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Stochastic nonzero-sum games: a new connection between singular control and optimal stopping

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Abstract. In this paper we establish a new connection between a class of 2-player nonzero-sum games of optimal stopping and certain 2-player nonzero-sum games of singular control. We show that whenever a Nash equilibrium in the game of stopping is attained by hitting times at two separate boundaries, then such boundaries also trigger a Nash equilibrium in the game of singular control. Moreover a differential link between the players' value functions holds across the two games.

Keywords: games of singular control, games of optimal stopping, Nash equilibrium, one-dimensional diffusion, Hamilton-Jacobi-Bellman equation, verification theorem.

MSC2010 subject classification: 91A15, 91A05, 93E20, 91A55, 60G40, 60J60, 91B76.

1 Introduction

Connections between some problems of singular stochastic control (SSC) and questions of optimal stopping (OS) are well known in control theory. In 1966 Bather and Chernoff [5] studied the problem of controlling the motion of a spaceship which must reach a given target within a fixed period of time, and with minimal fuel consumption. This problem of aerospace engineering was modeled in [5] as a singular stochastic control problem, and an unexpected link with optimal stopping was observed. The value function of the control problem was indeed differentiable in the direction of the controlled state variable, and its derivative coincided with the value function of an optimal stopping problem.

The result of Bather and Chernoff was obtained by using mostly tools from analysis. Later on, Karatzas [24, 25], and Karatzas and Shreve [26] employed fully probabilistic methods to perform a systematic study of the connection between SSC and OS for the so-called "monotone follower problem". The latter consists of tracking the motion of a stochastic process (a Brownian motion in [24], [25], [26]) by a nondecreasing control process in order to maximise (minimise) a performance criterion which is concave (convex) in the control variable. Further, a link to optimal stopping was shown to hold also for monotone follower problems of finite-fuel type; i.e. where the total variation of the control (the fuel available to the controller) stays bounded (see [17], [27], and also [4] for dynamic stochastic finite-fuel). More recent works provided extensions of the above results to diffusive settings in [6] and [7], to Brownian two-dimensional problems with state constraints in [11], and to non-Markovian processes in [3].

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It was soon realised that this kind of connections could be established in wider generality with admissible controls which are of bounded variation as functions of time (rather than just monotone). Indeed, under suitable regularity assumptions (including convexity or concavity of the objective functional with respect to the control variable) the value function of a bounded variation control problem is differentiable in the direction of the controlled state variable and its derivative equals the value function of a 2-player zero-sum game of optimal stopping (Dynkin game). To the best of our knowledge this link was noticed for the first time in [36] in a problem of controlling a Brownian motion, and then generalised in [8] and [29], and later on also in [20] via optimal switching.

It is important to observe that despite their appearance in numerous settings, connections between SSC and OS are rather "delicate" and should not be given for granted, even for monotone follower problems with very simple diffusion processes. Indeed counterexamples were recently found in [14] and [15] where the connection breaks down even if the cost function is arbitrarily smooth and the underlying processes are Ornstein-Uhlenbeck or Brownian motion.

The existing theory on the connection between SSC and OS is well established for *single agent* optimisation problems. However the latter are not suitable for the description of more complex systems where strategic interactions between several decision makers play a role. Problems of this kind arise for instance in economics and finance when studying productive capacity expansion in an oligopoly [35], the competition for the market-share control [30], or the optimal control of an exchange rate by a central bank (see the introduction of the recent [22] for such application).

In this paper we establish a new connection between a class of 2-player nonzero-sum games of optimal stopping (see [16] and references therein) and certain 2-player nonzero-sum games of singular stochastic control. In particular, we consider a game of control in which each one of the two players may exert a monotone control to adjust the trajectory of a one-dimensional controlled Itô-diffusion $\widetilde{X}$. The first player pushes the level of $\widetilde{X}$ up, whereas the second player pushes it down. If player 1 uses a unit of control at time $t > 0$ pays a cost $G_1(\widetilde{X}_t)$ and player 2 receives a reward $L_2(\widetilde{X}_t)$. A symmetric situation occurs if player 2 exerts control (see Section 2.2). Each player wants to maximise her own expected reward functional. To establish the link we also consider a 2-player nonzero-sum game of stopping on another diffusion process X suitably related to $\widetilde{X}$. Both players in the game aim at minimising their expected costs by optimally stopping the game. The i-th player can decide to stop first and pay $G_i(X_t)$ or to wait until the other player stops and then pay $L_i(X_t)$.

We show that if a Nash equilibrium in the game of stopping is attained by hitting times of two separate thresholds, i.e. the process X is stopped as soon as it leaves an interval (a_*, b_*) of the real line, then the couple of controls that keep $\widetilde{X}$ inside $[a_*, b_*]$ with minimal effort (i.e. according to a Skorokhod reflection policy) realises a Nash equilibrium in the game of singular control. Moreover, we also prove that the value functions of the two players in the game of singular control can be obtained by suitably integrating their respective ones in the game of optimal stopping. The existence of Nash equilibria of threshold type for the game of stopping holds in a large class of examples as it is demonstrated in the recent [16]. Here the proof of our main theorem (cf. Theorem 3.1 below) is based on a verification argument following an educated guess. In order to illustrate an application of our results we present a game of pollution control between a social planner and a firm representative of the productive sector.

Another important result of this paper is a simple explicit construction of closed-loop Nash equilibria[1] for a class of 2-player continuous time stochastic games of singular control. This is a problem in game theory which has not been solved in full generality yet (see the discussion in Section 2 of [2] and in [35]), and here we contribute to further improve results in that direction. We seek for Nash equilibria in the class of control strategies $\mathcal{M}$ which forbids the players to

[1] i.e. equilibria in which each player dynamically reacts to her opponent's decisions

exert simultaneous impulsive controls (i.e. simultaneous jumps of their control variables). On the one hand this is a convenient choice for technical reasons, but on the other hand we also show in Appendix A.1 that it induces no loss of generality in a large class of problems commonly addressed in the literature on singular stochastic control.

Our work marks a new step towards a global view on the connection between singular stochastic control problems and questions of optimal stopping by extending the existing results to multi-agent optimisation problems. A link between these two classes of optimisation problems is important not only from a purely theoretical point of view but also from a practical point of view. Indeed as it was pointed out in [26] (cf. p. 857) one may hope to "jump" from one formulation to the other in order to "pose and solve more favourable problems". As an example, one may notice that questions of existence and uniqueness of optimisers are more tractable in control problems, than in stopping ones; on the other hand, a characterisation of optimal control strategies is in general a harder task than the one of optimal stopping rules. Recent contributions to the literature (e.g., [12] and [13]) have already highlighted how the combined approach of singular stochastic control and optimal stopping is extremely useful to deal with investment/consumption problems for a single representative agent. It is therefore reasonable to expect that our work will increase the mathematical tractability of investment/consumption problems for multiple interacting agents.

The rest of the paper is organised as follows. In Section 2 we introduce the setting, the game of singular controls and the game of optimal stopping. In Section 3 we prove our main result and we discuss the assumptions needed. An application to a game of pollution control is considered in Section 4, whereas some proofs and a discussion regarding admissible strategies are collected in the appendix.

2 Setting

2.1 The underlying diffusions

Denote by $(\Omega, \mathcal{F}, \mathsf{P})$ a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ under usual hypotheses. Let $\widetilde{W} = (\widetilde{W}_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion adapted to $\mathbb{F}$, and $(\widetilde{X}^{\nu,\xi}_t)_{t\geq 0}$ the solution of the controlled stochastic differential equation (SDE)

$$d\widetilde{X}^{\nu,\xi}_t = \mu(\widetilde{X}^{\nu,\xi}_t)dt + \sigma(\widetilde{X}^{\nu,\xi}_t)d\widetilde{W}_t + d\nu_t - d\xi_t, \qquad \widetilde{X}^{\nu,\xi}_0 = x \in \mathcal{I}, \tag{2.1}$$

with $\mathcal{I} := (\underline{x}, \overline{x}) \subseteq \mathbb{R}$. Here $(\nu_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ belong to

$$\mathcal{S} := \big\{\eta : (\eta_t(\omega))_{t\geq 0} \text{ left-continuous, adapted, increasing, with } \eta_0 = 0, \ \mathsf{P}\text{-a.s.}\big\} \tag{2.2}$$

and we denote

$$\sigma_{\mathcal{I}} := \inf\{t \geq 0 \,:\, \widetilde{X}^{\nu,\xi}_t \notin \mathcal{I}\} \tag{2.3}$$

the first time the controlled process leaves $\mathcal{I}$.

Notice that ν and ξ can be expressed as the sum of their continuous part and pure jump part, i.e.

$$\nu_t = \nu^c_t + \sum_{s<t} \Delta\nu_s, \quad \xi_t = \xi^c_t + \sum_{s<t} \Delta\xi_s, \tag{2.4}$$

where $\Delta\nu_s := \nu_{s+} - \nu_s$ and $\Delta\xi_s := \xi_{s+} - \xi_s$. Throughout the paper we will consider the process $\widetilde{X}^{\nu,\xi}$ killed at $\sigma_{\mathcal{I}}$ and we make the following assumptions on μ and σ.

Assumption 2.1. *The functions μ and σ are in $C^1(\mathcal{I})$ and $\sigma(x) > 0$, $x \in \mathcal{I}$.*

To account for the dependence of $\widetilde{X}$ on its initial position, from now on we shall write $\widetilde{X}^{x,\nu,\xi}$ where appropriate. In the rest of the paper we use the notation $\mathsf{E}_x f(\widetilde{X}^{\nu,\xi}_t) = \mathsf{E} f(\widetilde{X}^{x,\nu,\xi}_t)$, for f Borel-measurable, since $(\widetilde{X}, \nu, \xi)$ is Markovian but the initial value of the controls is always zero. Here E_x is the expectation under the measure $\mathsf{P}_x(\,\cdot\,) := \mathsf{P}(\,\cdot\,|X_0 = x)$ on $(\Omega, \mathcal{F})$.

While (2.1) will be the underlying process in the game of control, we also introduce on $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$ another Brownian motion $(W_t)_{t\ge 0}$ and a diffusion $(X_t)_{t\ge 0}$ evolving according to

$$dX_t = \big(\mu(X_t) + \sigma(X_t)\sigma'(X_t)\big)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathcal{I}, \tag{2.5}$$

which will appear in the game of stopping. Notice that under Assumption 2.1 the above SDE admits a weak solution $(X, W, \mathbb{F})$ which is unique in law up to a possible explosion time [28, Ch. 5.5]. Indeed for every $x \in \mathcal{I}$ there exists $\varepsilon_o > 0$ such that

$$\int_{x-\varepsilon_o}^{x+\varepsilon_o} \frac{1 + |\mu(z)| + |\sigma(z)\sigma'(z)|}{|\sigma(z)|^2}\, dz < +\infty. \tag{2.6}$$

To account explicitly for the initial condition we denote by X^x the solution of (2.5) starting from $x \in \mathcal{I}$ at time zero. Due to (2.6) the diffusion X is regular in $\mathcal{I}$; that is, if $\tau_z := \inf\{t \ge 0 : X^x_t = z\}$ one has $\mathsf{P}(\tau_z < \infty) > 0$ for every x and z in $\mathcal{I}$ so that the state space cannot be decomposed into smaller sets from which X cannot exit (see [9, Ch. 2]).

For the diffusion X^x we take $\underline{x}$ and $\overline{x}$ either natural or entrance-not-exit, hence unattainable (see p. 15 in [9]). We also assume that $\underline{x}$ and $\overline{x}$ are unattainable for the uncontrolled process $\widetilde{X}^{0,0}$ and in the next remark we show that such condition holds provided that σ' is sufficiently integrable.

Remark 2.2. *Let us consider the uncontrolled dynamics $\widetilde{X}^{0,0}$ on the canonical space under the measure $\mathsf{P}^{\widetilde{X}}_x$ and the dynamics X on the canonical space under the measure P^X_x. Let us also define a new measure $\mathsf{Q}^{\widetilde{X}}_x$ by the Radon-Nikodym derivative*

$$Z_t := \frac{d\mathsf{Q}^{\widetilde{X}}_x}{d\mathsf{P}^{\widetilde{X}}_x}\bigg|_{\mathcal{F}_t} = \exp\Big\{ \int_0^t \sigma'(\widetilde{X}^{0,0}_s)d\widetilde{W}_s - \frac{1}{2}\int_0^t (\sigma')^2(\widetilde{X}^{0,0}_s)ds \Big\}, \qquad \mathsf{P}^{\widetilde{X}}_x - a.s.$$

which is an exponential martingale under suitable integrability conditions on σ'. Hence Girsanov theorem implies that the process $B_t := \widetilde{W}_t - \int_0^t \sigma'(\widetilde{X}^{0,0}_s)ds$ is a standard Brownian motion under $\mathsf{Q}^{\widetilde{X}}_x$ and it is not hard to verify that Law$(\widetilde{X}^{0,0}\big|\mathsf{Q}^{\widetilde{X}}_x) =$ Law$(X\big|\mathsf{P}^X_x)$.

It follows that denoting $\sigma^0_{\mathcal{I}} = \inf\{t > 0 : \widetilde{X}^{0,0}_t \notin \mathcal{I}\}$ and $\tau_{\mathcal{I}} = \inf\{t > 0 : X_t \notin \mathcal{I}\}$ we have that Law$(\sigma^0_{\mathcal{I}}|\mathsf{Q}^{\widetilde{X}}_x) =$ Law$(\tau_{\mathcal{I}}|\mathsf{P}^X_x)$. Notice also that the measures $\mathsf{Q}^{\widetilde{X}}_x$ and $\mathsf{P}^{\widetilde{X}}_x$ are equivalent on $\mathcal{F}^{\widetilde{W}}_t$ for all $0 \le t < +\infty$ (see [28], Chapter 3.5) and in particular $\{\sigma^0_{\mathcal{I}} \le t\} \in \mathcal{F}^{\widetilde{W}}_t$. Therefore using that $\underline{x}$ and $\overline{x}$ are unattainable for X we get

$$0 = \mathsf{P}^X_x(\tau_{\mathcal{I}} \le t) = \mathsf{Q}^{\widetilde{X}}_x(\sigma^0_{\mathcal{I}} \le t) \implies \mathsf{P}^{\widetilde{X}}_x(\sigma^0_{\mathcal{I}} \le t) = 0$$

for all $t > 0$. Hence $\mathsf{P}^{\widetilde{X}}_x(\sigma^0_{\mathcal{I}} < +\infty) = 0$ which proves that $\underline{x}$ and $\overline{x}$ are unattainable for the process $\widetilde{X}^{0,0}$ under $\mathsf{P}^{\widetilde{X}}_x$ for all $x \in \mathcal{I}$.

The infinitesimal generator of the uncontrolled diffusion $\widetilde{X}^{x,0,0}$ is denoted by $\mathcal{L}_{\widetilde{X}}$ and is defined as

$$\left(\mathcal{L}_{\widetilde{X}} f\right)(x) := \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x), \quad f \in C^2(\overline{\mathcal{I}}),\, x \in \mathcal{I}, \tag{2.7}$$

whereas the one for X is denoted by $\mathcal{L}_X$ and is defined as

$$\left(\mathcal{L}_X f\right)(x) := \frac{1}{2}\sigma^2(x)f''(x) + (\mu(x) + \sigma(x)\sigma'(x))f'(x), \quad f \in C^2(\overline{\mathcal{I}}), x \in \mathcal{I}. \tag{2.8}$$

Letting $r > 0$ be a fixed constant, we assume

Assumption 2.3. *$r > \mu'(x)$ for $x \in \overline{\mathcal{I}}$.*

We denote by ψ and ϕ the fundamental solutions of the ODE (see [9, Ch. 2, Sec. 10])

$$\mathcal{L}_X u(x) - (r - \mu'(x))u(x) = 0, \qquad x \in \mathcal{I}, \tag{2.9}$$

and we recall that they are strictly increasing and decreasing, respectively.

Finally we denote by $S'(x)$, $x \in \mathcal{I}$ the density of the scale function of $(X_t)_{t\ge 0}$ and by w the Wronskian

$$w := \frac{\psi'(x)\phi(x) - \phi'(x)\psi(x)}{S'(x)}, \quad x \in \mathcal{I}, \tag{2.10}$$

which is a positive constant.

Particular attention in this paper is devoted to solutions of (2.1) reflected inside intervals $[a, b] \subset \mathcal{I}$ and we recall here the following result on Skorokhod reflection whose proof can be found for instance in [37, Thm. 4.1] (notice that μ' and σ' are bounded on $[a, b]$).

Lemma 2.4. *Let Assumption 2.1 hold. For any $a, b \in \mathcal{I}$ with $a < b$ and any $x \in \mathcal{I}$ there exist a unique couple $(\nu^a, \xi^b) \in \mathcal{S} \times \mathcal{S}$ that solves the Skorokhod reflection problem $\boldsymbol{SP}(a, b)$ defined as:*

$$\text{Find } (\nu,\xi) \in \mathcal{S}\times\mathcal{S} \text{ s.t.} \begin{cases} \widetilde{X}^{x,\nu,\xi}_t \in [a,b], \mathsf{P}\text{-a.s. for } 0 < t \le \sigma_{\mathcal{I}}, \\ \int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbb{1}_{\{\widetilde{X}^{x,\nu,\xi}_t > a\}} d\nu_t = 0, \mathsf{P}\text{-a.s. for any } T > 0, \\ \int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbb{1}_{\{\widetilde{X}^{x,\nu,\xi}_t < b\}} d\xi_t = 0, \mathsf{P}\text{-a.s. for any } T > 0. \end{cases} \tag{$\mathbf{SP}(a,b)$}$$

It also follows that $supp\{d\nu^a_t\} \cap supp\{d\xi^b_t\} = \emptyset$.

For future frequent use we also recall the one-sided version of the above result.

Lemma 2.5. *Let Assumption 2.1 hold. For any $a \in \mathcal{I}$, $x \in \mathcal{I}$ and $\xi \in \mathcal{S}$ there exists a unique $\nu^a \in \mathcal{S}$ that solves the Skorokhod reflection problem $\boldsymbol{SP}^{\xi}_{a+}$ defined by*

$$\text{find } \nu \in \mathcal{S} \text{ s.t.} \begin{cases} \widetilde{X}^{x,\nu,\xi}_t \in [a,\overline{x}), \mathsf{P}\text{-a.s. for } 0 < t \le \sigma_{\mathcal{I}}, \\ \int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbb{1}_{\{\widetilde{X}^{x,\nu,\xi}_t > a\}} d\nu_t = 0, \mathsf{P}\text{-a.s. for any } T > 0. \end{cases} \tag{$\mathbf{SP}^{\xi}_{a+}$}$$

Similarly, for any $b \in \mathcal{I}$, $x \in \mathcal{I}$ and $\nu \in \mathcal{S}$ there exists a unique $\xi^b \in \mathcal{S}$ that solves the Skorokhod reflection problem $\boldsymbol{SP}^{\nu}_{b-}$ defined by

$$\text{find } \xi \in \mathcal{S} \text{ s.t.} \begin{cases} \widetilde{X}^{x,\nu,\xi}_t \in (\underline{x},b], \mathsf{P}\text{-a.s. for } 0 < t \le \sigma_{\mathcal{I}}, \\ \int_0^{T\wedge\sigma_{\mathcal{I}}} \mathbb{1}_{\{\widetilde{X}^{x,\nu,\xi}_t < b\}} d\xi_t = 0, \mathsf{P}\text{-a.s. for any } T > 0. \end{cases} \tag{$\mathbf{SP}^{\nu}_{b-}$}$$

The proof of the above lemma is based on a Picard iteration scheme. Although this derivation seems to be standard we could not find a precise reference for our particular setting, and we provide a short proof in Appendix A.2.

2.2 The game of controls

We introduce a 2-player nonzero-sum game of singular control, where player 1 (resp. player 2) can influence the dynamics (2.1) by exerting the control ν (resp. ξ). The game has the following structure: if player 1 uses a unit of control at time $t > 0$ pays a cost $G_1(\widetilde{X}^{\nu,\xi}_t)$ and player 2 receives a reward $L_2(\widetilde{X}^{\nu,\xi}_t)$. A symmetric situation occurs if player 2 exerts control. Both players want to maximise their own expected discounted reward functional Ψ_i defined by

$$\Psi_1(x;\nu,\xi) := \mathsf{E}\Big[\int_0^{\sigma_{\mathcal{I}}} e^{-rt} L_1(\widetilde{X}^{x,\nu,\xi}_t)\ominus d\xi_t - \int_0^{\sigma_{\mathcal{I}}} e^{-rt} G_1(\widetilde{X}^{x,\nu,\xi}_t)\oplus d\nu_t\Big], \tag{2.11}$$

$$\Psi_2(x;\nu,\xi) := \mathsf{E}\Big[\int_0^{\sigma_{\mathcal{I}}} e^{-rt} L_2(\widetilde{X}^{x,\nu,\xi}_t)\oplus d\nu_t - \int_0^{\sigma_{\mathcal{I}}} e^{-rt} G_2(\widetilde{X}^{x,\nu,\xi}_t)\ominus d\xi_t\Big], \tag{2.12}$$

where $r > 0$ is the discount rate and the integrals are defined below.

A definition of the integrals with respect to the controls in presence of state dependent costs requires some attention because simultaneous jumps of ξ and ν may be difficult to handle. An extended discussion on this matter is provided in Appendix A.1. Here we consider the class of admissible strategies (see Remark 2.8 below)

$$\mathcal{M} := \{(\nu,\xi)\in\mathcal{S}\times\mathcal{S} \,:\, \mathsf{P}_x(\Delta\nu_t\cdot\Delta\xi_t > 0) = 0 \text{ for all } t\geq 0 \text{ and } x\in\mathcal{I}\}. \tag{2.13}$$

Following [39] (see also [30, 31] among others) we define the discounted costs of controls by

$$\int_0^T e^{-rt} g(\widetilde{X}^{x,\nu,\xi}_t)\ominus d\xi_t = \int_0^T e^{-rt} g(\widetilde{X}^{x,\nu,\xi}_t) d\xi^c_t + \sum_{t<T}\int_0^{\Delta\xi_t} g(\widetilde{X}^{x,\nu,\xi}_t - z)dz\,, \tag{2.14}$$

$$\int_0^T e^{-rt} g(\widetilde{X}^{x,\nu,\xi}_t)\oplus d\nu_t = \int_0^T e^{-rt} g(\widetilde{X}^{x,\nu,\xi}_t) d\nu^c_t + \sum_{t<T}\int_0^{\Delta\nu_t} g(\widetilde{X}^{x,\nu,\xi}_t + z)dz\,, \tag{2.15}$$

for $T > 0$, $(\nu,\xi)\in\mathcal{M}$ and for any function g such that the integrals are well defined.

Throughout the paper we take functions G_i and L_i satisfying

Assumption 2.6. *G_i, $L_i : \overline{\mathcal{I}} \to \mathbb{R}\cup\{\pm\infty\}$, with $L_i < G_i$ on $\mathcal{I}$ and with $G_i \in C^1(\mathcal{I})$ and $L_i \in C(\mathcal{I})$. Moreover the following asymptotic behaviours hold*

$$\limsup_{x\to\underline{x}}\Big|\frac{G_i}{\phi}\Big|(x) = 0 \quad \textit{and} \quad \limsup_{x\to\overline{x}}\Big|\frac{G_i}{\psi}\Big|(x) = 0.$$

Nash equilibria for the game are defined in the following way.

Definition 2.7. *For $x\in\mathcal{I}$ we say that a couple $(\nu^*,\xi^*)\in\mathcal{M}$ is a Nash equilibrium if and only if*

$$\big|\Psi_i(x;\nu^*,\xi^*)\big| < +\infty, \quad i = 1,2,$$

and

$$\begin{cases} \Psi_1(x;\nu^*,\xi^*) \geq \Psi_1(x;\nu,\xi^*) & \textit{for any } \nu\in\mathcal{S} \textit{ s.t. } (\nu,\xi^*)\in\mathcal{M}, \\ \Psi_2(x;\nu^*,\xi^*) \geq \Psi_2(x;\nu^*,\xi) & \textit{for any } \xi\in\mathcal{S} \textit{ s.t. } (\nu^*,\xi)\in\mathcal{M}. \end{cases} \tag{2.16}$$

We also say that $V_i(x) := \Psi_i(x;\nu^*,\xi^*)$ is the value of the game for the i-th player relative to the equilibrium.

Remark 2.8. *In several problems of interest for applications, the functionals* (2.11) *and* (2.12) *may be rewritten as the sum of three terms: the integral in time of a state dependent running profit plus two integrals with respect to the controls with constant instantaneous costs (see, e.g., [13], [19] and [33] for similar functionals in the case of single agent optimisation problems). In such cases condition* (2.13) *on the admissible strategies is not needed. In fact we show in Appendix A.1 that if at least one player picks a control that reflects the process at a fixed boundary (i.e. solving one of the problems in Lemma 2.5) then the other player has no incentives in picking strategies outside of the class $\mathcal{M}$.*

Remark 2.9. *Nash equilibria could in principle exist in broader sets than $\mathcal{M}$. However this fact does not* per se *add useful information. In fact unless some additional optimality criterion is introduced (for example maximisation of the total profit of the two players) it is often impossible to rank multiple equilibria according to the players' individual preferences. In this paper we content ourselves with equilibria in $\mathcal{M}$ as these lead to explicit solutions and to the desired connection between OS and SSC.*

2.3 The game of stopping

In this section we introduce a 2-player nonzero-sum game of stopping where the underlying process is X^x as in (2.5). This is the game which we show is linked to the game of control introduced in the previous section.

Denote by $\mathcal{T}$ the set of $\mathbb{F}$-stopping times. The i-th player chooses a $\tau_i \in \mathcal{T}$ with the aim of minimising an expected cost functional $\mathcal{J}_i(x;\tau_1,\tau_2)$, and the game ends at $\tau_1 \wedge \tau_2$. This game has payoffs of immediate stopping given by the marginal costs of control G_i and L_i appearing in the functionals (2.11) and (2.12) of the game of control. More precisely for $i = 1, 2$ and $j \neq i$ we set

$$\mathcal{J}_i(\tau_1,\tau_2;x) := \mathsf{E}\Big[e^{-\int_0^{\tau_i}(r-\mu'(X_s^x))ds}G_i(X^x_{\tau_i})\mathbb{1}_{\{\tau_i<\tau_j\}} + e^{-\int_0^{\tau_j}(r-\mu'(X_s^x))ds}L_i(X^x_{\tau_j})\mathbb{1}_{\{\tau_i\geq\tau_j\}}\Big]. \quad (2.17)$$

As in the case of the game of controls also here we introduce the notion of Nash equilibrium.

Definition 2.10. *For $x \in \mathcal{I}$ we say that a couple $(\tau_1^*,\tau_2^*) \in \mathcal{T}\times\mathcal{T}$ is a Nash equilibrium if and only if*

$$\big|\mathcal{J}_i(x;\tau_1^*,\tau_2^*)\big| < +\infty, \quad i = 1, 2$$

and

$$\begin{cases} \mathcal{J}_1(\tau_1^*,\tau_2^*;x) \leq \mathcal{J}_1(\tau_1,\tau_2^*;x), & \forall\, \tau_1 \in \mathcal{T}, \\ \mathcal{J}_2(\tau_1^*,\tau_2^*;x) \leq \mathcal{J}_2(\tau_1^*,\tau_2;x), & \forall\, \tau_2 \in \mathcal{T}. \end{cases} \quad (2.18)$$

We also say that $v_i(x) := \mathcal{J}_i(\tau_1^*,\tau_2^*;x)$ is the value of the game for the i-th player relative to the equilibrium.

3 The main result

Here we prove the key result of the paper, i.e. a differential link between the value functions v_i, $i = 1, 2$ relative to Nash equilibria in the game of stopping and the value functions V_i, $i = 1, 2$ relative to Nash equilibria in the game of control. The result holds when the equilibrium stopping times for X are hitting times to suitable thresholds so that the related optimally controlled $\widetilde{X}$ is reflected at such thresholds.

Our main theorem relies on assumptions regarding the existence of a Nash equilibrium in the game of stopping and suitable properties of the associated values v_1 and v_2. It was shown in [16] that such requirements hold in a broad class of examples and we will summarise results of [16] in Proposition 3.5 below, for completeness.

Theorem 3.1. *Suppose there exist a_*, b_* with $\underline{x} < a_* < b_* < \overline{x}$ such that the following conditions hold:*

(a) The stopping times

$$\tau_1^* := \inf\{t > 0 \,:\, X_t^x \leq a_*\}, \qquad \tau_2^* := \inf\{t > 0 \,:\, X_t^x \geq b_*\} \quad (3.1)$$

form a Nash equilibrium for the game of stopping as in Definition 2.10;

(b) The value functions $v_i(x) := \mathcal{J}_i(x;\tau_1^*,\tau_2^*)$, $i=1,2$ *are such that* $v_i \in C(\mathcal{I})$, $i=1,2$ *with* $v_1 \in W^{2,\infty}_{loc}(\underline{x},b_*)$ *and* $v_2 \in W^{2,\infty}_{loc}(a_*,\overline{x})$ *(hence in particular* $v_1 \in C^1(\underline{x},b_*)$ *and* $v_2 \in C^1(a_*,\overline{x})$ *by Sobolev embedding [10, Ch. 9, Cor. 9.15]);*

(c) $v_1 = G_1$ *in* $(\underline{x},a_*]$, $v_1 = L_1$ *in* $[b_*,\overline{x})$ *and* $v_2 = G_2$ *in* $[b_*,\overline{x})$, $v_2 = L_2$ *in* $(\underline{x},a_*]$. *Moreover they solve the boundary value problem*

$$\big(\mathcal{L}_X v_i - (r-\mu')v_i\big)(x) = 0, \qquad a_* < x < b_*,\ i=1,2 \tag{3.2}$$

$$\big(\mathcal{L}_X v_1 - (r-\mu')v_1\big)(x) \geq 0, \qquad \underline{x} < x \leq a_* \tag{3.3}$$

$$\big(\mathcal{L}_X v_2 - (r-\mu')v_2\big)(x) \geq 0, \qquad b_* \leq x < \overline{x} \tag{3.4}$$

$$v_i \leq G_i, \qquad x \in \mathcal{I},\ i=1,2. \tag{3.5}$$

Then the couple of controls (ν^{a_*},ξ^{b_*}) *which uniquely solves Problem* $\boldsymbol{SP}(a_*,b_*)$ *forms a Nash equilibrium for the game of control as in Definition 2.7 and the value functions* $V_i(x) = \Psi_i(x;\nu^{a_*},\xi^{b_*})$, $i=1,2$ *are given by*

$$V_1(x) = \kappa_1 + \int_{a_*}^{x} v_1(z)dz, \qquad x \in \mathcal{I}, \tag{3.6}$$

$$V_2(x) = \kappa_2 + \int_{x}^{b_*} v_2(z)dz, \qquad x \in \mathcal{I}, \tag{3.7}$$

with

$$\kappa_1 := \frac{1}{r}\Big(\frac{\sigma^2}{2}\,G_1' + \mu\,G_1\Big)(a_*), \qquad \kappa_2 := -\frac{1}{r}\Big(\frac{\sigma^2}{2}\,G_2' + \mu\,G_2\Big)(b_*). \tag{3.8}$$

Proof. The proof is by direct check and it is performed in two steps.

Step 1. The functions

$$u_1(x) = \kappa_1 + \int_{a_*}^{x} v_1(z)dz, \qquad x \in \mathcal{I}, \tag{3.9}$$

$$u_2(x) = \kappa_2 + \int_{x}^{b_*} v_2(z)dz, \qquad x \in \mathcal{I}, \tag{3.10}$$

with κ_1 and κ_2 as in (3.8), are C^1 on $\mathcal{I}$ (by continuity of G_i and L_i on $\mathcal{I}$) with $u_1 \in C^2(\underline{x},b_*)$ since $v_1 \in C^1(\underline{x},b_*)$, and $u_2 \in C^2(a_*,\overline{x})$ since $v_2 \in C^1(a_*,\overline{x})$. We now show that u_1, u_2 and the boundaries a_*, b_* solve the system of coupled variational problems

$$\begin{cases} (\mathcal{L}_{\widetilde{X}} u_1 - ru_1)(x) = 0, & x \in (a_*,b_*) \\ (\mathcal{L}_{\widetilde{X}} u_1 - ru_1)(x) \leq 0, & x \in (\underline{x},b_*) \\ u_1'(x) \leq G_1(x), & x \in (\underline{x},b_*) \\ u_1'(x) = G_1(x), & x \in (\underline{x},a_*) \\ u_1'(x) = L_1(x), & x \in (b_*,\overline{x}) \end{cases} \tag{3.11}$$

and

$$\begin{cases} (\mathcal{L}_{\widetilde{X}} u_2 - ru_2)(x) = 0, & x \in (a_*,b_*) \\ (\mathcal{L}_{\widetilde{X}} u_2 - ru_2)(x) \leq 0, & x \in (a_*,\overline{x}) \\ u_2'(x) \geq -G_2(x), & x \in (a_*,\overline{x}) \\ u_2'(x) = -G_2(x), & x \in (b_*,\overline{x}) \\ u_2'(x) = -L_2(x), & x \in (\underline{x},a_*). \end{cases} \tag{3.12}$$

We will only give details about the derivation of (3.12) as the ones for (3.11) are analogous. The last three properties in (3.12) follow by observing that $u_2' = -v_2$ and by using $v_2 = G_2$ in $[b_*, \overline{x})$, $v_2 = L_2$ in $(\underline{x}, a_*]$ and (3.5) (cf. (c) in the statement of the theorem). For the first equation in (3.12) we use that

$$(\mathcal{L}_{\widetilde{X}} u_2 - r u_2)(x) = -\frac{\sigma^2(x)}{2} v_2'(x) - \mu(x) v_2(x) - r\kappa_2 - \int_x^{b_*} r v_2(z) dz \tag{3.13}$$

and that for $x \in (a_*, b_*)$

$$\int_x^{b_*} r v_2(z) dz = \int_x^{b_*} \big(\mathcal{L}_X v_2(z) + \mu'(z) v_2(z)\big) dz \tag{3.14}$$

by (3.2). Integrating by parts the latter, using $v_2(b_*) = G_2(b_*)$ and $v_2'(b_*) = G_2'(b_*)$ and substituting the result back into (3.13), the right-hand side of (3.13) equals zero upon recalling the definition of κ_2 (see (3.8)). Finally to prove the second line in (3.12) it is enough to notice that for $x \in [b_*, \overline{x})$

$$\int_x^{b_*} r v_2(z) dz \geq \int_x^{b_*} \big(\mathcal{L}_X v_2(z) + \mu'(z) v_2(z)\big) dz \tag{3.15}$$

by (3.4) and then argue as before.

Step 2. We now proceed to a verification argument to show that $u_i = V_i$, $i = 1, 2$ and that the controls (ν^{a_*}, ξ^{b_*}) form a Nash equilibrium. To simplify the notation we set $\nu^* := \nu^{a_*}$ and $\xi^* := \xi^{b_*}$. Since (ν^*, ξ^*) solves $\mathbf{SP}(a_*, b_*)$, it clearly belongs to $\mathcal{M}$. We provide again full details only for u_2 as the proof follows in the same way for u_1.

First we show that $u_2 \geq V_2$. Let $\xi \in \mathcal{S}$ be such that $(\nu^*, \xi) \in \mathcal{M}$. Since $u_2 \in C^2(a_*, \overline{x})$ and $\widetilde{X}^{x,\nu^*,\xi}_t \geq a_*$ for all $t > 0$ we can apply Itô-Meyer's formula up to a localising sequence of stopping times. The integral with respect to the continuous part of the bounded variation process $\nu^* - \xi$ is the difference of the integrals with respect to $d\nu^{*,c}$ and $d\xi^c$. For $x \in \mathcal{I}$ we obtain

$$\begin{aligned} u_2(x) =& e^{-r\theta_y} u_2(\widetilde{X}^{x,\nu^*,\xi}_{\theta_y}) - \int_0^{\theta_y} e^{-rs} (\mathcal{L}_{\widetilde{X}} - r) u_2(\widetilde{X}^{x,\nu^*,\xi}_s) ds - M_{\theta_y} \\ & - \int_0^{\theta_y} e^{-rs} u_2'(\widetilde{X}^{x,\nu^*,\xi}_s) d\nu^{*,c}_s + \int_0^{\theta_y} e^{-rs} u_2'(\widetilde{X}^{x,\nu^*,\xi}_s) d\xi^c_s \\ & - \sum_{s<\theta_y} e^{-rs} \big(u_2(\widetilde{X}^{x,\nu^*,\xi}_{s+}) - u_2(\widetilde{X}^{x,\nu^*,\xi}_s) \big), \end{aligned} \tag{3.16}$$

where M is

$$M_t := \int_0^t e^{-rs} \sigma(\widetilde{X}^{x,\nu^*,\xi}_s) u_2'(\widetilde{X}^{x,\nu^*,\xi}_s) d\widetilde{W}_s \tag{3.17}$$

and θ_y is the stopping time

$$\theta_y := \inf\{u > 0 \,:\, \widetilde{X}^{x,\nu^*,0}_u \geq y\}, \quad \text{for}\, y > b_*. \tag{3.18}$$

Notice that for any $t \in (0, \theta_y]$ we have $a_* \leq \widetilde{X}^{x,\nu^*,\xi}_t \leq \widetilde{X}^{x,\nu^*,0}_t \leq y$, hence continuity of σ and of u_2' imply that $(M_t)_{t \leq \theta_y}$ is a martingale.

Since $(\nu^*, \xi) \in \mathcal{M}$, the process $\widetilde{X}^{x,\nu^*,\xi}$ is left-continuous and we have

$$\begin{aligned} & \sum_{s<\theta_y} e^{-rs} \big(u_2(\widetilde{X}^{x,\nu^*,\xi}_{s+}) - u_2(\widetilde{X}^{x,\nu^*,\xi}_s) \big) \\ & = \sum_{s<\theta_y} e^{-rs} \big(u_2(\widetilde{X}^{x,\nu^*,\xi}_{s+}) - u_2(\widetilde{X}^{x,\nu^*,\xi}_s) \big) \big[\mathbb{1}_{\{\Delta\nu^*_s > 0\}} + \mathbb{1}_{\{\Delta\xi_s > 0\}} \big], \end{aligned} \tag{3.19}$$

where

$$\sum_{s<\theta_y} e^{-rs}\big(u_2(\widetilde{X}^{x,\nu^*,\xi}_{s+}) - u_2(\widetilde{X}^{x,\nu^*,\xi}_{s})\big)\mathbb{1}_{\{\Delta\nu^*_s>0\}} = \sum_{s<\theta_y} e^{-rs}\int_0^{\Delta\nu^*_s} u_2'(\widetilde{X}^{x,\nu^*,\xi}_{s}+z)dz,$$

$$\sum_{s<\theta_y} e^{-rs}\big(u_2(\widetilde{X}^{x,\nu^*,\xi}_{s+}) - u_2(\widetilde{X}^{x,\nu^*,\xi}_{s})\big)\mathbb{1}_{\{\Delta\xi_s>0\}} = -\sum_{s<\theta_y} e^{-rs}\int_0^{\Delta\xi_s} u_2'(\widetilde{X}^{x,\nu^*,\xi}_{s}-z)dz.$$

Hence (3.16) may be written in a more compact form as (cf. (2.14), (2.15))

$$\begin{aligned} u_2(x) =& e^{-r\theta_y}u_2(\widetilde{X}^{x,\nu^*,\xi}_{\theta_y}) - \int_0^{\theta_y} e^{-rs}(\mathcal{L}_{\widetilde{X}}-r)u_2(\widetilde{X}^{x,\nu^*,\xi}_{s})ds - M_{\theta_y} \\ &- \int_0^{\theta_y} e^{-rs}u_2'(\widetilde{X}^{x,\nu^*,\xi}_{s})\oplus d\nu^*_s + \int_0^{\theta_y} e^{-rs}u_2'(\widetilde{X}^{x,\nu^*,\xi}_{s})\ominus d\xi_s. \end{aligned} \tag{3.20}$$

Now, using the fact that $u_2' \geq -G_2$ on $\mathcal{I}$ and that $u_2'(\widetilde{X}^{x,\nu^*,\xi}_{s}) = -L_2(\widetilde{X}^{x,\nu^*,\xi}_{s})$ for all s in the support of $d\nu^*_s$ (i.e. for all $s \geq 0$ s.t. $\widetilde{X}^{x,\nu^*,\xi}_{s} \leq a_*$), and employing the second expression in (3.12) jointly with the fact that $\widetilde{X}^{x,\nu^*,\xi}_{s} \geq a_*$ for $s > 0$ we get

$$\begin{aligned} u_2(x) \geq& e^{-r\theta_y}u_2(\widetilde{X}^{x,\nu^*,\xi}_{\theta_y}) - M_{\theta_y} \\ &+ \int_0^{\theta_y} e^{-rs}L_2(\widetilde{X}^{x,\nu^*,\xi}_{s})\oplus d\nu^*_s - \int_0^{\theta_y} e^{-rs}G_2(\widetilde{X}^{x,\nu^*,\xi}_{s})\ominus d\xi_s. \end{aligned} \tag{3.21}$$

By taking expectations we end up with

$$u_2(x) \geq \mathsf{E}_x\Big[e^{-r\theta_y}u_2(\widetilde{X}^{\nu^*,\xi}_{\theta_y}) + \int_0^{\theta_y} e^{-rs}L_2(\widetilde{X}^{\nu^*,\xi}_{s})\oplus d\nu^*_s - \int_0^{\theta_y} e^{-rs}G_2(\widetilde{X}^{\nu^*,\xi}_{s})\ominus d\xi_s\Big]. \tag{3.22}$$

We aim at taking limits as $y \to \overline{x}$ in (3.22) and we preliminarily notice that $\theta_y \uparrow \sigma_{\mathcal{I}}$ as $y \to \overline{x}$, P_x-a.s.

(i) By (3.10) it is easy to see that

$$\begin{aligned} |u_2(\widetilde{X}^{\nu^*,\xi}_{\theta_y})| \leq& \kappa_2 + \int_{a_*}^{b_*} |v_2(z)|dz + \int_{b_*}^{b_*\vee \widetilde{X}^{\nu^*,\xi}_{\theta_y}} |G_2(z)|dz \\ \leq& C_2 + \int_{b_*}^{b_*\vee \widetilde{X}^{\nu^*,0}_{\theta_y}} |G_2(z)|dz \leq C_2 + \int_{b_*}^{y} |G_2(z)|dz, \end{aligned}$$

for some $C_2 > 0$, and where we have used $v_2 = G_2$ on $[b_*, \overline{x})$ and $\widetilde{X}^{\nu^*,\xi}_{\theta_y} \leq \widetilde{X}^{\nu^*,0}_{\theta_y} \leq y$ P_x-a.s. Hence we have

$$\mathsf{E}_x\big[e^{-r\theta_y}u_2(\widetilde{X}^{\nu^*,\xi}_{\theta_y})\big] \geq -\mathsf{E}_x\big[e^{-r\theta_y}\big]\Big(C_2 + \int_{b_*}^{y}|G_2(z)|dz\Big). \tag{3.23}$$

Lemma A.2 in Appendix guarantees

$$\limsup_{y\uparrow\overline{x}} \mathsf{E}_x\big[e^{-r\theta_y}\big]\Big(C_2 + \int_{b_*}^{y}|G_2(z)|dz\Big) \leq 0, \tag{3.24}$$

so that (3.23) yields

$$\liminf_{y\uparrow\overline{x}} \mathsf{E}_x\big[e^{-r\theta_y}u_2(\widetilde{X}^{\nu^*,\xi}_{\theta_y})\big] \geq 0. \tag{3.25}$$

(ii) Denoting by h^+ (resp. h^-) the positive (resp. negative) part of any Borel-measurable function h, the integrals with respect to the controls can be rewritten as

$$\begin{aligned}&\mathsf{E}_x\Big[\int_0^{\theta_y} e^{-rs}L_2(\widetilde{X}^{\nu^*,\xi}_s)\oplus d\nu^*_s-\int_0^{\theta_y} e^{-rs}G_2(\widetilde{X}^{\nu^*,\xi}_s)\ominus d\xi_s\Big]\\&=\mathsf{E}_x\Big[\int_0^{\theta_y} e^{-rs}(L_2)^+(\widetilde{X}^{\nu^*,\xi}_s)\oplus d\nu^*_s-\int_0^{\theta_y} e^{-rs}(G_2)^+(\widetilde{X}^{\nu^*,\xi}_s)\ominus d\xi_s\Big]\\&-\mathsf{E}_x\Big[\int_0^{\theta_y} e^{-rs}(L_2)^-(\widetilde{X}^{\nu^*,\xi}_s)\oplus d\nu^*_s-\int_0^{\theta_y} e^{-rs}(G_2)^-(\widetilde{X}^{\nu^*,\xi}_s)\ominus d\xi_s\Big].\end{aligned}$$

Recall that $\theta_y \uparrow \sigma_{\mathcal{I}}$ as $y \uparrow \infty$ and apply monotone convergence theorem to each term in the right hand-side of the above expression to obtain

$$\begin{aligned}&\lim_{y\to\overline{x}}\mathsf{E}_x\Big[\int_0^{\theta_y} e^{-rs}L_2(\widetilde{X}^{\nu^*,\xi}_s)\oplus d\nu^*_s-\int_0^{\theta_y} e^{-rs}G_2(\widetilde{X}^{\nu^*,\xi}_s)\ominus d\xi_s\Big]\\&=\mathsf{E}_x\Big[\int_0^{\sigma_{\mathcal{I}}} e^{-rs}L_2(\widetilde{X}^{\nu^*,\xi}_s)\oplus d\nu^*_s-\int_0^{\sigma_{\mathcal{I}}} e^{-rs}G_2(\widetilde{X}^{\nu^*,\xi}_s)\ominus d\xi_s\Big].\end{aligned}$$

Finally we combine items (i) and (ii) and take limits in (3.22) as $y \to \overline{x}$ to get

$$u_2(x)\geq\mathsf{E}_x\Big[\int_0^{\sigma_{\mathcal{I}}} e^{-rs}L_2(\widetilde{X}^{\nu^*,\xi}_s)\oplus d\nu^*_s-\int_0^{\sigma_{\mathcal{I}}} e^{-rs}G_2(\widetilde{X}^{\nu^*,\xi}_s)\ominus d\xi_s\Big]. \tag{3.26}$$

Hence $u_2(x) \geq \Psi_2(x;\nu^*,\xi)$ for any $\xi \in \mathcal{S}$ such that $(\nu^*,\xi) \in \mathcal{M}$.

Now repeating the steps above with $\xi = \xi^*$, the inequalities in (3.21) and (3.22) become strict equalities due to the fact that $\widetilde{X}^{x,\nu^*,\xi^*}_t \in [a_*, b_*]$ for all $t > 0$ and $u_2'(\widetilde{X}^{x,\nu^*,\xi^*}_t) = -G_2(\widetilde{X}^{x,\nu^*,\xi^*}_t)$ on $\text{supp}\{d\xi^*_t\}$. Hence passing to the limit as $y \to \overline{x}$ dominated convergence theorem gives (3.25) so that $u_2(x) = \Psi(x;\nu^*,\xi^*) = V_2(x)$. □

Remark 3.2. *From the game-theoretic point of view, Nash equilibria of Theorem 3.1 above are Markov perfect [32] (also called Nash equilibria in closed-loop strategies), i.e. equilibria in which players' actions only depend on the "payoff-relevant" state variable $\widetilde{X}$. Our result provides a simple construction of closed-loop Nash equilibria for specific continuous time stochastic games of singular control. Since this problem is yet to be solved in game theory in its full generality (see the discussion in Section 2 of [2] and in [35]), our work contributes to fill this gap.*

3.1 On the assumptions of Theorem 3.1.

In this section we give sufficient conditions under which a_* and b_* as in Theorem 3.1 exist. Moreover in Remark 3.6 we provide algebraic equations for a_* and b_* which can be solved at least numerically. Recall ϕ and ψ, i.e. the fundamental decreasing and increasing solutions to (2.9), and recall that $r > \mu'(x)$ for $x \in \overline{\mathcal{I}}$ by Asssumption 2.3. We need the following set of functions:

Definition 3.3. *Let $\mathcal{A}$ be the class of real valued functions $H \in C^2(\mathcal{I})$ such that*

$$\limsup_{x\to\underline{x}}\Big|\frac{H}{\phi}\Big|(x)=0,\quad \limsup_{x\to\overline{x}}\Big|\frac{H}{\psi}\Big|(x)=0 \tag{3.27}$$

$$\text{and}\quad \mathsf{E}_x\bigg[\int_0^{\sigma_{\mathcal{I}}} e^{-\int_0^t(r-\mu'(X_s))ds}\big|h(X_t)\big|dt\bigg]<\infty \tag{3.28}$$

for all $x \in \mathcal{I}$ and with $h(x) := (\mathcal{L}_X H - (r-\mu')H)(x)$. We denote by $\mathcal{A}_1$ (respectively $\mathcal{A}_2$) the set of all $H \in \mathcal{A}$ such that h is strictly positive (resp. negative) on $(\underline{x}, x_h)$ and strictly negative (resp. positive) on $(x_h, \overline{x})$ for some $x_h \in \mathcal{I}$ with $\liminf_{x\to\underline{x}} h(x) > 0$ (resp. $\limsup_{x\to\underline{x}} h(x) < 0$) and $\limsup_{x\to\overline{x}} h(x) < 0$ (resp. $\liminf_{x\to\overline{x}} h(x) > 0$).

We also need the following

Assumption 3.4. *For $i = 1, 2$, it holds $G_i \in \mathcal{A}_i$ with $\inf_{x\in\mathcal{I}} G_i(x) < 0$ and*

$$\limsup_{x\to\underline{x}} \Big|\frac{L_i}{\phi}\Big|(x) < +\infty \quad \textit{and} \quad \limsup_{x\to\overline{x}} \Big|\frac{L_i}{\psi}\Big|(x) < +\infty. \tag{3.29}$$

Moreover letting $\hat{x}_1$ and $\hat{x}_2$ in $\mathcal{I}$ be such that

$$\{x : (\mathcal{L}_X G_1 - (r-\mu')G_1)(x) > 0\} = (\underline{x}, \hat{x}_1), \tag{3.30}$$
$$\{x : (\mathcal{L}_X G_2 - (r-\mu')G_2)(x) > 0\} = (\hat{x}_2, \overline{x}), \tag{3.31}$$

we assume $\hat{x}_1 < \hat{x}_2$.

The above conditions have a simple interpretation for the game of stopping (see the introduction of [16]): payoffs G_i which are negative for at least some values of x guarantee that stopping in finite time is incentivised relative to waiting indefinitely; the fact that $\hat{x}_1 < \hat{x}_2$ implies that for any value of the process X at least one player has a running benefit from waiting.

The proofs of the next two propositions are given in Appendix A.2. In their statements we denote

$$\vartheta_i(x) := \frac{G_i'(x)\phi(x) - G_i(x)\phi'(x)}{w\, S'(x)}, \qquad i = 1, 2, \tag{3.32}$$

with $w > 0$ as in (2.10).

Proposition 3.5. *Let Assumptions 2.6 and 3.4 hold, then each one of the conditions below is sufficient for the existence of a_* and b_* fulfilling (a), (b) and (c) of Theorem 3.1:*

1. *$\underline{x}$ and $\overline{x}$ are natural boundaries for $(X_t)_{t\ge0}$.*
2. *$\underline{x}$ is an entrance boundary and $\overline{x}$ is a natural boundary for $(X_t)_{t\ge0}$; moreover the following hold*
   - *(2.i)* $\vartheta_1(\underline{x}+) := \lim_{x\downarrow\underline{x}} \vartheta_1(x) < (L_1/\psi)(x_2^\infty)$*, where x_2^∞ uniquely solves $\vartheta_2(x) = (G_2/\psi)(x)$ in $(\hat{x}_2, \overline{x})$;*
   - *(2.ii)* $\sup\{x > \underline{x} : L_1(x) = \vartheta_1(\underline{x}+)\psi(x)\} \le \hat{x}_2$*;*
   - *(2.iii)* $\lim_{x\uparrow\overline{x}}(L_1/\phi)(x) > -\infty$.

Remark 3.6. *An important byproduct of our connection between nonzero-sum games of control and nonzero-sum games of stopping is that the equilibrium thresholds a_* and b_* of Theorem 3.1 are a solution of a system of algebraic equations which can be computed at least numerically. In the terminology of singular control theory these equations correspond to the smooth-fit conditions $V_1''(a_*+) = G_1'(a_*)$ and $V_2''(b_*-) = -G_2'(b_*)$ and were obtained via a geometric constructive approach in [16] (see Theorem 3.4). We recall the system here for completeness*

$$\begin{cases} \dfrac{G_1}{\phi}(a_*) - \dfrac{L_1}{\phi}(b_*) - \vartheta_1(a_*)\Big(\dfrac{\psi}{\phi}(a_*) - \dfrac{\psi}{\phi}(b_*)\Big) = 0\,, \\ \dfrac{G_2}{\phi}(b_*) - \dfrac{L_2}{\phi}(a_*) - \vartheta_1(b_*)\Big(\dfrac{\psi}{\phi}(b_*) - \dfrac{\psi}{\phi}(a_*)\Big) = 0\,, \end{cases} \tag{3.33}$$

where $a_* < \hat{x}_1$ and $b_* > \hat{x}_2$.

Uniqueness of the solution to (3.33) *is discussed in [16, Thm. 3.7].*

4 A game of pollution control

In order to understand the nature of our Assumptions 2.6 and 3.4 and illustrate an application of our results we present here a game version of a pollution control problem.

A social planner wants to keep the level of pollution low while the productive sector of the economy (modeled as a single representative firm) wants to increase production capacity. If we assume that the pollution level is proportional to the firm's production capacity (see for example [23, 38]) then the problem translates into a game of capacity expansion. Indeed the representative firm aims at maximising profits by investing to increase the production level, whereas the social planner aims at keeping the pollution level under control through environmental regulations which effectively cap the maximum production rate.

For the production capacity we consider a controlled geometric Brownian motion as in [12, 13, 19], amongst others,

$$d\widetilde{X}^{\nu,\xi}_t = \mu\widetilde{X}^{\nu,\xi}_t dt + \sigma\widetilde{X}^{\nu,\xi}_t d\widetilde{W}_t + d\nu_t - d\xi_t, \quad \widetilde{X}^{\nu,\xi}_0 = x \in \mathbb{R}_+, \tag{4.1}$$

for some $\mu \in \mathbb{R}$ and $\sigma > 0$. The firm has running operating profit $\pi(x)$, which is C^1 and strictly concave and a positive cost per unit of investment $\alpha_1(x)$. The social planner has an instantaneous utility function $u(x)$ which is C^1, decreasing and strictly concave[2]. Since imposing a reduction of production might also have some negative impact on social welfare (e.g., it might cause an increase in the level of unemployment) we introduce a positive 'cost' (in terms of the expected total utility) associated to the social planner's policies and we denote it by $\alpha_2(x)$. For simplicity here we assume $\alpha_i(x) \equiv \alpha_i > 0$, $i = 1, 2$, and the objective functionals for the firm, denoted by Ψ_1, and the social planner, denoted by Ψ_2, are given by

$$\Psi_1(x;\nu,\xi) := \mathsf{E}_x\Big[\int_0^{\sigma_\mathcal{I}} e^{-rt}\pi(\widetilde{X}^{\nu,\xi}_t)dt - \alpha_1\int_0^{\sigma_\mathcal{I}} e^{-rt}d\nu_t\Big], \tag{4.2}$$

$$\Psi_2(x;\nu,\xi) := \mathsf{E}_x\Big[\int_0^{\sigma_\mathcal{I}} e^{-rt}u(\widetilde{X}^{\nu,\xi}_t)dt - \alpha_2\int_0^{\sigma_\mathcal{I}} e^{-rt}d\xi_t\Big]. \tag{4.3}$$

Both players want to maximise their respective functional by picking admissible strategies from $\mathcal{M}$. As explained in Lemma A.1 below, in this context there is no loss of generality for our scopes in considering $\mathcal{M}$ rather than $\mathcal{S}\times\mathcal{S}$.

The game with functionals (4.2)–(4.3) could be tackled directly with the same methods developed in the previous sections. This however would require some lengthy repetitions on the side of the OS game to account for the running profit/utility terms. Nevertheless results of [16] would still apply in this setting. To avoid such repetitions we prefer to reduce (4.2)–(4.3) to our original formulation and for that it is indeed convenient to deal with $\mathcal{M}$.

Here $\mathcal{I} = \mathbb{R}_+$ and we define the functions Π and U via the ODEs

$$(\mathcal{L}_{\widetilde{X}} - r)\Pi(x) = \pi(x), \qquad (\mathcal{L}_{\widetilde{X}} - r)U(x) = u(x) \tag{4.4}$$

with finiteness conditions at zero, $\Pi(0+) < +\infty$ and $U(0+) < +\infty$, and possibly growth conditions at infinity as needed.

Then for any $(\nu,\xi) \in \mathcal{M}$ an application of Itô-Meyer formula shows that finding a Nash equilibrium for the functionals (4.2)–(4.3) is equivalent to finding one for (2.11)–(2.12) with

$$G_1(x) = \alpha_1 + \Pi'(x), \qquad G_2(x) = \alpha_2 - U'(x),$$
$$L_1(x) = \Pi'(x), \qquad L_2(x) = -U'(x).$$

[2]The social planner's utility decreases with increasing pollution levels. Moreover, if the pollution is high the marginal benefit from decreasing it is large, whereas if the pollution is low a further contraction of the economy has very little or no benefit.

It is not hard to verify that

$$\zeta_1(x) := (\mathcal{L}_X G_1 - (r-\mu)G_1)(x) = \pi'(x) - (r-\mu)\alpha_1 \tag{4.5}$$

$$\zeta_2(x) := (\mathcal{L}_X G_2 - (r-\mu)G_2)(x) = -u'(x) - (r-\mu)\alpha_2, \tag{4.6}$$

and we notice that ζ_1 is decreasing by concavity of π whereas ζ_2 is increasing by concavity of u. For instance assuming Inada conditions

$$\lim_{x\to\infty} \pi'(x) = 0, \qquad \lim_{x\to 0} \pi'(x) = +\infty,$$

$$\lim_{x\to\infty} u'(x) = -\infty, \qquad \lim_{x\to 0} u'(x) = 0,$$

we have that (3.30) and (3.31) hold for some $\hat{x}_i$, $i = 1, 2$, which depend on the specific choice of π and u.

A more detailed discussion regarding Assumption 3.4 can be easily addressed in the case of $\pi(x) = x^\lambda$ and $u(x) = -x^\delta$ where $\lambda \in (0, 1)$ and $\delta > 1$. For $r > \mu$ and sufficiently large we can guarantee (3.28). Moreover, denoting by γ_1 (resp. γ_2) the positive (resp. negative) root of the second order equation $\frac{1}{2}\sigma^2\gamma(\gamma-1) + (\mu+\sigma^2)\gamma - (r-\mu) = 0$, conditions (3.27) on G_1 and G_2 are satisfied if $\lambda > \max\{0, \gamma_2 + 1\}$ and $1 < \delta < 1 + \gamma_1$. Clearly (3.29) holds by the same arguments. In all cases G_1 assumes negative values for x in suitable intervals, whereas we guarantee $\inf_x G_2(x) < 0$ by additionally requiring, e.g., $\gamma_1 < \delta < \gamma_1 + 1$. Finally we have

$$\hat{x}_1 = \left(\frac{r\alpha_1}{\lambda}\right)^{-\frac{1}{1-\lambda}}, \qquad \hat{x}_2 = \left(\frac{r\alpha_2}{\delta}\right)^{\frac{1}{\delta-1}},$$

so that a suitable choice of α_1 and α_2 ensures that $\hat{x}_1 < \hat{x}_2$.

A Appendix

A.1 Cost integrals and the set of strategies $\mathcal{M}$

It is well known in the singular stochastic control literature that state dependent instantaneous costs of control give rise to questions concerning the definition of integrals representing the cumulative cost of exercising control.

Zhu in [39] provided a definition consistent with the classical verification argument used in SSC for the solution of an HJB equation derived by the Dynamic Programming Principle. This definition has been adopted in several other papers concerning explicit solutions of SSC problems (see [30, 31] among others), and this is also the one that we use in our (2.14) and (2.15). Another, perhaps more natural, possibility is instead to define the integral as a Riemann-Stieltjes' integral as for example it was done by Alvarez in [1].

Despite this formal difference it is remarkable that the two definitions for the cost of exercising control lead essentially to the same optimal strategies for problems of monotone follower type. In particular it is possible to obtain Zhu's integral from the Riemann-Stieltjes' one by taking the limit as $n \to \infty$ of a sequence of controls that at a given time t make n instantaneous jumps of length h/n for a fixed $h > 0$. The optimality of this behaviour is illustrated for example by Alvarez in Corollary 1 of [1] and it is often referred to as "chattering policy". The inconvenient with this approach is that the control obtained in the limit is not admissible in our $\mathcal{S}$ and therefore optimisers can only be obtained in a larger class.

Zhu's integral has proved to work very well in problems with monotone controls (representing for instance irreversible investments) or with controls of bounded variation (representing for instance partially reversible investment policies). In particular the latter are often chosen in such a way that the controller's decisions to invest/disinvest reflect the minimal decomposition of the control process (cf. [13], [18] and [21], among others). In other words, investment and disinvestment do not occur at the same time, and this assumption is often justified by conditions on the absence of arbitrage opportunities.

Here instead we have agents who use their controls independently and it is unclear why *a priori* they should decide not to contrast each other's moves by acting simultaneously. To elaborate more on this point and understand our choice of the set $\mathcal{M}$, it is convenient to look at particular cases of our problem.

In some instances an application of Itô-Tanaka formula allows to rewrite our functionals (2.11) and (2.12) in terms of equivalent ones with a state-dependent running cost π_i plus constant costs of control α_i, β_i (see our example in Section 4 or problems studied in [13], [19] or [33]). The new functionals read as follows

$$\widehat{\Psi}_1(x;\nu,\xi) := \mathsf{E}\Big[\int_0^{\sigma_\mathcal{I}} e^{-rt}\pi_1(\widetilde{X}^{x,\nu,\xi}_t)dt + \int_0^{\sigma_\mathcal{I}} e^{-rt}\alpha_1\, d\xi_t - \int_0^{\sigma_\mathcal{I}} e^{-rt}\beta_1\, d\nu_t\Big], \tag{A-1}$$

$$\widehat{\Psi}_2(x;\nu,\xi) := \mathsf{E}\Big[\int_0^{\sigma_\mathcal{I}} e^{-rt}\pi_2(\widetilde{X}^{x,\nu,\xi}_t)dt + \int_0^{\sigma_\mathcal{I}} e^{-rt}\alpha_2\, d\nu_t - \int_0^{\sigma_\mathcal{I}} e^{-rt}\beta_2\, d\xi_t\Big]. \tag{A-2}$$

In these cases the integrals with respect to the controls are simply understood as a Riemann-Stieltjes' integrals. For $\alpha_i < \beta_i$ we prove that if one of the two players opts for a control that reflects the process at a threshold, then the other player's best response avoids simultaneous jumps of the controls. The condition $\alpha_i < \beta_i$ is the analogue in this context of the absence of arbitrage in papers like [13], [19] and [33]. The result is illustrated in the next lemma.

Lemma A.1. *Consider the game with functionals* (A-1)–(A-2)*. Recall Lemma 2.5 and assume* $\alpha_i < \beta_i$*,* $i = 1,2$*. If player 1 (resp. player 2) chooses* ν^a *(resp.* ξ^b*) that solves* $\boldsymbol{SP}^\xi_{a+}$ *for* $a \in \mathcal{I}$ *(resp.* $\boldsymbol{SP}^\nu_{b-}$ *for* $b \in \mathcal{I}$*) then the best reply* $\hat{\xi}_a := \mathrm{argmax}\, \widehat{\Psi}_2(x;\nu^a,\xi)$ *is such that* $(\nu^a, \hat{\xi}_a) \in \mathcal{M}$ *(resp.* $(\hat{\nu}_b, \xi^b) \in \mathcal{M}$ *with* $\hat{\nu}_a := \mathrm{argmax}\, \widehat{\Psi}_1(x;\nu,\xi^b)$*).*

Proof. Let $x, a \in \mathcal{I}$ and $\xi \in \mathcal{S}$ and consider ν^a solving $\mathbf{SP}^\xi_{a+}$. We want to perform a pathwise comparison of the cost functional for player 2 under two different controls. In particular we fix $\omega \in \Omega$ and assume that there exists (a stopping time) $t_0 = t_0(\omega) > 0$ such that $\big(\Delta\nu^a_{t_0} \cdot \Delta\xi_{t_0}\big)(\omega) > 0$. With no loss of generality we may assume that $X^{x,\nu^a,\xi}_{t_0}(\omega) > a$ and that the downward jump $\Delta\xi_{t_0}$ is trying to push the process below a, i.e.

$$\Delta\xi_{t_0}(\omega) > [X^{x,\nu^a,\xi}_{t_0} - a](\omega).$$

This push causes the immediate reaction of the control ν^a and therefore a simultaneous jump of the two controls. The case in which $X^{x,\nu^a,\xi}_{t_0}(\omega) \le a$ can be dealt with in the same way up to trivial changes.

We denote by ξ^0 a control in $\mathcal{S}$ such that

$$\xi^0_t(\omega) = \begin{cases} \xi_t(\omega)\,, & t \le t_0 \\ \xi_t(\omega) - [\Delta\xi_{t_0} - (X^{x,\nu^a,\xi}_{t_0} - a)](\omega)\,, & t > t_0, \end{cases}$$

i.e. $\xi^0(\omega)$ is the same as $\xi(\omega)$ but the jump size at $t_0(\omega)$ is reduced so that the process is not pushed below a. For ν^a solving $\mathbf{SP}^{\xi^0}_{a+}$ the jump at t_0 is not triggered and $X^{x,\nu^a,\xi^0}_{t_0+}(\omega) = a$ due only to the downward push given by ξ^0. Now we observe that the (random) Borel measure $d\nu^a$ induced by ν^a is the same for ξ and ξ^0 with the only exception of a mass at time t_0 in reaction to the jump of ξ. Moreover, since ν^a solves $\mathbf{SP}^\xi_{a+}$ for any ξ, then $X^{x,\nu^a,\xi}_t(\omega) = X^{x,\nu^a,\xi^0}_t(\omega)$ for all $t > 0$, since $X^{x,\nu^a,\xi}_{t_0+}(\omega) = X^{x,\nu^a,\xi^0}_{t_0+}(\omega) = a$ and nothing else has changed for $t \neq t_0$.

It is now easy to see that the couple (ν^a,ξ) requires an additional cost for player 2 compared to the couple (ν^a,ξ^0) and therefore cannot be optimal. For the sake of clarity here we denote by $\nu^{a,\xi}$ the solution of $\mathbf{SP}^{\xi}_{a+}$ and by ν^{a,ξ^0} the solution of $\mathbf{SP}^{\xi^0}_{a+}$. So we obtain

$$\begin{aligned}&\int_0^{\sigma_{\mathcal{I}}} e^{-rt}\pi_2(\widetilde{X}^{x,\nu^{a,\xi},\xi}_t)dt + \int_0^{\sigma_{\mathcal{I}}} e^{-rt}\alpha_2\, d\nu^{a,\xi}_t - \int_0^{\sigma_{\mathcal{I}}} e^{-rt}\beta_2\, d\xi_t\\ &= \int_0^{\sigma_{\mathcal{I}}} e^{-rt}\pi_2(\widetilde{X}^{x,\nu^{a,\xi^0},\xi^0}_t)dt + \int_0^{\sigma_{\mathcal{I}}} e^{-rt}\alpha_2\, d\nu^{a,\xi^0}_t - \int_0^{\sigma_{\mathcal{I}}} e^{-rt}\beta_2\, d\xi^0_t\\ &\quad + e^{-rt_0}(\alpha_2-\beta_2)[\Delta\xi_{t_0} - (X^{x,\nu^{a,\xi},\xi}_{t_0} - a)]\end{aligned}$$

and the last term is negative as $\alpha_2 < \beta_2$. Since the above argument can be repeated for any simultaneous jump of ν^a and ξ, and any $\omega\in\Omega$ the proof is complete. □

The point of the above lemma is that if costs of control are constant then a simple condition for the absence of arbitrage opportunities implies that if one player picks a reflecting strategy then the other one will pick a control such that $(\nu,\xi)\in\mathcal{M}$. Therefore under such assumptions the equilibria constructed in Theorem 3.1 are also equilibria in the larger class $\mathcal{S}\times\mathcal{S}$.

A.2 Auxiliary results

We recall here the fundamental solutions ϕ and ψ of (2.9), and recall also that $\underline{x}$ and $\overline{x}$ are unattainable for X of (2.5) and for the uncontrolled diffusion $\widetilde{X}^{0,0}$ of (2.1).

Lemma A.2. *Let $a_*\in\mathcal{I}$ be arbitrary but fixed and denote by $\widetilde{X}^{\nu_*,0}$ the solution of the Skorokhod reflection problem $\boldsymbol{SP}^0_{a_*+}$ of Lemma 2.5. For $y>a_*$, $y\in\mathcal{I}$ set $\theta_y := \inf\{t>0 : \widetilde{X}^{\nu_*,0}\ge y\}$ and*

$$q(x,y) := \mathsf{E}_x\big[e^{-r\theta_y}\big],\qquad x\in[a_*,\overline{x}).$$

Then for $i=1,2$ we have

$$\lim_{y\uparrow\overline{x}} q(x,y)\left(1+\int_{a_*}^{y}|G_i(z)|dz\right)=0. \tag{A-3}$$

Similarly let $b_*\in\mathcal{I}$ be arbitrary but fixed and denote by $\widetilde{X}^{0,\xi_*}$ the solution of the Skorokhod reflection problem $\boldsymbol{SP}^0_{b_*-}$ of Lemma 2.5. For $y<b_*$, $y\in\mathcal{I}$ set $\eta_y := \inf\{t>0 : \widetilde{X}^{0,\xi_*}\le y\}$ and

$$p(x,y) := \mathsf{E}_x\left[e^{-r\eta_y}\right],\qquad x\in(\underline{x},b_*].$$

Then for $i=1,2$ we have

$$\lim_{y\downarrow\underline{x}} p(x,y)\left(1+\int_{b_*}^{y}|G_i(z)|dz\right)=0. \tag{A-4}$$

Proof. We provide a full proof only for the first claim as the one for the second claim follows by similar arguments. Existence of a solution to $\mathbf{SP}^0_{a_*+}$ is a well known result. It is shown in Lemma 2.1 and Corollary 2.2 of [34] that the function $q(\,\cdot\,,y)$ solves

$$(\mathcal{L}_{\widetilde{X}}-r)q(x,y)=0,\qquad x\in(a_*,\overline{x}) \tag{A-5}$$

with boundary conditions

$$q(y-,y) := \lim_{x\uparrow y} q(x,y)=1,\qquad q_x(a_*+,y) := \lim_{x\downarrow a_*} q_x(x,y)=0.$$

In particular we refer to the condition at a_* as the reflecting boundary condition.

Since $q(\,\cdot\,,y)$ solves (A-5) then it may be written as

$$q(x,y) = A(y)\widetilde{\psi}(x) + B(y)\widetilde{\phi}(x), \qquad x \in \mathcal{I}$$

where $\widetilde{\psi}$ and $\widetilde{\phi}$ denote the fundamental increasing and decreasing solutions, respectively, of $(\mathcal{L}_{\widetilde{X}} - r)u = 0$ on $\mathcal{I}$. By imposing the reflecting boundary condition we get

$$B(y) = -A(y)\frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}$$

which plugged back into the expression for q gives

$$q(x,y) = A(y)\left(\widetilde{\psi}(x) - \frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\widetilde{\phi}(x)\right). \tag{A-6}$$

Now imposing the boundary condition at y we also obtain

$$A(y) = \left(\widetilde{\psi}(y) - \frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\widetilde{\phi}(y)\right)^{-1}. \tag{A-7}$$

Notice that $-\widetilde{\psi}'(a_*)/\widetilde{\phi}'(a_*) > 0$ thus implying $A(y), B(y) > 0$ and $q(x,y) > 0$ as expected. Since the sample paths of $\widetilde{X}^{\nu_*,0}$ are continuous for all $t > 0$ then $y \mapsto q(x,y)$ must be strictly decreasing. Hence

$$q_y(x,y) = A'(y)\left(\widetilde{\psi}(x) - \frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\widetilde{\phi}(x)\right) < 0$$

which implies $A'(y) < 0$ since the term in brackets is positive. From (A-7) and direct computation we get

$$A'(y) = -\frac{1}{(A(y))^2}\left(\widetilde{\psi}'(y) - \frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\widetilde{\phi}'(y)\right)$$

and $A'(y) < 0$ implies

$$\left(\widetilde{\psi}'(y) - \frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\widetilde{\phi}'(y)\right) > 0. \tag{A-8}$$

The latter inequality is important to prove (A-3).

The assumed regularity of μ and σ (see Assumption 2.1) implies that $\widetilde{\psi}'$ solves $\mathcal{L}_X u(x) - (r - \mu'(x))u(x) = 0$ in $\mathcal{I}$ (cf. (2.9)), and it can therefore be written as a linear combination of the fundamental increasing and decreasing functions ψ and ϕ. That is,

$$\widetilde{\psi}'(x) = \alpha\psi(x) + \beta\phi(x), \tag{A-9}$$

for some $\alpha, \beta \in \mathbb{R}$. Analogously

$$\widetilde{\phi}'(x) = \gamma\psi(x) + \delta\phi(x). \tag{A-10}$$

Moreover since $\widetilde{\psi}' > 0$ and $\widetilde{\phi}' < 0$ in $\mathcal{I}$, and $\underline{x}$ and $\overline{x}$ are unattainable for X, then it must be $\alpha, \beta \geq 0$ and $\gamma, \delta \leq 0$. Noticing that $y > a_*$ was arbitrary, the inequality (A-8) now reads

$$\left(\alpha - \gamma\frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\right)\psi(y) + \left(\beta - \delta\frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\right)\phi(y) > 0 \qquad y > a_*. \tag{A-11}$$

We aim at showing that $\alpha > 0$ and we can do it by considering separately two cases.

Case 1. Assume $\gamma < 0$. Since the second term in (A-11) can be made arbitrarily small by letting $y \to \overline{x}$ then it must be $\alpha > \gamma\widetilde{\psi}'(a_*)/\widetilde{\phi}'(a_*) > 0$.

Case 2. Assume $\gamma = 0$. If $\alpha = 0$ then the first term on the left-hand side of (A-11) is zero and by using (A-9) and (A-10) we get from (A-11)

$$0 < \left(\beta - \delta\frac{\widetilde{\psi}'(a_*)}{\widetilde{\phi}'(a_*)}\right)\phi(y) = \left(\beta - \delta\frac{\beta\phi(a_*)}{\delta\phi(a_*)}\right)\phi(y) = 0,$$

hence a contradiction. So it must be $\alpha > 0$.

Finally, for fixed $x \in [a_*, y)$, there exists a constant $C = C(a_*, x) > 0$ such that (A-6) and (A-7) give

$$0 \leq q(x,y)\left(1 + \int_{a_*}^{y}|G_i(z)|dz\right) \leq \frac{C}{\widetilde{\psi}(y)}\left(1 + \int_{a_*}^{y}|G_i(z)|dz\right). \tag{A-12}$$

Now letting $y \to \overline{x}$ we have $\widetilde{\psi}(y) \to \infty$ as $\overline{x}$ is unattainable for $\widetilde{X}^{0,0}$. We have two possibilities:

(a) $\int_{a_*}^{\overline{x}}|G_i(z)|dz < +\infty$ and therefore (A-3) holds trivially from (A-12);

(b) $\int_{a_*}^{\overline{x}}|G_i(z)|dz = +\infty$ so that by using de l'Hôpital rule in (A-12), (A-9) and Assumption 2.6 we get

$$\lim_{y\to\overline{x}}\frac{1}{\widetilde{\psi}(y)}\int_{a_*}^{y}|G_i(z)|dz = \lim_{y\to\overline{x}}\frac{|G_i(y)|}{\alpha\psi(y)} = 0.$$

□

Proof of Lemma 2.5. We provide here a short proof of the existence of a unique solution to the Skorokhod reflection problem $\mathbf{SP}^{\xi}_{a+}$.

Notice that the drift and diffusion coefficients in the dynamics (2.1) are locally Lipschitz-continuous due to our Assumption 2.1. So we first prove the result for Lipschitz coefficients and then extend it to locally Lipschitz ones. Notice that here we are not assuming sublinear growth of μ and σ but we rely on non attainability of $\underline{x}$ and $\overline{x}$ for the uncontrolled process $\widetilde{X}^{0,0}$. Existence of a unique solution to problem $\mathbf{SP}^{\nu}_{b-}$ can be shown by analogous arguments.

Step 1 - Lipschitz coefficients. Here we assume $\mu, \sigma \in \mathrm{Lip}(\mathcal{I})$ with constant smaller than $L > 0$. Let $a \in \mathcal{I}$, $x \in \mathcal{I}$ and $\xi \in \mathcal{S}$, and consider the sequence of processes defined recursively by $X^{[0]}_t = x$, $\nu^{[0]}_t = 0$ and

$$\begin{cases} X^{[k+1]}_t = x + \int_0^t \mu(X^{[k]}_u)du + \int_0^t \sigma(X^{[k]}_u)dW_u + \nu^{[k+1]}_t - \xi_t, \\ \nu^{[k+1]}_t = \sup_{0\leq s\leq t}\big[a - x - \int_0^s \mu(X^{[k]}_u)du - \int_0^s \sigma(X^{[k]}_u)dW_u + \xi_s\big], \end{cases} \tag{A-13}$$

for any $k \geq 0$ and $t \geq 0$. Notice that at any step the process $X^{[k+1]}$ is kept above the level a by the process $\nu^{[k+1]}$ with minimal effort, i.e. according to a Skorokhod reflection at a. The Lipschitz-continuity of μ and σ allows to obtain from (A-13) the estimate

$$\mathsf{E}_x\Big[\sup_{0\leq s\leq t}\big|X^{[k+1]}_s - X^{[k]}_s\big|^2\Big] \leq C\mathsf{E}_x\Big[\int_0^t \big|X^{[k]}_s - X^{[k-1]}_s\big|^2 ds\Big], \tag{A-14}$$

for $k \geq 1$ and for some positive $C := C(x, a, L)$. Since for $k = 0$ one has $\mathsf{E}_x[\sup_{0\leq s\leq t}|X^{[1]}_s - x|^2] \leq Rt$ for some $R := R(x, a, L) > 0$, then an induction argument together with (A-14) yield

$$\mathsf{E}_x\Big[\sup_{0\leq s\leq t}\big|X^{[k+1]}_s - X^{[k]}_s\big|^2\Big] \leq \frac{(R_0 t)^{k+1}}{(k+1)!}, \quad k \geq 0, \tag{A-15}$$

for some other positive $R_0 := R_0(x, a, L)$. Analogously,

$$\mathsf{E}_x\Big[\sup_{0\leq s\leq t}\big|\nu^{[k+1]}_s - \nu^{[k]}_s\big|^2\Big] \leq \frac{(R_1 t)^{k+1}}{(k+1)!}, \quad k \geq 0, \tag{A-16}$$

with $R_1 := R_1(x, a, L) > 0$.

Thanks to (A-15) and (A-16) we can now proceed with an argument often used in SDE theory for the proof of existence of strong solutions (see, e.g., the proof of [28, Ch. 5, Thm. 2.9]). That is, we use Chebyshev inequality and Borel-Cantelli's lemma to find that $(X^{[k+1]}, \nu^{[k+1]})_{k\geq 0}$ converges a.s., locally uniformly in time, as $k \uparrow \infty$. We denote this limit by $(\widetilde{X}^{\nu^a,\xi}, \nu^a)$. By Lipschitz continuity of μ and σ and the same arguments as above we also obtain that the sequences $(\int_0^t \mu(X^{[k]}_u)du)_{k\geq 0}$ and $(\int_0^t \sigma(X^{[k]}_u)dW_u)_{k\geq 0}$ converge a.s., locally uniformly in time. Then we have a.s. (up to a possible subsequence)

$$\begin{aligned}\nu^a = \lim_{k\uparrow\infty}\nu^{[k+1]}_t &= \lim_{k\uparrow\infty}\sup_{0\leq s\leq t}\big[a - x - \int_0^s \mu(X^{[k]}_u)du - \int_0^s \sigma(X^{[k]}_u)dW_u + \xi_s\big]\\ &= \sup_{0\leq s\leq t}\big[a - x - \int_0^s \mu(\widetilde{X}^{\nu^a,\xi}_u)du - \int_0^s \sigma(\widetilde{X}^{\nu^a,\xi}_u)dW_u + \xi_s\big].\end{aligned}$$

It thus follows that $(\widetilde{X}^{\nu^a,\xi}, \nu^a)$ solves $\mathbf{SP}^{\xi}_{a+}$. Finally, uniqueness can be proved as, e.g., in the proof of [37, Thm. 4.1].

Step 2 - locally Lipschitz coefficients. Here we assume μ and σ as in Assumption 2.1. Let $x_n \uparrow \overline{x}$ and define

$$\mu_n(x) = \mu(x)\mathbb{1}_{\{x\leq x_n\}} + \mu(x_n)\mathbb{1}_{\{x>x_n\}}, \quad \sigma_n(x) = \sigma(x)\mathbb{1}_{\{x\leq x_n\}} + \sigma(x_n)\mathbb{1}_{\{x>x_n\}}.$$

For each n we denote by $\mathbf{SP}^{\xi\,(n)}_{a+}$ the Skorokhod problem $\mathbf{SP}^{\xi}_{a+}$ but for the dynamics

$$dX_t = \mu_n(X_t)dt + \sigma_n(X_t)dW_t + d\nu_t - d\xi_t$$

rather than for (2.1).

Since for each n we have μ_n and σ_n uniformly Lipschitz on $[a, \overline{x})$ then Step 1 guarantees that there exists a unique $(X^{(n)}, \nu^{(n)})$ that solves $\mathbf{SP}^{\xi\,(n)}_{a+}$. We denote $\tau_n := \inf\{t > 0 : X^{(n)}_t \geq x_n\}$ and for all $t \leq \tau_n$ we have

$$\begin{aligned}X^{(n)}_t &= x + \int_0^t \mu_n(X^{(n)}_u)du + \int_0^t \sigma_n(X^{(n)}_u)dW_u + \nu^{(n)}_t - \xi_t\\ &= x + \int_0^t \mu(X^{(n)}_u)du + \int_0^t \sigma(X^{(n)}_u)dW_u + \nu^{(n)}_t - \xi_t \qquad \text{(A-17)}\\ \nu^{(n)}_t &= \sup_{0\leq s\leq t}\big[a - x - \int_0^s \mu_n(X^{(n)}_u)du - \int_0^s \sigma_n(X^{(n)}_u)dW_u + \xi_s\big]\\ &= \sup_{0\leq s\leq t}\big[a - x - \int_0^s \mu(X^{(n)}_u)du - \int_0^s \sigma(X^{(n)}_u)dW_u + \xi_s\big]. \qquad \text{(A-18)}\end{aligned}$$

Since the coefficients above do not depend on n, by construction the process $(X^{(n)}_t, \nu^{(n)}_t)$ also solves $\mathbf{SP}^{\xi}_{a+}$ for $t \leq \tau_n$. Uniqueness of the solution for $\mathbf{SP}^{\xi\,(n)}_{a+}$ implies that $(X^{(n)}_t, \nu^{(n)}_t)$ is also the solution of $\mathbf{SP}^{\xi\,(m)}_{a+}$ for $t \le \tau_m$, for each $m \le n$ and therefore the unique solution of $\mathbf{SP}^{\xi}_{a}$ up to the stopping time τ_n.

Fix an arbitrary $T > 0$. For all $\omega \in \{\tau_n > T\}$ and all $t \le T$ we can define $(\widetilde{X}^{\nu^a,\xi}_t, \nu^a_t) := (X^{(n)}_t, \nu^{(n)}_t)$ so that the couple $(\widetilde{X}^{\nu^a,\xi}_t, \nu^a_t)$ is the unique solution of $\mathbf{SP}^{\xi}_{a+}$ for $t \le T$. It remains to show that $\lim_{n\to\infty} \mathsf{P}(\tau_n > T) \to 1$ so that we have constructed a unique solution of $\mathbf{SP}^{\xi}_{a+}$ for a.e. $\omega \in \Omega$ up to time T.

Let us consider first the case $\xi \equiv 0$. It follows from Lemma A.2 that $\mathsf{E}_x[e^{-r\theta_{x_n}}] \to 0$ as $n \to \infty$ with $\theta_{x_n} = \inf\{t > 0 : \widetilde{X}^{\nu^a,0}_t \ge x_n\}$ and hence $\theta_{x_n} \to \infty$ P_x-a.s. Hence

$$\lim_{n\to\infty} \mathsf{P}(\tau_n > T) = \lim_{n\to\infty} \mathsf{P}(\theta_{x_n} > T) = 1$$

since $\tau_n = \theta_{x_n}$ P-a.s. To conclude it is sufficient to notice that $\widetilde{X}^{\nu^a,\xi}_t \le \widetilde{X}^{\nu^a,0}_t$, for all $t > 0$, and arbitrary $\xi \in \mathcal{S}$. Then $\overline{x}$ is unattainable for $\widetilde{X}^{\nu^a,\xi}$ as well. □

Proof of Proposition 3.5. The proofs are contained in [16] and here we provide precise references to the relevant results in each case. In particular one must notice that Appendix A.2 of [16] addresses the specific setting of the state dependent discount factor $r - \mu'(x)$ that appears in our stopping functional (2.17).

1. It follows from Theorem 3.4 (and Appendix A.2) of [16].

2. It follows from Proposition 3.11 (and Appendix A.2) of [16]. For the sake of completeness here we notice that to prove that x_2^∞ uniquely solves $\vartheta_2(x) = (G_2/\psi)(x)$ in $(\hat{x}_2, \overline{x})$ it is useful to change variables. Defining $y = (\psi/\phi)(x) =: F(x)$, where F is strictly increasing, and introducing $\hat{G}_2(y) := \big[(G_2/\phi) \circ F^{-1}\big](y)$, $y > 0$, it follows from simple algebra (cf. Appendix A.1 of [16]) that $\vartheta_2(x) = (G_2/\psi)(x)$ is equivalent to $\hat{G}'_2(y)y = \hat{G}_2(y)$. It is shown on page 13 of [16] that the latter equation has a unique root y_2^∞ in the interval $(\hat{y}_2, \infty)$, with $\hat{y}_2 := F(\hat{x}_2)$ and $\infty = \lim_{x\uparrow\overline{x}} F(x)$. Therefore $x_2^\infty = F^{-1}(y_2^\infty)$ solves the initial problem in $(\hat{x}_2, \overline{x})$.

□

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