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A solvable two-dimensional degenerate singular stochastic control problem with non convex costs*

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Abstract. In this paper we provide a complete theoretical analysis of a two-dimensional degenerate non convex singular stochastic control problem. The optimisation is motivated by a storage-consumption model in an electricity market, and features a stochastic real-valued spot price modelled by Brownian motion. We find analytical expressions for the value function, the optimal control and the boundaries of the *action* and *inaction* regions. The optimal policy is characterised in terms of two monotone and discontinuous repelling free boundaries, although part of one boundary is constant and and the smooth fit condition holds there.

Keywords: finite-fuel singular stochastic control; optimal stopping; free boundary; Hamilton-Jacobi-Bellmann equation; irreversible investment; electricity market.

MSC2010 subsject classification: 91B70, 93E20, 60G40, 49L20.

1 Introduction

In this paper we study a two-dimensional degenerate problem of singular stochastic control (SSC) with monotone, bounded controls and a non convex performance criterion that was introduced in [10] in the context of electricity markets. Here the first component of the state process is the electricity spot price, represented by a one-dimensional Brownian motion $B := (B_t)_{t\geq 0}$ carried by a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and the optimisation problem detailed in [10] reads

$$U(x,c) = \inf_{\nu} \mathsf{E}\bigg[\int_0^\infty e^{-\lambda t}\lambda X_t^x \Phi(c+\nu_t)dt + \int_0^\infty e^{-\lambda t} X_t^x \, d\nu_t\bigg], \qquad (x,c) \in \mathbb{R}\times[0,1], \tag{1.1}$$

with $X_t^x := x + B_t$, $t \geq 0$, and where the infimum is taken over a suitable class of nondecreasing controls ν such that $c + \nu_t \leq 1$, P-a.s. for all $t \geq 0$. The constant λ denotes a positive discount factor and Φ is a strictly convex, twice continuously differentiable, decreasing function.

As discussed in Appendix A of [10], problem (1.1) is a *non convex* optimisation problem arising naturally from storage-consumption problems for electricity, when the spot price X is modelled by a continuous strong Markov process taking negative values with positive probability. In this problem $c + \nu_t$ represents the inventory level at time t of an electricity storage facility such as a battery, so that ν_t is the cumulative amount of energy purchased up to time t. A *finite*

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fuel constraint $c + \nu_t \leq 1$, $c \in [0,1]$, P-a.s. for all $t \geq 0$, reflects the fact that electricity storage has limited total capacity.

The study in [10], where the uncontrolled process X is of Ornstein-Uhlenbeck (OU) type, reveals how the non-convexity impacts in a complex way on the structure of the optimal control and on the connection between SSC problems and associated optimal stopping (OS) problems (as standard references on the subject see [19] and [20]). The analysis in [10] identifies three regimes, two of which are solved and the third of which is left as open problem under the OU dynamics. Here, with the aim of a full theoretical investigation, we take a more canonical example letting X be a Brownian motion. The complete solution that we provide also gives some insight in the open case of [10] since Brownian motion is a special case of OU with null rate of mean reversion. The methodology we employ here is different from that of [10], as we employ the characterisation via concavity of excessive functions for Brownian motion introduced in [11], Chapter 3 (for Brownian motion, later expanded in [7]) to study a parameterised family of OS problems in Section 3.3 below. This characterisation allows us to obtain the necessary monotonicity and regularity results for the optimal boundaries in (1.1). In contrast to the OU case, the Laplace transforms of the hitting times of Brownian motion are available in closed form and it is this feature which enables the method of the present paper.

From the mathematical point of view (1.1) falls into the class of finite fuel, singular stochastic control problems of monotone follower type (see, e.g., [4], [6], [12], [13], [21] and [22] as classical references on finite fuel monotone follower problems). As noted in [10] the total expected cost functional we aim at minimising in (1.1) is not convex in the control variable. In particular, by simply writing X as the difference of its positive and negative part, it is easy to see that the total expected cost functional in (1.1) can be written as a d.c. functional, i.e. as the difference of two convex functionals (see [16] or [17] for a reference on d.c. functions). As a consequence the standard connection between singular stochastic control and optimal stopping as addressed, for example, in [12], [19] and [21] among others, does not provide an approach to solving problem (1.1). To the best of our knowledge, when the connection to optimal stopping cannot be used as in our case, the explicit solution of two-dimensional problems of this kind becomes much more complex and it has to be performed on a case by case basis.

We will show that due to this non convexity, the control policy even in this Brownian case is quite complex. While the action region is disconnected as expected from [10] the two free boundaries $c \mapsto \hat{\beta}(c)$ and $c \mapsto \hat{\gamma}(c)$ are *discontinuous*, the former being non-increasing everywhere but at a single jump and the latter being non-decreasing with a vertical asymptote. Through a verification argument we are able to show that control is always exercised discontinuously, that is, by inducing jumps in the state process.

The free boundaries $\hat{\beta}$ and $\hat{\gamma}$ are therefore *repelling* (in the terminology of [8] or [23]). However, in contrast with most known examples of repelling boundaries, if the optimally controlled process hits $\hat{\beta}$ the controller does not immediately exercise all available control but, rather, causes the inventory level to jump to a *critical level* $\hat{c} \in (0,1)$ (which coincides with the point of discontinuity of the upper boundary $c \mapsto \hat{\beta}(c)$). After this jump the optimally controlled process continues to diffuse until hitting the lower boundary $\hat{\gamma}$ (the upper boundary is then formally infinite; for details see Sections 3 and 4).

This optimal process is unexpected in light of [10], whose results might suggest the presence of a continuously reflecting boundary. However the present solution can in part be related to the usual connection between SSC and OS as addressed in [12], [19] and [21], among others. In particular when the initial inventory level c is strictly larger than the critical value $\hat{c}$ there is a single lower boundary $\hat{\gamma}$ which is constant and the optimal policy consists in exercising all the available control when the process X hits this boundary. However the so-called *smooth fit* condition holds at $\hat{\gamma}$ (for $c > \hat{c}$), i.e. U_{xc} is continuous across it, and U_c coincides with the value function of an associated optimal stopping problem on $\mathbb{R} \times (\hat{c}, 1]$. This constant boundary can therefore be considered *discontinuously* reflecting, as a non increasing counterpart of the more canonical strictly decreasing reflecting boundaries. On the other hand, when the initial inventory level c is smaller than the critical value $\hat{c}$ we establish through solving the family of OS problems in Section 3.3 and examining their free boundaries that the value function U of problem (1.1) coincides itself (and not through its derivative U_c) with the value function of an associated optimal stopping problem. In this case we confirm that U_{xc} is discontinuous across the optimal boundaries so that the smooth fit condition breaks down.

The rest of the paper is organised as follows. In Section 2 we set up the problem and in Section 3 we construct a candidate value function. A candidate optimal control for problem (1.1) and the candidate value function from Section 3 are then validated in Section 4 through a verification argument. Finally, proofs of some results needed in Section 3 are collected in Appendix A.

2 Setting and basic assumptions

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space carrying a one-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ adapted to its natural filtration augmented by P-null sets $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. We denote by X^x the Brownian motion starting from $x \in \mathbb{R}$ at time zero; i.e.

$$X^x_t = x + B_t, \quad t \geq 0. \tag{2.1}$$

It is well known that X^x is a recurrent process with infinitesimal generator $\mathbb{L}_X := \frac{1}{2}\frac{d^2}{dx^2}$ and with fundamental decreasing and increasing solutions of the characteristic equation $(\mathbb{L}_X - \lambda)u = 0$ given by $\phi_\lambda(x) := e^{-\sqrt{2\lambda}x}$ and $\psi_\lambda(x) := e^{\sqrt{2\lambda}x}$, respectively.

Letting $c \in [0, 1]$ be constant, we denote by $C^{c,\nu}$ the purely controlled process evolving according to

$$C^{c,\nu}_t = c + \nu_t, \quad t \geq 0, \tag{2.2}$$

where ν is a control process belonging to the set

$$\mathcal{A}_c := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, (\nu_t(\omega))_{t\geq 0} \text{ is nondecreasing, left-continuous, adapted} \\ \text{with } c + \nu_t \leq 1 \ \forall t \geq 0,\ \nu_0 = 0 \ \ \mathsf{P} - \text{a.s.}\}.$$

From now on controls belonging to $\mathcal{A}_c$ will be called admissible.

Given a positive discount factor λ and a convex running cost function Φ, the problem is to find

$$U(x, c) := \inf_{\nu \in \mathcal{A}_c} \mathcal{J}_{x,c}(\nu), \tag{2.3}$$

with

$$\mathcal{J}_{x,c}(\nu) := \mathsf{E}\bigg[\int_0^\infty e^{-\lambda s}\lambda X^x_s \Phi(C^{c,\nu}_s)ds + \int_0^\infty e^{-\lambda s} X^x_s \, d\nu_s\bigg], \tag{2.4}$$

and the minimising control policy ν^*.

Notice that throughout this paper we make use of the notation $\int_0^t e^{-\lambda s} X^x_s d\nu_s$, $t \geq 0$, to indicate the Stieltjes integral $\int_{[0,t)} e^{-\lambda s} X^x_s d\nu_s$, $t \geq 0$, with respect to any $\nu \in \mathcal{A}_c$. Moreover, from now on the following standing assumption on the running cost factor Φ will hold.

Assumption 2.1. $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$ *lies in* $C^2(\mathbb{R})$ *and is decreasing and strictly convex with* $\Phi(1) = 0$.

For frequent future use it is also convenient to introduce the following quantities. We denote by $c_o \in (0, 1)$ the unique solution of

$$R(c) := 1 - c - \Phi(c) = 0 \tag{2.5}$$

in $(0,1)$ should one exist. Note that $R(1)=0$ and R is strictly concave, hence if c_o exists then $1-\Phi(0)<0$, R is negative on $[0,c_o)$ and positive on $(c_o,1)$. As in [10] the sign of the function

$$k(c) := \lambda + \lambda\,\Phi'(c) \tag{2.6}$$

over $c\in[0,1]$ will also play a fundamental role in the solution of problem (2.3). Since $c\mapsto k(c)$ is strictly increasing by the strict convexity of Φ (cf. Assumption 2.1) $\hat{c}\in\mathbb{R}$ is the unique solution in $(0,1)$ of

$$k(c) = 0 \tag{2.7}$$

should one exist. Notice that if both $\hat{c}$ and c_o exist in $(0,1)$ then $\hat{c}>c_o$ since $k(c)=-\lambda R'(c)$, $R(1)=0$ and R is strictly concave, so that at $\hat{c}$ the function R attains a positive maximum.

From now on we make the following standing assumption.

Assumption 2.2. *Both c_o and $\hat{c}$ exist in $(0,1)$ with $0<c_o<\hat{c}<1$.*

Such assumption guarantees the most general setting for our problem and the cases where either c_o or both c_o and $\hat{c}$ do not exist in $(0,1)$ are also covered by the results that we present in the next sections.

3 Construction of a candidate value function

The next preliminary result shows that under our assumptions problem (2.3) is well posed with finite value function.

Proposition 3.1. *Let U be as in (2.3). Then there exists $K>0$ such that $|U(x,c)|\le K(1+|x|)$ for any $(x,c)\in\mathbb{R}\times[0,1]$.*

Proof. We take $\nu\in\mathcal{A}_c$ and integrate by parts the cost term $\int_0^\infty e^{-\lambda s}X^x_s d\nu_s$ in (2.4) noting that the martingale $M_t := \int_0^t e^{-\lambda s}\nu_s dB_s$ is uniformly integrable and hence its expectation vanishes. Then by well known estimates for the Brownian motion we get

$$|\mathcal{J}_{x,c}(\nu)| \le \mathsf{E}\Big[\int_0^\infty e^{-\lambda s}\lambda|X^x_s|\big[\Phi(C^{c,\nu}_s)+\nu_s\big]ds\Big] \le K(1+|x|), \tag{3.1}$$

for some suitable $K>0$, since $\Phi(c)\le\Phi(0)$, $c\in[0,1]$ by Assumption 2.1 and $\nu\in\mathcal{A}_c$ is bounded from above by 1. By (3.1) and arbitrariness of $\nu\in\mathcal{A}_c$ the proposition is proved. □

The aim of our study is to find analytical expressions for the value function U of problem (2.3) and the associated optimal control ν^*. That will be achieved by constructing in this section a suitable solution, W, of the Hamilton-Jacobi-Bellman (HJB) equation naturally associated with U of (2.3) (cf. (3.2) below). The function W will be our candidate value function of the optimisation problem in (2.3) and in Section 4 we will use a generalised version of Itô's formula to prove that $W=U$ provided that suitable regularity results are obtained for W beforehand. The optimal control will be specified by relying on geometric properties of suitable free boundaries which we associate to the *action* and *inaction* region of the control problem.

To be more precise, for $\mathcal{O}:=\mathbb{R}\times(0,1)$, we aim at finding $W\in C^1(\mathcal{O})\cap C(\overline{\mathcal{O}})$ with $W_{xx}\in L^\infty_{loc}(\mathcal{O})$ such that it solves the variational problem

$$\max\big\{(-\tfrac{1}{2}W_{xx}+\lambda W)(x,c)-\lambda x\Phi(c)\,,\,-W_c(x,c)-x\big\}=0,\quad \text{for a.e. }(x,c)\in\mathcal{O} \tag{3.2}$$

with $W(x,1)=0$, $x\in\mathbb{R}$. The candidate action and inaction regions associated to W are denoted $\mathcal{D}_W$ and $\mathcal{I}_W$, respectively and are defined by

$$\mathcal{D}_W := \big\{(x,c)\in\mathcal{O} : W_c(x,c)=-x\big\} \quad\text{and}\quad \mathcal{I}_W := \big\{(x,c)\in\mathcal{O} : W_c(x,c)>-x\big\}. \tag{3.3}$$

3.1 Heuristic study of the optimal policy

Here we provide an initial, heuristic analysis of the geometry of the action and inaction regions in problem (2.3). For this and for the subsequent solution of the problem it is convenient to consider separately the intervals $[0,\hat{c})$ and $(\hat{c},1]$ of possible values for the controlled state variable.

We begin by comparing two strategies when $c \in (\hat{c},1]$, hence $k(c) > 0$. If control is never exercised, i.e. $\nu_t \equiv 0$, $t \geq 0$, one obtains from (2.4) an overall cost $\mathcal{J}_{x,c}(0) = x\Phi(c)$ by an application of Fubini's theorem. If instead at time zero one increases the inventory by a small amount $\delta > 0$ and then does nothing for the remaining time, i.e. $\nu_t = \nu_t^\delta := \delta$ for $t > 0$ in (2.4), the total cost is $\mathcal{J}_{x,c}(\nu^\delta) = x(\delta + \Phi(c+\delta))$. By approximating $\Phi(c+\delta) = \Phi(c) + \Phi'(c)\delta + o(\delta^2)$ we find that $\mathcal{J}_{x,c}(\nu^\delta) = \mathcal{J}_{x,c}(0) + \delta x(1+\Phi'(c)) + o(\delta^2)$ so that exercising a small amount of control reduces future costs relative to a complete inaction strategy only if $xk(c)/\lambda < 0$, i.e. $x < 0$ since $k(c) > 0$. It is then natural to expect that for each $c \in (\hat{c},1]$ there should exists $\gamma(c) < 0$ such that it is optimal to exercise control only when the process X falls below such a threshold. We now want to understand whether a small control increment is more efficient than a large one and for that we consider a strategy where at time zero one exercises all available control, i.e. $\nu_t = \nu_t^f := 1 - c$ for $t > 0$. The latter produces a total expected cost equal to $\mathcal{J}_{x,c}(\nu^f) = x(1-c)$, so that for $x < 0$ and recalling that k is increasing one has

$$\mathcal{J}_{x,c}(\nu^f) - \mathcal{J}_{x,c}(\nu^\delta) = \frac{x}{\lambda}\Big(\int_c^1 k(y)dy - \delta k(c)\Big) + o(\delta^2) \leq \frac{x}{\lambda}k(c)(1-c-\delta). \tag{3.4}$$

Since $k(c) > 0$ the last expression is negative whenever $1-c > \delta$, so it is reasonable to expect that large control increments are more profitable than small ones. This suggests that the threshold γ introduced above should not be of the reflecting type (see for instance [12] or [14]) but rather of repelling type as observed in [1], [2] and [8] among others.

Now consider the case $c \in [0,\hat{c})$, i.e. $k(c) < 0$ and argue similarly. If again we compare the cost associated with complete inaction to that associated with the strategy ν^δ we find that the latter is convenient if and only if $xk(c)/\lambda < 0$, i.e. $x > 0$ since now $k(c) < 0$. Hence we expect that for fixed $c \in [0,\hat{c})$ one should act when the process X exceeds a positive upper threshold $\beta(c)$. Then compare small control increment with a large one, in particular consider a policy $\nu^{\hat{c}}$ that immediately exercises an amount $\hat{c} - c$ of control and then acts optimally for problem (2.3) with initial conditions $(x,\hat{c})$. The expected cost associated to $\nu^{\hat{c}}$ is $\mathcal{J}_{x,c}(\nu^{\hat{c}}) = x(\hat{c}-c) + U(x,\hat{c})$ and one has

$$\mathcal{J}_{x,c}(\nu^{\hat{c}}) - \mathcal{J}_{x,c}(\nu^\delta) \leq \frac{x}{\lambda}\Big(\int_c^{\hat{c}} k(y)dy - \delta k(c)\Big) + o(\delta^2) \tag{3.5}$$

where we have used that $U(x,\hat{c}) \leq x\Phi(\hat{c})$. If we fix $c \in [0,\hat{c})$ and $x > 0$, then for $\delta > 0$ sufficiently small the right-hand side of (3.5) becomes negative, which suggests that a reflection strategy at the upper boundary β would be less efficient than the strategy described by $\nu^{\hat{c}}$. We can interpret this observation as an effect of the "proximity" to the action/inaction set of the state space's region $\mathbb{R} \times (\hat{c},1]$ discussed in the previous paragraph. For $x > 0$ large enough the controller finds it convenient to increase the inventory by the amount needed to push the process (X,C) inside the inaction region of the subset $\mathbb{R} \times (\hat{c},1]$ described in the previous paragraph, rather than optimising with smaller purchases. In fact such proximity affects the geometry of action/inaction set in a deeper way and it turns out that it may be sometimes convenient to act also in the region $(-\infty,0) \times [0,\hat{c})$. To make this claim clearer let us compare therein the strategies $\nu_t \equiv 0$, $t \geq 0$ and $\nu^{\hat{c}}$. Fix $x < 0$, $c \in [0,\hat{c})$ and note that $U(x,\hat{c}) \leq x(1-\hat{c})$ to obtain

$$\mathcal{J}_{x,c}(\nu^{\hat{c}}) - \mathcal{J}_{x,c}(0) \leq \frac{x}{\lambda}\int_c^1 k(y)dy = \frac{x}{\lambda}\Big(\int_c^{\hat{c}} k(y)dy + \int_{\hat{c}}^1 k(y)dy\Big). \tag{3.6}$$

Now, the first integral on the right-hand side of (3.6) is negative but its absolute value could be made arbitrarily small by taking c close to $\hat{c}$. Instead the second integral is positive and its value is not affected by the choice of c. Thus, given that $x < 0$, the overall expression becomes negative when c approaches $\hat{c}$ from the left. This suggests that when the inventory is smaller than but close enough to the critical value $\hat{c}$ and $x < 0$ an investment sufficiently large to increase the inventory to a critical level $\hat{c}$ is a better choice than complete inaction. This is a remarkable effect and we will see in Sections 3.3 how this leads to an efficient method of solution through an auxiliary optimal stopping problem.

3.2 Step 1: $c \in [\hat{c}, 1]$.

Recall that $\hat{c}$ denotes the unique solution in $(0, 1)$ of (2.7) (cf. Assumption 2.2) and take $c \in [\hat{c}, 1]$. Since $\hat{c} > c_o$ (cf. (2.5) and Assumption 2.2) we have $R(c) > 0$ for $c \in [\hat{c}, 1)$. In the portion of plane $[\hat{c}, 1] \times \mathbb{R}$ we expect to find that the inaction region is of the form $x > \gamma^o(c)$ where γ^o is a *repelling* boundary such that once the process X hits γ^o the optimal strategy is to exert all the available fuel. Therefore we write (3.2) as a free-boundary problem where we want to find the couple of functions (u, γ) solving

$$
\begin{cases}
\frac{1}{2}u_{xx}(x,c) - \lambda u(x,c) = -\lambda x \Phi(c) & \text{for } x > \gamma(c),\ c \in [\hat{c}, 1) \\
\frac{1}{2}u_{xx}(x,c) - \lambda u(x,c) \geq -\lambda x \Phi(c) & \text{for a.e. } (x,c) \in \mathbb{R} \times [\hat{c}, 1) \\
u_c(x,c) \geq -x & \text{for } x \in \mathbb{R},\ c \in [\hat{c}, 1) \\
u(x,c) = x(1-c) & \text{for } x \leq \gamma(c),\ c \in [\hat{c}, 1] \\
u_x(x,c) = (1-c) & \text{for } x \leq \gamma(c),\ c \in [\hat{c}, 1) \\
u(x,1) = 0 & \text{for } x \in \mathbb{R}.
\end{cases}
\tag{3.7}
$$

Proposition 3.2. *The couple (W^o, γ^o) defined by $\gamma^o := -\frac{1}{\sqrt{2\lambda}}$ and*

$$
W^o(x,c) := \begin{cases} -\frac{1}{\sqrt{2\lambda}} e^{-1} R(c) \phi_\lambda(x) + x\Phi(c), & x > \gamma^o, \\ x(1-c), & x \leq \gamma^o, \end{cases}
\tag{3.8}
$$

solves (3.7) with $W^o \in C^1(\mathbb{R} \times [\hat{c}, 1])$ and $W^o_{xx} \in L^\infty_{loc}(\mathbb{R} \times (\hat{c}, 1))$.

Proof. A general solution to the first equation in (3.7) is given by

$$
u(x,c) = A^o(c)\psi_\lambda(x) + B^o(c)\phi_\lambda(x) + x\Phi(c), \qquad x > \gamma(c),
$$

with A^o, B^o and γ to be determined. Since $\psi_\lambda(x)$ diverges with a superlinear trend as $x \to \infty$ and U has sublinear growth by Proposition 3.1, we set $A^o(c) \equiv 0$. Imposing the fourth and fifth of (3.7) for $x = \gamma(c)$ and recalling the expression for R as in (2.5) it is easy to find

$$
B^o(c) := -\frac{1}{\sqrt{2\lambda}} e^{-1} R(c), \qquad \gamma(c) = \gamma^o = -\frac{1}{\sqrt{2\lambda}}.
\tag{3.9}
$$

This way the function W^o of (3.8) clearly satisfies $W^o(x,1) = 0$, W^o_x is continuous by construction and by some algebra it is not difficult to see that W^o_c is continuous on $\mathbb{R} \times [\hat{c}, 1]$ with $W^o_c(\gamma^o, c) = -\gamma^o$, $c \in [\hat{c}, 1]$. Moreover one also has

$$
W^o_{cx}(x,c) + 1 = (1 + \Phi'(c))\big(1 - e^{-1}\phi_\lambda(x)\big) \geq 0, \qquad x > \gamma^o, c \in [\hat{c}, 1],
\tag{3.10}
$$

and hence $W^o_{cx}(\gamma^o, c) = -1$, for $c \in [\hat{c}, 1]$, i.e. the smooth-fit holds, and $W^o_c(x,c) \geq -x$ on $\mathbb{R} \times [\hat{c}, 1]$ as required. It should be noticed that W^o_{xx} fails to be continuous across the boundary although it remains bounded on any compact subset of $\mathbb{R} \times [\hat{c}, 1]$.

Finally we observe that

$$\tfrac{1}{2}W^o_{xx}(x,c) - \lambda W^o(x,c) = -\lambda x(1-c) \geq -\lambda x \Phi(c) \quad \text{for } x \leq \gamma^o,\ c \in [\hat{c},1], \tag{3.11}$$

since $\gamma^o < 0$ and $R(c) \geq 0$ on $c \in [\hat{c},1]$. □

Remark 3.3. *In this setting the classical connection between SSC and OS holds as in the case of convex problems (see [19]). Direct derivation of the first and third equations in* (3.7) *(or alternatively of* (3.8)*) easily allow to show that* W^o_c *solves a free-boundary problem which is naturally associated to the following family of OS problems parametric in* $c \in [\hat{c},1)$

$$w(x,c) := \sup_{\tau \geq 0} \mathsf{E}\Big[\lambda\Phi'(c)\int_0^\tau e^{-\lambda t}X^x_t dt - e^{-\lambda\tau}X^x_\tau\Big], \quad x \in \mathbb{R}. \tag{3.12}$$

Moreover $W^o_c(\,\cdot\,,c) \in C^1(\mathbb{R})$ *for all* $c \in [\hat{c},1)$*, as proven above and hence from standard verification arguments it follows* $W^o_c = w$*. Details are omitted here since they can be found in the very wide existing literature on infinite time-horizon OS problems (see for instance [24] and references therein).*

The analysis conducted so far provides us with a candidate analytical expression, W^o, for the function U of (2.3) and a candidate optimal control of *bang-bang* type triggered by the repelling boundary γ^o. Clearly W^o needs to be suitably pasted with the solution of the HJB equation that we will find in the next section for the portion of state space $\mathbb{R} \times [0,\hat{c})$.

3.3 An auxiliary problem of optimal stopping

We now consider $c \in [0,\hat{c})$ and as it will become clear in what follows our study of this region goes through two subcases, namely $c \in (c_o,\hat{c})$ and $c \in [0,c_o]$ with c_o the unique solution in $(0,1)$ of (2.5) (cf. Assumption 2.2). For $c \in [0,\hat{c})$ we expect again an optimal control of *bang-bang* type so that, once the uncontrolled process X enters the action region, the optimal policy is to increase the capacity up to $\hat{c}$ and then to continue optimally in the region $\mathbb{R} \times [\hat{c},1]$. This structure of the expected optimal control and results obtained in the previous section imply that for fixed $c \in [0,\hat{c})$ the function U of (2.3) should coincide with the value function of an infinite time-horizon, one-dimensional, parameter dependent (where c enters only as a parameter) optimal stopping problem. More precisely we aim at proving that $U(\,\cdot\,,c)$ equals

$$W^1(x,c) := \inf_{\tau \geq 0} \mathsf{E}\Big[\int_0^\tau e^{-\lambda t}\lambda X^x_t \Phi(c)dt + e^{-\lambda\tau}X^x_\tau(\hat{c}-c) + e^{-\lambda\tau}W^o(X^x_\tau,\hat{c})\Big], \tag{3.13}$$

for $x \in \mathbb{R}$ and where the optimisation is taken over the set of $(\mathcal{F}_t)$-stopping times valued in $[0,\infty]$. The rest of this section is devoted to the study of analytical properties of W^1 and of the associated optimal stopping regions.

From now on we will adopt the convention

$$\begin{cases} e^{-\lambda\tau}X^x_\tau := \lim_{t\uparrow\infty} e^{-\lambda t}X^x_t = 0 & \text{on } \{\tau = +\infty\} \\[2ex] e^{-\lambda\tau}W^o(X^x_\tau,\hat{c}) := \lim_{t\uparrow\infty} e^{-\lambda t}W^o(X^x_t,\hat{c}) = 0 & \text{on } \{\tau = +\infty\} \end{cases}$$

where the equalities follow from the law of iterated logarithm and the fact that $|W^o(x,c)| \leq C(1+|x|)$ for suitable $C > 0$ (cf. (3.8)).

Notice that the integral term in (3.13) may be rewritten by using Itô's formula so that (3.13) reads as

$$W^1(x,c) := x\Phi(c) + V(x,c), \tag{3.14}$$

where

$$V(x,c) := \inf_{\tau \geq 0} \mathsf{E}\big[e^{-\lambda\tau} G(X^x_\tau, c)\big] \tag{3.15}$$

is the value function of an optimal stopping problem (again, parametric in c) with

$$G(x,c) := x(\hat{c} - c - \Phi(c)) + W^o(x,\hat{c}). \tag{3.16}$$

According to the standard optimal stopping theory (see, e.g., [24]), for any fixed $c \in [0,\hat{c})$, we define the continuation region $\mathcal{C}_c$ and the stopping region $\mathcal{S}_c$ of problem (3.15) by

$$\mathcal{C}_c := \{x \in \mathbb{R} : V(x,c) < G(x,c)\} \quad \text{and} \quad \mathcal{S}_c := \{x \in \mathbb{R} : V(x,c) = G(x,c)\}, \tag{3.17}$$

respectively. When the observed process X is in $\mathcal{C}_c$ then the value function of (3.15) is strictly smaller than the value of immediate stopping, i.e. G, therefore it is optimal to continue the observation. On the other hand if X lies in $\mathcal{S}_c$ then the value of immediate stopping equals the value of continuation and hence it is optimal stop the observation.

3.3.1 Step 2.1: $c \in (c_o, \hat{c})$.

Recall (2.5), (2.7), (3.8) and Assumption 2.2 and take $c \in (c_o, \hat{c})$ so that $R(c) > 0$ for any c in such interval. An initial insight into the shapes of $\mathcal{C}_c$ and $\mathcal{S}_c$ is obtained by studying the sign of $\frac{1}{2}G_{xx} - \lambda G$ as indeed standard arguments based on exit times from small intervals guarantee that for every $c \in (c_o,\hat{c})$ one has $\mathcal{S}_c \subset \{x : (\frac{1}{2}G_{xx} - \lambda G)(x,c) \geq 0\}$ and $\mathcal{C}_c \supset \{x : (\frac{1}{2}G_{xx} - \lambda G)(x,c) < 0\}$. From simple calculations one finds from (3.16) and (3.8)

$$(\tfrac{1}{2}G_{xx} - \lambda G)(x,c) = \begin{cases} -\lambda x \int_c^{\hat{c}} k(y)dy, & x > \gamma^o, \\ -\lambda x R(c), & x \leq \gamma^o. \end{cases} \tag{3.18}$$

Hence, recalling that $R(y) > 0$ and $k(y) < 0$ for $y \in (c_o,\hat{c})$ we get $\mathcal{S}_c \subset (-\infty, \gamma^o] \cup [0,\infty)$. We thus expect a non-connected stopping set and two optimal stopping boundaries.

Since $|G(x,c)| \leq C(1+|x|)$ for suitable $C > 0$ it is clear that $|V(x,c)| < +\infty$ for all $(x,c) \in \mathbb{R}\times(c_o,\hat{c})$ and since $x \mapsto \mathsf{E}\big[e^{-\lambda\tau}G(X^x_\tau,c)\big]$ is continuous for any fixed $\tau \geq 0$ and $c \in (c_o,\hat{c})$ it follows that $x \mapsto V(x,c)$ is upper-semi-continuous (one can in fact prove that it is continuous). Then it follows from standard theory (cf. for instance [24]) that $\tau_* := \inf\{t \geq 0 : X^x_t \in \mathcal{S}_c\}$ is an optimal stopping time and V solves the variational problem

$$\max\big\{-\tfrac{1}{2}u_{xx} + \lambda u\,,\, u - G\big\} = 0 \quad \text{for a.e. } (x,c) \in \mathbb{R}\times(c_o,\hat{c}). \tag{3.19}$$

A standard way of characterising V analytically would be to proceed as in Section 3.2 by writing down and solving (3.19) as a free-boundary problem with boundary conditions at the endpoints of an interval (γ,β) with $\gamma < \gamma^o$ and $\beta > 0$ to be determined. Natural boundary conditions are $u = G$ (continuous-fit) and $u_x = G_x$ (smooth-fit). Those would lead to a system of non-linear equations involving ϕ_λ and ψ_λ that we would need to solve (to some extent) in order to prove that the treble of functions (u,γ,β) has suitable properties. Namely, to prove that $u = V$ we require $u \in C^1$ with locally bounded second derivative with respect to x and $u \leq G$ everywhere. Moreover to substantiate our initial claim that $W^1 = U$ and that the optimal strategy in the control problem is of bang-bang type we also need to verify that $c \mapsto \gamma(c)$ is increasing and $c \mapsto \beta(c)$ is decreasing. It turns out that this work-plan is not feasible due to the difficulty in handling the resulting system of non-linear equations.

Therefore we tackle the optimal stopping problem (3.15) via a different approach. That is, we adapt to our parameter-dependent setting the geometric approach originally introduced in [11], Chapter 3, for Brownian motion (see also [7] for further extensions and details) which in this case proves particularly handy due to the nature of the uncontrolled process X. As in [7], eq. (4.6), we define

$$F_\lambda(x) := \frac{\psi_\lambda(x)}{\phi_\lambda(x)} = e^{2\sqrt{2\lambda}x}, \qquad x \in \mathbb{R}, \tag{3.20}$$

together with its inverse

$$F_\lambda^{-1}(y) := \frac{1}{2\sqrt{2\lambda}} \ln(y), \qquad y > 0, \tag{3.21}$$

and, for fixed $c \in (c_o, \hat{c})$, the function

$$H(y,c) := \begin{cases} \frac{G(F_\lambda^{-1}(y),c)}{\phi_\lambda(F_\lambda^{-1}(y))}, & y > 0 \\ 0 & y = 0. \end{cases} \tag{3.22}$$

We can now restate part of Proposition 5.12 and Remark 5.13 of [7] as follows.

Proposition 3.4. *Fix $c \in (c_o, \hat{c})$ and let $Q(\,\cdot\,, c)$ be the largest non-positive convex minorant of $H(\,\cdot\,, c)$ (cf. (3.22)), then $V(x,c) = \phi_\lambda(x) Q(F_\lambda(x), c)$ for all $x \in \mathbb{R}$. Moreover $\mathcal{S}_c = F_\lambda^{-1}(\mathcal{S}_c^Q)$, where $\mathcal{S}_c^Q := \{y > 0 : Q(y,c) = H(y,c)\}$ (cf. (3.17)).*

To solve problem (3.15), and hence (3.13), we have now to determine H and Q as above and characterise the contact set $\mathcal{S}_c^Q$. From now on we fix $c \in (c_o, \hat{c})$, then recalling (3.8) and (3.16) we get

$$G(x,c) = \begin{cases} xR(c), & x \le \gamma^o \\ -\frac{1}{\sqrt{2\lambda}} e^{-1} R(\hat{c}) \phi_\lambda(x) + x(R(c) - R(\hat{c})), & x > \gamma^o, \end{cases} \tag{3.23}$$

with R as in (2.5). Noting that $\phi_\lambda(F_\lambda^{-1}(y)) = y^{-\frac{1}{2}}$, $y > 0$, we obtain from (3.21), (3.22) and (3.23)

$$H(y,c) = \begin{cases} 0, & y = 0 \\ \frac{1}{2\sqrt{2\lambda}} R(c) y^{\frac{1}{2}} \ln y, & 0 < y \le e^{-2} \\ -\frac{1}{\sqrt{2\lambda}} e^{-1} R(\hat{c}) + \frac{1}{2\sqrt{2\lambda}} (R(c) - R(\hat{c})) y^{\frac{1}{2}} \ln y, & y > e^{-2}. \end{cases} \tag{3.24}$$

In the next lemma we collect some elementary properties of H. The proof is trivial and it is moved to Appendix A for completeness.

Lemma 3.5. *The function $H(\,\cdot\,, c)$ belongs to $C^1(0,\infty) \cap C([0,\infty))$ it is strictly decreasing and $H_{yy}(\,\cdot\,, c) \in L^\infty_{loc}(\delta, \infty)$ for all $\delta > 0$ (with a single discontinuity at $y = e^{-2}$). Moreover $H(\,\cdot\,, c)$ is convex in the intervals $[0, e^{-2}) \cup (1, \infty)$ and it is concave in $[e^{-2}, 1]$.*

To get a geometric intuition of the meaning of Proposition 3.4 we may say, roughly speaking, that in order to find the largest non-positive convex minorant of $H(\,\cdot\,, c)$, i.e. $Q(\,\cdot\,, c)$, we must put a rope below the new obstacle function H with both ends pulled to the sky (see, e.g., Section 8.1 of Chapter IV in [24] for such geometric interpretation). Mathematically, in our case, this corresponds to finding two points, $y_1 := y_1(c)$ and $y_2 := y_2(c)$, with $y_1 < e^{-2}$ and $y_2 > 1$, such that the tangent straight lines to H in y_1 and y_2, denoted respectively r_{y_1} and r_{y_2}, coincide. Then $Q = H$ on $[0, y_1] \cup [y_2, \infty)$ and $Q = r_{y_1} = r_{y_2}$ on (y_1, y_2). Namely, y_1 and y_2 must solve the system

$$\begin{cases} H_y(y_1, c) = H_y(y_2, c) & \text{(same slopes)} \\ H(y_1, c) - H_y(y_1, c) y_1 = H(y_2, c) - H_y(y_2, c) y_2 & \text{(same intercepts)} \end{cases} \tag{3.25}$$

and they will be the boundaries of the stopping region $\mathcal{S}_c^Q = [0, y_1] \cup [y_2, \infty)$.

A geometric proof of the following existence and uniqueness result is provided in Appendix A.

Proposition 3.6. *There exists a unique couple $(\hat{y}_1(c), \hat{y}_2(c))$ solving system (3.25) for any $c \in (c_o, \hat{c})$ with $\hat{y}_1(c) \in (0, e^{-2})$ and $\hat{y}_2(c) > 1$.*

We also defer to the Appendix the proof of the following properties.

Proposition 3.7. *The functions $\hat{y}_1$ and $\hat{y}_2$ of Proposition 3.6 belong to $C^1(c_o, \hat{c})$ with $c \mapsto \hat{y}_1(c)$ increasing and $c \mapsto \hat{y}_2(c)$ decreasing on $(c_o, \hat{c})$. Moreover, one has*

1. $\lim_{c \uparrow \hat{c}} \hat{y}_1(c) = e^{-2}$;
2. $\lim_{c \uparrow \hat{c}} \hat{y}_1'(c) = 0$;
3. $\lim_{c \downarrow c_o} \hat{y}_1(c) = 0$;
4. $\hat{y}_2(c) < e^2$ *for all* $c \in (c_o, \hat{c})$.

We are now ready to construct the largest non-positive convex minorant Q (see Proposition 3.4) of our function H (cf. (3.24)) by setting, for any fixed $c \in (c_o, \hat{c})$,

$$Q(y,c) = \begin{cases} H(y,c), & y \in [0, \hat{y}_1(c)], \\ H_y(\hat{y}_2(c), c)(y - \hat{y}_2(c)) + H(\hat{y}_2(c), c), & y \in (\hat{y}_1(c), \hat{y}_2(c)), \\ H(y,c), & y \in [\hat{y}_2(c), \infty). \end{cases} \tag{3.26}$$

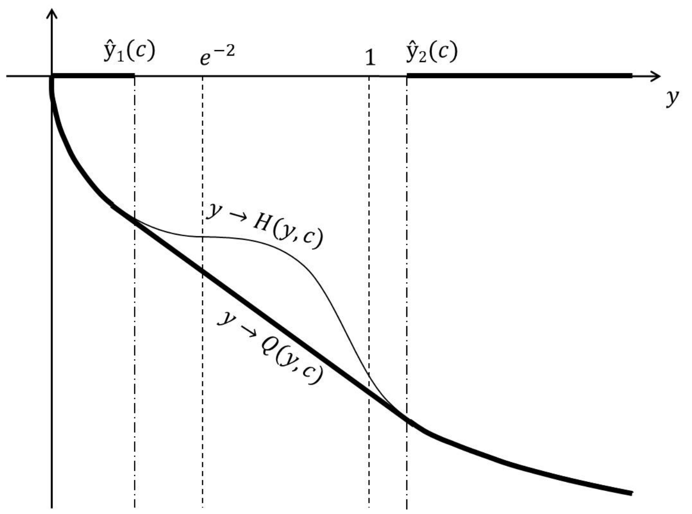

Figure 1: An illustrative plot of the functions $y \mapsto H(y,c)$ and $y \mapsto Q(y,c)$ (bold) of (3.24) and (3.26), respectively, for fixed $c \in (c_o, \hat{c})$. The bold interval $[0, \hat{y}_1(c)] \cup [\hat{y}_2(c), \infty)$ on the y-axis is the stopping region $\mathcal{S}_c^Q$.

From the above expression, using Proposition 3.4 and setting

$$\hat{\gamma}(c) := F_\lambda^{-1}(\hat{y}_1(c)) \quad \text{and} \quad \hat{\beta}(c) := F_\lambda^{-1}(\hat{y}_2(c)) \tag{3.27}$$

with F_λ^{-1} as in (3.21) we find the expression for V of (3.15)

$$V(x,c) = \begin{cases} G(x,c), & x \in (-\infty, \hat{\gamma}(c)] \\ \phi_\lambda(x)\Big[H_y(F_\lambda(\hat{\beta}(c)), c)\Big(F_\lambda(x) - F_\lambda(\hat{\beta}(c))\Big) + H(F_\lambda(\hat{\beta}(c)), c)\Big], & x \in (\hat{\gamma}(c), \hat{\beta}(c)) \\ G(x,c), & x \in [\hat{\beta}(c), \infty). \end{cases} \tag{3.28}$$

Remark 3.8. *Note that since $\hat{y}_1$ and $\hat{y}_2$ solve* (3.25)*, the second expression in* (3.26) *may be equivalently rewritten in terms of $\hat{y}_1$, i.e. $Q(y,c) = H_y(\hat{y}_1(c),c)(y-\hat{y}_1(c)) + H(\hat{y}_1(c),c)$ for $y \in (\hat{y}_1(c),\hat{y}_2(c))$ and analogously* (3.28) *may be equivalently rewritten in terms of $\hat{\gamma}$, that is $V(x,c) = \phi_\lambda(x)\Big[H_y(F_\lambda(\hat{\gamma}(c)),c)\Big(F_\lambda(x) - F_\lambda(\hat{\gamma}(c))\Big) + H(F_\lambda(\hat{\gamma}(c)),c)\Big]$ for $x \in (\hat{\gamma}(c),\hat{\beta}(c))$.*

Before proving some other crucial properties of V we consider the case of $c \in [0,c_o)$.

3.3.2 Step 2.2: $c \in [0,c_o)$.

Recall (2.5) and take $c \in [0,c_o)$ arbitrary but fixed so to have $R(c) < 0$. Proceeding as at the beginning of Section 3.3.1, an initial insight into the geometry of the continuation and the stopping region $\mathcal{C}_c$ and $\mathcal{S}_c$ (cf. (3.17)) may be obtained also in this case by looking at the sign of $\frac{1}{2}G_{xx} - \lambda G$. From (3.18), recalling that $R(y) < 0$ and $k(y) < 0$ for $y \in [0,c_o)$ we get $\mathcal{S}_c \subset [0,\infty)$ and we thus expect a connected stopping set and one optimal stopping boundary.

Again we address the optimal stopping problem (3.15) via a geometric approach in the spirit of [7] and [11]. Recalling (3.24), for any $c \in [0,c_o)$ the following result easily follows by arguments similar to those employed in the proof of Lemma 3.5 performed in Appendix A.

Lemma 3.9. *The function $H(\,\cdot\,,c)$ of* (3.24) *belongs to $C^1(0,\infty) \cap C([0,\infty))$. It is strictly increasing in $(0,e^{-2})$ and strictly decreasing in (e^{-2},∞). Moreover, $H_{yy}(\,\cdot\,,c) \in L^\infty_{loc}(\delta,\infty)$ for all $\delta > 0$ (with a single discontinuity at $y = e^{-2}$), $H_{yy}(1,c) = 0$ and $H(\,\cdot\,,c)$ is (strictly) concave in the interval $(0,1)$ and it is (strictly) convex in $(1,\infty)$.*

The strict concavity of H in $(0,1)$ suggest that there should exist a unique point $y_2^*(c) > 1$ solving

$$H_y(y,c)y = H(y,c). \tag{3.29}$$

This way the straight line $r_{y_2^*} : [0,\infty) \mapsto (-\infty,0]$

$$r_{y_2^*}(y) := H(y_2^*(c),c) + H_y(y_2^*(c),c)(y - y_2^*(c))$$

is tangent to H at $y_2^*(c)$ and $r_{y_2^*}(0) = 0$. Notice that by (3.24) and (A-1), equation (3.29) may be rewritten in the equivalent form

$$F_3(y;c) = 0, \tag{3.30}$$

where we define the jointly continuous function $F_3 : (0,\infty) \times [0,1] \mapsto \mathbb{R}$ as

$$F_3(y;c) := y^{\frac{1}{2}}\big(1 - \tfrac{1}{2}\ln y\big) - \frac{2e^{-1}R(\hat{c})}{R(\hat{c}) - R(c)}. \tag{3.31}$$

The proof of the next result may be found in Appendix A.

Proposition 3.10. *For each $c \in [0,c_o)$ there exists a unique point $y_2^*(c) \in (1,e^2)$ solving* (3.30)*. The function $c \mapsto \hat{y}_2^*(c)$ is decreasing and belongs to $C^1([0,c_o))$. Moreover, for $\hat{y}_2$ as in Proposition 3.6 one has*

$$y_2^*(c_o-) := \lim_{c\uparrow c_o} y_2^*(c) = \lim_{c\downarrow c_o} \hat{y}_2(c) =: \hat{y}_2(c_o+) \tag{3.32}$$

and

$$(y_2^*)'(c_o-) := \lim_{c\uparrow c_o}(y_2^*)'(c) = \lim_{c\downarrow c_o}(\hat{y}_2)'(c) =: (\hat{y}_2)'(c_o+). \tag{3.33}$$

Thanks to Proposition 3.10 we see that the curve y_2^* is actually the C^1-extension in the set $c \in [0, c_o]$ of $\hat{y}_2$ of Proposition 3.6. Therefore from now on, with a slight abuse of notation, we will simply refer to y_2^* and $\hat{y}_2$ by using uniquely $\hat{y}_2$, with the understanding that it is the upper boundary of the optimal stopping problem (3.15) for any value of $c \in [0, \hat{c})$. This notation will be fully justified in the following section by also proving regularity of the value function V, obtained constructively in $[0, c_o) \cup (c_o, \hat{c})$ (see (3.28) and (3.35) below), across $c = c_o$.

As in Section 3.3.1 we can construct the largest non-positive convex minorant Q (see Proposition 3.4) of our function H (cf. (3.24)) by setting, for any fixed $c \in [0, c_o)$,

$$Q(y,c) = \begin{cases} H_y(\hat{y}_2(c), c)y, & y \in [0, \hat{y}_2(c)), \\ H(y,c), & y \in [\hat{y}_2(c), \infty). \end{cases} \tag{3.34}$$

Recalling (3.27) we set $\hat{\beta}(c) := F_\lambda^{-1}(\hat{y}_2(c))$ so that the expression for V of (3.15) reads (cf. Proposition 3.4)

$$V(x,c) = \begin{cases} \phi_\lambda(x) H_y(F_\lambda(\hat{\beta}(c)), c) F_\lambda(x), & x \in (-\infty, \hat{\beta}(c)) \\ G(x,c), & x \in [\hat{\beta}(c), \infty). \end{cases} \tag{3.35}$$

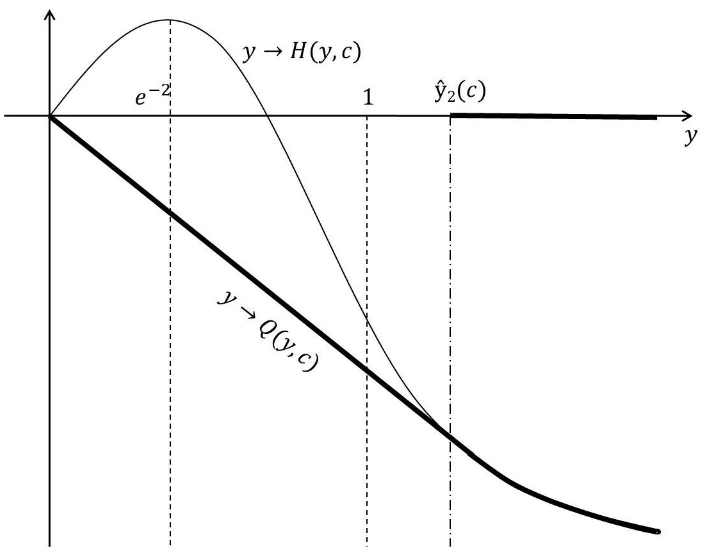

Figure 2: An illustrative plot of the functions $y \mapsto H(y,c)$ and $y \mapsto Q(y,c)$ (bold) of (3.24) and (3.34), respectively, for fixed $c \in [0, c_o)$. The bold interval $[\hat{y}_2(c), \infty)$ on the y-axis is the stopping region $\mathcal{S}_c^Q$.

3.3.3 Regularity of V and its offsprings

Recalling (3.17) we see that, by construction, $\mathcal{C} := \cup_{c \in [0,\hat{c})} \mathcal{C}_c$ and $\mathcal{S} := \mathbb{R} \times [0, \hat{c}) \setminus \mathcal{C}$ are such that

$$\mathcal{C} = \{(x,c) \in \mathbb{R} \times [0, \hat{c}) : x \in (\hat{\gamma}(c), \hat{\beta}(c))\} \tag{3.36}$$

$$\mathcal{S} = \{(x,c) \in \mathbb{R} \times [0, \hat{c}) : x \in (-\infty, \hat{\gamma}(c)] \cup [\hat{\beta}(c), +\infty)\} \tag{3.37}$$

with the convention that $\hat{\gamma} = -\infty$ on $[0, c_o]$. From Propositions 3.7 and 3.10 and from (3.27) follows the regularity of $\hat{\gamma}$ and $\hat{\beta}$.

Corollary 3.11. *It holds*

i) $\hat{\beta} \in C^1([0, \hat{c}))$ *monotone strictly decreasing and* $\hat{\beta}(c) \in (0, 1/\sqrt{2\lambda})$ *for all* $c \in [0, \hat{c})$*;*

ii) $\hat{\gamma} \in C^1((c_o, \hat{c}])$ *monotone strictly increasing with* $\hat{\gamma}(c) \leq \gamma^o$ *for all* $c \in [0, \hat{c})$ *and* $\hat{\gamma} = -\infty$ *on* $[0, c_o]$.

We now address the question of the C^1 regularity of V across the two boundaries and on $[0, \hat{c})$.

Proposition 3.12. *The value function* V *of* (3.15) *belongs to* $C^1(\mathbb{R} \times [0, \hat{c}))$ *with* $V_{xx} \in L^\infty_{loc}(\mathbb{R} \times (0, \hat{c}))$. *Moreover it is such that*

$$\begin{cases} \frac{1}{2}V_{xx}(x,c) - \lambda V(x,c) = 0 & \text{for } \hat{\gamma}(c) < x < \hat{\beta}(c),\ c \in [0,\hat{c}) \\ \frac{1}{2}V_{xx}(x,c) - \lambda V(x,c) \geq 0 & \text{for a.e. } (x,c) \in \mathbb{R} \times [0,\hat{c}) \\ V(x,c) = G(x,c) & \text{for } x \leq \hat{\gamma}(c),\ x \geq \hat{\beta}(c),\ c \in [0,\hat{c}) \\ V_x(x,c) = G_x(x,c) & \text{for } x \leq \hat{\gamma}(c),\ x \geq \hat{\beta}(c),\ c \in [0,\hat{c}) \\ V_c(x,c) = G_c(x,c) & \text{for } x \leq \hat{\gamma}(c),\ x \geq \hat{\beta}(c),\ c \in [0,\hat{c}). \end{cases} \tag{3.38}$$

Remark 3.13. *As one may note in* (3.38) *the boundary condition at* $\hat{c}$ *is missing. This will be retrieved at a later stage when we will show that there is a* C^1 *pasting of* W^1 *and* W^o *at* $c = \hat{c}$.

Proof. The proof will be divided in a number of steps.

Step 1. First we show that $V \in C(\mathbb{R} \times [0, \hat{c}))$. Note that $V \in C(\mathbb{R} \times [0, \hat{c}))$ if and only if $Q \in C((0, \infty) \times [0, \hat{c}))$ by Proposition 3.4. For the sake of clarity of notation we denote $Q^- := Q$ restricted to $[0, \infty) \times [0, c_o)$ and $Q^+ := Q$ restricted to $[0, \infty) \times (c_o, \hat{c})$. From (3.26) and (3.34) it is easy to see that $Q^- \in C([0, \infty) \times [0, c_o))$ and $Q^+ \in C([0, \infty) \times (c_o, \hat{c}))$ and moreover, for any $y > 0$ there exist limits $Q^-(y, c_o-) := \lim_{c \uparrow c_o} Q^-(y, c)$ and $Q^+(y, c_o+) := \lim_{c \downarrow c_o} Q^+(y, c)$ and such limits are locally uniform with respect to y in bounded subsets of $[0, \infty)$. It is also easy to see that $Q^\pm(\cdot\,, c_o\pm)$ are continuous as well by Propositions 3.7 and 3.10 and therefore we can continuously extend Q^- and Q^+ respectively to $[0, \infty) \times [0, c_o]$ and $[0, \infty) \times [c_o, \hat{c})$ and for simplicity we denote such extensions again by Q^- and Q^+. If now we can prove $Q^-(y, c_o) = Q^+(y, c_o)$ for all $y > 0$ then $V \in C(\mathbb{R} \times [0, \hat{c}))$.

Note that for $y \geq \hat{y}_2(c_o)$ one has $Q^-(y, c_o) = Q^+(y, c_o) = H(y, c_o)$ hence the proof is trivial. On the other hand for any $\delta \in (0, \hat{y}_2(c_o))$ and $y \in [\delta, \hat{y}_2(c_o))$, there always exists $c_\delta > c_o$ such that $(y, c) \in (\hat{y}_1(c), \hat{y}_2(c))$ for all $c \in [c_o, c_\delta)$ by (3) of Proposition 3.7 and (3.32). Hence by using (3.26), (3.34) and Proposition 3.10 one has

$$Q^-(y, c_o) = Q^+(y, c_o) = H_y(\hat{y}_2(c_o), c_o)(y - \hat{y}_2(c_o)) + H(\hat{y}_2(c_o), c_o). \tag{3.39}$$

By arbitrariness of $\delta > 0$ continuity of Q in $(0, \infty) \times [0, \hat{c})$ follows.

Step 2. We now employ arguments similar to those above to prove that $V \in C^1(\mathbb{R} \times [0, \hat{c}))$. Again, from (3.26) and (3.28) it is clear that $V \in C^1(\mathbb{R} \times [0, \hat{c}))$ if and only if $Q \in C^1((0, \infty) \times [0, \hat{c}))$.

First we show that Q is C^1 on $(0, \infty) \times [0, c_o)$ and on $(0, \infty) \times (c_o, \hat{c})$ by proving that $Q^\pm_y = H_y$ and $Q^\pm_c = H_c$ across $\hat{y}_i$, $i = 1, 2$ (where we also adopt the convention $\hat{y}_1 = 0$ on $[0, c_o]$). Let us start by considering $\bar{c} \in (c_o, \hat{c})$, and define the open set $\Lambda_{\bar{c},h} := \{(y, c) : y \in (\hat{y}_1(c), \hat{y}_2(c)), c \in (\bar{c} - h, \bar{c} + h)\}$ with suitable $h > 0$. Inside $\Lambda_{\bar{c},h}$ we can take the derivative of $Q = Q^+$ with respect to c and use Remark 3.8 to obtain

$$Q^+_c(y, c) = \Big[H_{yc}(\hat{y}_1(c), c) + H_{yy}(\hat{y}_1(c), c)\hat{y}_1'(c)\Big](y - \hat{y}_1(c)) + H_c(\hat{y}_1(c), c). \tag{3.40}$$

Moreover, Q^+_c is uniformly continuous in $\Lambda_{\bar{c},h}$ and hence can be continuously extended to the closure of $\Lambda_{\bar{c},h}$ with $Q^+_c(\hat{y}_1(c), c) = H_c(\hat{y}_1(c), c)$. Using the expression of Q in terms of $\hat{y}_2$

(cf. (3.26)) we can perform calculations analogous to those that led to (3.40) and show that it also holds $Q^+_c(\hat{y}_2(c), c) = H_c(\hat{y}_2(c), c)$ for $c \in (\bar{c} - h, \bar{c} + h)$. Similarly we take the derivative of Q^+ with respect to y and find

$$Q^+_y(y, c) = H_y(\hat{y}_i(c), c), \quad i = 1, 2. \tag{3.41}$$

By arbitrariness of $\bar{c}$ we conclude that $Q^+ \in C^1((0, \infty) \times (c_o, \hat{c}))$ and the very same arguments may be used to prove that $Q^- \in C^1((0, \infty) \times (0, c_o))$

Now we aim at proving that Q is also C^1 across $c = c_o$. From (3.40), (3.41) (where in (3.40) we replace $\hat{y}_1$ by $\hat{y}_2$ according to (3.34)) and noting that $Q = H$ on $[0, \hat{y}_1(c)] \cup [\hat{y}_2(c), \infty)$ for $c \in [0, \hat{c})$ we conclude that for all $y > 0$ the limits $Q^\pm_c(y, c_o\pm)$ and $Q^\pm_y(y, c_o\pm)$ are well defined, they are uniform for y in bounded intervals, and the resulting functions $Q^\pm_c(\,\cdot\,, c_o\pm)$ and $Q^\pm_y(\,\cdot\,, c_o\pm)$ are continuous. Then we can extend $Q^\pm_c$ and $Q^\pm_y$ up to $c = c_o$ (see also *Step 1* above). Such extensions will be denoted again by $Q^\pm_c$ and $Q^\pm_y$. If now we can prove that $Q^-_c(y, c_o) = Q^+_c(y, c_o)$ and $Q^-_y(y, c_o) = Q^+_y(y, c_o)$ for $y > 0$ then that will imply $V \in C^1(\mathbb{R} \times [0, \hat{c}))$.

For $y \geq \hat{y}_2(c_o)$ again $Q^\pm = H$ and the proof is trivial. On the other hand for any $\delta \in (0, \hat{y}_2(c_o))$ and $y \in [\delta, \hat{y}_2(c_o))$, there always exists $c_\delta > c_o$ such that $(y, c) \in (\hat{y}_1(c), \hat{y}_2(c))$ for all $c \in [c_o, c_\delta)$ by (3) of Proposition 3.7 and (3.32). Hence, from (3.26) and (3.34), for $y \in [\delta, \hat{y}_2(c_o))$ we have

$$Q^+_c(y, c) = \Big[H_{yc}(\hat{y}_2(c), c) + H_{yy}(\hat{y}_2(c), c)\hat{y}_2'(c)\Big](y - \hat{y}_2(c)) + H_c(\hat{y}_2(c), c) \quad c \in [c_o, c_\delta) \tag{3.42}$$

$$Q^-_c(y, c) = \Big[H_{yc}(\hat{y}_2(c), c) + H_{yy}(\hat{y}_2(c), c)\hat{y}_2'(c)\Big](y - \hat{y}_2(c)) + H_c(\hat{y}_2(c), c) \quad c \in [0, c_o] \tag{3.43}$$

and clearly Q_c is continuous across $c = c_o$ by Proposition 3.10. Similarly we can conclude that Q_y is continuous as well.

Step 3. To prove that V_{xx} is locally bounded it suffices to show it for Q_{yy}. One has $Q_{yy} = H_{yy}$ on $\mathcal{S}$ and $Q_{yy} = 0$ on $\mathcal{C}$ since $Q_y(y, c) = H_y(\hat{y}_2(c), c)$ for $c \in [0, \hat{c})$, then the claim follows.

Step 4. The fact that V solves (3.38) is a consequence of its regularity and derives from standard Markovian arguments which are well known in the optimal stopping theory (see for example [24], Sec. 7). The last equations in (3.38) provide the so-called smooth-fit condition and we have verified them in *Step 2* above. □

As a straightforward consequence of Proposition 3.12 and (3.14) we have

Corollary 3.14. *$W^1 \in C^1(\mathbb{R} \times [0, \hat{c}))$, with $W^1_{xx} \in L^\infty_{loc}(\mathbb{R} \times (0, \hat{c}))$ and in particular it holds*

$$W^1_c(x, c) = -x \quad \text{and} \quad W^1_x(x, c) = \hat{c} - c + W^o_x(x, \hat{c}) \tag{3.44}$$

for $x \in (-\infty, \hat{\gamma}(c)] \cup [\hat{\beta}(c), +\infty)$ and $c \in [0, \hat{c})$.

Since we are trying to argue that W^1 solves (3.2) in $\mathbb{R} \times [0, \hat{c})$ we need a suitable lower bound for W^1_c. This is provided in the next proposition. Its proof is quite long and technical and it is given in Appendix A.

Proposition 3.15. *One has $W^1_c(x; c) \geq -x$ for all $(x, c) \in \mathbb{R} \times [0, \hat{c})$.*

Before concluding this section and providing a verification theorem we want to show that W^1 and W^o fulfill a C^1 pasting across $c = \hat{c}$.

Proposition 3.16. *Let*

$$W(x, c) := \begin{cases} W^1(x, c), & \text{for } (x, c) \in \mathbb{R} \times [0, \hat{c}) \\ W^o(x, c), & \text{for } (x, c) \in \mathbb{R} \times [\hat{c}, 1], \end{cases} \tag{3.45}$$

then $W \in C^1(\mathbb{R} \times [0, 1])$ and $W_{xx} \in L^\infty_{loc}(\mathbb{R} \times [0, 1])$.

Proof. From (3.14), (3.26) and (3.28), Corollaries 3.11 and 3.14 and by using Remark 3.8 and (1) − (2) of Proposition 3.7 we observe that for all $x \in \mathbb{R}$

$$W^1(x,\hat{c}-) := \lim_{c\uparrow\hat{c}} W^1(x,c), \quad W^1_c(x,\hat{c}-) := \lim_{c\uparrow\hat{c}} W^1_c(x,c) \quad \text{and} \quad W^1_x(x,\hat{c}-) := \lim_{c\uparrow\hat{c}} W^1_x(x,c)$$

exist and they are uniform with respect to x in bounded subsets of $\mathbb{R}$. It is also easy to see that $W^1_c(\,\cdot\,,\hat{c}-)$ and $W^1_x(\,\cdot\,,\hat{c}-)$ are continuous and therefore W^1 has a C^1 extension to $\mathbb{R}\times[0,\hat{c}]$ which we denote again by W^1.

For $x \in (-\infty,\gamma^o] \cup [\hat{\beta}(\hat{c}-),+\infty)$ we have $W^1(x,\hat{c}) = W^o(x,\hat{c})$, $W^1_c(x,\hat{c}) = W^o_c(x,\hat{c})$ and $W^1_x(x,\hat{c}) = W^o_x(x,\hat{c})$ since $V = G$, $V_c = G_c$ and $V_x = G_x$ in that set (cf. (3.15), (3.16) and (3.8)). For $x \in (\gamma^o,\hat{\beta}(\hat{c}-))$ we have

$$W^1(x,\hat{c}) = x\Phi(\hat{c}) + \phi_\lambda(x)Q(F_\lambda(x),\hat{c}-) \tag{3.46}$$

$$W^1_c(x,\hat{c}) = x\Phi'(\hat{c}) + \phi_\lambda(x)Q_c(F_\lambda(x),\hat{c}-) \tag{3.47}$$

$$W^1_x(x,\hat{c}) = \Phi(\hat{c}) + \phi_\lambda(x)\Big[Q_x(F_\lambda(x),\hat{c}-)F'_\lambda(x) - \sqrt{2\lambda}Q(F_\lambda(x),\hat{c}-)\Big] \tag{3.48}$$

by (3.14) and Proposition 3.4. To find an explicit expression of (3.46) we study $Q(y,\hat{c}-)$ for $y \in (e^{-2},\hat{y}_2(\hat{c}-))$ (see (1) of Proposition 3.7). In particular from (3.26), Remark 3.8 and Proposition 3.7 (noting that $\hat{y}_1(c) < e^{-2}$ for $c < \hat{c}$) we find

$$Q(y,\hat{c}-) = H_y(e^{-2}-,\hat{c})(y-e^{-2}) + H(e^{-2}-,\hat{c}) = -\tfrac{1}{\sqrt{2\lambda}}R(\hat{c})e^{-1} \tag{3.49}$$

by (A-1). It then follows $W^1(x,\hat{c}) = W^o(x,\hat{c})$ by simple calculations, (3.8) and (3.20).

For (3.47) we consider $Q_c(y,\hat{c}-)$ for $y \in (e^{-2},\hat{y}_2(\hat{c}-))$ and arguing as above we obtain

$$Q_c(y,\hat{c}-) = \Big[H_{yc}(e^{-2}-,\hat{c}) + H_{yy}(e^{-2}-,\hat{c})\hat{y}'_1(\hat{c}-)\Big](y-e^{-2}) + H_c(e^{-2}-,\hat{c}) = 0 \tag{3.50}$$

by (3.24), (A-1) and (A-2), hence $V_c(x,\hat{c}-) = 0$ and $W^1_c(x,\hat{c}) = W^o_c(x,\hat{c}) = -x$ by recalling that $\Phi'(\hat{c}) = -1$ (cf. (2.7)).

To conclude the proof we observe that $Q_y(y,\hat{c}-) = H_y(\hat{y}_1(\hat{c}-),\hat{c}) = 0$ for $y \in (e^{-2},\hat{y}_2(\hat{c}-))$, hence (3.49) and (3.48) give $W^1_x(x,c) = W^o_x(x,\hat{c}) = \Phi(\hat{c}) + \phi_\lambda(x)R(\hat{c})e^{-1}$. ☐

Remark 3.17. *It is not hard to see that the geometry of $H(y,\hat{c})$ (cf.* (3.24)*) is such that $V(x,\hat{c}) = G(x,\hat{c})$ for all $x \in \mathbb{R}$. In other words, according to our definitions* (3.17) *the continuation set $\mathcal{C}_{\hat{c}}$ is empty whereas $\mathcal{S}_{\hat{c}} = \mathbb{R}$. This happens despite the upper boundary $\hat{\beta}$ has a strictly positive left limit and may be interpreted in terms of a discontinuity of such boundary, i.e. $\hat{\beta}(\hat{c}-) > \hat{\beta}(\hat{c}) = \gamma^o$.*

However one should also notice that $(\mathbb{L}_X - \lambda)V(x,\hat{c}) = 0$ for $x > \gamma^o$ so that the region $x > \gamma^o$ is a region of "indifference" since continuing the observation would produce the same performance as stopping at once (by a standard martingale argument). With this in mind one may also redefine $\mathcal{C}_{\hat{c}}$ and $\mathcal{S}_{\hat{c}}$ by setting $\hat{\beta}(\hat{c}) = +\infty$. Doing so one would still obtain the same analytical properties of V and $\tau_o := \inf\{t \geq 0 : X_t \leq \gamma^o\}$ would be an optimal stopping time.

In the next section we will see how these considerations affect the construction of the optimal control.

4 The verification theorem and the optimal solution

In this section we perform a verification argument which will allow us to conclude that the candidate value function W of (3.45) we have constructed in previous Section 3 is indeed the value function of our problem (2.3). As a byproduct, we also provide the explicit expression of the optimal control policy for problem (2.3) which is shown to be of *bang-bang* type triggered by the boundaries γ^o of Proposition 3.2 and $\hat{\beta}$ and $\hat{\gamma}$ of Corollary 3.11.

From now on, by noting thanks to (1) and (2) of Proposition 3.7 and (3.27) that the curve $\hat{\gamma}$ is actually the C^1-extension in the set $c \in (c_o, \hat{c}]$ of γ^o of Proposition 3.2, to simplify exposition, with a slight abuse of notation, we will simply refer to γ^o and $\hat{\gamma}$ by using uniquely $\hat{\gamma}$.

Proposition 4.1. *The function W of* (3.45) *solves the variational problem* (3.2). *Moreover, $|W(x,c)| \leq K(1+|x|)$ for some $K > 0$ and $W(x,1) = U(x,1) = 0$.*

Proof. Recalling (3.8), (3.14) and (3.45), and collecting the results of Proposition 3.2, Proposition 3.12, Corollary 3.14, Proposition 3.15 and Proposition 3.16 it easily follows that W as in (3.45) solves (3.2).

Moreover, (3.45), together with (3.8), (3.13) (see also (3.28) and (3.35)), shows that W has sub-linear growth. Finally, the fact that $W(x,1) = 0$ can be easily seen thanks to (3.45) and (3.8). □

From now on, according to Corollary 3.11 and to the discussion in Remark 3.17, we formally set $\hat{\gamma}(c) = -\infty$ on $[0, c_o]$ and $\hat{\beta}(c) = +\infty$ on $[\hat{c}, 1]$.

Let $(x,c) \in \mathbb{R} \times [0,1]$ be arbitrary but fixed and consider the two dimensional dynamics $(X^x, C^{c,\nu})$ for an arbitrary admissible control ν. Define the stopping times

$$\tau_{\hat{\beta}} := \inf\{t \geq 0 : X^x_t \geq \hat{\beta}(C^{c,\nu}_t)\}, \qquad \tau_{\hat{\gamma}} := \inf\{t \geq 0 : X^x_t \leq \hat{\gamma}(C^{c,\nu}_t)\}, \tag{4.1}$$

and

$$\tau^* := \tau_{\hat{\beta}} \wedge \tau_{\hat{\gamma}}, \qquad \sigma^* := \inf\{t \geq \tau_{\hat{\beta}} : X^x_t \leq \hat{\gamma}(C^{c,\nu}_t)\}. \tag{4.2}$$

Here we notice that such stopping times could be P-a.s. infinite depending on the particular choice of ν but in general $\sigma^* \geq \tau_{\hat{\beta}} \geq \tau^*$ P-a.s. Then, recalling that $\hat{c}$ is the unique solution in $(0,1)$ of (2.7) we introduce the admissible purely discontinuous control

$$\nu^*_t := (1-c)\mathbb{1}_{\{t > \tau^*\}}\mathbb{1}_{\{\tau^* = \tau_{\hat{\gamma}}\}} + \left[(\hat{c} - c)\mathbb{1}_{\{t \leq \sigma^*\}} + (1-\hat{c})\mathbb{1}_{\{t > \sigma^*\}}\right]\mathbb{1}_{\{t > \tau^*\}}\mathbb{1}_{\{\tau^* = \tau_{\hat{\beta}}\}}. \tag{4.3}$$

The control ν^* prescribes to do nothing until the uncontrolled process X^x leaves the interval $(\hat{\gamma}(c), \hat{\beta}(c))$, where $c \in [0,1)$ is the initial capacity. Then, if $\tau_{\hat{\gamma}} < \tau_{\hat{\beta}}$ one should immediately exert all the available fuel after hitting $\hat{\gamma}(c)$, if instead $\tau_{\hat{\gamma}} > \tau_{\hat{\beta}}$ one should initially increase the capacity to $\hat{c}$ after hitting $\hat{\beta}(c)$ and then wait until X hits $\hat{\gamma}(\hat{c})$ before exerting all the control left. We remark that under the strategy ν^* the stopping time $\tau_{\hat{\gamma}}$ is P-a.s. finite. In Figure 3 below we provide an illustrative diagram of the boundaries and of the behaviour of control ν^* of (4.3). Optimality of ν^* for problem (2.3) is proved in the next theorem.

Theorem 4.2. *The admissible control ν^* of* (4.3) *is optimal for problem* (2.3) *and $W \equiv U$.*

Proof. The proof is based on a verification argument and, as usual, it splits into two steps.

Step 1. Fix $(x,c) \in \mathbb{R} \times [0,1]$ and take $R > 0$. Set $\tau_R := \inf\big\{t \geq 0 : X^x_t \notin (-R,R)\big\}$, take an admissible control ν, and recall the regularity results for W by Proposition 3.16. Then we can use Itô's formula in the weak version of [15], Chapter 8, Section VIII.4, Theorem 4.1, up to the stopping time $\tau_R \wedge T$, for some $T > 0$, to obtain

$$\begin{aligned} W(x;c) =& \mathsf{E}\left[e^{-\lambda(\tau_R \wedge T)} W(X^x_{\tau_R \wedge T}, C^{c,\nu}_{\tau_R \wedge T})\right] - \mathsf{E}\left[\int_0^{\tau_R \wedge T} e^{-\lambda s}(\mathbb{L}_X - \lambda)W(X^x_s, C^{c,\nu}_s)ds\right] \\ &- \mathsf{E}\left[\int_0^{\tau_R \wedge T} e^{-\lambda s} W_c(X^x_s, C^{c,\nu}_s)d\nu_s\right] \\ &- \mathsf{E}\Big[\sum_{0 \leq s < \tau_R \wedge T} e^{-\lambda s}\left(W(X^x_s, C^{c,\nu}_{s+}) - W(X^x_s, C^{c,\nu}_s) - W_c(X^x_s, C^{c,\nu}_s)\Delta\nu_s\right)\Big] \end{aligned}$$

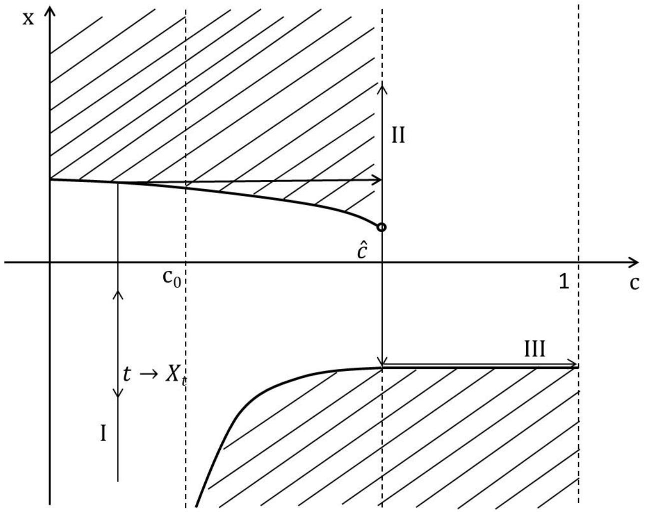

Figure 3: An illustrative diagram of the the free-boundaries and of the optimal control ν^* of (4.3). The upper boundary $\hat{\beta}$ and the lower boundary $\hat{\gamma}$ split the state space into the inaction region (white) and action region (hatched). When the initial state is (x, c) with $c \in [0, c_o)$ and $x < \hat{\beta}(c)$ one observes the following three regimes: in regime (I) the process X diffuses until hitting $\hat{\beta}(c)$, then the capacity is increased by $\Delta\nu = \hat{c} - c$ pushing horizontally the process (X, C) to the regime (II); there X continues to diffuse until it hits γ^o and at that point all the fuel is spent and (X, C) is pushed horizontally to regime (III) where the optimisation is concluded.

where $\Delta\nu_s := \nu_{s+} - \nu_s$ and the expectation of the stochastic integral vanishes since W_x is bounded on $(x, c) \in [-R, R] \times [0, 1]$.

Now, recalling that any $\nu \in \mathcal{A}_c$ can be decomposed into the sum of its continuous part and of its pure jump part, i.e. $d\nu = d\nu^{cont} + \Delta\nu$, one has (see [15], Chapter 8, Section VIII.4, Theorem 4.1 at pp. 301-302)

$$\begin{aligned} W(x,c) =& \mathsf{E}\left[e^{-\lambda(\tau_R \wedge T)} W(X^x_{\tau_R \wedge T}, C^{c,\nu}_{\tau_R \wedge T})\right] - \mathsf{E}\left[\int_0^{\tau_R \wedge T} e^{-\lambda s} (\mathbb{L}_X - \lambda) W(X^x_s, C^{c,\nu}_s) ds\right] \\ & - \mathsf{E}\left[\int_0^{\tau_R \wedge T} e^{-\lambda s} W_c(X^x_s, C^{c,\nu}_s) d\nu^{cont}_s + \sum_{0 \le s < \tau_R \wedge T} e^{-\lambda s} \left(W(X^x_s, C^{c,\nu}_{s+}) - W(X^x_s, C^{c,\nu}_s)\right)\right]. \end{aligned}$$

Since W satisfies the Hamilton-Jacobi-Bellman equation (3.2) (cf. Proposition 4.1) and by noticing that

$$W(X^x_s, C^{c,\nu}_{s+}) - W(X^x_s, C^{c,\nu}_s) = \int_0^{\Delta\nu_s} W_c(X^x_s, C^{c,\nu}_s + u) du, \tag{4.4}$$

we obtain

$$\begin{aligned} W(x,c) \le& \mathsf{E}\left[e^{-\lambda(\tau_R \wedge T)} W(X^x_{\tau_R \wedge T}, C^{c,\nu}_{\tau_R \wedge T})\right] + \mathsf{E}\left[\int_0^{\tau_R \wedge T} e^{-\lambda s} \lambda X^x_s \Phi(C^{c,\nu}_s) ds\right] \\ & + \mathsf{E}\left[\int_0^{\tau_R \wedge T} e^{-\lambda s} X^x_s d\nu^{cont}_s\right] + \mathsf{E}\Big[\sum_{0 \le s < \tau_R \wedge T} e^{-\lambda s} X^x_s \Delta\nu_s\Big] \\ =& \mathsf{E}\left[e^{-\lambda(\tau_R \wedge T)} W(X^x_{\tau_R \wedge T}, C^{c,\nu}_{\tau_R \wedge T}) + \int_0^{\tau_R \wedge T} e^{-\lambda s} \lambda X^x_s \Phi(C^{c,\nu}_s) ds + \int_0^{\tau_R \wedge T} e^{-\lambda s} X^x_s d\nu_s\right]. \end{aligned} \tag{4.5}$$

When taking limits as $R \to \infty$ we have $\tau_R \wedge T \to T$, P-a.s. By standard properties of Brownian motion it is easy to prove that the integral terms in the last expression on the right hand side of (4.5) are uniformly bounded in $L^2(\Omega, \mathsf{P})$, hence uniformly integrable. Moreover, W has sub-linear growth by Proposition 4.1. Then we also take limits as $T \uparrow \infty$ and it follows that

$$W(x,c) \leq \mathsf{E}\Big[\int_0^\infty e^{-\lambda s}\lambda X^x_s \Phi(C^{c,\nu}_s)ds + \int_0^\infty e^{-\lambda s} X^x_s d\nu_s\Big], \tag{4.6}$$

due to the fact that $\lim_{T\to\infty} \mathsf{E}[e^{-\lambda T} W(X^x_T, C^{c,\nu}_T)] = 0$. Since the latter holds for all admissible ν we have $W(x,c) \leq U(x;c)$.

Step 2.

If $c = 1$ then $W(x,1) = 0 = U(x,1)$. Then take $c \in [0,1)$ and define $C^*_t := C^{c,\nu^*}_t = c + \nu^*_t$, with ν^* as in (4.3). Applying Itô's formula again (possibly using localisation arguments as in Step 1.) up to time $\tau_{\hat\gamma}$ (cf. (4.1)) we find

$$\begin{aligned} W(x,c) =& \mathsf{E}\Big[e^{-\lambda\tau_{\hat\gamma}} W(X^x_{\tau_{\hat\gamma}}, C^*_{\tau_{\hat\gamma}}) - \int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds\Big] \\ &- \mathsf{E}\Big[\sum_{0\leq s<\tau_{\hat\gamma}} e^{-\lambda s}\big(W(X^x_s, C^*_{s+}) - W(X^x_s, C^*_s)\big)\Big], \end{aligned} \tag{4.7}$$

where we have used that ν^* does not have a continuous part. We also recall, as already observed, that $\tau_{\hat\gamma} < +\infty$, P-a.s. under the control policy of ν^*.

From (4.2) one has $\tau^* \leq \tau_{\hat\gamma}$, P-a.s. and therefore we can always write

$$\begin{aligned} &\int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds \\ &= \int_0^{\tau^*} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds + \int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds \\ &= -\int_0^{\tau^*} e^{-\lambda s}\lambda X^x_s\Phi(C^*_s)ds + \int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds \end{aligned} \tag{4.8}$$

where the second inequality follows by recalling that $(\mathbb{L}_X - \lambda)W(x,c) = -\lambda x\Phi(c)$ for $\hat\gamma(c) < x < \hat\beta(c)$ and hence it holds in the first integral for $s \leq \tau^*$. To evaluate the last term of (4.8) we study separately the events $\{\tau^* = \tau_{\hat\beta}\}$ and $\{\tau^* = \tau_{\hat\gamma}\}$. We start by observing that under the control strategy ν^* one has $\{\tau^* = \tau_{\hat\beta}\} = \{\tau_{\hat\gamma} = \sigma^*\}$ and we get

$$\mathbb{1}_{\{\tau^*=\tau_{\hat\beta}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = -\mathbb{1}_{\{\tau^*=\tau_{\hat\beta}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\lambda X^x_s\Phi(C^*_s)ds \tag{4.9}$$

by Proposition 3.2 since $(X^x_s, C^*_s) = (X^x_s, \hat c)$ for any $\tau^* < s \leq \tau_{\hat\gamma} = \sigma^*$ on $\{\tau^* = \tau_{\hat\beta}\}$. On the other hand

$$\mathbb{1}_{\{\tau^*=\tau_{\hat\gamma}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = 0 = \mathbb{1}_{\{\tau^*=\tau_{\hat\gamma}\}}\int_{\tau^*}^{\tau_{\hat\gamma}} e^{-\lambda s}\lambda X^x_s\Phi(C^*_s)ds. \tag{4.10}$$

Then it follows from (4.8), (4.9) and (4.10) that

$$\int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = -\int_0^{\tau_{\hat\gamma}} \lambda X^x_s\Phi(C^*_s)ds. \tag{4.11}$$

Moreover $\Phi(C^*_s) = 0$ for any $s > \tau_{\hat\gamma}$ because $C^*_s = 1$ for any such s and thus we finally get from (4.11)

$$\int_0^{\tau_{\hat\gamma}} e^{-\lambda s}\big(\mathbb{L}_X - \lambda\big)W(X^x_s, C^*_s)ds = -\int_0^{\infty} \lambda X^x_s\Phi(C^*_s)ds. \tag{4.12}$$

Note that under the control strategy ν^* it also holds $\{\tau_{\hat\gamma} < \tau_{\hat\beta}\} = \{\tau_{\hat\gamma} < \sigma^*\}$ and $\{\tau_{\hat\gamma} > \tau_{\hat\beta}\} = \{\tau_{\hat\gamma} = \sigma^*\}$, then from (4.3) we have

$$\begin{aligned}
&\mathsf{E}\Big[e^{-\lambda\tau_{\hat\gamma}} W(X^x_{\tau_{\hat\gamma}}, C^*_{\tau_{\hat\gamma}})\Big] \\
=&\mathsf{E}\Big[\mathbb{1}_{\{\tau_{\hat\gamma}>\tau_{\hat\beta}\}} e^{-\lambda\tau_{\hat\gamma}} W(\hat\gamma(\hat c),\hat c)\Big] + \mathsf{E}\Big[\mathbb{1}_{\{\tau_{\hat\gamma}<\tau_{\hat\beta}\}} e^{-\lambda\tau_{\hat\gamma}} W(\hat\gamma(c),c)\Big] \\
=&\mathsf{E}\Big[\mathbb{1}_{\{\tau_{\hat\gamma}>\tau_{\hat\beta}\}} e^{-\lambda\tau_{\hat\gamma}} \hat\gamma(\hat c)(1-\hat c)\Big] + \mathsf{E}\Big[\mathbb{1}_{\{\tau_{\hat\gamma}<\tau_{\hat\beta}\}} e^{-\lambda\tau_{\hat\gamma}} \hat\gamma(c)(1-c)\Big] \\
=&\mathsf{E}\Big[\int_{\tau_{\hat\gamma}}^{\infty} e^{-\lambda s} X^x_s d\nu^*_s\Big]
\end{aligned} \tag{4.13}$$

by using that $W(\hat\gamma(c),c) = \hat\gamma(c)(1-c)$ for all $c \in [0,1)$ as proved in Section 3 (see also Figure 3).

For the jump part of the control, i.e. for the last term in (4.7), again we argue in a similar way as above and use that on the event $\{\tau^* = \tau_{\hat\gamma}\}$ there is no jump strictly prior to $\tau_{\hat\gamma}$ and the sum in (4.7) is zero, whereas on the event $\{\tau^* = \tau_{\hat\beta}\}$ a single jump occurs prior to $\tau_{\hat\gamma}$, precisely at $\tau_{\hat\beta}$. This gives

$$\begin{aligned}
&\mathsf{E}\Big[\sum_{0\le s<\tau_{\hat\gamma}} e^{-\lambda s}\big(W(X^x_s, C^*_{s+}) - W(X^x_s, C^*_s)\big)\Big] \\
=&\Big[\mathbb{1}_{\{\tau^*=\tau_{\hat\gamma}\}}\cdot 0 + \mathbb{1}_{\{\tau^*=\tau_{\hat\beta}\}} e^{-\lambda\tau_{\hat\beta}} X^x_{\tau_{\hat\beta}}(\hat c - c)\Big] = \mathsf{E}\Big[\int_0^{\tau_{\hat\gamma}} e^{-\lambda s} X^x_s d\nu^*_s\Big].
\end{aligned} \tag{4.14}$$

Combining (4.12), (4.13) and (4.14) it follows from (4.7) that

$$W(x,c) = \mathsf{E}\Big[\int_0^{\infty} e^{-\lambda s}\lambda X^x_s\,\Phi(C^*_s)ds + \int_0^{\infty} e^{-\lambda s} X^x_s d\nu^*_s\Big] \ge U(x,c), \tag{4.15}$$

which together with Step 1. implies $W(x,c) = U(x,c)$, $(x,c) \in \mathbb{R}\times[0,1]$ and ν^* of (4.3) is optimal. □

Remark 4.3. *Recalling Remark 3.17 we note here that for $c = \hat c$ the optimal control of* (4.3) *prescribes to do nothing as long as X stays above γ^o; however, from* (3.8) *we see that $W^o_c(x,\hat c) = -x$ for all $x \in \mathbb{R}$ and therefore there seems to be a contradiction with our definition of action and inaction regions* (3.3)*. The point here is that although $(x,\hat c) \in \mathcal{D}_W$ for all $x \in \mathbb{R}$, one also has $(\mathbb{L}_X - \lambda)W(x,\hat c) = -\lambda\Phi(\hat c)x$ for $x > \gamma^o$ and therefore, as long as X stays above γ^o, an inaction strategy does not increase the overall costs. A general principle of "minimality" stands in stochastic control, that is one should only use the minimal effort to achieve the optimal performance. In our case this translates into acting only in the region of the state space where $(\mathbb{L}_X - \lambda)W(x,\hat c) > -\lambda\Phi(\hat c)x$ so that we can effectively redefine action and inaction region accordingly.*

A Some proofs needed in Section 3

Proof of Lemma 3.5

Recall that $R(c) - R(\hat c) < 0$ since $c \in (c_o, \hat c)$. One has $\lim_{y\downarrow 0} H(y,c) = 0$, hence $H(\,\cdot\,,c)$ is continuous on $[0,\infty)$ with $\lim_{y\uparrow\infty} H(y,c) = -\infty$. On the other hand, simple algebra leads to

$$H_y(y,c) = \frac{1}{2\sqrt{2\lambda}} y^{-\frac{1}{2}}\big(1 + \frac{1}{2}\ln y\big) \times \begin{cases} R(c), & 0 < y \le e^{-2} \\ R(c) - R(\hat c), & y > e^{-2}, \end{cases} \tag{A-1}$$

hence $H(\,\cdot\,,c) \in C^1(0,\infty)$, it is decreasing with $\lim_{y\downarrow 0} H_y(y,c) = -\infty$ and $\lim_{y\uparrow\infty} H_y(y,c) = 0$. Taking the derivative of (A-1) with respect to y we obtain

$$H_{yy}(y;c) = -\frac{y^{-\frac{3}{2}}}{8\sqrt{2\lambda}}\ln(y) \times \begin{cases} R(c), & 0 < y \le e^{-2} \\ R(c) - R(\hat{c}), & y > e^{-2}. \end{cases} \tag{A-2}$$

Therefore $H_{yy}(\,\cdot\,c)$ is continuous on $(0,\infty)\setminus\{e^{-2}\}$, it is positive in the interval $(0,e^{-2})\cup(1,\infty)$ and negative elsewhere.

Proof of Proposition 3.6

Fix $c \in (c_o,\hat{c})$, for $y > 1$ denote by $r_y(\,\cdot\,)$ the straight line tangent to $H(y,c)$, i.e.

$$r_y(z) = H_y(y,c)(z-y) + H(y,c), \quad z > 0 \tag{A-3}$$

and define

$$\hat{y}_2 := \hat{y}_2(c) = \inf\{y > 1 : P_r(y,c) < 0\}, \tag{A-4}$$

where $P_r(y,c) := \sup_{z\in[0,e^{-2}]}\big(r_y(z) - H(z,c)\big)$ for $y > 1$. Clearly the set to the right of (A-4) is not empty by properties of H listed in Lemma 3.5 and we claim that $\hat{y}_2 > 1$ (this will be proved later). The map $y \mapsto P_r(y,c)$ is continuous and decreasing on $(1,\infty)$ and hence $P_r(\hat{y}_2,c) = 0$ and $\hat{y}_2$ is unique. Since $z \mapsto r_{\hat{y}_2}(z) - H(z,c)$ is concave on $[0,e^{-2}]$ (cf. Lemma 3.5) then there exists a unique $z_* = z_*(\hat{y}_2)$ such that $r_{\hat{y}_2}(z_*) - H(z_*,c) = 0$ and $\hat{y}_1 = z_*$. It thus follows that, by construction, $(\hat{y}_1,\hat{y}_2)$ uniquely solves system (3.25) and to complete the proof we only need to show that $\hat{y}_2 > 1$.

We argue by contradiction and assume that $\hat{y}_2 = 1$. Then $r_{\hat{y}_2}(z) = r_1(z) = H_y(1,c)(z-1) + H(1,c)$ and by the explicit formulae in the proof of Lemma 3.5 we find

$$r_1(e^{-2}) - H(e^{-2},c) = \tfrac{1}{\sqrt{2\lambda}}(R(c) - R(\hat{c}))\big(\tfrac{1}{2}e^{-2} + e^{-1} - \tfrac{1}{2}\big) > 0$$

which contradicts the definition of $\hat{y}_2$ given the continuity of $P_r(\,\cdot\,,c)$. Note that $\hat{y}_1$ must be strictly positive since any straight line passing through the origin cannot be tangent to H as $\lim_{z\downarrow 0} H_y(z,c) = -\infty$ for all $c \in (c_o,\hat{c})$. Similarly $\hat{y}_1(c) < e^{-2}$ since the tangent line to H is horizontal for $y = e^{-2}$.

Proof of Proposition 3.7

We rewrite (3.25) as

$$F_1(\hat{y}_1(c),\hat{y}_2(c);c) = 0 \quad \text{and} \quad F_2(\hat{y}_1(c),\hat{y}_2(c);c) = 0, \tag{A-5}$$

with the two functions $F_i : (0,\infty)\times(0,\infty)\times[0,1]$, $i = 1,2$, defined by

$$F_1(x,y;c) := x^{-\frac{1}{2}}(1 + \tfrac{1}{2}\ln x)R(c) - y^{-\frac{1}{2}}(1 + \tfrac{1}{2}\ln y)(R(c) - R(\hat{c})), \tag{A-6}$$

$$F_2(x,y;c) := x^{\frac{1}{2}}(1 - \tfrac{1}{2}\ln x)R(c) - y^{\frac{1}{2}}(1 - \tfrac{1}{2}\ln y)(R(c) - R(\hat{c})) - 2e^{-1}R(\hat{c}). \tag{A-7}$$

We formally take total derivatives with respect to c in both equations in (A-5) to find

$$\begin{cases} \hat{y}_1'(c)\frac{\partial F_1}{\partial x}(\hat{y}_1(c),\hat{y}_2(c);c) + \hat{y}_2'(c)\frac{\partial F_1}{\partial y}(\hat{y}_1(c),\hat{y}_2(c);c) = -\frac{\partial F_1}{\partial c}(\hat{y}_1(c),\hat{y}_2(c);c), \\ \hat{y}_1'(c)\frac{\partial F_2}{\partial x}(\hat{y}_1(c),\hat{y}_2(c);c) + \hat{y}_2'(c)\frac{\partial F_2}{\partial y}(\hat{y}_1(c),\hat{y}_2(c);c) = -\frac{\partial F_2}{\partial c}(\hat{y}_1(c),\hat{y}_2(c);c), \end{cases} \tag{A-8}$$

which takes the form of 2×2 system in the unknowns $(\hat{y}_1'(c), \hat{y}_2'(c))$. Defining

$$\begin{cases} D_o(x,y;c) := \Big[\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}\Big](x,y;c), \\ D_1(x,y;c) := \Big[-\frac{\partial F_1}{\partial c}\frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial c}\frac{\partial F_1}{\partial y}\Big](x,y;c), \\ D_2(x,y;c) := \Big[-\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial c} + \frac{\partial F_1}{\partial c}\frac{\partial F_2}{\partial x}\Big](x,y;c), \end{cases} \tag{A-9}$$

and applying Cramer's method it is easy to see that the unique solution of (A-8) is given by

$$\hat{y}_1'(c) = \frac{D_1(\hat{y}_1(c), \hat{y}_2(c); c)}{D_o(\hat{y}_1(c), \hat{y}_2(c); c)} \quad \text{and} \quad \hat{y}_2'(c) = \frac{D_2(\hat{y}_1(c), \hat{y}_2(c); c)}{D_o(\hat{y}_1(c), \hat{y}_2(c); c)}. \tag{A-10}$$

Now, if for a given $\bar{c} \in (c_o, \hat{c})$ one has $D_o(\hat{y}_1(\bar{c}), \hat{y}_2(\bar{c}); \bar{c}) \neq 0$ then the implicit function theorem allows us to make the above derivation rigorous in a suitable interval $I_h := (\bar{c} - h, \bar{c} + h)$ for some $h > 0$, so that $\hat{y}_1$ and $\hat{y}_2$ are C^1 in I_h. As it will be shown below $D_o(\hat{y}_1(c), \hat{y}_2(c); c) \neq 0$ for all $c \in (c_o, \hat{c})$ hence $\hat{y}_1$ and $\hat{y}_2$ are C^1 as claimed.

Simple but tedious algebra gives

$$\begin{cases} D_o(x,y;c) = \frac{1}{16}\big[R(c) - R(\hat{c})\big]R(c)\frac{1}{\sqrt{xy}}\Big(\frac{1}{y} - \frac{1}{x}\Big)\ln x \ln y, \\ D_1(x,y;c) = \frac{1}{4}R'(c)\big[R(c) - R(\hat{c})\big]y^{-1}\Big[\sqrt{\frac{x}{y}}(1 - \frac{1}{2}\ln x) + \ln y - \sqrt{\frac{y}{x}}(1 + \frac{1}{2}\ln x)\Big]\ln y, \\ D_2(x,y;c) = -\frac{1}{4}x^{-\frac{3}{2}}R(c)R'(c)\Big[x^{\frac{1}{2}}\ln x - xy^{-\frac{1}{2}}(1 + \frac{1}{2}\ln y) + y^{\frac{1}{2}}(1 - \frac{1}{2}\ln y)\Big]\ln x. \end{cases} \tag{A-11}$$

Since $\hat{y}_2(c) > 1$, $\hat{y}_1(c) < e^{-2}$ and $R(c) < R(\hat{c})$ we obtain that $D_o(\hat{y}_1(c), \hat{y}_2(c); c) < 0$ and hence implicit function theorem applies. Moreover one has $D_1(\hat{y}_1(c), \hat{y}_2(c); c) < 0$, hence from the first of (A-10) we get $c \mapsto \hat{y}_1(c)$ increasing.

In order to prove that $\hat{y}_2'(c) < 0$ it remains to show that $D_2(\hat{y}_1(c), \hat{y}_2(c); c) > 0$ in (A-10). Notice that since $0 < \hat{y}_1(c) < e^{-2}$, $R(c) > 0$ and $R'(c) < 0$, the sign of $D_2(\hat{y}_1(c), \hat{y}_2(c); c)$ is the same as the one of $\hat{D}_2(\hat{y}_1(c), \hat{y}_2(c); c)$ where

$$\hat{D}_2(x,y;c) := x^{\frac{1}{2}}\ln x - xy^{-\frac{1}{2}}(1 + \frac{1}{2}\ln y) + y^{\frac{1}{2}}(1 - \frac{1}{2}\ln y). \tag{A-12}$$

Recalling now (A-5), (A-6) and (A-7), we obtain

$$\hat{y}_2^{\frac{1}{2}}(c)\big(1 - \frac{1}{2}\ln \hat{y}_2(c)\big) = \frac{R(c)}{R(c) - R(\hat{c})}\hat{y}_1^{\frac{1}{2}}(c)\big(1 - \frac{1}{2}\ln \hat{y}_1(c)\big) - \frac{2e^{-1}}{R(c) - R(\hat{c})}R(\hat{c}),$$

and

$$\hat{y}_2^{-\frac{1}{2}}(c)\big(1 + \frac{1}{2}\ln \hat{y}_2(c)\big) = \frac{R(c)}{R(c) - R(\hat{c})}\hat{y}_1^{-\frac{1}{2}}(c)\big(1 + \frac{1}{2}\ln \hat{y}_1(c)\big),$$

which plugged into (A-12) give

$$\hat{D}_2(\hat{y}_1(c), \hat{y}_2(c); c) = -\frac{R(\hat{c})}{(R(c) - R(\hat{c})}(\sqrt{\hat{y}_1(c)}\ln \hat{y}_1(c) + 2e^{-1}) =: -\frac{R(\hat{c})}{(R(c) - R(\hat{c})}q(\hat{y}_1(c)).$$

It is now easy to see that $x \mapsto q(x)$ is strictly decreasing on $(0, e^{-2})$ and such that $q(e^{-2}) = 0$ and $\lim_{x \downarrow 0} q(x) = 2e^{-1} > 0$. Hence $q(\hat{y}_1(c)) > 0$ as $\hat{y}_1(c) \in (0, e^{-2})$ and therefore $\hat{D}_2(\hat{y}_1(c), \hat{y}_2(c); c) > 0$, implying $D_2(\hat{y}_1(c), \hat{y}_2(c); c) > 0$. It thus follows from (A-10) and the fact that $D_o(\hat{y}_1(c), \hat{y}_2(c); c) < 0$ that $\hat{y}_2'(c) < 0$; i.e. $c \mapsto \hat{y}_2(c)$ decreasing.

To complete the proof we need to show properties $(1) - (4)$ and for that we observe that due to monotonicity of $\hat{y}_i(\cdot)$, $i = 1, 2$, on $(c_o, \hat{c})$ there exist limits at any point of such interval.

(1) Taking limits as $c \uparrow \hat{c}$ in the second of (A-5), using (A-7) and defining $\hat{y}_1(\hat{c}-) := \lim_{c\uparrow\hat{c}} \hat{y}_1(c)$ we get

$$\hat{y}_1^{\frac{1}{2}}(\hat{c}-)\big(1 - \frac{1}{2}\ln \hat{y}_1(\hat{c}-)\big) = 2e^{-1},$$

which is uniquely solved by $\hat{y}_1(\hat{c}-) = e^{-2}$.

(2) We take limits as $c \uparrow \hat{c}$ in the first of (A-10) and note that $\lim_{c\uparrow\hat{c}} D_1(y_1(c), y_2(c); c) = 0$ by (A-11).

(3) We argue by contradiction and assume that $\lim_{c\downarrow c_o} \hat{y}_1(c) = \overline{y}_1 > 0$. Then taking limits as $c \downarrow c_o$ in the first of (A-5) and recalling that $R(c_o) = 0$ we find

$$R(\hat{c})\sqrt{\hat{y}_2(c_o+)}\big[1 + \frac{1}{2}\ln \hat{y}_2(c_o+)\big] = 0,$$

which is clearly impossible since $\hat{y}_2(c_o+) \geq 1$ due to the fact that $\hat{y}_2(c) > 1$ for any $c \in (c_o, \hat{c})$.

(4) From the second of (A-5) and (A-7) one finds

$$\sqrt{\hat{y}_2(c)}\big[1 - \frac{1}{2}\ln \hat{y}_2(c)\big] = \frac{2e^{-1}R(\hat{c}) - R(c)\sqrt{\hat{y}_1(c)}\big[1 - \frac{1}{2}\ln \hat{y}_1(c)\big]}{R(\hat{c}) - R(c)} \geq 2e^{-1} > 0 \tag{A-13}$$

where the first lower bound follows by the fact that $x \mapsto \sqrt{x}[1 - \frac{1}{2}\ln(x)]$ is strictly increasing and positive on $[0, e^{-2}]$, with maximum value $2e^{-1}$. Since also $\hat{y}_2(c) > 0$, from (A-13) we conclude that $\big[1 - \frac{1}{2}\ln \hat{y}_2(c)\big] > 0$, thus implying $\hat{y}_2(c) < e^2$.

Proof of Proposition 3.10

The proof is carried out in three steps.

Step 1. It is easy to see that the function $f(y) := \sqrt{y}(1 - \frac{1}{2}\ln(y))$, $y > 0$, is such that $f(1) = 1$, $f(e^2) = 0$ and it is strictly decreasing on $(1, e^2)$. Since the absolute value of the second term of (3.31) is smaller than one, then there exists a unique $y_2^*(c) \in (1, e^2)$ solving (3.30). Moreover since

$$\frac{\partial F_3}{\partial y}(y, c) = -\tfrac{1}{4}y^{-\frac{1}{2}}\ln y < 0 \quad \text{for } (y, c) \in (1, e^2) \times [0, c_o) \tag{A-14}$$

we can use the implicit function theorem to conclude that

$$(y_2^*)'(c) = -\Big(\frac{\partial F_3}{\partial y}\Big/\frac{\partial F_3}{\partial c}\Big)(y, c) = -\frac{8e^{-1}R(\hat{c})R'(c)}{\big(R(\hat{c}) - R(c)\big)^2}\frac{y^{\frac{1}{2}}}{\ln y} < 0 \quad \text{on } [0, c_o) \tag{A-15}$$

and $(y_2^*)' \in C([0, c_o))$.

Step 2. Taking limits for $c \uparrow c_o$ in the equation $F_3(y_2^*(c); c) = 0$ one finds

$$0 = \lim_{c\uparrow c_o} F_3(y_2^*(c); c) = F_3(y_2^*(c_o-); c_o), \tag{A-16}$$

where $y_2^*(c_o-) := \lim_{c\uparrow c_o} y_2^*(c)$ clearly exists by monotonicity; that is

$$\sqrt{y_2^*(c_o-)}\big(1 - \tfrac{1}{2}\ln y_2^*(c_o-)\big) = 2e^{-1}. \tag{A-17}$$

We now take limits as $c \downarrow c_o$ in (A-7) and use (3.) of Proposition 3.7 to conclude that

$$\sqrt{\hat{y}_2(c_o+)}\big(1 - \tfrac{1}{2}\ln \hat{y}_2(c_o+)\big) = 2e^{-1}, \tag{A-18}$$

where $\hat{y}_2(c_o+) := \lim_{c\downarrow c_o} \hat{y}_2(c)$ exists by monotonicity of $\hat{y}_2$ (cf. Proposition 3.7). Hence from (A-17), (A-18) and uniqueness of the solution to $F_3(y; c_o) = 0$ we obtain (3.32). From now on we simply set $y_2^*(c_o-) = \hat{y}_2(c_o+) = \hat{y}_2(c_o)$.

Step 3. To prove (3.33) requires a bit more work. Taking limits as $c \uparrow c_o$ in (A-15) and using (3.32) we obtain

$$(y_2^*)'(c_o-) = -\frac{8e^{-1}R'(c_o)\sqrt{\hat{y}_2(c_o)}}{R(\hat{c})\ln \hat{y}_2(c_o)}. \tag{A-19}$$

We now turn to study the limit of $\hat{y}_2'(c)$ when $c \downarrow c_o$. Recall (A-10) and D_o and D_2 as in (A-11). We have $\hat{y}_1(c) \downarrow 0$ and $R(c) \downarrow 0$ for $c \downarrow c_o$, however from point (3) in the proof of Proposition 3.7 and by taking limits in the first equation of (A-5), it turns out that it must be

$$\lim_{c\downarrow c_o} R(c)\hat{y}_1^{-\frac{1}{2}}(c)\ln \hat{y}_1(c) = -\ell \quad \text{for some } \ell > 0. \tag{A-20}$$

Therefore as c approaches c_o from above we have the following asymptotic behaviours

$$D_o(\hat{y}_1(c), \hat{y}_2(c); c) \approx \frac{1}{16}R(\hat{c})R(c)\hat{y}_2^{-\frac{1}{2}}(c)\ln \hat{y}_2(c)\hat{y}_1^{-\frac{3}{2}}(c)\ln \hat{y}_1(c),$$

and

$$D_2(\hat{y}_1(c), \hat{y}_2(c); c) \approx -\tfrac{1}{4}R'(c)R(c)\hat{y}_2^{\frac{1}{2}}(c)\big(1 - \tfrac{1}{2}\ln \hat{y}_2(c)\big)\hat{y}_1^{-\frac{3}{2}}(c)\ln \hat{y}_1(c)$$

so that

$$\frac{D_2(\hat{y}_1(c), \hat{y}_2(c); c)}{D_o(\hat{y}_1(c), \hat{y}_2(c); c)} \approx -4\frac{R'(c)}{R(\hat{c})}\left[\frac{\hat{y}_2^{\frac{1}{2}}(c)\Big(1 - \frac{1}{2}\ln \hat{y}_2(c)\Big)}{\hat{y}_2^{-\frac{1}{2}}(c)\ln \hat{y}_2(c)}\right]. \tag{A-21}$$

Hence, taking limits as $c \downarrow c_o$ in the second equation of (A-10), using (A-21) and recalling (A-18) (A-19) we obtain (3.33).

Proof of Proposition 3.15

Recalling (3.14), (3.15), (3.16) and Proposition 3.12 we see that it suffices to show that $V_c(x, c) \geq G_c(x, c)$ for any $x \in (\hat{\gamma}(c), \hat{\beta}(c))$ and $c \in [0, \hat{c})$, i.e. inside the continuation set $\mathcal{C}$ (see (3.36)). The proof is performed in two steps.

Step 1. Fix $c \in [0, c_o]$ and recall that (cf. Section 3.3.2 and (3.17)) for any such c the continuation set is of the form $\mathcal{C}_c = (-\infty, \hat{\beta}(c))$. Define $u := V_c - G_c$, then it is not hard to see by (3.16), Proposition 3.12 and (3.38) that $u \in C(\mathbb{R} \times [0, c_o])$ and it is the unique classical solution of

$$(\mathbb{L}_X - \lambda)u(x, c) = -\lambda x(1 + \Phi'(c)), \quad \text{for } x < \hat{\beta}(c) \text{ with } u(\hat{\beta}(c), c) = 0. \tag{A-22}$$

Therefore, setting $\tau_\beta := \inf\{t \geq 0 : X_t^x \geq \hat{\beta}(c)\}$ and using the Feynmann-Kac representation formula (possibly up to a standard localisation argument), we get

$$u(x, c) = \mathsf{E}\left[e^{-\lambda\tau_\beta}u(X_{\tau_\beta}^x, c) + \lambda(1 + \Phi'(c))\int_0^{\tau_\beta} e^{-\lambda t}X_t^x dt\right] = (1 + \Phi'(c))\mathsf{E}\left[\int_0^{\tau_\beta} \lambda e^{-\lambda t}X_t^x dt\right], \tag{A-23}$$

where we have used that $u(X_{\tau_\beta}^x, c) = 0$ P-a.s. since $\tau_\beta < \infty$ P-a.s. by the recurrence property of Brownian motion. Recalling that $X_t^x = x + B_t$ (cf. (2.1)) we can write

$$x = \mathsf{E}\left[\int_0^\infty \lambda e^{-\lambda t}X_t^x dt\right] \tag{A-24}$$

and then by strong Markov property and standard formulae for the Laplace transform of τ_β it easily follows from (A-23)

$$u(x,c) = (1+\Phi'(c))\Big(x - \mathsf{E}\big[e^{-\lambda\tau_\beta}X^x_{\tau_\beta}\big]\Big) = (1+\Phi'(c))\left[x - \hat{\beta}(c)\frac{\psi_\lambda(x)}{\psi_\lambda(\hat{\beta}(c))}\right]. \tag{A-25}$$

Since $(1+\Phi'(c)) \leq 0$ for $c \in [0, c_o]$, we have $u(x,c) = V_c(x,c) - G_c(x,c) \geq 0$ if and only if

$$\theta(x,c) := x - \hat{\beta}(c)\frac{\psi_\lambda(x)}{\psi_\lambda(\hat{\beta}(c))} \leq 0 \quad \text{for } x < \hat{\beta}(c). \tag{A-26}$$

From Proposition 3.6 we obtain $1 < \hat{y}_2(c) < e^2$ and hence $0 < \hat{\beta}(c) < 1/\sqrt{2\lambda}$. Therefore, also recalling that $\psi_\lambda(x) = e^{\sqrt{2\lambda}x}$, one has for any $x < \hat{\beta}(c)$

$$\theta_x(x,c) = 1 - \hat{\beta}(c)\sqrt{2\lambda}e^{\sqrt{2\lambda}(x-\hat{\beta}(c))} \geq 1 - \hat{\beta}(c)\sqrt{2\lambda} \geq 0.$$

We can now conclude that (A-26) is fulfilled by noting that $\theta(\hat{\beta}(c),c) = 0$, hence $u \geq 0$ in $(-\infty, \hat{\beta}(c)) \times [0, c_o]$.

Step 2. Fix now $c \in (c_o, \hat{c})$, take $x \in (\hat{\gamma}(c), \hat{\beta}(c))$ and denote again $u := V_c - G_c$. As in *Step 1* it is not hard to see that

$$(\mathbb{L}_X - \lambda)u(x,c) = -\lambda x(1+\Phi'(c)) \quad \text{for } x \in (\hat{\gamma}(c), \hat{\beta}(c)) \text{ and } u(\hat{\gamma}(c),c) = u(\hat{\beta}(c),c) = 0. \tag{A-27}$$

Set $\tau_{\gamma,\beta} := \tau_\gamma \wedge \tau_\beta$ with τ_β as in *Step 1* above and $\tau_\gamma := \inf\{t \geq 0 : X^x_t \leq \hat{\gamma}(c)\}$. Then u is continuous and it admits the Feynmann-Kac representation

$$u(x,c) = (1+\Phi'(c))\mathsf{E}\left[\int_0^{\tau_{\gamma,\beta}} \lambda e^{-\lambda t}X^x_t\,dt\right] \tag{A-28}$$

where we have used that $u(X^x_{\tau_{\gamma,\beta}}, c) = 0$ P-a.s. due to (A-27) and to the fact that $\tau_{\gamma,\beta} < \infty$ P-a.s. by the recurrence property of Brownian motion. Since $(1+\Phi'(c)) < 0$ on $[c_o, \hat{c})$ then $u(x,c) \geq 0$ on $(\hat{\gamma}(c), \hat{\beta}(c))$ (i.e. $V_c \geq G_c$) if and only if $\mathsf{E}[\int_0^{\tau_{\gamma,\beta}} \lambda e^{-\lambda t}X^x_t dt] \leq 0$ for $x \in (\hat{\gamma}(c), \hat{\beta}(c))$. Thanks to (A-24), strong Markov property and Green's formula (cf. also [7], eq. (4.3))

$$\begin{aligned}
\mathsf{E}\left[\int_0^{\tau_{\gamma,\beta}} \lambda e^{-\lambda t}X^x_t\,dt\right] &= x - \mathsf{E}\big[e^{-\lambda\tau_{\gamma,\beta}}X^x_{\tau_{\gamma,\beta}}\big] \\
&= x - \Big\{\hat{\gamma}(c)\mathsf{E}\big[e^{-\lambda\tau_\gamma}\mathbb{1}_{\{\tau_\gamma<\tau_\beta\}}\big] + \hat{\beta}(c)\mathsf{E}\big[e^{-\lambda\tau_\beta}\mathbb{1}_{\{\tau_\beta<\tau_\gamma\}}\big]\Big\} \\
&= x - \left\{\hat{\gamma}(c)\frac{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c)-x)\big)}{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c)-\hat{\gamma}(c))\big)} + \hat{\beta}(c)\frac{\sinh\big(\sqrt{2\lambda}(x-\hat{\gamma}(c))\big)}{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c)-\hat{\gamma}(c))\big)}\right\} \\
&= \frac{1}{\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c)-\hat{\gamma}(c))\big)}\Theta(x,c;\hat{\beta}(c),\hat{\gamma}(c)),
\end{aligned} \tag{A-29}$$

where we define

$$\begin{aligned}
&\Theta(x,c;\hat{\gamma}(c),\hat{\beta}(c)) \\
&:= \Big[x\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c)-\hat{\gamma}(c))\big) - \hat{\gamma}(c)\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c)-x)\big) - \hat{\beta}(c)\sinh\big(\sqrt{2\lambda}(x-\hat{\gamma}(c))\big)\Big].
\end{aligned} \tag{A-30}$$

To simplify notation we set $\vartheta(x,c) := \Theta(x,c;\hat{\gamma}(c),\hat{\beta}(c))$. The right-hand side of (A-29) is negative for any $x \in (\hat{\gamma}(c), \hat{\beta}(c))$ if and only if $\vartheta(x,c) \leq 0$ therein. To study the sign of ϑ we first note that $\vartheta(\hat{\gamma}(c), c) = 0 = \vartheta(\hat{\beta}(c), c)$ and

$$\begin{cases} \vartheta_x(x,c) = \sinh\big(\sqrt{2\lambda}(\hat{\beta} - \hat{\gamma})(c)\big) \\ \qquad\qquad +\sqrt{2\lambda}\Big[\hat{\gamma}(c)\cosh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big) - \hat{\beta}(c)\cosh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big)\Big] \\ \vartheta_{xx}(x,c) = -2\lambda\hat{\gamma}(c)\sinh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big) - 2\lambda\hat{\beta}(c)\sinh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big) \\ \vartheta_{xxx}(x,c) = 2\lambda\sqrt{2\lambda}\Big[\hat{\gamma}(c)\cosh\big(\sqrt{2\lambda}(\hat{\beta}(c) - x)\big) - \hat{\beta}(c)\cosh\big(\sqrt{2\lambda}(x - \hat{\gamma}(c))\big)\Big]. \end{cases} \tag{A-31}$$

From (A-31) it is easy to see that *i)* $\vartheta_x(\hat{\gamma}(c), c) < 0$, since $\hat{\gamma}(c) \leq -1/\sqrt{2\lambda}$, *ii)* $\vartheta_{xx}(\hat{\gamma}(c), c) > 0$, $\vartheta_{xx}(\hat{\beta}(c), c) < 0$ and *iii)* $\vartheta_{xxx}(x, c) < 0$. Hence $x \mapsto \vartheta_{xx}(x, c)$ is strictly decreasing and there exists a unique point $x_* := x_*(c)$ such that $\vartheta_{xx}(x_*, c) = 0$. Clearly x_* is a maximum of $x \mapsto \vartheta_x(x, c)$ in $(\hat{\gamma}(c), \hat{\beta}(c))$. We claim now and we will prove it later that $\vartheta_x(\hat{\beta}(c), c) > 0$. Then $\vartheta_x(x, c) > 0$ for $x \in (x_*, \hat{\beta}(c))$. Moreover since $\vartheta_x(\hat{\gamma}(c), c) < 0$, then there exists a unique point $x'_* := x'_*(c) < x_*$ such that $\vartheta_x(x'_*, c) = 0$. Such x'_* is the unique stationary point of $\vartheta(\,\cdot\,, c)$ in $(\hat{\gamma}(c), \hat{\beta}(c))$ and it is a negative minimum due to the fact that $\vartheta_{xx}(x, c) > 0$ for any $x < x_*$. Therefore, recalling also $\vartheta(\hat{\gamma}(c), c) = 0 = \vartheta(\hat{\beta}(c), c)$, we conclude that $\vartheta(x, c) < 0$ for any $x \in (\hat{\gamma}(c), \hat{\beta}(c))$. From (A-28) and (A-29) we thus get $u(x, c) \geq 0$ for any $x \in (\hat{\gamma}(c), \hat{\beta}(c))$.

To complete the proof it remains to show that $\vartheta_x(\hat{\beta}(c), c) > 0$. For that it is convenient to rewrite the first of (A-31) in terms of $\hat{y}_1(c)$ and $\hat{y}_2(c)$ (cf. (3.27)) so to have

$$\begin{aligned} \vartheta_x(\hat{\beta}(c), c) &= \Theta_x(F_\lambda^{-1}(\hat{y}_2(c)), c; F_\lambda^{-1}(\hat{y}_1(c)), F_\lambda^{-1}(\hat{y}_2(c))) \\ &= \hat{y}_2^{\frac{1}{2}}(c)\hat{y}_1^{-\frac{1}{2}}(c)\big(1 - \tfrac{1}{2}\ln\hat{y}_2(c)\big) - \hat{y}_2^{-\frac{1}{2}}(c)\hat{y}_1^{\frac{1}{2}}(c)\big(1 + \tfrac{1}{2}\ln\hat{y}_2(c)\big). \end{aligned} \tag{A-32}$$

From system (3.25) (see also (A-5), (A-6) and (A-7)) we obtain

$$\begin{cases} \hat{y}_2^{\frac{1}{2}}(c)\big(1 - \frac{1}{2}\ln\hat{y}_2(c)\big) = \dfrac{-2e^{-1}R(\hat{c}) + R(c)\hat{y}_1^{\frac{1}{2}}(c)\big(1 - \frac{1}{2}\ln\hat{y}_1(c)\big)}{R(c) - R(\hat{c})} \\ \hat{y}_2^{-\frac{1}{2}}(c)\big(1 + \frac{1}{2}\ln\hat{y}_2(c)\big) = \dfrac{R(c)}{R(c) - R(\hat{c})}\hat{y}_1^{-\frac{1}{2}}(c)\big(1 + \frac{1}{2}\ln\hat{y}_1(c)\big), \end{cases} \tag{A-33}$$

which plugged into (A-32) give

$$\begin{aligned} 2\vartheta_x(\hat{\beta}(c), c) &= 2\Theta_x(F_\lambda^{-1}(\hat{y}_2(c)), c; F_\lambda^{-1}(\hat{y}_1(c)), F_\lambda^{-1}(\hat{y}_2(c))) \\ &= \frac{\hat{y}_1^{-\frac{1}{2}}(c)}{R(\hat{c}) - R(c)}\Big[2e^{-1}R(\hat{c}) + R(c)\sqrt{\hat{y}_1(c)}\ln\hat{y}_1(c)\Big]. \end{aligned} \tag{A-34}$$

Recalling now that $0 < \hat{y}_1(c) < e^{-2}$, $R(\hat{c}) > R(c) > 0$ and noting that the function $\sqrt{x}\ln(x)$ is nonnegative on $[0, e^{-2}]$, we conclude by (A-34) that $\vartheta_x(\hat{\beta}(c), c) > 0$ for all $c \in (c_o, \hat{c})$ as claimed.

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