Fractal analysis of singularly continuous measures generated by Cantor series expansions

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Contents

Introduction

By Lebesgue's decomposition theorem, any finite Borel measure on $\mathbb{R}$ can be decomposed into the absolutely continuous, singular continuous and pure point parts. Namely,

$$d\mu = d\mu_d + d\mu_{ac} + d\mu_{sc}.$$

Absolutely continuous here is considered with respect to Lebesgue measure such that $d\mu_{ac} = f(x)dx$ for some measurable function f. The pure discrete part, $d\mu_d$, is a countable sum of atomic measures. The singular continuous part, $d\mu_{sc}$, is supported on some set of zero Lebesgue measure, and does not give weight to any individual points $\mu(\{x\}) = 0, \forall x \in \mathbb{R}$. We will concentrate our attention on the last class of pure measures.

The classical analysis such as an integration theory does not give a comprehensive information about singular continuous measures (specification, local description, measurement, classification, geometrical properties). In particular, sets with zero Lebesgue measure are neglected. One can give the following question: "What is the right approach for the investigation of the singular continuous measures?".

A first systematic study of singular probability distributions was done by P. Lévy, A. Wintner, B. Jessen, R. Kershner, R. Salem. Starting from 1990's a big activity both in the fractal analysis and in the analysis of singularly continuous probability distributions arose. It was natural to use Fractal geometry as sensitive tool to the measures with non-trivial supports.

Recent investigations show that singularity is main for many classes of random variables, and absolutely continuous and discrete distributions arise only in exceptional cases (see, e.g., [Zam82], [Pra98], for details). For instance, for the subfamily of infinite Bernoulli convolutions:

$$\psi(\lambda) = \sum_{k=1}^{\infty} \psi_k \cdot \lambda^k, \tag{1}$$

where independent, identically distributed random variables ψ_k take values 0 and 1 with probabilities $\frac{1}{2}$, $\lambda \in (0, \frac{1}{2})$, the corresponding probability measure $\mu_{\psi(\lambda)}$ is singularly continuous.

Possible applications in the spectral theory of self-adjoint operators ([Tri10]) is an extra reason in the intensive investigation of singularly continuous measures. It was proved that Schrödinger type operators with singular continuous spectra are generic for special classes of potentials ([DRJMS94]). Moreover, by using the fractal analysis of the corresponding spectral singularly continuous measures, it is possible to analyze the dynamical properties of the corresponding quantum systems ([Las96]). The number theory, fractal geometry itself provide motivations for intensive investigations of such measures.

The thesis is devoted to the development of the fractal analysis of singularly continuous probability measures as well as to the implementation of such an analysis for special classes of probability distributions, in particular, connecting with generalized infinite Bernoulli convolutions and distributions of random variables with independent $\widetilde{Q}$-symbols.

The first step of the analysis is the study of metric, topological and fractal properties of the spectrum (minimal closed support) of a distribution. It should be mentioned here that usually, it is rather difficult to determine (or even estimate) the Hausdorff dimension for sets from a given family or even for a given set is a rather non-trivial problem (see, e.g., [Bil61, Fal04, Bar07] and references therein).

On the other hand, the topological support is a rather "rough" characteristic for a measure with a complicated local structure. For instance, the random variables $\tau(p) = \sum_{k=1}^{\infty} \frac{\tau_k(p)}{2^k}$ where independent, identically distributed random variables $\tau_k(p)$ take values 0 and 1 with probabilities p and $1-p$, the corresponding probability measures are mutually singular and they are singular with respect to the Lebesgue measure ($p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$). Nevertheless, the spectrum of every distribution $\psi(p)$ coincides with $[0, 1]$. So, it will be interesting to analyze the Haudorff dimension of the measure, i. e., minimal borel support of the measure.

Fractal analysis. The definition of the dimension of a set is central to fractal geometry ([Fal04]). Roughly, dimension indicates how much space a set occupies near to each of its points. Of the big variety of "fractal dimensions" in use, the definition of Hausdorff, based on a construction of Carathdodory, is the oldest and probably the most important. The Hausdorff dimension has the advantage of being defined for any set, and is mathematically convenient, as it is based on measures, which are relatively easy to manipulate.

In this project, we work with an important concept of faithful/nonfaithful covering families for Hausdorff dimension calculation (special relatively narrow families of coverings leading to the classical Hausdorff dimension of an arbitrary subset, see Section 1.3 for details). This concept is very useful for fractal analysis of singularly continuous probability measures, in particular, the determination or estimation of the Hausdorff dimension of sets and probability measures. Chapter 1 contains important definitions, facts and notations for general metric spaces. For the simplicity we shall speak here about subsets from the unit interval. Let Φ be a *fine family of coverings* on $[0, 1]$, i.e., a family of subsets of $[0, 1]$ such that for any $\varepsilon > 0$ there exists an at most countable ε-covering $\{E_j\}$ of $[0, 1]$ with $E_j \in \Phi$. Let us shortly recall that *the α-dimensional Hausdorff measure* of a set $E \subset [0,1]$ w.r.t. a given fine family of coverings Φ is defined by

$$H^{\alpha}(E,\Phi) = \lim_{\varepsilon\to 0}\left[\inf_{|E_j|\le\varepsilon}\left\{\sum_j |E_j|^{\alpha}\right\}\right] = \lim_{\varepsilon\to 0} H^{\alpha}_{\varepsilon}(E,\,\Phi),$$

where the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of E, $E_j \in \Phi$. We remark that, generally speaking, $H^{\alpha}(E,\Phi)$ depends on the family Φ. The family of all subsets of $[0,1]$ and the family of all closed (open) subintervals of $[0,1]$ give rise to the same α-dimensional Hausdorff measure, which will be denoted by $H^{\alpha}(E)$. The quantity

$$\dim_H(E,\,\Phi) = \inf\{\alpha:\ H^{\alpha}(E,\,\Phi) = 0\}$$

is called the Hausdorff dimension of the set $E \subset [0,1]$ w.r.t. a family Φ. If Φ is the family of all subsets of $[0,1]$, or Φ coincides with the family of all closed (open) subintervals of [0,1], then $\dim_H(E,\,\Phi)$ equals to the classical Hausdorff dimension $\dim_H(E)$ of the subset $E \subset [0,1]$.

The notion of comparable net measures are also well known. Roughly speaking, net measures are special cases of $H^{\alpha}(\cdot,\Phi)$, where the family Φ satisfies the following properties: 1) if A_1 and A_2 belong to Φ, then $A_1 \subset A_2$ or $A_2 \subset A_1$ or $A_1 \bigcap A_2 = \emptyset$; 2) Φ is countable; 3) at most a finite number of sets from Φ contain any given set from Φ. Then the corresponding net measure $H^{\alpha}(E,\Phi)$ is said to be comparable to Hausdorff measure if the ratios of measures are bounded above and below. Comparable net measures proved to be very useful in the study of Hausdorff measures (see, e.g., [Bes52, Rog98, Fal04] and references therein).

Let us consider a $\mathbf{N}_k \times \mathbb{N}$ – Cartesian products ("matrix") $\widetilde{Q} = ||q_{ik}||$, where $i \in \mathbf{N}_k = \{0,1,...,N_k-1\}$, $k \in \mathbb{N}$ and $1 < N_k \in \mathbb{N} \cup +\infty$. Let

1. $q_{ik} > 0$, for all $i \in \mathbf{N}_k$ and $k \in \mathbb{N}$;

2. $\prod_{k=1}^{\infty} \max_{i\in\mathbf{N}_k} \{q_{ik}\} = 0$;

3. $\sum\limits_{i\in\mathbf{N}_k} q_{ik} = 1$.

There are several well-known subclasses of Cartesian products $\widetilde{Q}$. The $\widetilde{Q}$ – "matrix" is denoted by:

1. S if $N_k = s-1$ and $q_{ik} = \frac{1}{s}$, where $i \in \mathbf{N}_k,\ k \in \mathbb{N},\ s \in \mathbb{N}\backslash\{1\}$;

2. Q if $N_k = s-1$ and $q_{ik} = q_i$, where $i \in \mathbf{N}_k,\ k \in \mathbb{N},\ s \in \mathbb{N}\backslash\{1\}$;

3. Q^* if $N_k = s-1$, where $i \in \mathbf{N}_k,\ k \in \mathbb{N},\ s \in \mathbb{N}\backslash\{1\}$.

4. Q_∞ if $N_k = \infty$ and $q_{ik} = q_i$, where $i \in \mathbf{N}_k,\ k \in \mathbb{N}$.

5. $\widetilde{Q}^*$ if $N_k \in \mathbb{N}$, where $k \in \mathbb{N}$.

It is known (see Section 1.2) that for any point $x \in [0,1]$ there is a sequence $i_k(x) \in \mathbf{N}_k$ such that

$$x = a_{i_1(x)} + \sum_{k=2}^{\infty} \left[a_{i_k(x)} \prod_{j=1}^{k-1} q_{i_j(x)j} \right] =: \Delta^{\widetilde{Q}}_{i_1(x)i_2(x)\ldots i_k(x)\ldots}, \tag{2}$$

where $a_{i_k(x)} = \sum\limits_{s=0}^{i_k(x)-1} q_{sk}$ under conditions $i_k(x) > 0$ and $a_0 = 0$.

The representation of the real number x in the form (2) is said to be the $\widetilde{Q}$ - expansion (representation) of the point $x \in [0,1]$. The $\widetilde{Q}$ - expansion is a broad generalization of classical binary representation of real numbers.

Conditions for a fine covering family to be faithful were studied by many authors (see, e.g., [Bil61, Cut88, AT05] and references therein). First steps in this direction have been done by A. Besicovitch ([Bes52]), who showed the faithfulness for the family of cylinders of binary expansion. His result was extended by P. Billingsley ([Bil61]) to the family of S-adic cylinders, by M. Pratsiovytyi ([TP92]) to the family of Q-cylinders, and by S. Albeverio and G. Torbin ([AT04]) to the family of Q^*-cylinders for those matrices Q^* whose elements are bounded from zero $\inf_k \{q_{0k}, q_{(s-1)k}\} > 0$. Some general sufficient conditions for the faithfulness of a given family of coverings are also known ([Cut88]).

Let us mentioned here that all these results were obtained by using the "standard approach": if for a given family Φ there exist positive constants $\beta \in \mathbb{R}$ and $N^* \in \mathbb{N}$ such that for any interval $B = (a, b)$ there exist at most N^* sets $B_j \in \Phi$ which cover (a, b) and $|B_j| \leq \beta \cdot |B|$, then the family Φ is faithful. It is clear that all above mentioned families of net-coverings are even comparable.

The family of cylinders of the classical continued fraction expansion can be considered as a rather unexpected example of non-faithful one-dimensional net-family of coverings ([PT]). By using the approach which has been invented by Yuval Peres to prove non-faithfulness of the family of continued fraction cylinders (see [PT]), in [AKNT] the authors have proven the non-faithfulness for the family of cylinders of Q_∞-expansion with polynomially decreasing elements $\{q_i\}$. The latter families of coverings give examples of non-comparable net measures. So, it is natural to ask about the existence of faithful covering families which are not comparable (see Section 2.3).

DP-transformations.

Erlangen program (Klein) of the group theoretic approach to geometry is well known. What is the "fractal geometry" from this point of view? The monograph [Fal04] contains an attempt to answer the question saying that "... one approach to fractal geometry is to regard two sets as "the same" if there is a bi-Lipschitz mapping between them", i.e., fractal geometry is in this sense the study of invariants of bi-Lipschitz transformations (and, thus, affine geometry may be considered as a part of fractal geometry). In [APT04] a view on fractal geometry was proposed in the same spirit, but with a more general definition of allowable mappings. It was shown that the group G of all DP-transformations (one to one mappings which preserve the Hausdorff dimension of every subset) is essentially larger than the group of bi-Lipschitz transformations, and the smoothness and bi-Lipschitz properties of transformations are very rough sufficient conditions for dimension preservation. A series of papers (see, e.g., [APT04], [APT08], and references therein) is devoted to the development of a general theory of DP-transformations and to the finding of conditions for the Hausdorff dimension preservation in special classes of transformations. It can be proven (see, e.g., [APT08]) that a one-dimensional transformation g is a DP-transformation of R^1 if and only if g preserves the Hausdorff dimension of every subset of any interval. So, without loss of generality it is enough to study only DP-transformations of the unit interval. It is also clear that an arbitrary continuous transformation g of $[0,1]$ is either a strictly increasing distribution function F_θ of some random variable θ or it is of the form $g = 1 - F_\theta$. Because of this reason it is enough to investigate DP-properties of the distribution functions of random variables θ whose spectra S_θ coincide with $[0,1]$. Earlier such DP-transformations g were studied where both sets $N_0 = \left\{x : g^{'}(x) = 0\right\}$ and $N_\infty = \left\{x : \lim\limits_{\varepsilon \to 0} \frac{g\,(x+\varepsilon) - g(x)}{\varepsilon} = +\infty\right\}$ are either finite or they form an at most countable set.

A class of distribution functions of random variables with independent S-adic digits was analyzed in detail in [APT08], where necessary conditions and sufficient conditions for dimension preservation under corresponding probability distribution functions were found. Relations between the Hausdorff dimension of the corresponding probability measures, the entropy of probability distributions, and their DP-properties were also discussed in [APT08]. In particular, it was proved that the superfractality ($\dim_H \mu = 1$) of a probability distribution μ is a necessary condition for the Hausdorff dimension preservation under the corresponding probability distribution function. Paper [Tor07] contains a generalization of these results to the case of random variables with independent Q-symbols.

Besides of pure theoretical reasons for the development of the general theory of DP-transformations (for instance, for the creation of an axiomatic theory of fractal geometry), there exists an additional reason for such a study connected with the application of DP-transformations to the construction of new methods for the determination of the Hausdorff dimension of concrete subsets (see, e.g., [APT04]).

Main results of Chapter 1. In Section 1.3.3 we give an equivalent conditions for the Hausdorff dimension faithfulness of covering families. Section 1.3.4 is devoted to general necessary conditions for a Hausdorff dimension faithfulness of covering families:

Theorem 1.2 *Let W be a bounded subset of a metric space $(\mathbf{M}, \rho)$. Let $\Phi := \Phi_W$ and $\Psi := \Psi_W$ be fine covering families on W and Ψ be a faithful on W for calculation of Hausdorff - Besicovitch dimension. Assume that there exists a positive constant C and a function $f(x) : R_+ \to \mathbb{N}$ such that*

1. for any set $I \in \Psi$ there exist at most $f(|I|)$ subsets

$$\triangle_1^I,\ \triangle_2^I,\ ...,\ \triangle_{l(I)}^I \in \Phi,$$

$$l(I) \leq f(|I|),\ |\triangle_j^I| \leq |I| \textit{ and } I \subset \bigcup_{j=1}^{l(I)} \triangle_j^I;$$

2. for any $\delta \in (0, \alpha)$ there exists $\varepsilon_1(\delta) > 0$ such that

$$f(|I|) \cdot (|I|)^{\delta} \leq C, \textit{ for any set } I \in \Psi \textit{ with diameter less then } \varepsilon_1(\delta).$$

Then the family Φ is faithful on W for calculation of Hausdorff - Besikovich dimension, i. e., $\dim_H E = \dim_H(E, \Phi),\ \forall E \subset W$.

Main results of Chapter 2. Chapter 2 is devoted to the case of $\widetilde{Q}$ - expansion, where a sequence $\{n_k\}_{k\in\mathbb{N}}$ satisfies $1 < n_k \in \mathbb{N}$ and $q_{ik} = \frac{1}{n_k}$, $\forall i \in \{0,\ 1,\ ...\ n_k - 1\}$. In this case $\widetilde{Q}$ - expansion coincides with classical Cantor expansion (see [Can69, ER59, Man10]). So, let $\{n_k\}_{k\in\mathbb{N}}$ with $n_k \in \mathbb{N}\backslash\{1\}$, $k \in \mathbb{N}$. Then the expansion of $x \in [0, 1]$ in the following form

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k}, \ \alpha_k \in \{0, \ 1, \ ..., \ n_k - 1\}$$

is called *Cantor expansion of* x.

Let $\mathcal{A}$ be the family of all possible semi - closed intervals (cylinders), i.e.,

$$\mathcal{A} := \{E : E = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}, \ k \in \mathbb{N}, \ \alpha_i \in \{0, ..., n_i - 1\}, \ i = 1, \ 2, \ ..., \ k\},$$

where

$$\Delta_{\alpha_1 \alpha_2 \ldots \alpha_k} := \left\{ x : x \in \left[\sum_{i=1}^{k} \frac{\alpha_i}{n_1 n_2 \ldots n_i}, \frac{1}{n_1 n_2 \ldots n_k} + \sum_{i=1}^{k} \frac{\alpha_i}{n_1 n_2 \ldots n_i} \right) \right\}.$$

The main result of Section 2.1 is the sharp condition for the Hausdorff dimension faithfulness of the Cantor series expansion coverings $\mathcal{A}$.

Theorem 2.1 *The family* $\mathcal{A}$ *of Cantor coverings of the unit interval is faithful for the Hausdorff dimension calculation if and only if*

$$\lim_{k \to \infty} \frac{\ln n_k}{\ln (n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})} = 0.$$

To the best of our knowledge this theorem gives the first sharp condition of the faithfulness for a class of covering families containing both faithful and non-faithful ones.

Applying the latter theorem and methods from [AT05], we get the Hausdorff dimension of the probability distribution μ_τ of the random variable τ with independent digits of the Cantor series expansion (*Random Cantor expansion*), i.e.,

$$\tau = \sum_{k=1}^{\infty} \frac{\tau_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k},$$

where independent random variables τ_k take values $0, 1, ..., n_k - 1$ with probabilities p_{0k},

p_{1k}, ..., $p_{n_k-1,k}$, respectively ($\sum_{i=0}^{n_k-1} p_{i,k} = 1$ and $1 < n_k \in \mathbb{N}$, $\forall k \in \mathbb{N}$).

Theorem 2.2 *Let*

$$\sum_{k=1}^{\infty} \left(\frac{\ln n_k}{\ln \prod_{i=1}^{k} n_i} \right)^2 < \infty, \tag{3}$$

then the Hausdorff dimension of the probability distribution μ_τ of the random variable τ with independent digits of the Cantor series expansion is equal to

$$\dim_H(\mu_\tau) = \lim_{k\to\infty} \frac{H_k}{\ln(n_1 n_2 ... n_k)},$$

where $H_k = \sum_{j=1}^{k} h_j$, $\forall k \in \mathbb{N}, h_j = -\sum_{i=0}^{n_j-1} p_{ij} \ln p_{ij}$, $\forall j \in \mathbb{N}$ *and* $0 \ln 0 := 0$.

Applying our results, we show that a class of faithful net-coverings essentially wider that the class of comparable ones. We construct, in particular, rather simple examples of faithful families $\mathcal{A}$ of net-coverings which are "extremely non-comparable" to the Hausdorff measure (see Section 2.3).

In Section 2.5, we find conditions for the distribution functions of Random Cantor series to be DP-transformations.

Theorem 2.5 *Let* $\sup n_k < \infty$. *Then the distributional function F_τ of random Cantor series τ preserves the Hausdorff dimension of any subset of the unit interval iff*

$$\begin{cases} \dim_H \mu_\tau = 1; \\ \overline{\lim_{k\to\infty}} \dfrac{\sum_{j \in T_k^{(1)}} \ln \frac{1}{p_j}}{k} = 0. \end{cases}$$

Generalized infinite Bernoulli convolutions. Let $\mu_\xi = \mu$ – be the distribution of the random variable

$$\xi = \sum_{k=1}^{\infty} \xi_k a_k, \tag{4}$$

where $\sum_{k=1}^{\infty} a_k$ is a convergent series whose terms are nonnegative and where ξ_k are independent random variables assuming two values 0 and 1 with probabilities p_{0k} and p_{1k} respectively. The distribution μ_ξ is called a generalized infinite Bernoulli convolution.

A theorem due to Jessen and Wintner [JW35] says that the distribution of ξ is pure. A theorem due to Lévy [Lev31] provides necessary and sufficient conditions for μ_ξ to be purely discrete, namely the measure μ_ξ is discrete if and only if

$$M = \prod_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} > 0. \tag{5}$$

The criteria for ξ to be purely absolutely continuous with respect to the Lebesgue measure (or purely singular) are not known yet even in the case of random power series ($a_k = \lambda^k$ and $p_{0k} = \frac{1}{2}$). The probability measure μ_λ, which corresponds to such a random variable ξ_λ is known as "*infinite symmetric Bernoulli convolution*". Measures of this form have been studied since 1930's from the pure probabilistic point of view as well as for their applications in harmonic analysis, in fractal analysis and in the theory of dynamical systems.

Surveys of problems and solutions in this field are given in [PSS00, GPT09]. Some applications of infinite Bernoulli convolutions are discussed in [AZ91, PSS00]. If the series $\sum_{k=1}^{\infty} a_k$ converges fast enough that is, if

$$a_k \geq r_k := \sum_{i=k+1}^{\infty} a_i$$

for all sufficiently large k, then the Lebesgue structure and fractal properties of Bernoulli convolutions are studied rather well (see [Coo98, AT08]). In contrast, if the inequality $a_k < r_k$ occurs for an infinite number of indices k, then these problems are studied much less. The main problem in this case is how to obtain fine properties of the Bernoulli convolutions for which almost all (with respect to the Lebesgue measure or in the sense of the

Hausdorff dimension) points of the spectrum have continuum many different expansions of the form $\sum\limits_{k=1}^{\infty}\omega_k a_k$, where $\omega_k \in \{0,1\}$. The probability measures of this type belong to the class of the so-called Bernoulli convolutions with essential intersections([GPT09]). The main aim of Chapter 3 is to prove the singularity of the distribution of the random variable ξ and to investigate its fine properties for the case where the sequence $\{a_k\}$ is such that

$$\forall k \in N,\ \exists s_k \in N_0 := N \cup \{0\}: \quad a_k = a_{k+1} = ... = a_{k+s_k} \geq r_{k+s_k}, \tag{6}$$

and moreover $s_k > 0$ for an infinite number of indices k. We shall say that a generalized infinite Bernoulli convolution (4) is a LT - Bernoulli convolution if condition (6) is satisfied.

Let us stress that the case when $a_k < r_k$ occurs for an infinite number of indices k is essentially more complicated. Nevertheless, we perform a complete fractal analysis of LT-Bernoulli convolution based on the developed the $\widetilde{P}-\widetilde{Q}$ approach (see, e.g., [AT05, AKPT06]).

Main results of Chapter 3. In Section 3.2, we present a complete description of Lebesgue structure of LT-Bernoulli convolution. Due to results of [AKPT11] and by using the following two technical lemmas, we show the singularity of correspondent infinite Bernoulli convolutions. Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative integer numbers such that $i \in \{k_n\}_{n\in\mathbb{N}}$ if and only if $s_i = 0$. Also let $l_n = k_n - k_{n-1}$, $k_0 = 0$.

Lemma 3.1 *Let* $R_{l_n} := \{0,1\}^{l_n}$ *and* $\delta := (\delta_1, \delta_2, ..., \delta_{l_n}) \in R_{l_n}$, *where* $|\delta| = \sum_{k=k_{n-1}+1}^{k_n}\delta_k$ *for all* $n \in \mathbb{N}$. *Then there is a function* $\varphi(n)$ *such that*

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n}p_{\delta_k,k}\right)} \leq \varphi(n) \to 0\ (n\to\infty),$$

where $0 < p_{0k} < 1,\ p_{1k} = 1 - p_{0k}$ *for all* $k \in \mathbb{N}$.

Lemma 3.2 *Let* $R_n = \{0,1\}^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in R_n$, $|\delta| = \sum_{k=1}^{n} \delta_k$ *for all* $n \in \mathbb{N}\backslash\{1\}$. *Let* $(p_{01},\ p_{02},\ ...,\ p_{0n}) \in [0,1]^n$, $(p_{11},\ p_{12},\ ...,\ p_{1n}) = (1-p_{01},\ 1-p_{02},\ ...,\ 1-p_{0n})$. *Then*

$$\upsilon(p_{01}, ..., p_{0n}) = \sqrt{\frac{1}{n+1}} \sum_{i=0}^{n} \sqrt{\sum_{\delta \in R_n, |\delta|=i} \left(\prod_{k=1}^{n} p_{\delta_k, k} \right)} \leq K_n < 1,$$

where K_n *is a constant that depends on* n.

Section 3.3 is devoted to the fractal faithfulness of covering families (see the definition below) on ξ-spectrum. In Section 3.4, we determine the Hausdorff dimension of the set of those points for which there exist continuum many of different representations (Theorem 3.4). We also determine the dimension of the set of points that have a finite number of representations.

Fine fractal properties of LT-Bernoulli convolutions are shown in Section (3.5).

Theorem 3.5 *If*

$$\sum_{n=1}^{\infty} \left(\frac{\ln r_{k_{n-1}}}{\ln r_{k_n}} - 1 \right)^2 < \infty,$$

then the Hausdorff dimension of the probability distribution μ_ξ *of the random variable* ξ *is equal to*

$$\dim_H(\mu_\xi) = \varliminf_{n \to \infty} \frac{H_n}{-\ln r_{k_n}},$$

where $H_n = \sum\limits_{j=1}^{n} h_j$, $h_j = -\sum\limits_{i=0}^{m_j - 1} \tilde{p}_{ij} \ln \tilde{p}_{ij}$.

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Chapter 1

Singular probability measures and Hausdorff faithfulness of covering families

1.1 Hausdorff measure and Hausdorff dimension

First, let us recall some basic definitions, facts and notations [Rog98, Fal04, Edg08].

Definition 1.1. Let W be a bounded subset of a metric space $(\mathbf{M}, \rho)$. By $d(W)$ denote the diameter of the set W, i. e.,

$$d(W) := \sup\{\rho(x, y) : x, y \in E\}.$$

Definition 1.2. Let ε be a positive constant. A finite or countable family $\{E_j\}$ of sets is called an *ε-covering* of a set E if $E \subset \bigcup\limits_j E_j$, where

$$d\,(E_j) \leq \varepsilon,\ E_j \subset \mathbf{M},\ \forall j \in \mathbb{N}.$$

Definition 1.3. A family of subsets Φ_W is called a *fine family of coverings* of a bounded subset W if for any $\varepsilon > 0$ there exists an ε-covering $\{E_j\}$ of the set W with

$$E_j \in \Phi_W, \forall j \in \mathbb{N}.$$

Remark 1.1. A fine family of coverings does not exists for every metric space. We give a transparent example. Let $(\mathbf{M}, \rho)$ be a metric space such that $\mathbf{M} = (-\infty, +\infty)$ and

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

If E is a continuum set and $E \subset [0, 1]$, then an ε - covering of E with $\varepsilon < 1$ does not exist.

In the sequel, only metric spaces with fine covering families are considered.

Definition 1.4. Let α and ε are positive numbers. The $\alpha - \varepsilon$ - *Hausdorff measure* of bounded set E is defined by

$$H_\varepsilon^\alpha(E) := \inf_{d(E_j) \leq \varepsilon} \left\{ \sum_j d\left(E_j\right)^\alpha \right\},$$

where the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of E, $E_j \subset \mathbf{M}$.

Definition 1.5. Let α be a positive number. The α - *dimensional Hausdorff measure* (*Hausdorff measure*) of a bounded set E is defined by

$$H^\alpha(E) := \lim_{\varepsilon \to 0} H_\varepsilon^\alpha(E).$$

Obviously, the limit $H^\alpha(E)$ is well defined (see Remark 1.1).

Let us recall some basic properties of the α - dimensional Hausdorff measure (see for details [Rog98, Ch. 2], [Fal04, Ch. 2]). Fix $\beta > \alpha > 0$.

1. If $H^{\alpha}(E) < \infty$, then $H^{\beta}(E) = 0$.

2. If $H^{\beta}(E) > 0$, then $H^{\alpha}(E) = \infty$.

3. Suppose a function $f : \mathbf{M} \to \mathbf{M}$ is similarity transformation

$$\rho(f(x), f(y)) = c \cdot \rho(x, y)$$

with a scale factor $c > 0$. Then $H^{\alpha}(f(E)) = c^{\alpha} H^{\alpha}(E)$.

4. Suppose a function $f : \mathbf{M} \to \mathbf{M}$ satisfies a following Holder condition

$$\rho(f(x), f(y)) \leq c(\rho(x, y))^{s}$$

for all $x, y \in E$ and some fixed $c > 0, s > 0$. Then

$$H^{\alpha/s}(f(E)) \leq c^{\alpha/s} H^{\alpha}(E).$$

5. If $E \subset E'$, then $H^{\alpha}(E) \leq H^{\alpha}(E')$.

6. If $E \subset \bigcup\limits_{j \in \mathbb{N}} E_j$, then $H^{\alpha}(E) \leq \sum\limits_{j \in \mathbb{N}} H^{\alpha}(E_j)$.

The following definition was introduced by F. Hausdorff in 1918 ([Hau18]).

Definition 1.6. The nonnegative number α is called the Dimension of a set E if

$$0 < H^{\alpha}(E) < \infty. \tag{1.1}$$

F. Hausdorff calculated the Dimension of the Cantor set

$$C = \left\{ x : x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k},\ \alpha_k \in \{0, 2\},\ \forall k \in \mathbb{N} \right\}.$$

However Definition 1.6 is not well defined for all sets (see Section 1.4). A. Besicovitch constructed the first example of such sets. A. Besicovitch proposed a following definition of a metric dimension.

Definition 1.7. The nonnegative number

$$\dim_H(E) := \inf\{\alpha : H^\alpha(E) = 0\}$$

is called the *Hausdorff dimension* (Hausdorff dimension) of a set $E \subset W$.

This definition is well known in modern mathematics.

Remark 1.2. The Hausdorff dimension of a set $E \subset W$ is equal to

$$\dim_H(E) = \sup\{\alpha : H^\alpha(E) = +\infty\}$$

except the case when $\dim_H(E) = 0$.

Let us recall some properties of the Hausdorff dimension:

1. If E_1 and E_2 are geometrically similar sets then $\dim_H(E_1) = \dim_H(E_2)$;
2. If $E_1 \subset E_2$ then $\dim_H(E_1) \leq \dim_H(E_2)$;
3. $\dim_H(E) = 0$ if E is finite or countable set;
4. $\dim_H(\bigcup\limits_n E_n) = \sup\limits_n \dim_H(E_n)$.

It is well known that the α - dimensional Hausdorff measure in a case when $(\mathbf{M}, \rho) = \mathbb{R}^1$ and $\alpha = 1$ coincides with Lebesgue outer measure. Hence a set with positive Lebesgue measure has the Hausdorff dimension 1.

1.2 Singular probability measures

Let $(\Omega, \mathcal{A})$ be a measurable space with probability measures μ and ν. Let $\{\omega\} \in \mathcal{A}, \forall \omega \in \Omega$.

Definition 1.8. The measure μ is said to be discrete if there is at most countable set $S \in \Omega$ with $\mu(S) = 1$.

Definition 1.9. The measure μ is said to be continuous if $\forall \omega \in \Omega$ we have $\mu(\{\omega\}) = 0$.

Definition 1.10. The measure μ is said to be absolutely continuous with respect to the measure ν ($\mu \ll \nu$) if $\mu(E) = 0$ for all E from

$$\{E \in \mathcal{A} : \nu(E) = 0\}.$$

Definition 1.11. The measure μ is said to be singular with respect to the measure λ ($\mu \perp \nu$) if there is a set $E \in \mathcal{A}$ such that

$$\nu(E) = 0 \text{ and } \mu(E) = 1.$$

In this section we will speak about sets and measures on the unit interval $[0, 1]$. That is $\Omega = [0, 1]$, λ is a Lebesgue measure on $[0, 1]$, $\mathcal{A}$ is a σ - algebra of Lebesgue measurable sets on $[0, 1]$.

We will discuss singular and absolute continuous measures with respect to Lebesgue measure λ.

Definition 1.12. The measure μ is said to be singular continuous if μ is continuous and there is a set S such that $\lambda(S) = 0$ and $\mu(S) = 1$.

Remark 1.3. If ξ is a random variable with probability distribution function F_ξ, then the corresponding probability measure μ_ξ is singular with respect to Lebesgue measure λ if and only if $F_\xi^{'}(x) = 0$ for λ-almost all x.

Theorem (Lebesgue) *Let μ be a probability measure on $[0,1]$. Then there is a unique decomposition of μ*

$$\mu = \alpha_1\mu_d + \alpha_2\mu_{ac} + \alpha_3\mu_{sc}, \tag{1.2}$$

where μ_d is a discrete probability measure, μ_{ac} is a absolutely continuous probability measure, μ_{sc} is a singularly continuous probability measure and

$$\alpha_i \geq 0,\ \forall i \in \{1,\ 2,\ 3\},\ \alpha_1 + \alpha_2 + \alpha_3 = 1.$$

The class of singular continuous probability measures is less studied. However there are a significant number of papers devoted to this class (see [Cat82, Cha63, Cha64, Coo98, Erd39, ES91, Gar62, Gar63, Gil31, HK38, Hen92, HT29, Hof95, Hu97, HL90, JW35, KS58, Kin58, KP64, KP65, KP66a, KP66b, Lau92, Lau93, LN99b, LP96a, LN99a, LP96b, Lev31, Lit36, MR97, MR98, Ren59, RT59, RT63, Sal42, Sal43a, Sal43b, Sal52, Sie11a, Sie11b, Sie13, Sie14, Str91, STZ95, Tuc64, WW38, Win34, Win35, Zam82]).

The study and applications of special classes of singular measures are also important ([MR98, Pra95, Rei82a, Rei82b, Rei86, Tak78, TU95, TP92, AZ91, Coo98, Hu97, HL90, Lau93, LN98, LN99a, LP94, LP96a, MS98, PSS00, PS96, PS98]).

In particular, we will discuss the class of probability measures generated by random variables with independent symbols over dynamic alphabets. From one point of view, this class contains measures with not trivial fractal properties; from another point of view, there is direct connection with generalized Bernoulli convolutions.

We will need the following notations, assumptions and definitions.

Let us consider a $\mathbf{N}_k \times \mathbb{N}$ – Cartesian products ("matrix") $\widetilde{Q} = ||q_{ik}||$, where $i \in \mathbf{N}_k = \{0, 1, ..., N_k - 1\}$, $k \in \mathbb{N}$ and $1 < N_k \in \mathbb{N}$. Let

$$q_{ik} > 0,\ \forall i \in \mathbf{N}_k,\ k \in \mathbb{N}; \tag{1.3}$$

$$\prod_{k=1}^{\infty} \max_{i \in \mathbf{N}_k} \{q_{ik}\} = 0; \tag{1.4}$$

$$\sum_{i \in \mathbf{N}_k} q_{ik} = 1. \tag{1.5}$$

There are several well-known subclasses of Cartesian products $\widetilde{Q}^*$. The $\widetilde{Q}^*$ – "matrix" is denoted by:

1. S if $\mathbf{N}_k = \{0, \ ..., \ s-1\}$, $\forall k \in \mathbb{N}$, and $q_{ik} = \frac{1}{s}, \ s \in \mathbb{N} \backslash \{1\}$;

2. Q if $\mathbf{N}_k = \{0, \ ..., \ s-1\}$, $\forall k \in \mathbb{N}$, and $q_{ik} = q_i, s \in \mathbb{N} \backslash \{1\}$;

3. Q^* if $\mathbf{N}_k = \{0, \ ..., \ s-1\}$, $\forall k \in \mathbb{N}, \ s \in \mathbb{N} \backslash \{1\}$.

With a $\widetilde{Q}^*$ – "matrix" we consecutively perform decompositions of the segment $[0,1)$ and unit interval $[0,1]$ as follows.

Step 1. We decompose unit interval $[0,1]$ (from left to right) into the union of closed intervals $\Delta_{i_1}^{\widetilde{Q}^*}$, $i_1 \in \mathbf{N}_1$ of the length $\left|\Delta_{i_1}^{\widetilde{Q}^*}\right| = q_{i_1 1}$,

$$[0,1] = \bigcup_{i_1 \in \mathbf{N}_1} \Delta_{i_1}^{\widetilde{Q}^*},$$

without common interior points. Each interval $\Delta_{i_1}^{\widetilde{Q}^*}$ is called a *closed 1-rank interval* (1-rank cylinder).

In the same way we can decompose unit semi-interval $[0,1)$ (from left to right) into the union of semi-closed intervals without common points. Each interval is called a *semi-closed 1-rank interval* (1-rank cylinder). We will use the same notations $\Delta_{i_1}^{\widetilde{Q}^*}$ for simplicity.

Step $k \geq 2$. We decompose (from left to right) each closed $(k-1)$-rank interval

$\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_{k-1}}$ into the union of closed intervals $\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_k}$,

$$\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_{k-1}} = \bigcup_{i_k \in \mathbf{N}_k} \Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_k},$$

where their lengths

$$\left|\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_k}\right| = q_{i_1 1} \cdot q_{i_2 2} \cdots q_{i_k k} = \prod_{s=1}^{k} q_{i_s s} \tag{1.6}$$

are related as follows

$$\left|\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_{k-1} 0}\right| : \left|\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_{k-1} 1}\right| : \cdots : \left|\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_{k-1} i_k}\right| = q_{0k} : q_{1k} : \cdots : q_{i_k k}.$$

Each closed interval

$$\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_k} \tag{1.7}$$

is called a *closed k-rank interval* (k-rank cylinder).

As well we can decompose semi-closed interval $\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_{k-1}}$ (from left to right) into the union of semi-closed intervals without common points. Each interval is called a *semi-closed k-rank interval* (k-rank cylinder). We will use the same notations $\Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_k}$ for simplicity.

By assumptions (1.4) and (1.6), for any sequence of indices $\{i_k\}_{k \in \mathbb{N}}$, $i_k \in \mathbf{N}_k$, there corresponds the sequence of embedded cylinders

$$\Delta^{\widetilde{Q}^*}_{i_1} \supset \Delta^{\widetilde{Q}^*}_{i_1 i_2} \supset \cdots \supset \Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_k} \supset \cdots$$

such that $|\Delta^{\widetilde{Q}^*}_{i_1 \ldots i_k}| \to 0$ as $k \to \infty$. Therefore, there exists a unique point $x \in [0,1]$ (except for a case of semi-closed cylinders with $i_k = N_k - 1$ under the condition $\forall k > k_0 \in \mathbb{N}$) belonging to all intervals

$$\Delta^{\widetilde{Q}^*}_{i_1}, \Delta^{\widetilde{Q}^*}_{i_1 i_2}, \ldots, \Delta^{\widetilde{Q}^*}_{i_1 i_2 \ldots i_k}, \ldots .$$

Conversely, for any point $x \in [0,1)$ there exists a unique sequence of embedded semi - closed intervals

$$\Delta^{\widetilde{Q}^*}_{i_1} \supset \Delta^{\widetilde{Q}^*}_{i_1 i_2} \supset ... \supset \Delta^{\widetilde{Q}^*}_{i_1 i_2 ... i_k} \supset ... ,$$

containing x and for any point $x \in [0,1]$ there exists a sequence of embedded closed intervals

$$\Delta^{\widetilde{Q}^*}_{i_1} \supset \Delta^{\widetilde{Q}^*}_{i_1 i_2} \supset ... \supset \Delta^{\widetilde{Q}^*}_{i_1 i_2 ... i_k} \supset ... ,$$

containing x i.e.,

$$x = \bigcap_{k=1}^{\infty} \Delta^{\widetilde{Q}^*}_{i_1 i_2 ... i_k} = \bigcap_{k=1}^{\infty} \Delta^{\widetilde{Q}^*}_{i_1(x) i_2(x) ... i_k(x)} =: \Delta^{\widetilde{Q}^*}_{i_1(x) i_2(x) ... i_k(x) ...}. \tag{1.8}$$

By the above, for any point $x \in [0,1]$ there is a sequence $i_k(x)$, $i_k(x) \in \mathbf{N}_k$ such that

$$x = a_{i_1(x)} + \sum_{k=2}^{\infty} \left[a_{i_k(x)} \prod_{j=1}^{k-1} q_{i_j(x) j} \right], \tag{1.9}$$

where $a_{i_k(x)} = \sum\limits_{s=0}^{i_k(x)-1} q_{sk}$ when $i_k(x) > 0$ and $a_0 = 0$.

The following definition is given for the case of closed embedded intervals.

Definition 1.13. The expressions (1.8) and (1.9) are called the $\widetilde{Q}^*$ – *expansion* (representation) of the point $x \in [0,1]$.

the $\widetilde{Q}^*$ – expansion allows to construct in a convenient way a wide classes of fractals on R^1, R^n and other mathematical objects with fractal properties (see [Pra98, AKPT06, AKPT11, Tor05]).

There are some special cases of $\widetilde{Q}^*$ - expansions:

1. If $\widetilde{Q}^* = S$, then the $\widetilde{Q}^*$ - expansion coincides with classical s - adic expansion or s-adic representation.

2. If $\widetilde{Q}^* = Q$, then the $\widetilde{Q}^*$ - expansion coincides with Q - expansion.

3. If $\widetilde{Q}^* = Q^*$, then the $\widetilde{Q}^*$ - expansion coincides with Q^* - expansion.

Chapter 2 is devoted to the case where a sequence $\{n_k\}_{k\in\mathbb{N}}$ satisfies $1 < n_k \in \mathbb{N}$ and $q_{ik} = \frac{1}{n_k}$, $\forall i \in \{0,\ 1,\ ...\ n_k - 1\}$. In this case $\widetilde{Q}^*$ - expansion coincides with classical Cantor expansion(see [Can69, ER59, Man10]). We now recall the definition of Cantor expansion. Let $\{n_k\}_{k=1}^{\infty}$ with $n_k \in \mathbb{N}\backslash\{1\}$, $k \in \mathbb{N}$; then the expansion of $x \in [0, 1]$ in the following form

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k},\ \alpha_k \in \{0,\ 1,\ ...,\ n_k - 1\}$$

is called *Cantor expansion of* x.

Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of independent random variables with the following distributions:

$$P(\xi_k = i) := p_{ik} \geq 0 \quad \text{and} \quad \sum_{i\in\mathbf{N}_k} p_{ik} = 1, \quad k \in \mathbb{N}.$$

Let us consider the random variable ξ:

$$\xi := \Delta^{\widetilde{Q}^*}_{\xi_1\xi_2...\xi_k...}. \tag{1.10}$$

Definition 1.14. ξ is said to be a *random variable with independent symbols over dynamic alphabets* or *a random variable with independent* $\widetilde{Q}^*$ *– digits.*

So, the distribution ξ is defined by "matrix" $\widetilde{Q}^* = ||q_{ik}||$ and $\widetilde{P} = ||p_{ik}||$. We will denote by μ_ξ the correspondent probability measure.

If $q_{ik} = q_i$ and $p_{ik} = p_i$ $\forall j \in \mathbb{N}$, $i \in \{0,\ 1,\ ...,\ s-1\}$ (i.e., ξ is a random variable with independent identically distributed Q - digits), then the measure μ_ξ is the self-similar measure associated with the list $(S_0, ..., S_{s-1}, p_0, ..., p_{s-1})$, where S_i is a similarity with the ratio q_i ($\sum\limits_{i=0}^{s-1} q_i = 1$), and the list $(S_0, ..., S_{s-1})$ satisfies the open set condition. More precisely, μ_ξ is the unique Borel probability measure on $[0, 1]$ such that

$$\mu_\xi = \sum_{i=0}^{s-1} p_i \cdot \mu_\xi \circ S_i^{-1},$$

(see, e.g., [Fal04, Ch. 9] Iterated function systems).

by the following theorem, distribution of the random variable ξ is of pure type.

Theorem[AKPT11] *The distribution of the random variable ξ is of pure type such that 1) μ_ξ is of absolutely continuous type iff*

$$\rho := \prod_{k=1}^{\infty} \left\{ \sum_{i \in \mathbf{N}_k} \sqrt{p_{ik} \cdot q_{ik}} \right\} > 0; \tag{1.11}$$

2) μ_ξ is of the discrete type iff

$$P_{max} := \prod_{k=1}^{\infty} \max_{i \in \mathbf{N}_k} \{p_{ik}\} > 0; \tag{1.12}$$

3) μ_ξ is of singularly continuous type iff

$$\rho = 0 = P_{max}. \tag{1.13}$$

1.3 Hausdorff dimension faithfulness of covering families for the determination of the Hausdorff dimension

1.3.1 Basic definitions and facts

Definition 1.15. Let α and ε are positive numbers. The $\alpha - \varepsilon$ - *Hausdorff measure* of a bounded set $E \subset W$ with reference to a given fine family of coverings Φ_W of a bounded set $W \subset \mathbf{M}$ is defined by

$$H_{\varepsilon}^{\alpha}(E,\Phi_W) = \inf_{d(E_j)\leq\varepsilon}\left\{\sum_j d\left(E_j\right)^{\alpha}\right\},$$

where the infimum is taken over all at most countable ε-coverings $\{E_j\}_{j\in\mathbb{N}}$ of E, $E_j \in \Phi_W,\ \forall j \in \mathbb{N}$.

Definition 1.16. Let α be a positive number. The α - *dimensional Hausdorff measure* (*Hausdorff measure*) of a bounded set $E \subset W$ with reference to a given fine family of coverings Φ_W of a bounded set $W \subset \mathbf{M}$ is defined by

$$H^{\alpha}(E,\Phi_W) = \lim_{\varepsilon\to 0} H_{\varepsilon}^{\alpha}(E,\Phi_W).$$

Definition 1.17 ([AT04])**.** The nonnegative number

$$\dim_H(E,\Phi_W) := \inf\{\alpha : H^{\alpha}(E,\Phi_W) = 0\}$$

is called the *Hausdorff dimension* of a set $E \subset W$ with reference to a given fine family of coverings Φ_W of a bounded set $W \subset \mathbf{M}$.

Definition 1.18. A fine covering family Φ_W of a set W is said to be *faithful family of coverings* for the Hausdorff dimension calculation on W if

$$\dim_H(E,\Phi_W) = \dim_H(E), \forall E \subset W.$$

Definition 1.19. A fine covering family Φ_W of a set W is said to be *non-faithful family of coverings* for the Hausdorff dimension calculation on W if

$$\exists E \subseteq W : \dim_H(E,\Phi_W) \neq \dim_H(E).$$

Remark 1.4. The family of cylinders of the classical continued fraction expansion can probably be considered as a rather unexpected example of non-faithful one-dimensional net-family of coverings ([PT]). By using the approach which has been invented by Y. Peres to prove non-faithfulness of the family of continued fraction cylinders, in [AKNT] the authors have proven the non-faithfulness for the family of cylinders of Q_∞-expansion with polynomially decreasing elements $\{q_i\}_{i\in\mathbb{N}}$. The latter two families of coverings give examples of non-comparable net measures (see next section). So, it is natural to ask about the existence of faithful covering families which are not comparable.

Conditions for a fine covering family to be faithful for the Hausdorff dimension calculation on $W = [0,1]$ were studied by many authors (see, e.g., [Bil61, Cut88, AT05] and references therein). First steps in this direction have been done by A. Besicovitch ([Bes52]), who proved the faithfulness for the family of cylinders of binary expansion. His result was extended by P. Billingsley ([Bil61]) to the family of s-adic cylinders, by S. Albeverio and G. Torbin ([AT04]) to the family of Q^*-cylinders for those matrices Q^* whose elements satisfy the following restriction $\inf\limits_k\{q_{0k}, q_{(n-1)k}\} > 0$. Some general sufficient conditions for the faithfulness of a given family of coverings are also known ([Cut88]).

Let us mentioned here that all these results were obtained by using the standard approach:

Proposition 1.1. *If for a given family $\Phi_{[0,1]}$ on $[0,1]$ there exist positive constants $\beta \in \mathbb{R}$ and $N^* \in \mathbb{N}$ such that for any interval $B = (a,b)$ there exist at most N^* sets $B_j \in \Phi$ which cover (a,b) and $|B_j| \le \beta \cdot |B|$, then the family Φ is faithful.*

1.3.2 Comparable and non-comparable Hausdorff net measures

Definition 1.20. A family of subsets Φ is called *net* on W if:

(a) if A_1 and A_2 belong to Φ, then $A_1 \subset A_2$ or $A_2 \subset A_1$ or $A_1 \bigcap A_2 = \emptyset$;

(b) every element $\omega \in W$ belongs to $C \in \Phi$ with $d(C) = 0$ or subfamily of sets Φ with arbitrary small diameters;

(c) Φ is countable;

(d) at most a finite number of sets from Φ contain any given set from Φ;

(e) every element from Φ is F_σ - set.

$H^\alpha(\cdot, \Phi)$ is called Hausdorff net measure if Φ is a net ([Rog98, Ch. 2]).

Definition 1.21. Let Φ_W be a fine family of covering Φ_W of a bounded set $W \subset \mathbf{M}$ and $\alpha > 0$. α - dimensional Hausdorff measure $H^\alpha(\ \cdot\ , \Phi_W)$ is called *comparable* to the Hausdorff measure $H^\alpha(\cdot)$ if there is a constant $C = C(\alpha) > 0$ such that

$$H^\alpha(E, \Phi) \leq C H^\alpha(E), \forall E \subset W.$$

Definition 1.22. Let Φ_W be a fine family of covering Φ_W of a bounded set $W \subset \mathbf{M}$ and $\alpha > 0$. α - dimensional Hausdorff measure $H^\alpha(\ \cdot\ , \Phi_W)$ is called *non-comparable* to the Hausdorff measure $H^\alpha(\cdot)$ there is a set $E \subset W$ such that $(H^\alpha(E) = 0$ and $H^\alpha(E, \Phi) > 0)$ or $(H^\alpha(E) \in (0, +\infty)$ or $H^\alpha(E, \Phi) = +\infty)$.

The two families of coverings mentioned above (The family of cylinders of the classical continued fraction expansion; the family of cylinders of Q_∞-expansion with polynomially decreasing elements $\{q_i\}_{i\in\mathbb{N}}$) give examples of non-comparable net measures. So, it is natural to ask about the existence of faithful covering families which are not comparable.

1.3.3 Hausdorff dimension faithfulness of covering families

We start with a very useful theorem, which can be proven easily, and, nevertheless, presents general necessary and sufficient conditions for the faithfulness.We will need the following convention

$$+\infty \leq C \cdot (+\infty),$$

where C is a positive constant.

Theorem 1.1. *Let W be a bounded subset of a metric space $(\mathbf{M}, \rho)$. Let $\Phi := \Phi_W$ be a fine covering family on W. Then Φ is faithful on W if and only if there exists a positive constant C such that for any $E \subset W$, any $\alpha > 0$ and any $\delta \in (0, \alpha)$ the following inequality holds:*

$$H^{\alpha}(E, \Phi) \leq C \cdot H^{\alpha-\delta}(E). \tag{1.14}$$

Proof. Suppose (1.14) holds. It is clear that

$$\dim_H(E) \leq \dim_H(E, \Phi), \ \forall E \subset W.$$

Let us prove the opposite inequality. Let

$$\alpha^* := \dim_H(E), \ \alpha^{**} := \dim_H(E, \Phi).$$

Suppose that $\alpha^* < \alpha^{**}$. Let

$$\alpha' := \frac{\alpha^* + \alpha^{**}}{2}.$$

Then

$$H^{\alpha'}(E, \Phi) = +\infty \tag{1.15}$$

and

$$H^{\alpha'}(E) = 0.$$

Let δ be an arbitrary positive real number such that $\alpha^* < \alpha' - \delta$. Then

$$H^{\alpha'-\delta}(E) = 0.$$

On the other hand, from (1.14) it follows that

$$H^{\alpha'}(E,\Phi) \le C \cdot H^{\alpha'-\delta}(E),$$

and, therefore,

$$H^{\alpha'}(E,\Phi) = 0,$$

which contradicts to (1.15) and proves the first part of the theorem.

To prove the second part let us assume that Φ is faithful on W , i.e.,

$$\dim_H(E) = \dim_H(E,\Phi), \quad \forall E \in W,$$

and consider all three possible cases:

1) if $\alpha < \alpha^*$, then

$$+\infty = H^{\alpha}(E,\Phi) \le C_1 \cdot H^{\alpha-\delta}(E) = +\infty, \quad \forall C_1 \in (0,+\infty);$$

2) if $\alpha > \alpha^*$, then

$$0 = H^{\alpha}(E,\Phi) \le C_1 \cdot H^{\alpha-\delta}(E), \quad \forall C_1 \in (0,+\infty);$$

3) if $\alpha = \alpha^*$, then

$$H^{\alpha}(E,\Phi) \le C_1 \cdot H^{\alpha-\delta}(E) = +\infty, \quad \forall C_1 \in (0,+\infty).$$

So, in all these cases condition (1.14) holds. □

Remark 1.5. The previous theorem is moderately interesting: it is difficult directly to verify condition (1.14) for a concrete covering family. So we will give a more practically useful statement in the following section.

1.3.4 General necessary conditions for a Hausdorff dimension faithfulness of covering families

The next theorem generalizes and enhances Statement 1.1. We will use the notion $|E|$ for diameter of a set E for convenience.

Theorem 1.2. *Let W be a bounded subset of a metric space $(\mathbf{M}, \rho)$. Let $\Phi := \Phi_W$ and $\Psi := \Psi_W$ be fine covering families on W and Ψ be a faithful on W for calculation of Hausdorff - Bezikovich dimension.*

Assume that there exists a positive constant C and a function $f(x) : R_+ \to \mathbb{N}$ such that

1) for any set $I \in \Psi$ there exist at most $f(|I|)$ subsets

$$\triangle_1^I,\ \triangle_2^I,\ \ldots,\ \triangle_{l(I)}^I \in \Phi,$$

$$l(I) \leq f(|I|),\ |\triangle_j^I| \leq |I| \text{ and } I \subset \bigcup_{j=1}^{l(I)} \triangle_j^I;$$

2) for any $\delta \in (0, \alpha)$ there exists $\varepsilon_1(\delta) > 0$ such that

$$f(|I|) \cdot (|I|)^\delta \leq C, \text{ for any set } I \in \Psi \text{ with diameter less then } \varepsilon_1(\delta).$$

Then the family Φ is faithful on W for calculation of Hausdorff - Bezikovich dimension, i. e., $\dim_H E = \dim_H(E, \Phi),\ \forall E \subset W$.

Proof. It is clear that $\dim_H(E) \leq \dim_H(E, \Phi),\ \forall E \subset W$. Let us prove $\dim_H(E) \geq \dim_H(E, \Phi)$.

Let α and δ be arbitrary real numbers with $0 < \delta < \alpha$. Let $\{I_j\}_{j\in\mathbb{N}}$ be an arbitrary ε-covering of E by subsets from Ψ with $\varepsilon \leq \varepsilon_1(\delta)$. From assumptions of the theorem it follows that there exist no more then $f(|I_j|)$ subsets

$$\Delta_1^{I_j},\ \Delta_2^{I_j},\ \ldots,\ \Delta_{l(I_j)}^{I_j}$$

from Φ with $|\Delta_i^{I_j}| \leq |I_j|$ for

$$i \in \{1, ..., l(I_j)\},\ I_j \cap E \subset \bigcup_{i=1}^{l(I_j)} \Delta_i^{I_j}.$$

Therefore, we have

$$|\Delta_i^{I_j}|^\alpha \leq |I_j|^\alpha \text{ with } i \in \{1, ..., l(I_j)\} \text{ and}$$

$$\sum_{i=1}^{l(I_j)} |\Delta_i^{I_j}|^\alpha \leq f(|I_j|)|I_j|^\alpha = f(|I_j|)|I_j|^\delta |I_j|^{\alpha-\delta}.$$

So, we have

$$\sum_j \sum_{i=1}^{l(I_j)} \left(\Delta_i^{I_j}\right)^\alpha \leq C \cdot \sum_j |I_j|^{\alpha-\delta}$$

for any $\alpha > 0$, $\delta \in (0, \alpha)$, and for an arbitrary ε-covering $\{I_j\}_{j\in\mathbb{N}}$ of E, $\varepsilon \leq \varepsilon_1(\delta)$.

This gives

$$H_\varepsilon^\alpha(E, \Phi) \leq \sum_j \sum_{i=1}^{l(I_j)} \left(|\Delta_i^{I_j}|\right)^\alpha \leq C \cdot \sum_j |I_j|^{\alpha-\delta},$$

for any $\delta \in (0, \alpha)$, and for an arbitrary ε-covering$\{I_j\}_{j\in\mathbb{N}}$ of set E, $\varepsilon \leq \varepsilon_1(\delta)$. Therefore

$$H_\varepsilon^\alpha(E, \Phi) \leq C H_\varepsilon^{\alpha-\delta}(E, \Psi), \quad \forall\alpha > 0,\ \forall\delta \in (0, \alpha),\ \forall\varepsilon \leq \varepsilon_1(\delta).$$

Hence

$$H^\alpha(E, \Phi) \leq C H^{\alpha-\delta}(E, \Psi),\ \forall\alpha > 0,\ \forall\delta \in (0, \alpha). \tag{1.16}$$

By the faithfulness of Ψ,

$$H^\alpha(E, \Phi) \leq C_1 \cdot H^{\alpha-\delta}(E), \ \forall \alpha > 0, \ \forall \delta \in (0, \alpha).$$

By Theorem (1.1), the family Φ is faithful.

□

1.4 Hausdorff - Billingsley dimension

In section we review some of the standard definitions and facts about the Billingsley dimension ([Bil60, Bil61, Bil65]).

Let $(\Omega, \mathfrak{B}, \mu)$ be an arbitrary probability space with continuous measure μ (see definition 1.9). Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be discrete stochastic process that is defined on $(\Omega, \mathfrak{B}, \mu)$ and have finite or countable state space σ.

Definition 1.23. The set

$$\{\omega : \omega \in \Omega, x_1(\omega) = \alpha_1, x_2(\omega) = \alpha_2, \ldots, x_n(\omega) = \alpha_n\}$$

is called n - *rank cylinder* with base $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in \sigma$.

By Φ denote a family of cylinders of all ranks.

Definition 1.24. Let ε be a positive constant. A finite or countable subset $\{V_i\}_{i\in\mathbb{N}} \subset \Phi$ is called *μ-ε-covering of a set E* if $E \subset \bigcup_i V_i$ and $\mu(V_i) \leq \varepsilon$, $\forall i \in \mathbb{N}$.

Definition 1.25. Let α and ε be positive numbers. The *α - μ - ε - dimensional Hausdorff - Billingsley measure* of a set E with reference to Φ is defined by

$$H_\mu(E, \alpha, \varepsilon, \Phi) = \inf \sum_i (\mu(V_i))^\alpha,$$

where the infimum is taken over all μ - ε - coverings $\{V_j\}_{j\in\mathbb{N}}$ of E, $V_i \in \Phi$, $\forall i \in \mathbb{N}$.

Definition 1.26. Let α be a positive number. The *α - μ - dimensional Hausdorff - Billingsley measure* (*Hausdorff - Billingsley measure*) of set a E with reference to the Φ is defined by

$$H_\mu(E, \alpha, \Phi) := \lim_{\varepsilon \to 0} H_\mu(E, \alpha, \varepsilon, \Phi).$$

Remark 1.6. α - μ - ε - dimensional Hausdorff - Billingsley measure is monotonically non-decreasing under the condition that ε becomes smaller. Hence the α - μ Hausdorff - Billingsley measure exists whenever μ-ε-covering exists .

Let us recall some properties of the α - μ Hausdorff - Billingsley measure [Bil61]:

1. If $H_\mu(E, \alpha, \Phi) < +\infty$, then $\forall \delta > 0 : H_\mu(E, \alpha + \delta, \Phi) = 0$;
2. If $H_\mu(E, \alpha, \Phi) > 0$, then $\forall \delta > 0 : H_\mu(E, \alpha - \delta, \Phi) = +\infty$.

Definition 1.27. Let $(\Omega, \mathfrak{B}, \mu)$ be an arbitrary probability space with continuous measure μ (see definition 1.9). Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a discrete stochastic process that is defined on $(\Omega, \mathfrak{B}, \mu)$ and has finite or countable state space σ. Let Φ be a family of cylinders of all ranks. The *Hausdorff - Billingsley dimension* of set the E with reference to the Φ and μ is defined by

$$\dim_\mu(E, \Phi) = \inf\{\alpha : H_\mu(E, \alpha, \Phi) = 0\}.$$

From now on we make the assumption:

$$\mu \{\omega : x_n(\omega) = \alpha_n, n = 1,\ 2,\ ...\} = 0,\ \forall \{\alpha_n\}_{n \in \mathbb{N}}. \tag{1.17}$$

Remark 1.7. Using (1.17), we see that $H_\mu(E, \alpha, \Phi) = 0,\ \forall \alpha > 1,\ \forall E \in \Omega$. By the previous statement

$$0 \le \dim_\mu(E, \Phi) \le 1, \forall E \in \Omega.$$

Let $\omega \in \Omega$ and

$$\Delta_n(\omega) = \{\omega' : x_k(\omega') = x_k(\omega), k = 1, 2, \dots, n\}, \ n \in \mathbb{N},$$

in other words, $\{\Delta_n(\omega)\}_{n\in\mathbb{N}}$ is a family of cylinders with point ω. P. Billingsley proved the following too results. These results will be intensively used in a sequel.

Theorem 1.3 ([Bil61]). *Let $\delta \geq 0$. Then*

$$\dim_\mu \left(\left\{ \omega : \liminf_{n\to\infty} \frac{\ln \nu(\Delta_n(\omega))}{\ln \mu(\Delta_n(\omega))} \leq \delta \right\}, \Phi \right) \leq \delta.$$

Theorem 1.4 ([Bil61]). *Let $\delta \geq 0$. If*

$$E \subset \left\{ \omega : \liminf_{n\to\infty} \frac{\ln \nu(\Delta_n(\omega))}{\ln \mu(\Delta_n(\omega))} \geq \delta \right\},$$

Then

$$\dim_\mu (E, \Phi) \geq \delta \dim_\nu (E, \Phi).$$

We propose "the almost Cantor set" below as an example of the problem with Definition 1.6. Let us stress that the Hausdorff measure of the Cantor set equals to 1 (see [Fal04, Example 2.7]).

Example 1.1. Suppose $(\mathbf{M}, \rho) = R^1$ and $W = [0, 1]$,

$$C_* = \left\{ x : x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}; \ \alpha_k \in \{0, 2\} \text{ if } k \in \mathbb{N} \backslash \{i : i = 10^n, \ n \in \mathbb{N}\} \right.$$

$$\left. \text{and } \alpha_k = 0 \text{ if } k \in \{i : i = 10^n, \ n \in \mathbb{N}\} \right\};$$

Then

$$H^{\alpha}(C_*) = \begin{cases} 0, \text{ if } \alpha \geq \log_3 2; \\ +\infty, \text{ if } \alpha < \log_3 2. \end{cases} \tag{1.18}$$

Proof. Let μ_ξ be the probability measure corresponding to the distribution of the random variable

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{3^k},$$

where ξ_k are independent random variables with following distributions:

if $k \in \mathbb{N} \backslash \{i : i = 10^n,\ n \in \mathbb{N}\}$ then ξ_k are equal to 0 or 1 with probability $\frac{1}{2}$;

if $k \in \{i : i = 10^n,\ n \in \mathbb{N}\}$ then ξ_k are equal to 0 with probability 1.

Let $\Delta_n(x)$ be 3-adic cylinder (see the next section, formula 1.7) of rank n that contains a point x. By construction, we have

$$\mu_\xi(\Delta_n(x)) = 2^{-(n-[\log_{10}(n)])} \quad \text{and} \quad \lambda(\Delta_n(x)) = 3^{-n},$$

where $x \in C_*$. Hence

$$\lim_{n\to\infty} \frac{\ln \mu_\xi(\Delta_n(x))}{\ln \lambda(\Delta_n(x))} = \frac{\log 2}{\log 3}, \forall x \in C_*. \tag{1.19}$$

Using Theorem 1.3 and Theorem 1.4, we get

$$H^{\alpha}(C_*) = \begin{cases} 0, \text{ if } \alpha > \log_3 2; \\ +\infty, \text{ if } \alpha < \log_3 2. \end{cases}$$

Suppose $\varepsilon > 0$ and $k_0 := \inf\left\{k \in \mathbb{N} : \frac{1}{3^k} \leq \varepsilon\right\}$; then the set C_* can be covered by $2^{k_0-[\log_2 k_0]}$ cylinders with length $\frac{1}{3^{k_0}}$. The α-volume of this ε-covering is equal to

$$A(\varepsilon) := 2^{k_0-[\log_2 k_0]} \left(\frac{1}{3^{k_0 \cdot \alpha_0}}\right) = 2^{-[\log_2 k_0]} \leq \frac{1}{k_0 - 1}.$$

By definition of the $\alpha - \varepsilon$ - Hausdorff measure (see definition 1.4), we have

$$H_{\varepsilon}^{\alpha_0}(C_*) \leq A(\varepsilon) \leq \frac{1}{k_0 - 1} \to 0(\varepsilon \to 0).$$

Taking into account the previous inequality and the definition of Hausdorff measure (see definition 1.5), we obtain

$$H^{\alpha_0}(C_*) = 0.$$

This completes the proof.

□

Chapter 2

Fine fractal properties of probability measures generated by Cantor series expansions and their applications

2.1 Sharp conditions for the Hausdorff dimension faithfulness of the Cantor series coverings

Let us recall the definition of Cantor series expansion.

Definition 2.1. For a given sequence $\{n_k\}_{k=1}^{\infty}$ with $n_k \in \mathbb{N}\backslash\{1\}$, $k \in \mathbb{N}$ the expression of $x \in [0, 1]$ in the following form

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k} =: \Delta_{\alpha_1\alpha_2\ldots\alpha_k\ldots}, \ \alpha_k \in \{0, \ 1, \ \ldots, \ n_k - 1\}$$

is said to be the Cantor series expansion of x.

These expansions, which have been initially studied by G. Cantor in 1869 (see, e.g., [Can69]), are natural generalizations of the classical s-adic expansion for reals. Cantor series expansions have been intensively studied from different points of view during the last century (see, e.g., [Man10, Sch95] and references therein).

We will denote by $\mathcal{A}_k$ the family of the k-th rank semi-closed intervals (cylinders), i.e.,

$$\mathcal{A}_k := \{E : E = \Delta_{\alpha_1\alpha_2\ldots\alpha_k},\ \alpha_i \in \{0, ..., n_i - 1\},\ i = 1,\ 2,\ ...,\ k\}$$

with

$$\Delta_{\alpha_1\alpha_2\ldots\alpha_k} := \left\{ x : x \in \left[\sum_{i=1}^{k} \frac{\alpha_i}{n_1 n_2 \ldots n_i}, \frac{1}{n_1 n_2 \ldots n_k} + \sum_{i=1}^{k} \frac{\alpha_i}{n_1 n_2 \ldots n_i} \right) \right\}.$$

Let $\mathcal{A}$ be the family of all possible semi-closed intervals (cylinders), i.e.,

$$\mathcal{A} := \{E : E = \Delta_{\alpha_1\alpha_2\ldots\alpha_k},\ k \in \mathbb{N},\ \alpha_i \in \{0, ..., n_i - 1\},\ i = 1,\ 2,\ ...,\ k\}.$$

Remark 2.1. The conditions of proposition 1.1 are not satisfied even in the case of "fractional" covering, i.e., the family of $\tilde{Q}^*$-cylinders (see 1.2) such that

$$\widetilde{Q}^* = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} & \cdots \\ & \frac{1}{3} & \cdots & \frac{1}{n+1} & \cdots \\ & & \cdots & \cdots & \cdots \\ & & & \frac{1}{n+1} & \cdots \\ & & & & \cdots \end{pmatrix}.$$

Main result of the present section is a sharp condition for the Hausdorff dimension faithfulness of the Cantor series coverings $\mathcal{A}$. To the best of our knowledge this theorem gives the first necessary and sufficient condition of the faithfulness for a class of covering families containing both faithful and non-faithful ones.

Theorem 2.1. *The family $\mathcal{A}$ of Cantor coverings of the unit interval is faithful for the Hausdorff dimension calculation if and only if*

$$\lim_{k\to\infty} \frac{\ln n_k}{\ln n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1}} = 0. \tag{2.1}$$

2.1.1 Sufficient condition

Proof. Let (2.1) holds. It is enough to prove that

$$\dim_H(E) \geq \dim_H(E, \mathcal{A}), \ \forall E \subset [0,1).$$

Let I be an arbitrary interval. Then there exists a cylinder of $k = k(I)$-th rank $\Delta(k(I)) = \Delta_{\alpha_1 \ldots \alpha_{k(I)}} \in \mathcal{A}$ such that:

1) $\Delta_{\alpha_1 \ldots \alpha_{k(I)}} \subset I$;

2) any interval of $(k(I) - 1)$-th rank is not a subset of I.

The interval I contains at most $2 \cdot n_{k(I)}$ cylinders from $\mathcal{A}_k$. Hence I can be covered by $f(|I|) := 2 \cdot n_{k(I)} + 2$ cylinders from $\mathcal{A}_k$. Therefore,

$$|\Delta(k(I))| \leq |I| < f(|I|) \cdot |\Delta(k(I))|. \tag{2.2}$$

Let us prove the following additional lemma.

Lemma 2.1. *Let C be an arbitrary positive constant. Then the equality*

$$\lim_{k\to\infty} \frac{\ln n_k}{\sum_{i=1}^{k-1} \ln n_i} = 0$$

holds if and only if for any positive δ there exists $k_0(\delta) \in \mathbb{N}$ such that $\forall k > k_0(\delta)$:

$$(2 \cdot n_k + 2) \cdot \left(\frac{2 \cdot n_k + 2}{n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1} \cdot n_k} \right)^{\delta} \leq C.$$

Proof. Let C be a positive constant. If the equality

$$\lim_{k\to\infty} \frac{\ln n_k}{\sum_{i=1}^{k-1} \ln n_i} = 0$$

holds, then for every real number $\delta > 0$, there exists a natural number $k_1(\delta)$ such that for all $k > k_1(\delta)$ we have

$$\frac{\ln n_k}{\sum_{i=1}^{k-1} \ln n_i} \le \frac{\delta}{2}.$$

This gives

$$\ln n_k \le \ln \left(\prod_{i=1}^{k-1} n_i \right)^{\frac{\delta}{2}}$$

and

$$\frac{n_k}{(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})^{\frac{\delta}{2}}} \le 1,$$

for all $k > k_1(\delta)$. Therefore, we have

$$(2n_k + 2) \cdot \left(\frac{2 \cdot n_k + 2}{n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1} \cdot n_k} \right)^{\delta}$$

$$\le 4 \cdot n_k \cdot \left(\frac{4 \cdot n_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1} \cdot n_k} \right)^{\delta} =$$

$$= 4^{1+\delta} \left(\frac{n_k}{(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})^{\delta}} \right)$$

$$\le 4^{1+\delta} \left(\frac{n_k}{(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})^{\frac{\delta}{2}}} \cdot \frac{1}{(n_1 \le \cdot n_2 \cdot \ldots \cdot n_{k-1})^{\frac{\delta}{2}}} \right)$$

$$\le 4^{1+\delta} \left(\frac{1}{(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})^{\frac{\delta}{2}}} \right) \to 0 \ (k \to \infty).$$

Hence, for a given real number C>0, there exists $k_0(\delta) > k_1(\delta)$ such that for all $k > k_0(\delta)$ we have

$$(2 \cdot n_k + 2) \cdot \left(\frac{2 \cdot n_k + 2}{n_1 \cdot n_2 \cdot ... \cdot n_{k-1} \cdot n_k} \right)^{\delta}$$

$$\leq 4^{1+\delta} \left(\frac{1}{(n_1 \cdot n_2 \cdot ... \cdot n_{k-1} \cdot n_k)^{\frac{\delta}{2}}} \right) \leq C,$$

which proves the first part of the lemma.

Suppose that for a given positive constant C and for all $\delta > 0$, there exists $k_0(\delta)$ such that $\forall k > k_0(\delta)$:

$$(2 \cdot n_k + 2) \cdot \left(\frac{2 \cdot n_k + 2}{n_1 \cdot n_2 \cdot ... \cdot n_{k-1} \cdot n_k} \right)^{\delta} \leq C.$$

Hence, we have

$$n_k \cdot \left(\frac{2 \cdot n_k}{n_1 \cdot n_2 \cdot ... \cdot n_{k-1} \cdot n_k} \right)^{\delta} \leq C.$$

Therefore, for all $\delta > 0$, there exists $k_0(\delta)$ such that for all $k > k_0(\delta)$ we have

$$n_k \leq \frac{1}{2^{\delta}} C (n_1 \cdot n_2 \cdot ... \cdot n_{k-1})^{\delta} \quad \Leftrightarrow \quad \ln n_k \leq \ln \left(C \left(n_1 \cdot n_2 \cdot ... \cdot n_{k-1} \right)^{\delta} \right).$$

Hence,

$$\frac{\ln n_k}{\ln(n_1 \cdot n_2 \cdot ... \cdot n_{k-1})} \leq \frac{\ln C}{\ln(n_1 \cdot n_2 \cdot ... \cdot n_{k-1})} + \delta, \ \forall \delta > 0, \ \forall k > k_0(\delta).$$

Clearly,

$$\frac{\ln C}{\ln(n_1 \cdot n_2 \cdot ... \cdot n_{k-1})} \to 0 \ (k \to \infty),$$

and, therefore,

$$\lim_{k \to \infty} \frac{\ln n_k}{\sum_{i=1}^{k-1} \ln n_i} = 0,$$

which proves Lemma 2.1 .

□

Therefore, for every $\delta > 0$, there exists a natural number $k_0(\delta)$ such that for all $k > k_0(\delta)$ we have

$$f(|I|) \cdot |I|^{\delta} \leq (2 \cdot n_k + 2) \cdot \left(\frac{2 \cdot n_k + 2}{n_1 \cdot n_2 \cdot ... \cdot n_{k-1} \cdot n_k} \right)^{\delta} \leq C.$$

Hence from Lemma 1.2 it follows that $\mathcal{A}$ is faithful for the Hausdorff dimension calculation.

□

2.1.2 Necessary condition

Proof. We will prove the following statement. If

$$\overline{\lim_{k\to\infty}} \frac{\ln n_k}{\ln n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1}} =: C > 0, \tag{2.3}$$

then $\mathcal{A}$ is non-faithful for the Hausdorff dimension calculation. We shall construct a set $T = T(C)$ with the following properties:

$$1) \dim_H(T) \leq \frac{2}{2+C};$$

$$2) \dim_H(T, \mathcal{A}) \geq \frac{4+C}{4+3C}.$$

From (2.3) it follows that there exists a subsequence $\{k_i\}_{i\in\mathbb{N}}$ such that for every $\delta \in (0, C)$, there exists natural number $N_0(\delta)$ such that for all $k_i > N_0(\delta)$ we have

$$(n_1 n_2 \ldots n_{k_i-1})^{C-\delta} \leq n_{k_i} \leq (n_1 n_2 \ldots n_{k_i-1})^{C+\delta}. \tag{2.4}$$

It is clear that for $\varepsilon > 0$, there is natural number $N_1(\varepsilon)$ such that for all $k > N_1(\varepsilon)$ we have

$$\frac{1}{n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1}} < \varepsilon. \tag{2.5}$$

Let $N_2(\varepsilon, \delta) := \max\{N_0(\delta),\ N_1(\varepsilon)\}$. Let us choose a subsequence $\{k'_j\}_{j\in\mathbb{N}}$ from the sequence $\{k_i\}_{i\in\mathbb{N}}$ with the following property:

$$\frac{\ln(n_{k'_{j-1}+1} \cdots n_{k'_j-1})}{\ln(n_1 n_2 \ldots n_{k'_{j-1}-1} n_{k'_{j-1}} n_{k'_{j-1}+1} \cdots n_{k'_j-1})} > 1 - \frac{C}{4}. \tag{2.6}$$

Since for every natural number j there is i such that $j < i$ and

$$\frac{\ln(n_1 \cdot n_2 \cdot \ldots \cdot n_{k_j})}{\ln(n_1 \cdot n_2 \cdot \ldots \cdot n_{k_j} \cdot n_{k_j+1} \cdot n_{k_j+2} \cdot \ldots \cdot n_{k_i})} \leq \frac{C}{4},$$

it follows that a sequence $\{k'_j\}_{j\in\mathbb{N}}$ with condition (2.6) always exists. We will use the following set T:

$$T := \left\{ x : x \in [0,1],\ x = \sum_{k=1}^{\infty} \frac{\alpha_k(x)}{\prod_{i=1}^{k} n_i},\ \alpha_k(x) \in \left\{0, ..., \left[\sqrt{n_k}\right]\right\} \right.$$

$$\left. \text{if } k \in \{k'_j\}, \quad \text{and} \quad \alpha_k(x) \in \{0, ..., n_k - 1\} \quad \text{if } \ , k \notin \{k'_j\} \right\}.$$

Firstly let us show that

$$\dim_H(T) \leq \frac{2}{2+C}. \tag{2.7}$$

Let $k'_j > N_2(\varepsilon, \delta)$. The set T can be covered by $n_1 \cdot n_2 \cdot \ldots \cdot n_{k'_j-1}$ semi-closed intervals and each of them is a union of $\left[\sqrt{n_{k'_j}}\right] + 1$ sets from $\mathcal{A}_{k'_j}$. The α-volume of this ε-covering is equal to

$$n_1 n_2 \dots n_{k'_j - 1} \left(\frac{\left[\sqrt{n_{k'_j}}\right] + 1}{n_1 n_2 \dots n_{k'_j}} \right)^{\alpha}.$$

From (2.4) it follows that

$$n_1 n_2 \dots n_{k'_j - 1} \left(\frac{\left[\sqrt{n_{k'_j}}\right] + 1}{n_1 n_2 \dots n_{k'_j}} \right)^{\alpha} \le 2^{\alpha} (n_1 n_2 \dots n_{k'_j - 1})^{1 - \frac{1}{2}\alpha(C-\delta) - \alpha}.$$

Suppose

$$1 - \frac{1}{2}\alpha(C-\delta) - \alpha < 0,$$

then

$$H^{\alpha}_{\varepsilon}(T) \le \lim_{j \to \infty} 2^{\alpha} (n_1 n_2 ... n_{k'_j - 1})^{1 - \frac{1}{2}\alpha(C-\delta) - \alpha} = 0.$$

Therefore

$$H^{\alpha}_{\varepsilon}(T) = 0, \ \forall \alpha > \frac{2}{C - \delta + 2}, \ \forall \varepsilon > 0, \ \forall \delta > 0.$$

Hence

$$\dim_H T \le \frac{2}{C - \delta + 2}, \ \forall \delta > 0.$$

Therefore

$$\dim_H T \le \frac{2}{C + 2}.$$

Now let us show that

$$\dim_H (T, \mathcal{A}) \ge \frac{4 + C}{4 + 3C}.$$

Let

$$\{k''_j\} = \{k'_j\} \bigcap \{N_2(\varepsilon, \delta) + 1, \ N_2(\varepsilon, \delta) + 2, \ ...\}.$$

Let $\mu = \mu_{N_2(\varepsilon,\delta)}$ be the probability measure corresponding to the random variable

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{\prod_{i=1}^{k} n_i},$$

where ξ_k are independent random variables; if $k \in \{k''_j\}$, then ξ_k takes values $0,\ 1,\ ...,\ [\sqrt{n_k}]$ with probabilities $\frac{1}{[\sqrt{n_k}]+1}$; if $k \notin \{k''_j\}_{j\in\mathbb{N}}$, then ξ_k takes values $0,\ 1,\ ...,\ n_k - 1$ with probabilities $\frac{1}{n_k}$. So

$$|\Delta_{\alpha_1\alpha_2...\alpha_k}| = \frac{1}{n_1 n_2 ... n_k}$$

for any $\Delta_{\alpha_1\alpha_2...\alpha_k}$ from $\mathcal{A}_k$, and

$$\mu(\Delta_{\alpha_1\alpha_2...\alpha_k}) = \frac{1}{\varphi_1\varphi_2...\varphi_k},$$

where $\varphi_t = n_t$ if $t \notin \{k''_j\}_{j\in\mathbb{N}}$ and $\varphi_t = \left[\sqrt{n_t}\right] + 1$ if $t \in \{k''_j\}_{j\in\mathbb{N}}$, $\forall t \in \mathbb{N}$.

Let us show that

$$\frac{\ln(\mu(\Delta_{\alpha_1\alpha_2...\alpha_k}))}{\ln(|\Delta_{\alpha_1\alpha_2...\alpha_k}|)} \geq \frac{4+C-2\delta}{4+3C+4\delta},\ \forall k \in \mathbb{N}. \tag{2.8}$$

Taking into account properties of $\{k''_j\}_{j\in\mathbb{N}}$, one can prove by induction on j that

$$\frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_j}\right)}{\ln\left(n_1 n_2 n_3 ... n_{k''_j}\right)} \geq \frac{4+C-2\delta}{4+3C+4\delta},\ \forall j \in \mathbb{N}. \tag{2.9}$$

For the case $j = 1$:

$$\frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_1-1}\varphi_{k''_1}\right)}{\ln\left(n_1n_2n_3...n_{k''_1-1}n_{k''_1}\right)} = \frac{\ln\left(n_1n_2n_3...n_{k''_1-1}\left(\left[\sqrt{n_{k''_1}}\right]+1\right)\right)}{\ln\left(n_1n_2n_3...n_{k''_1-1}n_{k''_1}\right)} \geq$$
$$\geq \frac{\ln\left(n_1n_2n_3...n_{k''_1-1}\right)^{1+\frac{C}{2}-\frac{\delta}{2}}}{\ln\left(n_1n_2n_3...n_{k''_1-1}\right)^{1+C+\delta}} = \frac{1+\frac{C}{2}-\frac{\delta}{2}}{1+C+\delta},$$

where (in the inequalities) we have used (2.4).

Therefore, we have

$$\frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_1-1}\varphi_{k''_1}\right)}{\ln\left(n_1n_2n_3...n_{k''_1-1}n_{k''_1}\right)} - \frac{4+C-2\delta}{4+3C+4\delta} \geq$$
$$\geq \frac{1+\frac{C}{2}-\frac{\delta}{2}}{1+C+\delta} - \frac{4+C-2\delta}{4+3C+4\delta} = \frac{C(C+3d)}{2(1+C+d)(4+3C+4d)} > 0.$$

Let us now assume that (2.9) holds for $j = p$ and prove that it is also holds for the case $j = p+1$:

$$\frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_{p+1}-1}\varphi_{k''_{p+1}}\right)}{\ln\left(n_1n_2n_3...n_{k''_{p+1}-1}n_{k''_{p+1}}\right)} =$$

$$\frac{\ln\left(\left(\varphi_1\varphi_2...\varphi_{k''_p-1}\varphi_{k''_p}\right)\left(n_{k''_p+1}...n_{k''_{p+1}-1}\right)\left(\left[\sqrt{n_{k''_{p+1}}}\right]+1\right)\right)}{\ln\left(\left(n_1n_2n_3...n_{k''_p-1}n_{k''_p}\right)\left(n_{k''_p+1}...n_{k''_{p+1}-1}\right)n_{k''_{p+1}}\right)}. \tag{2.10}$$

Using (2.6) and the elementary statement: if $0 < a < b$, and $0 < c < d$, then $\frac{a+c}{b+c} < \frac{a+d}{b+d}$ we get that the right hand side of (2.10) is larger or equal to

$$\frac{\ln\left(\left(\varphi_1\varphi_2...\varphi_{k''_p-1}\varphi_{k''_p}\right)\left(n_1...n_{k''_{p+1}-1}\right)^{1-\frac{C}{4}}\left(\left[\sqrt{n_{k''_{p+1}}}\right]+1\right)\right)}{\ln\left(\left(n_1n_2n_3...n_{k''_p-1}n_{k''_p}\right)\left(n_1...n_{k''_{p+1}-1}\right)^{1-\frac{C}{4}}n_{k''_{p+1}}\right)} \geq$$

$$\geq \frac{\ln\left(\left(\varphi_1\varphi_2...\varphi_{k''_p-1}\varphi_{k''_p}\right)\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{C}{4}-\frac{\delta}{2}}\right)}{\ln\left(\left(n_1n_2n_3...n_{k''_p-1}n_{k''_p}\right)\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{3}{4}C+\delta}\right)} =$$

$$\frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_p-1}\varphi_{k''_p}\right)+\ln\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{C}{4}-\frac{\delta}{2}}}{\ln\left(n_1n_2n_3...n_{k''_p-1}n_{k''_p}\right)+\ln\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{3}{4}C+\delta}}.$$

From now on, set

$$W := \frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_p-1}\varphi_{k''_p}\right)+\ln\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{C}{4}-\frac{\delta}{2}}}{\ln\left(n_1n_2n_3...n_{k''_p-1}n_{k''_p}\right)+\ln\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{3}{4}C+\delta}}.$$

From the induction assumption it follows that

$$\frac{\ln\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{C}{4}-\frac{\delta}{2}}}{\ln\left(n_1...n_{k''_{p+1}-1}\right)^{1+\frac{3}{4}C+\delta}} = \frac{1+\frac{C}{4}-\frac{\delta}{2}}{1+\frac{3}{4}C+\delta} \leq \frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_p-1}\varphi_{k''_p}\right)}{\ln\left(n_1n_2n_3...n_{k''_p-1}n_{k''_p}\right)}.$$

Combing the above and a property of mediant we get

$$W \leq \frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_p-1}\varphi_{k''_p}\right)}{\ln\left(n_1n_2n_3...n_{k''_p-1}n_{k''_p}\right)} .$$

Consequently, we have

$$\frac{\ln\left(\varphi_1\varphi_2...\varphi_{k''_j}\right)}{\ln\left(n_1n_2n_3...n_{k''_j}\right)} \geq \frac{4+C-2\delta}{4+3C+4\delta},\ \forall j\in\mathbb{N},$$

which completes the proof of (2.9).

Let $k\in(k''_j,k''_{j+1})$. Then

$$\frac{\ln\left(\mu\left(\Delta_{\alpha_1\alpha_2...\alpha_k}\right)\right)}{\ln\left(\left|\Delta_{\alpha_1\alpha_2...\alpha_k}\right|\right)} \geq \frac{\ln\left(\mu\left(\Delta_{\alpha_1\alpha_2...\alpha_{k''_j}}\right)\right)}{\ln\left(\left|\Delta_{\alpha_1\alpha_2...\alpha_{k''_j}}\right|\right)} \geq \frac{4+C-2\delta}{4+3C+4\delta},\ \forall k\in\mathbb{N}.$$

Let $\{\Delta'_i\}_{i\in\mathbb{N}}$ be an arbitrary ε-covering of T and $\Delta'_i\in\mathcal{A}$, $\forall i\in\mathbb{N}$. Then, using (2.9) we get

$$\frac{4+C-2\delta}{4+3C+4\delta} \leq \frac{\ln(\mu(\Delta'_i))}{\ln(|\Delta'_i|)} < 1$$

which implies that

$$\mu(\Delta'_i) \leq |\Delta'_i|^{\frac{4+C-2\delta}{4+3C+4\delta}}.$$

Let $\alpha\in\left[0,\frac{4+C-2\delta}{4+3C+4\delta}\right)$. Then we have

$$1=\mu(T)\leq\bigcup_i\mu(\Delta'_i)\leq\sum_i|\Delta'_i|^{\frac{4+C-2\delta}{4+3C+4\delta}}\leq\sum_i|\Delta'_i|^{\alpha}.$$

Hence for any real numbers $\delta>0$, $\varepsilon>0$ and $\alpha\in\left[0,\frac{4+C-2\delta}{4+3C+4\delta}\right)$, and for any ε-covering $\{\Delta'_i\}$ of the set T by cylinders $\Delta'_i\in\mathcal{A}$ we have

$$\sum_i|\Delta'_i|^{\alpha}\geq 1.$$

Therefore

$$H^{\alpha}_{\varepsilon}(T,\mathcal{A})\geq 1,\quad \forall\delta>0,\ \forall\varepsilon>0,\ \forall\alpha\in\left[0,\frac{4+C-2\delta}{4+3C+4\delta}\right).$$

Consequently

$$H^{\alpha}(T, \mathcal{A}) \geq 1, \quad \forall \delta > 0,\ \forall \alpha < \frac{4 + C - 2\delta}{4 + 3C + 4\delta}.$$

Hence, we have

$$\dim_H(T, \mathcal{A}) \geq \frac{4 + C - 2\delta}{4 + 3C + 4\delta},\ \forall \delta > 0,$$

and an inequality

$$\dim_H(T, \mathcal{A}) \geq \frac{4 + C}{4 + 3C},$$

which completes the proof. □

2.2 Hausdorff dimension of the probability distributions of the Random Cantor expansions

Fractal analysis of singularly continuous distributions helps to get essential properties of such distributions. The first step of an analysis is the study of metric, topological and fractal properties of the spectrum (minimal closed support of a distribution) of a distribution. It should be mentioned here that the determination of the Hausdorff-Besocovitch dimension even for the spectrum is often a non-trivial problem.

On the other hand, the topological support is a rather "rough" characteristic for a measure with a complicated local structure. For instance, the subfamily of infinite Bernoulli convolutions: $\xi(p) = \sum\limits_{k=1}^{\infty} \frac{\xi_k}{2^k}$, where ξ_k is a sequence of independent random variables taking the values 0 and 1 with probabilities $p \in (0, \frac{1}{2})$ and $1-p$ correspondingly. Two distributions of random variables $\xi(p_1)$ and $\xi(p_2)$ $(p_1 \neq p_2)$ are mutually singular and they are singular with respect to Lebesgue measure. Nevertheless, the spectrum of every distribution $\xi(p)$ coincides with $[0, 1]$.

Let us recall that for a given probability measure μ the number

$$\dim_H \mu = \inf\{\dim_H(E): \ \mu(E) = 1\}$$

is said to be the Hausdorff dimension of the measure μ. In the case of singularity this number is a rather important characteristic of a probability measure (see, e.g., [AT05]).

Applying the latter theorem and methods from [AT05], we will get the Hausdorff dimension of the probability distribution μ_ξ of the random variable ξ with independent digits of the Cantor series expansion (*Random Cantor expansion*), i.e.,

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k}, \tag{2.11}$$

where independent random variables ξ_k take values $0, 1, ..., n_k - 1$ with probabilities p_{0k}, p_{1k}, ..., $p_{n_k-1,k}$, respectively ($\sum_{i=0}^{n_k-1} p_{i,k} = 1$ and $1 < n_k \in \mathbb{N}$, $\forall k \in \mathbb{N}$).

We will need the following convention: $0 \ln 0 := 0$.

Theorem 2.2. *Let*

$$\sum_{k=1}^{\infty} \left(\frac{\ln n_k}{\ln \prod\limits_{i=1}^{k} n_i} \right)^2 < \infty, \tag{2.12}$$

then the Hausdorff dimension of the probability distribution μ_ξ *of the random variable* ξ *with independent digits of the Cantor series expansion is equal to*

$$\dim_H(\mu_\xi) = \varliminf_{k\to\infty} \frac{H_k}{\ln(n_1 n_2 ... n_k)},$$

where $H_k = \sum\limits_{j=1}^{k} h_j$, $\forall k \in \mathbb{N}$ *and* $h_j = -\sum\limits_{i=0}^{n_j-1} p_{ij} \ln p_{ij}$, $\forall j \in \mathbb{N}$.

Proof. By Jessen-Wintner's theorem ([JW35]), the random variable ξ has a pure type. Without loss of generality we can assume that μ_ξ is a continuous measure (otherwise equation (2.12) is true).

Let x be an arbitrary point from the set $S_{\mu_\xi}\setminus\{1\}$. Then there exists a cylinder $\Delta_k(x) = \Delta_{\alpha_1(x)\alpha_2(x)...\alpha_k(x)} \in \mathcal{A}_k$ such that $x \in \Delta_k(x)$. Let λ be Lebesgue's measure on $[0,1]$. Then we have

$$\mu_\xi(\Delta_k(x)) = p_{\alpha_1(x)1} \cdot p_{\alpha_2(x)2} \cdot ... \cdot p_{\alpha_k(x)k} > 0,$$

$$\lambda(\Delta_k(x)) = \frac{1}{n_1 n_2 ... n_k}.$$

Let us consider the following expression

$$\frac{\ln \mu_\xi(\Delta_k(x))}{\ln \lambda(\Delta_k(x))} = \frac{\sum\limits_{j=1}^{k} \ln p_{\alpha_j(x)j}}{-\ln(n_1 n_2 ... n_k)}.$$

Throughout the proof, $\{\eta_k\}_{k\in\mathbb{N}}$ denotes an auxiliary sequence of independent discrete random variables on probability space $([0,1],\ \mathcal{B}([0,1]), \mu_\xi)$ ($\mathcal{B}([0,1])$ is a Borel σ-algebra). Let

$$\{\eta_k\}_{k\in\mathbb{N}} = \{\eta_k(y)\}_{k\in\mathbb{N}} := \{\ln p_{\alpha_k(y)k}\}_{k\in\mathbb{N}},$$

i.e., η_j takes values

$$\ln p_{0j},\ \ln p_{1j}, ...,\ \ln p_{n_j-1,j}$$

with probabilities $p_{0,j},\ p_{1,j},\ ...,\ p_{n_j-1,j}$. It is clear that

$$E\eta_j = \sum_{i=0}^{n_j-1} p_{ij} \ln p_{ij} = -h_j$$

and

$$|E\eta_j| \le \ln n_j.$$

Let us show that

$$E\eta_j^2 = \sum_{i=0}^{n_j-1} p_{ij} \ln^2 p_{ij} \le \max\{4, \ln^2 n_j\}.$$

To this end let $z_0 \in \mathbb{R} \setminus \{1\}$ be the non-trivial root of equation

$$\ln(z) - 2z + 2 = 0.$$

Here $\varphi : [0,1] \to \mathbb{R}$ denotes the function such that

$$\varphi(z) = \begin{cases} z \ln^2 z, & \text{if} \quad z \in [0, z_0); \\ -z_0 \ln^2 z_0 \cdot \frac{z - z_0}{1 - z_0} + z_0 \ln^2 z_0, & \text{if} \quad z \in [z_0, 1]. \end{cases}$$

From the definition of $\varphi(z)$ it follows that

$$z \ln^2 z \leq \varphi(z), \quad \forall z \in [0,1].$$

The function $\varphi(z)$ is convex on $[0,1]$. Therefore,using the Jensen's inequality we have

$$E\eta_j^2 \leq \sum_{i=0}^{n_j - 1} \varphi(p_{ij}) \leq n_j \varphi\left(\frac{1}{n_j}\right) \leq \max\{4, \ln^2 n_j\}.$$

Hence,

$$D(\eta_j) = E\eta_j^2 - (E\eta_j)^2 \leq 2\max\{4, \ln^2 n_j\}.$$

Applying Kolmogorov's theorem ([Shi96, Ch IV, §3.2]) and the assumption (2.12) of the theorem, we get for μ_ξ-almost all points $x \in [0,1]$:

$$\lim_{k \to \infty} \frac{(\eta_1(x) + \eta_2(x) + ... + \eta_k(x)) - E\left(\eta_1(x) + \eta_2(x) + ... + \eta_k(x)\right)}{\ln(n_1 n_2 ... n_k)} = 0. \tag{2.13}$$

We remark that

$$E(\eta_1 + \eta_2 + ... + \eta_k) = -H_k,$$

and

$$\lambda(\Delta_k(x)) = \frac{1}{n_1 n_2 ... n_k}.$$

Let $D := \underline{\lim}_{k\to\infty} \frac{H_k}{\ln(n_1 n_2 ... n_k)}$ and let us consider the set

$$T = \left\{ x : \lim_{k\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} - \frac{H_k}{-\ln \lambda(\Delta_k(x))} \right) = 0 \right\}$$

$$= \left\{ x : \lim_{k\to\infty} \left(\frac{\eta_1 + \eta_2 + ... + \eta_k - M(\eta_1 + \eta_2 + ... + \eta_k)}{\ln(n_1 n_2 ... n_k)} \right) = 0 \right\}.$$

Since $\mu_\xi(T) = 1$, we deduce that $\dim_{\mu_\xi}(T, \mathcal{A}) = 1$. Let

$$T_1 = \left\{ x : \underline{\lim}_{k\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} - \frac{H_k}{-\ln \lambda(\Delta_k(x))} \right) = 0 \right\};$$

$$T_2 = \left\{ x : \underline{\lim}_{k\to\infty} \frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} \leq \underline{\lim}_{k\to\infty} \frac{H_k}{-\ln \lambda(\Delta_k(x))} \right\}$$

$$= \left\{ x : \underline{\lim}_{k\to\infty} \frac{\ln \mu_\xi(\Delta_k(x))}{\ln \lambda(\Delta_k(x))} \leq D \right\};$$

$$T_3 = \left\{ x : \underline{\lim}_{k\to\infty} \frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} \geq \underline{\lim}_{k\to\infty} \frac{H_k}{-\ln \lambda(\Delta_k(x))} \right\}$$

$$= \left\{ x : \underline{\lim}_{k\to\infty} \frac{\ln \mu_\xi(\Delta_k(x))}{\ln \lambda(\Delta_k(x))} \geq D \right\}.$$

It is obvious that $T \subset T_1$. Let us show inclusions $T_1 \subset T_3$ and $T \subset T_2$.

We will use the well known inequality

$$\underline{\lim}_{k\to\infty} (x_k - y_k) \leq \underline{\lim}_{k\to\infty} (x_k) - \underline{\lim}_{n\to\infty} (y_k),$$

(except the cases "$\infty - \infty$" and "$-\infty + \infty$").

If $x \in T_1$, then

$$\underline{\lim}_{k\to\infty} \frac{\ln \mu_\xi(\Delta_k(x))}{\ln \lambda(\Delta_k(x))} - D =$$

$$= \underline{\lim}_{k\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\ln \lambda(\Delta_k(x))} \right) - \underline{\lim}_{k\to\infty} \frac{H_k}{-\ln \lambda(\Delta_k(x))} \geq$$

$$\geq \quad \underline{\lim}_{k\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} - \frac{H_k}{- \ln \lambda(\Delta_k(x))} \right) = 0.$$

Therefore, $x \in T_3$. If $x \in T$, then

$\lim_{k\to\infty} \left(\frac{\eta_1(x)+\eta_2(x)+...+\eta_k(x)}{\ln \lambda(\Delta_k(x))} - \frac{H_k}{- \ln \lambda(\Delta_k(x))} \right) = 0$ and

$$\underline{\lim}_{k\to\infty} \frac{H_k}{- \ln \lambda(\Delta_k(x))} - \underline{\lim}_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} \right) \geq$$

$$\geq \underline{\lim}_{k\to\infty} \left(\frac{H_k}{- \ln \lambda(\Delta_k(x))} - \frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} \right) =$$

$$= - \overline{\lim}_{k\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_k(x)}{\ln \lambda(\Delta_k(x))} - \frac{H_k}{- \ln \lambda(\Delta_k(x))} \right) = 0$$

Hence $x \in T_2$.

Since $T \subset T_2$, we have

$$\dim_\lambda(T, \mathcal{A}) \leq \dim_\lambda(T_2, \mathcal{A}).$$

From Theorem 1.3 it follows that $\dim_\lambda(T_2, \mathcal{A}) \leq D$. So,

$$\dim_\lambda(T, \mathcal{A}) \leq D.$$

From Theorem 1.4 and the inclusion $T \subset T_3$, we deduce that

$$\dim_\lambda(T, \mathcal{A}) \geq D \cdot \dim_{\mu_\xi}(T, \mathcal{A}) = D \cdot 1 = D.$$

So,

$$\dim_\lambda(T, \mathcal{A}) = D.$$

Since λ is Lebesgue measure on $[0, 1]$, we have $\dim_H(T, \mathcal{A}) = \dim_\lambda(T, \mathcal{A}) = D$.

From our assumption (2.12) it follows that

$$\lim_{k\to\infty} \frac{\ln n_k}{\ln (n_1 n_2 ... n_k)} = 0.$$

According to theorem 2.1, we have that the family $\mathcal{A}$ of Cantor coverings is faithful for the Hausdorff-Besicovitch dimension calculation. So,

$$\dim_H(T, \mathcal{A}) = \dim_H(T) = D.$$

We now prove that the above constructed set T is the "smallest" support of the measure μ_ξ in the sense of the Hausdorff-Besicovitch dimension. Let M be an arbitrary support of the measure μ_ξ. It is easily seen that the set $M_1 := M \cap T$ is also a support of the same measure μ_ξ, and $M_1 \subset M$. So,

$$\dim_H(M_1) \le \dim_H(M)$$

and $M_1 \subset T$. We shall now prove that

$$\dim_H(M_1) = \dim_H(T).$$

From $M_1 \subset T$ it follows that $\dim_H(M_1) \le \dim_H(T) = D$. On the other hand, we have

$$M_1 \subset T \subset T_3 = \left\{ x : \lim_{k\to\infty} \frac{\ln \mu_\xi(\Delta_k(x))}{\ln \lambda(\Delta_k(x))} \ge D \right\}.$$

Therefore, by using the faithfulness of the family of Cantor coverings and Theorem 1.4, we conclude

$$\dim_H(M_1) = \dim_\lambda(M_1, \mathcal{A}) \ge D \cdot \dim_{\mu_\xi}(M_1, \mathcal{A}) = D \cdot 1 = D.$$

So, $\dim_H(M_1) = D = \dim_H(T)$.

□

2.3 Faithfulness and comparability of covering families

The following two definitions provide a natural connection between the faithfulness property of covering families and comparability property of Hausdorff measures.

Definition 2.2. A fine family of coverings Φ_W of bounded set W is called *comparable* if $\forall\alpha > 0$ the correspondent Hausdorff measure $H^\alpha(\cdot, \Phi_W)$ is comparable with Hausdorff measure (Definition 1.21) .

Definition 2.3. A fine family of coverings Φ_W of bounded set W is called *non-comparable* if there is $\alpha > 0$ such that the correspondent Hausdorff measure $H^\alpha(\cdot, \Phi_W)$ is non-comparable with Hausdorff measure (Definition 1.22).

Remark 2.2. By the definition, a arbitrary comparable family of covering Φ_W is faithful for the calculation of Hausdorff dimension on W.

Remark 2.3. Under the assumptions of proposition 1.1, a fine family of coverings is not only faithful for the calculation of Hausdorff dimensionon $[0, 1]$ but also comparable.

Now let us consider examples which show essential differences between the notions of *faithful family of coverings* and *comparable family of coverings.*

Theorem 2.3. *Let $n_k = 4^k$, $\forall k \in \mathbb{N}$ and let Φ be the fine family of coverings generated by the corresponding Cantor series expansion. Then Φ is faithful for the calculation of Hausdorff dimensionon $[0, 1)$ and non-comparable.*

Proof. Let us consider a set

$$A = \left\{ x \in [0, 1] : \ x = \sum_{k=1}^{\infty} \frac{\alpha_k(x)}{\prod_{i=1}^{k} n_i}, \ \alpha_k(x) \in \{0, 1, ..., 2^k - 1\}, \quad \forall k \in \mathbb{N} \right\},$$

and prove that

$$\dim_H A = \frac{1}{2},\ H^{\frac{1}{2}}(A, \Phi) \geq 1 \text{ and } H^{\frac{1}{2}}(A) = 0.$$

Let λ be the Lebesgue measure on the unit interval and let μ_ξ be the probability measure of the random variable

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{\prod_{i=1}^{k} n_i},$$

where ξ_k are independent random variables taking values $0,\ 1,\ \ldots,\ 2^k - 1$ with probabilities $\frac{1}{2^k}$. by Theorems 1.3 and 1.4 and since A is speculum of the measure μ_ξ, we see that $\dim_H(A, \Phi) = \frac{1}{2}$. By Theorem 2.1, the family Φ faithful for the calculation of Hausdorff dimensionon $[0, 1)$. So,

$$\dim_H(A) = \dim_H(A, \Phi) = \frac{1}{2}.$$

Let $\{E_j\}_{j \in \mathbb{N}}$ be an arbitrary ε-covering of the set A by cylinders from Φ. Without loss of generality we may assume that $E_j \cap A \neq \emptyset$, i.e., $E_j = \Delta_{n_j}(x)$ for some $x \in A$. Applying the mass distributional principle, we have

$$1 = \mu(A) \leq \mu(\bigcup_j E_j) \leq \sum_j \mu(E_j) = \sum_j |E_j|^{\frac{1}{2}}$$

for any ε-covering of A by cylinders from Φ. Therefore, $H^{\frac{1}{2}}(A, \Phi) \geq 1$.

The set A can be covered by $2^1 \cdot 2^2 \cdot \ldots \cdot 2^{k-1} \cdot 1$ intervals (each of them is a union of 2^k k-th rank cylinders) with length 2^{-k^2}. The $\frac{1}{2}$-volume of this covering is equal to $2^{\frac{(k-1)k}{2}} \cdot \left(2^{-k^2}\right)^{\frac{1}{2}}$, which tends to 0 as $k \to \infty$. Therefore, $H^{\frac{1}{2}}(A) = 0$, and the proof is complete.

□

The following example shows that a faithful family of covering can be "extremely non-comparable".

Theorem 2.4. *Let $n_k = 4^k$ and let Φ be the family of coverings generated by the corresponding Cantor series expansion. Let*

$$T = \left\{ x \in [0,1]: \; x = \sum_{k=1}^{\infty} \frac{\alpha_k(x)}{\prod_{i=1}^{k} 4^i} \quad \text{with} \;\; \alpha_k(x) \in \{0, ..., \sqrt{n_k} - 1\} \;\; \text{if } k \neq 2^s \quad \text{and} \right.$$

$$\left. \alpha_k(x) \in \{0, ..., k \cdot \sqrt{n_k} - 1\} \quad \text{if} \;\; k = 2^s, \;\; s \in \mathbb{N} \right\}.$$

Then the family Φ is faithful for the Hausdorff dimension calculation on $[0,1]$ and

$$\dim_H T = \frac{1}{2}, \;\; H^{\frac{1}{2}}(T, \Phi) = +\infty, \;\; H^{\frac{1}{2}}(T) = 0.$$

Proof. Let μ_ξ be the probability measure with respect to the random variable

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{\prod_{i=1}^{k} n_i},$$

where ξ_k are independent random variables with following distributions:

if $k \neq 2^s$, then ξ_k takes values $0, \; 1, \; ..., \; 2^k - 1$ with probabilities $\frac{1}{2^k}$;

if $k = 2^s$, then ξ_k takes values $0, \; 1, \; ..., \; k \cdot 2^k - 1$ with probabilities $\frac{1}{k \cdot 2^k}$.

Let $\Delta_n(x)$ be the n-th rank cylinder of the Cantor series expansion containing x. From the construction of ξ it follows that for any $x \in T$ one has

$$\mu_\xi(\Delta_n(x)) = 2^{-\left(\frac{n(n+1)}{2} + \frac{([\log_2 n]+1)[\log_2 n]}{2}\right)} \quad \text{and} \quad \lambda(\Delta_n(x)) = 4^{-\frac{n(n+1)}{2}}.$$

So,

$$\lim_{n \to \infty} \frac{\ln \mu_\xi(\Delta_n(x))}{\ln \lambda(\Delta_n(x))} = \frac{1}{2}, \forall x \in T. \tag{2.14}$$

By Theorems 1.3 and 1.4 and the fact that T is the spectrum of the measure μ_ξ, we have

$$\dim_H(T,\Phi) = \frac{1}{2}$$

and the equation

$$\dim_H(T) = \dim_H(T,\Phi) = \frac{1}{2}$$

(see Theorem 2.1). For a given $m \in \mathbb{N}$ let us consider 2^m probability measures $\mu^j,\ j \in \{0, ..., 2^m - 1\}$ corresponding to the random variables

$$\xi^j = \sum_{k=1}^{\infty} \frac{\xi_k^j}{\prod_{i=1}^{k} n_i},$$

whose independent digits ξ_k^j have the following distributions:

if $k \neq 2^s$, then ξ_k^j takes values $0,\ 1,\ ...,\ 2^k - 1$ with probabilities $\frac{1}{2^k}$;

if $k = 2^s,\ s \neq m$, then ξ_k^j takes values $0,\ 1,\ ...,\ k \cdot 2^k - 1$ with probabilities $\frac{1}{k \cdot 2^k}$;

if $k = 2^m$, then ξ_k^j takes values $j \cdot 2^k + 0,\ j \cdot 2^k + 1,\ ...,\ (j+1) \cdot 2^k - 1$ with probabilities $\frac{1}{2^k}$.

Taking into account inequality $\frac{\ln \mu^j(\Delta_n(x))}{\ln \lambda(\Delta_n(x))} \geq \frac{1}{2}, \forall x \in S_j$, and applying the mass distribution principle simultaneously for all measures m^j, we get $H^{\frac{1}{2}}(T, \mathcal{A}) \geq 2^m$.

The length of cylinders of 2^m-th rank is $2^{-2^m(2^m+1)}$. Let

$$\varepsilon(m) \leq 2^{-2^m(2^m+1)}.$$

Then, any set $\Delta \in \Phi$ with $\lambda(\Delta) < \varepsilon(m)$ and $\Delta \cap S^j \neq \emptyset$, has an empty intersection with the spectrum of all other S_l when $l \neq j$.

Let $\{E_v\}_{v \in \mathbb{N}}$ be some $\varepsilon(m)$ - covering of T and $E_v \in \Phi,\ \forall v \in \mathbb{N}$. Without loss of generality let every set of the family $\{E_v\}_{v \in \mathbb{N}}$ has not-empty intersection with the interior of the set T. For any

$$E_v,\ \exists!\ j \in \{0, ..., 2^m - 1\} :\ E_v \cap S_j \neq 0.$$

So $\{E_v\}_{v\in\mathbb{N}}$ can be split into 2^m groups such that $\{E_v^j\}_{v\in\mathbb{N}}$ forms a covering of S_j.

Let

$$\Delta \in \Phi \text{ and } \lambda(\Delta) < \varepsilon(m),\ \Delta \cap T \neq \emptyset.$$

Let $n := n(\Delta)$ be the rank of Δ. If $\Delta \in \{E_v^j\}_{v\in\mathbb{N}}$, then

$$\mu^j(\Delta) = 2^{-\left(\frac{n(n+1)}{2} + \frac{([\log_2 n]+1)[\log_2 n]}{2} - m\right)}$$

and

$$\frac{\ln(\mu^j(\Delta))}{\ln(\lambda(\Delta))} = \frac{\ln\left(2^{-\left(\frac{n(n+1)}{2} + \frac{([\log_2 n]+1)[\log_2 n]}{2} - m\right)}\right)}{\ln\left(2^{-n(n+1)}\right)}$$

$$= \frac{1}{2} + \frac{\frac{([\log_2 n]+1)[\log_2 n]}{2} - m}{n(n+1)} \geq \frac{1}{2}.$$

So,

$$\mu^j(\Delta) \leq (\lambda(\Delta))^{\frac{1}{2}} \text{ with } \Delta \in \{E_v^j\}_{v\in\mathbb{N}}.$$

Let $W_j = \{v : E_v \cap S_j \neq \emptyset\}$. Therefore,

$$\sum_{v=1}^{\infty} \lambda(E_v)^{\frac{1}{2}} = \sum_{j=0}^{2^m-1} \sum_{v:v\in W_j} \lambda(E_v)^{\frac{1}{2}}$$

$$\geq \sum_{j=0}^{2^m-1} \sum_{v:v\in W_j} \mu^j(E_v) \geq \sum_{j=0}^{2^m-1} 1 = 2^m.$$

So, for any $\{E_v\}_{v\in\mathbb{N}} - \varepsilon(m)$ - covering we have

$$\sum_{v=1}^{\infty} \lambda(E_v)^{\frac{1}{2}} \geq 2^m,$$

which results $H^{\frac{1}{2}}_{\varepsilon(m)}(T, \Phi) \geq 2^m$ and $H^{\frac{1}{2}}(T, \Phi) \geq 2^m$. So,

$$H^{\frac{1}{2}}(T, \Phi) = +\infty.$$

On the other hand, the set T can be covered by

$$2^1 \cdot 2^2 \cdot \ldots \cdot 2^{2^s-1} \cdot 2^1 \cdot 2^2 \ldots 2^{s-1} \cdot 1 = 2^{\frac{(2^s-1)2^s}{2} + \frac{(s-1)s}{2}}$$

semi-intervals, each of them is a union of $2^s 2^{2^s}$ cylinders from Φ_{2^s} with length

$$\left(\frac{1}{4}\right)^{\frac{2^s(2^s-1)}{2}} \cdot \frac{2^s 2^{2^s}}{4^{2^s}} = \left(\frac{1}{2}\right)^{2^{2s}-s}.$$

The $\frac{1}{2}$-volume of this covering is equal to

$$2^{\frac{(2^s-1)2^s}{2} + \frac{(s-1)s}{2}} \left(2^{-2^{2s}+s}\right)^{\frac{1}{2}} = 2^{-\frac{1}{2}(2^s - s^2)} \to 0, \quad (s \to \infty).$$

Therefore, $H^{\frac{1}{2}}(T) = 0$.

□

Remark 2.4. The last proposition shows extreme differences between comparable and faithful net-coverings and demonstrates that the class of faithful net-coverings is essentially wider then the class of comparable ones. The relation between these two classes is similar to the relation between bi-Lipshitz transformations and transformations preserving the Hausdorff dimension (see, e.g., [APT04, APT08] for details). Deeper connections between faithfulness of net-coverings and the theory of transformations preserving the

Hausdorff dimension will also be discussed in the forthcoming Section 2.5 .

2.4 Hausdorff dimension of the spectrum of the Random Cantor expansions

Let us recall the definition of the Random Cantor expansion. The random variable ξ

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k}, \tag{2.15}$$

where independent random variables ξ_k take values $0, 1, ..., n_k - 1$ with probabilities p_{0k}, p_{1k}, ..., $p_{n_k-1,k}$, respectively, ($\sum_{i=0}^{n_k-1} p_{i,k} = 1$ and $1 < n_k \in \mathbb{N}$, $\forall k \in \mathbb{N}$) is called *Random Cantor expansion.*

Proposition 2.1. *Let m_k be the number of non-zero elements p_{ij}, $i \in \{0, \ ..., \ n_k - 1\}$. Assume that*

$$\sum_{k=1}^{\infty} \left(\frac{\ln n_k}{\ln \prod_{i=1}^{k} n_i} \right)^2 < \infty. \tag{2.16}$$

Then the Hausdorff dimension of the spectrum S_ξ of the Random Cantor expansion ξ is equal to

$$\dim_H(S_\xi) = \varliminf_{k\to\infty} \frac{\ln(m_1 m_2 ... m_k)}{\ln(n_1 n_2 ... n_k)}. \tag{2.17}$$

Proof. The lower estimate of the Hausdorff dimension of the spectrum S_ξ follows from Theorem 2.2. Indeed, let us construct additional measure μ_{ξ^*} with the spectrum of μ_ξ.

So, we consider a random variable

$$\xi^* = \sum_{k=1}^{\infty} \frac{\xi_k^*}{n_1 \cdot n_2 \cdot ... \cdot n_k},$$

where $\{\xi_k^*\}_{k\geq 1}$ are independent random variables taking values $0, 1, ..., n_k - 1$ with probabilities

$$p_{ik}^* = \begin{cases} 0, \text{ if } \ p_{ik} = 0; \\ \frac{1}{m_k}, \text{ if } p_{ik} > 0. \end{cases}.$$

By Theorem 2.2 and equality $h_k^* = -\sum\limits_{i=0}^{n_k-1} p_{ik}^* \ln p_{ik}^* = \ln m_k$, the Hausdorff dimension of measure μ_{ξ^*} is equal to $\dim_H \mu_{\xi^*} = \varliminf\limits_{k\to\infty} \frac{\ln(m_1\cdot\ldots\cdot m_k)}{\ln(n_1\cdot\ldots\cdot n_k)}$. Therefore

$$\dim_H S_\xi \geq \varliminf_{k\to\infty} \frac{\ln(m_1\cdot\ldots\cdot m_k)}{\ln(n_1\cdot\ldots\cdot n_k)}.$$

On the other hand, the spectrum S_ξ can be covered by $m_1 \cdot m_2 \cdot \ldots \cdot m_k$ interval with length $\frac{1}{n_1\cdot\ldots\cdot n_k}$.

The α-volume of this covering is equal to $m_1 \cdot m_2 \cdot \ldots \cdot m_k \frac{1}{(n_1\cdot\ldots\cdot n_k)^\alpha}$. Hence $H_\varepsilon^\alpha(S_\xi) \leq m_1 \cdot m_2 \cdot \ldots \cdot m_k \frac{1}{(n_1\cdot\ldots\cdot n_k)^\alpha}$, $\forall \varepsilon > \frac{1}{(n_1\cdot\ldots\cdot n_k)}$. If $\alpha > B := \varliminf\limits_{k\to\infty} \frac{\ln(m_1\cdot\ldots\cdot m_k)}{\ln(n_1\cdot\ldots\cdot n_k)}$ then there is subsequence $\{k_s\}_{s\geq 1}$ such that

$$\frac{\ln(m_1\cdot\ldots\cdot m_{k_s})}{\ln(n_1\cdot\ldots\cdot n_{k_s})} < \frac{B+\alpha}{2}, \ \forall s \in \mathbb{N}.$$

It follows that

$$\frac{m_1\cdot\ldots\cdot m_{k_s}}{(n_1\cdot\ldots\cdot n_{k_s})^{\frac{B+\alpha}{2}}} < 1, \ \forall s \in \mathbb{N}.$$

Consequently

$$\lim_{s\to\infty} \frac{m_1\cdot m_2\cdot\ldots m_{k_s}}{(n_1\cdot n_2\cdot\ldots\cdot n_{k_s})^\alpha} = 0.$$

So,

$$H_\varepsilon^\alpha(S_\xi) = 0, \ \forall \varepsilon > 0, \forall \alpha > B,$$

and it follows that $H^\alpha(S_\xi) = 0, \forall \alpha > B$. Thus $\dim_H S_\xi \leq B$.

$\square$

Remark 2.5. From the above it follows that the Hausdorff dimension of the spectrum of random variable of the Random Cantor expansion ξ equals

$$\lim_{k\to\infty} \frac{\ln(m_1 m_2 ... m_k)}{\ln(n_1 n_2 ... n_k)}, \tag{2.18}$$

when

$$\sup_{k\in\mathbb{N}} n_k < \infty.$$

Moreover, formula (2.18) is true even in the case when (2.16) holds. One can think that the formula is true without any additional restrictions on $\{n_k\}_{k\in\mathbb{N}}$.

Proposition 2.2. *The Hausdorff dimension of the spectrum S_ξ of the Random Cantor expansion ξ does not equal to*

$$\lim_{k\to\infty} \frac{\ln(m_1 m_2 ... m_k)}{\ln(n_1 n_2 ... n_k)}$$

in general.

Proof. Set

$$n_i = 2^{4\cdot 5^{i-1}},\ \forall i \in \mathbb{N}.$$

Let

$$p_{ij} = \frac{1}{\sqrt{n_j}}, \forall i \in \left\{0,\ ...,\ \sqrt{n_j} - 1\right\}$$

and

$$p_{ij} = 0, \forall i \in \left\{\sqrt{n_j},\ ...,\ n_j - 1\right\},$$

the spectrum of of the Random Cantor expansion ξ is

$$T = \left\{ x : x \in [0,1],\ x = \sum_{k=1}^{\infty} \frac{\alpha_k(x)}{\prod_{i=1}^{k} n_i},\ \alpha_k(x) \in \{0, ..., \sqrt{n_k} - 1\},\ \forall k \in \mathbb{N} \right\}.$$

Moreover,

$$\overline{\lim_{k\to\infty}} \frac{\ln n_k}{\ln n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1}} =$$

$$= \lim_{k\to\infty} \frac{\ln 2^{4\cdot 5^{k-1}}}{\ln 2^4 \cdot 2^{4\cdot 5} \cdot \ldots \cdot 2^{4\cdot 5^{k-2}}} = \lim_{k\to\infty} \frac{4\cdot 5^{k-1}}{4\cdot(1+5+...+5^{k-2})} =$$

$$= \lim_{k\to\infty} \frac{4\cdot 5^{k-1}}{(5^{k-1}-1)} = 4.$$

By the proof of inequality (2.7), the Hausdorff dimension of the set T satisfies the equality

$$\dim_H(T) \leq \frac{2}{2+4} = \frac{1}{3}.$$

However

$$\underline{\lim_{k\to\infty}} \frac{\ln(m_1 m_2 ... m_k)}{\ln(n_1 n_2 ... n_k)}$$

equals

$$\underline{\lim_{k\to\infty}} \frac{\ln\left(\sqrt{2^4}\sqrt{2^{4\cdot 5}}...\sqrt{2^{4\cdot 5^{k-1}}}\right)}{\ln\left(2^4\cdot 2^{4\cdot 5}\cdot\ldots\cdot 2^{4\cdot 5^{k-1}}\right)} = \frac{1}{2}.$$

So,

$$\dim_H(S_\xi) = \dim_H(T) \leq \frac{1}{3} < \frac{1}{2} = \underline{\lim_{k\to\infty}} \frac{\ln(m_1 m_2 ... m_k)}{\ln(n_1 n_2 ... n_k)}.$$

□

2.5 Transformations preserving the Hausdorff dimension and Random Cantor expansions

The group theoretic approach to geometry (Klein's programm) is well known. What is the "fractal geometry" from this point of view? The monograph [Fal04] contains an attempt to answer the question saying that "... one approach to fractal geometry is to regard two sets as "the same" if there is a bi-Lipschitz mapping between them", i.e., fractal geometry is in this sense the study of invariants of bi-Lipschitz transformations (and, thus, affine geometry may be considered as a part of fractal geometry). In [APT04] a view on fractal geometry was proposed in the same spirit, but with a more general definition of allowable mappings. It was shown that the group G of all DP-transformations (one to one mappings which preserve the Hausdorff dimension of every subset) is essentially larger than the group of bi-Lipschitz transformations, and the smoothness and bi-Lipschitz properties of transformations are very rough sufficient conditions for dimension preservation. A series of papers (see, e.g., [APT04], [APT08], and references therein) is devoted to the development of a general theory of DP-transformations and to the finding of conditions for the Hausdorff dimension preservation of special classes of transformations. It can be proven (see, e.g., [APT08]) that a one-dimensional transformation f is a DP-transformation of R^1 if and only if f preserves the Hausdorff dimension of every subset of any intervals. So, without loss of generality it is enough to study only DP-transformations of the unit interval. It is also clear that an arbitrary continuous transformation f of $[0,1]$ is either a strictly increasing distribution function F_ξ of some random variable ξ or it is of the form $f = 1 - F_\xi$. Because of this reason it is enough to investigate DP-properties of the distribution functions of random variables ξ whose spectra S_ξ coincide with $[0,1]$. Earlier such DP-transformations f were studied where both sets $N_0 = \{x : f'(x) = 0\}$ and $N_\infty = \left\{x : \lim\limits_{\varepsilon\to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} = +\infty\right\}$ are either finite or they form an at most countable set.

A class of distribution functions of random variables with independent s-adic digits was analyzed in details in [APT08], where necessary conditions and sufficient conditions for dimension preservation under corresponding probability distribution functions were found. Relations between the Hausdorff dimension of the corresponding probability measures, the entropy of probability distributions, and their DP-properties also were discussed in [APT08]. In particular, it was proven that the superfractality ($\dim_H \mu = 1$) of a probability distribution μ is a necessary condition for the Hausdorff dimension preservation under the corresponding probability distribution function. Paper [Tor07] contains a generalization of these results to the case of random variables with independent Q-symbols.

Besides of pure theoretical reasons for the development of the general theory of DP-transformations (for instance, for the creation of an axiomatic theory of fractal geometry), there exists an additional reason for such a study connected with the application of DP-transformations to the construction of new methods for the determination of the Hausdorff dimension of concrete sets (see, e.g., [APT04]).

In this section we proceed with the study of distribution functions of the Random Cantor series, i.e.,

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k},$$

where independent random variables ξ_k take values $0, 1, ..., n_k - 1$ with probabilities p_{0k}, p_{1k}, ..., $p_{n_k-1,k}$, respectively ($\sum_{i=0}^{n_k-1} p_{i,k} = 1$ and $1 < n_k \in \mathbb{N}$, $\forall k \in \mathbb{N}$) Our main aim is to find conditions for the distribution functions of the Random Cantor series to be DP-transformations.

We will need the following assumptions: 1)The "matrix" $\widetilde{P}^* = \|p_{ik}\|$ does not contain zeros;

2) $\prod_{k=1}^{\infty} \max_i p_{ik} = 0$. (In the converse case, the correspondent distributional function is not a bijection of $[0, 1]$).

Define

$$n^* = \sup_{k \in \mathbb{N}} n_k < \infty.$$

Set $p_j := \min_i p_{ij}, \ \forall j \in \mathbb{N}$ and

$$T^{(1)} = \left\{ k : \ k \in \mathbb{N}, \ p_k < \frac{1}{2n^*} \right\}, \quad T_k^{(1)} = T^{(1)} \cap \{1, 2, ..., k\}.$$

Let

$$A := \overline{\lim_{k\to\infty}} \frac{\sum\limits_{j\in T_k^{(1)}} \ln \frac{1}{p_j}}{k}.$$

Theorem 2.5. *Let* $\sup n_k < \infty$*. Then the distributional function* F_ξ *of random Cantor series* ξ *preserves the Hausdorff dimension of any subset of the unit interval iff*

$$\begin{cases} \dim_H \mu_\xi = 1; \\ A = 0. \end{cases} \tag{2.19}$$

Proof. **Sufficient condition.** Set $\dim_H \mu_\xi = 1$ and $A = 0$. We will need the following property of entropy

$$h_k \le \ln n_k. \tag{2.20}$$

Hence the equality $\dim_H \mu_\xi = 1$ is equivalent to

$$\lim_{k\to\infty} \frac{h_1 + h_2 + ... + h_k}{\ln(n_1 \cdot n_2 \cdot ... \cdot n_k)} = 1, \tag{2.21}$$

where $h_j = -\sum\limits_{i=0}^{n_j-1} p_{ij} \ln p_{ij}$ (see Theorem 2.2).

Let ε be an arbitrary positive number such that $\varepsilon < \frac{1}{2n^*}$. Let us consider the following sets:

$$T^+_{\varepsilon,k} = \left\{ j : \quad j \in \mathbb{N},\ j \le k, \quad \left| p_{ij} - \frac{1}{n_j} \right| \le \varepsilon, \quad \forall i \in \{0, ..., n_k - 1\} \right\},$$

$$T^-_{\varepsilon,k} = \{1, 2, ..., k\} \setminus T^+_{\varepsilon,k}.$$

The following lemma helps to analyze the "density" of the set $T^+_{\varepsilon,k}$ in $\mathbb{N}$. Let $| E |$ be a number of elements in a subset E of natural numbers ($E \subset \mathbb{N}$).

Lemma 2.2. *If condition (2.19) holds, then* $\lim\limits_{k\to\infty} \frac{|T^+_{\varepsilon,k}|}{k} = 1$*.*

Proof. Suppose, contrary to our claim, that $\lim\limits_{k\to\infty} \frac{|T^+_{\varepsilon,k}|}{k} \ne 1$.By the above, there is a subsequence $\{k_m\}_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\frac{|T^+_{\varepsilon,k_m}|}{k_m}=C<1.$$

By the inequality (2.20), for every $\varepsilon>0$ there is a positive constant $\delta=\delta(\varepsilon)$ such that $h_j\le(1-\delta)\ln k_j$ for every $j\in T^-_{\varepsilon,k}$. Consequently

$$\frac{\sum\limits_{j=1}^{k_m}h_j}{\ln(n_1\cdot\ldots\cdot n_{k_m})}=\frac{\sum\limits_{j\in T^+_{\varepsilon,k_m}}h_j+\sum\limits_{j\in T^-_{\varepsilon,k_m}}h_j}{\ln(n_1\cdot\ldots\cdot n_{k_m})}$$

$$\le\frac{\sum\limits_{j\in T^+_{\varepsilon,k_m}}\ln n_j+(1-\delta)\sum\limits_{j\in T^-_{\varepsilon,k_m}}\ln n_j}{\ln(n_1\cdot\ldots\cdot n_{k_m})}\le 1-\delta\frac{\sum\limits_{j\in T^-_{\varepsilon,k}}\ln n_j}{\ln(n_1\cdot\ldots\cdot n_{k_m})}.$$

So,

$$\frac{\sum\limits_{j=1}^{k_m}h_j}{\ln(n_1\cdot\ldots\cdot n_{k_m})}\le 1-\delta\frac{|T^-_{\varepsilon,k_m}|\ln 2}{k_m\ln n^*}. \tag{2.22}$$

By the inequality (2.22), there is

$$1=\lim_{n\to\infty}\frac{h_1+h_2+\ldots+h_{k_m}}{\ln(n_1\cdot\ldots\cdot n_{k_m})}\le 1-\delta\frac{\ln 2}{\ln n^*}\left(1-C\right).$$

So, we get a contradiction.

□

The set $T^-_{\varepsilon,k}$ can be represented as a union :

$$T^-_{\varepsilon,k}=T^{(1)}_k\cup T_{\varepsilon,k},$$

where $T^{(1)}_k$ is defined above and $T_{\varepsilon,k}=T^-_{\varepsilon,k}\setminus T^{(1)}_k$. By Lemma 2.2, it follows that

$$\lim_{k\to\infty}\frac{|T^-_{\varepsilon,k}|}{k}=\lim_{k\to\infty}\frac{|T^{(1)}_k|}{k}=\lim_{k\to\infty}\frac{|T_{\varepsilon,k}|}{k}=0.$$

Let λ be Lebesgue measure on the unit interval. Let $x\in[0,1)$. Let $\Delta_{\alpha_1(x)\ldots\alpha_k(x)}$ be a cylinder such that $x \in \Delta_{\alpha_1(x)\ldots\alpha_k(x)}$ and $\lambda\left(\Delta_{\alpha_1(x)\ldots\alpha_k(x)}\right) \le \varepsilon$. So,

$$-\ln\mu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)}) = -\left(\ln[\prod_{j=1}^{k} p_{\alpha_j(x)j}]\right)$$

$$= -\left(\sum_{j\in T_k^{(1)}} \ln p_{\alpha_j(x)j} + \sum_{j\in T_{\varepsilon,k}} \ln p_{\alpha_j(x)j} + \sum_{j\in T_{\varepsilon,k}^{+}} \ln p_{\alpha_j(x)j}\right).$$

Obviously,

$$\sum_{j\in T_{\varepsilon,k}} \ln\frac{1}{p_{\alpha_j(x)j}} \le |T_{\varepsilon,k}|\ln(2n^*)$$

and

$$\sum_{j\in T_{\varepsilon,k}^{+}} \ln\frac{1}{p_{\alpha_j(x)j}} \le \sum_{j\in T_{\varepsilon,k}^{+}} \ln\frac{1}{\frac{1}{n_j}-\varepsilon}$$

$$= \sum_{j\in T_{\varepsilon,k}^{+}} \left(\ln n_j + \ln\left(1+\frac{n_j\varepsilon}{1-n_j\varepsilon}\right)\right) \le$$

$$\le \sum_{j\in T_{\varepsilon,k}^{+}} \ln n_j + |T_{\varepsilon,k}^{+}|\frac{\varepsilon n^*}{1-\varepsilon n^*}.$$

By the above, it follows that

$$\overline{\lim_{k\to\infty}} \frac{\ln\mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln\lambda(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} \le 1 + \frac{\varepsilon n^*}{(1-\varepsilon n^*)\ln 2},$$

where $x \in [0,1)$ and $\varepsilon < \frac{1}{n^*}$.

However,

$$\sum_{j\in T_{\varepsilon,k}} \ln\frac{1}{p_{\alpha_j(x)j}} > |T_{\varepsilon,k}|\ln\frac{2n^*}{2n^*-1}$$

and

$$\sum_{j\in T_{\varepsilon,k}^{+}} \ln\frac{1}{p_{\alpha_j(x)j}} \ge \sum_{j\in T_{\varepsilon,k}^{+}} \ln\left(\frac{1}{\frac{1}{n_j}+\varepsilon}\right)$$

$$= \sum_{j \in T^+_{\varepsilon,k}} \left(\ln n_j + \ln \frac{\frac{1}{n_j}}{\frac{1}{n_j} + \varepsilon} \right) \geq \sum_{j \in T^+_{\varepsilon,k}} \ln n_j - |T^+_{\varepsilon,k}| \left(1 + n^* \varepsilon\right).$$

Therefore, for any $x \in [0,1)$ and for any $\varepsilon < \frac{1}{2n^*}$ we have

$$\varliminf_{k \to \infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} \geq 1 - \frac{\ln(1 + n^*\varepsilon)}{\ln 2}.$$

Hence for every $x \in [0,1)$ we have

$$\lim_{k \to \infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = 1. \tag{2.23}$$

The next lemma connects the property of faithfulness of coverings

$$\mathcal{A}' := \{F_\xi(E) : E \in \Phi\}$$

for the Hausdorff dimension of any subset of the unit interval $[0,1)$ and the property "to be DP - transformation" of the distribution function F_ξ.

Let us recall that: 1) F_ξ is a distributional function of Random Cantor expansion and $\widetilde{P}^* = ||p_{ik}||$ is a correspondent “matrix” with $p_{ik} > 0$ and

$$\prod_{k=1}^{\infty} \max_i p_{ik} = 0.$$

2) $\mathcal{A}$ be the family of cylinders of Cantor expansion.

Let $\mathcal{A}'$ be a family of $\widetilde{Q}^*$- cylinders such that

$$\mathcal{A}' := \{F_\xi(E) : E \in \mathcal{A}\}$$

(or $\widetilde{Q}^* = \widetilde{P}^*$).

Lemma 2.3. *Assume that*

$$\lim_{k\to\infty}\frac{\ln\mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln\lambda(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}=1, \tag{2.24}$$

for all $x\in[0,1)$. Then

1. *$\mathcal{A}'$ is faithful for the Hausdorff dimension calculation on $[0,1)$;*

2. *F_ξ is DP - trunsformation of $[0,1]$;*

3. *first and second items are equivalent.*

Proof. By the condition (2.24), Theorem 1.3 and Theorem 1.4, we have

$$\dim_\lambda(E,\mathcal{A})=1\cdot\dim_{\mu_\xi}(E,\mathcal{A}),\forall E\subset[0,1),$$

where $\dim_\lambda(E,\mathcal{A})$ and $\dim_{\mu_\xi}(E,\mathcal{A})$ are Hausdorff - Billingsley dimension (1.4) with respect to measures λ and μ_ξ.

Since

$$\dim_H(E)=\dim_H(E,\mathcal{A})=\dim_\lambda(E,\mathcal{A}),$$

$$\dim_{\mu_\xi}(E,\mathcal{A})=\dim_H\left(F_\xi(E),\mathcal{A}'\right),\forall E\subset[0,1)$$

and the above remark, we have

$$\dim_H(E)=\dim_H\left(F_\xi(E),\mathcal{A}'\right),\ \forall E\subset[0,1). \tag{2.25}$$

If $\mathcal{A}'$ is faithful for the Hausdorff dimension calculation on $[0,1)$, then

$$\dim_H\left(E',\mathcal{A}'\right)=\dim_H\left(E'\right),\ \forall E'\subset[0,1).$$

By (2.25) and assumption of faithfulness of $\mathcal{A}'$, it follows that

$$\dim_H(E) = \dim_H\left(F_\xi(E)\right), \ \forall E \subset [0,1],$$

i.e., F_ξ is a DP - transform of $[0,1]$.

If F_ξ is DP - transform of $[0,1]$, then

$$\dim_H\left(E'\right) = \dim_H\left(F_\xi^{-1}(E')\right), \ \forall E' \subset [0,1].$$

By equation (2.25), it follows that

$$\dim_H\left(E', \mathcal{A}'\right) = \dim_H(F_\xi^{-1}(E')), \ \forall E' \subset [0,1)$$

and we get

$$\dim_H\left(E'\right) = \dim_H\left(E', \mathcal{A}'\right), \ \forall E' \subset [0,1).$$

Hence $\mathcal{A}'$ is faithful for the Hausdorff dimension calculation on $[0,1)$.

Let us show that $\mathcal{A}'$ is faithful (the second part (2.) of the lemma follows immediately). Let E' be an arbitrary set of unit interval $[0,1)$ and $E := F_\xi^{-1}\left(E'\right)$. Let $x \in E$ and $\delta > 0$. If the condition (2.24) is true, there exists the minimal number $n_0 := n_0(\delta, x)$ such that $\forall n > n_0$ we have

$$\left|\Delta^{\mathcal{A}}_{\alpha_1(x)\ldots\alpha_n(x)}\right|^{1+\delta} \leq \left|\Delta^{\mathcal{A}'}_{\alpha_1(x')\ldots\alpha_n(x')}\right| \leq \left|\Delta^{\mathcal{A}}_{\alpha_1(x)\ldots\alpha_n(x)}\right|^{1-\delta}, \tag{2.26}$$

where $x' := F_\xi(x)$. Set $\Delta_n(x) := \Delta^{\mathcal{A}}_{\alpha_1(x)\ldots\alpha_n(x)}$ and $\Delta'_n(x') := \Delta^{\mathcal{A}'}_{\alpha_1(x')\ldots\alpha_n(x')}$ for simplicity. Inequality (2.26) can be rewritten in following form

$$\left|\Delta_n(x)\right|^{1+\delta} \leq \left|\Delta'_n(x')\right| \leq \left|\Delta_n(x)\right|^{1-\delta}. \tag{2.27}$$

Let $m \in \mathbb{N}$ and $\delta > 0$ are fixed. Set

$$W_{m,\delta} := \left\{ x : x \in E \wedge |\Delta_n(x)|^{1+\delta} \le \left|\Delta_n^{'}(x^{'})\right| \le |\Delta_n(x)|^{1-\delta},\ \forall n > m \right\}.$$

and

$$W_{m,\delta}^{'} := F_\xi \left(W_{m,\delta} \right).$$

From this

$$W_{1,\delta} \subset W_{2,\delta} \subset ... \subset W_{m,\delta} \subset ...,$$

where

$$E := \bigcup_{m=1}^{\infty} W_{m,\delta}, \forall \delta > 0.$$

Since F_ξ is continuous on $[0,1]$, then F_ξ and F_ξ^{-1} are uniformly continuous on $[0,1]$. Hence $\forall \varepsilon > 0$ there is

$$\varepsilon^{'}(\varepsilon) > 0, \tag{2.28}$$

such that $|I^{'}| \le \varepsilon^{'}(\varepsilon)$ when $|F_\xi^{-1}(I^{'})| \le \varepsilon, \forall I^{'} \subset [0,1]$.

Let us choose ε such that $\left(\frac{1}{n^*}\right)^m = \varepsilon$. Let us consider arbitrary $\varepsilon^{'}-$ covering $\left\{E_j^{'}\right\}_{j \in \mathbb{N}}$ of the set $W_{m,\delta}^{'}$ with $E_j^{'} := [a_j^{'}, b_j^{'})$, $\forall j \in \mathbb{N}$, with $\varepsilon^{'} \le \varepsilon^{'}(\varepsilon)$ (see 2.28). Without loss of generality we will supposed that $E_j^{'} \cap W_{m,\delta}^{'} \ne \emptyset$. Let $E_j := F_\xi^{-1}(E_j^{'}) = [a_j, b_j)$, where $a_j = F_\xi^{-1}(a_j^{'})$, $b_j = F_\xi^{-1}(b_j^{'})$. Then $\{E_j\}_{j \in \mathbb{N}}$ is a ε - covering of the set $W_{m,\delta}$. For fixed $j \in \mathbb{N}$, there exists the cylinder $\Delta_{v_j} \in \mathcal{A}_{v_j}$ such that v_j is minimal rank and $\Delta_{v_j} \subset E_j$. Then the correspondent cylinder $\Delta_{v_j}^{'} := F_\xi \left(\Delta_{v_j}\right) \in \mathcal{A}^{'}$ is a subset of $E_j^{'}$. From $\Delta_{v_j} \subset E_j$ it follows that $|\Delta_{v_j}| \le \varepsilon$ and $v_j \ge m$.

The set $E_j \cap W_{m,\delta}$ can be covered by $2n^*$ cylinders

$$\Delta_{v_j}^0, \Delta_{v_j}^1, ..., \Delta_{v_j}^{l_j}$$

of v_j- rank such that $\Delta^k_{v_j} \cap W_{m,\delta} = \emptyset, \forall k \in \{0, ..., l_j\}$. Also, we have

$$\left|\Delta^0_{v_j}\right| = \left|\Delta^1_{v_j}\right| = ... = \left|\Delta^{l_j}_{v_j}\right| = \frac{1}{\prod\limits_{i=1}^{v_j} n_i}.$$

Since $\Delta^i_{v_j} \cap W_{m,\delta} \neq \emptyset,\ \forall i \in \{0,\ ...,\ l_j\}$ and $\Delta^{'}_{v_j} \subset E^{'}_j$, we have

$$\left|\Delta'^i_{v_j}\right| \leq \left|\Delta^i_{v_j}\right|^{1-\delta} \leq \left|\Delta'^i_{v_j}\right|^{\frac{1-\delta}{1+\delta}} \leq \left|E^{'}_j\right|^{\frac{1-\delta}{1+\delta}}, \forall i \in \{0,\ ...,\ l_j\},$$

where $\Delta'^i_{v_j} := F_\xi\left(\Delta^i_{v_j}\right),\ \forall i \in \{0,\ ...,\ l_j\}$. Hence

$$\left|\Delta'^i_{v_j}\right| \leq \left|E^{'}_j\right|^{\frac{1-\delta}{1+\delta}} \leq \left(\varepsilon^{'}\right)^{\frac{1-\delta}{1+\delta}}, \forall i \in \{0,\ ...,\ l_j\}.$$

Therefore

$$\sum_{i=0}^{l_j} \left|\Delta'^i_{v_j}\right|^{\alpha} \leq 2n^* \cdot \left|E^{'}_j\right|^{\alpha \cdot \frac{1-\delta}{1+\delta}},\ \alpha > 0.$$

Consequently

$$\sum_j \sum_{i=0}^{l_j} \left|\Delta'^i_{v_j}\right|^{\alpha} \leq 2n^* \cdot \sum_j \left|E^{'}_j\right|^{\alpha \cdot \frac{1-\delta}{1+\delta}},\ \alpha > 0. \tag{2.29}$$

Hence $\forall \varepsilon > 0$ and $\forall \varepsilon^{'}$ - covering $\left\{E^{'}_j := [a^{'}_j, b^{'}_j)\right\}_{j \in \mathbb{N}}$ of the set $W^{'}_{m,\delta}$, where $\varepsilon^{'} \leq \varepsilon^{'}(\varepsilon)$, there exists a set of cylinders $\Delta^i_{v_j},\ \forall j \in \mathbb{N},\ i \in \{0, ..., l_j\}$ such that

1. $\left|\Delta^i_{v_j}\right| \leq \left(\varepsilon^{'}\right)^{\frac{1-\delta}{1+\delta}}$;
2. $\sum_j \sum_{i=0}^{l_j} \left|\Delta'^i_{v_j}\right|^{\alpha} \leq 2n^* \cdot \sum_j \left|E^{'}_j\right|^{\alpha \cdot \frac{1-\delta}{1+\delta}},\ \alpha > 0$.

Therefore,

$$H^{\alpha}_{\left(\varepsilon^{'}\right)^{\frac{1-\delta}{1+\delta}}}\left(W^{'}_{m,\delta}, \mathcal{A}^{'}\right) \leq 2n^* \cdot \sum_j \left|E^{'}_j\right|^{\alpha \cdot \frac{1-\delta}{1+\delta}},\ \alpha > 0.$$

Consequently

$$H^{\alpha}_{\left(\varepsilon'\right)^{\frac{1-\delta}{1+\delta}}}\left(W'_{m,\delta},\mathcal{A}'\right) \le 2n^{*}\cdot H^{\alpha\cdot\frac{1-\delta}{1+\delta}}_{\varepsilon'}\left(W'_{m,\delta}\right),\ \alpha>0.$$

We have

$$H^{\alpha}\left(W'_{m,\delta},\mathcal{A}'\right) \le 2n^{*}\cdot H^{\alpha\cdot\frac{1-\delta}{1+\delta}}\left(W'_{m,\delta}\right),\ \alpha>0, \tag{2.30}$$

as $\varepsilon' \to 0$.

Let $\alpha_0 = \inf\left\{\alpha : H^{\alpha\cdot\frac{1-\delta}{1+\delta}}\left(W'_{m,\delta}\right)=0\right\}$, then $\alpha_0\cdot\frac{1-\delta}{1+\delta} = \dim_H\left(W'_{m,\delta}\right)$. Therefore, $\beta>\alpha_0:\ H^{\beta}\left(W'_{m,\delta},\mathcal{A}'\right)=0$. Hence

$$\dim_H\left(W'_{m,\delta},\mathcal{A}'\right) \le \frac{1+\delta}{1-\delta}\cdot\dim_H\left(W'_{m,\delta}\right).$$

Consequently, we have

$$\dim_H\left(E',\mathcal{A}'\right) = \dim_H\left(\bigcup_{m=1}^{\infty} W'_{m,\delta},\mathcal{A}'\right) = \sup_m \dim_H\left(W'_{m,\delta},\mathcal{A}'\right)$$

$$\le \frac{1+\delta}{1-\delta}\sup_m \dim_H\left(W'_{m,\delta}\right) = \frac{1+\delta}{1-\delta}\dim_H\left(E'\right), \forall\delta>0.$$

It follows that

$$\dim_H\left(E',\mathcal{A}'\right) \le \frac{1+\delta}{1-\delta}\dim_H\left(E'\right), \forall\delta>0.$$

By the above,

$$\dim_H\left(E',\mathcal{A}'\right) \le \dim_H\left(E'\right).$$

This proves first part (1.) of the lemma $\dim_H\left(E',\mathcal{A}'\right) = \dim_H\left(E'\right),\ \forall E' \subset [0,1)$.

□

By the lemma 2.3 and (2.23), F_ξ is a DP - transform of the unit interval, when the conditions $\dim_H \mu_\xi = 1$ and $A = 0$ are satisfied.

Necessary condition. Let F_ξ is a DP - transform of the unit interval . Let us show that $\dim_H \mu_\xi = 1$ and $A = 0$.

Let us assume that $\dim_H \mu_\xi < 1$. Then there is a Borel support E of the measure μ_ξ such that $\dim_H(E) < 1$. If $\mu_\xi(E) = 1$, then

$$\dim_H(F_\xi(E)) = 1 \neq \dim(E).$$

That contradicts the assumption $\dim_H \mu_\xi < 1$. Therefore, the condition $\dim_H \mu_\xi = 1$ is necessary for the function F_ξ be a DP - transform.

Let us assume that $A > 0$. Let

$$L = \Big\{x : \ x = \Delta_{\alpha_1 \ldots \alpha_k \ldots}; \ \alpha_k \in \{0, 1, ..., n_k - 1\} \ \text{ if } \ k \notin T^{(1)};$$

$$\alpha_k = f_k \text{ if } k \in T^{(1)}, \ \ \text{with } p_{f_k k} = \min_i p_{ik} \Big\}.$$

The set L is an element of family $C[\widetilde{Q}^*, V_k]$ (see [AKPT11]), where

$$q_{ik} = \frac{1}{n_k}, \ \forall i \in \{0, 1, ..., n_k - 1\}$$

and $V_k = \{0, 1, ..., n_k - 1\}$ when $k \notin T^{(1)}$; $V_k = \{f_k\}$ when $k \in T^{(1)}$.

It is well known that the sets from $C[\widetilde{Q}^*, \{V_k\}]$ have a zero Lebesgue measure iff $\sum\limits_{k=1}^{\infty} W_k = +\infty$, where $W_k = \sum\limits_{i: i \notin V_k} q_{ik}$. By the equalities

$$W_k = \frac{n_k - 1}{n_k} \geq \frac{1}{2}, \ \forall k \in T^{(1)} \text{ and } \ | \, T^{(1)} \, | = +\infty,$$

the set L has a zero Lebesgue measure: $\lambda(L) = 0$.

Let us show that $\dim_H L = 1$. We will need an additional random variable η:

$$\eta = \sum_{k=1}^{\infty} \frac{\eta_k}{n_1 \cdot n_2 \cdot ... n_k},$$

where $\{\eta_k\}_{k\geq 1}$ are independent random variables; if $k \in T^{(1)}$, then $\eta_k = f_k$ with a probability 1; if $k \notin T^{(1)}$, then $\eta_k = i$ with a probability $\frac{1}{n_k}$, $\forall i \in \{0, 1, ..., n_k - 1\}$. It is clear that the set L is a spectrum of the random variable η. Hence

$$\dim_H L \geq \dim_H \mu_\eta.$$

By Theorem 2.2,

$$\dim_H \mu_\eta = \underline{\lim}_{n\to\infty} \frac{h_1 + h_2 + ... + h_k}{\ln (n_1 \cdot n_2 \cdot ... \cdot n_k)},$$

where $h_j = -\sum\limits_{i=0}^{n_j-1} p_{ij} \ln p_{ij}$. So, we have

$$h_j = \begin{cases} \ln n_j, \text{ if } j \notin T^{(1)}; \\ 0, \text{ if } j \in T^{(1)}. \end{cases}$$

Consequently

$$\dim_H \mu_\xi = \underline{\lim}_{k\to\infty} \frac{\sum\limits_{j\in T_k^{+}\cup T_{\varepsilon,k}} \ln n_j}{\ln (n_1 \cdot ... \cdot n_k)}$$

$$= \underline{\lim}_{k\to\infty} \left(1 - \frac{\sum\limits_{j\in T_k^{(1)}} \ln n_j}{\ln (n_1 \cdot ... \cdot n_k)} \right) \geq \underline{\lim}_{k\to\infty} \left(1 - \frac{| T_k^{(1)} | \cdot \ln n^*}{\ln (n_1 \cdot ... \cdot n_k)} \right) = 1.$$

By the equality

$$\overline{\lim}_{k\to\infty} \frac{\sum\limits_{j\in T_k^{(1)}} \ln p_j}{-k} = A,$$

there is a subsequence $\{k_m\}_{m\in\mathbb{N}}$ such that the limit

$$\lim_{m\to\infty} \frac{\sum\limits_{j\in T_{k_m}^{(1)}} \ln p_j}{-k_m}$$

exists and equals to A. Therefore, we have

$$\lim_{m\to\infty}\frac{\ln\mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)})}{\ln\lambda(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)})}=$$

$$=\lim_{m\to\infty}\left(\frac{\sum\limits_{j\in T^{(1)}_{k_m}}\ln\frac{1}{p_{\alpha_j(x)j}}}{\sum\limits_{j=1}^{k_m}\ln n_j}+\frac{\sum\limits_{j\in T_{\varepsilon,k_m}}\ln\frac{1}{p_{\alpha_j(x)j}}}{\sum\limits_{j=1}^{k_m}\ln n_j}+\frac{\sum\limits_{j\in T^{+}_{\varepsilon,k_m}}\ln\frac{1}{p_{\alpha_j(x)j}}}{\sum\limits_{j=1}^{k_m}\ln n_j}\right),\ \forall x\in L. \tag{2.31}$$

Let us estimate each element of the sum (2.31). If $x\in L$, then

$$\frac{\sum\limits_{j\in T^{(1)}_{\varepsilon,k_m}}\ln\frac{1}{p_{\alpha_j(x)j}}}{\sum\limits_{j=1}^{k_m}\ln n_j}=\frac{\sum\limits_{j\in T^{(1)}_{\varepsilon,k_m}}\ln\frac{1}{p_j}}{\sum\limits_{j=1}^{k_m}\ln n_j}$$

$$\geq\frac{\sum\limits_{j\in T^{(1)}_{\varepsilon,k_m}}\ln\frac{1}{p_j}}{k_m\ln n^*}\to\frac{A}{\ln n^*}\ (m\to\infty).$$

By the following inequality

$$0\leq\lim_{m\to\infty}\frac{\sum\limits_{j\in T_{\varepsilon,k_m}}\ln\frac{1}{p_{\alpha_j(x)j}}}{\sum\limits_{j=1}^{k_m}\ln n_j}$$

$$\leq\lim_{m\to\infty}\frac{|T_{\varepsilon,k_m}|\ln 2n^*}{k_m\ln 2}=\lim_{m\to\infty}\frac{|T_{\varepsilon,k_m}|\ln 2n^*}{k_m\ln 2}=0,$$

there is

$$\lim_{m\to\infty}\frac{\sum\limits_{j\in T_{\varepsilon,k_m}}\ln\frac{1}{p_{\alpha_j(x)j}}}{\sum\limits_{j=1}^{k_m}\ln n_j}=0,\ \forall x\in L.$$

Let us estimate third element of the sum (2.31):

$$\frac{\sum\limits_{j\in T^+_{\varepsilon,k_m}} \ln \frac{1}{p_{\alpha_j(x)j}}}{\sum\limits_{j=1}^{k_m} \ln n_j} \geq \frac{\sum\limits_{j\in T^+_{\varepsilon,k_m}} \ln \frac{1}{\frac{1}{n_j}+\varepsilon}}{\sum\limits_{j=1}^{k_m} \ln n_j} = \frac{\sum\limits_{j\in T^+_{\varepsilon,k_m}} \ln n_j - \sum\limits_{j\in T^+_{\varepsilon,k_m}} \ln(1+\varepsilon n_j)}{\sum\limits_{j=1}^{k_m} \ln n_j}$$

$$\geq \frac{\sum\limits_{j=1}^{k_m} \ln n_j - \sum\limits_{j\in T^-_{\varepsilon,k_m}} \ln n_j - \sum\limits_{j\in T^+_{\varepsilon,k_m}} \ln(1+\varepsilon n_j)}{\sum\limits_{j=1}^{k_m} \ln n_j}$$

$$\geq 1 - \frac{\mid T^-_{\varepsilon,k_m} \mid \cdot \ln n^* + \mid T^+_{\varepsilon,k_m} \mid \ln(1+\varepsilon n^*)}{k_m \ln 2} \to 1 - \frac{\ln(1+\varepsilon n^*)}{\ln 2}(m\to\infty).$$

Hence

$$1 - \frac{1+\varepsilon n^*}{\ln 2} + \frac{A}{\ln n^*} \leq \varliminf_{m\to\infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)})}{\ln \lambda(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)})}, \ \forall\varepsilon>0.$$

Therefore

$$1 + \frac{A}{\ln n^*} \leq \varliminf_{m\to\infty} \frac{\ln \mu_\xi(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)})}{\ln \lambda(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)})}.$$

Hence, for every real number $\delta>0$, there exists $m(\delta)$ such that $\forall m > m(\delta)$:

$$1 + \frac{A}{\ln n^*} - \delta \leq \frac{\ln \mu_\xi\left(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)}{\ln \lambda\left(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)}, \ \forall x\in L.$$

The last inequality is equivalent to

$$\mu\left(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right) \leq \lambda\left(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)^{1+\frac{A}{\ln n^*}-\delta}.$$

Let $d(\cdot)$ be a diameter of a set. Therefore, we have

$$d\left(\Delta'_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right)^{\frac{1}{1+c\cdot A-\delta}} \leq d\left(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right), \forall x\in L,\ \delta>0,\ m>m(\delta), \tag{2.32}$$

where

$$\Delta^{'}_{\alpha_1(x)\ldots\alpha_{k_m}(x)} = F_{\mu_\xi}\left(\Delta_{\alpha_1(x)\ldots\alpha_{k_m}(x)}\right).$$

Let us take $\delta \in \left(0, \frac{A}{\ln n^*}\right)$. By the equality $\lambda(L) = 0$, The Hausdorff measure equals

$$H^1_\varepsilon(L) = 0, \ \forall \varepsilon > 0.$$

Hence for some $\varepsilon > 0$ and some $t > 0$ there exists an ε-covering $\{E_i\}$ of a set L with k_m-rank cylinders (m depends on an ε and a t) such that $\sum_i d(E_i) < t$. The family of sets $\{E^{'}_i\} = \{F_\xi(E_i)\}$ is $\varepsilon^{'}$-covering of $L^{'} = F_\xi(L)$. It follows immediately that $\varepsilon^{'} \to 0 \Leftrightarrow \varepsilon \to 0$ (F_ξ is a uniformly continuous on the unit interval).

Without loss of generality we can consider only sets E_i such that $E_i \cap L \neq \emptyset$. By the inequality (2.32), we have

$$\sum_i \left[d(E^{'}_i)\right]^{\frac{1}{1+\frac{A}{\ln n^*}-\delta}} \leq \sum_i d(E_i) < t.$$

We can take ε and t such that

$$H^{\frac{1}{1+\frac{A}{\ln n^*}-\delta}}_{\varepsilon^{'}}(L') = 0, \qquad \forall \varepsilon^{'} > 0.$$

Hence

$$H^{\frac{1}{1+\frac{A}{\ln n^*}-\delta}}(L') = 0.$$

Therefore

$$\dim_H(L') \leq \frac{1}{1+\frac{A}{\ln n^*}-\delta} < 1, \ \forall \delta > 0.$$

Consequently $\dim_H L^{'} \leq \frac{1}{1+\frac{A}{\ln n^*}}$. Therefore, we have a contradiction. □

Chapter 3

Generalized infinite Bernoulli convolutions

3.1 Introduction

Let $\mu_\xi = \mu$ be the distribution of the random variable

$$\xi = \sum_{k=1}^{\infty} \xi_k a_k, \tag{3.1}$$

where $\sum\limits_{k=1}^{\infty} a_k$ is a convergent series whose terms are nonnegative and where ξ_k are independent random variables assuming two values 0 and 1 with probabilities p_{0k} and $p_{bk} = 1 - p_{1k}$, respectively.

Definition 3.1. The distribution μ_ξ is called a *generalized infinite Bernoulli convolution.*

It is shown in the paper [AT08] that, when studying the Lebesgue structure and the fractal properties of the measure μ_ξ, one can restrict consideration without loss of generality to the case where the "matrix" $\|p_{ik}\|$ does not contain zeros (that is, $p_{0k} \in (0,1)$ for all $k \in \mathbb{N}$) and where the sequence $\{a_k\}$ is nondecreasing (that is, $a_k \geq a_{k+1}$ for all $\forall k \in \mathbb{N}$) and such that $\sum\limits_{k=1}^{\infty} a_k = 1$.

A theorem due to Levy (1931) provides necessary and sufficient conditions for μ to be purely discrete

Theorem [Lev31]. *The measure μ is discrete if and only if*

$$\prod_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} = 0.$$

A theorem due to Jessen and Wintner says that the distribution of ξ is pure.

Theorem [JW35]. *The measure μ_ξ is either purely discrete, or purely absolutely continuous with respect to the Lebesgue measure, or purely singularly continuous.*

Remark 3.1. The criteria for ξ to be purely absolutely continuous with respect to the Lebesgue measure (or purely singular) are not known yet even in the case of random power series ($a_k = \lambda^k$ and $p_{0k} = \frac{1}{2}$), despite the problem having been actively studied over last 80 years or so (see, for example, [Pra98, AT04, AZ91, Erd39, Gar62, PSS00, PS96, PS98, Sol95]). Surveys of problems in this field are given in [PSS00]. Some applications of infinite Bernoulli convolutions are discussed in [AZ91, PSS00].

If the series $\sum_{k=1}^{\infty} a_k$ converges "fast enough", that is, if

$$a_k \geq r_k := \sum_{n=k+1}^{\infty} a_n$$

for all sufficiently large k, then the Lebesgue structure and fractal properties of generalized infinite Bernoulli convolutions are studied rather well (see [Coo98, AT08]). In contrast, if the inequality $a_k < r_k$ occurs for an infinite number of indices k, then these problems are studied much less. The main problem in this case is how to obtain appropriate properties of the Bernoulli convolutions for which almost all (with respect to the Lebesgue measure or in the sense of the Hausdorff - Besicovitch dimension) points of the spectrum have continuum many different expansions of the form $\sum_{k=1}^{\infty} \omega_k a_k$, where $\omega_k \in \{0, 1\}$. The probability measures of this type belong to the class of the so-called Bernoulli convolutions with essential intersections ([GPT09]). The main aim of the section is to prove the singularity of the distribution (in most cases) of the random variable ξ and to investigate its fine fractal properties for the case where the sequence $\{a_k\}$ is such that

$(*)$ $\forall k \in \mathbb{N},\ \exists s_k \in \mathbb{N} \cup \{0\} : a_k = a_{k+1} = ... = a_{k+s_k} \geq r_{k+s_k}$,

and moreover $s_k > 0$ for an infinite number of indices k.

We introduce some auxiliary notation.

Definition 3.2. We shall say that a generalized infinite Bernoulli convolution 3.1 is a LT - Bernoulli convolution if condition $(*)$ is satisfied.

Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative integer numbers such that $i \in \{k_n\}_{n\in\mathbb{N}}$ if and only if $s_i = 0$. Also let $l_n = k_n - k_{n-1}$, $k_0 = 0$.

3.2 Lebesgue structure of the LT - Bernoulli convolutions

Let $\Omega = \{0, 1\}^{\infty}$. For a fixed series $\sum_{k=1}^{\infty} a_k$ consider the mapping $\varphi : \Omega \to R$ defined as follows

$$\forall \omega = (\omega_1, \omega_2, ..., \omega_k, ...) \in \Omega : \quad \varphi(\omega) = \sum_{k=1}^{\infty} \omega_k a_k.$$

Definition 3.3. The set

$$\Delta' = \Delta'(\{a_k\}) = \varphi(\Omega) = \{x : \ \exists\ \omega \in \Omega \ \wedge \ \varphi(\omega) = x\}$$

is called the set of *incomplete sums* of the series $\sum_{k=1}^{\infty} a_k$.

Since $p_{ik} > 0$ for all $i \in \{0, 1\}$ and $k \in \mathbb{N}$, the set Δ' is the spectrum (in other words, the minimal closed support) of the distribution of the random variable ξ.

Following paper [GPT09], the set of real numbers of the form

$$\Delta'_{c_1...c_m} := \left\{ \sum_{n=1}^{m} c_n a_n + \sum_{n=m+1}^{\infty} \omega_n a_n : \omega_n \in \{0,1\}, \forall n \in \mathbb{N} \right\},$$

is called the cylinder of rank m with the base

$$c_1...c_m \ (c_i \in \{0,1\}).$$

It is clear that the set $\Delta'_{c_1...c_m}$ is the image under the mapping φ of the cylinder belonging to Ω whose base is

$$c_1...c_m \ (c_i \in \{0,1\}).$$

The interval

$$\Delta_{c_1...c_m} := \left[\sum_{n=1}^{m} c_n a_n, \ r_m + \sum_{n=1}^{m} c_n a_n \right]$$

is called the *cylindrical interval* of rank m with the base $c_1...c_m$. Note that $\Delta'_{c_1...c_m} \subset \Delta_{c_1...c_m}$.

Some general properties of cylinders and cylindrical intervals follow directly from their definitions, namely

1) $\inf \Delta_{c_1...c_m} = \inf \Delta'_{c_1...c_m}$; $\sup \Delta_{c_1...c_m} = \sup \Delta'_{c_1...c_m}$;

2) $\Delta'_{c_1...c_m} = \Delta'_{c_1...c_m 0} \bigcup \Delta'_{c_1...c_m 1}$;

3) $\inf \Delta_{c_1...c_m} = \inf \Delta_{c_1...c_m 0}$, $\sup \Delta_{c_1...c_m} = \sup \Delta_{c_1...c_m 1}$;

4) $|\Delta_{c_1...c_m}| = r_m \to 0 \, (m \to \infty)$;

5) $\bigcap_{m=1}^{\infty} \Delta_{c_1...c_m} = \bigcap_{m=1}^{\infty} \Delta'_{c_1...c_m} \equiv \Delta_{c_1...c_m...} = x \in \Delta' \subset [0,1]$.

The following property is a consequence of condition (*):

6) $\Delta_{c_1 c_2 ... c_{k_1} \ c_{k_1+1} ... c_{k_2} \ ... \ c_{k_{n-1}+1} \ ... \ c_{k_n}} = \Delta_{d_1 d_2 ... d_{k_1} \ d_{k_1+1} \ ... \ d_{k_2} \ ... \ d_{k_{n-1}+1} \ ... \ d_{k_n}}$ if and only if

$$\begin{cases} \sum_{i=1}^{k_1} c_i = \sum_{i=1}^{k_1} d_i \\ \sum_{i=k_1+1}^{k_2} c_i = \sum_{i=k_1+1}^{k_2} d_i \\ \vdots \\ \sum_{i=k_{n-1}+1}^{k_n} c_i = \sum_{i=k_{n-1}+1}^{k_n} d_i \end{cases} \tag{3.2}$$

Below is the description of those points of the spectrum that have only a finite number of representations. Let

$$x = \Delta_{c_1(x)c_2(x)...c_{k_1}(x) \quad c_{k_1+1}(x) \ ... \ c_{k_2}(x) \ ... \quad c_{k_{n-1}+1}(x) \ ... \ c_{k_n}(x) \quad ...} \tag{3.3}$$

be one of the representations of a point x. If $\sum\limits_{i=k_{n-1}+1}^{k_n} c_i(x) \notin \{0, l_n\}$ for an infinite number of indices n, then the point x has continuum many representations in the form $\sum\limits_{i=1}^{\infty} c_i a_i$. Indeed, property 6) implies that if $\sum\limits_{i=k_{n-1}+1}^{k_n} c_i(x) \notin \{0, l_n\}$ then the equation

$$x_1 + x_2 + ... + x_{l_n} = \sum_{i=k_{n-1}+1}^{k_n} c_i(x)$$

has at least two different solutions $(x_1^{(1)}, x_2^{(1)}, ..., x_{l_n}^{(1)})$ and $(x_1^{(2)}, x_2^{(2)}, ..., x_{l_n}^{(2)})$, where $x_i^{(j)} \in \{0,1\}$ for all $i \in \{1, \ ..., \ l_n\}$ and $j \in \{1,2\}$. Applying this reasoning to those numbers n for which $\sum\limits_{i=k_{n-1}+1}^{k_n} c_i(x) \in \{1, 2, ..., l_n - 1\}$ and taking into account that $l_n > 1$ for infinitely many indices n, we prove the result desired.

Note also that there are points belonging to the spectrum that have a unique representation in the form of (3.3). If $a_{k_n} > r_{k_n}$ for infinitely many indices n, then all points of the form

$$\Delta_{c_1(x)c_2(x)...c_{k_1}(x) \quad c_{k_1+1}(x) \ ... \ c_{k_2}(x) \ ... \quad c_{k_{n-1}+1}(x) \ ... \ c_{k_n}(x) \quad ...}, \tag{3.4}$$

where

$$c_{k_{n-1}+1}(x) + c_{k_{n-1}+2}(x) + ... + c_{k_n}(x) \in \{0, l_n\}, \quad \forall n \in \mathbb{N},$$

has a unique form (3.3). If $a_{k_n} = r_{k_n}$ for an infinite number of indices n, then some points of the form 3.4 have exactly two different representations (note that the set of such points is countable): one of these representations has „0 “in period, while the other one has „1 “in period. Thus the set of those points that have a unique representation in the form of 3.3 is uncountable. It is clear that if

$$c_{k_{n-1}+1}(x) + c_{k_{n-1}+2}(x) + ... + c_{k_n}(x) \in \{0, l_n\}$$

for all sufficiently large $\forall n \in \mathbb{N}$, then the point x has a finite number of representations in the form of 3.3.

In Section 3.4 we determine the dimension of the set of those points for which there exist continuum many different representations. We also determine the dimension of the set of points that have a finite number of representations.

Our current goal is to show that the distribution of the random variable ξ is a probability measure with independent $\widetilde{Q}^*$ -symbols.

Having this goal in mind, we introduce the sequence $\{m_n\}_{n \in \mathbb{N}}$ by

$$m_n = \begin{cases} l_n + 1, \text{if } \ a_{k_n} = r_{k_n}; \\ 2l_n + 1, \ \text{if } \ a_{k_n} > r_{k_n}. \end{cases}$$

For every n, define the stochastic vector column

$$q_n = (q_{0n}, q_{1n}, ..., q_{m_n-1,n})$$

as follows:

$1) m_n = l_n + 1$

$$q_{in} = \frac{1}{l_n + 1}, i \in \{0, 1, 2, ..., m_n - 1\} = B_n;$$

$2) m_n = 2l_n + 1$

$$q_{in} = \frac{r_{k_n}}{r_{k_{n-1}}}, i \in \{0, 2, 4, ..., \ m_n - 1\} = B_n;$$

$$q_{in} = \frac{a_{k_n} - r_{k_n}}{r_{k_{n-1}}}, i \in \{1, 3, 5, ..., m_n - 2\}.$$

The stochastic "matrix" $\widetilde{Q}^* = ||q_{in}||$, whose column n coincides with the stochastic vector q_n, generates a $\widetilde{Q}^*$-representation of numbers of the interval $[0, 1]$ in the following way. Let $A_n = \{0, 1, ..., m_n - 1\}$ and $\gamma_n \in A_n$. Consider the mapping

$f : A_1 \times A_2 \times ... \times A_n \times ... \ \mapsto [0, 1]$, given by

$$f(\{\gamma_n\}) = x = \beta_{\gamma_1 1} + \sum_{n=2}^{\infty} \beta_{y_n n} \prod_{i=1}^{n-1} q_{\gamma_i i},$$

where $\beta_{\gamma_n n} = \sum_{j=0}^{\gamma_n - 1} q_{jn}$. We also write

$$x = \Delta^{\widetilde{Q}^*}_{\gamma_1 \gamma_2 ... \gamma_n \ ...}, \ \gamma_n \in A_n.$$

The latter expression is $\widetilde{Q}^*$ representation of a number x.

Let

$$\Delta^{\widetilde{Q}^*}_{\gamma_1 ... \gamma_m} = \left[\beta_{\gamma_1 1} + \sum_{n=2}^{m} \beta_{y_n n} \prod_{i=1}^{n-1} q_{\gamma_i i}, \beta_{\gamma_1 1} + \sum_{n=2}^{m} \beta_{y_n n} \prod_{i=1}^{n-1} q_{\gamma_i i} + \prod_{i=1}^{m} q_{\gamma_i i} \right].$$

Since the cylindrical intervals $\Delta_{c_1 ... c_{k_n}}$ of rank k_n are either disjoint or coincide, there exists a correspondence between the set of cylindrical intervals $\Delta_{c_1 c_2 ... c_{k_n}}$ of rank k_n and the set of $\Delta^{\widetilde{Q}^*}_{\gamma_1 \gamma_2 ... \gamma_n}$, $\gamma_i \in A_i$. The correspondence mentioned above is generated by the mapping

$$\gamma_i = \begin{cases} c_{k_{i-1}+1} + ... + c_{k_i}, \text{ if } a_{k_i} = r_{k_i}; \\ 2(c_{k_{i-1}+1} + ... + c_{k_i}), \text{ if } a_{k_i} > r_{k_i}. \end{cases}$$

This means, for a fixed series $\sum_{k=1}^{\infty} a_k$ and for an arbitrary collection

$$c_1c_2...c_{k_1}\ c_{k_1+1}...c_{k_2}\ ...\ c_{k_{n-1}+1}...c_{k_n}, \quad c_i \in \{0,1\},$$

that there exists a unique set $\gamma_1, \gamma_2, ..., \gamma_n(\gamma_i \in A_i)$, such that

$$\Delta_{c_1c_2...c_{k_n}} = \Delta^{\widetilde{Q}^*}_{\gamma_1\gamma_2...\gamma_n},$$

where γ_i is defined by l_i symbols $c_{k_{i-1}+1}...c_{k_i}$ according to the condition $a_{k_n} = r_{k_n}$.

We will need the following notations. Let

$$R_{l_n} := \{0,1\}^{l_n}$$

and

$$\delta := (\delta_1, \delta_2, ..., \delta_{l_n}) \in R_{l_n}$$

where $|\delta| = \sum_{k=k_{n-1}+1}^{k_n} \delta_k$ for all $n \in \mathbb{N}$.

Let $\{\tilde{\xi}_n\}$ be a sequence of independent random variables assuming the values

$$0, 1, \dots, m_n - 1$$

with probabilities

$$\tilde{p}_{0n},\ \tilde{p}_{1n},\ ...,\ \tilde{p}_{m_n-1,n},$$

respectively, where

$$\tilde{p}_{in} = \sum_{\delta \in R_{l_n}, |\delta|=i} \left(\prod_{k=k_{n-1}+1}^{k_n} p_{\delta_k,k} \right)$$

for $a_{k_n} = r_{k_n}$, and let

$$\tilde{p}_{in} = \begin{cases} \sum\limits_{\delta \in R_{l_n}, |\delta|=\frac{i}{2}} \left(\prod\limits_{k=k_{n-1}+1}^{k_n} p_{\delta_k,k} \right), \text{ if } i \text{ is even} \\ 0, \text{ if i is odd} \end{cases},$$

for $a_{k_n} > r_{k_n}$.

The random variable $\tilde{\xi}$ with independent $\widetilde{Q}^*$-symbols,

$$\tilde{\xi} = \beta_{\tilde{\xi}_1 1} + \sum_{n=2}^{\infty} \beta_{\tilde{\xi}_n n} \prod_{i=1}^{n-1} q_{\tilde{\xi}_i i}, \beta_{\gamma_n n} = \sum_{j=0}^{\gamma_n - 1} q_{jn}$$

is determined by the stochastic “matrix” $\|q_{in}\|$ and the sequence of independent random variables $\{\tilde{\xi}_n\}$.

Remark 3.2. The random variables ξ and $\tilde{\xi}$ are identically distributed.

Indeed, it is sufficient to show that

$$\mathrm{P}_{\xi}(\Delta_{c_1 c_2 \ldots c_{k_n}}) = \mathrm{P}_{\tilde{\xi}}\ (\Delta^{\widetilde{Q}^*}_{\gamma_1 \gamma_2 \ldots \gamma_n}),\ \forall n \in \mathbb{N},$$

where

$$\gamma_i = \begin{cases} c_{k_{i-1}+1} + \ldots + c_{k_i},\ \text{if}\ a_{k_i} = r_{k_i}; \\ 2(c_{k_{i-1}+1} + \ldots + c_{k_i}),\ \text{if}\ a_{k_i} > r_{k_i}. \end{cases}$$

This equality is obvious in view of the construction of the random variable $\tilde{\xi}$ and in view of the properties of the binomial distribution, since the random variables

$$\xi_1,\ \xi_2,\ \ldots,\ \xi_{k_n}$$

are independent and identically distributed.

Theorem 3.1. *The Lebesgue measure of the spectrum of the random variable ξ is equal to*

$$\lim_{n\to\infty} r_{k_n}\left(\prod_{j=1}^{n}(l_j+1)\right).$$

Proof. The spectrum of the random variable $\tilde{\xi}$ is an infinite intersection of unions of $\widetilde{Q}^*$ - cylindrical intervals (each being of a nonzero measure μ_{ξ})of all ranks. Such $\widetilde{Q}^*$ - cylindrical intervals of rank n coincide coincide, and the total number of these cylindrical intervals is equal to l_n+1. Hence

$$S_{\tilde{\xi}} = \bigcap_{n=1}^{\infty}\bigcup_{\gamma_1\in B_1}\dots\bigcup_{\gamma_{n-1}\in B_{n-1}}\bigcup_{\gamma_n\in B_n}\Delta^{\widetilde{Q}^*}_{\gamma_1\gamma_2\dots\gamma_{n-1}\gamma_n}.$$

Since

$$\mu_{\tilde{\xi}}(\Delta^{\widetilde{Q}^*}_{\gamma_1\gamma_2\dots\gamma_{n-1}\gamma_n}) = \widetilde{p}_{\gamma_1 1}\cdot\widetilde{p}_{\gamma_2(x)2}\cdot\ldots\cdot\widetilde{p}_{\gamma_n(x)n} > 0 \text{ and } \lambda(\Delta^{\widetilde{Q}^*}_{\gamma_1\gamma_2\dots\gamma_{n-1}\gamma_n}) = r_{k_n},$$

the continuity of the Lebesgue measure implies that

$$\lambda(S_{\bar{\xi}}) = \lambda\left(\bigcap_{n=1}^{\infty}\bigcup_{\gamma_1\in B_1}\dots\bigcup_{\gamma_{n-1}\in B_{n-1}}\bigcup_{\gamma_n\in B_n}\Delta^{\widetilde{Q}^*}_{\gamma_1\gamma_2\dots\gamma_{n-1}\gamma_n}\right) = \lim_{n\to\infty} r_{k_n}\prod_{j=1}^{n}(l_j+1),$$

and this completes the proof of Theorem. □

Lemma 3.1. *Let $R_{l_n} := \{0,1\}^{l_n}$ and $\delta := (\delta_1,\delta_2,\dots,\delta_{l_n}) \in R_{l_n}$, where $|\delta| = \sum_{k=k_{n-1}+1}^{k_n}\delta_k$ for all $n\in\mathbb{N}$. Then there is a function $\varphi(n)$ such that*

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n}p_{\delta_k,k}\right)} \le \varphi(n)\to 0\ (n\to\infty), \tag{3.5}$$

where $0<p_{0k}<1,\ p_{1k}=1-p_{0k}$ for all $k\in\mathbb{N}$.

Proof. Let $\{\zeta_j\}_{j\in\{1,\ldots,n\}}$ – be a sequence of independent random variables assuming the values 0 and 1 with probabilities p_{0j} and p_{1j} respectively ($p_{0j}+p_{1j}=1 \;\; \forall j\in\mathbb{N}$) and

$$S_n := \zeta_1+\zeta_2+...+\zeta_n.$$

Then

$$\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n} p_{\delta_k,k}\right)=\mathrm{P}\{S_n=i\}=\tilde{p}_{in}.$$

Hence condition (3.5) is true iff

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\tilde{p}_{in}}\to 0(n\to\infty).$$

By Chebyshev's inequality,

$$\mathrm{P}\{\left|\frac{S_n-ES_n}{n}\right|\geq\varepsilon\}\leq\frac{D(\frac{S_n}{n})}{\varepsilon^2},\forall\varepsilon>0,$$

where $E(\tau)$ is a mathematical expectation of a random variable τ and $D(\tau)$ is a variance of a random variable τ. Therefore,

$$\mathrm{P}\{\left|\frac{S_n-ES_n}{n}\right|\geq\varepsilon\}=\mathrm{P}\{|S_n-ES_n|\geq\varepsilon n\}\leq\frac{D(\frac{S_n}{n})}{\varepsilon^2}=\frac{D(S_n)}{n^2\varepsilon^2},$$

Since the random variables $\{\zeta_j\}_{j\in\{1,\ ...,\ n\}}$ are independent, it follows that

$$D(S_n)=D\zeta_1+D\zeta_2+...+D\zeta_n=p_{01}p_{11}+p_{02}p_{12}+...+p_{0n}p_{1n}\leq\frac{n}{4}.$$

Therefore,

$$\mathrm{P}\{\left|\frac{S_n-ES_n}{n}\right|\geq\varepsilon\}\leq\frac{1}{4n\varepsilon^2}. \tag{3.6}$$

The Cauchy - Schwarz inequality,

$$(1 \cdot a_1 + ... + 1 \cdot a_m)^2 \leq m \cdot (a_1^2 + ... + a_m^2),$$

together with (3.6) implies that

$$\sum_{i:\left|\frac{i-ES_n}{n}\right|\geq\varepsilon}^{n} \sqrt{\tilde{p}_{in}} \leq \sqrt{n \cdot \sum_{i:\left|\frac{i-ES_n}{n}\right|\geq\varepsilon}^{n} \tilde{p}_{in}} \leq \sqrt{n \cdot \frac{1}{4n \cdot \varepsilon^2}} = \frac{1}{2\varepsilon}. \tag{3.7}$$

We will estimate the sum $\sum \sqrt{\tilde{p}_{in}}$ for i such that

$$\left|\frac{i - ES_n}{n}\right| \leq \varepsilon,$$

i.e.,

$$i \in [n \cdot ES_n - \varepsilon \cdot n,\ n \cdot ES_n + \varepsilon \cdot n] .$$

Since there exist at most $2n \cdot \varepsilon + 1$ positive integer numbers i such that

$$i : \left|\frac{i - ES_n}{n}\right| \leq \varepsilon.$$

We again use The Cauchy - Schwarz inequality and the condition $\sum_{i=0}^{n} \tilde{p}_{in} = 1$ to prove that

$$\sum_{i:\left|\frac{i-ES_n}{n}\right|\leq\varepsilon}^{n} \sqrt{\tilde{p}_{in}} \leq \sqrt{(2n\varepsilon + 1) \cdot \sum_{i:\left|\frac{i-ES_n}{n}\right|\leq\varepsilon}^{n} \tilde{p}_{in}} \leq \sqrt{2n\varepsilon + 1}. \tag{3.8}$$

Inequalities (3.7) and (3.8) imply that

$$\sum_{i=0}^{n} \sqrt{\tilde{p}_{in}} \leq \frac{1}{2\varepsilon} + \sqrt{2n\varepsilon + 1}, \quad \forall \varepsilon > 0.$$

Therefore $\forall\varepsilon > 0,$ and $\forall n \in \mathbb{N}$:

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\tilde{p}_{in}} \leq \sqrt{\frac{1}{n+1}}(\frac{1}{2\varepsilon} + \sqrt{2n\varepsilon + 1}).$$

It $\varepsilon = \frac{1}{\sqrt[4]{n}}$, then

$$\sqrt{\frac{1}{n+1}}\frac{1}{2\varepsilon} \to 0(n \to \infty)$$

and

$$\sqrt{\frac{1}{n+1}}\sqrt{(2n\varepsilon + 1)} \to 0(n \to \infty).$$

Let

$$\varphi(n) = \frac{1}{\sqrt{n+1}}\left(\frac{\sqrt[4]{n}}{2} + \sqrt{2n^{\frac{3}{4}} + 1}\right).$$

Then we get inequality

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\tilde{p}_{in}} \leq \varphi(n).$$

which completes the proof Lemma 3.1.

□

By Lemma 3.1, we have the following corollary.

Corollary 3.1. *Let*

$$R_{l_n} := \{0,1\}^{l_n}$$

and let

$$\delta := (\delta_1, \delta_2, ..., \delta_{l_n}) \in R_{l_n}$$

where $|\delta| = \sum_{k=k_{n-1}+1}^{k_n} \delta_k$ *for all* $n \in \mathbb{N}$*. Then there is* n_0 *such that* $\forall n > n_0$*:*

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n, |\delta|=i}\left(\prod_{k=1}^{n} p_{\delta_k,k}\right)} \leq \frac{1}{2},$$

where $0 < p_{0k} < 1,\ p_{1k} = 1 - p_{0k},\ \forall k \in \mathbb{N}.$

Let us study the following expression

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n}p_{\delta_k,k}\right)}$$

as a function on

$$(p_{01},\ p_{02},\ ...,\ p_{0n}) \in [0,1]^n.$$

Lemma 3.2. *Let*

$$R_n = \{0,1\}^n,$$

$$\delta = (\delta_1, \delta_2, ..., \delta_n) \in R_n, \quad |\delta| = \sum_{k=1}^{n}\delta_k$$

for all $n \in \mathbb{N}\backslash\{1\}$. *Let*

$$(p_{01},\ p_{02},\ ...,\ p_{0n}) \in [0,1]^n,$$

$$(p_{11},\ p_{12},\ ...,\ p_{1n}) = (1-p_{01},\ 1-p_{02},\ ...,\ 1-p_{0n}).$$

Then

$$\upsilon(p_{01},...,p_{0n}) = \sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n}p_{\delta_k,k}\right)} \le K_n < 1,$$

where K_n *is a constant that depends on* n.

Proof. Consider the function

$$\varphi(x_0,x_1,...,x_n) = \sqrt{x_0} + \sqrt{x_1} + ... + \sqrt{x_n}$$

in the domain G of the hyperplane $x_0 + x_1 + ... + x_n = 1$ that belongs to the $(n+1)$ -dimensional cube $[0,1]^{\ n+1}$. Since

$$\overrightarrow{(\sqrt{x_0},\sqrt{x_1},...,\sqrt{x_n})}\overrightarrow{(1,1,...,1)} \le \left|\overrightarrow{(\sqrt{x_0},\sqrt{x_1},...,\sqrt{x_n})}\right|\left|\overrightarrow{(1,1,...,1)}\right| =$$

$$= \sqrt{n+1},$$

the function $\varphi(x_0, x_1, ..., x_n)$ (being continuous in the domain G) attains its maximal value $\sqrt{n+1}$ at the point with equal coordinates

$$x_0 = x_1 = ... = x_n = \frac{1}{n+1}$$

and

$$\varphi(x_0, x_1, ..., x_n) < \sqrt{n+1}$$

for all other points of the set G.

Let

$$\upsilon(p_{01}, ..., p_{0n}) = \sqrt{\frac{1}{n+1}} \sum_{i=0}^{n} \sqrt{\sum_{\delta \in R_n, |\delta|=i} \left(\prod_{k=1}^{n} p_{\delta_k, k} \right)}$$

where

$$(p_{01},\ p_{02},\ ...,\ p_{0n}) \in [0,1]^n.$$

This function is continuous in $[0,1]^n$ and is bounded from above:

$$\upsilon(p_{01}, ..., p_{0n}) \leq \max \left(\frac{\varphi(x_0, x_1, ..., x_n)}{\sqrt{n+1}} \right) = 1.$$

The inequality becomes an equality only for the case of

$$\sum_{\delta \in R_n, |\delta|=i} \left(\prod_{k=1}^{n} p_{\delta_k, k} \right) = \frac{1}{n+1}$$

for $i \in \{0, 1, ..., n\}$.

Since

$$(p_{01} \cdot p_{02} \cdot ... \cdot p_{0n}) \cdot (p_{11} \cdot p_{12} \cdot ... \cdot p_{1n}) \leq \left(\frac{1}{4} \right)^n,$$

there is at least one product such that $\left(\frac{1}{2}\right)^n < \frac{1}{n+1}, \forall n \geq 2$. Therefore,

$$\upsilon(p_{01},\ ...,\ p_{0n}) < 1,\ \forall n \geq 2. \tag{3.9}$$

Since the function $\upsilon(p_{01},\ ...,\ p_{0n})$ is defined and continuous in a compact set $[0,1]^n$, it attains its maximal value K_n.

By the inequality (3.9), we have

$$\upsilon(p_{01}, ..., p_{0n}) = \sqrt{\frac{1}{n+1}} \sum_{i=0}^{n} \sqrt{\sum_{\delta \in R_n, |\delta|=i} \left(\prod_{k=1}^{n} p_{\delta_k,k} \right)} \leq K_n < 1, \forall n \geq 2,$$

which completes the theorem.

□

Theorem 3.2. *The random variable ξ has a singularly continuous distribution.*

Proof. Let

$$M = \prod_{n=1}^{\infty} \max_i \{\tilde{p}_{in}\}.$$

According to Levy's theorem ([Lev31]), the random variable $\tilde{\xi}$ has either a pure discrete distribution $(M > 0)$ or a pure continuous $(M = 0)$ distribution.

The random variable $\tilde{\xi}$, as a random variable with independent $\widetilde{Q}^*$-symbols, has a pure absolutely continuous distribution if and only iff

$$\prod_{n=1}^{\infty} (\sum_{i=0}^{m_n-1} \sqrt{q_{in}\tilde{p}_{in}}) > 0$$

(see[AKPT11]).

By the construction of random variable $\tilde{\xi}$, it follows that

$$\sum_{i=0}^{m_n-1} \sqrt{q_{in}\widetilde{p}_{in}} = \sqrt{\frac{r_{k_n}}{r_{k_{n-1}}}} \cdot \sum_{i=0}^{l_n} \sqrt{\sum_{\delta \in R_{l_n}, |\delta|=i} \left(\prod_{k=k_{n-1}+1}^{k_n} p_{\delta_k,k} \right)}.$$

Since $\sqrt{\frac{r_{k_n}}{r_{k_{n-1}}}} \leq \sqrt{\frac{1}{l_n+1}}, \forall n \in \mathbb{N}$, we get

$$\sum_{i=0}^{m_n-1}\sqrt{q_{in}\tilde{p}_{in}} \le \sqrt{\frac{1}{l_n+1}}\cdot\sum_{i=0}^{l_n}\sqrt{\sum_{\delta\in R_{l_n},|\delta|=i}\left(\prod_{k=k_{n-1}+1}^{k_n} p_{\delta_k,k}\right)}.$$

The necessary condition for the convergence of the product

$$\prod_{n=1}^{\infty}(\sum_{i=0}^{m_n-1}\sqrt{q_{in}\tilde{p}_{in}})$$

is given by

$$\sqrt{\frac{1}{l_n+1}}\cdot\sum_{i=0}^{l_n}\sqrt{\sum_{\delta\in R_{l_n},|\delta|=i}\left(\prod_{k=k_{n-1}+1}^{k_n} p_{\delta_k,k}\right)} \to 1\ (n\to\infty). \tag{3.10}$$

By the corollary 3.1, there is a number n_0 such that $\forall n > n_0$:

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n} p_{\delta_k,k}\right)} \le \frac{1}{2}.$$

If $2 \le n \le n_0$, then the Lemma 3.1 imply that $\forall k \in \{2,\ ...,\ n_0\}$, $\exists K_0$:

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n} p_{\delta_k,k}\right)} \le K_0 = \max\{K_i : i \in \{2,\ ...,\ n_0\}\} < 1.$$

From the above it follows that $\forall n \in \mathbb{N}$, $\exists K$:

$$\sqrt{\frac{1}{n+1}}\sum_{i=0}^{n}\sqrt{\sum_{\delta\in R_n,|\delta|=i}\left(\prod_{k=1}^{n} p_{\delta_k,k}\right)} \le K = \max\{\frac{1}{2}, K_0\} < 1.$$

Therefore,

$$\sqrt{\frac{1}{l_n+1}}\cdot\sum_{i=0}^{l_n}\sqrt{\sum_{\delta\in R_{l_n},|\delta|=i}\left(\prod_{k=k_{n-1}+1}^{k_n} p_{\delta_k,k}\right)} \nrightarrow 1(n\to\infty).$$

Therefore the distribution of the random variable $\tilde{\xi}$ cannot be absolutely continuous, which proves the theorem. Since the distribution of the random variable ξ is pure, the measure μ_ξ is singular in view of Remark 3.2.

□

3.3 Faithfulness of covering family on the distribution spectrum S_ξ

Let $\widetilde{\mathcal{A}}_n$ be the family of cylindrical intervals of rank k_n, that is,

$$\widetilde{\mathcal{A}}_n = \{E : \; E = \Delta_{\alpha_1 \ldots \alpha_{k_n}},\, \alpha_i \in \{0,1\}, \; i = 1,\, 2,\, ...,\, k_n\},$$

where

$$\Delta_{c_1 \ldots c_m} = \left[\sum_{n=1}^{m} c_n a_n, \; r_m + \sum_{n=1}^{m} c_n a_n \right), \forall m \in \mathbb{N},$$

and let

$$\widetilde{\mathcal{A}} = \bigcup_{n=1}^{\infty} \widetilde{\mathcal{A}}_n. \tag{3.11}$$

Let us recall the

Definition 3.4. A fine covering family Φ_W of a set W is said to be *faithful family of coverings* for the Hausdorff-Besicovitch dimension calculation on W if

$$\dim_H(E, \Phi_W) = \dim_H(E), \forall E \subset W.$$

We can now formulate sufficient conditions for the faithfulness of covering $\widetilde{\mathcal{A}}$ on the spectrum S_ξ.

Theorem 3.3. *If*

$$\lim_{n\to\infty} \frac{\ln r_{k_{n-1}}}{\ln r_{k_n}} = 1,$$

the $\widetilde{\mathcal{A}}$ is a faithful family of coverings on the spectrum $S_{\mu_{\tilde{\xi}}}\backslash\{1\}$.

Proof. It is easily seen that $\widetilde{\mathcal{A}}$ fine family of covering on $S_{\mu_{\tilde{\xi}}}\backslash\{1\}$ and $S_{\mu_{\tilde{\xi}}} \subset [0,1]$. Let $x \in (0,1)$. Then there is $n(x) \in \mathbb{N}$ such that $x \in (r_{k_{n(x)}},\ \ r_{k_{n(x)-1}}]$. Let a function $f : R_+ \to R_+$ equals $3\frac{r_{k_{n(x)-1}}}{r_{k_{n(x)}}}$, where $x \in (r_{k_{n(x)}},\ \ r_{k_{n(x)-1}}]$ and $f(x)$ is arbitrary defined on $x \in [1,+\infty)$.

Let I be an arbitrary closed interval. Then there exists a number $n(|I|)$ such that $|I| \in (r_{k_{n(|I|)}},\ r_{k_{n(|I|)-1}}]$, where $|\cdot|$ is a diameter of a set. A set $I \cap S_{\mu_{\tilde{\xi}}}$ can be covered by 3 cylindrical intervals from $\widetilde{\mathcal{A}}_{n(|I|)-1}$ and A set $I \cap S_{\mu_{\tilde{\xi}}}$ can be covered by at most $[f(|I|)]([x]$ is a floor function of $x)$ cylindrical intervals from $\widetilde{\mathcal{A}}_{n(|I|)}$. Therefore

$$I \cap S_{\mu_{\tilde{\xi}}} \subset \bigcup_{j=1}^{l(I)} \triangle_j(I),$$

where $|\triangle_j(I)| \leq |I|, j \in \{1,...,l(I)\}$ and $l(I) \leq f(|I|)$. Hence the condition 1) of Theorem 1.2 is satisfied under the convention $C = 3$.

Let us check condition 2) of Theorem 1.2. Let $\delta \in (0,1]$. By assumption of the theorem, there exists $n_0(\delta)$ such that

$$\forall n \geq n_0(\delta):\ 3\frac{r_{k_{n-1}}}{r_{k_n}} \cdot \left(r_{k_{n-1}}\right)^{\delta} \leq C.$$

Set $\varepsilon_1(\delta) = r_{k_{n_0(\delta)}}$. Therefore, for arbitrary $\delta \in (0,1]$, there exists $\varepsilon_1(\delta) > 0$ such that $f(|I|) \cdot |I|^{\delta} \leq C$, for $|I| \leq \varepsilon_1(\delta)$. By Theorem 1.2, the family $\widetilde{\mathcal{A}}$ is faithful for the Hausdorff-Besicovitch calculation on the spectrum of $\mu_{\tilde{\xi}}$. □

3.4 Hausdorff - Besicovitch dimension of the spectrum of LT - Bernoulli convolutions

We will use stochastic "matrix" $\widetilde{Q}^* = \|q_{in}\|$ for determination of spectrum (minimal closed support of a distribution) of random variable $\tilde{\xi}$.

Let us recall the definitions of sets and sequences from Section 3.2. A sequence $\{m_n\}_{n\in\mathbb{N}}$ is determined as follows:

$$m_n = \begin{cases} l_n + 1, \text{ if } a_{k_n} = r_{k_n} \\ 2l_n + 1, \text{ if } a_{k_n} > r_{k_n} \end{cases}.$$

For every n, define the stochastic vector column

$$q_n = (q_{0n}, q_{1n}, ..., q_{m_n-1,n})$$

as follows: 1) $m_n = l_n + 1$

$$q_{in} = \frac{1}{l_n+1}, i \in \{0, 1, 2, ..., m_n - 1\} = B_n;$$

2) $m_n = 2l_n + 1$

$$q_{in} = \frac{r_{k_n}}{r_{k_{n-1}}}, i \in \{0, 2, 4, ..., m_n - 1\} = B_n;$$

$$q_{in} = \frac{a_{k_n} - r_{k_n}}{r_{k_{n-1}}}, i \in \{1, 3, 5, ..., m_n - 2\}.$$

Step 1. We decompose unit interval $[0, 1]$ (from the left to the right) into the union of closed intervals $\Delta_{i_1}^{\widetilde{Q}^*}$, $i_1 \in \{0, ..., m_1 - 1\}$ (without common interior points) of the length $\left|\Delta_{i_1}^{\widetilde{Q}^*}\right| = q_{i_1 1}$,

$$[0, 1] = \bigcup_{i_1 \in \{0,...,m_1-1\}} \Delta_{i_1}^{\widetilde{Q}^*}.$$

We shall say that the family of cylindrical intervals

$$\mathcal{A}_1^{\mu} := \left\{\Delta : \Delta = \Delta_{i_1}^{\widetilde{Q}^*}, i_1 \in B_1\right\}$$

is called *spectral cylinders of first rank.* Hence there are $l_1 + 1$ spectral cylinders of first rank with the length r_{k_1}.

Step $n \geq 2$. We decompose (from the left to the right) each closed $(n-1)$-rank interval $\Delta_{i_1 i_2 \ldots i_{n-1}}^{\widetilde{Q}^*}$ from the set $\mathcal{A}_{n-1}^{\mu}$ into the union of closed intervals $\Delta_{i_1 i_2 \ldots i_{n-1} i_n}^{\widetilde{Q}^*}$, $i_n \in \{0,\ ...,\ m_n - 1\}$ (without common interior points) of the length $\left|\Delta_{i_1 i_2 \ldots i_{n-1} i_n}^{\widetilde{Q}^*}\right| = r_{k_{n-1}} \cdot q_{i_n n}$,

$$\Delta_{i_1 i_2 \ldots i_{n-1}}^{\widetilde{Q}^*} = \bigcup_{i_n \in \{0,\ ...,\ m_n - 1\}} \Delta_{i_1 i_2 \ldots i_n}^{\widetilde{Q}^*}.$$

We shall say that the family of cylindrical intervals

$$\mathcal{A}_n^{\mu} := \left\{\Delta : \Delta = \Delta_{i_1 i_2 \ldots i_{n-1} i_n}^{\widetilde{Q}^*},\ i_t \in B_t,\ t \in \{1,\ ...,\ n\}\right\}$$

is called *spectral cylinders of* n *th rank.*Hence, there are $l_n + 1$ spectral cylinders of n-th rank with the length r_{k_1}.

Let S_n be a union of spectral cylinders of n-th rank S_{μ}, i.e.,

$$S_n := \bigcup_{I \in \mathcal{A}_n^{\mu}} I.$$

Hence the spectrum of random variable $\tilde{\xi}$ can be seen as an intersection of sets S_n:

$$S_{\mu} = \bigcap_{n=1}^{\infty} S_n.$$

Let us construct an auxiliary family of sets. Set

$$\mathcal{T}_n := \{T : T = \bigcup_{i=1}^{l} \Delta_i,\ \Delta_i \in \mathcal{A}_n^\mu, 1 \le l \le l_n + 1 \text{ and } \exists\, T' \in \mathcal{A}_{n-1}^\mu : T \subset T'\},$$

i.e., $\mathcal{T}_n$ is the family of sets and every set of this family is a union of spectral cylinders of n-th rank of S_μ (these n-th rank cylinders are subsets of one spectral cylinder with $(n-1)$-th rank of S_μ). Let

$$\mathcal{T} = \bigcup_{n=1}^{\infty} \mathcal{T}_n.$$

To determine the Hausdorff dimension of the set of those points for which there exist continuum many of different representations, we will use an approach developed by Feng D., Wen Z. and Wu J. in [FRW96].

Lemma 3.3 ([FRW96])**.** *If* $\alpha \in (0, 1]$, *then*

$$\frac{1}{6} H^\alpha(S_\mu, \mathcal{T}) \le H^\alpha(S_\mu).$$

Proof. Let $\{E_i\}_{i\in\mathbb{N}}$ be an arbitrary ε-covering of the spectrum S_μ by intervals $E_i = (a_i, b_i)$. Without loss of generality we assume $E_i \bigcap S_\mu \ne \emptyset$ (one can calculate the $\alpha - \varepsilon$ Hausdorff measure $H_\varepsilon^\alpha(S_\mu)$ using the sets with condition $E_i \bigcap S_\mu \ne \emptyset$). There is a spectral cylinder I_i of n-th rank for an arbitrary interval E_i such that $I_i \subset E_i$ and E_i do not contain spectral cylinders of $(n-1)$-th rank.

We must have that E_i can not intersect with more than two spectral cylinders of $n-1$-th rank, for otherwise E_i contains spectral cylinder of $n-1$-th rank. We will denote by I_i^1, I_i^2 these cylinders. Let T_i^1 and T_i^2 be unions of spectral cylinders of n-th rank and these spectral cylinders is subsets of I_i^1 and I_i^2 respectively. It is assumed that $T_i^1 \cap E \ne \emptyset$ and $T_i^2 \cap E \ne \emptyset$. Of course T_i^1, $T_i^2 \in \mathcal{T}$. Without loss of generality we will make the assumption: $|T_i^1| \ge |T_i^2|$. According to the above assumption T_i^1 contains at least one spectral cylinder of n-th rank. Hence $|T_i^1|^\alpha \le (|E_i| + 2r_{k_n})^\alpha \le (3|E_i|)^\alpha$ and

$$|T_i^1|^\alpha + |T_i^2|^\alpha \le 2(3)^\alpha |E_i|^\alpha \le 6|E_i|^\alpha$$

By the construction of the sets T_i^1 and T_i^2, it follows that $S_\mu \cap E_i \subset (T_i^1 \cap E) \cup (T_i^2 \cap E)$ and $|T_i^1|,\ |T_i^2| \le 3|E_i|$.

Therefore $\{T_i^1\}_{i>0} \cup \{T_i^2\}_{i>0}$ is a 3ε - covering of S_μ by the sets from $\mathcal{T}$. From this for arbitrary ε - covering $\{E_i\}$ of S_μ by intervals $E_i = (a_i, b_i)$ and $\forall \alpha > 0$ it follows that

$$\sum_i |E_i|^\alpha \ge \frac{1}{6} \sum_i (|T_i^1|^\alpha + |T_i^2|^\alpha),$$

which completes the proof.

□

This lemma gives important

Corollary 3.2. *$\mathcal{T}$ is a faithful family of coverings for the Hausdorff-Besicovitch dimension calculation on the spectrum S_μ.*

Lemma 3.4 ([FRW96])**.** *Let $\alpha \in (0,1]$ and $\mathcal{E} = \{E_i\}$ be an arbitrary ε - covering of the spectrum S_μ by the sets from $\mathcal{T}$, then there exists a covering of S_μ by spectral cylinders of $n(\varepsilon)$ -th rank such that*

$$\sum_i |E_i|^\alpha \ge \frac{1}{4} \sum_{I \in \mathcal{A}^\mu_{n(\varepsilon)}} |I|^\alpha.$$

Proof. Let $\mathcal{E} = \{E_i\}$ be an arbitrary ε - covering of spectrum S_μ and $\mathcal{E} \subset \mathcal{T}$. Because S_μ is a compact set, we can make an assumption that $\mathcal{E}$ is finite (see [Fal04]). Let n_1 and n_2 be the minimum and maximum ranks of "forming" spectral cylinders of $\mathcal{E}$ (every set from $\mathcal{T}$ is a union of spectral cylinders of some rank).

Let us construct some ε-covering $\mathcal{P}$ by sets from $\mathcal{E}$:

$$\mathcal{P}_1 := \{I : I \in \mathcal{E} \ \text{ and } \ I \in \mathcal{T}_{n_1}\};$$

$$\mathcal{P}_2 := \{I : I \in \mathcal{A}^{\mu}_{n_1} \ \text{ and } \ I \not\subset E, \forall E \in \mathcal{E}\};$$

$$\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2.$$

By the definition, the family $\mathcal{P}$ is a covering of the set S_{n_1}, where elements of covering are union of spectral cylinders of n_1-th rank S_μ or simply spectral cylinders of n_1-th rank.

Let us consider the following function of sets $I \in \mathcal{P}$:

$$f(I,\alpha) = \begin{cases} \frac{|I|^\alpha}{N(I)}\ , \text{ if } I \in \mathcal{P}_1; \\ \sum\limits_{E\in\mathcal{E},\ E\subset I} |E|^\alpha\ , \text{ if } I \in \mathcal{P}_2; \end{cases},$$

where $N(I)$ is a number of spectral cylinders of n_1-th rank which and these spectral cylinders formed the set $I \in \mathcal{P}_1$. By construction of $\mathcal{T}$, it follows that: if $E_i \in \mathcal{E}$ and $E_i \notin \mathcal{P}_1$, then E_i must be a subset of spectral cylinder of n_1-th rank. Let the function $f(\alpha, I)$ get minimum in some element $I_{min} \subset \mathcal{P}$, i.e.,

$$f(I_{min},\alpha) = \min_{I\in\mathcal{P}} f(\alpha, I)$$

(I_{min} always exists, since $\mathcal{P}$ is a finite family of sets).

We have

$$\sum_i |E_i|^\alpha = \sum_{I\in\mathcal{P}_1} |I|^\alpha + \sum_{I\in\mathcal{P}_2} \left(\sum_{E_i\in\mathcal{E},\ E_i\subset I} |E_i|^\alpha \right) =$$

$$= \sum_{I\in\mathcal{P}_1} (N(I) f(I,\alpha)) + \sum_{I\in\mathcal{P}_2} f(I,\alpha) \geq$$

$$\geq \sum_{I \in \mathcal{P}_1} (N(I) f(I_{min}, \alpha)) + \sum_{I \in \mathcal{P}_2} f(I_{min}, \alpha) =$$

$$= \left(\sum_{I \in \mathcal{P}_1} (N(I)) + \sum_{I \in \mathcal{P}_2} (1) \right) f(I_{min}, \alpha). \tag{3.12}$$

Let us mention that the expression

$$\left(\sum_{I \in \mathcal{P}_1} (N(I)) + \sum_{I \in \mathcal{P}_2} (1) \right) \tag{3.13}$$

is the number of n_1-th rank spectral cylinders (i.e., number of elements $\mathcal{A}^{\mu}_{n_1}$). Expression (3.13) equals to $\prod_{j=1}^{n_1} (l_j + 1)$. By inequality (3.12), we have

$$\sum_i |E_i|^{\alpha} \geq \prod_{j=1}^{n_1} (l_j + 1) f(I_{min}, \alpha). \tag{3.14}$$

Let us consider two possible cases (i) and (ii).

(i) Let $I_{min} \in \mathcal{P}_1$. If $N(I_{min}) = 1$, then $I_{min} \in \mathcal{A}^{\mu}_{n_1}$ and

$$\sum_i |E_i|^{\alpha} \geq \prod_{j=1}^{n_1} (l_j + 1) f(I_{min}, \alpha)$$

$$= \prod_{j=1}^{n_1} (l_j + 1) |I_{min}|^{\alpha} = \sum_{I \in \mathcal{A}^{\mu}_{n_1}} |I|^{\alpha},$$

which proves the lemma. If

$$N(I_{min}) \geq 2,$$

then

$$2 \left(\frac{l_{n_1} + 1}{N(I_{min})} + 1 \right) |I_{min}| \geq |I|, \ I \in \mathcal{A}^{\mu}_{n_1 - 1}. \tag{3.15}$$

(The number of gaps between unions of cylinders of n_1-th rank. The maximum number is less thet $\frac{l_{n_1}+1}{N(I_{min})} + 1$).

According to the above inequality, we have

$$|I_{min}| \geq \frac{1}{4}\frac{N(I_{min})}{(l_{n_1}+1)}|I|, \ I \in \mathcal{A}^{\mu}_{n_1-1}. \tag{3.16}$$

By (3.14) and (3.16), we get

$$\sum_i |E_i|^\alpha \geq \prod_{j=1}^{n_1}(l_j+1)f(I_{min},\alpha)$$

$$= \prod_{j=1}^{n_1-1}(l_j+1)\left(\frac{l_{n_1}+1}{N(I_{min})}|I_{min}|^\alpha\right) \geq$$

$$\geq \prod_{j=1}^{n_1-1}(l_j+1)\left(\frac{l_{n_1}+1}{N(I_{min})}\left(\frac{1}{4}\frac{N(I_{min})}{(l_{n_1}+1)}|I|\right)^\alpha\right) \geq$$

$$\geq \frac{1}{4}\prod_{j=1}^{n_1-1}(l_j+1)|I|^\alpha = \frac{1}{4}\sum_{I\in\mathcal{A}^{\mu}_{n_1-1}}|I|^\alpha.$$

Hence the first statement of the lemma is true in the case $N(I_{min}) \geq 2$.

(ii) Let $I_{min} \in \mathcal{P}_2$. In this case I_{min} is a spectral cylinder of n_1-th rank. Let

$$\mathcal{Q}_0 = \{I : I \in \mathcal{E}, \ I \subset I_{min}\}.$$

Let l be the distance from the left site of cylinder I_{min} and point 0, i.e.,

$$l = \inf\{|x| : x \in I_{min}\}.$$

Define the family of sets $\mathcal{Q}_1$ by the shifting of all sets from the family $\mathcal{Q}_1$ by l , i. e.,

$$\mathcal{Q}_1 = \{\{\{x-l\} : x \in I\} : I \in \mathcal{Q}_0\}.$$

Let

$$\mathcal{Q}_{i+1} = \{\{x + i|I_{min}| : x \in I\} : I \in \mathcal{Q}_1\}\, m$$

for $i \in \{1,\ ...,\ \prod_{j=1}^{n_1}(l_j+1) - 1\}$. We can build a covering of the spectrum S_μ

$$\mathcal{Q} = \bigcup_{i=1}^{\prod_{j=1}^{n_1}(l_j+1)} \mathcal{Q}_i.$$

By construction of the covering we have

$$\sum_{I\in\mathcal{Q}} |I|^\alpha = \prod_{j=1}^{n_1}(l_j+1) \sum_{E\in\mathcal{E},\ E\subset I_{min}} |E|^\alpha.$$

By the above and inequality (3.14), we get

$$\sum_i |E_i|^\alpha \geq \prod_{j=1}^{n_1}(l_j+1) f(I_{min}, \alpha)$$

$$= \prod_{j=1}^{n_1}(l_j+1) \sum_{E\in\mathcal{E},\ E\subset I_{min}} |E|^\alpha = \sum_{I\in\mathcal{Q}} |I|^\alpha,$$

i.e.,

$$\sum_i |E_i|^\alpha \geq \sum_{I\in\mathcal{Q}} |I|^\alpha. \tag{3.17}$$

It should be mentioned that if n_1' and n_2' are minimum and maximum "forming" spectral cylinders of covering $\mathcal{Q}$, then $n_1 > n_1' \geq n_2' \geq n_2$. Now we can repeat the procedure of (i) using inequality (3.17). After a finite number of steps one can find $n(\varepsilon)$ such that $n_1 - 1 \geq n(\varepsilon) \geq n_2$ and

$$\sum_i |E_i|^\alpha \geq \frac{1}{4} \sum_{I\in\mathcal{A}^\mu_{n(\varepsilon)}} |I|^\alpha.$$

□

Remark 3.3. By a standard procedure, one can proof the faithfulness of a covering family

$$\mathcal{A}^{\mu} := \bigcup_{i=1}^{\infty} \mathcal{A}_i^{\mu}$$

on the spectrum of the random variable ξ for the case $\sup\{l_n\} < \infty$. But the family $\mathcal{A}^{\mu}$ can be non-faithful $\sup\{l_n\} = \infty$ (see Theorem 2.1).

However, one can prove that the Hausdorff - Besicovitch dimension of the spectrum of the random variable ξ equals to

$$\varliminf_{n\to\infty} \left(\frac{\sum_{j=1}^{n} \ln(l_j+1)}{-\ln r_{k_n}} \right).$$

Lemma 3.5 ([FRW96])**.**

$$\dim_H S_{\mu} = \varliminf_{n\to\infty} \frac{\log \prod\limits_{j=1}^{n} (l_j+1)}{-\log r_{k_n}}$$

Proof. It is easily seen that $\mathcal{A}_n^{\mu}$ is a covering of spectrum S_{μ}. By the above,

$$H^{\alpha}(S_{\mu}) \le H^{\alpha}(S_{\mu}, \mathcal{T}) \le H^{\alpha}(S_{\mu}, \mathcal{A}_n^{\mu})$$

$$\le \varliminf_{n\to\infty} \sum_{I \in \mathcal{A}_n^{\mu}} I^{\alpha} = \varliminf_{n\to\infty} r_{k_n}^{\alpha} \cdot \prod_{j=1}^{n} (l_j+1).$$

Hence

$$H^{\alpha}(S_{\mu}) \le \varliminf_{n\to\infty} r_{k_n}^{\alpha} \cdot \prod_{j=1}^{n} (l_j+1). \tag{3.18}$$

Let $\mathcal{E} = \{E_i\}_{i \in mathbbN}$ be some ε-covering of the spectrum S_{μ} and $\mathcal{E} \subset \mathcal{T}$. Let $\alpha \in (0,1]$. By Lemma 3.4, it follows that

$$H^{\alpha}(S_{\mu}, \mathcal{T}) \geq \frac{1}{4} \varliminf_{n\to\infty} \sum_{I \in \mathcal{A}_n^{\mu}} I^{\alpha}.$$

By the above and Lemma 3.3, one can get

$$H^{\alpha}(S_{\mu}) \geq \frac{1}{24} \varliminf_{n\to\infty} \sum_{I \in \mathcal{A}_n^{\mu}} I^{\alpha} = \frac{1}{24} \varliminf_{n\to\infty} r_{k_n}^{\alpha} \cdot \prod_{j=1}^{n} (l_j + 1).$$

Therefore

$$\frac{1}{24} \varliminf_{n\to\infty} r_{k_n}^{\alpha} \prod_{j=1}^{n} (l_j + 1) \leq H^{\alpha}(S_{\mu}) \leq \varliminf_{n\to\infty} r_{k_n}^{\alpha} \prod_{j=1}^{n} (l_j + 1), \alpha \in (0, 1]. \tag{3.19}$$

If

$$\alpha > \varliminf_{n\to\infty} \frac{\log \prod_{j=1}^{n} (l_j + 1)}{-\log r_{k_n}},$$

then there exists subsequence $\{n(i)\}_{i \geq 1}$ such that

$$\alpha > \frac{\log \prod_{j=1}^{n(i)} (l_j + 1)}{-\log r_{k_{n(i)}}}.$$

Hence

$$r_{k_{n(i)}}^{\alpha} \prod_{j=1}^{n(i)} (l_j + 1) \leq 1, \ \forall i \in \mathbb{N}.$$

Therefore

$$\varliminf_{n\to\infty} r_{k_n}^{\alpha} \prod_{j=1}^{n} (l_j + 1) \leq 1.$$

By the above inequality and (3.19), one can get

$$H^{\alpha}(S_{\mu}) \leq 1.$$

Hence

$$\dim_H S_\mu \leq \varliminf_{n\to\infty} \frac{\log \prod_{j=1}^{n}(l_j+1)}{-\log r_{k_n}}.$$

Let

$$\alpha < \varliminf_{n\to\infty} \frac{\log \prod_{j=1}^{n}(l_j+1)}{-\log r_{k_n}},$$

then for all subsequences $\{n(i)\}_{i\geq 1}$

$$\alpha < \frac{\log \prod_{j=1}^{n(i)}(l_j+1)}{-\log r_{k_{n(i)}}}.$$

Therefore $\forall \{n(i)\}_{i\geq 1}$ we have

$$\prod_{j=1}^{n(i)}(l_j+1) r_{k_{n(i)}}^{\alpha} \geq 1,\ \forall i \in \mathbb{N}.$$

So,

$$\varliminf_{n\to\infty} \prod_{j=1}^{n}(l_j+1) r_{k_n}^{\alpha} \geq 1.$$

By the above inequality and (3.19), we have

$$H^\alpha(S_\mu) \geq \frac{1}{24}.$$

Therefore

$$\dim_H S_\mu \geq \varliminf_{n\to\infty} \frac{\log \prod_{j=1}^{n}(l_j+1)}{-\log r_{k_n}},$$

which proves the theorem, i.e.,

$$\dim_H S_\mu = \varliminf_{n\to\infty} \frac{\log \prod_{j=1}^{n}(l_j+1)}{-\log r_{k_n}}.$$

□

Remark 3.4. The formula for $\dim_H S_\xi$ was proved without any restrictions on$\{l_k\}_{k\in\mathbb{N}}$.

Theorem 3.4. *1) the Hausdorff - Besicovitch dimension of the set of points that have a finite number of representations (3.3) is equal to*

$$\varliminf_{n\to\infty}\left(\frac{n\ln 2}{-\ln r_{k_n}}\right).$$

2) the Hausdorff - Besicovitch dimension of the set of points that have continuum many representations (3.3) is equal to

$$\varliminf_{n\to\infty}\left(\frac{\sum_{j=1}^{n}\ln(l_j+1)}{-\ln r_{k_n}}\right).$$

Proof. We recall some main properties of the Hausdorff - Besicovitch dimension (see [Fal04] for details):

B1) if $E_1\subset E_2$, then $\dim_H(E_1)\le\dim_H(E_2)$;

B2) $\dim_H(\bigcup\limits_n E_n)=\sup\limits_n\dim_H(E_n)$;

B3) if E_1 and E_2 are homothetic, then $\dim_H(E_1)=\dim_H(E_2)$.

We construct an auxiliary sequence of sets $\{L_j\}_{j\in\mathbb{N}}$ such that

$$L_j:=\Bigg\{x:\Delta^{\widetilde{Q}^*}_{\gamma_1\gamma_2\ldots\gamma_n\ \ldots},\gamma_n\in B_n,\quad\text{if}\ \ n\in\{1,2,...,j-1\},$$

$$\text{and}\ \ \gamma_n\in\{0,m_n-1\},\quad\text{if}\ \ n\in\mathbb{N}\backslash\{1,2,...,j-1\}\Bigg\}.$$

The set L_1 coincides with the set of those points that have a unique representation (3.3). The reasoning similar to that used in the proof of Theorem 3.5 shows that

$$\dim_H(L_1)=\varliminf_{n\to\infty}\left(\frac{n\ln 2}{-\ln r_{k_n}}\right).$$

Now we are going to show that $\dim_H(L_j) = \dim_H(L_1), \forall j \in \mathbb{N}$.

Let $j \in \mathbb{N}\backslash\{1\}$. Then

$$L_1 = \bigcup_{i}^{2^{j-1}} L_1^{(i)},$$

where $\{L_1^{(i)}\}_{i\in\{1\ ,...,\ 2^{j-1}\}}$ are isometric sets whose intersection consists of at most one point and where

$$L_1^{(1)} := \Bigg\{x : x = \Delta^{\widetilde{Q}^*}_{\gamma_1\gamma_2...\gamma_n\ \ ...}, \gamma_n = 0, \quad \text{if} \quad n \in \{1, 2, ..., j-1\},$$

$$\text{i} \quad \gamma_n \in \{0, m_n - 1\}, \quad \text{if} \quad n \in \mathbb{N}\backslash\{1, 2, ..., j-1\}\Bigg\}.$$

The equality

$$\dim_H(L_1^{(1)}) = \dim_H(L_1) \tag{3.20}$$

follows from the properties B2) and B3).

The set L_j can be represented in the form

$$L_j = \bigcup_{t=1}^{\prod\limits_{i=1}^{j-1}(l_i+1)} L_j^{(t)},$$

where

$$\{L_j^{(t)}\}_{t\in\left\{1,\ ...,\ \prod\limits_{i=1}^{j-1}(l_i+1)\right\}}$$

are isometric sets whose intersection contains at most one point, and

$$L_1^{(1)} \in \{L_j^{(t)}\}_{t\in\left\{1,\ ...,\ \prod\limits_{i=1}^{j-1}(l_i+1)\right\}}.$$

Then properties B2) and B3) together with equality (3.20) imply that

$$\dim_H(L_j) = \dim_H(L_1), \ \forall j \in \mathbb{N}.$$

The preceding equality together with B2) yields

$$\dim_H(\bigcup_i L_i) = \sup_i \dim_H(L_i) = \dim_H(L_1). \tag{3.21}$$

The set of points that have a finite number of representations in the form of (3.3) coincides with the set $\bigcup\limits_{j=1}^{\infty} L_j$. This completes the proof of the first statement of the Theorem.

Let L^* be the set of points for which there exist continuum many different representations in the form of (3.3). If

$$\dim_H S_\mu > \dim_H L_1 = \varliminf_{n\to\infty} \left(\frac{n \ln 2}{-\ln r_{k_n}} \right),$$

then the equality $S_\mu = L^* \cup \left(\bigcup\limits_{j=1}^{\infty} L_j \right)$ and property B2) imply that $\dim_H L^* = \dim_H S_\mu$.

Hence almost all points (in the sense of the Hausdorff - Besicovitch dimension) of the spectrum S_μ have continuum many different representations in the form of (3.3).

Now we show that $\dim_H L^* = \dim_H S_\mu$, even in the case where

$$\varliminf_{n\to\infty} \left(\frac{\sum\limits_{j=1}^{n} \ln(l_j+1)}{-\ln r_{k_n}} \right) = \varliminf_{n\to\infty} \left(\frac{n \ln 2}{-\ln r_{k_n}} \right),$$

that is where $\dim_H S_\mu = \dim_H L_1$ (this is the case, in particular, if $l_n = 1$ $(n \neq 2^s)$ and $l_n = 2$ $(n = 2^s)$).

Since $l_n > 1$ for infinitely many indices n, one can choose a sufficiently "sparse" subsequence n_t in such a way that the sets B_{n_t} contain at least three elements. In each of the sets $B_{n_t} \setminus \{0, m_{n_t} - 1\}$ (see the beginning of the Section for the definition of B_{n_t}) we choose an arbitrary element and denote it by θ_t. Consider the set

$$K_1 = \Bigg\{ x : \ x := \tilde{\Delta}_{\gamma_1 \gamma_2 \ldots \gamma_n \ldots},$$

$$\text{where} \quad \gamma_k \in \{0, m_k - 1\} \quad \text{when} \quad k \notin \{n_t\} \quad \text{and} \quad \gamma_{n_t} = \theta_t, \ \forall t \in \mathbb{N} \Bigg\}.$$

Each point of the set K_1 has continuum many representations of the form (3.3), that is, $K_1 \subset L^*$. The reasoning similar to that used in the proof of Theorem 3.4, proves that

$$\dim_H(K_1) = \varliminf_{n \to \infty} \left(\frac{\sum\limits_{i=1}^{n} \ln(z_i + 1)}{- \ln r_{k_n}} \right),$$

where $z_i = 0$ for $i \in \{n_t\}$, and $z_i = 1$ for $i \notin \{n_t\}$. Then

$$\dim_H(K_1) = \varliminf_{n \to \infty} \left(\frac{(n - \tau(n)) \ln 2}{- \ln r_{k_n}} \right) = \varliminf_{n \to \infty} \left(\frac{(n - \tau(n))}{n} \ \frac{n \ln 2}{- \ln r_{k_n}} \right),$$

where $\tau(n)$ is the number of members of the sequence $\{n_t\}$ which is less than n. Since the sequence $\{n_t\}$ is sufficiently "sparse" in the sense that $\frac{\tau(n)}{n} \to 0 (n \to \infty)$, we have $\dim_H K_1 = \dim_H L_1$. Hence $\dim_H L^* = \dim_H L_1$.

□

Corollary 3.3. *If $\lambda(S_\mu) > 0$, the almost all (with respect to the Lebesgue measure) spectrum points have continuum many different representations (3.3).*

3.5 Hausdorff dimension of the probability distributions of LT - Bernoulli convolutions

Recall that the number

$$\dim_H(\tau) = \inf \left\{ \dim_H(E),\ E \in \mathcal{B}_\tau \right\},$$

is called the Hausdorff dimension of the distribution of a random variable τ, where $\mathcal{B}_\tau$ is the class of all Borel supports ($\mathcal{B}_\tau$ needs not necessarily be closed) of a random variable τ; that is,

$$\mathcal{B}_\tau = \left\{ E:\ E \in \mathcal{B},\ P_\tau(E) = 1 \right\}.$$

We will need the following notations. Let

$$h_j = -\sum_{i=0}^{m_j-1} \tilde{p}_{ij} \ln \tilde{p}_{ij}, \quad H_n = \sum_{j=1}^{n} h_j.$$

Theorem 3.5. *If*

$$\sum_{n=1}^{\infty} \left(\frac{\ln r_{k_{n-1}}}{\ln r_{k_n}} - 1 \right)^2 < \infty, \tag{3.22}$$

then the Hausdorff dimension of the probability distribution μ_ξ of the random variable ξ is equal to

$$\dim_H(\mu_\xi) = \varliminf_{n\to\infty} \frac{H_n}{-\ln r_{k_n}},$$

Proof. Let $\widetilde{\Delta}_{[n]}(x) = \Delta^{\widetilde{Q}^*}_{a_1(x)a_2(x)\ldots a_n(x)}$ be a $\widetilde{Q}^*$ cylindrical interval of rank n that contains a point x of the spectrum $S_\xi \backslash \{1\}$. Note that the class of all cylindrical intervals coincides with $\widetilde{\mathcal{A}}$ (see (3.11)). Let μ be the probability measure of the random variable ξ, that is,

$$\forall E \in \mathcal{B}:\ \mu(E) = P\{\xi \in E\}.$$

Let λ - denote the Lebesgue measure in $[0,1]$. Then

$$\mu(\widetilde{\Delta}_{[n]}(x)) = \tilde{p}_{a_1(x)1} \cdot \tilde{p}_{a_2(x)2} \cdot \ldots \cdot \tilde{p}_{a_n(x)n},$$

$$\lambda(\widetilde{\Delta}_{[n]}(x)) = q_{a_1(x)1} \cdot q_{a_2(x)2} \cdot \ldots \cdot q_{a_n(x)n} = r_{k_n}.$$

Consider

$$\frac{\ln \mu(\widetilde{\Delta}_{[n]}(x))}{\ln \lambda(\widetilde{\Delta}_{[n]}(x))} = \frac{\sum_{j=1}^{n} \ln \tilde{p}_{a_j(x)j}}{\ln r_{k_n}}.$$

If $x = \widetilde{\Delta}_{a_1(x)a_2(x)\ldots a_n(x)\ldots}$ is chosen randomly such that

$$P(a_j(x) = i) = \tilde{p}_{ij}$$

(in other words, the distribution of the random variable x corresponds to the measure μ), then

$$\{\eta_j\} = \{\eta_j(x)\} := \{\ln \tilde{p}_{a_j(x)j}\}$$

is a sequence of independent random variables with the following distributions:

$$P\{\eta_j = \ln \tilde{p}_{ij}\} = \tilde{p}_{ij},\ i \in \{0,\ \ldots,\ m_j - 1\}\,.$$

It is also clear that

$$E\eta_j = \sum_{i=0}^{m_j-1} \tilde{p}_{ij} \ln \tilde{p}_{ij} = -h_j,\ \text{and}\ |E\eta_j| \leq \ln{(l_j + 1)}.$$

Let us show that

$$E\eta_j^2 = \sum_{i=0}^{m_j-1} \tilde{p}_{ij} \ln^2 \tilde{p}_{ij} \leq \max\{4, \ln^2{(l_j + 1)}\}.$$

To this end we put $\{x_0\} := \{x : \ln(x) - 2x + 2 = 0\} \backslash \{1\}$. Here $\varphi : [0,1] \to \mathbb{R}$ denotes the function such that

$$\varphi(x) = \begin{cases} x\ln^2 x, & \text{if} \quad x \in [0, x_0); \\ -x_0 \ln^2 x_0 \cdot \frac{x-x_0}{1-x_0} + x_0 \ln^2 x_0, & \text{if} \quad x \in [x_0, 1]. \end{cases}$$

From the definition of $\varphi(x)$ it follows that

$$x \ln^2 x \le \varphi(x), \quad \forall x \in [0, 1].$$

The function $\varphi(x)$ is convex on $[0, 1]$. Therefore,using the Jensen's inequality we have

$$E\eta_j^2 \le \sum_{i=0}^{m_j-1} \varphi(\tilde{p}_{ij}) \le (l_j + 1)\varphi\left(\frac{1}{l_j+1}\right) \le \max\{4, (\ln(l_j + 1))^2\}.$$

Therefore

$$D(\eta_j) = E\eta_j^2 - (E\eta_j)^2 \le 2\max\{4, \ln^2(l_j + 1)\}.$$

By the inequality

$$\left(\frac{\ln(l_n+1)}{\ln r_{k_n}}\right)^2 \le \left(\frac{\ln r_{k_{n-1}}}{\ln r_{k_n}} - 1\right)^2$$

and Kolmogorov's theorem ([Shi96, Ch IV, §3.2]) we get for $x \in [0, 1]$ μ_ξ-almost all points $x \in [0, 1]$:

$$\lim_{n\to\infty} \frac{(\eta_1(x) + \eta_2(x) + ... + \eta_n(x)) - E\left(\eta_1(x) + \eta_1(x) + ... + \eta_n(x)\right)}{\ln r_{k_n}} = 0. \tag{3.23}$$

Set

$$D = \underline{\lim}_{n\to\infty} \frac{H_n}{-\ln r_{k_n}}.$$

Consider

$$T = \left\{x : \lim_{n\to\infty}\left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\ln \lambda(\Delta_n(x))} - \frac{H_n}{-\ln \lambda(\Delta_n(x))}\right)\right\}$$

$$= \left\{ x : \lim_{n\to\infty} \left(\frac{\eta_1 + \eta_2 + ... + \eta_n - E(\eta_1 + \eta_2 + ... + \eta_n)}{\ln r_{k_n}} \right) = 0 \right\}.$$

Since $\mu(T) = 1$, it follows that $\dim_\mu(T, \widetilde{\mathcal{A}}) = 1$. Let

$$T_1 = \left\{ x : \lim_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\ln r_{k_n}} - \frac{H_n}{-\ln r_{k_n}} \right) = 0 \right\};$$

$$T_2 = \left\{ x : \underline{\lim}_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\ln r_{k_n}} \le \underline{\lim}_{n\to\infty} \frac{H_n}{-\ln r_{k_n}} \right\}$$

$$= \left\{ x : \underline{\lim}_{n\to\infty} \frac{\ln \mu(\widetilde{\Delta}_{[n]}(x))}{\ln \lambda(\widetilde{\Delta}_{[n]}(x))} \le \underline{\lim}_{n\to\infty} \frac{H_n}{-\ln r_{k_n}} \right\};$$

$$T_3 = \left\{ x : \underline{\lim}_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + ... + \eta_n(x)}{\ln r_{k_n}} \ge \underline{\lim}_{n\to\infty} \frac{H_n}{-\ln r_{k_n}} \right\}$$

$$= \left\{ x : \underline{\lim}_{n\to\infty} \frac{\ln \mu(\widetilde{\Delta}_{[n]}(x))}{\ln \lambda(\widetilde{\Delta}_{[n]}(x))} \ge \underline{\lim}_{n\to\infty} \frac{H_n}{-\ln r_{k_n}} \right\}.$$

One can prove that $T \subset T_1$, $T_1 \subset T_3$ and $T \subset T_2$.

By Theorem 1.3, we get

$$\dim_\lambda(T_2, \widetilde{\mathcal{A}}) \le D.$$

By the inclusion $T \subset T_2$, we have

$$\dim_\lambda(T, \widetilde{\mathcal{A}}) \le D.$$

Since

$$T \subset T_3 = \left\{ x : \underline{\lim}_{n\to\infty} \frac{\ln \mu(\widetilde{\Delta}_{[n]}(x))}{\ln \lambda(\widetilde{\Delta}_{[n]}(x))} \ge D \right\}$$

and by Theorem 1.4, we have

$$\dim_\lambda(T, \widetilde{\mathcal{A}}) \geq D \cdot \dim_\mu(T, \widetilde{\mathcal{A}}) = D \cdot 1 = D.$$

Therefore $\dim_\lambda(T, \widetilde{\mathcal{A}}) = D$. Since λ is the Lebesgue measure on $[0, 1]$, we have

$$\dim_H(T, \widetilde{\mathcal{A}}) = \dim_\lambda(T, \widetilde{\mathcal{A}}) = D.$$

By assumption (3.22), we have

$$\lim_{n\to\infty} \frac{\ln r_{k_{n-1}}}{\ln r_{k_n}} = 1.$$

From Theorem 3.3 it follows that the family $\widetilde{\mathcal{A}}$ is faithful for Hausdorff - Bezikovich dimension calculation on $S_\mu \backslash \{1\}$. Hence

$$\dim_H(T, \widetilde{\mathcal{A}}) = \dim_H(T) = D.$$

We now prove that the above constructed se T is the "smallest" support of the measure μ in the sense of Hausdorff - Besicovitch dimension. Let C be an arbitrary support of the measure μ, that is, $\mu(C) = 1$. It is easily seen that the set $C_1 := C \bigcap T$ is also a support of the same measure μ , and $C_1 \subset C$. Hence $\dim_H(C_1) \leq \dim_H(C)$ and $C_1 \subset T$. We shall prove that $\dim_H(C_1) = \dim_H(T)$. From $C_1 \subset T$ it follows that

$$\dim_H(C_1) \leq \dim_H(T) = D.$$

On the other hand, we have

$$C_1 \subset T \subset T_3 = \left\{ x : \underline{\lim}_{n\to\infty} \frac{\ln \mu(\widetilde{\Delta}_{[n]}(x))}{\ln \lambda(\widetilde{\Delta}_{[n]}(x))} \geq D \right\}.$$

Therefore, by using the faithfulness of the family of Cantor coverings and Theorem 1.4 and (3.3) we conclude

$$\dim_H(C_1) = \dim_\lambda(C_1, \widetilde{\mathcal{A}})$$

$$\geq D \cdot \dim_\mu(T, \widetilde{\mathcal{A}}) \geq D \cdot \dim_\mu(C_1, \widetilde{\mathcal{A}}) = D \cdot 1 = D.$$

□

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