$\tilde{\beta}^{All}(x) = \begin{cases} 0, & x < \tilde{x} \\ a + \int\limits_{\tilde{x}}^{x} y g(y) dy, & x \geq \tilde{x} \end{cases}, \tag{4.10}$$

where $\tilde{x}$ is the solution of the following equation:

$$G(x)x = a.$$

Note that such $\tilde{x}$ always exists, because $G(x)x$ is a strictly increasing continuous function of x on the interval $[0, \omega]$ and a is assumed to belong to this interval.

Let us now suppose that $\mu^{All} < 1$, that is, the bidders believe there is some chance that they encounter the honest seller. Suppose that all buyers except buyer 1 are using a feasible strategy $\tilde{\beta}^{All} \neq 0$, but instead of being non-decreasing we assume that the strategy is strictly increasing. The strict monotonicity allows us to define the "quasi-inverse" function of the bidding strategy, denoted by $\left(\tilde{\beta}^{All}\right)^{-1}$, in the following way: for any $b \in [0, \beta(\omega)]$, let $\left(\tilde{\beta}^{All}\right)^{-1}(b) = \inf\left\{ x \in [0, \omega] | \, \tilde{\beta}^{All}(x) \geq b \right\}$. Specifically, let $\tilde{x} = \left(\tilde{\beta}^{All}\right)^{-1}(a)$ if $\tilde{\beta}^{All}(\omega) \geq a$; otherwise, we set $\tilde{x} = \omega$. Let us look at the optimization problem of buyer 1. Suppose that her private value is x. Let us calculate her expected payoff if she bids b, which we denote by $\Pi^{All}(b, x)$. Note that b must be chosen such that $\tilde{\beta}^{All}(\omega) \geq b \geq 0$, that is, $\left(\tilde{\beta}^{All}\right)^{-1}(b) \in [0, \omega]$. Given that the other buyers use $\tilde{\beta}^{All}$ and the agent is believed to bid a, $\Pi^{All}(b, x)$ is equal to:

$$\begin{aligned}\Pi^{All}(b,x) &= (1-\mu^{All})\cdot\left[-b+x\cdot Prob\left(\max_{2\le j\le N}\tilde{\beta}^{All}(x_j)\le b\right)\right]\\ &+ \mu^{All}\cdot\left[-b+x\cdot Prob\left(\max_{2\le j\le N}\tilde{\beta}^{All}(x_j)\le b\right)\cdot 1_{\{a\le b\}}\right]\end{aligned}$$

$$\Pi^{All}(b,x) = -b + xG\left(\left(\tilde{\beta}^{All}\right)^{-1}(b)\right)\left[(1-\mu^{All})+\mu^{All}\cdot 1_{\{a\le b\}}\right],$$

where $G(\cdot)$ is the cumulative distribution function of the random variable $\max_{2\le j\le N} X_j$. Due to the strict monotonicity of the bidding strategy, we have that $\max_{2\le j\le N}\tilde{\beta}^{All}(x_j)\le b$ if and only if $\max_{2\le j\le N} x_j \le \left(\tilde{\beta}^{All}\right)^{-1}(b)$, which led to the last equality.

Let us interpret terms on the right-hand side of the last equality. The term $-b$ stands for the payment rule of an all-pay auction: a buyer will certainly pay her bid. Once facing the honest seller (an event with $1-\mu^{All}$ probability) she will get the object (whose value is x) if her bid is the highest (that happens with chance equal to $G\left(\left(\tilde{\beta}^{All}\right)^{-1}(b)\right)$. If bidder 1 faces the manipulative seller (an event with μ^{All} chance to happen) , in order to win the object, it is not sufficient that her bid is the highest among others' bids ($G\left(\left(\tilde{\beta}^{All}\right)^{-1}(b)\right)$), but her bid has to be higher than the agent's bid a (the indicator function$1_{\{a\le b\}}$).

Thus, the optimization problem for the buyer 1 is the following:

$$\max_{0\le b\le\tilde{\beta}^{All}(\omega)}\left\{-b+x\left[(1-\mu^{All})G\left(\left(\tilde{\beta}^{All}\right)^{-1}(b)\right)+\mu^{All}G\left(\left(\tilde{\beta}^{All}\right)^{-1}(b)\right)\cdot 1_{\{a\le b\}}\right]\right\}. \tag{4.11}$$

In order to solve the problem there are two cases to be considered; bid b is less than believed agent's bid a and the opposite case.

In the case where $b < a$, the first order condition for the last equation gives rise to the following equation:

$$\begin{aligned}\frac{d}{db}\left\{-b^*+x\left[(1-\mu^{All})G\left(\left(\tilde{\beta}^{All}\right)^{-1}(b^*)\right)\right]\right\} &= 0\\ (1-\mu^{All})xg\left(\left(\tilde{\beta}^{All}\right)^{-1}(b^*)\right)\frac{1}{\left(\tilde{\beta}^{All}\right)'\left(\left(\tilde{\beta}^{All}\right)^{-1}(b^*)\right)} &= 1.\end{aligned}$$

Due to the symmetry of bidders, we have that $\left(\tilde{\beta}^{All}\right)^{-1}(b^*) = x$, thus, we obtain the following ordinary differential equation (ODE):

$$\left(\tilde{\beta}^{All}\right)'(t) \quad = (1-\mu^{All}) \quad tg(t). \tag{4.12}$$

Integrating the last equation with respect to t on the interval $[0, x]$ and plugging in the initial condition $\tilde{\beta}^{All}(0) = 0$ we obtain

$$\begin{aligned} \tilde{\beta}^{All}(x) - \tilde{\beta}^{All}(0) &= (1-\mu^{All})\int_0^x tg(t)\,dt \\ \tilde{\beta}^{All}(x) &= (1-\mu^{All})\int_0^x tg(t)\,dt. \end{aligned} \tag{4.13}$$

Because we assumed that $b^* < a$, it must be that (4.13) is the expression for the bidding strategy in the case of $x < \tilde{x}$. Thus, let us define

$$\beta_{left}(x) = (1-\mu^{All})\int_0^x tg(t)\,dt.$$

If $\tilde{\beta}^{All}(\omega) < a$ then we have derived the optimal strategy, i.e. $\tilde{\beta}^{All} = \beta_{left}$.

We continue the derivation for the case where $\tilde{\beta}^{All}(\omega) \geq a$. In this case, it is left to see how buyer 1 bids when her value is greater than $\tilde{x}$. Thus, for $b \geq a$ the first order condition for the optimization problem (4.11) leads to the following equation:

$$\begin{aligned} \frac{d}{db}\left\{-b^* + xG\left(\left(\tilde{\beta}^{All}\right)^{-1}(b^*)\right)\right\} &= 0 \\ xg\left(\left(\tilde{\beta}^{All}\right)^{-1}(b^*)\right)\frac{1}{\left(\tilde{\beta}^{All}\right)'\left(\left(\tilde{\beta}^{All}\right)^{-1}(b^*)\right)} &= 1. \end{aligned} \tag{4.14}$$

Similarly to process of solving the earlier ODE, we obtain that

$$\beta_{right}(x) = \int_{\tilde{x}}^x tg(t)\,dt + a$$

solves the last ODE with initial condition $\tilde{\beta}^{All}(\tilde{x}) = a$, for any $x \geq \tilde{x}$.

Finally, we obtained that for the values lower than $\tilde{x}$, bidder 1 will bid according to β_{left} and, otherwise, the bidder will follow β_{right}. Thus, $\tilde{\beta}^{All}$ has a cutoff shape with $\tilde{x}$ being the cutoff value. In order to complete the explicit formula for the bidding strategy, it is left to determine the cutoff value. For this purpose, we use the expected payoff equivalence at the cutoff point $x = \tilde{x}$:

$$\begin{aligned} -\beta_{left}(\tilde{x}) + (1-\mu^{All})\tilde{x}G(\tilde{x}) &= -\beta_{right}(\tilde{x}) + \tilde{x}G(\tilde{x}) \\ -(1-\mu^{All})\int_0^{\tilde{x}} tg(t)\,dt + (1-\mu^{All})\tilde{x}G(\tilde{x}) &= -a + \tilde{x}G(\tilde{x}) \\ (1-\mu^{All})\int_0^{\tilde{x}} tg(t)\,dt + \mu^{All}\tilde{x}G(\tilde{x}) &= a. \end{aligned}$$

Overall, we may summarize the derived bidding pattern. Let us consider the following equation (where the unknown variable is x) :

$$(1-\mu^{All})\int_0^{x} tg(t)\,dt + \mu^{All}xG(x) = a. \tag{4.15}$$

If there exists $\tilde{x} \in [0,\omega]$ which solves the last equation, then the optimal symmetric best reply in an all-pay auction, given the belief that the manipulative seller sends an agent who bids a, is given by

$$\tilde{\beta}^{All}(x) = \begin{cases} (1-\mu^{All})\int_0^{x} tg(t)\,dt, & x < \tilde{x} \\ a + \int_{\tilde{x}}^{x} tg(t)\,dt, & x \geq \tilde{x} \end{cases}. \tag{4.16}$$

Otherwise, that is, if there is no private value satisfying equation (4.15), then the best symmetric bidding strategy in an all-pay auction, given the belief $B^{All} \equiv a$, is

$$\tilde{\beta}^{All}(x) = (1-\mu^{All})\int_0^{x} tg(t)\,dt,\ x \in [0,\omega].$$

This is exactly the bidding strategy proposed in the lemma. Note the last formula may be extended for the case $\mu^{All} = 1$, once we relaxed the requirement that the bidding strategy is strictly increasing.

In order to prove that the above heuristics is correct, it is left to prove that $\tilde{\beta}^{All}(x)$ is feasible, satisfies the assumptions imposed at the beginning of its derivation and that it is, indeed, the optimal choice of a bidder. Whether it has the cutoff shape (in the case of the existence of $\tilde{x}$) or not, it follows straightforwardly that $\tilde{\beta}^{All}(0) = 0$ and $\tilde{\beta}^{All}$ is differentiable almost everywhere. The requirement of being non-decreasing (from feasibility definition) follows trivially from the assumption on F being strictly increasing. Note that the same argument guarantees the $\tilde{\beta}^{All}$ is strictly increasing in the case where $\mu^{All} < 1$, which was a requirement from heuristics. The fact that $\tilde{\beta}^{All}(x) \leq x$ for any private value x, if there is no cutoff value $\tilde{x}$ or for values $x < \tilde{x}$, is satisfied, because:

$$\begin{aligned}
\tilde{\beta}^{All}(x) &= (1-\mu^{All})\int_0^x tg(t)\,dt \\
&\leq \int_0^x tg(t)\,dt \\
&\leq x\int_0^x g(t)\,dt \\
&= xG(x) \\
&\leq x.
\end{aligned}$$

Let us establish the inequality in the case where $\tilde{x}$ exists and $x \geq \tilde{x}$. In this particular case, we have that

$$\begin{aligned}
\tilde{\beta}^{All}(x) &= a + \int_{\tilde{x}}^x tg(t)\,dt \\
&= (1-\mu^{All})\int_0^{\tilde{x}} tg(t)\,dt + \mu^{All}\tilde{x}G(\tilde{x}) + \int_{\tilde{x}}^x tg(t)\,dt \\
&\leq \tilde{x}G(\tilde{x}) + x(G(x) - G(\tilde{x})) \\
&= xG(x) - G(\tilde{x})(x-\tilde{x}) \\
&\leq xG(x) \\
&\leq x.
\end{aligned}$$

The second equality comes from the equation which defines the cutoff value $\tilde{x}$ of the bidding strategy (see equation (4.15)).

Thus, we have just shown that the function is feasible and satisfies the assumptions imposed at the beginning of this section. It is left to show that $\tilde{\beta}^{All}$ is indeed optimal. For the case where $\mu^{All} = 1$, we use the fact that the bidding strategy and our model coincide with the framework of an all-pay auction with a reserve payment a and from literature we know that $\tilde{\beta}^{All}$ is optimal. Thus, we concentrate on the case where $\mu^{All} < 1$. It is sufficient to prove that$\tilde{\beta}^{All}(x)$ is a global maximum of the payoff function, for any value x. Let us fix an arbitrarily chosen private value x. We shall consider the expected payoff $\Pi^{All}(b,x)$ as a function of the bid value b in a neighborhood of the point $b_x = \tilde{\beta}^{All}(x)$. For our goal, it suffices to show that $\Pi^{All}(b,x)$ increases as $b \to b_x^-$ (that is, as b approaches b_x from the left side on the real line) and decreases as b increases and goes away from b_x. From above exposed derivations, we have that the first derivative of the function $\Pi^{All}(b,x)$, with respect to b, has the following form:

$$\frac{\partial}{\partial b}\Pi^{All}(b,x) = \begin{cases} -1 + (1-\mu^{All})\frac{xg(z)}{\left(\tilde{\beta}^{All}\right)'(z)} & , b < \tilde{\beta}^{All}(\tilde{x}) \\ -1 + \frac{xg(z)}{\left(\tilde{\beta}^{All}\right)'(z)} & , b \geq \tilde{\beta}^{All}(\tilde{x}) \end{cases},$$

where $z = \left(\tilde{\beta}^{All}\right)^{-1}(b)$. From definition of the $\tilde{\beta}^{All}$ as the solution of ODEs (4.12) and (4.14), we have that

$$\left(\tilde{\beta}^{All}\right)'(z) = \begin{cases} \left(1-\mu^{All}\right) zg(z) & , z < \tilde{x} \\ zg(z) & , z \geq \tilde{x} \end{cases}.$$

Therefore, let us assume that $z < x$. Irrespective of comparison between z and the cutoff value $\tilde{x}$ and even, in the case of the non-existence of the cutoff value, we have that

$$\begin{aligned} \frac{\partial}{\partial b}\Pi^{All}(b,x) &= -1 + \frac{x}{z} \\ &> -1 + 1 = 0. \end{aligned}$$

Thus, for $z < x \Leftrightarrow b < \tilde{\beta}^{All}(x)$ we have that $\frac{\partial}{\partial b}\Pi^{All}(b,x) > 0$. In a similar manner, one can show that $\frac{\partial}{\partial b}\Pi^{All}(b,x) < 0$ for $b > b_x$ which proves the required claim.

4.A.2 Proof of Lemma 4.3

Let us suppose that the manipulative seller chooses to sell the good via a second-price auction. We aim to find the optimal agent's bid in this case, that is, the bid that maximizes the expected payoff to the seller. For this purpose, let us look at the expected payoff of the manipulative seller once he announces the second-price auction and sends an agent who bids $a \in [0,\omega]$. We shall denote the seller's expected payoff as $u_M(II, a\,|\tilde{\sigma}_B)$. It is important to note that we assume that the bidders bid truthfully (since it is the optimal choice, as Lemma 4.1 shows). Due to the truth telling, note that if the highest of the bid is less than the agent's bid a then the seller will keep the object. Otherwise, a regular second-price auction takes place. Hence, the expected payoff has the following form:

$$\begin{aligned} u_M(II, a\,|\tilde{\sigma}_B) &= \underbrace{x_0 \times F^N(a)}_{\text{"Fake sale" (the seller keeps the good due to the agent's bid)}} + \underbrace{N\int_a^\omega m_M^{II}(x)\, f(x)\, dx}_{\text{Sale}} \\ &= x_0 F^N(a) + N\int_a^\omega \left[aG(a) + \int_a^x yg(y)dy \right] f(x)\, dx \\ &= x_0 F^N(a) + N\int_a^\omega aG(a) f(x)\, dx + N\int_a^\omega \left[\int_a^x yg(y)dy \right] f(x)\, dx \\ &= x_0 F^N(a) + NaG(a)\,(1 - F(a)) + N\int_a^\omega \left[\int_a^x yg(y)dy \right] f(x)\, dx, \end{aligned} \tag{4.17}$$

where $m_M^{II}(x)$ is the expected payment from a bidder with private value higher than a. Let us consider a bidder with such private value $x \geq a$. The bidder makes positive payment to the seller if she wins the auction. She is the winner if and only if the bids of the other bidders are lower than her own bid. Since the bidders bid truthfully, the bidder is the winner if and only if her value is the highest. Given that the bidder is the winner, the price she pays to the seller depends whether the highest others' bid is lower or higher than a, as the agent will certainly bid a. In the former case, the winner pays always a, whereas in the latter case the price is equal to the highest bid of the others. This is exactly the reasoning, which was behind the equation:

$$m_M^{II}(x) = \underbrace{aG(a)}_{\text{Others' highest bid is lower than } a} + \underbrace{\int_a^x yg(y)dy}_{\text{"Real" sale}}.$$

Reordering the terms and using the expression for the expected revenue in a standard second-price auction, we may rewrite $u_M(II, a\,|\tilde{\sigma}_B)$ as follows:

$$
\begin{aligned}
u_M(II, a\,|\tilde{\sigma}_B) &= x_0 F^N(a) + NaG(a)\,(1 - F(a)) + N\int_a^{\omega}\left[\int_a^x yg(y)dy\right] f(x)\,dx \\
&= x_0 F^N(a) + a\left(NF^{N-1}(a) - NF^N(a)\right) + N\int_a^{\omega}\left[\int_0^x yg(y)dy - \int_0^a yg(y)dy\right] f(x)\,dx \\
&= x_0 F^N(a) + a\left(H(a) - F^N(a)\right) + N\int_a^{\omega}\left[\int_0^x yg(y)dy\right] f(x)\,dx \\
&\quad - N\int_a^{\omega}\left[\int_0^a yg(y)dy\right] f(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= x_0 F^N(a) + aH(a) - aF^N(a) + N\left\{\int_0^{\omega}\left[\int_0^x yg(y)dy\right] f(x)\,dx - \int_0^a\left[\int_0^x yg(y)dy\right] f(x)\,dx\right\} \\
&\quad - N\left[\int_0^a yg(y)dy\right][1 - F(a)] \\
&= aH(a) - F^N(a)(a - x_0) + \mathbb{E}R^{II} - N\int_0^a\left[\int_y^a f(x)\,dx\right] yg(y)dy - N\left[\int_0^a yg(y)dy\right][1 - F(a)] \\
&= \mathbb{E}R^{II} + aH(a) - F^N(a)(a - x_0) - N\int_0^a (F(a) - F(y) + 1 - F(a))\, yg(y)dy \\
&= \mathbb{E}R^{II} + aH(a) - F^N(a)(a - x_0) - \int_0^a N(1 - F(y))\, yg(y)dy \\
&= \mathbb{E}R^{II} + aH(a) - F^N(a)(a - x_0) - \int_0^a yh(y)\,dy.
\end{aligned}
$$

Or, equivalently,

$$
u_M(II, a\,|\tilde{\sigma}_B) = \mathbb{E}R^{II} + \tilde{\psi}(x_0, a)
$$

with

$$
\begin{aligned}
\tilde{\psi}(x_0, a) &= aH(a) - \int_0^a yh(y)\,dy - F^N(a)(a - x_0) \\
&= \int_0^a H(y)\,dy - F^N(a)(a - x_0).
\end{aligned}
$$

Remember that H and h are the c.d.f. and d.f., respectively, of the second-highest value given by (4.4) and (4.5). Therefore, the whole benefit of the manipulation in the second-price auction is expressed by the function $\tilde{\psi}(x_0, a)$. We are interested in the optimal behavior of the seller and accordingly shall proceed with determining the optimal choice of the agent's bid a - in the further text denoted as $\tilde{a}^{II}$. Hence, we are heading to solve the following optimization problem:

$$\max_{a\in[0,\omega]} u_M(II, a\,|\tilde{\sigma}_B)$$
$$\Updownarrow$$
$$\max_{a\in[0,\omega]} \left\{\mathbb{E}R^{II} + \tilde{\psi}(x_0, a)\right\}$$
$$\Updownarrow$$
$$\mathbb{E}R^{II} + \max_{a\in[0,\omega]} \tilde{\psi}(x_0, a)\,.$$

Since $\mathbb{E}R^{II}$ does not depend on a, it is left to find global maximum of the function $\tilde{\psi}(x_0, a)$. The necessary first order condition (F.O.C.) might be written as:

$$\begin{aligned}
\frac{\partial}{\partial a}\tilde{\psi}(x_0, a) &= 0 \\
&\Updownarrow \\
\frac{\partial}{\partial a}\left\{\int_0^a H(y)\,dy - F^N(a)(a - x_0)\right\} &= 0 \\
&\Updownarrow \\
H(a) - F^N(a) - NF^{N-1}(a)\,f(a)(a - x_0) &= 0 \\
&\Updownarrow \\
NF^{N-1}(a) - (N-1)F^N(a) - F^N(a) - NF^{N-1}(a)\,f(a)(a - x_0) &= 0 \\
&\Updownarrow \\
NF^{N-1}(a) - NF^N(a) - NF^{N-1}(a)\,f(a)(a - x_0) &= 0 \\
&\Updownarrow \\
NF^{N-1}(a)\left[1 - F(a) - f(a)(a - x_0)\right] &= 0.
\end{aligned}$$

Thus, they are two stationary points or candidates for the optimal choice $\tilde{a}^{II}$. Either

$$F^{N-1}(\tilde{a}^{II}) = 0 \Leftrightarrow \tilde{a}^{II} = 0,$$

or $\tilde{a}^{II}$ is the solution of the following equation:

$$\tilde{a}^{II} = x_0 + \frac{1 - F\left(\tilde{a}^{II}\right)}{f\left(\tilde{a}^{II}\right)}. \tag{4.18}$$

It is clear that $\tilde{a}^{II} = 0$ is a local minimum because, for all $a \leq x_0$, we have that

$$\begin{aligned}
\frac{\partial}{\partial a} u_M(II, a\,|\sigma_B) &= NF^{N-1}(a)\left[1 - F(a) - f(a)(a - x_0)\right] \\
&= NG(x_a)\left[1 - F(a) + f(x_a)(x_0 - a)\right] \\
&\geq 0.
\end{aligned}$$

Let us take a closer look at the second candidate - the solution of equation (4.18). Actually from Myerson (1981), one can see that this candidate is indeed the optimal reserve price in a regular second-price auction. Thus, as F is supposed to have strictly increasing hazard rate, $\tilde{a}^{II}$ is well-defined and is the optimal agent's bid, which proves the claim of the lemma.

Let us now determine the optimal agent's bid once the manipulative seller runs an all-pay auction and sends an agent to bid a. His expected payoff, denoted by $u_M(All, a\,|\tilde{\sigma}_B)$, is equal to:

$$\begin{aligned}
u_M(All, a\,|\tilde{\sigma}_B) &= \underbrace{x_0 \times F^N(x_a)}_{\text{"Fake sale" (sellers keeps the good due to agent)}} + \underbrace{N\int_0^{\omega} \tilde{m}^{All}(x) f(x)\,dx}_{\text{Sale}} \\
&= x_0 F^N(x_a) + N\int_0^{\omega} \tilde{\beta}^{All}(x) f(x)\,dx,
\end{aligned}$$

where x_a is the maximal "no-sale" private value, that is, $x_a = \max\left\{x \,\middle|\, \tilde{\beta}^{All}(x) \leq a\right\}$. Thus, the manipulative seller here always collects money from all bidders and, in addition, with probability $F^N(x_a)$, he keeps the object. Consequently, it is clear that the seller will send an agent, who will bid higher that all other bids, that is, the optimal agent's bid is ω - because, in this case, the seller keeps always the object and gets $N\int_0^{\omega} \tilde{\beta}^{All}(x) f(x)\,dx$, on average.

4.A.3 Proof of Lemma 4.4

In order to prove the claim of the lemma let us define several functions. First of all, let γ be a function that associates to any x_0 the optimal price $\tilde{a}^{II}$ in the second-price auction, that is, $\gamma(x_0) = \tilde{a}^{II}$. According to equation (4.8), its inverse function is defined as:

$$(\gamma)^{-1}\left(\tilde{a}^{II}\right)=\tilde{a}^{II}-\frac{1-F\left(\tilde{a}^{II}\right)}{f\left(\tilde{a}^{II}\right)}$$

Note that from the assumption that F has a strictly increasing hazard rate, it follows that $(\gamma)^{-1}$ is strictly increasing, and, hence, γ itself is a strictly increasing function.

Furthermore, let us define Λ as a function of the seller's value x_0 in the following way:

$$\begin{aligned}\Lambda\left(x_0\right) &= \tilde{\psi}\left(x_0,\gamma\left(x_0\right)\right)-x_0\\ &\Updownarrow\\ \Lambda\left(x_0\right) &= \int_0^{\gamma(x_0)} H\left(y\right)dy-F^N\left(\gamma\left(x_0\right)\right)\left(\gamma\left(x_0\right)-x_0\right)-x_0.\end{aligned}$$

Moreover, note that for any x_0, by definition of the function $\tilde{\psi}$ (see the proof of Lemma 4.3) we have that

$$\tilde{\psi}\left(x_0,\gamma\left(x_0\right)\right)=\max_{a\in[0,\omega]}\tilde{\psi}\left(x_0,a\right).$$

The claims of the lemma are easily expressed in terms of the function Λ. In order to prove the claims, it is sufficient to show that the function Λ changes sign at the endpoints of its domain, is a continuous and strictly decreasing function on $[0,\omega]$. From its definition, it is clear that the function is continuous. Let us look at the sign that the function takes at the endpoints of its domain: 0 and ω. Let us denote with $a_0=\gamma\left(0\right)$, that is, a_0 is the optimal price in the second price auction once $x_0=0$. First of all, note that $a_0\neq 0$. Indeed, $(\gamma)^{-1}\left(0\right)=-\frac{1}{f(0)}<0$, $(\gamma)^{-1}\left(a_0\right)=0$ and the strict monotonicity of the function γ^{-1} imply that $a_0\neq 0$.[7] Overall, we have that

$$\begin{aligned}\Lambda\left(0\right) &= \tilde{\psi}\left(0,a_0\right)-0\\ &= \tilde{\psi}\left(0,a_0\right)\\ &= \max_{a\in[0,\omega]}\tilde{\psi}\left(0,a\right)\\ &> \tilde{\psi}\left(0,0\right)\\ &= \int_0^0 H\left(y\right)dy-F^N\left(0\right)\left(0-x_0\right)\\ &= 0.\end{aligned}$$

[7]Here we actually use the extension of $(\gamma)^{-1}$ on the whole interval $[0,\omega]$ and not only on the image set $\gamma\left([0,\omega]\right)$.

Let us now look at the value of the function at the other endpoint, that is, at ω. Note that $\gamma(\omega) = \omega$ (since $(\gamma)^{-1}(\omega) = \omega$). Thus, we have that:

$$\begin{aligned} \Lambda(\omega) &= \int_0^\omega H(y)\,dy - F^N(\omega)(\omega - \omega) - \omega \\ &= \int_0^\omega H(y)\,dy - \omega \\ &= -\mathbb{E}R^{II} \\ &< 0. \end{aligned}$$

The last equality in the previous calculation is obtained from the fact that $\mathbb{E}R^{II} = \int_0^\omega y h(y)\,dy$. Integrating by parts, we obtain $\mathbb{E}R^{II} = \omega - \int_0^\omega H(y)\,dy$. Thus, by continuity of the function Λ and the intermediate value theorem, we can conclude that there exists $\bar{x}_0$ such that $\Lambda(\bar{x}_0) = 0$. The sufficient condition for the rest of the claim in the lemma (the uniqueness of $\bar{x}_0$ and the strict monotonicity of Λ) is a proof that Λ is a strictly decreasing function. Thus, we aim to prove that

$$\Lambda'(x_0) = \frac{d}{dx_0}\left(\tilde{\psi}(x_0, \gamma(x_0)) - x_0\right) < 0$$

for any x_0.

Let x_0 be arbitrarily chosen and fixed. Let us have a closer look at $\Lambda'(x_0)$.

$$\begin{aligned} \Lambda'(x_0) &= \frac{d}{dx_0}\left(\tilde{\psi}(x_0, \gamma(x_0)) - x_0\right) \\ &= \frac{\partial}{\partial x_0}\tilde{\psi}(x_0, \gamma(x_0)) + \frac{\partial}{\partial a}\tilde{\psi}(x_0, \gamma(x_0))\,\gamma'(x_0) - 1 \\ &= \frac{\partial}{\partial x_0}\tilde{\psi}(x_0, \gamma(x_0)) + 0 \cdot \gamma'(x_0) - 1 \\ &= \frac{\partial}{\partial x_0}\tilde{\psi}(x_0, \gamma(x_0)) - 1. \end{aligned}$$

Note that we considered $\tilde{\psi}$ as a function of two variables, x_0 and a, as defined in (4.9). Since we have that $\gamma(x_0) = \max_{a \in [0,\omega]} \tilde{\psi}(x_0, a)$,the necessary first order conditions imply $\frac{\partial}{\partial a}\tilde{\psi}(x_0, \gamma(x_0)) = 0$. So, it is left to calculate $\frac{\partial}{\partial x_0}\tilde{\psi}(x_0, \gamma(x_0))$. Using the definition of $\tilde{\psi}$, we obtain that

$$\frac{\partial}{\partial x_0}\tilde{\psi}\left(x_0, \gamma\left(x_0\right)\right) = F^N\left(\gamma\left(x_0\right)\right).$$

Finally, we get:

$$\begin{aligned} \Lambda'\left(x_0\right) &= \frac{\partial}{\partial x_0}\tilde{\psi}\left(x_0, \gamma\left(x_0\right)\right) - 1 \\ &= F^N\left(\gamma\left(x_0\right)\right) - 1. \end{aligned}$$

Thus, $\Lambda'\left(x_0\right) < 0$ for $x_0 < \omega$ and $\Lambda'\left(x_0\right) = 0$ for $x_0 = \omega$. Consequently, the claims of the lemma hold true.

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