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A full characterization of all deterministic dominant strategy incentive compatible, ex-post individually rational, and ex-post budget balanced direct mechanisms in the public good provision problem with independent private values

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Abstract

In this note I give a full characterization of all deterministic direct mechanisms in the public good provision problem with independent private values that are dominant strategy incentive compatible, ex-post individually rational, and ex-post budget balanced.

Keywords: public good provision, asymmetric information, dominant strategy
JEL codes: C72, D82, H41

1 Introduction

In this note I give a full characterization of all deterministic direct mechanisms in the public good provision problem with independent private values that are dominant strategy incentive compatible, ex-post individually rational, and ex-post budget balanced.

2 Setup

The following is as in Börgers (2013), a special case of the more general d'Aspremont and Gerard-Varet (1979).

A *public good problem* with independent private values is a tuple consisting of the following ingredients: A set I of N agents; for each agent $i \in I$, a set of possible private

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values (for the indivisible non-excludable public good) $\theta_i \in \Theta_i = [\underline{\theta}_i; \overline{\theta}_i] \subset [0, \infty)$, which is private information to the agent; the cost of providing the public good $c > 0$. Let $\Theta = \times_{i \in I} \Theta_i$ and, for all $i \in I$ let $\Theta_{-i} = \times_{j \in I, j \neq i} \Theta_j$ with typical element θ_{-i}.

For a public good problem an *allocation rule* can be written as a function q from the set of value-profiles, Θ, to the set $\{0, 1\}$, where a 1 indicates the provision of the public good and a 0 indicates that the public good is not provided.

A direct mechanism for a public good problem consists of an allocation rule and a set of transfer functions, t_i, one for each agent $i \in I$, where the transfer (possibly negative) is a money amount that is taken from the agent and given to the mechanism designer. The transfer functions are functions from the set of value-profiles to $\mathbb{R}^N$. A direct mechanism for a public good problem is *ex-post budget balanced* (EPBB) if, for all value-profiles, the sum of all transfers to the designer is equal to the cost of providing the public good if the public good is provided and equal to zero otherwise.

A direct mechanism is *dominant strategy incentive compatible* (DSIC) if "truth-telling" (i.e. stating ones type) is a (weakly) dominant strategy. It is *ex-post individually rational* (EPIR) if, for any value-profile, any agent expects a weakly higher payoff from participating in the mechanism than from not participating.

3 Useful Known Results

The following is, almost verbatim, Proposition 4.5 of Börgers (2013).

Proposition 1 *A direct mechanism is dominant strategy incentive compatible (DSIC) if and only for every $i \in I$ and for every $\theta_{-i} \in \Theta_{-i}$, there are functions $\hat{\theta}_i, \tau_i$ and $\hat{\tau}_i$ from the set Θ_{-i} to the set of real numbers $\mathbb{R}$ such that:*

$$\begin{aligned}
\theta_i < \hat{\theta}_i(\theta_{-i}) \quad &\Rightarrow \quad q(\theta_i, \theta_{-i}) = 0 \text{ and } t_i(\theta_i, \theta_{-i}) = \tau_i(\theta_{-i}); \\
\theta_i > \hat{\theta}_i(\theta_{-i}) \quad &\Rightarrow \quad q(\theta_i, \theta_{-i}) = 1 \text{ and } t_i(\theta_i, \theta_{-i}) = \hat{\tau}_i(\theta_{-i}); \\
\theta_i = \hat{\theta}_i(\theta_{-i}) \quad &\Rightarrow \quad q(\theta_i, \theta_{-i}) = 0 \text{ and } t_i(\theta_i, \theta_{-i}) = \tau_i(\theta_{-i}) \text{ or} \\
&\quad\quad\; q(\theta_i, \theta_{-i}) = 1 \text{ and } t_i(\theta_i, \theta_{-i}) = \hat{\tau}_i(\theta_{-i}); \\
\hat{\tau}_i(\theta_{-i}) - \tau_i(\theta_{-i}) \quad &= \quad \hat{\theta}_i(\theta_{-i})
\end{aligned}$$

The proof is in Börgers (2013).

The following is, almost verbatim, Proposition 4.6 of Börgers (2013).

Proposition 2 *A dominant strategy incentive compatible direct mechanism is ex post individually rational (EPIR) if and only for every $i \in I$ and for every $\theta_{-i} \in \Theta_{-i}$:*

$$t_i(\underline{\theta}_i, \theta_{-i}) \leq \underline{\theta}_i q(\underline{\theta}_i, \theta_{-i}).$$

4 Full Characterization

Lemma 1 *Consider a dominant strategy incentive compatible (DSIC) direct mechanism. For all* $i \in I$*, let* $\hat{\theta}_i$ *be defined as in Proposition 1. Then* $\hat{\theta}_i$*, is a weakly decreasing function in all its arguments (i.e. in all* θ_j *with* $j \neq i$*).*

Proof: W.l.o.g. consider agent 1 and consider an arbitrary profile $\theta_{-1} = (\theta_2, ..., \theta_N)$. Now let $\theta_1 = \hat{\theta}_1(\theta_{-1})$. Then, by definition, we have that $q(\theta_1', \theta_{-1}) = 0$ for all $\theta_1' < \theta_1$ and $q(\theta_1', \theta_{-1}) = 1$ for all $\theta_1' > \theta_1$. Now assume that $q(\theta_1, \theta_{-1}) = 1$. W.l.o.g. consider now agent 2. Let $\tilde{\theta}_2 > \theta_2$ and let $\tilde{\theta}_{-1} = (\tilde{\theta}_2, \theta_3, ..., \theta_N)$, or in the case of $N = 2$ simply $\tilde{\theta}_{-1} = \tilde{\theta}_2$. By DSIC (Proposition 1) for agent 2 we must have that $q(\theta_1, \tilde{\theta}_{-1}) = 1$ also. By DSIC (Proposition 1) for agent 1 we then obtain that for all $\theta_1' > \theta_1$ we must have that $q(\theta_1', \tilde{\theta}_{-1}) = 1$ as well. Thus, $\hat{\theta}_1(\theta_2', \tilde{\theta}_{-1}) \leq \theta_1 = \hat{\theta}_1(\theta_{-1})$, which is what we wanted to show.

For the case that $q(\theta_1, \theta_{-1}) = 0$ a similar argument applies. Instead of $\tilde{\theta}_2 > \theta_2$ we need to choose $\tilde{\theta}_2 < \theta_2$ and then go through the appropriate steps. QED

Now to the main result, a version of which has been proven for $N = 2$ in Börgers (2013, Proposition 4.8).

Proposition 3 *Consider a direct mechanism* (q, t) *with the property that there is a* $\theta \in \Theta$ *such that* $q(\theta) = 1$*. This mechanism is dominant strategy incentive compatible (DSIC), ex post individually rational (EPIR), and ex post budget balanced (EPBB) if and only if there are payments* $\hat{\tau}_i \in \mathbb{R}$ *with* $\sum_{i \in I} \hat{\tau}_i = c$ *such that* $q(\theta) = 1$ *and* $t_i(\theta) = \hat{\tau}_i$ *for all* $i \in I$ *if* $\theta_i \geq \hat{\tau}_i$ *for all* $i \in I$*, and* $q(\theta) = 0$ *and* $t_i(\theta) = 0$ *for all* $i \in I$ *otherwise.*

Proof: It is easy to see that the given mechanisms satisfy DSIC, EPIR, and EPBB. In what follows I prove the reverse.

Let $\theta \in \Theta$ be such that $q(\theta) = 0$. Then EPIR (Proposition 2) implies that $t_i(\theta) \leq 0$ for all $i \in I$. EPBB implies that $\sum_{i \in I} t_i(\theta) = 0$. Together this implies that $t_i(\theta) = 0$ for all $i \in I$.

Together with Proposition 1 this implies that, using the terminology of Proposition 1, we have that $\hat{\tau}_i \equiv \hat{\theta}_i$ for all $i \in I$.

Denote by $\overline{\theta} = (\overline{\theta}_1, ..., \overline{\theta}_N)$ the vector of maximum values. Now suppose first that $q(\theta) = 1$ only if $\theta = \overline{\theta}$. Then the result is trivially satisfied.

Thus, suppose that there is a $\theta \in \Theta$ with $\theta \neq \overline{\theta}$ such that $q(\theta) = 1$ and let this θ be otherwise arbitrary. By DSIC (Lemma 1 and the fact that $\hat{\tau}_i \equiv \hat{\theta}_i$) we have that $\hat{\tau}_i(\theta_{-i})$ is weakly decreasing in all its arguments.

DSIC (Proposition 1) implies that $q(\overline{\theta}) = 1$ also. Now EPBB requires that $\sum_{i \in I} \hat{\tau}_i(\theta_{-i}) = \sum_{i \in I} \hat{\tau}_i(\overline{\theta}_{-i}) = c$. But as $\overline{\theta}_i \geq \theta_i$ for all $i \in I$ and as all functions $\hat{\tau}_i$ are weakly decreasing in all its arguments we must have that all $\hat{\tau}_i(\theta_{-i}) = \hat{\tau}_i(\overline{\theta}_{-i})$. As θ with $q(\theta) = 1$ was chosen arbitrarily, this implies that for any such θ we must have that $\hat{\tau}_i(\theta_{-i}) = \hat{\tau}_i(\overline{\theta}_{-i})$. Thus, all payments are equal to the thresholds and all are constant. QED

References

Börgers, T. (2013): "Introduction to the Theory of Mechanism Design," Book Manuscript, Version May 24, 2013.

d'Aspremont, C., and L.-A. Gerard-Varet (1979): "Incentives and incomplete information," *Journal of Public Economics*, 11, 25–45.