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Respect for experts or respect for unanimity? The liberal paradox in probabilistic opinion pooling

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Abstract. Amartya Sen (1970) has shown that three natural desiderata for social choice rules are inconsistent: universal domain, respect for unanimity, and respect for some minimal rights — which can be interpreted as either expert rights or liberal rights. Dietrich and List (2008) have generalised this result to the setting of binary judgement aggregation. This paper proves that the liberal paradox holds even in the framework of probabilistic opinion pooling and discusses options to circumvent this impossibility result: restricting the aggregator domain to profiles with no potential for conflicting rights, or considering agendas whose issues are not all mutually interdependent.

Key words: probabilistic opinion pooling, Sen's liberal paradox, expert rights, liberal rights, unanimity, general aggregation theory

Journal of Economic Literature classification: D71, D82, C11

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1. Introduction

Suppose expert A is a specialist for the treatment of some medical condition α, while expert B specialises on therapies for condition β. Based on substantial expertise, expert A asserts that a certain drug D cures patients with the α diagnosis with probability 75±3%, while expert B testifies that the chances for D to cure β patients are 63±4%. However, both experts A and B agree that — for theoretical pharmacological reasons — D should be more effective in the treatment of β than of α. If we accept the unanimous theoretical testimony of both experts on the pharmacology of drug D, we cannot also accept both of their empirical findings.

Unfortunately, this is not an exceptional case. Consider a panel of experts each making probabilistic judgements about a given set of propositions, according to their expertise. All of the experts have their specific areas

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of competence where they have superior expertise to their colleagues. Is it possible to aggregate their probabilistic judgements in a way that preserves unanimous agreement and follows each expert in her respective area of competence? This paper shows that this is not generally possible — even if one allows for full rationality of the experts (in the sense that they each have a fully specified probability measure on the algebra of propositions under consideration). We shall identify a set of necessary and sufficient conditions, showing that the predicament in the above example is not exceptional. For the case of binary preferential judgements this phenomenon is well-known and commonly referred to as Sen's [**7**] *liberal paradox*. For the case of general binary judgements, Dietrich and List [**2**] have formulated and demonstrated an analogue.

The last decade has seen the emergence of a new strand of literature on general, logical aggregation problems, commonly referred to as *judgement aggregation* (surveyed, e.g., by List and Puppe [**4**] and Mongin [**6**]). In distinction from Arrow's [**1**] classical theory of preference aggregation, this new current of research investigates the aggregation of more general data, such as sets of propositions in some formal logic.

One of the more recent developments is a unified approach to judgement aggregation and probabilistic opinion pooling (in McConway's [**5**] sense), first proposed by Dietrich and List [**3**] and subsequently investigated by other authors.

Now this kind of unification suggests a new set of research questions: Which (im)possibility results from (preference and) judgement aggregation theory possess an analogue in the framework of probabilistic pooling? The present paper could also be regarded as a contribution to this line of research.

The structure of the paper is as follows. We first propose a simplified framework for probabilistic opinion pooling which can be seen as a natural extension of judgement aggregation — with continuous rather than binary truth values. In this simplified framework for probabilistic opinion pooling, we formulate and prove an analogue of Sen's liberal paradox.

2. Formal framework

To introduce the formal framework, let $\mathcal{A}$ be a Boolean algebra, called *agenda*. We shall refer to the elements of $\mathcal{A}$ as *propositions*. Let $\Delta(\mathcal{A})$ be the set of all finitely-additive probability measures on $\mathcal{A}$, called *complete probabilistic judgements*. Let $\mathcal{C}$ be a subset of $\Delta(\mathcal{A})$, called the set of *admissible probability judgements*; $\mathcal{C}$ can be viewed as encoding a common background theory shared by the electorate (which may include the requirement of σ-additivity for the individual probability measures). A subset $\mathcal{C}' \subseteq \mathcal{C}$ is called *cylindrical* or *generated by proposition-wise constraints* if and only if there is a subset $\mathcal{B} \subseteq \mathcal{A}$ and some $\Gamma : \mathcal{B} \to 2^{[0,1]}$ such that $\mathcal{C}'$ is the set of all $P \in \mathcal{C}$ satisfying $P(A) \in \Gamma(A)$ for all $A \in \mathcal{B}$.

Let N be a finite or infinite set, the *electorate*, whose elements will be called *individuals*. The elements of $\mathcal{C}^N$ will be called *profiles*. An *aggregator* is a map with domain $\subseteq \mathcal{C}^N$ and range $\subseteq \mathcal{C}$.

Let $A, B \in \mathcal{A}$. We say that an individual i is *decisive* under an aggregator F for proposition A if and only if $F(\underline{P})(A) = P_i(A)$ for all $\underline{P}$ in the domain of F.

The agenda $\mathcal{A}$ is said to be $\mathcal{C}$-*connected* if and only if for all $A, B \in \mathcal{A}$, one can find $\alpha, \beta \in [0,1]$ and some cylindrical $\mathcal{C}' \subseteq \mathcal{C}$ such that there exist $P, P' \in \mathcal{C}'$ with $P(A) = \alpha$ and $P'(B) = \beta$, but there is no $P^* \in \mathcal{C}'$ satisfying both $P^*(A) = \alpha$ and $P^*(B) = \beta$. In other words, the agenda is $\mathcal{C}$-connected if and only if for every pair of propositions one can find a pair of truth-degree assignments to the propositions and a set of $\mathcal{C}$-admissible constraints which can be satisfied independently but not jointly.)

An aggregator F is said to *preserve unanimity* if and only if one has $F(\underline{P})(A) = \alpha$ whenever $P_i(A) = \alpha$ for all $i \in N$. An aggregator F is said to *respect minimal rights* if and only if there are at least two individuals that are decisive under F on at least one proposition each. An aggregator F has $\mathcal{C}$-*universal domain* if and only if $F : \mathcal{C}^N \to \mathcal{C}$.

3. Main result

Within this simplified formal framework of probabilistic opinion pooling, the liberal paradox can be formulated as follows:

THEOREM 1. *There is no aggregator with $\mathcal{C}$-universal domain respecting minimal rights and preserving unanimity if and only if $\mathcal{A}$ is $\mathcal{C}$-connected.*

4. Discussion

In view of the Theorem, there are two natural ways of harmonising respect for minimal rights with preservation of unanimity. One approach is to ensure that the agenda in question is not connected given the commonly shared theory (expressed by the set $\mathcal{C}$ of admissible probabilistic judgements). Suppose one can find at least two propositions in the agenda which are not connected given the common theory $\mathcal{C}$. Then, in light of our Theorem, there are aggregators which have universal domain, preserve unanimous judgements and respect minimal rights.

Alternatively, one may relinquish the assumption of universal domain, and only allow for profiles where everyone respects the rights of their fellows. If one restricts the aggregator domain to the set of such profiles (called "deferring/empathetic" profiles by Dietrich and List [**2**]), one can easily construct aggregators that preserve unanimity and respect rights.

As our initial motivating example (involving experts on different diagnoses) shows, there are situations where deferral is impossible because one of the propositions falls within the scope of competency of multiple experts. In such cases, a modification of either the agenda $\mathcal{A}$ or the commonly accepted theory $\mathcal{C}$ appears to be the only possibility for enabling rational aggregation. In the said example, one might exclude the commonly accepted theoretical assertion (about the comparative effectiveness of drug D across diagnoses) from the agenda, counting their testimonies as evidence against that theoretical statement. Simultaneous respect for expert rights and unanimity requires careful phrasing of the collective decision problem.

5. Proof

The proof is elementary, as in Sen [**7**] and Dietrich and List [**2**].

Proof. "$\Leftarrow$". Let $\mathcal{A}$ be $\mathcal{C}$-connected, and suppose there were some aggregator F with $\mathcal{C}$-universal domain which respects minimal rights and preserves unanimity. Let $i, j \in N$ and A, B such that under F, i is decisive on A and j is decisive on B. The agenda being connected, we can find $\alpha, \beta \in [0,1]$, some cylindrical $\mathcal{C}'$ and $P, P' \in \mathcal{C}'$ such that $P(A) = \alpha$ and $P'(B) = \beta$, while there is no $P^* \in \mathcal{C}'$ satisfying both $P^*(A) = \alpha$ and $P^*(B) = \beta$. Consider now any profile $\underline{P}$ that satisfies

$$P_i = P, \quad P_j = P', \qquad \forall k \in N \setminus \{i, j\} \qquad P_k \in \{P, P'\}.$$

Since F has $\mathcal{C}$-universal domain, $\underline{P}$ is in the domain of F. By the choice of $\mathcal{C}'$, there is a subset $\mathcal{B} \subseteq \mathcal{A}$ and some $\Gamma : \mathcal{B} :\rightarrow 2^{[0,1]}$ such that $\mathcal{C}'$ is the set of all $Q \in \mathcal{C}$ satisfying $Q(C) \in \Gamma(C)$ for all $C \in \mathcal{B}$. In particular, $P_i(C) \in \{P(C), P'(C)\} \subseteq \Gamma(C)$ for all $i \in N$ and all $C \in \mathcal{B}$. Since F preserves unanimity, it follows that $F(\underline{P})(C) \in \Gamma(C)$ for all $C \in \mathcal{B}$ whence $F(\underline{P}) \in \mathcal{C}'$. On the other hand, by the choice of i, j, A, B as witnesses of the aggregator's respect for minimal rights, we have $F(\underline{P})(A) = P_i(A) = P(A) = \alpha$ and $F(\underline{P})(B) = P_j(B) = P'(B) = \beta$. Hence $P^* = F(\underline{P})$ yields some $P^* \in \mathcal{C}'$ such that $P^*(A) = \alpha$ and $P^*(B) = \beta$, contradiction.

"$\Rightarrow$". Now suppose $\mathcal{A}$ is not $\mathcal{C}$-connected. We have to construct some $F : \mathcal{C}^N \rightarrow \mathcal{C}$ which respects minimal rights and preserves unanimity.

Consider any $\underline{P} \in \mathcal{C}^N$. In order to construct $F(\underline{P})$, let $\mathcal{B}_{\underline{P}} \subseteq \mathcal{A}$ be the set of all propositions on which all P_i agree, i.e.

$$\mathcal{B}_{\underline{P}} = \left\{ C \in \mathcal{A} \ : \ \exists \gamma_{\underline{P},C} \in [0,1] \quad \forall i \in N \quad P_i(C) = \gamma_{\underline{P},C} \right\},$$

and let $\Gamma_{\underline{P}} : \mathcal{B} \rightarrow 2^{[0,1]}$ be such that

$$\forall C \in \mathcal{B}_{\underline{P}} \quad \forall i \in N \qquad \{P_i(C)\} = \Gamma_{\underline{P}}(C).$$

(In other words, $\Gamma_{\underline{P}}(C)$ is the singleton containing $\gamma_{\underline{P},C}$.)

Let $\mathcal{C}'_{\underline{P}}$ be the set of all $P \in \mathcal{C}$ which satisfy $P(C) \in \Gamma_{\underline{P}}(C)$ for all $C \in \mathcal{B}$. By our choice of $\mathcal{B}_{\underline{P}}$ and $\Gamma_{\underline{P}}$ this means that $\mathcal{C}'$ is the set of all $P \in \mathcal{C}$ which adopt the consensus of $\underline{P}$ whenever there is one: $\mathcal{C}'_{\underline{P}}$ is the set of all $P \in \mathcal{C}$ that satisfy

$$(1) \qquad \forall \gamma \in [0,1] \quad \forall C \in \mathcal{A} \qquad ((\forall i \in N \quad P_i(C) = \gamma) \Rightarrow P(C) = \gamma)$$

If we construct F in such a way that $F(\underline{P}) \in \mathcal{C}'_{\underline{P}}$ for all $\underline{P} \in \mathcal{C}^N$, it will preserve unanimity.

Since $\mathcal{A}$ is not connected, there are $A, B \in \mathcal{A}$ such that for all $\alpha, \beta \in [0,1]$ and all cylindrical $\mathcal{C}' \subseteq \mathcal{C}$ the following holds: If there are $P, P' \in \mathcal{C}'$ with $P(A) = \alpha$ and $P'(B) = \beta$, then there is some $P^* \in \mathcal{C}'$ satisfying both $P^*(A) = \alpha$ and $P^*(B) = \beta$.

Fix such $A, B \in \mathcal{A}$, and for an arbitrary $\underline{P} \in \mathcal{C}^N$ let $\mathcal{C}' = \mathcal{C}'_{\underline{P}}$. Given any $j, k \in N$, choose $\alpha, \beta \in [0,1]$ such that $P_j(A) = \alpha$ and $P_k(B) = \beta$. Clearly $P_j, P_k \in \mathcal{C}'_{\underline{P}}$. Thus, there is some $P^*_{\underline{P}} \in \mathcal{C}'_{\underline{P}}$ satisfying both $P^*_{\underline{P}}(A) = \alpha$ and $P^*_{\underline{P}}(B) = \beta$. Define $F(\underline{P}) = P^*_{\underline{P}}$. Then by Equation (1), F preserves unanimity, and it preserves minimal rights, j being decisive for A and k being decisive for B.

□

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