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Controlling Public Debt without Forgetting Inflation*

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Abstract. Consider the problem of a government that wants to control its debt-to-GDP (gross domestic product) ratio, while taking into consideration the evolution of the inflation rate of the country. The uncontrolled inflation rate follows an Ornstein-Uhlenbeck dynamics and affects the growth rate of the debt ratio. The level of the latter can be reduced by the government through fiscal interventions. The government aims at choosing a debt reduction policy which minimises the total expected cost of having debt, plus the total expected cost of interventions on debt ratio. We model such problem as a two-dimensional singular stochastic control problem over an infinite time-horizon. We show that it is optimal for the government to adopt a policy that keeps the debt-to-GDP ratio under an inflation-dependent ceiling. This curve is the free-boundary of an associated fully two-dimensional optimal stopping problem, and it is shown to be the unique solution of a nonlinear integral equation.

Key words: debt-to-GDP ratio; inflation rate; debt ceiling; singular stochastic control; optimal stopping; free-boundary; nonlinear integral equation.

MSC2010 subsject classification: 93E20, 60G40, 91B64, 45B05, 60J60.

JEL classification: C61, H63.

1 Introduction

Controlling debt-to-GDP ratio (also called the "debt ratio") and keeping it below some desirable level is of fundamental importance for all countries. It has been shown by different authors by means of different statistical and methodological approaches, that high government debt has a negative effect on long-term economic growth. The usual outcome is that when government debt grows, private investment shrinks, and future growth and future wages lower (see, e.g., [43]). In [38] it is shown that high government debt hurts growth even in the absence of a crisis. This negative effect on economic growth from high debt levels has been observed also in [8] in 18 different advanced economies.

It is common practice that different countries, or communities of states, apply different policies to control their debt ratio. For example, the Maastricht Treaty in 1992 has imposed to all member countries of the European Economic and Monetary Union to have a debt-to-GDP

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ratio less than 60% (according to [11] this value was simply chosen as the median of the debt ratio of some european countries). In the USA the congress sets a ceiling on the nominal debt and changes it whenever it is needed. As the debt ceiling problem of 2011 has shown, it might be a complex and lengthy political problem to reach an agreement on such a level. As a result of the delay in the decision, in 2011 Standard and Poor's downgraded USA debt from AAA to AA^+.

In this paper we propose a continuous-time stochastic model for the control of debt-to-GDP ratio. The problem we have in mind is that of a government aiming to answer the question: *How much is too much?*[1] We have already seen that in practice governments usually answer this question by choosing a suitable level (the so-called debt ceiling) under which keeping the debt ratio. In this paper such threshold is endogenously determined: it is part of the solution to the singular stochastic control problem that we introduce to model the government debt management problem. The optimal debt ceiling is therefore obtained by solving an optimisation problem, and in this sense its value is justified at a theoretical level. This is different to some recommendations on sustainable debt ratio levels that can be found in the literature, which seem to have been deduced only on the basis of empirical facts (see, e.g., [16] and [37]).

Following classical macroeconomic theory (see, e.g., [6]), in any given period the debt ratio stock grows by the existing debt stock multiplied by the difference between real interest rate and GDP growth, less the primary budget balance. We assume that the government can influence the level of debt-to-GDP ratio by adjusting the primary budget balance, e.g. through fiscal interventions like raising taxes or reducing expenses. We therefore interpret the cumulative interventions on debt ratio as the government's control variable, and we model it as a nonnegative and nondecreasing stochastic process.

Uncertainty in our model comes through the inflation rate of the country, whose level is not chosen by the government. As it is common in advanced economies, the inflation rate is regulated by an autonomous Central Bank, whose action, however, is not modeled in this paper (see, e.g., [9], [10] and [24] for problems related to the optimal control of inflation). The inflation rate directly affects real interest rate and, as a result, the dynamics of debt ratio. Indeed, by Fisher law [20] the real interest rate is given by the difference of nominal interest rate and inflation rate. In reality these variables are all time-dependent, stochastic and related. In this paper we are interested in developing a continuous-time model for public debt control which takes into account the role of inflation. We therefore assume that the evolution of inflation rate is described by a stochastic process Y, whereas we keep nominal interest rate and GDP growth rate as constant (see also Remark 2.1 below). In particular, we assume that Y evolves according to a one-dimensional Ornstein-Uhlenbeck process so to capture mean-reversion and stationarity of inflation rate usually observed in empirical studies (see, e.g., [33]).

Since high debt-to-GDP ratios can constrain economic growth making it more difficult to break the burden of the debt, we assume that debt ratio generates an instantaneous cost/penalty. This is a quadratic function of debt ratio level, that government would like to keep as close to zero as possible. However, at any time the government decides to intervene in order to reduce the level of debt ratio, it incurs into a cost which is proportional to the amount of debt reduction. The government thus aims at choosing a (cumulative) debt reduction policy minimising the sum of the total expected cost of having debt, and of the total expected cost of interventions on debt

[1] cf. *The Economist*, June 3rd 2015.

ratio.

Mathematically, our model leads to a two-dimensional singular stochastic control problem (see [41] for an introduction) where the drift of a degenerate controlled component (debt ratio) is instantaneously affected by the uncontrolled diffusive component (inflation rate). This setting is different with respect to that of several papers in the literature solving multi-dimensional degenerate singular stochastic control problems arising from questions of optimal capacity expansion (expansion/contraction) under uncertainty (see [13], [18], [19], [21], [35] and [40], among many others). There the dynamics of a purely controlled state, modeling the production capacity of a firm, is independent of uncontrolled diffusive processes, representing the demand of a produced good or other factors influencing the company's running profit. In our problem we have interaction between the two components of the state process, and we still obtain a complete characterisation of the optimal policy and of the value function.

We show that it is optimal to keep the debt ratio X always below an inflation-dependent level $b(Y)$. If the level of debt at time t is below $b(Y_t)$, there is no need for interventions. The government should intervene to reduce its debt generating fiscal surpluses only at those (random) times for which $X_t \geq b(Y_t)$, any other intervention being sub-optimal. At those instants the amount of the reduction should be minimal, in the sense that the government should reduce only enough to prevent a level of debt ratio above the current inflation-dependent ceiling.

As already noticed, the curve b is part of the solution to the problem. We characterise it in terms of the (generalised) inverse of the unique continuous, nondecreasing solution $\hat{y}$ to a nonlinear integral equation of Fredholm type. $\hat{y}$ is actually the free-boundary of an associated fully two-dimensional optimal stopping problem that we also solve in our paper by providing a probabilistic representation of its value function. In the optimal stopping problem the state variable is given by the inflation rate process Y and its time-integral Z. It is well known that (Z, Y) is a time-homogeneous, strong Markov process, whose first component is of bounded-variation (being a time integral). It therefore turns out that the free-boundary formulation of the optimal stopping problem involves a second-order linear partial differential equation of (local) parabolic type. Since characterisations of solutions to multi-dimensional optimal stopping problems are quite rare in the literature (see [29] for a parabolic problem, and Sections 3-4 of [13] for an elliptic one), we believe that also such result represents a valuable contribution to the literature.

Our result suggests a debt ratio ceiling depending on the inflation rate of the country, and in this sense it is somehow consistent with the recommendation of [44]. There it is observed that it is not reasonable to apply to different countries the same upper bound on the debt ratio (by the way, this is exactly what it has been decided in 1992 within the Maastricht Treaty) since the debt-to-GDP ratio of each country depends on their individual values of the interest rate on debt, economic growth rate, and different countries can very well deal with different values of debt ratio without necessarily having debt problems.

Our interest in stochastic control methods for public debt management started reading the recent and very nice [7], which, to the best of our knowledge, is the only other paper dealing with a mathematical rigorous analysis of the debt ceiling problem. In [7] the debt ratio evolves according to a linearly controlled one-dimensional geometric Brownian motion, and the government aims at minimising the total expected costs arising from having debt and intervening on it. Although the government cost functional we consider in this paper is similar to the one in [7], our singular control problem is fully two-dimensional, whereas that of [7] is one-dimensional.

Our debt ratio dynamics is indeed degenerate, and its growth rate is affected by the inflation rate modeled as a linear diffusion. This allows us to obtain an optimal debt ceiling which is a function of the current level of inflation rate, whereas that of [7] is a constant.

The rest of the paper is organised as follows. In Section 2 we set up the model and introduce the problem. Our main results are then presented in Section 3, whose proofs are developed in the remaining sections of the paper. In particular, Section 4 is devoted to the study of the two-dimensional optimal stopping problem associated to the control one, whereas in Section 5 we construct the control problem's value function and provide the optimal control policy. Finally, Appendix A collects some proofs, whereas Appendix B contains some auxiliary results.

2 The Model and the Control Problem

Let X_t be the level of public debt-to-gross domestic product (GDP) ratio at time $t \geq 0$; that is,

$$X_t := \frac{\text{gross public debt at time } t}{\text{GDP at time } t}.$$

According to classical macroeconomic theory (see, e.g., [6]), in any given period the debt stock grows by the existing debt stock multiplied by the difference between real interest rate and GDP growth, less the primary budget balance. By Fisher law [20] the real interest rate is given by the difference of nominal interest rate and inflation rate. In reality these variables are all time-dependent, stochastic and related. In this paper we are interested in developing a continuous-time model for public debt control which takes into account the role of inflation. We therefore assume that the evolution of the inflation is described by a stochastic process $Y := \{Y_t, t \geq 0\}$, whereas we keep nominal interest rate and GDP growth rate constant (see also Remark 2.1 for comments on this). The dynamics of X then takes the form

$$dX_t = (\delta - Y_t - g)X_t dt - d\nu_t, \quad t \geq 0, \qquad X_{0-} = x > 0, \tag{2.1}$$

where $g \in \mathbb{R}$ is the (constant) GDP growth rate, $\delta \geq 0$ is the (constant) nominal interest rate, and ν_t is the cumulative primary balance up to time t. The primary balance is the variable that the government can control, e.g. through fiscal interventions, in order to reduce the public debt level. Notice that the initial level of public debt x is assumed to be strictly positive, meaning that the government might initially want to intervene.

We observe from (2.1) that inflation rate level Y directly affects the dynamics of public debt. Supported by empirical evidences (see, e.g., [33] and references therein) we model Y as a stationary and mean-reverting process. In particular, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ it evolves as an Ornstein-Uhlenbeck process

$$dY_t = (a - \theta Y_t)dt + \sigma dW_t, \qquad Y_0 = y \in \mathbb{R}, \tag{2.2}$$

with $W := \{W_t, t \geq 0\}$ a one-dimensional Brownian motion. We donote by $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ the Brownian filtration, as usual augmented by $\mathbb{P}$-null sets of $\mathcal{F}$. $\mathbb{F}$ is the flow of information available to the government. In (2.2) $a/\theta \in \mathbb{R}$ is the equilibrium level, $\sigma > 0$ the volatility and the parameter $\theta > 0$ is the speed at which Y asymptotically converges in average towards its equilibrium. Notice that we assume that the inflation rate is not under government control.

It is regulated by an autonomous Central Bank, whose optimal management problem is not investigated in this paper.

The system of stochastic differential equations given by (2.1) and (2.2) can be explicitly solved yielding for any $t \geq 0$ and $(x, y) \in (0, \infty) \times \mathbb{R}$

$$\begin{cases} X_t = e^{(\delta-g)t - \int_0^t Y_s ds}\Big[x - \int_0^t e^{-(\delta-g)s - \int_0^s Y_u du} d\nu_s\Big], \\ Y_t = ye^{-\theta t} + \frac{a}{\theta}(1 - e^{-\theta t}) + \sigma e^{-\theta t} \int_0^t e^{\theta s} dW_s. \end{cases} \tag{2.3}$$

Remark 2.1. *There are several directions towards which our model might be extended.*

1. *It might be interesting to introduce a stochastic dynamics for the nominal interest rate, for example still diffusive, mean-reverting, and correlated with the inflation rate. This will lead to an intricate optimisation problem with three-dimensional state space.*
2. *Also, it would definitely deserve consideration allowing for a GDP growth rate g following a Markov regime switching model à la Hamilton [23], so to capture business cycles in the GDP dynamics. Allowing for a regime-switching model for the GDP rate, the uncertainty in the model will be driven not only by the Brownian motion W, but also by a continuous-time Markov chain $\varepsilon := \{\varepsilon_t, t \geq 0\}$, independent of W. The dimensionality of the optimisation problem will then increase and the mathematical analysis of the problem will become much more challenging.*
3. *One might also allow the autonomous Central Bank to choose, according to a certain optimality criterion, the level of inflation rate. This would lead to an interesting nonzero-sum game between the government, controlling the debt ratio, and the Central Bank, controlling the inflation rate and therefore influencing the debt ratio level via* (2.3).

We leave all these interesting extensions for future research.

In the rest of this paper we will often write $(X^{x,y,\nu}, Y^y)$ to account for the dependence of (X, Y) on the initial levels $(x, y) \in (0, \infty) \times \mathbb{R}$ and on the primary balance policy ν. For $(x, y) \in (0, \infty) \times \mathbb{R}$, the admissible policies that the government can employ to decrease the level of debt ratio are drawn from the set

$$\mathcal{A}(x, y) := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, (\nu_t(\omega) := \nu(\omega, t))_{t \geq 0} \text{ is nondecreasing, right-continuous, } \mathbb{F}\text{-adapted, such that } X_t^{x,y,\nu} \geq 0 \ \forall\, t \geq 0,\ \nu_{0-} = 0 \ \ \mathbb{P} - \text{a.s.}\}. \tag{2.4}$$

Notice that in (2.4) we do not allow for policies that let debt ratio become negative, i.e. that make the government a net lender. This is somehow a realistic requirement, as a situation with negative debt is less relevant in real world economies.

For any admissible ν, the process (X, Y, ν) is a Markov process, but since any $\nu \in \mathcal{A}(x, y)$ always starts from zero, we shall also denote by $\mathbb{E}_{(x,y)}$ the expectation under the measure on $(\Omega, \mathcal{F})$ $\mathbb{P}_{(x,y)}(\,\cdot\,) = \mathbb{P}(\,\cdot\,|X_0^\nu = x, Y_0 = y)$ and equivalently use the notation $\mathbb{E}[f(X_t^{x,y,\nu}, Y_t^y)] = \mathbb{E}_{(x,y)}[f(X_t^\nu, Y_t)]$, for any Borel-measurable function for which the expectation is well defined.

The government aims at reducing the level of debt ratio. Having a debt level X_t at time $t \geq 0$ the government incurs into an instantaneous cost $h(X_t) = (X_t)^2/2$. This may be interpreted as a measure of the resulting losses for the country due to the debt, as, e.g., a tendency to suffer low subsequent growth (see [8], [38], [43], among others, for empirical studies).

On the other hand, whenever the government decides to reduce the level of public debt, it incurs into an intervention cost that is proportional to the amount of debt reduction. Fiscal adjustments as raising taxes or reducing expenses may generate such a cost. Assuming that the government discounts at a rate $\rho > 0$, its goal is to choose a policy $\nu^* \in \mathcal{A}(x,y)$, $(x,y) \in (0,\infty) \times \mathbb{R}$, minimising the total expected cost

$$\mathcal{J}_{x,y}(\nu) := \mathbb{E}_{(x,y)}\bigg[\int_0^\infty e^{-\rho t}\frac{1}{2}(X_t^\nu)^2 dt + \int_0^\infty e^{-\rho t} d\nu_t\bigg]. \tag{2.5}$$

Notice that for any $\nu \in \mathcal{A}(x,y)$, $\mathcal{J}_{x,y}(\nu)$ is well defined but possibly infinite. Denoting by $\mathcal{O} := (0,\infty) \times \mathbb{R}$, the government's value function is therefore

$$v(x,y) := \inf_{\nu \in \mathcal{A}(x,y)} \mathcal{J}_{x,y}(\nu), \qquad (x,y) \in \mathcal{O}. \tag{2.6}$$

Problem (2.6) takes the form of a singular stochastic control problem, i.e. of a problem in which control processes may be singular with respect to Lebesgue measures, as functions of time (see [41] for an introduction, and [30] and [31] as classical references).

Remark 2.2. *To some extent, problem* (2.6) *shares common mathematical features with problems of optimal consumption under Hindy-Huang-Kreps (HHK) preferences (cf. [25], [26], [4] and references therein). Indeed, with regard to* (2.3)*, the process X might be related to what it is usually referred to the "level of satisfaction" in the literature on HHK preferences. The main difference is that the weighting function is here stochastic and given by* $e^{-\int_0^\cdot (Y_s - \delta + g)ds}$.

In the rest of this paper we make the standing assumption that the government's discount factor ρ is sufficiently large. Namely,

Assumption 2.3.

$$\rho > 8\Big(\delta - g - \frac{a}{\theta} + \frac{2\sigma^2}{\theta^2}\Big) \vee 0.$$

Assumption 2.3 is reasonable in light of the fact that usually governments run only for a finite number of years and are therefore more concerned about present than future. Mathematically, the previous requirement implies in particular that v as defined in (2.6) is finite. This is shown in the next proposition, whose proof can be found in Appendix A.

Proposition 2.4. *Set* $C := 2\big[\rho - 2\big(\delta - g - \frac{a}{\theta} + \frac{\sigma^2}{\theta^2}\big)\big] > 0$. *Then for any* $(x,y) \in \mathcal{O}$ *one has*

$$0 \le v(x,y) \le Cx^2 e^{\frac{2}{\theta}|y - \frac{a}{\theta}|}. \tag{2.7}$$

Moreover, $v(0,y) = 0$ *and the mapping* $x \mapsto v(x,y)$ *is convex for any* $y \in \mathbb{R}$.

3 The Optimal Solution and its Economic Implications

In this section we present the solution to problem (2.6). We provide the expression of the optimal cumulative primary balance and of the minimal cost function. The proofs of the next theorems are developed in the following sections of this paper.

In the problem's solution the two-dimensional process $(Z^{z,y}, Y^y)$, with Y^y as in (2.2) and

$$Z^{z,y}_t := z + (\delta - g)t - \int_0^t Y^y_u du, \quad (z,y) \in \mathbb{R}^2, \tag{3.1}$$

plays an important role. Its properties can be easily obtained from p. 287 of [32] and we recall them here for the sake of completeness.

Lemma 3.1. *The process $(Z^{z,y}, Y^y) := \{(Z^{z,y}_t, Y^y_t), t \geq 0\}$ is strong Markov, time-homogeneous, and its infinitesimal generator is given by the second-order differential operator*

$$\mathbb{L}_{Z,Y} := \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} + (a - \theta y)\frac{\partial}{\partial y} + (\delta - g - y)\frac{\partial}{\partial z}. \tag{3.2}$$

Moreover, it has transition density

$$\begin{aligned} &p_t(z,y;v,u) := \mathbb{P}\big((Z_t, Y_t) \in (dv, du) \big| Z_0 = z, Y_0 = y\big)/dvdu \\ &= \frac{1}{2\pi\sqrt{\Delta_t}} \exp\Big\{ - \frac{\theta^2}{2\big(t - \frac{2}{\theta}\tanh(\frac{\theta t}{2})\big)} \Big[v + (\delta - g)t - z - \frac{1}{\theta}\tanh\big(\frac{\theta t}{2}(u + y - a)\big)\Big]^2 \\ &- \frac{\theta}{1 - e^{-\theta t}}\big(u - e^{-\theta t}(y - a)\big)^2 \Big\}, \end{aligned} \tag{3.3}$$

where we have set $\Delta_t := \frac{1}{2\theta^3}(1 - e^{-2\theta t})(t - \frac{2}{\theta}\tanh(\frac{\theta t}{2}))$.

In the following we shall denote by $\mathbb{E}_{(z,y)}$ the expectation under the measure on $(\Omega, \mathcal{F})$ $\mathbb{P}_{(z,y)}(\,\cdot\,) = \mathbb{P}(\,\cdot\,|Z_0 = z, Y_0 = y)$, $(z,y) \in \mathbb{R}^2$, and equivalently use the notation $\mathbb{E}[f(Z^{z,y}_t, Y^y_t)] = \mathbb{E}_{(z,y)}[f(Z_t, Y_t)]$, for any Borel-measurable function for which the expectation is well defined. Also, the following convention is adopted

$$e^{-\rho\tau + Z_\tau} := \liminf_{t\uparrow\infty} e^{-\rho t + Z_t} = 0 \quad \text{on } \{\tau = +\infty\},$$

where the last equality above is proved in Lemma B.2 in Appendix B.

The next theorem provides the expression of the optimal primary balance policy and of the minimal cost function. These are given in terms of the stopping boundary and of the value of an associated optimal timing problem.

Theorem 3.2. *Define the optimal stopping problem*

$$u(z,y) := \inf_{\tau \geq 0} \mathbb{E}_{(z,y)}\Big[\int_0^\tau e^{-\rho t + 2Z_t} dt + e^{-\rho\tau + Z_\tau}\Big], \quad (z,y) \in \mathbb{R}^2, \tag{3.4}$$

where the optimisation is taken over all $\mathbb{F}$-stopping times. Also, introduce

$$\hat{y}(z) := \inf\{y \in \mathbb{R} : \, u(z,y) < e^z\}, \quad z \in \mathbb{R}, \tag{3.5}$$

with the convention $\inf \emptyset = \infty$*, and define*

$$b(y) := \sup\{x \in (0,\infty) : \, y > \hat{y}(\ln(x))\}, \quad y \in \mathbb{R}, \tag{3.6}$$

with the convention $\sup \emptyset = 0$*. Then setting for* $(x,y) \in \mathcal{O}$

$$\overline{\nu}^*_t = \Big[x - \inf_{0\le s\le t}\Big(b(Y^y_s)e^{-(\delta-g)s+\int_0^s Y^y_u du}\Big)\Big] \vee 0, \quad t \ge 0, \qquad \overline{\nu}^*_{0-} = 0, \tag{3.7}$$

the process

$$\nu^*_t := \int_0^t e^{(\delta-g)s-\int_0^s Y^y_u du} d\overline{\nu}^*_s, \quad t \ge 0, \qquad \nu^*_{0-} = 0,$$

is optimal for (2.6)*. Moreover, one has*

$$v(x,y) = \int_{-\infty}^{\ln(x)} u(q,y)dq, \quad (x,y) \in (0,\infty)\times\mathbb{R}. \tag{3.8}$$

The boundary b of (3.6) triggers the optimal debt reduction policy. It is defined in terms of the generalised inverse of stopping boundary $\hat{y}$ of (3.5). In turn, $\hat{y}$ is given in terms of (3.4). Problem (3.4) is a fully two-dimensional optimal stopping problem for the Markov process (Z,Y) and it is worth noting that characterisations of the solutions to multi-dimensional optimal stopping problems are quite rare in the literature (see, e.g., [29] and Sections 3-4 of [13] for recent contributions). In the next theorem we show that the optimal stopping boundary $\hat{y}$ is the unique solution (within a certain functional class) of a nonlinear integral equation. This result is of interest on its own.

Theorem 3.3. *Recall* (3.3) *and introduce the set of functions*

$$\mathcal{M} := \Big\{f : \mathbb{R} \mapsto \mathbb{R} \text{ continuous, nondecreasing, dominated from above by } \vartheta(z) := e^z + \delta - g - \rho\Big\}. \tag{3.9}$$

Then the boundary $\hat{y}(\,\cdot\,)$ *is the unique function in* $\mathcal{M}$ *solving the nonlinear integral equation*

$$\begin{aligned} e^z = &\int_0^\infty e^{-\rho t}\Big(\int_{\mathbb{R}^2} e^{2v}\mathbb{1}_{\{u>f(v)\}}p_t(z,f(z);v,u)dudv\Big)dt \\ &- \int_0^\infty e^{-\rho t}\Big(\int_{\mathbb{R}^2} e^{v}(\delta-g-\rho-u)\mathbb{1}_{\{u\le f(v)\}}p_t(z,f(z);v,u)dudv\Big)dt. \end{aligned} \tag{3.10}$$

Moreover, u *as in* (3.4) *admits the representation*

$$\begin{aligned} u(z,y) = &\int_0^\infty e^{-\rho t}\Big(\int_{\mathbb{R}^2} e^{2v}\mathbb{1}_{\{u>\hat{y}(v)\}}p_t(z,y;v,u)dudv\Big)dt \\ &- \int_0^\infty e^{-\rho t}\Big(\int_{\mathbb{R}^2} e^{v}(\delta-g-\rho-u)\mathbb{1}_{\{u\le \hat{y}(v)\}}p_t(z,y;v,u)dudv\Big)dt, \end{aligned} \tag{3.11}$$

and the stopping time

$$\tau^* := \inf\{t \ge 0 : \, Y_t \le \hat{y}(Z_t)\}, \quad \mathbb{P}_{(z,y)} - a.s.$$

is optimal for (3.4)*.*

Remark 3.4. *Notice that* (3.11) *is equivalent to*

$$u(z,y) = \mathbb{E}_{(z,y)}\left[\int_0^\infty e^{-\rho s+2Z_s}\mathbb{1}_{\{Y_s>\hat{y}(Z_s)\}}ds\right] - \mathbb{E}_{(z,y)}\left[\int_0^\infty e^{-\rho s+Z_s}\big(\delta-g-\rho-Y_s\big)\mathbb{1}_{\{Y_s\le\hat{y}(Z_s)\}}ds\right].$$

The latter is actually the representation of u that we are going to derive in the next section.

From the first of (2.3) and recalling that $\nu^*_t := \int_0^t e^{(\delta-g)s-\int_0^s Y^y_u du} d\overline{\nu}^*_s$ we can write

$$X^{x,y,\nu^*}_t = e^{(\delta-g)t-\int_0^t Y_s ds}\big[x-\overline{\nu}^*_t\big],$$

which, with regard to (3.7), shows that

$$0 \le X^{x,y,\nu^*}_t \le b(Y^y_t), \qquad t\ge 0,\ \mathbb{P}-a.s. \tag{3.12}$$

Moreover, it can also be shown that the policy ν^* is minimal, in the sense that it satisfies

$$\int_0^\infty \mathbb{1}_{\{X^{x,y,\nu^*}_t<b(Y^y_t)\}}d\nu^*_s = 0, \qquad \mathbb{P}-a.s. \tag{3.13}$$

Equations (3.7), (3.12) and (3.13) allow us to draw some interesting conclusions about the optimal debt management policy of our model.

(i) Reducing public debt to zero is not optimal since too costly. The optimal policy does not indeed prescribe that.

(ii) If at initial time the level of public debt x is above $b(y)$, then an immediate lump-sum reduction of amplitude $(x-b(y))$ is optimal.

(iii) At any other instant, employing the optimal primary balance policy, the government keeps the public debt level below the inflation-dependent ceiling b.

(iv) If the level of debt at time t is below $b(Y_t)$, there is no need for interventions. The government should intervene to reduce its debt generating fiscal surpluses only at those (random) times t for which $X_t \ge b(Y_t)$, any other intervention being sub-optimal. These times do actually solve the optimal timing problem (3.4). At those instants the amount of the reduction should be minimal, in the sense that the government should reduce only enough to prevent a level of public debt above the current inflation-dependent ceiling (cf. (3.13)).

(v) The optimal cumulative debt reduction (3.7) is given in terms of the minimal level that the inflation-dependent debt ceiling, suitably discounted, has reached over $[0,t]$. The discount rate is dynamic and does depend on the history of inflation. It is indeed $\int_0^\cdot(\delta-g-Y^y_u)du$.

In the Maastricht Treaty in 1992 the upper bound on debt ratio for countries willing to become members of the European Economic Community is set 60%, independently of the inflation level of the state. In the USA the congress sets a ceiling on the nominal debt and changes it whenever it is needed. As the debt ceiling problem of 2011 has shown, it might be difficult to reach a political agreement on such a level. In our model we suggest a simple rule for the management of debt ratio recommending each country to define *dynamically in time* its own debt ceiling, keeping track of the level of its *inflation rate*. Such policy clearly differs from the one established, e.g., in the Maastricht Treaty. In our model the latter turns out to be not necessarily the optimal one!

4 The Solution to the Auxiliary Optimal Timing Problem

In this section we study the optimal stopping problem defined in (3.4). In particular we characterise its solution in terms of the optimal stopping boundary (3.5). According to Theorem 3.3, the latter curve will be shown to uniquely solve a nonlinear integral equation (cf. (3.10)).

Recalling (3.4), it is easy to see that $u(z, y) \leq e^z$ for any $(z, y) \in \mathbb{R}^2$. As usual in optimal stopping theory (see, e.g., [36]), we can define the *continuation region*

$$\mathcal{C} := \{(z, y) \in \mathbb{R}^2 : u(z, y) < e^z\}, \tag{4.1}$$

and the *stopping region*

$$\mathcal{S} := \{(z, y) \in \mathbb{R}^2 : u(z, y) = e^z\}. \tag{4.2}$$

Because $y \mapsto Y^y$ is increasing (cf. (2.2)), the mapping $y \mapsto Z^{z,y}$ is decreasing (cf. (3.1)). It thus follows from (3.4) that $y \mapsto u(z, y)$, $z \in \mathbb{R}$, is decreasing and therefore

$$\mathcal{C} := \{(z, y) \in \mathbb{R}^2 : y > \hat{y}(z)\}, \qquad \mathcal{S} := \{(z, y) \in \mathbb{R}^2 : y \leq \hat{y}(z)\}, \tag{4.3}$$

for $\hat{y}$ as defined in (3.5); that is,

$$\hat{y}(z) := \inf\{y \in \mathbb{R} : u(z, y) < e^z\}, \quad z \in \mathbb{R},$$

with the convention $\inf \emptyset = \infty$.

Denoting by $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times, an integration by parts gives

$$e^{-\rho\tau + Z_\tau} = e^z + \int_0^\tau e^{-\rho t + Z_t}\big(\delta - g - Y_t - \rho\big)dt, \quad \mathbb{P}_{(z,y)} - a.s.,$$

for any $\tau \in \mathcal{T}$, and (3.4) may be rewritten as

$$u(z, y) := e^z + \inf_{\tau \geq 0} \mathbb{E}_{(z,y)}\bigg[\int_0^\tau e^{-\rho t + Z_t}\Big(e^{Z_t} + \delta - g - Y_t - \rho\Big)dt\bigg], \tag{4.4}$$

where the optimisation is taken again over all the stopping times in $\mathcal{T}$. In the rest of this paper, we will make use of both the equivalent representations (3.4) and (4.4).

Proposition 4.1. *The value function u of* (3.4) *(equivalently, of* (4.4)*) is such that $(z, y) \mapsto u(z, y)$ is continuous on $\mathbb{R}^2$.*

Proof. It is enough to show that:

(i) $z \mapsto u(z, y)$ is continuous at z_o, uniformly over $y \in [y_o - \eta, y_o + \eta]$;

(ii) $y \mapsto u(z_o, y)$ is continuous at y_o,

for every given and fixed $(z_o, y_o) \in \mathbb{R}^2$ and for some $\eta > 0$ sufficiently small.

(i) Let $(z_o, y_o) \in \mathbb{R}^2$ be given and fixed. Because $z \mapsto Z^{z,y}$ is nondecreasing (cf. (3.1)), it is not hard to be convinced that $z \mapsto u(z, y)$ is nondecreasing for any $y \in \mathbb{R}$. Then taking $z_2 \leq z_1$ in $[z_o - \eta, z_o + \eta]$ and $y \in [y_o - \eta, y_o + \eta]$, for some $\eta > 0$, we have

$$\begin{aligned}
0 &\geq u(z_2, y) - u(z_1, y) \geq \inf_{\tau \geq 0} \mathbb{E}\bigg[\int_0^\tau e^{-\rho t + 2Z_t^{0,y}}\big(e^{2z_2} - e^{2z_1}\big)dt + e^{-\rho\tau + Z_\tau^{0,y}}\big(e^{z_2} - e^{z_1}\big)\bigg] \\
&= \inf_{\tau \geq 0} \mathbb{E}\bigg[\int_0^\tau e^{-\rho t + 2Z_t^{0,y}}\big(e^{z_2} - e^{z_1}\big)\big(e^{z_2} + e^{z_1}\big)dt + e^{-\rho\tau + Z_\tau^{0,y}}\big(e^{z_2} - e^{z_1}\big)\bigg] \\
&\geq \big(e^{z_2} - e^{z_1}\big)\sup_{\tau \geq 0} \mathbb{E}\bigg[\int_0^\tau 2e^{z_1} e^{-\rho t + 2Z_t^{0,y}}dt + e^{-\rho\tau + Z_\tau^{0,y}}\bigg] \\
&\geq \big(e^{z_2} - e^{z_1}\big)\sup_{\tau \geq 0} \mathbb{E}\bigg[2e^{z_o + \eta}\int_0^\tau e^{-\rho t + 2Z_t^{0,y_o - \eta}}dt + e^{-\rho\tau + Z_\tau^{0,y_o - \eta}}\bigg],
\end{aligned} \tag{4.5}$$

where in the last equality we have used that $z_1 \leq z_o + \eta$, $y \geq y_o - \eta$ and $y \mapsto Z^{z,y}$ is decreasing, and that $z_2 \leq z_1$. Notice that the supremum above is independent of z_1, z_2 and y. If it is finite, we conclude that $z \mapsto u(z, y)$ is continuous at z_o uniformly over $y \in [y_o - \eta, y_o + \eta]$.

To show finiteness, observe that for any stopping time $\tau \in \mathcal{T}$ we can write

$$\begin{aligned}
0 &\leq \mathbb{E}\bigg[2e^{z_o + \eta}\int_0^\tau e^{-\rho t + 2Z_t^{0,y_o - \eta}}dt + e^{-\rho\tau + Z_\tau^{0,y_o - \eta}}\bigg] \\
&= \mathbb{E}\bigg[\int_0^\tau e^{-\rho t + Z_t^{0,y_o - \eta}}\Big(2e^{z_o + \eta}e^{Z_t^{0,y_o - \eta}} + \delta - g - Y_t^{y_o - \eta} - \rho\Big)dt\bigg] + 1. \\
&\leq 2e^{z_o + \eta}\mathbb{E}\bigg[\int_0^\infty e^{-\rho t + 2Z_t^{0,y_o - \eta}}dt\bigg] + \mathbb{E}\bigg[\int_0^\infty e^{-\rho t + Z_t^{0,y_o - \eta}}\big|\delta - g - Y_t^{y_o - \eta} - \rho\big|dt\bigg] + 1.
\end{aligned}$$

Then by Lemma B.3 of Appendix B the last term above is finite and the proof is therefore completed.

(ii) Let $z_o \in \mathbb{R}$ be given and fixed, take $y_2 \geq y_1$ in $[y_o - \eta, y_o + \eta]$, $\eta > 0$, and notice that we can write (cf. (2.3)) $Y_t^y = ye^{-\theta t} + \Xi_t$, where we have set, for notational simplicity, $\Xi_t := \frac{a}{\theta}(1 - e^{-\theta t}) + \sigma e^{-\theta t}\int_0^t e^{\theta s}dW_s$. Because $y \mapsto u(z, y)$ is nonincreasing, we can write

$$\begin{aligned}
0 \geq u(z_o, y_2) - u(z_o, y_1) &\geq \inf_{\tau \geq 0} \mathbb{E}\bigg[\int_0^\tau e^{-\rho t + 2z_o + 2(\delta - g)t - 2\int_0^t \Xi_s ds}\Big(e^{-2y_2\int_0^t e^{-\theta s}ds} - e^{-2y_1\int_0^t e^{-\theta s}ds}\Big)dt \\
&\quad + e^{-\rho\tau + z_o + (\delta - g)\tau - \int_0^\tau \Xi_s ds}\Big(e^{-y_2\int_0^\tau e^{-\theta s}ds} - e^{-y_1\int_0^\tau e^{-\theta s}ds}\Big)\bigg] \\
\geq \inf_{\tau \geq 0} \mathbb{E}\bigg[\int_0^\tau & e^{-\rho t + 2z_o + 2(\delta - g)t - 2\int_0^t \Xi_s ds}e^{-2y_2\int_0^t e^{-\theta s}ds}\Big(1 - e^{-2(y_1 - y_2)\int_0^\infty e^{-\theta s}ds}\Big)dt \\
&\quad + e^{-\rho\tau + z_o + (\delta - g)\tau - \int_0^\tau \Xi_s ds}e^{-y_2\int_0^\tau e^{-\theta s}ds}\Big(1 - e^{-(y_1 - y_2)\int_0^\infty e^{-\theta s}ds}\Big)\bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{\tau\geq 0}\mathbb{E}\bigg[\int_0^\tau e^{-\rho t+2z_o+2(\delta-g)t-2\int_0^t \Xi_s ds}e^{-2y_2\int_0^t e^{-\theta s}ds}\Big(1-e^{\frac{2}{\theta}(y_2-y_1)}\Big)dt \\
&\qquad\qquad + e^{-\rho\tau+z_o+(\delta-g)\tau-\int_0^\tau \Xi_s ds}e^{-y_2\int_0^\tau e^{-\theta s}ds}\Big(1-e^{\frac{1}{\theta}(y_2-y_1)}\Big)\bigg] \qquad (4.6)\\
&\geq \Big(1-e^{\frac{1}{\theta}(y_2-y_1)}\Big)\sup_{\tau\geq 0}\mathbb{E}\bigg[e^{2\eta}\int_0^\tau e^{-\rho t+2Z_t^{z_o,y_2}}dt+e^{-\rho\tau+2Z_\tau^{z_o,y_2}}\bigg]\\
&\geq \Big(1-e^{\frac{1}{\theta}(y_2-y_1)}\Big)\sup_{\tau\geq 0}\mathbb{E}\bigg[e^{2\eta}\int_0^\tau e^{-\rho t+2Z_t^{z_o,y_o-\eta}}dt+e^{-\rho\tau+2Z_\tau^{z_o,y_o-\eta}}\bigg]
\end{aligned}
$$

where in the fourth inequality we have used that $y_2 \leq y_o + \eta$ and $y_1 \geq y_o - \eta$, and in the last step that $y_2 \geq y_o - \eta$. We then conclude by letting $y_1 \to y_2$, since the last supremum above is finite due to Lemma B.3 of Appendix B by the same arguments employed at the end of (i) above.

□

From Proposition 4.1 it follows that the stopping set $\mathcal{S}$ of (4.2) is closed, the continuation region $\mathcal{C}$ of (4.1) is open and that the stopping time

$$\tau^*(z,y) := \inf\{t \geq 0 : (Z_t^{z,y}, Y_t^y) \in \mathcal{S}\} = \inf\{t \geq 0 : Y_t^y \leq \hat{y}(Z_t^{z,y})\} \qquad (4.7)$$

is optimal for problem (3.4), whenever it is $\mathbb{P}$-a.s. finite (see Corollary 2.9 in [36]).

In the next proposition we rule out the possibility that the stopping set $\mathcal{S}$ is empty. Its proof is provided in Appendix A.

Proposition 4.2. *The stopping region $\mathcal{S}$ of* (4.2) *is not empty.*

Some preliminary properties of $\hat{y}$ are collected in the following proposition, whose proof can be found in Appendix A as well.

Proposition 4.3. *Let $\hat{y}$ be defined as in* (3.5)*. Then the following properties hold true:*

(i) $\hat{y}(z) \leq e^z + \delta - g - \rho$ for any $z \in \mathbb{R}$;

(ii) $z \mapsto \hat{y}(z)$ is nondecreasing.

We now continue by improving the regularity of the value function (3.4). Namely, we now show that the well known *smooth-fit principle* holds, by proving that $u \in C^1(\mathbb{R}^2)$. The proof relies of an application of an interesting result obtained by S.D. Jacka in [27] (cf. Corollary 7 in Section 4 of [27]).

Proposition 4.4. *The value function u of* (3.4) *(equivalently, of* (4.4)*) is such that $u \in C^1(\mathbb{R}^2)$.*

Proof. First of all we notice that an application of strong Markov property allows to write

$$u(z,y) = e^z + g(z,y) - f(z,y), \qquad (4.8)$$

where we have set

$$g(z,y) := \mathbb{E}_{(z,y)}\bigg[\int_0^\infty e^{-\rho t+Z_t}\Big(e^{Z_t}+\delta-g-\rho-Y_t\Big)dt\bigg], \qquad (4.9)$$

and

$$f(z,y) := \sup_{\tau\geq 0} \mathbb{E}_{(z,y)}\Big[e^{-\rho\tau} g(Z_\tau, Y_\tau)\Big]. \tag{4.10}$$

Hence, the C^1 property of u reduces to check that for g and f.

Because $Z_t^{z,y} = z + (\delta - g)t - \int_0^t Y_s^y ds$ and $Y_t^y = ye^{-\theta t} + \Xi_t$, with $\Xi_t := \frac{a}{\theta}(1 - e^{-\theta t}) + \sigma e^{-\theta t}\int_0^t e^{\theta s} dW_s$, we can write

$$g(z,y) := e^{2z}\mathbb{E}\bigg[\int_0^\infty e^{-(\rho-2(\delta-g))t - \frac{2y}{\theta}(1-e^{-\theta t}) - 2\int_0^t \Xi_s ds} dt\bigg]$$
$$+ e^{z}\mathbb{E}\bigg[\int_0^\infty e^{-(\rho-\delta+g)t - \frac{y}{\theta}(1-e^{-\theta t}) - \int_0^t \Xi_s ds}\Big(\delta - g - \rho - ye^{-\theta t} - \Xi_t\Big)dt\bigg],$$

and an application of dominated convergence theorem shows that $g \in C^1(\mathbb{R}^2)$ thanks to Assumption 2.3.

It thus remain to check for the C^1 property of f. With regard to the notation of [27] we set $\xi_t := (Z_t, Y_t)$,

$$X_t := e^{-\rho t} g(\xi_t) = \mathbb{E}\bigg[\int_t^\infty e^{-\rho s + Z_s}\Big(e^{Z_s} + \delta - g - \rho - Y_s\Big)ds\Big|\mathcal{F}_t\bigg],$$

and we can write $X_t = M_t + A_t$ where,

$$M_t := \mathbb{E}\bigg[\int_0^\infty e^{-\rho s + Z_s}\Big(e^{Z_s} + \delta - g - \rho - Y_s\Big)ds\Big|\mathcal{F}_t\bigg]$$

and

$$A_t := -\int_0^t e^{-\rho s + Z_s}\Big(e^{Z_s} + \delta - g - \rho - Y_s\Big)ds = \int_0^t \big(dA_s^+ + dA_s^-\big).$$

Notice M is a uniformly integrable martingale thanks to Lemma B.3, and dA^+ and dA^- above are given by

$$dA_s^+ := e^{-\rho s + Z_s}\Big(e^{Z_s} + \delta - g - \rho - Y_s\Big)^- ds \quad \& \quad dA_s^- := -e^{-\rho s + Z_s}\Big(e^{Z_s} + \delta - g - \rho - Y_s\Big)^+ ds,$$

which are clearly absolutely continuous with respect to Lebesgue measure $dm_2 := dt$. Moreover, the set $\partial\mathcal{D}$ in [27] reads in our case as $\{(z,y) \in \mathbb{R}^2 : y = \hat{y}(z,y)\}$, which has zero measure with respect to $dm_1 := dzdy$. Finally, the process $\xi := (Z,Y)$ has density with respect to m_1 which has spatial derivatives uniformly continuous in $\mathbb{R}^2 \times [t_0, t_1]$, for any $0 < t_0 < t_1 < \infty$ (see (3.3)). Hence, Corollary 7 in [27] holds and the proof is complete. □

Similarly to Corollary 14 in [29], we now exploit the fact that the process Z is of bounded variation to obtain additional regularity for u. The proof of the next result is provided in Appendix A.

Corollary 4.5. *$u \in C^{1,2}$ inside $\mathcal{C}$, and u_{yy} admits a continuous extension from $\mathcal{C}$ to $\overline{\mathcal{C}}$, where $\overline{\mathcal{C}} := \{(z,y) \in \mathbb{R}^2 : y \geq \hat{y}(z)\}$.*

From standard arguments based on strong Markov property (cf., e.g., Chapter III of [36]) and the results collected above it follows that u solves the free-boundary problem

$$\begin{cases} \big(\mathbb{L}_{Z,Y} - \rho\big)u(z,y) = -e^{2z}, & y > \hat{y}(z),\ z \in \mathbb{R} \\ u(z,y) = e^z, & y \leq \hat{y}(z),\ z \in \mathbb{R} \\ u_z(z,y) = e^z, & y = \hat{y}(z),\ z \in \mathbb{R} \\ u_y(z,y) = 0, & y = \hat{y}(z),\ z \in \mathbb{R}, \end{cases} \tag{4.11}$$

with $u \in C^{1,2}$ inside $\mathcal{C}$. We now show that the boundary $\hat{y}$ is in fact a continuous function.

Proposition 4.6. *The optimal stopping boundary $\hat{y}$ is such that $z \mapsto \hat{y}(z)$ is continuous.*

Proof. Right-continuity of $\hat{y}(\,\cdot\,)$ follows from standard arguments based on monotonicity of $z \mapsto \hat{y}(z)$ and on the fact that $\mathcal{S}$ is closed. We repeat them here for the sake of completeness. Fix $z \in \mathbb{R}$ and notice that for every $\varepsilon > 0$ we have by monotonicity of $\hat{y}$ that $\hat{y}(z+\varepsilon) \geq \hat{y}(z)$, which implies $\hat{y}(z) \leq \lim_{\varepsilon\downarrow 0}\hat{y}(z+\varepsilon) =: \hat{y}(z+)$. Consider now the sequence $\{(z+\varepsilon, \hat{y}(z+\varepsilon)) :\ \varepsilon > 0\} \subset \mathcal{S}$; one has $\{(z+\varepsilon, \hat{y}(z+\varepsilon)) :\ \varepsilon > 0\} \to (z, \hat{y}(z+))$ when $\varepsilon \downarrow 0$ and $(z, \hat{y}(z+)) \in \mathcal{S}$, since $\mathcal{S}$ is closed by continuity of u. It then follows that $\hat{y}(z+) \leq \hat{y}(z)$ from the definition (3.5) and the proof is complete.

We now prove that $\hat{y}(\,\cdot\,)$ is left-continuous by employing a contradiction scheme inspired by that in [12]. This is possible since the process Z is of bounded variation and therefore it behaves as a "time-like" variable. Assume that there exists some $z_o \in \mathbb{R}$ such that $\hat{y}(z_o-) < \hat{y}(z_o)$, where we have set $\hat{y}(z_o-) := \lim_{\varepsilon\downarrow 0}\hat{y}(z_o - \varepsilon)$. Such limit exists by monotonicity of $\hat{y}(\,\cdot\,)$. Then we can choose y_1, y_2 such that $\hat{y}(z_o-) < y_1 < y_2 < \hat{y}(z_o)$, $z_1 < z_o$ and define a rectangular domain with vertices (z_o, y_1), (z_o, y_2), (z_1, y_1), (z_2, y_2).

Noticing that $(z_1, z_o) \times (y_1, y_2) \subset \mathcal{C}$ and $\{z_o\} \times [y_1, y_2] \subset \mathcal{S}$, from (4.11) u is such that

$$\begin{cases} \big(\mathbb{L}_{Z,Y} - \rho\big)u(z,y) = -e^{2z}, & (z,y) \in (z_1, z_o) \times (y_1, y_2), \\ u(z_o, y) = e^{z_o}, & y \in [y_1, y_2], \end{cases} \tag{4.12}$$

Denote by $C_c^\infty([y_1, y_2])$ the set of functions with infinitely many continuous derivatives and compact support in $[y_1, y_2]$. Pick an arbitrary $\psi \geq 0$ from $C_c^\infty([y_1, y_2])$ such that $\int_{y_1}^{y_2}\psi(y)dy > 0$, multiply both sides of the first of (4.12) by ψ, and integrate over $[y_1, y_2]$ so to obtain

$$-\int_{y_1}^{y_2} e^{2z}\psi(y)dy = \int_{y_1}^{y_2}\big(\mathbb{L}_{Z,Y} - \rho\big)u(z,y)\psi(y)dy, \quad z \in [z_1, z_o).$$

Then recalling (3.2), integrating by parts (twice) the right-hand side of the latter one finds

$$-\int_{y_1}^{y_2} e^{2z}\psi(y)dy = \int_{y_1}^{y_2}\big(\mathbb{L}_Y^*\psi\big)(y)u(z,y)dy + \int_{y_1}^{y_2}\big(\delta - g - y\big)\psi(y)u_z(z,y)dy, \tag{4.13}$$

for $z \in [z_1, z_o)$, and where $\mathbb{L}_Y^*$ is the second-order differential operator which acting on a function $f \in C^2(\mathbb{R})$ yields

$$(\mathbb{L}_Y^* f)(y) := \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial y^2}(y) - \frac{\partial}{\partial y}\big((a - \theta y)f\big)(y) - \rho f(y).$$

Taking limits as $z \uparrow z_o$ on both sides of (4.13) above, invoking dominated convergence theorem and recalling continuity of u_z on $\mathbb{R}^2$ (cf. Proposition 4.4) one has

$$-\int_{y_1}^{y_2} e^{2z_o}\psi(y)dy = \int_{y_1}^{y_2} \big(\mathbb{L}_Y^*\psi\big)(y)u(z_o,y)dy + \int_{y_1}^{y_2} \big(\delta - g - y\big)\psi(y)u_z(z_o,y)dy. \quad (4.14)$$

Since $u_z(z_o,y) = e^{z_o} = u(z_o,y)$ for any $y \in [y_1,y_2]$, then rearranging terms in (4.14) gives

$$-\int_{y_1}^{y_2} e^{z_o}(\delta - g - \rho - y + e^{z_o})\psi(y)dy = \int_{y_1}^{y_2} e^{z_o}\Big(\frac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial y^2}(y) - (a-\theta y)\frac{\partial \psi}{\partial y}(y) + \theta\psi(y)\Big)dy.$$

Because $y_2 < \hat{y}(z_o) \leq \delta - g - \rho + e^{z_o}$ (cf. Proposition 4.3) and $\psi \geq 0$, the left-hand side of the last equation is strictly negative. On the other hand, an integration reveals that the right-hand side of (4.15) equals zero because $\psi \in C_c^\infty([y_1,y_2])$. Hence we reach a contradiction and the proof is complete. $\square$

The next result provides the integral representation of the value function u of problem (3.4), whose analytical formulation has been provided in (3.11) (see also Remark 3.4). Such representation will then allow us to obtain an integral equation for the stopping boundary $\hat{y}$ (cf. (3.10)).

Proposition 4.7. *Let $\hat{y}(\,\cdot\,)$ be the stopping boundary of* (3.5)*. Then for any $(z,y) \in \mathbb{R}^2$ the value function u of* (3.4) *can be written as*

$$\begin{aligned} u(z,y) = {} & \mathbb{E}_{(z,y)}\bigg[\int_0^\infty e^{-\rho s + 2Z_s}\mathbb{1}_{\{Y_s > \hat{y}(Z_s)\}}ds\bigg] \\ & - \mathbb{E}_{(z,y)}\bigg[\int_0^\infty e^{-\rho s + Z_s}\big(\delta - g - \rho - Y_s\big)\mathbb{1}_{\{Y_s \leq \hat{y}(Z_s)\}}ds\bigg]. \end{aligned} \quad (4.15)$$

Proof. The proof is based on an application of a generalised version of Itô's lemma. Let $R > 0$ and define $\tau_R := \inf\{t \geq 0 : |Y_t| \geq R \text{ or } |Z_t| \geq R\}$ under $\mathbb{P}_{(z,y)}$. Since $u \in C^1(\mathbb{R}^2)$ and $u_{yy} \in L^\infty_{loc}(\mathbb{R}^2)$ (cf. Proposition 4.4 and Corollary 4.5), we can apply a weak version of Itô's lemma (see, e.g., [5], Lemma 8.1 and Theorem 8.5, pp. 183–186) up to the stopping time $\tau_R \wedge T$, for some $T > 0$, so to obtain

$$u(z,y) = \mathbb{E}_{(z,y)}\bigg[e^{-\rho(\tau_R\wedge T)}u(Z_{\tau_R\wedge T}, Y_{\tau_R\wedge T}) - \int_0^{\tau_R\wedge T} e^{-\rho s}\big(\mathbb{L}_{Z,Y} - \rho\big)u(Z_s,Y_s)ds\bigg]. \quad (4.16)$$

Since u solves the free-boundary problem (4.11) and because

$$\big(\mathbb{L}_{Z,Y} - \rho\big)u(z,y) = -e^{2z}\mathbb{1}_{\{y > \hat{y}(z)\}} + (\delta - g - \rho - y)e^z\mathbb{1}_{\{y \leq \hat{y}(z)\}} \qquad \text{for a.a. } (z,y) \in \mathbb{R}^2,$$

equation (4.16) rewrites as

$$\begin{aligned} u(z,y) = {} & \mathbb{E}_{(z,y)}\bigg[e^{-\rho(\tau_R\wedge T)}u(Z_{\tau_R\wedge T}, Y_{\tau_R\wedge T}) + \int_0^{\tau_R\wedge T} e^{-\rho s + 2Z_s}\mathbb{1}_{\{Y_s > \hat{y}(Z_s)\}}ds\bigg] \\ & - \mathbb{E}_{(z,y)}\bigg[\int_0^{\tau_R\wedge T} e^{-\rho s + Z_s}\big(\delta - g - \rho - Y_s\big)\mathbb{1}_{\{Y_s \leq \hat{y}(Z_s)\}}ds\bigg] \end{aligned} \quad (4.17)$$

We now aim at taking limits in both sides of (4.17) as $R \uparrow \infty$, and later also as $T \uparrow \infty$. To this end we preliminary notice that $\tau_R \wedge T \uparrow T$ when $R \uparrow \infty$, and we analyse the three addends on the right-hand side of (4.17) separately.

(i) Notice that $0 \leq e^{-\rho(\tau_R \wedge T)} u(Z_{\tau_R \wedge T}, Y_{\tau_R \wedge T}) \leq e^{-\rho(\tau_R \wedge T) + Z_{\tau_R \wedge T}}$ $\mathbb{P}_{(z,y)}$-a.s., because $0 \leq u(z,y) \leq e^z$ for any $(z,y) \in \mathbb{R}^2$, and that

$$e^{-\rho(\tau_R \wedge T) + Z_{\tau_R \wedge T}} = e^{-\rho(\tau_R \wedge T)} \Big(e^z + \int_0^{\tau_R \wedge T} e^{Z_s} \big(\delta - g - Y_s\big) ds \Big),$$

by an integration by parts. Combining these two facts, and denoting by $C > 0$ a suitable constant independent of R and T, we can write

$$\begin{aligned}
0 &\leq \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} u(Z_{\tau_R \wedge T}, Y_{\tau_R \wedge T}) \Big] \leq \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} \Big(e^z + \int_0^{\tau_R \wedge T} e^{Z_s} \big(\delta - g - Y_s\big) ds \Big) \Big] \\
&\leq e^z \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} \Big] + \mathbb{E}_{(z,y)} \Big[e^{-\frac{\rho}{2}(\tau_R \wedge T)} \int_0^{\tau_R \wedge T} e^{-\frac{\rho}{2}(\tau_R \wedge T)} |\delta - g - Y_s| \, e^{Z_s} ds \Big] \\
&\leq e^z \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} \Big] + \mathbb{E}_{(z,y)} \Big[e^{-\frac{\rho}{2}(\tau_R \wedge T)} \int_0^{\tau_R \wedge T} e^{-\frac{\rho}{2}s} |\delta - g - Y_s| \, e^{Z_s} ds \Big] \\
&\leq e^z \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} \Big] + C \, \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} \Big]^{\frac{1}{2}} \mathbb{E}_{(z,y)} \Big[\int_0^\infty e^{-\frac{\rho}{2}s} |\delta - g - Y_s|^4 ds \Big]^{\frac{1}{4}} \\
&\qquad \times \mathbb{E}_{(z,y)} \Big[\int_0^\infty e^{-\frac{\rho}{2}s + 4Z_s} ds \Big]^{\frac{1}{4}},
\end{aligned} \tag{4.18}$$

where for the last step we have used Hölder inequality with respect to the measure $d\mathbb{P}_{(z,y)}$, Jensen inequality with respect to the measure $\frac{2}{\rho} e^{-\frac{\rho}{2}s} ds$, and again Hölder inequality, but now with respect to the measure $d\mathbb{P}_{(z,y)} \otimes \frac{2}{\rho} e^{-\frac{\rho}{2}s} ds$.

From the second of (2.3) it is easy to see that

$$\mathbb{E}_{(z,y)} \Big[\int_0^\infty e^{-\frac{\rho}{2}s} |\delta - g - Y_s|^4 ds \Big]^{\frac{1}{4}} \leq C_1(y), \tag{4.19}$$

for some $0 < C_1(y) < \infty$. On the other hand,

$$\begin{aligned}
\mathbb{E}_{(z,y)} \Big[\int_0^\infty e^{-\frac{\rho}{2}s + 4Z_s} ds \Big] &= \int_0^\infty e^{-\frac{\rho}{2}s} \mathbb{E}_{(z,y)} \big[e^{4Z_s} \big] ds \\
&= e^{4z} \int_0^\infty \exp \Big\{ -\frac{\rho}{2}s + 4(\delta - g)s - 4\mathbb{E}\Big[\int_0^s Y^y_u du \Big] + 8 \, \mathrm{Var} \Big[\int_0^s Y^y_u du \Big] \Big\} ds \\
&\leq C_2(z,y)
\end{aligned} \tag{4.20}$$

for some $0 < C_2(z,y) < \infty$, independent of R and T. The last inequality above is due to Assumption 2.3, upon using (B-3) and (B-4) from Appendix B and employing simple estimates.

Thanks to (4.19) and (4.20) we can then continue from (4.18) by writing

$$0 \leq \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} u(Z_{\tau_R \wedge T}, Y_{\tau_R \wedge T}) \Big] \leq e^z \, \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} \Big] + C_3(z,y) \, \mathbb{E}_{(z,y)} \Big[e^{-\rho(\tau_R \wedge T)} \Big]^{\frac{1}{2}},$$

for some $0 < C_3(z,y) < \infty$ and independent of R and T. Taking now limits as $R \uparrow \infty$, and invoking dominated convergence theorem we have from the latter

$$0 \leq \limsup_{R\uparrow\infty} \mathbb{E}_{(z,y)}\Big[e^{-\rho(\tau_R\wedge T)}u(Z_{\tau_R\wedge T}, Y_{\tau_R\wedge T})\Big] \leq e^{z}e^{-\rho T} + C_3(z,y)\, e^{-\frac{\rho}{2}T}. \quad (4.21)$$

(ii) By monotone convergence it follows that

$$\lim_{R\uparrow\infty} \mathbb{E}_{(z,y)}\bigg[\int_0^{\tau_R\wedge T} e^{-\rho s+2Z_s}\mathbb{1}_{\{Y_s>\hat{y}(Z_s)\}}ds\bigg] = \mathbb{E}_{(z,y)}\bigg[\int_0^{T} e^{-\rho s+2Z_s}\mathbb{1}_{\{Y_s>\hat{y}(Z_s)\}}ds\bigg]. \quad (4.22)$$

(iii) Also, writing $\big(\delta-g-\rho-Y_s\big) = \big(\delta-g-\rho-Y_s\big)^+ - \big(\delta-g-\rho-Y_s\big)^-$, $s\geq 0$, and applying monotone convergence to each of the resulting two integrals we find

$$\begin{aligned}&\lim_{R\uparrow\infty} \mathbb{E}_{(z,y)}\bigg[\int_0^{\tau_R\wedge T} e^{-\rho s+Z_s}\big(\delta-g-\rho-Y_s\big)\mathbb{1}_{\{Y_s\leq\hat{y}(Z_s)\}}ds\bigg]\\ &= \mathbb{E}_{(z,y)}\bigg[\int_0^{T} e^{-\rho s+Z_s}\big(\delta-g-\rho-Y_s\big)\mathbb{1}_{\{Y_s\leq\hat{y}(Z_s)\}}ds\bigg]. \quad (4.23)\end{aligned}$$

Then taking limits as $R\uparrow\infty$ in (4.17) and employing (4.21), (4.22) and (4.23) yield

$$\begin{aligned}&\mathbb{E}_{(z,y)}\bigg[\int_0^{T} e^{-\rho s+2Z_s}\mathbb{1}_{\{Y_s>\hat{y}(Z_s)\}}ds - \int_0^{T} e^{-\rho s+Z_s}\big(\delta-g-\rho-Y_s\big)\mathbb{1}_{\{Y_s\leq\hat{y}(Z_s)\}}ds\bigg]\\ &\leq u(z,y) \leq e^{z}e^{-\rho T} + C_3(z,y)\,e^{-\frac{\rho}{2}T} + \mathbb{E}_{(z,y)}\bigg[\int_0^{T} e^{-\rho s+2Z_s}\mathbb{1}_{\{Y_s>\hat{y}(Z_s)\}}ds\bigg] \quad (4.24)\\ &\quad - \mathbb{E}_{(z,y)}\bigg[\int_0^{T} e^{-\rho s+Z_s}\big(\delta-g-\rho-Y_s\big)\mathbb{1}_{\{Y_s\leq\hat{y}(Z_s)\}}ds\bigg].\end{aligned}$$

Finally, taking also limits as $T\uparrow\infty$ in (4.24), and arguing as in (ii) and (iii), we conclude that (4.15) holds true. □

Remark 4.8. *Proceeding similarly to the proof of Proposition 4.7, an application of Itô's lemma (together with standard localising arguments) to the process* $\{e^{-\rho t}u(Z_t^{z,y}, Y_t^y),\, t\geq 0\}$ *shows that*

$$u(z,y) = \mathbb{E}_{(z,y)}\bigg[\int_0^{\tau^*} e^{-\rho s+2Z_s}ds + e^{-\rho\tau^*+Z_{\tau^*}}\bigg], \quad (z,y)\in\mathbb{R}^2,$$

thus confirming the actual optimality of $\tau^* = \inf\{t\geq 0:\, Y_t^y \leq \hat{y}(Z_t^{z,y})\}$ *of* (4.7).

Setting

$$H_{\hat{y}}(z,y) := e^{2z}\mathbb{1}_{\{y>\hat{y}(z)\}} - (\delta-g-\rho-y)e^{z}\mathbb{1}_{\{y\leq\hat{y}(z)\}}, \quad (z,y)\in\mathbb{R}^2, \quad (4.25)$$

it follows that (4.15) takes the form

$$u(z,y) = \mathbb{E}_{(z,y)}\bigg[\int_0^{\infty} e^{-\rho s}H_{\hat{y}}(Z_s,Y_s)ds\bigg]. \quad (4.26)$$

Since $|H_{\hat{y}}(z,y)| \leq e^{2z} + |\delta - g - \rho - y|e^z$, $(z,y) \in \mathbb{R}^2$, it is a consequence of Lemma B.3 in Appendix B that $H_{\hat{y}}(Z_s, Y_s) \in L^1(\mathbb{P}_{(z,y)} \otimes e^{-\rho s}ds)$. This fact, together with strong Markov property and standard arguments based on conditional expectations applied to representation formula (4.15) (equivalently (4.26)) allow to conclude that for any stopping time $\tau \in \mathcal{T}$ and $(z,y) \in \mathbb{R}^2$

$$e^{-\rho\tau}u(Z^{z,y}_\tau, Y^y_\tau) + \int_0^\tau e^{-\rho s} H_{\hat{y}}(Z^{z,y}_s, Y^y_s)ds = \mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s} H_{\hat{y}}(Z_s, Y_s)ds \Big| \mathcal{F}_\tau\Big]. \tag{4.27}$$

In particular,

$$\Big\{e^{-\rho t}u(Z^{z,y}_t, Y^y_t) + \int_0^t e^{-\rho s} H_{\hat{y}}(Z^{z,y}_s, Y^y_s)ds,\, t \geq 0\Big\} \text{ is a uniformly integrable } \mathcal{F}_t\text{-martingale,} \tag{4.28}$$

and for any stopping time $\tau \in \mathcal{T}$

$$e^{-\rho\tau}u(Z^{z,y}_\tau, Y^y_\tau) \leq \mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s}\big|H_{\hat{y}}(Z_s, Y_s)\big|ds \Big| \mathcal{F}_\tau\Big], \quad (z,y) \in \mathbb{R}^2. \tag{4.29}$$

Hence the family of random variables $\{e^{-\rho\tau}u(Z^{z,y}_\tau, Y^y_\tau),\, \tau \in \mathcal{T}\}$ is uniformly integrable.

We now continue our analysis by providing the main result of this section (cf. also Theorem 3.3 in Section 3).

Theorem 4.9. *Recall* (3.3) *and let*

$$\mathcal{M} := \Big\{f : \mathbb{R} \mapsto \mathbb{R} \text{ continuous, nondecreasing, dominated from above by } \vartheta(z) := e^z + \delta - g - \rho\Big\}.$$

Then the boundary $\hat{y}(\,\cdot\,)$ is the unique function in $\mathcal{M}$ solving the nonlinear integral equation

$$\begin{aligned} e^z = &\int_0^\infty e^{-\rho t}\Big(\int_{\mathbb{R}^2} e^{2v}\mathbb{1}_{\{u>f(v)\}}p_t(z, f(z); v, u)dudv\Big)dt \\ -&\int_0^\infty e^{-\rho t}\Big(\int_{\mathbb{R}^2} e^{v}(\delta - g - \rho - u)\mathbb{1}_{\{u\leq f(v)\}}p_t(z, f(z); v, u)dudv\Big)dt. \end{aligned} \tag{4.30}$$

Proof. The proof is organised in several steps and it is inspired by arguments in Section 25 of [36] and references therein.

Step 1. To show existence of a solution to nonlinear integral equation (4.30) it suffices to show that $\hat{y}$ of (3.5) solves it. To see this we notice that by Propositions 4.3 and 4.6 $\hat{y} \in \mathcal{M}$. Moreover, evaluating both sides of (4.15) at $y = \hat{y}(z)$, $z \in \mathbb{R}$, one finds (3.10), upon using that $u(z, \hat{y}(z)) = e^z$ and expressing the expected value as an integral with respect to the probability density function (3.3) of the process (Z, Y).

We now move proving uniqueness and we argue by a contradiction scheme by supposing that there exists another function $b \in \mathcal{M}$ solving (4.30). In the following steps we will show that actually $\hat{y}(z) \leq b(z)$ for any $z \in \mathbb{R}$ (cf. *Step 2*), and also that $\hat{y}(z) \geq b(z)$ for any $z \in \mathbb{R}$ (cf. *Step 3*). To accomplish that for $(z,y) \in \mathbb{R}^2$ we define

$$w(z,y) := \mathbb{E}_{(z,y)}\bigg[\int_0^\infty e^{-\rho s} H_b(Z_s, Y_s) ds\bigg], \tag{4.31}$$

where

$$H_b(z,y) := e^{2z}\mathbb{1}_{\{y>b(z)\}} - (\delta - g - \rho - y)e^z \mathbb{1}_{\{y \leq b(z)\}}, \quad (z,y) \in \mathbb{R}^2. \tag{4.32}$$

Noticing that $|H_b(z,y)| \leq e^{2z} + e^z|\delta - g - \rho - y|$ and then arguing as in the proof of Lemma B.3 in Appendix B, one has that $H_b(Z_s, Y_s) \in L^1(\mathbb{P}_{(z,y)} \otimes e^{-\rho s} ds)$ under Assumption 2.3. Due to this fact one can then verify that

$$\Big\{e^{-\rho t} w(Z_t^{z,y}, Y_t^y) + \int_0^t e^{-\rho s} H_b(Z_s^{z,y}, Y_s^y) ds, \ t \geq 0\Big\} \text{ is an } \mathcal{F}_t\text{-martingale.} \tag{4.33}$$

In particular, it is an $\mathcal{F}_t$-uniformly integrable martingale. Moreover, for any stopping time $\tau \in \mathcal{T}$

$$e^{-\rho\tau} w(Z_\tau^{z,y}, Y_\tau^y) \leq \mathbb{E}_{(z,y)}\bigg[\int_0^\infty e^{-\rho s}\big|H_b(Z_s, Y_s)\big| ds \Big| \mathcal{F}_\tau\bigg], \tag{4.34}$$

and therefore the family of random variables $\{e^{-\rho\tau} w(Z_\tau^{z,y}, Y_\tau^y),\ \tau \in \mathcal{T}\}$ is uniformly integrable. Finally, in Lemma B.4 in Appendix B we show that $u \leq w$ on $\mathbb{R}^2$.

Step 2. We here prove that $\hat{y}(z) \leq b(z)$ for any $z \in \mathbb{R}$. Suppose that this is not true and that there exists a point $z_o \in \mathbb{R}$ such that $b(z_o) < \hat{y}(z_o)$. Then, taking $y < b(z_o)$ and setting $\sigma := \sigma(z_o, y) = \inf\{t \geq 0 : Y_t \geq \hat{y}(Z_t)\}$, $\mathbb{P}_{(z_o,y)}$-a.s., with regard to (4.31) and (4.33) one finds

$$\mathbb{E}_{(z_o,y)}\Big[e^{-\rho\sigma} w(Z_\sigma, Y_\sigma)\Big] = w(z_o, y) - \mathbb{E}_{(z_o,y)}\bigg[\int_0^\sigma e^{-\rho s} H_b(Z_s, Y_s) ds\bigg]. \tag{4.35}$$

On the other hand, (4.25) together with (4.28) imply

$$\mathbb{E}_{(z_o,y)}\Big[e^{-\rho\sigma} u(Z_\sigma, Y_\sigma)\Big] = u(z_o, y) + \mathbb{E}_{(z_o,y)}\bigg[\int_0^\sigma e^{-\rho s} e^{Z_s}\Big(\delta - g - \rho - Y_s\Big) ds\bigg]. \tag{4.36}$$

Substracting (4.35) from (4.36), noticing that $w(z_o, y) = e^{z_o} = u(z_o, y)$ since $y < b(z_o) < \hat{y}(z_o)$ by assumption, and considering that $u \leq w$ on $\mathbb{R}^2$ (cf. Lemma B.4 in Appendix B) we find

$$\begin{aligned} 0 &\geq \mathbb{E}_{(z_o,y)}\Big[e^{-\rho\sigma}\Big(u(Z_\sigma, Y_\sigma) - w(Z_\sigma, Y_\sigma)\Big)\Big] \\ &= \mathbb{E}_{(z_o,y)}\bigg[\int_0^\sigma e^{-\rho s} e^{Z_s}\Big(e^{Z_s} + \delta - g - \rho - Y_s\Big)\mathbb{1}_{\{b(Z_s) < Y_s < \hat{y}(Z_s)\}} ds\bigg]. \end{aligned} \tag{4.37}$$

However, $\sigma > 0$ $\mathbb{P}_{(z_o,y)}$-a.s. due to continuity of $(Z_\cdot, Y_\cdot)$ and continuity of $\hat{y}(\,\cdot\,)$, the set $\{(z,y) \in \mathbb{R}^2 : b(z) < y < \hat{y}(z)\}$ is open and not empty because of the continuity of $\hat{y}(\,\cdot\,)$ and $b(\,\cdot\,)$, and $\hat{y}(z) \leq e^z + \delta - g - \rho$. Hence the right-hand side of the latter is strictly positive and we reach a contradiction. Therefore, $\hat{y}(z) \leq b(z)$ for any $z \in \mathbb{R}$.

Step 3. Here we prove that $\hat{y}(z) \geq b(z)$ for any $z \in \mathbb{R}$. Assume by contradiction that there exists a point $z_o \in \mathbb{R}$ such that $\hat{y}(z_o) < b(z_o)$. Take $y \in (\hat{y}(z_o), b(z_o))$ and consider the stopping time $\tau^* := \tau^*(z_o, y) = \inf\{t \geq 0 : Y_t \leq \hat{y}(Z_t)\}$, $\mathbb{P}_{(z_o,y)}$-a.s. This is optimal for $u(z_o, y)$. Then by (4.25) and (4.26) one has

$$\mathbb{E}_{(z_o,y)}\Big[e^{-\rho\tau^*}u(Z_{\tau^*}, Y_{\tau^*})\Big] = u(z_o, y) - \mathbb{E}_{(z_o,y)}\Big[\int_0^{\tau^*} e^{-\rho s + 2Z_s} ds\Big], \tag{4.38}$$

whereas by (4.31) and (4.33) it follows

$$\mathbb{E}_{(z_o,y)}\Big[e^{-\rho\tau^*}w(Z_{\tau^*}, Y_{\tau^*})\Big] = w(z_o, y) - \mathbb{E}_{(z_o,y)}\Big[\int_0^{\tau^*} e^{-\rho s} H_b(Z_s, Y_s) ds\Big]. \tag{4.39}$$

Since by *Step 2* we already know that $\hat{y}(\,\cdot\,) \leq b(\,\cdot\,)$ then we can write

$$\mathbb{E}_{(z_o,y)}\Big[e^{-\rho\tau^*}u(Z_{\tau^*}, Y_{\tau^*})\Big] = \mathbb{E}_{(z_o,y)}\Big[e^{-\rho\tau^* + Z_{\tau^*}}\Big] = \mathbb{E}_{(z_o,y)}\Big[e^{-\rho\tau^*}w(Z_{\tau^*}, Y_{\tau^*})\Big]. \tag{4.40}$$

Then, substracting (4.39) from (4.38) and taking into account (4.40) and that $u \leq w$ on $\mathbb{R}^2$ (cf. Lemma B.4 in Appendix B) we find

$$0 \geq \mathbb{E}_{(z_o,y)}\Big[\int_0^{\tau^*} e^{-\rho s} e^{Z_s}\Big(e^{Z_s} + \delta - g - \rho - Y_s\Big)\mathbb{1}_{\{\hat{y}(Z_s) < Y_s \leq b(Z_s)\}} ds\Big]. \tag{4.41}$$

Now $\tau^* > 0$ $\mathbb{P}_{(z_o,y)}$-a.s. by continuity of $(Z_\cdot, Y_\cdot)$ and of $\hat{y}(\,\cdot\,)$. Moreover, the set $\{(z, y) \in \mathbb{R}^2 : \hat{y}(z) < y \leq b(z)\}$ is open and not empty because of the continuity of $\hat{y}(\,\cdot\,)$ and $b(\,\cdot\,)$, and $b(z) \leq e^z + \delta - g - \rho$ since $b \in \mathcal{M}$ by assumption. The right-hand side of (4.41) is therefore strictly positive and we reach a contradiction; that is, $\hat{y}(z) \geq b(z)$ for any $z \in \mathbb{R}$.

Step 4. By *Step 2* and *Step 3* we conclude that $\hat{y} \equiv b$ on $\mathbb{R}$ and we thus complete the proof. □

Remark 4.10. *Integral equation* (3.10) *belongs to the class of nonlinear Fredholm equations (see, e.g., [15] or [22]). Since the state space of* (Z, Y) *is unbounded,* (3.10) *is actually a singular nonlinear Fredholm equation of second kind. A survey of numerical methods for equations of this kind may be found in classical textbooks like [2] and [15]. Any of these methods is certainly non trivial, and we believe that such numerical computation falls outside the scopes of our work.*

5 The Optimal Cumulative Primary Balance and the Minimal Cost

In this section we provide the optimal debt reduction policy. As already anticipated in Theorem 3.2 this takes the form of a threshold policy, where the boundary triggering the optimal intervention rule is closely related to the stopping boundary $\hat{y}$ completely characterised in the last section.

5.1 The Action ad Inaction Regions

Recall that $\mathcal{O} := (0, \infty) \times \mathbb{R}$ and define

$$\mathcal{I} := \{(x, y) \in \mathcal{O} \mid u(\ln(x), y) < x\} \quad \text{and} \quad \mathcal{A} := \{(x, y) \in \mathcal{O} \mid u(\ln(x), y) = x\}. \tag{5.1}$$

The sets $\mathcal{I}$ and $\mathcal{A}$ are respectively the candidate *inaction* region and the candidate *action* region for the control problem (2.6). When the state variable belongs to $\mathcal{A}$ it should be optimal to reduce the level of public debt to GDP ratio, as it is too high. On the other hand, a non intervation policy should be applied in region $\mathcal{I}$. This will be verified in the following.

Thanks to Proposition 4.1, $\mathcal{I}$ and $\mathcal{A}$ are open and closed, respectively. Moreover, it is clear that they can be expressed also as $\mathcal{I} = \{(x,y) \in \mathcal{O} \mid \frac{1}{x}u(\ln(x),y) < 1\}$ and $\mathcal{A} := \{(x,y) \in \mathcal{O} \mid \frac{1}{x}u(\ln(x),y) = 1\}$, where a simple calculation from (3.4) reveals that

$$\frac{1}{x}u(\ln(x),y) := \inf_{\tau \geq 0} \mathbb{E}\bigg[\int_0^\tau xe^{-\rho t+2(\delta-g)t-2\int_0^t Y_s^y ds} + e^{-\rho\tau+(\delta-g)\tau-\int_0^\tau Y_s^y ds}\bigg], \quad (x,y) \in \mathcal{O}. \tag{5.2}$$

It follows from (5.2) that $y \mapsto \frac{1}{x}u(\ln(x),y)$ is nonincreasing and $x \mapsto \frac{1}{x}u(\ln(x),y)$ is nondecreasing. The latter property implies that for fixed $y \in \mathbb{R}$ the region $\mathcal{I}$ is below $\mathcal{A}$, and we define the boundary between these two regions by

$$b(y) := \sup\{x > 0 : \, u(\ln(x),y) < x\}, \qquad y \in \mathbb{R}, \tag{5.3}$$

with the convention $\sup\emptyset = 0$. Then $\mathcal{I}$ and $\mathcal{A}$ can be equivalently written as

$$\mathcal{I} := \{(x,y) \in \mathcal{O} \mid x < b(y)\} \quad \text{and} \quad \mathcal{A} := \{(x,y) \in \mathcal{O} \mid x \geq b(y)\}, \tag{5.4}$$

From the previous and from (4.1), (4.2) and (4.3) we also have

$$e^z \;<\; b(y) \iff u(z,y) \;<\; e^z \iff y \;>\; \hat{y}(z), \quad (z,y) \in \mathbb{R}^2. \tag{5.5}$$

Hence, for any $y \in \mathbb{R}$, b of (5.3) can be seen as the pseudo-inverse of the nondecreasing (cf. Proposition 4.3) function $z \mapsto \hat{y}(z)$ composed with the logarithmic function; that is,

$$b(y) = \sup\{x > 0 \mid y > \hat{y}(\ln(x))\}, \quad y \in \mathbb{R}. \tag{5.6}$$

It thus follows that the characterization of $\hat{y}$ of Theorem 4.9 is actually equivalent to a complete characterization of b thanks to (5.6).

The next proposition collects some properties of b, and its proof can be found in Appendix A.

Proposition 5.1. *The boundary b of* (5.3) *is such that*

1. *$y \mapsto b(y)$ is nondecreasing and left-continuous.*
2. *$y \mapsto b(y)$ is lower-semicontinuous.*
3. *$b(y) \geq y - (\delta - g - \rho)$ for any $y \in \mathbb{R}$.*

5.2 Optimal Control: a Verification Theorem

Recall (3.4) and define the function

$$U(x,y) := \int_{-\infty}^{\ln(x)} u(q,y)dq, \quad (x,y) \in \mathcal{O}. \tag{5.7}$$

Notice that U is well defined and finite as we can also write $U(x,y) = \int_0^x \frac{1}{z} u(\ln(z), y) dz$, $(x,y) \in \mathcal{O}$, with $\frac{1}{z}u(\ln(z),y)$ as in (5.2). Also, for b as in (5.3) introduce the nondecreasing process

$$\overline{\nu}^*_t = \Big[x - \inf_{0 \le s \le t} \Big(b(Y^y_s) e^{-(\delta - g)s + \int_0^s Y^y_u du} \Big) \Big] \vee 0, \quad t \ge 0, \qquad \overline{\nu}^*_{0-} = 0, \tag{5.8}$$

and then the process

$$\nu^*_t := \int_0^t e^{(\delta-g)s - \int_0^s Y^y_u du} d\overline{\nu}^*_s, \quad t \ge 0, \qquad \nu^*_{0-} = 0. \tag{5.9}$$

Proposition 5.2. *The process ν^* of* (5.9) *is an admissible control.*

Proof. Let $(x,y) \in \mathcal{O}$ be given and fixed, and recall the set of admissible controls $\mathcal{A}(x,y)$ of (2.4). It is clear that ν^* is nondecreasing. Moreover, since (cf. the first of (2.3))

$$X^{x,y,\nu^*}_t = e^{(\delta-g)t - \int_0^t Y^y_s ds}\big[x - \overline{\nu}^*_t\big],$$

it follows by (5.8) that ν^* is such that $X^{x,y,\nu^*}_t \ge 0$ a.s. for all $t \ge 0$.

To prove that $\nu^* \in \mathcal{A}(x,y)$ it then remains to show that: *i)* $t \mapsto \nu^*_t$ is right-continuous; *ii)* ν^* is $(\mathcal{F}_t)$-adapted. We will prove that these two properties are fulfilled by $\overline{\nu}^*$ of (5.8), since this implies that ν^* satisfies them as well.

We start by proving *i)* for $\overline{\nu}^*$. To this end, first notice that

$$\liminf_{s \downarrow t} b(Y^y_s) e^{-(\delta-g)s + \int_0^s Y^y_u du} \ge b(Y^y_t) e^{-(\delta-g)t + \int_0^t Y^y_u du} \tag{5.10}$$

by lower-semicontinuity of b (cf. Proposition 5.1) and continuity of $(\int_0^\cdot Y^y_u du, Y^y_\cdot)$. Moreover, from (5.10) we obtain

$$\begin{aligned}
&\lim_{s \downarrow t} \inf_{0 \le u \le s} \Big(b(Y^y_u) e^{-(\delta-g)u + \int_0^u Y^y_r dr} \Big) \\
&= \inf_{0 \le u \le t} \Big(b(Y^y_u) e^{-(\delta-g)u + \int_0^u Y^y_r dr} \Big) \wedge \lim_{s \downarrow t} \inf_{t < u \le s} \Big(b(Y^y_u) e^{-(\delta-g)u + \int_0^u Y^y_r dr} \Big) \\
&= \inf_{0 \le u \le t} \Big(b(Y^y_u) e^{-(\delta-g)u + \int_0^u Y^y_r dr} \Big) \wedge \liminf_{s \downarrow t} \Big(b(Y^y_s) e^{-(\delta-g)s + \int_0^s Y^y_u du} \Big) \\
&\ge \inf_{0 \le u \le t} \Big(b(Y^y_u) e^{-(\delta-g)u + \int_0^u Y^y_r dr} \Big) \wedge b(Y^y_t) e^{-(\delta-g)t + \int_0^t Y^y_u du} = \inf_{0 \le u \le t} \Big(b(Y^y_u) e^{-(\delta-g)u + \int_0^u Y^y_r dr} \Big),
\end{aligned} \tag{5.11}$$

which by (5.8) in turn gives $\lim_{s \downarrow t} \overline{\nu}^*_s \le \overline{\nu}^*_t$. Since $\lim_{s \downarrow t} \overline{\nu}^*_s \ge \overline{\nu}^*_t$ by monotonicity, we have proved right-continuity of $t \mapsto \overline{\nu}^*_t$.

As for *ii)*, notice that the process $\{e^{-\int_0^t Y^y_u du} b(Y^y_t),\, t \ge 0\}$ is progressively measurable since it is the product of two progressively measurable processes. Indeed, $e^{-\int_0^t Y^y_u du}$ is continuous and adapted, whereas $b(Y^y_t)$ is the composition of the Borel-measurable function b (which is lower-semicontinuous by Proposition 5.1) with the progressively measurable process Y^y_t. Therefore $\overline{\nu}^*$ is progressively measurable by Theorem IV.33, part (a), in [14], hence adapted. □

We can now prove the main result of this section, which shows optimality of (5.9) for the debt control problem (2.6). Its proof is probabilistic and relies on a change of variable formula already employed in the context of singular control problems (see, e.g., [3] and [17]).

Theorem 5.3. *Let U be as in* (5.7) *and v as in* (2.6). *Then one has $U = v$ on $\mathcal{O}$, and ν^* as in* (5.9) *is optimal for the control problem* (2.6).

Proof. Let $(x, y) \in \mathcal{O}$ be given and fixed. For $\nu \in \mathcal{A}(x, y)$, introduce the admissible control $\overline{\nu}$ such that $\overline{\nu}_t := \int_0^t e^{-(\delta-g)s+\int_0^s Y_u^y du} d\nu_s$, $t \geq 0$, and define its inverse (see, e.g., Chapter 0, Section 4 of [39]) by

$$\tau^{\overline{\nu}}(q) := \inf\{t \geq 0 \mid x - \overline{\nu}_t < e^q\}, \qquad q \leq \ln(x). \tag{5.12}$$

The process $\tau^{\overline{\nu}}(q) := \{\tau^{\overline{\nu}}(q),\ q \leq \ln(x)\}$ has decreasing, left-continuous sample paths and hence it admits right-limits

$$\tau_+^{\overline{\nu}}(q) := \inf\{t \geq 0 \mid x - \overline{\nu}_t \leq e^q\}, \qquad q \leq \ln(x). \tag{5.13}$$

Moreover, the set of points $q \in \mathbb{R}$ at which $\tau^{\overline{\nu}}(q)(\omega) \neq \tau_+^{\overline{\nu}}(q)(\omega)$ is a.s. countable for a.e. $\omega \in \Omega$.

Since $\overline{\nu}$ is right-continuous and $\tau^{\overline{\nu}}(q)$ is the first entry time of an open set, it is an $(\mathcal{F}_{t+})$-stopping time for any given and fixed $q \leq \ln(x)$. However, $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, hence $\tau^{\overline{\nu}}(q)$ is an $(\mathcal{F}_t)$-stopping time. Moreover, $\tau_-^{\overline{\nu}}(q)$ is the first entry time of the right-continuous process $\overline{\nu}$ into a closed set and hence it is an $(\mathcal{F}_t)$-stopping time as well for any $q \leq \ln(x)$.

With regard to (3.4) we can then write

$$\begin{aligned} U(x, y) = \int_{-\infty}^{\ln(x)} u(q, y) dq \leq \int_{-\infty}^{\ln(x)} \mathbb{E}\bigg[\int_0^{\tau^{\overline{\nu}}(q)} e^{-\rho t + 2q + 2(\delta-g)t - 2\int_0^t Y_s^y ds} dt \\ + e^{-\rho\tau^{\overline{\nu}}(q) + q + (\delta-g)\tau^{\overline{\nu}}(q) - \int_0^{\tau^{\overline{\nu}}(q)} Y_s^y ds}\bigg] dq, \end{aligned} \tag{5.14}$$

and we now consider the two terms in the last integral above separately.

Setting $\tau^{\overline{\nu}}(\xi) := \inf\{t \geq 0 \mid \overline{\nu}_t > x - \xi\}$, $\xi \in [0, x]$, we have

$$\begin{aligned} &\int_{-\infty}^{\ln(x)} \mathbb{E}\Big[e^{-\rho\tau^{\overline{\nu}}(q) + q + (\delta-g)\tau^{\overline{\nu}}(q) - \int_0^{\tau^{\overline{\nu}}(q)} Y_s^y ds}\Big] dq = \int_0^x \mathbb{E}\Big[e^{-\rho\tau^{\overline{\nu}}(\xi) + (\delta-g)\tau^{\overline{\nu}}(\xi) - \int_0^{\tau^{\overline{\nu}}(\xi)} Y_s^y ds}\Big] d\xi \\ &= \mathbb{E}_{(x,y)}\bigg[\int_0^\infty e^{-\rho t + (\delta-g)t - \int_0^t Y_s ds} d\overline{\nu}_t\bigg] = \mathbb{E}_{(x,y)}\bigg[\int_0^\infty e^{-\rho t} d\nu_t\bigg], \end{aligned} \tag{5.15}$$

where the second step follows from the change of variable formula in Chapter 0, Proposition 4.9 of [39] (see also eq. (4.7) in [3]) and Tonelli's theorem.

On the other hand, because for any $t \geq 0$ one has $t < \tau^{\overline{\nu}}(q)$ if and only if $\overline{\nu}_t < x - e^q$, by Tonelli's theorem

$$\begin{aligned} &\int_{-\infty}^{\ln(x)} \mathbb{E}\bigg[\int_0^{\tau^{\overline{\nu}}(q)} e^{-\rho t + 2q + 2(\delta-g)t - 2\int_0^t Y_s^y ds} dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^\infty \int_{-\infty}^{\ln(x)} e^{-\rho t + 2(\delta-g)t - 2\int_0^t Y_s^y ds} \mathbb{1}_{\{t < \tau^{\overline{\nu}}(q)\}} e^{2q} dq\, dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho t + 2(\delta-g)t - 2\int_0^t Y_s^y ds} \Big(\int_{-\infty}^{\ln(x)} \mathbb{1}_{\{\overline{\nu}_t < x - e^q\}} e^{2q} dq\Big) dt\bigg] \\ &= \mathbb{E}\bigg[\int_0^\infty e^{-\rho t + 2(\delta-g)t - 2\int_0^t Y_s^y ds} \frac{1}{2}(x - \overline{\nu}_t)^2 dt\bigg] = \mathbb{E}_{(x,y)}\bigg[\int_0^\infty e^{-\rho t} \frac{1}{2}(X_t^\nu)^2 dt\bigg], \end{aligned} \tag{5.16}$$

where the last equality follows from the first of (2.3).

Combining (5.15) and (5.16) we thus have $U(x,y) \leq \mathcal{J}_{x,y}(\nu)$. Hence, since ν was arbitrary

$$U(x,y) \leq v(x,y), \quad (x,y) \in \mathcal{O}. \tag{5.17}$$

Now we want to show that picking ν^* as in (5.9) (equivalently, picking $\overline{\nu}^*$ as in (5.8)) in the arguments above, all the inequalities become equalities. First of all, recall that the stopping time

$$\tau^*(q,y) = \inf\{t \geq 0 \mid Y^y_t \leq \hat{y}(Z^{q,y}_t)\}$$

is optimal for $u(q,y)$ (cf. (4.7)). Then, fix $(x,y) \in \mathcal{O}$, take $t \geq 0$ arbitrary, and note that, by (5.13) and (5.5), we have $\mathbb{P}$-a.s. the equivalences

$$\tau^{\overline{\nu}^*}_+(q) \leq t \iff \overline{\nu}^*_t \geq x - e^q \iff \Big[x - \inf_{0\leq s\leq t}\Big(b(Y^y_s)e^{-(\delta-g)s+\int_0^s Y^y_u du}\Big)\Big] \vee 0 \geq x - e^q$$
$$\iff b(Y^y_\theta) \leq e^{q+(\delta-g)\theta-\int_0^\theta Y^y_s ds} \text{ for some } \theta \in [0,t] \iff Y^y_\theta \leq \hat{y}(Z^{q,y}_\theta) \text{ for some } \theta \in [0,t]$$
$$\iff \tau^*(q,y) \leq t.$$

Hence, we can conclude that $\tau^{\overline{\nu}^*}_+(q) = \tau^*(q,y)$ $\mathbb{P}$-a.s. and for a.e. $q \leq \ln(x)$. However, by (5.12) and (5.13), we also have $\tau^{\overline{\nu}^*}_+(q) = \tau^{\overline{\nu}^*}(q)$ $\mathbb{P}$-a.s. and for a.e. $q \leq \ln(x)$; that is,

$$\tau^{\overline{\nu}^*}(q) = \tau^*(q,y) \ \mathbb{P}\text{-a.s. and for a.e. } q \leq \ln(x). \tag{5.18}$$

We can now take $\nu = \nu^*$ (equivalently, $\overline{\nu} = \overline{\nu}^*$) in (5.14) so to obtain equality there. Then, arguing as in (5.15) and (5.16) yields $U(x,y) = \mathcal{J}_{x,y}(\nu^*)$. Hence $U = v$ on $\mathcal{O}$, by (5.17), and ν^* is optimal. □

A direct byproduct of Theorem 5.3 is the following.

Corollary 5.4. *The identity* $v_x(x,y) = \frac{1}{x}u(\ln(x),y)$ *holds true on* $\mathcal{O}$.

This result is consistent with the fact that problems of singular stochastic control with performance criterion which is either convex or concave with respect to the control variable are related to questions of optimal stopping. In particular, the derivative of the control problem's value function in the direction of the controlled state variable equals the value of an optimal stopping problem. We refer to [1], [3] and [31], among others, as classical references.

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A Some Proofs

Proof of Proposition 2.4

We start obtaining the two bounds of (2.7). From (2.5), it is easy to see that $\mathcal{J}_{x,y}(\nu) \geq 0$ for any $\nu \in \mathcal{A}$. Hence $v(x,y) \geq 0$. On the other hand, being the admissible policy $\nu \equiv 0$ a priori suboptimal, we have

$$v(x,y) \leq \frac{1}{2}\mathbb{E}_{(x,y)}\Big[\int_0^\infty e^{-\rho t}(X^0_t)^2 dt\Big] = \frac{1}{2}\int_0^\infty e^{-\rho t}\mathbb{E}_{(x,y)}\big[(X^0_t)^2\big]dt, \tag{A-1}$$

where the last step follows by Tonelli's Theorem. To evaluate the last integral in the right-hand side of (A-1) notice that by (2.3) we can write for any $t \geq 0$

$$\begin{aligned}\mathbb{E}_{(x,y)}\big[(X^0_t)^2\big] &= x^2 e^{2(\delta-g)t}\mathbb{E}\Big[e^{-2\int_0^t Y^y_u du}\Big] \\ &= x^2 \exp\Big\{2(\delta-g)t - 2\mathbb{E}\Big[\int_0^t Y^y_u du\Big] + 2\,\mathrm{Var}\Big[\int_0^t Y^y_u du\Big]\Big\}, \end{aligned} \tag{A-2}$$

where the formula of the Laplace transform for the Gaussian random variable $\int_0^t Y^y_u du$ implies the final step. Plugging (B-3) and (B-4) of Lemma B.1 into (A-2), simple algebra and standard inequalities yield

$$\mathbb{E}_{(x,y)}\big[(X^0_t)^2\big] \leq x^2 e^{\frac{2}{\theta}|y-\frac{a}{\theta}|} \exp\Big\{-2t\Big(\frac{a}{\theta} - \frac{\sigma^2}{\theta^2} - \delta + g\Big)\Big\}.$$

Employing the inequality above in (A-1), and invoking Assumption 2.3 finally give the upper bound in (2.7).

The property $v(0,y) = 0$ for any $y \in \mathbb{R}$ follows by noticing that $\mathcal{A}(0,y) = \{\nu \equiv 0\}$ and therefore $v(0,y) = \mathcal{J}_{0,y}(0) = 0$.

We now turn to show convexity of $x \mapsto v(x,y)$. To this end, fix $y \in \mathbb{R}$, take $x_1 > 0$, $x_2 > 0$ and controls $\nu^{(1)} \in \mathcal{A}(x_1,y)$, $\nu^{(2)} \in \mathcal{A}(x_2,y)$. Then, for $\gamma \in [0,1]$, set $z := \lambda x_1 + (1-\lambda)x_2$ and $\nu := \lambda\nu^{(1)} + (1-\lambda)\nu^{(2)}$. Because (2.1) is linear, we have $X^{z,y,\nu}_t = \lambda X^{x_1,y,\nu^{(1)}}_t + (1-\lambda)X^{x_2,y,\nu^{(2)}}_t$, for any $t \geq 0$ and it is easy to see from (2.5) that

$$\mathcal{J}_{z,y}(\nu) \leq \lambda\mathcal{J}_{x_1,y}(\nu^{(1)}) + (1-\lambda)\mathcal{J}_{x_2,y}(\nu^{(2)}),$$

by convexity of $h(x) = x^2/2$. Since $\nu \in \mathcal{A}(z,y)$

$$v(\lambda x_1 + (1-\lambda)x_2) \leq \mathcal{J}_{z,y}(\nu) \leq \lambda\mathcal{J}_{x_1,y}(\nu^{(1)}) + (1-\lambda)\mathcal{J}_{x_2,y}(\nu^{(2)}),$$

which yields convexity of $v(\cdot,y)$ by arbitrariness of $\nu^{(1)}$ and $\nu^{(2)}$. □

Proof of Proposition 4.2

Assume by contradiction that

$$\mathcal{S} = \{(z,y) \in \mathbb{R}^2 : u(z,y) = e^z\} = \emptyset.$$

This would imply that for any $(z,y) \in \mathbb{R}^2$ one has

$$e^z > u(z,y) = \mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho t + 2Z_t}dt\Big] = e^{2z}\int_0^\infty e^{-(\rho-2(\delta-g))t}\mathbb{E}\Big[e^{-2\int_0^t Y^y_s ds}\Big]dt; \tag{A-3}$$

that is

$$1 > e^z \int_0^\infty e^{-(\rho-2(\delta-g))t}\mathbb{E}\Big[e^{-2\int_0^t Y^y_s ds}\Big]dt. \tag{A-4}$$

The expected value on the right-hand side of (A-4) can be evaluated by exploiting the fact that for any $t \geq 0$ the random variable $\int_0^t Y^y_s ds$ is Gaussian (see Lemma B.1 in Appendix B). In particular one has,

$$\mathbb{E}\Big[e^{-2\int_0^t Y^y_s ds}\Big] = \exp\Big\{-2\mathbb{E}\Big[\int_0^t Y^y_u du\Big] + 2\,\mathrm{Var}\Big[\int_0^t Y^y_u du\Big]\Big\}$$

and it thus follows from (B-3) and (B-4) in Appendix B and Assumption 2.3 that there exist positive (and finite) constants $C_1(y)$, $C_2(y)$ such that

$$C_1(y) \leq \int_0^{\infty} e^{-(\rho-2(\delta-g))t}\mathbb{E}\Big[e^{-2\int_0^t Y_s^y ds}\Big]dt \leq C_2(y). \tag{A-5}$$

We therefore reach a contradiction in (A-4) by taking z sufficiently big, for any given and fixed $y \in \mathbb{R}$. □

Proof of Proposition 4.3

(i) From (4.4) one can easily see that it is never optimal to stop the evolution of (Z, Y) in the region $\mathcal{U} := \{(z, y) \in \mathbb{R}^2 : e^z + \delta - g - y - \rho < 0\}$. That is $\mathcal{U} \subset \mathcal{C}$. Hence $\mathcal{S} \subseteq \mathcal{U}^c$ and $\hat{y}(z) \leq e^z + \delta - g - \rho$ for any $z \in \mathbb{R}$.

(ii) Let $(z_1, y) \in \mathcal{S}$ be given and fixed, take an arbitrary $z_2 \geq z_1$, and let $\tau^\varepsilon := \tau^\varepsilon(z_2, y)$ be ε-optimal for $u(z_2, y)$ (cf. (4.7)). Then we have

$$\begin{aligned}
0 \geq u(z_2, y) - e^{z_2} &\geq \mathbb{E}\bigg[\int_0^{\tau^\varepsilon} e^{-\rho t + z_2 + Z_t^{0,y}}\Big(e^{z_2 + Z_t^{0,y}} + \delta - g - \rho - Y_t^y\Big)dt\bigg] - \varepsilon \\
&= \mathbb{E}\bigg[\int_0^{\tau^\varepsilon} e^{-\rho t + z_1 + Z_t^{0,y}} e^{z_2 - z_1}\Big(e^{z_1 + Z_t^{0,y}} e^{z_2 - z_1} + \delta - g - \rho - Y_t^y\Big)dt\bigg] - \varepsilon \\
&= e^{z_2 - z_1}\mathbb{E}\bigg[\int_0^{\tau^\varepsilon} e^{-\rho t + Z_t^{z_1,y}}\Big(e^{Z_t^{z_1,y}} e^{z_2 - z_1} + \delta - g - \rho - Y_t^y\Big)dt\bigg] - \varepsilon \\
&\geq e^{z_2 - z_1}\mathbb{E}\bigg[\int_0^{\tau^\varepsilon} e^{-\rho t + Z_t^{z_1,y}}\Big(e^{Z_t^{z_1,y}} + \delta - g - \rho - Y_t^y\Big)dt\bigg] - \varepsilon \\
&\geq e^{z_2 - z_1}\big(u(z_1, y) - e^{z_1}\big) - \varepsilon = -\varepsilon,
\end{aligned} \tag{A-6}$$

where we have used that τ^ε is suboptimal for $u(z_1, y)$. It thus follows by arbitraryness of $\varepsilon > 0$ that $(z_2, y) \in \mathcal{S}$ for any $z_2 \geq z_1$, and therefore that $z \mapsto \hat{y}(z)$ is nondecreasing. □

Proof of Corollary 4.5

Results based on partial differential equations of parabolic type (cf. Chapter V of [34]), together with Itô's lemma and optional sampling theorem (cf. [36], p. 131) imply that $u \in C^{1,2}$ inside $\mathcal{C} \cap \{(z, y) \in \mathbb{R}^2 : y \neq \delta - g\}$, and there it uniquely solves

$$\big(\mathbb{L}_{Z,Y} - \rho\big)u = -e^{2z}; \tag{A-7}$$

that is (cf. (3.2))

$$\frac{1}{2}\sigma^2 u_{yy} = \rho u - (a - \theta y)u_y - (\delta - g - y)u_z - e^{2z}. \tag{A-8}$$

Since the right-hand side of the previous equation is continuous on $\mathbb{R}^2$ by Proposition 4.4, we conclude that u_{yy} is continuous at $(z, \delta - g)$ as well, for any $z \in \mathbb{R}$, and therefore $u \in C^{1,2}$ on the whole $\mathcal{C}$. But now again, since the right-hand side of (A-8) only involves u, its first derivatives and continuous functions, it is continuous in $\overline{\mathcal{C}}$ by Proposition 4.4. The claim thus follows. □

Proof of Proposition 5.1

1. The first claim follows from the fact that the mapping $y \mapsto \frac{1}{x}u(\ln(x), y)$ (cf. (5.2)) is nonincreasing. This indeed implies that if $(x, y_1) \in \mathcal{I}$, then $(x, y_2) \in \mathcal{I}$ for any $y_2 > y_1$. The fact that $b(\cdot)$ is left-continuous is implied by its monotonicity and by continuity of $(x, y) \mapsto \frac{1}{x}u(\ln(x), y)$ by following arguments similar to those employed in the first part of the proof of Proposition 4.6.

2. Notice that by (5.5) one has

$$\{y \in \mathbb{R} : x < b(y)\} = \{y \in \mathbb{R} : u(\ln(x), y) - x < 0\}, \tag{A-9}$$

for any given $x > 0$. The set on the right-hand side above is open since it is the preimage of an open set via the continuous mapping $y \mapsto u(\ln(x), y) - x$ (cf. Proposition 4.1). Hence the set on the left-hand side of (A-9) is open as well and $y \mapsto b(y)$ is therefore lower-semicontinuous.

3. The claim follows from (5.5) and the fact that $\hat{y}(z) \leq e^z + \delta - g - \rho$ for any $z \in \mathbb{R}$ (cf. Proposition 4.3). □

B Auxiliary Results

Lemma B.1. *Let Y be the Ornstein-Uhlenbeck process of (2.2). Then for any $t \geq 0$ one has*

$$\mathbb{E}[Y_t^y] = ye^{-\theta t} + \frac{a}{\theta}(1 - e^{-\theta t}) \tag{B-1}$$

and

$$\mathbb{E}[(Y_t^y)^2] = \big(ye^{-\theta t} + \frac{a}{\theta}(1 - e^{-\theta t})\big)^2 + \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}). \tag{B-2}$$

Moreover, the integral process $\{\int_0^t Y_u^y du, t \geq 0\}$ is Gaussian and such that

$$\mathbb{E}\bigg[\int_0^t Y_u^y du\bigg] = \frac{a}{\theta}t + \Big(y - \frac{a}{\theta}\Big)\left(\frac{1 - e^{-\theta t}}{\theta}\right). \tag{B-3}$$

and

$$\mathrm{Var}\left[\int_0^t Y_u^y du\right] = -\frac{\sigma^2}{2\theta^3}(1 - e^{-\theta t})^2 + \frac{\sigma^2}{\theta^2}\left(t - \left(\frac{1 - e^{-\theta t}}{\theta}\right)\right). \tag{B-4}$$

Proof. (B-1) and (B-2) are easily obtained from (2.2). On the other hand, for (B-3) and (B-4) we refer to page 122 of [28]. □

Lemma B.2. *For $Z^{z,y}$ as in (3.1) one has*

$$\liminf_{t \uparrow \infty} e^{-\rho t + Z_t} = 0, \quad \mathbb{P}_{(z,y)} - a.s. \tag{B-5}$$

Proof. By nonnegativity of $e^{-\rho t + Z_t}$, $t \geq 0$, we can invoke Fatou's lemma and obtain

$$0 \leq \mathbb{E}_{(z,y)}\Big[\liminf_{t \uparrow \infty} e^{-\rho t + Z_t}\Big] \leq \liminf_{t \uparrow \infty} \mathbb{E}_{(z,y)}\Big[e^{-\rho t + Z_t}\Big], \tag{B-6}$$

and the proof is complete if we show that $\liminf_{t\uparrow\infty}\mathbb{E}_{(z,y)}\Big[e^{-\rho t+Z_t}\Big]=0$.

To this end, notice that by (3.1) we can write

$$\begin{aligned}\mathbb{E}_{(z,y)}\Big[e^{-\rho t+Z_t}\Big]&=e^{z-(\rho-\delta+g)t}\mathbb{E}\Big[e^{-\int_0^t Y_s^y ds}\Big]\\&=\exp\Big\{z-(\rho-\delta+g)t-\mathbb{E}\Big[\int_0^t Y_s^y ds\Big]+\frac{1}{2}\operatorname{Var}\Big(\int_0^t Y_s^y ds\Big)\Big\},\end{aligned}\tag{B-7}$$

where we have used that for any given $t\geq 0$ the random variable $\int_0^t Y_s^y ds$ is Gaussian (cf. Lemma B.1). Then, employing (B-3) and (B-4) in (B-7) one has

$$\mathbb{E}_{(z,y)}\Big[e^{-\rho t+Z_t}\Big]\leq\exp\Big\{z-\Big(\rho-\delta+g+\frac{a}{\theta}-\frac{\sigma^2}{\theta^2}\Big)t-\big(y-\frac{a}{\theta}\big)\big(\frac{1-e^{-\theta t}}{\theta}\big)\Big\},$$

which clearly converges to zero as $t\uparrow\infty$ by Assumption 2.3. □

Lemma B.3. *For any $(z,y)\in\mathbb{R}^2$ one has*

$$\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s+2Z_s}ds\Big]+\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s+Z_s}|\delta-g-\rho-Y_s|ds\Big]<\infty.\tag{B-8}$$

Proof. We consider the two terms in (B-8) above separately.

By Tonelli's theorem, and by employing the fact that for any $s\geq 0$ the random variable $\int_0^s Y_u^y du$ is Gaussian (cf. Lemma B.1) we have

$$\begin{aligned}\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s+2Z_s}ds\Big]&=e^{2z}\int_0^\infty e^{-\rho s+2(\delta-g)s}\mathbb{E}\Big[e^{-2\int_0^s Y_u^y du}\Big]ds\\&=e^{2z}\int_0^\infty e^{-\rho s+2(\delta-g)s}\exp\Big\{-2\mathbb{E}\Big[\int_0^t Y_u^y du\Big]+2\operatorname{Var}\Big[\int_0^t Y_u^y du\Big]\Big\}.\end{aligned}\tag{B-9}$$

Recalling now (B-3) and (B-4) and performing simple estimates we have that there exists $0<C_1(z,y)<\infty$ such that

$$\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s+2Z_s}ds\Big]\leq C_1(z,y).$$

As for the second expectation in (B-8), Hölder inequality with respect the product measure $\mathbb{P}_{(z,y)}\otimes e^{-\rho s}ds$ gives

$$\begin{aligned}&\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s+Z_s}|\delta-g-\rho-Y_s|ds\Big]\leq\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s+2Z_s}ds\Big]^{\frac{1}{2}}\times\\&\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s}|\delta-g-\rho-Y_s|^2ds\Big]^{\frac{1}{2}}\leq\sqrt{C_1(z,y)}\mathbb{E}_{(z,y)}\Big[\int_0^\infty e^{-\rho s}|\delta-g-\rho-Y_s|^2ds\Big]^{\frac{1}{2}}.\end{aligned}$$

Since now by the second of (2.3) $\mathbb{E}_{(z,y)}[\int_0^\infty e^{-\rho s}|\delta-g-\rho-Y_s|^2ds]\leq C_2(1+|y|^2)$, for some $C_2>0$, the proof is complete. □

Lemma B.4. *Let w be defined by (4.25) and let u be given in terms of representation (4.26). Then $u\leq w$ on $\mathbb{R}^2$.*

Proof. The proof is organised in four steps.

Step 1. Since by assumption b solves integral equation (4.30), then $w(z,b(z)) = e^z$, $z \in \mathbb{R}$, and therefore

$$w(z,b(z)) = e^z \geq u(z,b(z)), \quad z \in \mathbb{R} \tag{B-10}$$

Step 2. Here we show that $w(z,y) = e^z$ for any $y < b(z)$ and $z \in \mathbb{R}$. This fact clearly implies that

$$w(z,y) = e^z \geq u(z,y), \quad y < b(z),\, z \in \mathbb{R}. \tag{B-11}$$

Let $z \in \mathbb{R}$ be given and fixed, take $y < b(z)$ and notice that by definition of H_b (cf. (4.32)) one has

$$H_b(Z_s,Y_s) = -(\delta - g - \rho - Y_s)e^{Z_s}, \quad \forall s \leq \sigma \;\; \mathbb{P}_{(z,y)} - a.s.$$

where we have defined $\sigma := \inf\{t \geq 0 : Y_t \geq b(Z_t)\}$, $\mathbb{P}_{(z,y)}$-a.s. Then martingale property of $\{e^{-\rho t}w(Z_t,Y_t) + \int_0^t e^{-\rho s}H_b(Z_s,Y_s)ds, t \geq 0\}$ (cf. (4.33)) and optional sampling theorem (cf. Chapter II, Theorem 3.2 in [39]) yield

$$w(z,y) = \mathbb{E}_{(z,y)}\left[e^{-\rho\sigma + Z_\sigma}\mathbb{1}_{\{\sigma<n\}} + e^{-\rho n}w(Z_n,Y_n)\mathbb{1}_{\{\sigma\geq n\}} + \int_0^{\sigma\wedge n} e^{-\rho s}H_b(Z_s,Y_s)ds\right], \tag{B-12}$$

for $y < b(z)$, $z \in \mathbb{R}$. Since now $\mathbb{E}_{(z,y)}[e^{-\rho n}w(Z_n,Y_n)\mathbb{1}_{\{\sigma\geq n\}}] = \mathbb{E}_{(z,y)}[\mathbb{1}_{\{\sigma\geq n\}}\int_n^\infty e^{-\rho s}H_b(Z_s,Y_s)ds]$ and $H_b(Z_s,Y_s) \in L^1(\mathbb{P}_{(z,y)} \otimes e^{-\rho s})$, we can take limits as $n \uparrow \infty$ in (B-12) and obtain

$$w(z,y) = \mathbb{E}_{(z,y)}\left[e^{-\rho\sigma + Z_\sigma} - \int_0^\sigma e^{-\rho s + Z_s}\Big(\delta - g - \rho - Y_s\Big)ds\right] = e^z, \tag{B-13}$$

for $y < b(z)$, $z \in \mathbb{R}$, and where the last equality is due to an integration by parts. Hence, (B-11) follows.

Step 3. Here we show that

$$w(z,y) \geq u(z,y), \quad y > b(z),\, z \in \mathbb{R}. \tag{B-14}$$

Let $z \in \mathbb{R}$ be given and fixed, take $y > b(z)$ and consider the stopping time $\tau := \inf\{t \geq 0 : Y_t \leq b(Z_t)\}$, $\mathbb{P}_{(z,y)}$-a.s. Then, arguing as in *Step 2* we find

$$w(z,y) = \mathbb{E}_{(z,y)}\left[e^{-\rho\tau + Z_\tau} + \int_0^\tau e^{-\rho s + 2Z_s}ds\right] \geq u(z,y). \tag{B-15}$$

Step 4. Combining *Step 1*, *Step 2* and *Step 3* we conclude that $w \geq u$ on $\mathbb{R}^2$. □

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