ELUCIDATION IN THE GARDEN OF EDEN

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We presuppose the terminology of Amoroso and Cooper [1]. The restriction of an array configuration c to a finite subset S of Z^2 will be called a <u>pattern</u> in the present paper. A pattern may alternatively be defined as a partial function from Z^2 into A. Note that what we call patterns are "configurations" in the sense of Moore [2] and Myhill [3,4].

An array configuration $\bar{c}$ is called <u>completion</u> of a pattern $(c)_S$ iff $(\bar{c})_S = (c)_S$.

A pattern $(c)_S$ is said to be a <u>Garden-of-Eden (GOE) pattern</u> iff there is no pattern $(d)_T$ $(S \subset T)$ such that the states of the cells of S are the result of mapping the states of the cells of T by the local transition function σ (see [2]). (T has to be at least N(S) because σ refers to the neighbors of each cell of S.)

What Moore [2] and Myhill [3] show is the following:

With respect to a given tesselation structure, the existence of mutually erasable patterns implies the existence of GOE patterns (Moore); and the existence of GOE patterns implies the existence of indistinguishable patterns (Myhill).

(The existence of mutually erasable patterns is equivalent to the existence of indistinguishable patterns.)

Our question is if it is possible to have a "global" Garden-of-Eden theorem stated in terms of (array) configurations as claimed by Amoroso and Cooper [1], where an array configuration c is called a <u>GOE configuration</u> iff no array configuration c' exists such that $\tau(c') = c$. (τ is the global transition function determined by σ.)

In their paper [1], Amoroso and Cooper state a Garden-of-Eden theorem for finite configurations of which the converse is not true. This is not an equivalent global theorem corresponding to the (local) theorem stated in terms of patterns as we will see later, yet it is possible to have an equivalent global theorem stated in terms of finite configurations.

In [5], Myhill claims that the Garden-of-Eden definition appearing in his paper [3] is different to the definition appearing in [2], and further he claims that in neither interpretation, both Moore's theorem and its converse are true. These statements are not correct. The following discussion will elucidate the confusion that has arisen in the Garden of Eden.

Moore [1] requires the tesselation to be such that a quiescent symbol is specified, but the proof of his Garden-of-Eden theorem is independent of such an assumption as well as the proof of the converse (see [4]); that is because these proofs depend on combinatorial considerations, and nowhere a special property of states is referred to [1]).

Lemma 1 : A pattern is GOE iff for all configurations d :

$$(\tau(d))_S \neq (c)_S$$

Proof: Suppose $(c)_S$ is a GOE pattern and there is a configuration d such that $(\tau(d))_S = (c)_S$.

Then obviously $(c)_S$ is obtained by mapping the states of the cells of $(d)_{N(S)}$ by σ. Hence $(c)_S$ is not GOE which is a contradiction.

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[1]) So for the Garden-of-Eden theorem it is not a necessary assumption to have at time $t = 0$ all but a finite number of cells in the quiescent state.

On the other hand, suppose $(\tau(d))_S \neq (c)_S$ for all d and $(c)_S$ not GOE. That means, there exists a pattern $(d)_T$ $(S \subset T)$ which gives rise to $(c)_S$. Define a configuration $\overline{d}$ such that

$$\overline{d}(\beta) = \begin{cases} d(\beta) & \beta \in N(S) \\ e(\beta) & \text{otherwise} \end{cases} \qquad (\beta \in Z^2)$$

where e is any configuration. Then $(\tau(\overline{d}))_S = (c)_S$ which gives a contradiction.

In the following, the set of all configurations of a tesselation shall be denoted by $\mathcal{C}$.

Lemma 2 : A configuration c is an element of $\tau(\mathcal{C})$ iff for all finite subsets T of Z^2 there exists a configuration d such that $(\tau(d))_T = (c)_T$.

Proof: Let c be an element of $\tau(\mathcal{C})$. That means, there exists a configuration d such that $\tau(d) = c$ and thus $(\tau(d))_T = (c)_T$ for all subsets T of Z^2.

Conversely we define subsets T_n of Z^2:

$T_o := \{\beta_o\}$ for some $\beta_o \in Z^2$

$T_{n+1} := N(T_n)$

The T_n $(n \in \mathbb{N})$ are well-ordered on Z^2 with respect to the inclusion. We suppose without loss of generality [2])

$$\bigcup_{n=o}^{\infty} T_n = Z^2$$

We wish to show that there exists a simultaneous d for which $(\tau(d))_{T_n} = (c)_{T_n}$

holds for all (finite) sets T_n, and $\tau(d) = c$.

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[2]) There are neighborhood templates which are not subject to this assumption. In this case, we define dummy neighbors in the way that the completed local transition function actually depends on the original neighbors. So the same proof holds for more general neighborhoods.

0. The assignment of states to the finite number of cells of T_o at time $t+1$ is determined by the assignment to the cells of $N(T_o) = T_1$ at time t.

   T_1 is finite, and so is the alphabet of available states, A. From this it follows that there are only finitely many patterns $(d)_{T_1}$ such that for any completion d

   $$(\tau(d))_{T_o} = (c)_{T_o}$$

   holds, in the maximum case as many as there are different assignments to T_1, and at least one according to the assumption.

1. For each of these finitely many patterns there are only finitely many continuing patterns $(\overline{d})_{T_2}$ such that for any completion $\overline{d}$ $(\tau(\overline{d}))_{T_1} = (c)_{T_1}$ because of the above-mentioned argument.

   This argument can be given for all n. In this way, an infinite but locally finite tree is obtained. By the Infinity lemma of König (see [7]), there exists an infinite path in this tree, i.e. a simultaneous d such that $\tau(d) = c$.

Using Lemma 2 we derive the following theorems:

Theorem 1 :

With respect to a given tesselation structure, there exist GOE patterns iff $\mathcal{C} \setminus \tau(\mathcal{C}) \neq \phi$.

Proof: (the contrapositions are proved)

$$\neg \exists (c)_S \text{ GOE pattern}$$

is equivalent to

$$\forall S \text{ finite } \forall (c)_S \ \exists d \ (d \in \mathcal{C} \ \& \ (\tau(d))_S = (c)_S)$$

This follows from Lemma 1 and is because of Lemma 2 equivalent to

$$\forall c \ (c \in \mathcal{C} \Longrightarrow \exists d \ (d \in \mathcal{C} \ \& \ \tau(d) = c))$$

which means $\mathcal{C} \setminus \tau(\mathcal{C}) = \phi$.

For tesselations with a quiescent symbol 0 specified, the set of all finite configurations shall be denoted by $\mathcal{C}_f$.

Theorem 2 :

With respect to an arbitrary tesselation structure for which a quiescent symbol is specified:

$$\mathcal{C} \setminus \tau(\mathcal{C}) \neq \phi \text{ iff } \mathcal{C}_f \setminus \tau(\mathcal{C}) \neq \phi$$

Proof: Let $c \in \mathcal{C} \setminus \tau(\mathcal{C})$. Then from Lemma 2 follows:

$$\exists T \text{ finite } \forall d \in \mathcal{C} \quad (\tau(d))_T \neq (c)_T$$

Define $\bar{c}$ such that
$$\bar{c}(\beta) = \begin{cases} c(\beta) & \beta \in N(T) \\ 0 & \text{otherwise} \end{cases} \qquad (\beta \in Z^2)$$

Then $\bar{c} \in \mathcal{C}_f$ and $(\bar{c})_T = (c)_T$. Hence $\bar{c} \in \mathcal{C}_f \setminus \tau(\mathcal{C})$.

The converse is obvious.

Because of Theorem 1 and Theorem 2 we can say:

An array configuration is GOE in the sense of Moore and Myhill if it has no predecessor, whether finite or not.

We can transmit the term "mutually erasable" to configurations in a canonical matter:

Two distinct configurations $c, d \in \mathcal{C}$ are called <u>mutually erasable configurations</u> iff there is a finite subset T of Z^2 such that
$$(c)_{Z^2 \setminus N(N(S))} = (d)_{Z^2 \setminus N(N(S))}$$
and $(c)_{N(N(S))}$, $(d)_{N(N(S))}$ are mutually erasable.

Lemma 3 : The following statements are equivalent:

There exist mutually erasable patterns.
There exist mutually erasable configurations.
There exist mutually erasable finite configurations.

Proof: Let $(c)_{N(N(S))}$, $(d)_{N(N(S))}$ be two mutually erasable patterns. We define configurations $\bar{c}$, $\bar{d}$ such that

$$\bar{c}(\beta) = \begin{cases} c(\beta) & \beta \in N(N(S)) \\ e(\beta) & \text{otherwise} \end{cases} \qquad \bar{d}(\beta) = \begin{cases} d(\beta) & \beta \in N(N(S)) \\ e(\beta) & \text{otherwise} \end{cases}$$

where e is an arbitrary (finite or infinite) configuration. From this construction it immediately follows that $\bar{c}$, $\bar{d}$ are mutually erasable configurations; they are finite if e is finite.

The existence of (finite or infinite) mutually erasable configurations implies the existence of mutually erasable patterns, as follows from the definition.

It is easy to see that mutually erasable configurations exist iff τ_f , the restriction of τ to $\mathcal{C}_f$, is not injective.

Now we are able to formulate a "global" Garden-of-Eden theorem:

Theorem 3 :

In a given tesselation structure there exist (finite or infinite) GOE configurations, i.e. $\mathcal{C} \setminus \tau(\mathcal{C}) \neq \phi$, iff (finite or infinite) mutually erasable configurations exist.

Corollary:

In a given tesselation structure the existence of (finite) mutually erasable configurations implies:

$$\mathcal{C}_f \setminus \tau_f(\mathcal{C}_f) \neq \phi$$

(The proof follows from Theorem 3 , Theorem 2, and the fact that $\mathcal{C}_f \setminus \tau(\mathcal{C}) \subseteq \mathcal{C}_f \setminus \tau_f(\mathcal{C}_f)$.)

The converse of this theorem is not true as shown by Amoroso and Cooper [1]. So this theorem is not an equivalent formulation of the Garden-of-Eden theorem.

We now proceed to establish an equivalent Garden-of-Eden theorem stated in terms of finite configurations.

A finite configuration is called <u>strongly GOE</u> iff $\exists S \subseteq Z^2$ such that

(i) S finite

(ii) $\mathrm{supp}(c) \subseteq S$

(iii) $\forall c' \in \mathcal{C}_f : \ \bar{c} \in \mathcal{C}_f \setminus \tau_f(\mathcal{C}_f)$ where $\bar{c}(\beta) = \begin{cases} c(\beta) & \beta \in S \\ c'(\beta) & \text{otherwise} \end{cases}$

Let G_S be the set of all strongly GOE configurations of a tesselation structure (see [6]).

The set G_S' shall be defined just like G_S but the condition (iii) shall be substituted by

(iii)' $\forall c' \in \mathcal{C}_f : \ \bar{c} \in \mathcal{C}_f \setminus \tau(\mathcal{C})$ where $\bar{c}(\beta) = \begin{cases} c(\beta) & \beta \in S \\ c'(\beta) & \text{otherwise} \end{cases}$

Lemma 4 : $(c)_S$ pattern:

$$\forall d \ (d \in \mathcal{C}_f \Longrightarrow (\tau_f(d))_S \neq (c)_S) \iff$$
$$\forall d \ (d \in \mathcal{C} \Longrightarrow (\tau(d))_S \neq (c)_S)$$

Proof: Suppose

$$\exists d \ (d \in \mathcal{C} \ \& \ (\tau(d))_S = (c)_S)$$

Define $\bar{d}$ such that $\bar{d}(\beta) = \begin{cases} d(\beta) & \beta \in N(S) \\ 0 & \text{otherwise} \end{cases}$

Then we have

$$\bar{d} \in \mathcal{C}_f \ \& \ (\tau_f(\bar{d}))_S = (\tau(\bar{d}))_S = (\tau(d))_S = (c)_S$$

which gives a contradiction.

The converse is obvious.

Lemma 5 : $G_S = G_S'$

Proof: $(G_S \subseteq G_S')$ Let c be an element of G_S. That means, there exists a subset S of Z^2 such that (i),(ii),(iii) holds.

Let $(c)_S$ be the restriction of c to S. The condition (iii) is then equivalent to:

$$\forall \bar{c}\ (\bar{c} \text{ finite completion of } (c)_S \Rightarrow \bar{c} \in \mathcal{C}_f \setminus \tau_f(\mathcal{C}_f))$$

This implies

$$\forall d\ (d \in \mathcal{C}_f \Rightarrow (\tau_f(d))_S \neq (c)_S)$$

and

$$\forall d\ (d \in \mathcal{C} \Rightarrow (\tau(d))_S \neq (c)_S)$$

This follows from Lemma 4 and implies

$$\forall \bar{c}\ (\bar{c} \text{ finite completion of } (c)_S \Rightarrow \bar{c} \in \mathcal{C}_f \setminus \tau(\mathcal{C}))$$

i.e.

$$\forall c' \in \mathcal{C}_f :\ \bar{c} \in \mathcal{C}_f \setminus \tau(\mathcal{C}) \quad \text{where} \quad \bar{c}(\beta) = \begin{cases} c(\beta) & \beta \in S \\ c'(\beta) & \text{otherwise} \end{cases} \qquad \text{(iii)'}$$

We conclude

$$c_1 \in G_S' \quad \text{where} \quad c_1(\beta) = \begin{cases} c(\beta) & \beta \in S \\ 0 & \text{otherwise} \end{cases}$$

$\mathrm{supp}(c) \subseteq S$ (ii) and thus $c_1 = c$. Hence $c \in G_S'$.

$G_S' \subseteq G_S$: obvious.

Lemma 6 : $G_S' = \mathcal{C}_f \setminus \tau(\mathcal{C})$

Proof: It is obvious that $G_S' \subseteq \mathcal{C}_f \setminus \tau(\mathcal{C})$.

On the other hand, let $c \in \mathcal{C}_f \setminus \tau(\mathcal{C})$. From Lemma 2 follows:

$$\exists T \subseteq Z^2,\ T \text{ finite}:$$

$$\forall d \in \mathcal{C}\ \exists \beta_0 \in T \quad \tau(d)(\beta_0) \neq c(\beta_0)$$

Choose $S \subseteq Z^2$ such that $T \subseteq S$ and $N(\mathrm{supp}(c)) \subseteq S$.
We infer immediately:

$$\forall c' \in \mathcal{C}_f :\ \bar{c} \in \mathcal{C}_f \setminus \tau(\mathcal{C}) \text{ where } \bar{c}(\beta) = \begin{cases} c(\beta) & \beta \in S \\ c'(\beta) & \text{otherwise} \end{cases}$$

Hence $c \in G_S'$.

We conclude:

Theorem 4 : (Garden-of-Eden theorem for finite configurations)

For a given tesselation structure, the existence of finite mutually erasable configurations is necessary and sufficient for the existence of strongly GOE configurations.

References:

[1] S. AMOROSO and G. COOPER, The Garden of Eden Theorem for Finite Configurations, Proc. Amer. Math. Soc. 26 (1970), pp. 158-164.

[2] E.F. MOORE, Machine Models of Self-Reproduction, Proc. of Symp. in Appl. Math. 14 (1962), pp. 17-33.

[3] J. MYHILL, The Converse of Moore's Garden-of-Eden Theorem, Proc. Amer. Math. Soc. 14 (1963), pp. 685-686.

[4] J. MYHILL, The Converse of Moore's Garden-of-Eden Theorem, in: W. BURKS, Essays on Cellular Automata, University of Illinois Press 1970, pp. 204-205.

[5] J. MYHILL, The Converse of Moore's Garden-of-Eden Theorem - A Correction, Private Communication, 1972.

[6] S. SKYUM, Confusion in the Garden of Eden, preprint, Aarhus University, 1972.

[7] K. WAGNER, Graphentheorie, Mannheim 1970.

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