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A Model of Quality Uncertainty with a Continuum of Quality Levels

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[Abstract:] This work takes a closer look on the predominant assumption in usual lemon market models of having finitely many or even only two different levels of quality. We model a situation which is close to the classical monopolistic setting but admits an interval of possible quality values. Additionally, to make the model interesting, the consumer receives a signal which is correlated to the quality level and is her private information. We introduce a new concept for the consumer reaction to the received information, encompassing rationality but also allowing for a certain degree of imperfection. We find that there is always a strictly positive price-quality relation in equilibrium but the classical adverse selection effects are not observed. In contrast, low quality levels do not make any sales. After applying a refinement to these equilibria, we show that when the additional signal is very precise, more low quality levels are excluded from the market. In the limit of perfect information, the market breaks down, a behavior completely opposed to the original perfect information case. These different and quite extreme results compared to the classical lemon market case should serve as a warning to have a closer look at the assumption of having finitely many quality levels.

Keywords: Quality uncertainty, Price signaling, Adverse selection, Two-sided incomplete information

JEL Classification Numbers: C72; D42; D82

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1 Introduction

Markets with quality uncertainty have been well discussed in the recent decades, starting from the famous paper by George Akerlof (1970). Since then, many articles have formalized the idea in different ways, most of which focused on a particular market feature to implement into the classical model. Some works like Bagwell and Riordan (1991) enriched the market by introducing multiple periods and thus letting the market price not only be determined by equilibrium posterior beliefs but also by past experience of the consumers. Others focused on advertising possibilities in terms of wasteful spending and thus costly signaling (Milgrom and Roberts (1986)) or on the possibility of the consumers to receive additional information before the purchase (Bester and Ritzberger (2001), Voorneveld and Weibull (2011), Martin (2012) and the previous chapter). Some efforts were made in transfering the monopolistic setting into a model with multiple sellers. See Adriani and Deidda (2011) for a case with finitely many sellers and buyers. Daughety and Reinganum (2007) consider a duopolistic setting in which the good differs in a "safety" aspect. Wilson (1980) introduced a setting with a continuum of sellers and buyers.

Most of the literature has an assumption in common which seems innocuous. While quality is modeled to be unknown to the consumer, it can only have finitely many different values in the real numbers. In most cases, there is only a "good" and a "bad" quality level. Two objections directly arise to this assumption. For one, when we think about the quality of a car, we think of many different aspects which are relevant and enter the computation. *Performance, safety, handling, comfort* are only some broad categories, each of which could be split into multiple characteristics of a car. Quality should thus intuitively be something multidimensional. However, it is widely known that under relatively mild assumptions, preferences over multidimensional objects can be expressed by a von Neumann utility function and thus the comparison can be made in the real numbers. One sure has to be careful of whether even these weak assumptions apply to all real-life situations but in this chapter we do not focus on relaxing this assumption.

The other objection, which is the more severe one, is the assumption of finitely many quality levels. Certainly, some characteristics, like the resolution of a TV screen, only have finitely many possible values but others, like its life period or the quality of its colors, would better be modeled on a continuous scale. Most of the literature ignores this aspect, the predominant reasons being the mathematical simplicity, expositional benefits and the idea that two quality levels are enough to capture the relevant market effects.

This chapter takes a closer view at the last point. Is it really the case that having a continuum of quality levels does not lead to qualitatively different phenomena compared to only two possible values? Is this true in every model or could some positive answers to this question hide other issues which occur only when the setting is enhanced?

We present a model with quality uncertainty and a continuum of quality levels that resembles the classical monopolistic model of quality uncertainty as similarly stated in Ellingsen (1997). We show two examples in which under "regular" assumptions, having many quality values either leads to undetermined behavior or does not add interesting phenomena to the comparable model with only two quality levels.

We then continue modifying the model by adding private information to the consumer. When receiving a free signal which is correlated to the true quality, there naturally arise mathematical problems when trying to update beliefs about the quality distribution in a mathematically correct way. The form of the objective function of a firm bears the problem that the type space can not directly be split into convex subsets, all in which types set the same price. Consequently, Bayesian updating can be impossible or at best highly complex for the consumer to realize.

To overcome this issue, we introduce an elegant generalization of building an expected quality level, demanding Bayesian updating only in the easiest cases and otherwise allowing for non-perfectness or (to some degree) irrationality of the consumer while at the same time preserving the possibility of full rationality.

Analyzing the structure of equilibria, we characterize their pricing function and find that there is always a positive prize-quality relationship in every equilibrium. Moreover, adverse selection phenomena do in general not occur. Since profits are non-decreasing in the quality, only low quality types can completely be excluded from trade. We further investigate the limit behavior when the consumer's information becomes perfect, i.e. the signal precision approaches perfect information. We show that in this case, the market breaks down uniformly over all existing equilibria. Furthermore, the proof shows that this effect is mainly caused by the interval structure of available quality levels.

The paper is structured as follows. We shortly present the model before we show two cases with a continuous quality set but with only one-sided asymmetric quality information. We show that these models do not provide interesting or previously not known behavior. We then proceed by analyzing the model with two-sided asymmetric information. After defining a generalization of expected quality with respect to Bayesian updating, we analyze the equilibria of the market. Interesting aspects of equilibria can be found already at this stage. Applying a refinement to these equilibria, we finally find that approaching the perfect information case drives low quality firms out of the market and leads to market breakdown in equilibrium.

2 The Model

We consider a minimalistic market with one firm and one consumer. The firm (or *seller*) produces and offers an indivisible good of random quality $q \in [0, 1]$, unobserved by the consumer. The consumer (or *buyer*) can buy this good for a certain price which is set by the firm as a take-it-or-leave it offer. For each quality, the buyer has a certain, publicly known utility $u(q)$. For simplicity, we normalize $u(q) = q$ and speak equivalently of the firm's *quality* or *type*.

This type q is drawn by nature by a distribution on $[0, 1]$ with a continuous, everywhere-positive density function f. This distribution is known by the consumer, while the realized quality is not. The price p is set by the firm after observing the quality q. The action set of a firm is the set of all price functions

$$\begin{aligned} \pi : [0,1] &\to [0,1] \\ q &\mapsto \pi(q). \end{aligned}$$

The consumer buys at most one unit of the good. In addition to the price, she observes a signal s before the purchase decision. This signal is costless and can be interpreted as the private observation of a test result or of the result of an own quality information acquisition process with a fixed cost.[1] Having the realized quality level q, the signal is uniformly distributed on the interval $[q-\kappa, q+\kappa]$ and hence depends on the true quality q. The error variable κ is fixed, strictly positive and known to the seller and the buyer. Denote $S := [-\kappa, 1+\kappa]$ the set of possible signal realizations.

The buyer is a risk-neutral utility maximizer. Observing the price and the signal and having built an expectation $E(p,s)$ of the realized type, her expected utility is

$$E(p,s) - p$$

from buying the good and 0 otherwise. Whenever these values are equal, she buys with some indifference probability $\alpha \in [0,1]$, chosen by her. The strategy of the consumer can thus be characterized by this value.

We need some notation for the analysis. We denote the complete Lebesgue measure on $\mathbb{R}$ by λ. In Particular, a set $A \subset \mathbb{R}$ is called a null set if and only if there exists a Borel set B with $\lambda(B) = 0$ and $A \subset B$. Having two sets A and B, we denote $A \triangle B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference of these two sets. If $A, B \neq \emptyset$, we use the notation

$$A < B \quad \Leftrightarrow \quad a < b \ \ \forall\, a \in A, b \in B.$$

An element a is a limit point of the set A if there exists a sequence (a_n) in A with $\lim_{n\to\infty} a_n = a$.

3 One-sided Asymmetric Information

Before we deal with the model, we consider the simpler case in which the consumer does not get the additional signal but only observes the price before making the buying decision. This would be the natural extension of the standard lemon market models. Two types with the same pricing strategy then have the same chance of selling since the buyer receives the identical information and hence behaves the same. From the optimality in an equilibrium[2], each type's pricing strategy must maximize the payoffs. Since there are

[1] For example, if you always do a test drive before buying a second hand car, the resulting information is available to you and the (fixed) cost of the test drive does not enter your utility maximization considerations.

[2] In this section, we speak of Bayesian equilibria without giving the formal definition. Updating behavior is rather easy in these cases (as long as the price function is well-behaved) and the optimality conditions of seller's and buyer's behavior is obvious. Since all the results in this section state *necessary* properties of equilibria and do not deal with existence, we do not have to worry about out-of-equilibrium beliefs.

no payoff differences between types, every strategy which is used by some type yields the same payoff. Note that for each price and without further information, the consumer reaction can only be “not buying”, “buying” or “buying with probability α” where α can not differ between prices. Since every price of every pricing strategy must yield the same payoff, this leaves only two possible prices for each equilibrium.

Proposition 3.1. *Let there be no extra signal for the consumer. Then in every equilibrium in which some type makes positive profit, there are at most two prices $p = \alpha p'$ where $\alpha \in (0, 1)$ is the consumer's indifference strategy.*

It is interesting to notice that the order of types setting these two prices is not clearly determined. From the consumer reaction it is clear that the set of types setting the high price p' must yield the expected quality p' because the buyer uses its indifference strategy α. In the same way, the expected quality from the set of types setting price p must be strictly above p. Each constellation which satisfies these assumptions constitutes an equilibrium. This, however, is not very restrictive and allows for many types of behavior, all of which only involve two prices but can have positive or negative price correlation. One example of such a setting is shown in Figure 1.

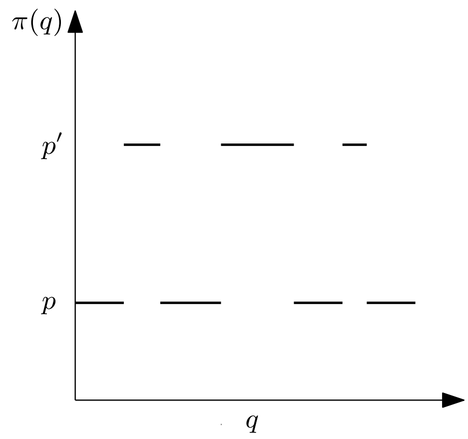

Figure 1: An example of a possible price function in the case without additional signal.

This behavior might actually stem from some of the other restrictions we make about the market. In particular, we assume one value α for all consumer reactions in which she is indifferent. Instead, one might think about allowing a different reaction for each price in which neither buying nor the absence from the purchase is the unique best reply. The result of only having two prices certainly stems from this restriction.

In the same spirit, quality-depending production costs (or outside options) could be present in the market which implies that the same price yields not only the same chance of selling but not the same profit for all types setting the price. This is what drives the high indeterminacy of the pricing function which was observed above. However, although getting rid of these restrictions does indeed help to overcome this behavior, it does not lead to new insights.

Proposition 3.2. *Let $c : [0,1] \to \mathbb{R}_+$ be a strictly increasing cost function and let the consumer strategy have the more general form $\gamma : [0,1] \to [0,1]$. Then in every equilibrium, if one exists, the price function is monotonically increasing and γ is strictly decreasing when being restricted to the equilibrium prices $\pi^{-1}([0,1])$.*

Knowing the results in the classical two-quality case, this statement is not very surprising and does not provide anything new to the matter. The monotonicity of the price function admits a positive price-quality relationship. This, in combination with the decreasing willingness of the consumer to buy with higher prices, also implies an adverse selection effect. Higher quality has a higher price and thus a lower chance of selling.

We could generalize this even more and allow the firm to have a mixed strategy. One can see in the proof that this modification would not change the result.

This detour shows that generalizing the standard model in a way *just* to incorporate a continuum of quality levels does not enrich the results in any way. Our model component of having the extra signal s is thus crucial for the following analysis and results. We now go back to the model presented in the previous section.

4 The Consumer

The notion of consumer's utility involves the building of an expectation based on the observed price and signal. The question, of course, is how this expectation is formed. If we followed classical Bayesian theory, a buyer would observe her information, in this case the price p and the signal s, and then hold a posterior belief $\mu(p,s) \in \Delta[0,1]$ about the actual product's quality. In an equilibrium, this probability distribution would be derived by Bayes' law whenever p and s correspond to at least one possible quality realization, given the signal distribution and the equilibrium price function π. While this works well in settings with finitely many quality levels, there are issues in our model that can not easily be overcome when sticking to this posterior belief assumption. In particular, the relatively unrestricted shape of the function π in the equilibrium definition below causes problems which are not easy to overcome.

Bayesian equilibria have of course been analyzed before, also in settings with continuous state spaces. A famous example is the signaling paper by Crawford and Sobel (1982). They analyze a sender-receiver setting in which the sender is biased and tries to induce a receiver's action which is not optimal for the receiver. In their setting, however, they show that no matter what the receiver's strategy, the optimal behavior of the sender is to divide the state space into (almost surely) convex sets and send messages depending on the set the state space is in. It is easy to show that Bayesian updating is always well-defined on these convex sets.[3] Similar arguments apply for extensions of this model

[3]Their definition of the posterior belief (the function p in point (2) on page 1434), is not well-defined

to the multi-dimensional case (Metzger, Jäger, Riedel (2011)) and for uncertainty about language competence (Blume, Board (2013)).

To approach this issue in our setting, imagine that the function π is fixed and known to the consumer and she observes a price p and a signal s. From the price p, and knowing the price function π, she infers that the true quality must be in the set

$$Q_p^\pi := \pi^{-1}(\{p\}) = \{q \in [0,1] | \pi(q) = p\} .$$

She also knows that the quality level is not more than κ away from the observed signal which yields

$$q \in Q_s := [s-\kappa, s+\kappa] \cap [0,1] = \{q \in [0,1] | s \in [q-\kappa, q+\kappa]\} .$$

If the quality level was outside of this set, the received signal would not be in the support of the signal distribution and could thus not be received. Altogether, she can infer that the true quality level must lie in the preimage

$$Q_{p,s}^\pi = Q_p^\pi \cap Q_s = \pi^{-1}(\{p\}) \cap [s-\kappa, s+\kappa].$$

If $Q_{p,s}^\pi$ is a Borel set with positive Lebesgue measure and with non-empty interior, a posterior distribution μ is given by the density function

$$g_\mu(q|p,s) = \begin{cases} \frac{f(q)}{\int_{Q_{p,s}^\pi} f(x)dx} & q \in Q_{p,s}^\pi \\ 0 & \text{else} \end{cases} \tag{1}$$

which is the normalized restriction of the original density function f to the set $Q_{p,s}^\pi$.[4] A similar expression is possible for the case in which this set is finite.[5]

In general, however, the set $Q_{p,s}^\pi$ can not be assumed to have this form and does not even have to be measurable. Even when assuming measurability, $Q_{p,s}^\pi$ could in theory be an infinite null set. Even if we excluded all these cases and agree on updating on finite sets, we would still be forced to distinguish situations in which we face a finite set or one of positive measure. We thus take a different, more general approach that allows us to keep the basic idea of a posterior distribution without having to further restrict the set of possible price functions π.

Note that if we had a posterior belief $\mu(p,s)$, the consumer would buy the product if the expected quality exceeds the price p, while there can be mixed behavior in the case of equality. In particular, the buying decision does not depend on the distribution μ itself but on the expected quality derived from this belief. Using this, we restrict ourselves to only consider expected quality instead of posterior beliefs.

if the integral $\int_0^1 q(n|t)f(t)dt$ is zero. The points (5),(6) and (7) in the proof of Lemma 1 show that they do not have to tackle this problem.

[4]Updating only f - and not the joint distribution of the type and the signal - is possible due to the uniform distribution of the signal.

[5]Voorneveld and Weibull (2011) use a version for the finite case in which the distribution over the set is just the normalized values of the density function. This can be justified as approximation from conditioning on environments around each point and letting these environments go to zero. In the strict sense, however, conditioning on null sets is problematic.

Definition 4.1. *Let a price function π be given. An expectation system with respect to π is a function $E : [0,1] \times S \to [0,1]$ such that*

(i) For every pair (p, s) for which $Q^{\pi}_{p,s}$ is not empty we have

$$E(p,s) \in \left[\inf Q^{\pi}_{p,s}, \sup Q^{\pi}_{p,s}\right].$$

(ii) The function is non-decreasing in s.

(iii) For each two pairs $(p, s), (p, s')$ with $Q^{\pi}_{p,s} = Q^{\pi}_{p,s'} \neq \emptyset$, we have $E(p, s) = E(p, s')$. If $Q^{\pi}_{p,s} = Q^{\pi}_{p,s'} = \emptyset$ and $s < s'$, $E(p, s) < E(p, s')$.

(iv) For two signals $s < s'$, if $Q^{\pi}_{p,s} \triangle Q^{\pi}_{p,s'}$ is not a null set, then $E(p, s) < E(p, s')$.

(v) Whenever $Q^{\pi}_{p,s}$ is a non-empty interval, $E(p, s)$ is the expectation of the distribution given in (1).

We say that E is an expectation system if there exists a price function $\tilde{\pi}$ so that E is an expectation system with respect to $\tilde{\pi}$.

Property (v) ensures that Bayesian updating is used at least in the simple case when we have an interval. The other items translate properties of this Bayesian updating to situations in which it can not be applied. Item (i) ensures that the consumer rationally does not assume a value outside the extremes of the set of possible quality levels. Property (ii) captures the fact that the induced quality distribution of a signal s, namely the uniform distribution on the interval $[s-\kappa, s+\kappa]$ first-order-stochastically dominated any other such distribution induced by any lower signal. Moreover, the signal is objective and not influenced by the firm. It is easy to check that when $Q^{\pi}_{p,s}$ is a Borel set with positive measure for two signals, Bayesian updating leads to this monotonic behavior in the signal. This effect is captured in an even stricter form by (iv). Whenever a signal increase removes or adds a set of qualities which is not a null set, the expectation must strictly increase, as it would in a Bayesian setting.

Property (iii) already contains an important refinement about the rationality of the consumer. On the one hand, having the same (non-empty) set of possible types for the same price should lead to the same expectation. Even if the signal s' is higher than s, the consumer rationally infers that there is no difference in the quality and thus the expectation is the same. This is different if $Q^{\pi}_{p,s}$ is empty. In this case it is clear that there was a deviation from the price function π. Although the definition is not very restrictive on these cases, we do need that a higher signal leads to a higher expectation when two of these deviations are observed for the same price. After all, the set of quality levels who could send the signal s is *strictly lower* (in an obvious sense) than the set for s'. While the information is proof for out-of-equilibrium behavior, the signal is the only objective, non-strategic information available to the consumer.

Overall, the concept of an expectation system not only allows to overcome measurability and Bayesian updating issues but also relaxes assumptions on the rationality of the consumer. She could be completely rational, using Bayesian updating whenever she can, or she can behave differently if the problem of updating is too complex. Heuristics or other forms of bounded rationality could be applied here.

Having introduced this new mathematical construct, one might wonder whether such an expectation system always exists or if one has to put assumptions on the price function.

Lemma 4.2. *For each price function π, there exists an expectation system.*

In particular, the concept of an expectation system does not impose a further restriction on the pricing function.

The proof is constructive, the first insight being that the definition of an expectation system does not contain restrictions across prices. We can thus define the value $E(p, s)$ for a fixed price. This is done by first using property (v) when it applies and then extend it to all signals for which $Q^{\pi}_{p,s}$ is not empty. The extension to the empty cases is then always possible.

Having this structure, there are some interesting consequences for the behavior of the consumer.

Lemma 4.3. *Let E be an expectation system and let p be a price. Then there exist unique values $\underline{s} \leq \overline{s}$ in S with*

$$E(p, s) \begin{cases} < p & s < \underline{s} \\ = p & \underline{s} < s < \overline{s} \\ > p & s > \overline{s}. \end{cases}$$

Moreover, we have $\overline{s} - \underline{s} \leq 2\kappa$.

In the situation of the lemma, define $\underline{q} = \overline{s} - \kappa$, $\overline{q} = \underline{s} + \kappa$, $\underline{\underline{q}} = \underline{s} - \kappa$, $\overline{\overline{q}} = \overline{s} + \kappa$, the quality levels which can just "reach" the signals $\underline{s}$ or $\overline{s}$. From $\overline{s} - \underline{s} \leq 2\kappa$ it also follows that we have $\overline{q} - \underline{q} \leq 2\kappa$. We say that the interval $[\underline{q}, \overline{q}]$ has *full length* if $\overline{q} - \underline{q} = 2\kappa$. This describes the special case $\underline{s} = \overline{s}$ so that the consumer is almost surely never indifferent between buying and not buying. Note that the order $\underline{\underline{q}} \leq \underline{q} \leq \overline{q} \leq \overline{\overline{q}}$ is always satisfied.

To illustrate these values, assume that Q^{π}_{p} is an interval $[a, b]$ of length smaller than 2κ and that the expected quality, restricted to that interval, matches the price p. This situation occurs regularly in equilibria as is shown in the equilibrium analysis below. If a signal is higher than the value $a + \kappa$, it can only have come from a certain fraction of the right side of the interval, which yield a higher expectation and thus must lead to sure buying. In the same way, a signal lower than $b - \kappa$ causes the buyer to not spend anything. Any signal between $b - \kappa$ and $a + \kappa$ would, on the other hand, give no further information to the consumer and she would thus stay indifferent. These boundary signals are the values of $\underline{s}$ and $\overline{s}$ from the lemma above.

It is worth mentioning that all these values are completely characterized by only knowing the pair $(\underline{s}, \overline{s})$ or the pair $(\underline{q}, \overline{q})$. Note also that in the example of the interval, $\underline{q}$ and $\overline{q}$ are the interval's end points a and b.

The values depend on the price p so we would have to write $\underline{s}(p), \overline{s}(p), \ldots$. For readability, we introduce a notation to leave out these arguments. A price denoted by p_q implies

Figure 2: The different values in the case of an interval

that the values $\underline{\underline{q}}, \underline{q}, \overline{q}$ and $\overline{\overline{q}}$ are determined with respect to this price. In the same way, prices p_r and p_t have the corresponding values $\underline{\underline{r}}, \ldots$ and $\underline{\underline{t}}, \ldots$, respectively. If only one price is considered at a certain point, the values $\underline{s}$ and $\overline{s}$ are taken with respect to that price.

Using the concept on an expectation system, we can analyze a basic property of what will later be an equilibrium. If we fix such an expectation system and assume that the firm knows it as well as the consumer indifference reaction α, every firm type should set a price that yields the highest profit of all prices.

Lemma 4.4. *Let E be an expectation system and $\alpha \in [0,1]$ be an indifference strategy. Define*

$$\phi(q, p; E, \alpha) := p \frac{1}{2\kappa} \int_{q-\kappa}^{q+\kappa} \alpha \mathbf{1}_{E(p,s)=p}(s) + \mathbf{1}_{E(p,s)>p}(s) ds$$

the profit of type q when setting price p. Moreover, let π be an optimal price system[6] *to the buyer's behavior. Then the function*

$$\phi_\pi(q; E, \alpha) := \phi(q, \pi(q); E, \alpha)$$

is continuous and non-decreasing.

Whenever E and α are given, we just write $\phi(q, p)$ instead of $\phi(q, p; E, \alpha)$. A short way of writing the profit function is by defining the probability γ of selling a product of

[6] A price system is optimal if for every type q the price $\pi(q)$ maximizes the type's profit, given the consumer reaction.

quality q for a certain price p

$$
\begin{aligned}
\gamma(q,p) :=& \frac{1}{2\kappa}\left(\int_{q-\kappa}^{q+\kappa} \alpha \mathbf{1}_{E(p,s)=p} + \mathbf{1}_{E(p,s)>p} ds\right) \\
=& \frac{1}{2\kappa}\left(\alpha\lambda\big([q-\kappa, q+\kappa] \cap [\underline{s}, \overline{s}]\big) + \lambda\big([q-\kappa, q+\kappa] \cap (\overline{s}, \infty)\big)\right) \\
=& \begin{cases} 0 & q+\kappa \leq \underline{s} \\ \frac{1}{2\kappa}\alpha(q+\kappa-\underline{s}) & q+\kappa \in (\underline{s}, \overline{s}) \\ \frac{1}{2\kappa}(\alpha(\overline{s}-\underline{s}) + (q+\kappa-\overline{s})) & q-\kappa \leq \underline{s}, \overline{s} \leq q+\kappa \\ \frac{1}{2\kappa}(\alpha(\overline{s}-(q-\kappa)) + (q+\kappa-\overline{s})) & q-\kappa \in (\underline{s}, \overline{s}) \\ 1 & q-\kappa \geq \overline{s} \end{cases} \qquad (2) \\
=& \begin{cases} 0 & q \leq \underline{\underline{q}} \\ \frac{1}{2\kappa}\alpha(2\kappa-(\overline{q}-q)) & q \in (\underline{\underline{q}}, \underline{q}) \\ \frac{1}{2\kappa}(\alpha(2\kappa-(\overline{q}-\underline{q})) + (q-\underline{q})) & q \in [\underline{q}, \overline{q}] \\ \frac{1}{2\kappa}(\alpha(2\kappa-(q-\underline{q})) + (q-\underline{q})) & q \in (\overline{q}, \overline{\overline{q}}) \\ 1 & q \geq \overline{\overline{q}} \end{cases}
\end{aligned}
$$

and writing $\phi(q,p) = p \cdot \gamma(q,p)$.

Given an expectation system E, an indifference strategie α and some price p, the form and slope of the profit function $\phi(q,p)$ is of high importance for the understanding of the proofs in the analysis. Note that we can have $E(p,s) < p$ for every signal, e.g. if no type is associated to the price p, so $Q_p^{\pi} = \emptyset$.[7]. If this happens, the profit of the firm is always zero whenever it sets the price p, regardless of its quality. In the other cases, however, the function looks as shown in Figure 3.[8]

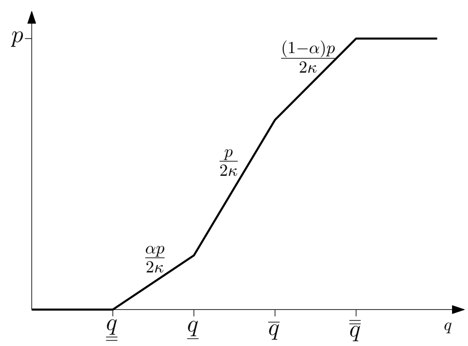

Figure 3: The typical form of $\phi(\cdot, p)$ and its slope for a non-trivial price.

This form of the profit function is why the classical concept of a Bayesian equilibrium is problematic in our setting and why the standard approach does not work. For two

[7]An example of such a construction is given in the proof of Lemma 4.2.

[8]Technically, this is not a special case but is equivalent to $\underline{s} = \overline{s} = 1 + \kappa$.

different prices p, p', it is possible to have types $q' < q < q''$ with q' and q'' preferring the price p' while the optimal price for q is p. This, given a fixed consumer reaction, allows for non-convexity of type regions Q_p^π setting the same price, even when the firm's behavior is optimal.[9]

Definition 4.5. *A tuple (π, E, α), consisting of a price function π, an expectation system E and an indifference strategy α is called an* equilibrium *if the price function π assumes finitely many values, E is an expectation system with respect to π and for every type $q \in [0, 1]$ the price $\pi(q)$ maximizes the firm's profit, given E and α.*

This definition is the natural adaptation of a Bayesian equilibrium, using the notion of expectation systems. The usual assumption of correct updating is replaced by the property of E being an expectation system for π. The optimality of the consumer's behavior is implicitly assumed, leaving her only α as choice variable. We assume that this price function can only take finitely many values, as is the case in most markets.[10]

Definition 4.6. *Let an equilibrium (π, E, α) be given. We call a price p an* equilibrium price *if there exists a type $q \in Q_p^\pi$ which makes positive profit in the equilibrium.*

For an equilibrium price p, denote $Q_p^* := Q_p \cap \{q \in [0, 1] | \phi_\pi(q) > 0\}$ the set of types setting this price and making positive profits in equilibrium. In this notation, we drop the superscript π for expositional reasons. We call a type q *profitable* if $q \in Q_{\pi(q)}^*$. Types that are in Q_p^π but have zero gains from the market are not bounded by incentive constraints and thus their behavior is quite arbitrary. Many statements about equilibrium behavior have to be restricted to profitable types.

5 Equilibrium Analysis

The obvious next step is to determine under which conditions a market equilibrium exists and what its main features are. The following result shows the structure of equilibrium price behavior.

Theorem 5.1. *An equilibrium exists. Let (π, E, α) be an equilibrium and let $q_{\min}$ be the infimum of all profitable types. Then π restricted to $(q_{\min}, 1]$ is almost surely a non-decreasing step function.*

In terms of price-quality relation, this is a strong statement, at least for the profitable types. One can argue that firms with a product of quality lower than $q_{\min}$ would not survive in the market and eventually drop out. Prices then monotonically increase with

[9] To see this in Figure 3, take some $p_r > \alpha p$ with $[\underline{r}, \overline{r}]$ having full length (so that $\overline{r} - \underline{r} = 2\kappa$ and the graph has only one increasing line, going from 0 to p_r) and $\underline{r} = \underline{\underline{r}} \in (\underline{\underline{q}}, \underline{q})$. One can calibrate this so that the new graph is above the existing one in $\underline{q}$ while it is below this graph in a point on the left and a point on the right side of $\underline{q}$.

[10] The most obvious example would be product prices in a supermarket. But it also applies to goods which can have even finer pricing like petrol at a gas station. Since the good we have is indivisible, it is also natural to assume a finite number of values for the pricing strategy.

quality which implies that a higher price corresponds to higher quality. Although the relation is not one-to-one (so some ambiguity is left to the consumer for every price), prices roughly signal the right quality.

This result, does not come natural. The formal proof involves a series of technical lemmas and is given in an extra section. Note that this statement also holds if κ is large, so that the additional signal does not convey much information. It is thus implied that even in the case of a rather uninformative signal, the indeterminate behavior which was shown in section 3 for the absence of a signal is prevented.

Having arrived at this result, our definition of an equilibrium and the construct of an expectation system may seem like overkill, considering that now the sets on which to update are well shaped. Nevertheless, we need the expectation system concept to reach this point of having convex sets of types setting the same price. This step was not easily given to us as it would be in other models, e.g. the classical signaling game of Crawford and Sobel (1982).

To give an intuition on the proof, we continue to state the informal version of the needed steps. The most important observation, fixing an equilibrium price p_q and having in mind the points $\underline{\underline{q}}, \underline{q}, \overline{q}$ and $\overline{\overline{q}}$, is to see that one of the types $\underline{q}$ and $\overline{\overline{q}}$ must have the price p_q as its optimal choice. They are the types which can just reach the signal $\overline{s}$ as upper or lower bound of the corresponding signal range. By the definition of $\overline{s}$, the expectation of the consumer must differ when receiving signals slightly above or below this value. In an equilibrium, this means that the information, i.e. the set of quality levels assigned to a signal, must differ between these signals. But the only difference in types can occur in environments of $\underline{q}$ and $\overline{\overline{q}}$. Applying a limit argument, we see that at least one of the points $\underline{q}$ and $\overline{\overline{q}}$ is a limit point of the set $Q^{\pi}_{p_q}$. Using continuity, setting price p_q must yield the optimal profit for this limit type. In the same way, this holds for the points $\overline{q}$ and $\underline{\underline{q}}$.

This observation is then extended to further statements. We show that $\overline{q}$ and $\overline{\overline{q}}$, if they are different, can not both be limit points at the same time. Moreover, in this case, there must be a type in an environment of $[\underline{q}, \overline{q}]$ actually setting the price p_q. Finally, we show that essentially no type in the sets $(\underline{\underline{q}}, \underline{q})$ and $(\overline{q}, \overline{\overline{q}})$ sets the price p_q. While the first points require rather technical arguments, the last property stems from item (iv) of the definition of an expectation system. If more than a null set of types in the two sets set the price p_q, it would contradict the definition of the signals $\underline{s}$ and $\overline{s}$.

Having these observations, we compare each two equilibrium prices $p_q > p_r$ for all different possible orders of the points $\underline{q}, \overline{q}, \underline{r}$ and $\overline{r}$. In each case we find that the situation is either impossible or the order $Q^*_{p_r} < Q^*_{p_q}$ holds almost surely which shows the monotonicity and thus the step function form of the equilibrium pricing behavior.

Existence of an equilibrium is shown quite easily by just noting that every single-price setting can be an equilibrium.

This equilibrium existence proof reveals a flaw of our so-far used equilibrium concept. Setting $E(p, s)$ low for all non-equilibrium prices, deviation is never profitable for the firm and thus every constant price function can be an equilibrium, independent of whether the market price is high or low. This phenomenon is not new and essentially the same as in regular Bayesian equilibria. To resolve these issues, we look closer at an equilibrium with a particularly low price. Consider the price function $\pi(q) = .1$ for all $q \in [0,1]$ in a setting with $\kappa = \frac{1}{10}$. The type $q = .8$ then sells for this very low price but with probability 1. The consumer, when facing such a type, observes the price p and a signal $s \in [.7, .9]$, indicating a far higher quality than the price would suggest. While it is not counter-intuitive that the consumer does not hesitate to buy the product for the price .1, it is harder to believe that for any slightly higher price p' she would assign a much lower expectation to any (also high) signal and never buy. Our next refinement captures this idea.

Definition 5.1 (Locally continuous equilibrium)**.** *An equilibrium (π, E, α) is called* locally continuous *if for every signal s the function $E(\cdot, s)$ is continuous in every equilibrium price.*

This refinement is in the same spirit as in the first chapter. It ensures that marginal price deviations do not cause a jump in equilibrium beliefs (and thus expected values). In the example above, the lowest possible signal coming from a type of quality .8 is $.8 - \kappa = .7$. Receiving this low signal, the consumer knows that the quality must be at least .6. Hence the value $E(.1, s)$ is at least .6 for every signal that could come from type .8. The local continuity of $E(\cdot, s)$ at the price $p = .1$ shows that for some marginally higher price the expectation must still be above p for every signal possibly induced by the quality level. The firm would thus still sell with probability 1 and this makes a deviation profitable. The constant-price equilibrium would then not be possible, at least for such low prices.

Lemma 5.2. *A locally continuous equilibrium exists. Let (π, E, α) be a* locally continuous *equilibrium. Then for every equilibrium price p_q - except for the lowest one - $Q^*_{p_q}$ is an interval with endpoints $\underline{q}$ and $\overline{q}$. For each of these intervals, the expected quality matches the price, i.e.*

$$p_q = Exp(q | q \in [\underline{q}, \overline{q}]) = \tfrac{1}{F(\overline{q}) - F(\underline{q})} \int_{\underline{q}}^{\overline{q}} q f(q) dq$$

This result shows how step function behavior is further enforced by the refinement. Although single-priced equilibria are still possible, the corresponding price can not be too far away from the highest possible quality level.[11] Moreover, the unrefined equilibrium definition in general allows for types that sell for sure in a way that every of their possible signals induces a consumer expectation strictly above the price. With local continuity, this "high reputation" can be used by the firm to demand a higher price, as described above. Note that even with this refinement, it is possible for a firm to sell with probability one but only in equilibria with $\alpha = 1$.

[11] This can be seen in the proof of Lemma 5.2.

To illustrate the market outcome, we can now look at such an equilibrium. We choose $\kappa = .25$ and a uniform quality distribution. From this, it follows that for each step of the price function (except the lowest one), the price is the middle point of the quality interval. Choosing the first discontinuity to be at .99, we get the following equilibrium price function. The value of $q_{\min}$ is positive in this example, as one can see in the profit

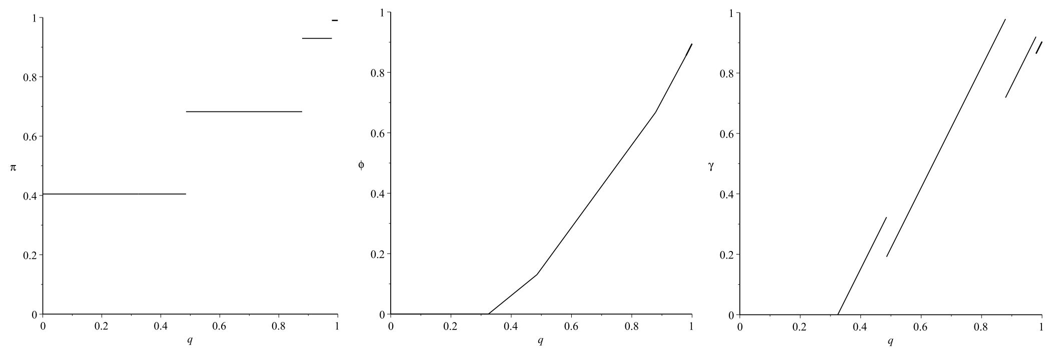

Figure 4: The equilibrium price, profit and selling probabilities in our example

.

function. Note that the price setting of types below $q_{\min}$ could be chosen differently to some extent. For expositional reasons, it is chosen to match the lowest price. The selling probabilities are increasing within the areas of same prices but are overall not continuous and not monotonic. One can hardly speak of an adverse selection effect in this equilibrium.

Adverse selection is thus not a big issue, anymore. Unlike in the classical model of Ellingsen (1997), high quality is in general not traded with a lower probability than low quality. Selling probabilities can go down but this is always compensated by a higher price so that profits still increase with quality. This result is partly driven by the missing production costs in this model. With such costs, this part of the result may be different. Note, however, that the existence of the lower bound $q_{\min}$ is not mainly caused by this assumption.

Regarding this cutoff value of profitable types, we did not yet say anything about its exact value and its dependence on the parameters. In particular, the signal precision variable κ does not appear in the so far established results. The example does not show the upper and lower bound of possible values of $q_{\min}$ over all equilibria. Clearly, choosing a different location for the last discontinuity (instead of .99) would change the point from which profits start to be positive.

Before we present the next result, we briefly think about the case of perfect information. With $\kappa = 0$, quality information would be public and hence the only equilibrium in such a market is that every type q sells its product for the "fair" price $p = q$ with probability one. The product would always be sold regardless of its quality. Of course, our assumption of only having a finite number of equilibrium price rules out this behavior. Nevertheless, looking at the previous result, one may expect the lower bound $q_{\min}$ to approach zero in a comparative static analysis when κ becomes small. Otherwise convergence to the full information case would not be possible in any sense.

The next result shows, however, that even the opposite phenomenon occurs. The result is stated for the special case in which the type's distribution is uniform.

Theorem 5.2 (Market breakdown on perfect information). *Let the firm type q be uniformly distributed over $[0,1]$. With signal precision approaching perfect information ($\kappa \to 0$), the maximal*[12] *expected amount of sold goods over all locally continuous equilibria converges to zero.*

The following proof of this theorem shows very nicely that the market breakdown is caused by the interplay of quality types who are close to each other. The incentive compatibility constraints for types on adjacent steps of the price function dictated that the length of these steps can not get arbitrarily large. This effect gets more extreme in a way that even the sum of these length is bounded with the bound going to zero as κ becomes small.

Proof. For fixed $\kappa > 0$, let (π, E, α) be a locally continuous equilibrium. Proposition 5.2 implies that for all equilibrium prices p_q the set $Q^*_{p_q}$ is an interval with endpoints $\underline{q}$ and $\overline{q}$ or p_q is the lowest equilibrium price. Using this, we have $\underline{q} - \overline{q} = 2\kappa$ or $E([\underline{q}, \overline{q}]) = p_q$. The former case of having full length is only possible for the lowest price. Otherwise, the profit of type $\underline{q}$ would be zero which is impossible for types strictly above $q_{\min}$.

Theorem 5.1 shows that π is almost surely a step function. Because of the profit's continuity, each type that lies on a discontinuity of the price function must be indifferent between setting either of the two adjacent prices.

In the case where π is a constant function above $q_{\min}$, note that we have[13] $q_{\min} \geq 1 - 2\kappa$ which converges to one with $\kappa \to 0$. In the same way, convergence of all price functions with two steps can be shown. In fact, for every fixed number of steps, the corresponding equilibria must yield uniform convergence of $q_{\min}$ to 1. But there is still an infinite number of possible steps and thus the convergence result does not follow from these thoughts. However, it shows that for the following proof we can assume the price function to have at least three different prices. This also implies $\alpha > 0$, otherwise the lowest type of each step would get zero profit which is a contradiction.

Let $q < r < t$ be three types that lay on adjacent discontinuities and denote $p_1 < p_2$ the corresponding prices as depicted in Figure 5. Assume that p_1 is not the lowest equilibrium price. For κ low enough we can choose these values so that r is above $\frac{1}{2} + \kappa$. The prices must be equal to the expected qualities over the intervals $[q, r]$ and $[r, t]$, respectively. From the uniform type distribution it follows that $p_1 = \frac{q+r}{2}$ and $p_2 = \frac{r+t}{2}$. Because of the continuity of the profit function, the type r is indifferent between setting price p_1 or

[12] Technically, the existence of a maximum is not guaranteed and we should speak of a supremum, here.

[13] This is shown in the existence proof for locally continuous equilibria. Intuitively, having steps of a size larger than 2κ, some types always send signals above $\overline{s}$. This is not compatible with locally continuous equilibria. The proof for any finite number of steps follows with the same argument.

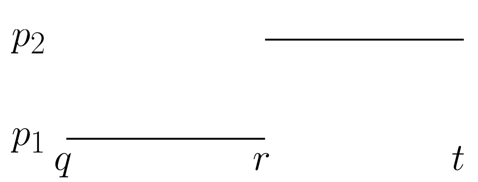

Figure 5: The situation of q,r and t in the proof

p_2. Hence the following equation holds.

$$
\begin{aligned}
& & \phi(r, p_1) &= \phi(r, p_2) \\
&\overset{(2)}{\Leftrightarrow} & p_1 \tfrac{1}{2\kappa}(r - q + \alpha(2\kappa - (r - q))) &= p_2 \tfrac{1}{2\kappa}(\alpha(2\kappa - (t - r))) \\
&\Leftrightarrow & \tfrac{q+r}{2}(r - q + \alpha(2\kappa - (r - q))) &= \tfrac{r+t}{2}(\alpha(2\kappa - (t - r))) \\
&\Leftrightarrow & r^2 - q^2 + \alpha(2\kappa(r + q) - (r^2 - q^2)) &= \alpha(2\kappa(t + r) - (t^2 - r^2))
\end{aligned}
$$

Reordering this, one gets

$$
\begin{aligned}
\alpha t^2 - 2\kappa\alpha t + (1 - 2\alpha)r^2 - (1 - \alpha)q^2 + 2\alpha\kappa q = 0 \\
t^2 - 2\kappa t + \tfrac{1-2\alpha}{\alpha} r^2 - \tfrac{1-\alpha}{\alpha} q^2 + 2\kappa q = 0
\end{aligned}
$$

and solving this for t yields

$$
\begin{aligned}
t =& \kappa \pm \sqrt{\kappa^2 - \tfrac{1-2\alpha}{\alpha} r^2 + \tfrac{1-\alpha}{\alpha} q^2 - 2\kappa q} \\
=& \kappa \pm \sqrt{(\kappa - r)^2 - \tfrac{1-\alpha}{\alpha} \underbrace{(r^2 - q^2)}_{>0} + 2\kappa(r - q)} \\
\overset{\alpha \in (0,1]}{\leq}& \kappa + \sqrt{(\kappa - r)^2 + 2\kappa(r - q)}.
\end{aligned}
$$

In other words, for each pair q, r we get an upper bound for the next discontinuity t which is independent of the parameter α.
For expositional purposes, we introduce the notation $t' := t - \kappa$ which we use similarly for the other variables. The inequality then becomes

$$
\begin{aligned}
t' \leq& \sqrt{r'^2 + 2\kappa(r' - q')} \\
=& \sqrt{r'^2} + \int_{r'^2}^{r'^2 + 2\kappa(r' - q')} \frac{1}{2\sqrt{z}} dz \\
\leq& r' + \int_{r'^2}^{r'^2 + 2\kappa(r' - q')} \frac{1}{2\sqrt{r'^2}} dz \\
=& r' + \frac{1}{2r'} 2\kappa(r' - q') \\
\overset{r' \geq \frac{1}{2}}{\leq}& r' + 2\kappa(r' - q')
\end{aligned}
$$

which shows that for the adjacent values q, r and t we have

$$t - r = t' - r' \leq 2\kappa(r' - q') = 2\kappa(r - q).$$

Take q_0 the smallest (satisfying $q_0 > \frac{1}{2} + \kappa$) such type that lays on a discontinuity of the price function and let $q_1, q_2, \dots$ be the following discontinuities. It follows that for all $n \in \mathbb{N}$ we get

$$\begin{aligned} q_n =& q_0 + (q_n - q_0) = q_0 + \sum_{i=1}^{n} (q_i - q_{i-1}) \leq q_0 + \sum_{i=1}^{n} (2\kappa)^{i-1} \underbrace{(q_1 - q_0)}_{\leq 2\kappa} \\ \leq& q_0 + \sum_{i=1}^{n} (2\kappa)^i \leq q_0 + \frac{2\kappa}{1 - 2\kappa}. \end{aligned}$$

Remember that q_n must be equal to 1 for some n. Letting κ go to zero forces q_0 to go to 1 uniformly for all equilibria.

Since all types below $q_{\min} \geq q_0 - 2\kappa$ are not able to sell their product, overall sales necessarily converge to zero uniformly over all equilibria when κ goes to zero and q_0 approaches one. □

6 The Proof of Theorem 5.1

This section presents lemmas and their proofs necessary for establishing the result in Theorem 5.1. They show how to use the properties of an expectation system and the optimality of the firm's behavior to determine the structure of an equilibrium price function.

As is shown below, the definition of an expectation system carries some properties similar to Bayesian updating, thus allowing for a similar analysis without assuming - but not excluding - perfect rationality on the consumer side.

The proofs of this section are presented directly after their corresponding statements. We use the shape of the profit function for a given equilibrium price, as depicted in Figure 3, very often. It is important to be familiar with the different areas of its slope to perfectly understand the proofs.

One of the main points we need to know about expectation systems in equilibria is formulated in the following lemma which generalizes a property from Bayesian updating.

Lemma 6.1. *In any equilibrium (π, E, α) and for each equilibrium price p_q, at least one of the points $\underline{q}$ and $\overline{\overline{q}}$ and at least one of the points $\overline{q}$ and $\underline{\underline{q}}$ are limit points of $Q^{\pi}_{p_q}$.*

The connection to the Bayesian case becomes clear if we remember the interval example. The points $\underline{q}$ and $\overline{q}$ are then the endpoints of the interval. The lemma shows this property in a weaker sense, only using the equilibrium system definition. Note that even in the case of regular Bayesian updating, it can happen that not $\underline{q}$ but $\overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$, e.g. if we have two intervals $[a,b] < [c,d]$ with $c - a < 2\kappa$ and $Exp(q|q \in [a,b]) = p_q$. Then the point[14] $c = \overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$ but $\underline{q} = c - 2\kappa < a$ is not.

Proof. We only show that $\underline{q}$ or $\overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$. If any of these two points are in $Q^{\pi}_{p_q}$, we are done. Assume now that this is not the case. We construct a sequence of types in $Q^{\pi}_{p_q}$, converging to either $\underline{q}$ or $\overline{\overline{q}}$.

Start with any $\varepsilon_0 > 0$ and observe that by definition of $\overline{s}$ the values $E(p_q, \overline{s} - \varepsilon_0)$ and $E(p_q, \overline{s} + \varepsilon_0)$ are not equal.[15]

Consider the corresponding sets $Q^{\pi}_{p_q,\overline{s}-\varepsilon_0}$ and $Q^{\pi}_{p_q,\overline{s}+\varepsilon_0}$. If one of them is not empty, they can not be equal due to Definition 4.1 (iii). This leaves two cases to consider.

<u>First case:</u> $Q^{\pi}_{p,\overline{s}-\varepsilon_0} = Q^{\pi}_{p,\overline{s}+\varepsilon_0} = \emptyset$

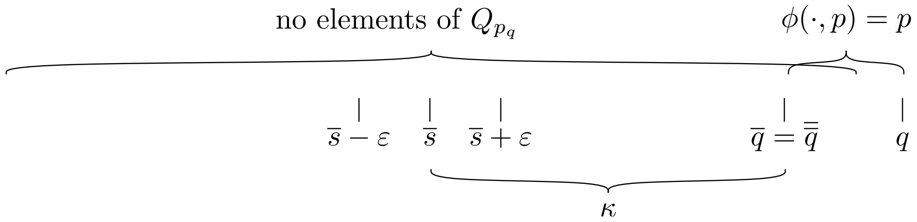

Figure 6: The situation of the first case

This situation is depicted in Figure 6. Because both sets are empty, we have $Q^{\pi}_{p_q,\overline{s}} \subset Q^{\pi}_{p_q,\overline{s}-\varepsilon_0} \cup Q^{\pi}_{p_q,\overline{s}+\varepsilon_0} = \emptyset$ and this is true for all smaller choices of $\varepsilon_0 > 0$. From 4.1 (iii), we know that $E(p_q, s)$ is strictly increasing in the signal within some interval around $\overline{s}$. Hence it follows that we have $\underline{s} = \overline{s}$ and thus $\overline{q} = \overline{\overline{q}}$. Since p_q is an equilibrium price, there must be some profitable type q with $\pi(q) = p_q$. The only way to make positive profit is if this type is above $\underline{s} - \kappa$ and thus above $\overline{q} + \varepsilon_0$. Hence the type q sells with probability one and we have $\phi_\pi(q) = p_q$. By the monotonicity of ϕ_π and since every type in the interval $(\overline{q}, q)$ can attain this profit, we know that $\phi_\pi(q') = p_q$ for all $q' \in (\overline{q}, q)$. Any two types q', q'' in this interval, not setting the price p_q, must have a selling probability in $(0,1)$ and the same profit $\phi_\pi(q') = \phi_\pi(q'') = p_q$. But $\phi(q', \pi(q')) = \phi(q'', \pi(q'')) = p_q$ is not possible if $\pi(q') = \pi(q'') \neq p_q > 0$ (see Figure 3, the same profit for the same price implies that this profit is either zero or matches the price). It follows that each type in the interval $(\overline{q}, \overline{q} + \varepsilon_0)$ sets a different price. Since there are only finitely many equilibrium prices, this is a contradiction. Hence only the following, second case can occur.

<u>Second case:</u> $Q^{\pi}_{p_q,\overline{s}-\varepsilon_0} \neq Q^{\pi}_{p_q,\overline{s}+\varepsilon_0}$

[14] To see that we have $c = \overline{\overline{q}}$, note that for a signal s slightly below $c - \kappa$, we have $Q^{\pi}_{p_q,s} = [a,b]$ so that the consumer is indifferent. For signals above $c - \kappa$, we must have $E(p_q, s) > p_q$. This is dictated by property (iv) of an expectation system. Hence $c - \kappa = \overline{s}$ and thus $c = \overline{\overline{q}}$.

[15] Since p_q is an equilibrium price, $\overline{s}$ can not be on the limit of $S = [-\kappa, 1 + \kappa]$. With ε_0 small enough, the expressions are well-defined.

Choose q_0 in the (non-empty) symmetric difference of these two sets and note that we have

$$q_0 \in [\overline{s} - \kappa - \varepsilon_0, \overline{s} - \kappa + \varepsilon_0] \cup [\overline{s} + \kappa - \varepsilon_0, \overline{s} + \kappa + \varepsilon_0] = [\underline{q} - \varepsilon_0, \underline{q} + \varepsilon_0] \cup [\overline{\overline{q}} - \varepsilon_0, \overline{\overline{q}} + \varepsilon_0]$$

By construction we have $q_0 \in Q^{\pi}_{p_q}$. Choose $\varepsilon_1 = \frac{1}{2} \min(|q_0 - \underline{q}|, |q_0 - \overline{\overline{q}}|) \in (0, \frac{\varepsilon_0}{2})$. Repeating these arguments[16], using the values $\varepsilon_1, \varepsilon_2, \ldots$, we obtain a sequence (q_n) in $Q^{\pi}_{p_q}$ whose elements satisfy

$$|q_n - \underline{q}| < \varepsilon_n \text{ or } |q_n - \overline{\overline{q}}| < \varepsilon_n$$

for all $n \in \mathbb{N}$. At least one of these two conditions is true for an infinite number of indices and hence there exists a subsequence of (q_n) such that either the left or right inequality is true for all of its elements. Since (ε_n) converges to zero, this subsequence converges to either $\underline{q}$ or $\overline{\overline{q}}$. This limit is thus a limit point of $Q^{\pi}_{p_q}$.

The proof for $\overline{q}$ or $\underline{\underline{q}}$ being a limit point uses the same arguments, starting with $\underline{s}$ instead of $\overline{s}$. We omit this part of the proof. □

Acknowledging this lemma, we say that a type is a p_q-limit point if it is a limit point of $Q^{\pi}_{p_q}$.

While this intermediate result may seem innocuous, it is very important for the analysis of the structure of equilibrium price systems. Knowing that these points are limit points, the continuity of the profit function ϕ_π implies that the corresponding profit of these types must attain its maximum in the price p_q. No other price can yield strictly higher profits to a firm with these quality levels. Hence we have[17]

$$\phi_\pi(\overline{q}) = \phi(\overline{q}, p_q) \text{ or } \phi_\pi(\underline{\underline{q}}) = \phi(\underline{\underline{q}}, p_q)$$

and

$$\phi_\pi(\underline{q}) = \phi(\underline{q}, p_q) \text{ or } \phi_\pi(\overline{\overline{q}}) = \phi(\overline{\overline{q}}, p_q),$$

depending on which of these types has the limit point property described above.

The next result is the first direct step to determining the equilibrium price function. It excludes two possible combinations of ordering p_q- and p_r-limit points when the order of these two prices is known. Its proof is a direct application of the previous lemma.

Lemma 6.2. *In an equilibrium, let $p_r < p_q$ be two equilibrium prices and assume $\underline{r} \geq \underline{q}$. Then we have $\overline{r} < \overline{q}$.*

[16] Since the first case leads to a contradiction, we always end up with the second case.

[17] Note, however, that for example the inequality $\pi(\overline{q}) = p_q$ does not follow from $\phi_\pi(\overline{q}) = \phi(\overline{q}, p_q)$. The type p_q may set a different price. However, there are arbitrarily close types which set the price p_q.

Figure 7: The two situations excluded by lemma 6.2

Proof. Assume $\underline{r} \geq \underline{q}$ and $\overline{r} \geq \overline{q}$ as shown in Figure 7. This implies

$$\begin{aligned}
\gamma(\underline{r}, p_q) &\overset{(2)}{=} \begin{cases} \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{q} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in [\underline{q}, \overline{q}] \\ \frac{1}{2\kappa}\left(\alpha(2\kappa - (\underline{r} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in (\overline{q}, \overline{\overline{q}}) \end{cases} \\
&= \begin{cases} \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \underline{r} - (\overline{q} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in [\underline{q}, \overline{q}] \\ \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \underline{r} - (\underline{r} - \underline{q})) + \underline{r} - \underline{q}\right) & \underline{r} \in (\overline{q}, \overline{\overline{q}}) \end{cases} \\
&= \begin{cases} \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \overline{q}) + (1 - \alpha)(\underline{r} - \underline{q})\right) & \underline{r} \in [\underline{q}, \overline{q}] \\ \frac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{r} - \underline{r})) + \alpha(\overline{r} - \underline{r}) + (1 - \alpha)(\underline{r} - \underline{q})\right) & \underline{r} \in (\overline{q}, \overline{\overline{q}}) \end{cases} \\
&\geq \frac{1}{2\kappa}\ \alpha(2\kappa - (\overline{r} - \underline{r})) \\
&\overset{(2)}{=} \gamma(\underline{r}, p_r)
\end{aligned}$$

in the case where $\underline{r} < \overline{\overline{q}}$. If $\underline{r} \geq \overline{\overline{q}}$, this inequality is simple to show.

$$\gamma(\underline{r}, p_q) \geq \gamma(\overline{\overline{q}}, p_q) = 1 \geq \gamma(\underline{r}, p_r)$$

The type $\underline{r}$ thus has a weakly higher change of selling for the high price p_q than for the price p_r. Note that only in the case where $\gamma(\underline{r}, p_q) = \gamma(\underline{r}, p_r) = 0$ this does not lead to a strictly higher profit when setting the high price. This case, however, would imply[18] that $\underline{q} = \underline{r} = \underline{\underline{q}} = \underline{\underline{r}}$. Setting p_r would thus be dominated by setting p_q in the sense that $\phi(q, p_q) > \phi(q, p_r)$ whenever $\phi(q, p_r) > 0$ for any type q. No profitable type could optimally set p_r; it would not be an equilibrium price.

Having $\phi(\underline{r}, p_q) > \phi(\underline{q}, p_r)$ shows that $\underline{r}$ is not a limit point of $Q^{\pi}_{p_r}$. It follows from $\underline{r} \geq \underline{q}$ that $\overline{\overline{r}} \geq \overline{\overline{q}}$ and thus $\gamma(\overline{\overline{r}}, p_r) = \gamma(\overline{\overline{r}}, p_q) = 1$. Since p_q is the higher price, we have $\phi(\overline{\overline{r}}, p_r) < \phi(\overline{\overline{r}}, p_q)$ so that $\overline{\overline{r}}$ is also not a p_r-limit point. This contradicts Lemma 6.1. □

This lemma excludes the most extreme cases of negative price-quality relation. The pairs $(\underline{q}, \underline{r})$ and $(\overline{q}, \overline{r})$ can not both be ordered opposite to the corresponding prices. Thinking about the interval example, this implies that there can not be two intervals $Q^{\pi}_{p_q} < Q^{\pi}_{p_r}$ so that the higher price is only set by lower types.

We continue to use this lemma to show two further equilibrium properties which help us to determine the form of equilibrium price functions.

[18] It is easy to see that a zero selling probability of $\underline{r}$ implies $\underline{r} = \underline{\underline{r}}$. The equality $\phi(\underline{r}, p_q) = 0$ implies the first inequality of $\underline{r} \leq \underline{\underline{q}} \leq \underline{q} \leq \underline{r}$.

Lemma 6.3. *In every equilibrium, for every equilibrium price* p_q *and corresponding values* $\underline{\underline{q}}, \underline{q}, \overline{q}, \overline{\overline{q}}$*, we have:*

(1) The set

$$Q^\pi_{p_q} \cap \Big((\underline{\underline{q}}, \underline{q}) \cup (\overline{q}, \overline{\overline{q}}) \Big)$$

is a null set.

(2) If $\overline{q} - \underline{q} < 2\kappa$ *and thus* $\overline{q} \neq \overline{\overline{q}}$*, the points* $\overline{q}$ *and* $\overline{\overline{q}}$ *are not both* p_q*-limit points.*

(3) If there exists $\varepsilon > 0$ *such that* $Q^\pi_{p_q} \cap [\underline{q} - \varepsilon, \overline{q} + \varepsilon] = \emptyset$*, the interval* $[\underline{q}, \overline{q}]$ *has full length.*

The last point may be a little surprising in that you may expect the set $[\underline{q} - \varepsilon, \overline{q} + \varepsilon]$ to always contain a type of $Q^\pi_{p_q}$. To see that this needs not always to be the case, imagine $p_q = .5, \kappa = .1$ and $Q^\pi_{p_q} = [.2, .3] \cup [.7, .8]$. We then have $E(p_q, .4) = .3 < p_q < .7 = E(p_q, .6)$. In what follows, it is possible to have $\overline{s} = \underline{s} = .5$ so that $[\underline{q}, \overline{q}] = [.4, .6]$ which has full length. A narrow environment of this interval contains no element of $Q^\pi_{p_q}$.

Proof. Proof of (1)

Note that this is trivial if $\underline{\underline{q}} = \underline{q}$ and thus also $\overline{q} = \overline{\overline{q}}$. If $\underline{\underline{q}} < \underline{q}$, we also have $\underline{s} < \overline{s}$ and thus $Q^\pi_{p_q,s}$ and $Q^\pi_{p_q,s'}$ are non-empty[19] and we have $E(p_q, s) = E(p_q, s')$ for every pair $s, s' \in (\underline{s}, \overline{s})$. By property (iv) of an expectation system, this implies that

$$Q^\pi_{p_q} \cap \Big((\underline{\underline{q}}, \underline{q}) \cup (\overline{q}, \overline{\overline{q}}) \Big) \subset \bigcup_{s,s' \in (\underline{s}, \overline{s}) \cap \mathbb{Q}} Q^\pi_{p,s} \triangle Q^\pi_{p,s'}$$

is a null set.

Proof of (2)

From $\overline{q} - \underline{q} < 2\kappa$ we know that $\overline{q} \neq \overline{\overline{q}} = \underline{q} + 2\kappa$. Assume that $\overline{q}$ and $\overline{\overline{q}}$ are p_q-limit points. Pick any type $r \in (\overline{q}, \overline{\overline{q}})$ with corresponding prize $p_r = \pi(r) \neq p_q$. This is possible due to the first point of this lemma. Note that because $\overline{q}$ and $\overline{\overline{q}}$ are limit points for p_q, we must have $\phi(\overline{q}, p_r) \leq \phi(\overline{q}, p_q)$ and $\phi(\overline{\overline{q}}, p_r) \leq \phi(\overline{\overline{q}}, p_q)$ while in r, the opposite is true: $\phi(r, p_r) \geq \phi(r, p_q)$. Since the slope of $\phi(\cdot, p_q)$ has the constant value $\frac{1-\alpha}{2\kappa} p_q$ in the whole interval $(\overline{q}, \overline{\overline{q}})$, it follows that the slope of $\phi(\cdot, p_r)$ must be weakly above this value in some point between $\overline{q}$ and r while it is weakly smaller than this value in $(r, \overline{\overline{q}})$.

If the slope of $\phi(\cdot, p_r)$ also had the constant value $\frac{1-\alpha}{2\kappa} p_q$ in the whole interval $(\overline{q}, \overline{\overline{q}})$ there are two options, either having $\alpha p_r = (1-\alpha) p_q$ or $p_r = (1-\alpha) p_q$. Refer to Figure 3 to see this.

In the first case, we had $\underline{\underline{r}} \leq \overline{q} < \overline{\overline{q}} \leq \underline{r}$ which implies via (1) that the set of types setting p_r in the interval $(\overline{q}, \overline{\overline{q}})$ is a null set and there is a different price that we could have

[19] Formally, there can not be two such empty sets over all possible values of s and s' (see by Definition 4.1 (iii)). It is trivial that, if at most one of these sets is empty, none of them are.

chosen in the beginning. We assume without loss of generality that this is the case.[20] The second possibility $p_r = (1-\alpha)p_q$ implies $p_r < p_q$ and $\overline{r} \geq \overline{\overline{q}}$ which is excluded by Lemma 6.2.

The slope of $\phi(\cdot, p_r)$ is hence weakly decreasing and not constant over the whole interval $(\overline{q}, \overline{\overline{q}})$. Again referring to Figure 3, we deduce that $\overline{\overline{r}} \in (\overline{q}, \overline{\overline{q}})$. To see this, note that $\overline{r}$ and $\overline{\overline{r}}$ are the only points at which the profit $\phi(\cdot, p_r)$ from setting the price p_r strictly decreases. One of these values thus has to be in the interval $(\overline{q}, \overline{\overline{q}})$. If this is not true for $\overline{\overline{r}}$, we had $\overline{q} < \overline{r} < \overline{\overline{q}} \leq \overline{\overline{r}}$ which also implies $\underline{q} \leq \underline{r}$. Moreover, comparing the slopes in the interval $(\overline{r}, \overline{\overline{q}})$ it yields $\frac{1-\alpha}{2\kappa}p_r < \frac{1-\alpha}{2\kappa}p_q$ and hence $p_r < p_q$. This constitutes a situation which is, again, excluded by Lemma 6.2.

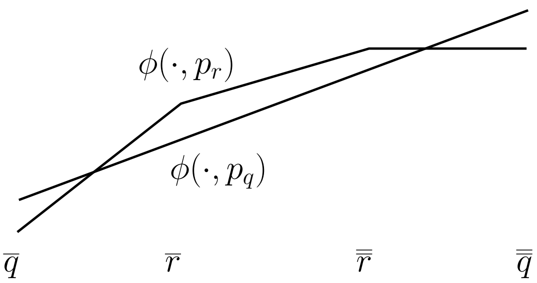

Figure 8: The situation of the proof, and the development of the different profit functions

We now know that $\overline{\overline{r}} < \overline{\overline{q}}$ and hence $p_r = \phi(\overline{\overline{q}}, p_r) \leq \phi(\overline{\overline{q}}, p_q) = p_q$. Since the prices are not equal, even the strict inequality is true. This shows that $\overline{q} < \overline{r}$, otherwise the slope of $\phi(\cdot, p_r)$ would never be below the one of $\phi(\cdot, p_q)$ in the interval $(\overline{q}, \overline{\overline{q}})$.

From the continuity and monotonicity of the profit functions, we know that there must be an interval (close to $\overline{\overline{q}}$) contained in $(\overline{\overline{r}}, \overline{\overline{q}})$ in which the profit from setting p_q is strictly higher than from setting p_r. Figure 8 shows the situation. Again by the first part of this lemma, we can find a type t in this interval that does not set the price p_q (and does not set q_r, as well, since it does not yield the highest profit). Using the same arguments as before, we end up with another equilibrium price p_t which is strictly below p_q (for the same arguments) but must be strictly above p_r since type $t > \overline{\overline{r}}$ sets this price and thus

$$p_r = \phi(t, p_r) < \phi(t, p_t) \leq p_t.$$

The relation of the functions $\phi(\cdot, p_t)$ and $\phi(\cdot, p_q)$ follow as before, using the same reasoning.

By further repeating these arguments, we end up with an infinite and strictly increasing sequence of equilibrium prices which are all below p_q. This contradicts the assumption that there can only be finitely many prices in an equilibrium.

Proof of (3)

[20] The new price $p_{r'}$ can not have the same property since then we would have $\alpha p_r = (1-\alpha)p_q = \alpha p_{r'}$. This contradicts $p_r \neq p_{r'}$.

Assume that $[\underline{q}, \overline{q}]$ does not have full length, i.e. $\underline{s} < \overline{s}$. The situation is given in Figure 9.

If there are $s < s'$ in $(\underline{s}, \overline{s})$ with $Q^{\pi}_{p_q,s} = \emptyset = Q^{\pi}_{p_q,s'}$, we have $E(p_q, s) < E(p_q, s')$ by property (iii) of an expectation system.

If there are no two such signals, define $\varepsilon' = \min\{\varepsilon, \frac{\overline{s}-\underline{s}}{4}\}$ (which is strictly greater than zero by the assumptions) and pick two points $s \in (\underline{s}, \underline{s} + \varepsilon')$ and $s' \in (\overline{s} - \varepsilon', \overline{s})$. Figure 9 shows the situation. By construction we now have $\underline{s} < s < s' < \overline{s}$. Note that $Q^{\pi}_{p_q,s}$ contains no element above $\underline{q} - \varepsilon$ while $Q^{\pi}_{p_q,s'}$ contains no element below $\overline{q} + \varepsilon$. By the assumption of this paragraph, we can choose s and s' so that these sets are not empty. Then we have

$$\sup Q^{\pi}_{p,s} < \underline{q} < \overline{q} < \inf Q^{\pi}_{p,s'}$$

which, by property (i), also implies $E(p_q, s) < E(p_q, s')$.

Figure 9: The situation in (3).

In both cases, the resulting inequality $E(p_q, s) < E(p_q, s')$, is a contradiction to $s, s' \in (\underline{s}, \overline{s})$. □

Having the monotonicity result of Lemma 6.2, one might think that this relation is even more extreme and that the ordering $\underline{q} \leq \underline{r}$ could never occur with $p_r < p_q$. The following lemma indeed shows that, although the case itself is not excluded, the implication for the order of profitable types setting the two prices is preserved.

Lemma 6.4. *In an equilibrium, let $p_r < p_q$ be two equilibrium prices. If $\underline{r} \geq \underline{q}$ and $\overline{r} > \overline{q}$ (the case of Lemma 6.2), we have*

$$Q^*_{p_r} < Q^*_{p_q}.$$

If additionally $\underline{r} > \underline{q}$, the interval $[\underline{q}, \overline{q}]$ has full length.

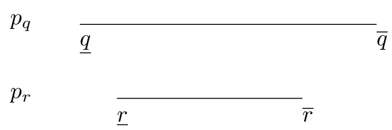

Figure 10: The situation of Lemma 6.4.

Proof. A picture of the situation at hand is given in Figure 10. First, consider the strict case $\underline{r} > \underline{q}$. For all $q \geq \overline{q}$ we have $\phi(q, p_q) > \phi(q, p_r)$ since

$$\begin{aligned}\gamma(\overline{q}, p_q) &= \tfrac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{q} - \underline{q})) + \overline{q} - \underline{q}\right) \\ &\geq \tfrac{1}{2\kappa}\left(\alpha(2\kappa - (\overline{q} - \underline{r})) + \overline{q} - \underline{r}\right) \\ &= \gamma(\overline{q}, p_r) > 0\end{aligned}$$

so that we have $\phi(\overline{q}, p_q) = p_q \cdot \gamma(\overline{q}, p_q) > p_r \cdot \gamma(\overline{q}, p_r) = \phi(\overline{q}, p_r)$. With higher types than $\overline{q}$, the left hand side of this inequality grows faster than the right hand side until the value $\overline{\overline{q}}$ from where we have $\phi(q, p_q) = p_q > p_r \geq \phi(q, p_r)$. This shows that $\overline{\overline{r}}$ (which is above $\overline{q}$) is not a p_r-limit point. We thus know that $\underline{r}$ is a p_r-limit point and hence $\phi(\underline{r}, p_r) \geq \phi(\underline{r}, p_q)$.

It follows that

$$\begin{aligned}\phi(\underline{q}, p_r) =& \phi(\underline{r}, p_r) - \int_{\underline{q}}^{\underline{r}} \phi'(t, p_r)dt \\ =& \phi(\underline{r}, p_r) - \int_{\underline{q}}^{\underline{r}} \frac{\alpha p_r}{2\kappa}dt \\ >& \phi(\underline{r}, p_q) - \int_{\underline{q}}^{\underline{r}} \frac{p_q}{2\kappa}dt \\ =& \phi(\underline{r}, p_q) - \int_{\underline{q}}^{\underline{r}} \phi'(t, p_q)dt \\ =& \phi(\underline{q}, p_q),\end{aligned}$$

using that both of these functions are differentiable in the non-empty interval $(\underline{q}, \underline{r})$. Finally, since $\overline{r} < \overline{q}$, we have $\underline{\underline{r}} < \underline{\underline{q}}$, thus $\phi(\underline{\underline{q}}, p_q) = 0 < \phi(\underline{\underline{q}}, p_r)$. These inequalities imply that neither $\underline{q}$ nor $\underline{\underline{q}}$ is a limit point of $\overline{Q^{\pi}_{p_q}}$. Using Lemma 6.3 (2), we see that $[\underline{q}, \overline{q}]$ has full length.

Observe now that the function $\phi(\cdot, p_q)$ always has a strictly higher slope than $\phi(\cdot, p_r)$ in the interval $(\underline{\underline{q}}, \overline{\overline{q}})$. Together with the inequalities

$$\phi(\underline{\underline{q}}, p_q) = 0 < \phi(\underline{\underline{q}}, p_r) \text{ and } \phi(\overline{\overline{q}}, p_q) = p_q > \phi(\overline{\overline{q}}, p_r)$$

this proves the existence of a "critical type" q_c with $\phi(q, p_q) < \phi(q, p_r)$ whenever $\underline{\underline{q}} < q < q_c$ and $\phi(q, p_q) > \phi(q, p_r)$ if $q > q_c$. Note that also no profitable type below $\underline{\underline{q}}$ sets the price p_q since then the profit would be zero. This proves $Q^*_{p_r} \leq q_c \leq Q^*_{p_q}$ and thus $Q^*_{p_r} < Q^*_{p_q}$ since the sets are disjoint.

The special case $\underline{r} = \underline{q}$ needs a different treatment. As before, having $\overline{r} < \overline{q}$ implies that $\overline{q}$ is a limit point of $\overline{Q^{\pi}_{p_q}}$. If $\overline{\overline{q}}$ also was such a limit point, the proof above works and we are done. Hence, we consider the case in which $\underline{q}$ is a limit point and thus $\phi(\underline{q}, p_q) \geq \phi(\underline{q}, p_r)$.

If this inequality was strict, $\underline{r} = \underline{q}$ could not be a limit point of $Q^{\pi}_{p_r}$. The same obviously holds for $\overline{\overline{r}} = \overline{\overline{q}}$ which gives a contradiction to Lemma 6.1.

Having $\phi(\underline{q}, p_q) = \phi(\underline{q}, p_r)$, it follows that $\phi(t, p_q) > \phi(t, p_r)$ for all $t > \underline{q} = \underline{r}$ and $\phi(t, p_q) < \phi(t, p_r)$ for $t \in (\underline{r}, \underline{q})$. This again can be seen by comparing the slopes of the profit functions. Hence $Q^*_{p_r} < Q^*_{p_q}$.

□

The previous lemmas deal with the counter-intuitive cases in which, although the price p_r is lower than p_q, the order $\underline{q} \leq \underline{r}$ holds and thus there could be a negative quality-price relation. In what follows, we show what happens if this relation has the "natural" order $\underline{r} < \underline{q}$.

Lemma 6.5. *Let $p_r < p_q$ be two equilibrium prices with $\underline{r} < \underline{q}$. Then we have $\overline{r} \leq \overline{q}$ and $Q^*_{p_r} \leq \underline{q} \leq Q^*_{p_q}$ a.s.*[21].

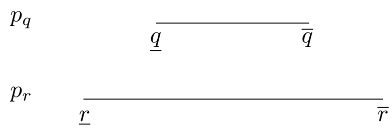

Figure 11: The situation excluded by Lemma 6.5

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To prove this statement, we use the following intermediary result.

Lemma 6.6. *Let $p_r < p_q$ be two equilibrium prices with $\underline{r} < \underline{q}$ and $\overline{q} < \overline{r}$. Then there exists another equilibrium price p_t such that either*

$$p_q < p_t, \overline{t} < \overline{r} \text{ and } \underline{t} > \underline{r}$$

or

$$p_t < p_r, \underline{t} < \underline{q} \text{ and } \overline{t} > \overline{q}$$

holds.

Proof. We have

$$\begin{aligned} \gamma(\underline{r}, p_q) =& \tfrac{1}{2\kappa}\alpha(2\kappa - (\overline{q} - \underline{r})) \\ \geq& \tfrac{1}{2\kappa}\alpha(2\kappa - (\overline{r} - \underline{r})) \\ =& \gamma(\underline{r}, p_r). \end{aligned}$$

Note that we always have $\gamma(\underline{r}, p_q) > 0$ since $\overline{q} < \overline{r}$ and thus $\underline{\underline{q}} < \underline{\underline{r}} \leq \underline{r}$. Hence the inequality above implies

$$\phi(\underline{r}, p_q) = p_q \cdot \gamma(\underline{r}, p_q) > p_r \cdot \gamma(\underline{r}, p_r) = \phi(\underline{r}, p_r)$$

[21]The "almost surely" notation is only necessary in a very special case, as one can see in the proof.

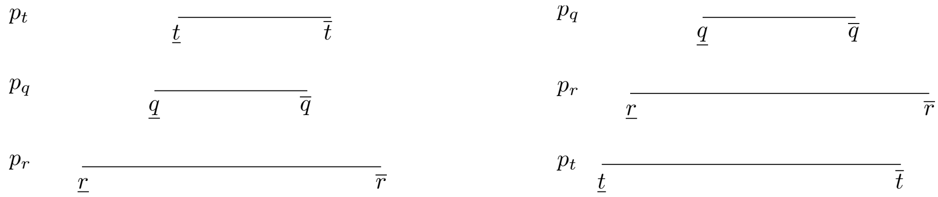

Figure 12: The two situations with the new equilibrium price p_t

so that $\underline{r}$ is not a p_r-limit point.

The condition $\overline{q} < \overline{r}$ implies $\phi(\underline{r}, p_q) > \phi(\underline{r}, p_r) = 0$. This also proves that $\overline{r}$ and $\overline{\overline{r}}$ must be limit points for p_r. Using Lemma 6.3 (2), this shows that the interval $[\underline{r}, \overline{r}]$ has full length, hence $\overline{r} = \overline{\overline{r}}$.

Assume now that $\phi(\overline{q}, p_q) \leq \phi(\overline{q}, p_r)$. From this it follows that $\phi(q, p_q) < \phi(q, p_r)$ for all $q \in [\underline{q}, \overline{q})$, implying that not $\underline{q}$ but $\overline{\overline{q}}$ is a limit point of $Q^{\pi}_{p_q}$. But since $\overline{q} - \underline{q} < \overline{r} - \underline{r} \leq 2\kappa$, Lemma 6.3 (2) then implies that $\overline{q}$ is not such a limit point. This proves the existence of $\varepsilon > 0$ such that $[\underline{q} - \varepsilon, \overline{q} + \varepsilon] \cap Q^{\varepsilon}_{p_q}$ is empty[22], implying by Lemma 6.3 (3) that $[\underline{q}, \overline{q}]$ has full length. This case is excluded in the situation at hand.

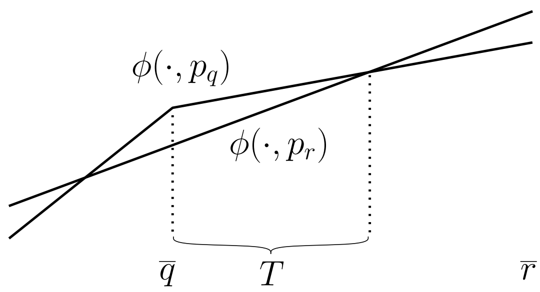

Figure 13: The type t is chosen from the open interval T.

We now know that $\phi(\overline{q}, p_q) > \phi(\overline{q}, p_r)$. By continuity, the same is true for an interval T of types above $\overline{q}$ (See Figure 13). Take any type t in this interval with $\pi(t) \neq p_q$. It exists by Lemma 6.3 (1) and since $\overline{q} < \overline{\overline{q}}$. We know that the corresponding price $p_t := \pi(t)$ is also not equal to p_r since it is not optimal for t to set p_r. We can assume from Lemma 6.3 (1) that t has been chosen with $t \notin (\overline{t}, \overline{\overline{t}}) \cup (\underline{\underline{t}}, \underline{t})$[23]. There are two cases left to consider.

<u>First case</u>: $t \in [\underline{t}, \overline{t}]$

Then we also have $\overline{t} \geq t > \overline{q}$.

[22]This is true even if we had, $\phi(\overline{q}, p_q) = \phi(\overline{q}, p_r)$. The non-existence of such an ε would make $\overline{q}$ a p_q-limit point, thus causing a contradiction.

[23]Otherwise, take a different t from the interval T. Since there are only finitely many prices and thus finitely many sets of the form $(\overline{t}, \overline{\overline{t}}) \cup (\underline{\underline{t}}, \underline{t}))$ which are all null sets, there exists a $t \in T$ which is not in any of these sets.

First, assume $p_t < p_q$. We know from Lemma 6.2 that $\underline{t} < \underline{q}$ which is the situation of the lemma. Hence with the same reasoning, we can show that $[\underline{t}, \overline{t}]$ has full length. Moreover, we have $p_t < p_r$. Otherwise, Lemma 6.2 shows $\overline{r} < \overline{q}$ and hence the inequality $\phi(t, p_t) > \phi(t, p_r)$ implies $\phi(\overline{r}, p_t) > \phi(\overline{r}, p_r)$ which is a contradiction to $\overline{r}$ being a p_r-limit point. We thus have $p_t < p_r$ and (via Lemma 6.2) $\underline{t} < \underline{r} < \underline{q}$ and $\overline{t} > \overline{q}$. This is the second case stated in the lemma.

Second, assume $p_t > p_q$ and thus also $p_t > p_r$. Because of the higher slope of $\phi(\cdot, p_t)$ compared to $\phi(\cdot, p_r)$ in the interval $(t, \overline{t})$, we have $\overline{t} < \overline{r} = \overline{\overline{r}}$. Otherwise $\overline{r}$ could not be a p_r-limit point. Lemma 6.2 now implies $\underline{t} > \underline{r}$. This satisfies the first of the two cases stated in the lemma.

Second case: $t \geq \overline{\overline{t}}$

From $\overline{\overline{t}} \leq t < \overline{r}$ we know

$$p_t = \phi(\overline{r}, p_t) \overset{\overline{r} \text{ is } p_r\text{-limit point}}{\leq} \phi(\overline{r}, p_r) \leq p_r$$

so that $p_t < p_r < p_q$. Note that we must have $\overline{t} > \overline{q}$. This can be seen by observing that otherwise the slope of $\phi(\cdot, p_q)$ is always higher than the slope of $\phi(\cdot, p_t)$ in the interval $(\underline{q}, t)$. Since $\phi(t, p_t) \geq \phi(t, p_q)$, the strict inequality would hold within this interval, making it impossible for any type in $[\underline{q} - \varepsilon, \overline{q} + \varepsilon]$ (for some $\varepsilon > 0$) to optimally set the price p_q which, again by Lemma 6.3 (3), gives a contradiction.

Knowing $\overline{t} > \overline{q}$ and $p_q > p_t$, Lemma 6.2 dictates $\underline{t} < \underline{q}$.

We thus have $\overline{t} > \overline{q}$, $\underline{t} < \underline{q}$ and $p_t < p_q$. This is the second of the two possibilities stated in the lemma. □

With this result we continue to prove the original statement.

Proof of Lemma 6.5. Assume that we had $\overline{q} < \overline{r}$. Denote $p_{\min} = p_r$, $p_{\max} = p_q$. Applying Lemma 6.6, the resulting price p_t is either higher or lower than both, $p_{\min}$ and $p_{\max}$. Redefine these values so that $p_{\min}$ and $p_{\max}$ are the most extreme of these three prices, note that the new values of $p_{\min}$ and $p_{\max}$ satisfy the assumptions of Lemma 6.6. We can thus repeat these arguments over and over, ending up with an infinite number of equilibrium prices. This contradicts the assumption of finitely many equilibrium prices and proves $\overline{q} \geq \overline{r}$.

It is left to show that the two sets $Q^*_{p_r}$ and $Q^*_{p_q}$ can be strictly separated as stated in the lemma. This again has to be done considering multiple cases.

First Case: $\overline{r} < \overline{q}$

It then follows that $\overline{q}$ is a limit point of $Q^\pi_{p_q}$ (since $\underline{\underline{r}} < \underline{\underline{q}}$, $\underline{\underline{q}}$ can not be a p_q-limit point). From Lemma 6.3 we know that $\overline{\overline{q}}$ is not a limit point but $\underline{q}$ is. We thus have $\phi(\underline{q}, p_q) \geq \phi(\underline{q}, p_r)$. In the whole interval $[\underline{q}, \overline{\overline{q}}]$, the slope of $\phi(\cdot, p_q)$ is greater than the one of $\phi(\cdot, p_r)$. Hence $\phi(q, p_q) > \phi(q, p_r)$ for all $q > \underline{q}$. In the other direction, note that $\overline{r} < \overline{q}$ implies $\underline{\underline{r}} < \underline{\underline{q}}$ so that $0 = \phi(\underline{\underline{q}}, p_q) < \phi(\underline{\underline{q}}, p_r)$. The slope of $\phi(\cdot, p_q)$ is constant while the slope of $\phi(\cdot, p_r)$ is non-decreasing in $[\underline{\underline{q}}, \underline{q}]$. This proves the existence of some $t \in (\underline{\underline{q}}, \underline{q})$ with $\phi(q, p_q) > \phi(q, p_r)$ if $q > t$ and $\phi(q, p_q) < \phi(q, p_r)$ if $\underline{\underline{q}} \leq q < t$. Thus $Q^*_{p_r} \leq t \leq Q^*_{p_q}$.

Second Case: $\overline{r} = \overline{q}, \underline{r} > \underline{\underline{q}}$
We show that this situation is not possible and leads to a contradiction. We have, since $\underline{r} < \underline{q}$,

$$\phi(\underline{r}, p_r) = \frac{\alpha p_r}{2\kappa}(\underline{r} - \underline{\underline{r}}) < \frac{\alpha p_q}{2\kappa}(\underline{r} - \underline{\underline{r}}) = \frac{\alpha p_q}{2\kappa}(\underline{r} - \underline{\underline{q}}) = \phi(\underline{r}, p_q).$$

From this, it follows that $\overline{\overline{r}}$ is a limit point of $Q^\pi_{p_r}$. Note that, since $\overline{q} = \overline{r}$, the slope of $\phi(\cdot, p_q)$ is higher than the one of $\phi(\cdot, p_r)$ in the whole interval $[\underline{q}, \overline{r}]$. For $\phi(\overline{\overline{r}}, p_r) \geq \phi(\overline{\overline{r}}, p_q)$ to be possible, we thus have $\phi(\overline{q}, p_q) < \phi(\overline{q}, p_r)$ and $\phi(\underline{q}, p_q) < \phi(\underline{q}, p_r)$. By continuity, this shows that

$$Q^\pi_{p_q} \cap [\underline{q} - \varepsilon, \overline{q} + \varepsilon] = \emptyset$$

for some $\varepsilon > 0$. Lemma 6.3 (3) then implies that $[\underline{q}, \overline{q}]$ has full length. But the assumptions of the second case imply

$$\overline{q} - \underline{q} < \overline{q} - \underline{r} < \overline{q} - \underline{\underline{q}} = 2\kappa.$$

Third case: $\overline{r} = \overline{q}, \underline{r} = \underline{\underline{q}}$
It follows that the interval $[\underline{r}, \overline{r}]$ has full length. If $p_r > \alpha p_q$, the claim $Q^*_{p_r} < Q^*_{p_q}$ automatically follows from observing that the slope of $\phi(\cdot, p_r)$ is strictly higher than $\phi(\cdot, p_q)$ before the point $\underline{q}$ and strictly lower afterwards. A similar argument holds if $p_r < \alpha p_q$, the slope always being lower and thus contradicting p_r being an equilibrium price. No type can make positive profit when setting this price. Both of these cases are shown in Figure 14.

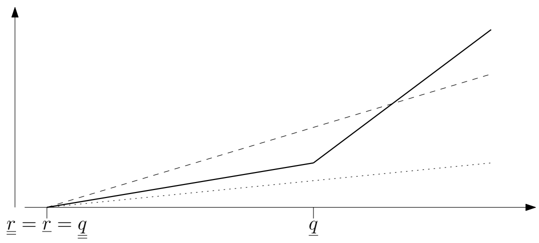

Figure 14: The situation of the third case when the price is high (dashed) or low (dotted).

A special case appears when $p_r = \alpha p_q$. All types in the interval $[\underline{\underline{q}}, \underline{q}]$ are then indifferent between setting p_q or p_r. While we know that only a null set of types in this interval can actually set p_q, this would still be enough for the inequality $Q^*_{p_r} < Q^*_{p_q}$ not to be true. However, it is enough to observe that this inequality holds almost surely. □

Lemma 6.7. *Let $p_r < p_q$ be two equilibrium prices with $\overline{r} \leq \underline{q}$. Then we have $Q^*_{p_r} < Q^*_{p_q}$ a.s..*

This lemma covers the intuitive case in which the intervals $[\underline{r}, \overline{r}]$ and $[\underline{q}, \overline{q}]$ are ordered according to their prices. The proof is rather easy, compared to the previous lemmas.

Proof. As before, $\overline{r} < \overline{q}$ implies that $\overline{q}$ is a p_q-limit point.
If $[\underline{q}, \overline{q}]$ has full length, the slope of $\phi(\cdot, p_q)$ is higher than the one of $\phi(\cdot, p_r)$ in the whole interval $[\underline{q}, \overline{q}]$. For values $q > \overline{q} = \overline{\overline{q}}$ we then have $\phi(q, p_q) = p_q > \phi(q, p_r)$. This proves the existence of $t \in (\underline{q}, \overline{q})$ so that

$$\phi(q, p_q) > \phi(q, p_r) \text{ if } q > t, \phi(q, p_q) < \phi(q, p_r) \text{ if } q < t,$$

proving the inequality $Q^*_{p_r} < Q^*_{p_q}$.
If $[\underline{q}, \overline{q}]$ does not have full length, Lemma 6.3 (2) shows that $\overline{\overline{q}}$ is not a p_q-limit points, so $\underline{q}$ is one. We thus have $\phi(\underline{q}, p_q) \geq \phi(\underline{q}, p_r)$ and $\phi(q, p_q) > \phi(q, p_r)$ for all types $q > \underline{q}$ (using the usual argument of $\phi(\cdot, p_q)$ growing faster than $\phi(\cdot, p_r)$). Hence $Q^\pi_{p_r} \leq \underline{q}$. Lemma 6.3 (1) tells us that only a null set of profitable types below $\underline{q}$ can set the price p_q so that we have $Q^*_{p_r} \leq \underline{q} \leq Q^*_{p_q}$ *a.s.* which concludes the proof. □

Finally, having these lemmas as preparation, we are able to proof our main theorem.

Proof of Theorem 5.1. First assume the existence of an equilibrium. Let $p_1 < \ldots < p_n$ be the equilibrium prices. The previous lemmas show that for each two indices $i < j$, the order of corresponding types setting the prices $p_i < p_j$ almost surely satisfy $Q^*_{p_i} < Q^*_{p_j}$.Using this, we have $Q^*_{p_1} < \ldots < Q^*_{p_n}$ *a.s.*. Every type t in the non-empty set $Q^*_{p_1}$ is profitable by definition. Since the profit function is monotone in the type, all higher types also make positive profit and must hence set an equilibrium price. This shows that $\bigcup_{i=1}^n Q^*_{p_i} \supset (\inf Q^*_{p_1}, 1]$ so that all types above $q_{\min} := \inf Q^*_{p_1}$ set one of the prices $p_1, \ldots, p_n$. Thus π is almost surely a non-decreasing step function when being restricted to types above $q_{\min}$.

The existence of an equilibrium is easy to show, noting that every constant price function $\pi(q) = p$ constitutes an equilibrium, independent of the indifference strategy α. This can easily be seen by noting that $E(p, s)$ is uniquely determined by regular Bayesian updating and that for every other price, $E(\cdot, s)$ can be set low enough like in the existence proof of Lemma 4.2 to not allow beneficial deviations. The construction of an expectation system in the proof of Lemma 4.2 is done in this way. By this construction, the price p always maximizes the firm's profit and we have an equilibrium. Note that the parameter α can be chosen arbitrarily. □

7 Conclusion and Discussion

We studied a model of quality uncertainty, modified in such a way to admit a continuum of possible quality types and a costless extra quality signal for the consumer. The analysis shows that the price of the good depends on the quality in a positively correlated way in that a firm with a certain quality level never sets a higher price than if it would with any higher quality product. Hence in every equilibrium, the price behavior is a step function.

An interesting aspect of the model is the result of having a clear equilibrium pricing structure which is not unique but always takes the form of a step function, at least in those regions where actual trade takes place. A result obtained in a context which does not require - but allows - full rationality and high computational capabilities on the consumer side. Instead, our concept of an expectation system, skipping the step of Bayesian updating in most settings and thus generalizing the concept of Bayesian equilibria, gives an answer to the criticism on the "homo economicus" assumption present in the vast majority of economic literature. At the same time, the class of consumers for which this result holds contains the completely rational behavior.

Of course, once the monotonicity of price behavior is established, the form of the price function follows from our assumption of having only finitely many equilibrium prices. As explained in the text, this assumption is not unrealistic in many settings. There are, for example, only finitely many prices that you can encounter in a supermarket (assuming there is an upper bound for how much an item can cost). But it is worth mentioning that even without this assumption, pricing behavior must leave some ambiguity. If the pricing function π was one-to-one, prices would perfectly signal the quality and thus the extra signal does (at least in equilibrium) not convey any information. If the signal had a marginal cost, the consumer would not choose to acquire it, thus only rely on price information and give lower quality levels an incentive to deviate. Similar arguments to the ones in Section 3 would apply to this situation. Moreover, in classical signaling games, the result of imperfect signaling even in the case where there are "enough" signals for perfect signaling, is common in the presence of a sender and a receiver with different objectives. The first to show this were Crawford and Sobel (1982). It is thus not at all clear whether the step function price behavior disappears if we relax our assumptions.

One of the most remarkable features of this model is certainly the fact that in the limit of perfect information, the maximum trade amount over all possible equilibria uniformly converges to zero, thus admitting an entirely different limit behavior than in the limit case of perfect information where the only equilibrium admits perfect trade for all quality types. Moreover, the proof of this phenomenon shows that it is indeed the continuum of types which causes this result.

While having more than two and even a whole interval of possible types is certainly more realistic than in many of other discussed models, our model is farther from the true situation in different aspects. We do not even claim that the case discussed here is closer to reality than certain other models with two quality levels (as for example given in the first chapter). However, this work should serve as a warning that in lemon markets, some simplification assumptions may not at all be innocuous.

8 Appendix

Proof of Proposition 3.2. For two equilibrium prices $p > p'$, we must have $\gamma(p) < \gamma(p')$, otherwise the price p' would never be set by any type which makes positive profit, since the higher price p has a higher chance of selling and thus dominates setting p'. Hence the consumer strategy γ must be strictly decreasing in the equilibrium prices.

Take any firm type q and let p be the price set by that type. For any other equilibrium price p', we thus have

$$\gamma(p)(p - c(q)) \geq \gamma(p')(p' - c(q)).$$

Take such a price p' with $p' < p$ and let q' be a higher quality level than q with a strictly higher production cost. We have $\gamma(p) < \gamma(p')$ and thus

$$\begin{aligned}\gamma(p)(p - c(q')) =& \gamma(p)(p - c(q)) + \gamma(p)(c(q) - c(q')) \\ >& \gamma(p')(p' - c(q)) + \gamma(p')(c(q) - c(q')) \\ =& \gamma(p')(p' - c(q'))\end{aligned}$$

so that higher quality level than q never sets a lower price than p. Hence we have monotonicity in the price function. □

Proof of Lemma 4.2. We perform the proof by construction of a function E, given any price function π. It is enough to define the function value for a fixed price p since all of the properties only involve one price. If $Q_p^\pi = \emptyset$, we can just choose $E(p, s) = p\frac{s+\kappa}{2(1+2\kappa)} \in [0, \frac{p}{2}]$. This obviously satisfies the definition and ensures that the consumer always strictly prefers to abstain from buying which later becomes important for the out-of-equilibrium consumer reaction in equilibria. In the case of $p = 0$, choose any increasing function. If $Q_p^\pi \neq \emptyset$, the property (v) of the definition determines the value of $E(p, s)$ for every signal in the set

$$\mathcal{S}_I := \left\{ s \in S | Q_{p,s}^\pi \text{ is a non-empty interval (including singletons)} \right\}.$$

It is clear, since $Q_{p,s}^\pi$ is "increasing" (in an obvious sense) in s and the Bayesian posterior depends only on the set $Q_{p,s}^\pi$ (not on p and s itself), that these values do not violate the other properties of the definition.

In what follows, we extend the function $E(p, \cdot)$ to the whole space S. Define the non-empty set

$$\mathcal{S}_{\neq\emptyset} := \left\{ s \in S | Q_{p,s}^\pi \neq \emptyset \right\}.$$

On $\mathcal{S}_{\neq\emptyset}$, define the non-decreasing functions

$$\begin{aligned}\overline{E}(s) &:= \sup Q_{p,s}^\pi \\ \underline{E}(s) &:= \inf Q_{p,s}^\pi.\end{aligned}$$

We can easily extend this function to the set $S_\emptyset := S \setminus s_{\neq\emptyset}$ in a way that ensures $\underline{E}(s) < \overline{E}(s)$ for all $s \notin S_I$.[24]

Take any $s \in \mathcal{S}_{\neq\emptyset} \setminus \mathcal{S}_I$. Define

$$\underline{\sigma} := \sup\left\{s' \in \mathcal{S}_I, s' < s\right\}, \overline{\sigma} := \inf\left\{s' \in \mathcal{S}_I, s' > s' \geq s\right\}$$
$$\text{and } \underline{e} := \sup\left\{E(p,s')|s' \in \mathcal{S}_I, s' < s\right\}, \overline{e} := \inf\left\{E(p,s')|s' \in \mathcal{S}_I, s' > s\right\},$$

the endpoints of the maximum interval of types around s for which E is not yet defined and the maximal and minimal value until these points. If one or both of these sets are empty, set $(\underline{\sigma}, \underline{e}) = (-\kappa, 0)$ or $(\overline{\sigma}, \overline{e}) = (1+\kappa, 1)$, respectively. Figure 15 shows the situation.

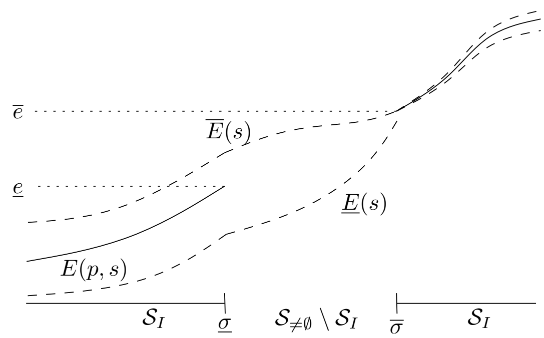

Figure 15: The situation of the proof after defining $E(p,s)$ on the set $\mathcal{S}_I$. The situation at $\overline{\sigma}$ shows the special case in which $E(p,\overline{\sigma}) = \overline{E}(\overline{\sigma}) = \underline{E}(\overline{\sigma})$.

To find values for the expectation system in the interval $(\underline{\sigma}, \overline{\sigma})$[25], it suffices to show that we have $\overline{E}(s) > \underline{e}$ and $\underline{E}(s) < \overline{e}$ for all signals s in this interval. We can then choose values for $E(p,s)$ in the never-empty corridor $\left(\max\{\underline{e}, \underline{E}(s)\}, \min\left\{\overline{e}, \overline{E}(s)\right\}\right)$ (for $s \in (\underline{\sigma}, \overline{\sigma})$) in accordance to the expectation system definition (Letting it be constant when the sets $Q^\pi_{p,s}$ do not change and strictly increasing when they are empty). We only show the left inequality, the other direction using the same argument.

Note that we have $\underline{e} \leq \overline{E}(\underline{\sigma}) \leq \overline{E}(s)$ for all signals in the interval. The first inequality is not strict if and only if $\overline{E}(\underline{\sigma}) = \underline{E}(\underline{\sigma})$[26]. But in this case, $\underline{\sigma}$ is in $\mathcal{S}_I$, while signals s' slightly above $\underline{\sigma}$ are not. This implies $\underline{e} = \overline{E}(\underline{\sigma}) = \underline{E}(\underline{\sigma}) \leq \underline{E}(s') < \overline{E}(s')$, the strict inequality using the property from our extension of $\underline{E}, \overline{E}$ to $S_\emptyset$.

□

[24] If $s \in \mathcal{S}_{\neq\emptyset}$, we know that $Q^\pi_{p,s}$ contains at least two elements. For the extension to $s \in S_\emptyset$, note that

$$\sup Q^\pi_{p,s'} < s - \kappa < s + \kappa \inf Q^\pi_{p,s''} \quad \forall\, s', s'' \in \mathcal{S}_{\neq\emptyset}, s' < s < s''.$$

Any two strictly increasing extensions with values in $[s-\kappa, s+\kappa] \cap [0,1]$ does the job.

[25] This interval could be empty in special cases. For single points like $\underline{\sigma}$ and $\overline{\sigma}$ one can just choose an appropriate value in $[\underline{e}, \overline{e}]$, keeping in mind the monotonicity assumptions of Definition 4.1.

[26] To see this, note that the density f has a positive minimum value so that there is $\beta \in (0,1)$ with $E(p,s) < \beta\overline{E}(s) + (1-\beta)\underline{E}(s)$. A picture of such a special case (for $\overline{\sigma}$) is given in Figure 15. The statement is trivially true in the "border" cases when $\underline{\sigma} = -\kappa$ or $\overline{\sigma} = 1+\kappa$.

Proof of Lemma 4.3. The existence and uniqueness follows just from item (ii) of an expectation system. Note that we can have $\underline{s} = \overline{s} \in \{-\kappa, 1+\kappa\}$ in the case where the expectation is never or always higher than the price.

Let now π be a price function to which E is an expectation system. Assume that $\overline{s} - \underline{s} > 2\kappa$. Then there exist $\underline{s} < s < s' < \overline{s}$ with $s' - s > 2\kappa$ and $E(p,s) = E(p,s') = p$. Definition 4.1 (iii) implies that one of the sets $Q^{\pi}_{p,s}$ and $Q^{\pi}_{p,s'}$ is not empty. If none of them is empty, we have

$$E(p,s) \leq \sup Q^{\pi}_{p,s} < \inf Q^{\pi}_{p,s'} \leq E(p,s')$$

which is a contradiction.

If $Q^{\pi}_{p,s}$ is empty, choose some s'' with $s < s'' < s' - 2\kappa < s'$. Then either $Q^{\pi}_{p,s''}$ is not empty (hence the argument above applies) or it is empty and we have by Definition 4.1 (iii) and (ii)

$$E(p,s) < E(p,s'') \leq E(p,s'),$$

again contradicting the equality of the left and the right expression.

The case of $Q^{\pi}_{p,s'} = \emptyset$ uses the same arguments. ☐

Proof of Lemma 4.4. Let $q < q'$ be two types. It then follows that

$$\begin{aligned}
\phi_{\pi}(q') &\geq \phi(q', \pi(q)) \\
&= \pi(q)\frac{1}{2\kappa}\left(\int_{q'-\kappa}^{q'+\kappa} \underbrace{\alpha \mathbf{1}_{E(\pi(q),s)=\pi(q)}(s) + \mathbf{1}_{E(\pi(q),s)>\pi(q)}(s)}_{=:\beta(s)}\, ds\right) \\
&= \phi_{\pi}(q) + \frac{\pi(q)}{2\kappa}\left(\int_{q+\kappa}^{q'+\kappa}\beta(s)ds - \int_{q-\kappa}^{q'-\kappa}\beta(s)ds\right) \\
&\geq \phi_{\pi}(q)
\end{aligned}$$

where the first inequality comes from optimality. To see the last inequality, let $s < s'$ be two signals. We then have the implications

$$\begin{aligned}
E(\pi(q), s) > \pi(q) &\Rightarrow E(\pi(q), s') > \pi(q) \\
E(\pi(q), s) = \pi(q) &\Rightarrow E(\pi(q), s') \geq \pi(q)
\end{aligned}$$

by using the monotonicity of E. In what follows, since $\alpha \leq 1$, $\beta(s) \leq \beta(s')$. The left of the two integrals is thus larger since the integration area contains higher signals.

We now prove the continuity of ϕ^{π}. Although this looks like a standard envelope theorem application, the function $\phi(q,p)$ is not continuous in the price component.

Let $q \in (0,1]$ be some type and let (q_n) be a sequence of types below q, converging to q. We have

$$\phi^{\pi}(q) \overset{\text{Monot.}}{\geq} \phi^{\pi}(q_n) \overset{\text{Optimality}}{\geq} \phi(q_n, \pi(q)) \;\forall\; n.$$

Since the right hand side converges to $\phi(q, \pi(q)) = \phi^\pi(q)$ ($\phi(q, p)$ is continuous in q), ϕ^π is left-continuous.

For $q \in [0, 1)$, let (q_n) now be a sequence converging to $q \in [0, 1)$ from below. For all $n \in \mathbb{N}$ we have

$$\phi_\pi(q_n) \overset{\text{Monot.}}{\geq} \phi_\pi(q) \overset{\text{Optimality}}{\geq} \phi(q, \pi(q_n)) \geq \phi_\pi(q_n) - C \cdot (q_n - q)$$

where C is an upper bound for the slope of $\phi(\cdot, p), p \in [0, 1]$.[27] Taking the limit shows $\lim_{n\to\infty} \phi_\pi(q_n) = \phi_\pi(q)$ and thus right-continuity. □

Proof of Lemma 5.2 (The Existence Part). Fix a price $\hat{p} \in (1 - \kappa, 1)$. We claim that the constant price function

$$\pi(q) = \hat{p} \ \forall \ q \in [0, 1]$$

can be part of a locally continuous equilibrium. For this, we have to construct an expectation system. Note that without the local continuity assumption, out-of-equilibrium beliefs can just be taken low enough so that the buyer would never buy for any price other than $\hat{p}$. Now, we have to define the values for $E(p, s)$ for all signals s in an environment of $\hat{p}$ in a continuous way. In what follows, the construction of out-of-equilibrium beliefs is taken not only in a locally but even in a globally continuous way, without the need to restrict ourselfs to an environment of $\hat{p}$.

Because $Q_{\hat{p},s}$ is always an interval, $E(\hat{p}, s)$ is given by Bayesian updating and is thus strictly increasing in s. So there exists a pivotal signal $\hat{s} \in (1 - 2\kappa, 1 + \kappa)$ with

$$E(\hat{p}, s) < \hat{p} \ \forall s < \hat{s} \quad E(\hat{p}, s) > \hat{p} \ \forall s > \hat{s}.$$

The existence and range of the signal comes from noting that signals close to $1 + \kappa$ prove a quality above $\hat{p}$ and that signals below $1 - 2\kappa$ induce an expectation below $1 - \kappa$ which is below the price $\hat{p}$.

For lower prices than $\hat{p}$, we set

$$E(p, s) = E(\hat{p}, s) \cdot \frac{p}{\hat{p}}, s \in S, p < \hat{p}.$$

This construction preserves the strict monotonicity (demanded by definition 4.1 (iii)) to the lower prices and ensures $E(p, s) > p \Leftrightarrow E(\hat{p}, s) > \hat{p}$ so that for all prices p no signal can give a higher sale probability than the price $\hat{p}$. Hence, deviation to a lower price is not profitable.

The case of higher prices is a bit trickier. Not only do we have to ensure that sale probabilities do not increase when setting a higher price, they have to fall fast enough to nullify the positive price effect.

Claim: There exists $C > 0$ such that $E(\hat{p}, s) \leq E(\hat{p}, \hat{s}) + C \cdot (s - \hat{s})$ for all $s \in (\hat{s}, 1 + \kappa)$. To proof this statement, note that we have

$$E(\hat{p}, s) = \text{Exp}(q|q \geq s - \kappa) = \frac{\int_{s-\kappa}^{1} q f(q) dq}{\int_{s-\kappa}^{1} f(q) dq}$$

[27] This upper bound can be chosen to be $\frac{1}{2\kappa}$, see Figure 3.

by Baye's law, using Definition 4.1 (v). Differentiating this expression with respect to s, we get

$$\begin{aligned}
\frac{\partial}{\partial s}E(\hat{p},s) =& \frac{1}{\left(\int_{s-\kappa}^{1} f(q)dq\right)^2}\cdot\left(-(s-\kappa)f(s-\kappa)\int_{s-\kappa}^{1} f(q)dq + f(s-\kappa)\int_{s-\kappa}^{1} qf(q)dq\right)\\
=& \frac{f(s-\kappa)}{\left(\int_{s-\kappa}^{1} f(q)dq\right)^2}\cdot\int_{s-\kappa}^{1}(q-(s-\kappa))f(q)dq\\
=& \frac{f(s-\kappa)}{\int_{s-\kappa}^{1} f(q)dq}\cdot \mathrm{Exp}(q-(s-\kappa)|q\geq s-\kappa)\\
\leq& \frac{f_{\max}}{f_{\min}\cdot(1-(s-\kappa))}\cdot(1-(s-\kappa)) = \frac{f_{\max}}{f_{\min}} =: C.
\end{aligned}$$

The values $f_{\max}$ and $f_{\min}$ refer to the maximum and minimum values of f. They exist and are positive due to our assumptions. We now have

$$E(\hat{p},s) = E(\hat{p},\hat{s}) + \int_{\hat{s}}^{s}\frac{\partial}{\partial t}E(\hat{p},t)dt \leq E(\hat{p},\hat{s}) + C\cdot(s-\hat{s})$$

which proves the claim.

Having this parameter C, we define the expectation for higher prices than $\hat{p}$ as follows.

$$E(p,s) = E(\hat{p},s)\frac{p}{\hat{p}+C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)}$$

This is a continuous expression in p and preserves the strict monotonicity in s for every price. Now, we have

$$\begin{aligned}
E(p,\hat{s}+\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})) &= E\left(\hat{p},\hat{s}+\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)\frac{p}{\hat{p}+C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)}\\
&\leq \left(E(\hat{p},\hat{s})+C\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)\frac{p}{\hat{p}+C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)}\\
&= \left(\hat{p}+C\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)\frac{p}{\hat{p}+C\left(\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)}\\
&= p
\end{aligned}$$

which, because of the strict monotonicity, implies that a firm can only sell for a price p if the signal is above $\hat{s}+\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})$. Hence, for every quality type q

$$\begin{aligned}
\phi(q,p) = p\cdot\gamma(q,p) &\leq p\left(q+\kappa-\left(\hat{s}+\frac{1+\kappa-\hat{s}}{\hat{p}}(p-\hat{p})\right)\right)\\
&\leq p\left(q+\kappa-\left(\hat{s}+\frac{1+\kappa-\hat{s}}{p}(p-\hat{p})\right)\right)\\
&= p(q-1)+\hat{p}(1+\kappa-\hat{s}) \leq \hat{p}(q-1)+\hat{p}(1+\kappa-\hat{s})\\
&= \hat{p}(q+\kappa-\hat{s}) = \hat{p}\cdot\gamma(q,\hat{p}) = \phi(q,\hat{p})
\end{aligned}$$

in the case where $\gamma(q,p) > 0$. Otherwise we trivially have $\gamma(q,p) = 0 \leq \phi(q,\hat{p})$. This shows that a deviation to a higher price is not profitable and we have an equilibrium. □

The rest of the proof of Lemma 5.2. We first apply Theorem 5.1. Knowing that prices are set due to a step function, the set $Q^*_{p_q}$ is an interval for all equilibrium prices p_q. Lemma 6.3 (1) shows that in this case the interval $[\underline{q},\overline{q}]$ has full length or we have $Q^*_{p_q} \subset [\underline{q},\overline{q}]$.
In the latter case, Lemma 6.1 implies that $Q^*_{p_q}$ must be an interval with endpoints $\underline{q}$ and $\overline{q}$.

We thus only have to show that the former case can not occur. Note that, if p_q is not the lowest equilibrium price, the case of $[\underline{q},\overline{q}]$ having full length implies that, $\overline{q} = \overline{\overline{q}}$ is a p_q-limit point but $\underline{q} = \underline{\underline{q}}$ is not. Otherwise $\phi_\pi(\underline{\underline{q}}) = \phi(\underline{\underline{q}}, p_q) = 0$ so that there can be no lower profitable type. From this it follows that there must be a type $q > \overline{\overline{q}}$ which sets the price p_q. Otherwise let

$$t := \inf Q^*_{p_q} > \underline{\underline{q}}.$$

All signals in $(\overline{\overline{q}} - \kappa, t + \kappa) \neq \emptyset$ yield the same expectation due to property (iii) of an expectation system. This expectation must be equal to the price. If it was lower, profits would be zero. If it was higher, the local continuity condition implies that type $\overline{\overline{q}}$ could set a marginally higher price and still sell with full probability, making it profitable to deviate. Having $E(p_q, s) = p_q$ for all signals $s \in (\overline{\overline{q}} - \kappa, t + \kappa) \neq \emptyset$ is a contradiction to $[\underline{q},\overline{q}]$ having full length.
The existence of the type $q > \overline{\overline{q}}$ setting the price p_q implies that $E(q,s) > p_q$ for all signals $s \in [q - \kappa, q + \kappa]$ so that for the same reason as before a higher price could be demanded by type q under the local continuity condition. □

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