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Robustness of Intermediate Agreements for the Discrete Raiffa Solution*,**

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Abstract

First via a counter example it is shown that the Proposition 3 of Anbarci & Sun (2013) is false. Then a gap and a mistake in their proof are identified. Finally, a modified version of their Proposition 3 is stated and proved.

*Counterexample and correction for Proposition 3 in N. Anbarci and C.-j. Sun (2013): Robustness of Intermediate Agreements and Bargaining Solutions, Games and Economic Behavior, 77, 367-376

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1 Basic Definitions and Axioms

This section is mainly an extract of relevant parts of the respective section in Anbarci and Sun (2013) supplemented by some remarks and an axiom from Salonen (1988). For simplicity I consider only the case $n = 2$. That suffices for the counter example. The extension of my Proposition to general $n \in \mathbb{N}$ is straight forward.

1.1 Basic Definitions

An 2-person (bargaining) problem is a pair (S, d), where $S \subset \mathbb{R}^2$ is the set of utility possibilities that the players can achieve through cooperation and $d \in S$ is the disagreement point, which is the utility allocation that results if no agreement is reached. For all S, let $IR(S, d) := \{x \in S | x \geq d\}$ be the set of individually rational utility allocations.

Let $\sum$ be the class of all 2-person problems satisfying the following:

(1) The set S is compact, convex and comprehensive.

(2) $x > d$ for some $x \in S$

Denote the ideal point of $(S, d) \in \sum$ as $b(S, d) = (b_i(S, d))_{i=1,\ldots,n}$ where $b_i(S, d) := max\{x_i \in \mathbb{R} | x \in IR(S, d)\}$; the midpoint of $(S, d) \in \sum$ is $m(S, d) := 1/2\ (b(S, d) + d)$.

A solution on $\sum$ is a function $f : \sum \longrightarrow \mathbb{R}^2$ such that for all $(S, d) \in \sum$ we have $f(S, d) \in S$.

Consider any bargaining problem $(S, d) \in \sum$. The game $(H^{S,d}, d) \in \sum$ defined by $H^{S,d} := co\ \{d,\ b_1(S, d)\ e_1,\ b_2(S, d)\ e_2\}$ with $e_i,\ i = 1, 2$, the canonical unit vectors of $\mathbb{R}^2$, is the "largest individually rational hyperplane game contained" in (S, d).

Given any bargaining problem $(S, d) \in \sum$ and a solution $f : \sum \longrightarrow \mathbb{R}^2$ the disagreement point set $D(S, d, f) := \{d' \in IR(S, d) \mid f(S, d') = f(S, d)\}$ collects all possible disagreement points d' that leave the solution $f(S, d)$ unaffected when (S, d) is replaced by $(S, d') \in \sum$.

Notice, that this definition employed by Anbarci and Sun (2013) makes use of the assumption that $(S, d') \in \sum$. Therefore, for the game $(H^{S,d}, d) \in \sum$ the set $D(H^{S,d}, d, m)$ does not contain $d' := m(H^{S,d}, d) = m(S, d)$ as an element. Therefore, the solution $m : \sum \longrightarrow \mathbb{R}^n$ on $\sum$ does not have the property that $\forall (S, d) \in \sum : m(S, d) \in D(S, d, m)$.

1.2 Axioms

First I introduce the three axioms of Anbarci and Sun (2013) relevant for their analysis. Secondly, I will discuss in detail the axioms MD and $INMD^*$. Then I will introduce a stronger version of $INMD^*$ that I denote $INMD$. Finally, I formulate for the present context and in the present terminology of Anbarci and Sun an axiom due to Salonen (1988) that is crucial for the assessment of the announced correction later in this note. Let $f : \sum \longrightarrow \mathbb{R}^n$ be a solution on $\sum$.

Robustness of Intermediate Agreements in the (d, b)-Box (RIA-Box):

For all $(S, d), (T, d) \in \sum$ such that $S \subset T$ and $b(S, d) = b(T, d)$:

$(D(S, d, f) \cup \{f(S, d)\}) \cap (D(T, d, f) \cup \{f(T, d)\}) \backslash \{d\} \neq \emptyset$.

Midpoint Domination (MD)

For any $(S, d) \in \sum : f(S, d) \geq m(S, d)$

Independence of Non-Midpoint-Dominating Alternatives (INMD*):

For all $(S, d), (T, d) \in \sum$ if $IR(S, m(S, d)) = IR(T, m(T, d))$ then $f(S, d) = f(T, d)$.

According to Proposition 3 of Anbarci and Sun (2013) the three axioms MD, $INMD^*$, RIA-Box determine uniquely the Discrete Raiffa Solution DR on $\sum$, which they define following Raiffa (1953) as follows:

For any $(S, d) \in \sum$ consider the non-decreasing sequence $(m_t)_{t \in \mathbb{N}_0}$ with $m_t \in S$ for all $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, m_0 := m(S, d)$ and $m_t := m(S, m_{t-1})$ for all $t \in \mathbb{N}$. Then $DR(S, d) := lim_{t \to \infty} m_t$.

Definition:

$DR : \sum \longrightarrow \mathbb{R}^2 : (S, d) \mapsto DR(S, d)$ is the Discrete Raiffa Solution.

As Anbarci and Sun (2013) stress the hypothesis of INMD* implies:

$b(S,d) = b(T,d)$ and $m(S,d) = m(T,d)$. Therefore the solution $m : \sum \longrightarrow \mathbb{R}^2$ on $\sum$ satisfies INMD*. I consciously deviate by my notation INMD* from Anbarci and Sun (2013) who call this assumption INMD.

INMD* and, what I will introduce as INMD, are both derived by replacing the disagreement point d of (S,d) by the middle point $m(S,d)$ of (S,d) from two logically equivalent formulations of Independence of Non-Individually-Rational Outcomes. This axiom has been first introduced by Peters (1986) and is discussed in Peters and van Damme (1991). Anbarci and Sun (2013) follow Peters and van Damme (1991) using the following formulation:

$INIR : \forall (S,d) \in \sum : f(S,d) = f(IR(S,d),d)$

Anbarci and Sun (2013) base on this definition their Independence of Non-Midpoint-Dominating Alternatives. But their formulation mimics, with $m(S,d)$ instead of d not $INIR$ but the following equivalent version:

$INIR^* : \forall (S,d),(T,d) \in \sum : IR(S,d) = IR(T,d) \Longrightarrow f(S,d) = f(T,d)$

We can define now:

$INMD :\ \forall (S,d) \in \sum$ with $H^{S,d} \neq IR(S,d) : f(S,d) = f(IR(S,m(S,d)),m(S,d))$

$INMD^*$ is what Anbarci and Sun (2013) termed $INMD$. I decided to change the notation because of the relations to $INIR$ and $INIR^*$

The following observations are crucial:

1. $INIR^* \Longleftrightarrow INIR$

   Proof:

   “$\Longleftarrow$”: $Let (S,d),(T,d) \in \sum$ with $IR(S,d) = IR(T,d)$

   Clearly, we have: $f(IR(S,d),d) = f(IR(T,d),d)$

   By $INIR$ we get: $f(S,d) = f(T,d)$, which proves $INIR^*$

   “$\Longrightarrow$” (by contraposition): to prove: $\neg INIR \Longrightarrow \neg INIR^*$

   Assume: $\neg INIR$. Therefore, $\exists (S,d) \in \sum : f(S,d) \neq f(IR(S,d),d)$

   Now define $T := IR(S,d)$. Then $\exists (S,d),(T,d) \in \sum$ such that:

   $(IR(S,d),d) = (T,d) = (IR(T,d),d)$, but $f(S,d) \neq f(T,d)$, which proves $\neg INIR^*$

2. $INMD \Longrightarrow INMD^*$

   Proof:

   Let $(S,d),(T,d) \in \sum$ such that $IR(S,m(S,d)) = IR(T,m(T,d))$

   Then $m(T,d) = m(S,d)$ and $f(IR(S,m(S,d)),m(S,d)) = f(IR(T,m(T,d)),m(T,d))$. Hence by $INMD$ we get $f(S,d) = f(T,d)$.

3. $INMD^* \not\Longrightarrow INMD$

   This is immediate as the mid-point solution m obviously satisfies $INMD^*$ but violates $INMD$.

   It will turn out later that $INMD^*$ used by Anbarci and Sun in their proof is not strong enough even combined with RIA-Box and MD to yield the desired result while $INMD$ with MD but without RIA-Box will work.

4. DR satisfies $INMD$

   Proof: By definition of DR we have:

   $DR(IR(S,m(S,d)),m(S,d)) \equiv DR(IR(S,m_0),m_0) = \lim_{t\in\mathbb{N}_0}\ m_t = \lim_{t\in\mathbb{N}}\ m_t = DR(S,d)$.

Salonen (1988) was the first article to my best knowledge that provided in his Theorem 2 an axiomatization of the Discrete Raiffa Solution on the set $\sum$ of bargaining problems.

By his definition $(S,d) \in \tilde{\sum}$ if and only if it is compact, convex, comprehensive and individually rational (i.e. $IR(S,d) = S$). That allows in particular bargainig problems with $|S| = 1$, i.e. $(S,d) = (\{d\},d)$.

The three axioms Salonen is using are anonymity, covariance under affine transformations and an axiom, that he called Second Decomposability axiom.

In the context of $\sum$ rather than $\tilde{\sum}$ and the terminology of Anbarci and Sun this axiom can be restated as:

$SD:$ For all $(S,d),(T,d) \in \sum$ such that $S \subset T$ and $b(S,d) = b(T,d)$
there exists a bargaining problem $(A,d) \in \sum$ such that:
$b(A,d) = b(S,d)(= b(T,d))$ and $f(A,d) \in D(S,d,f) \cap D(T,d,f)$

.

In fact, $(H^S,d) = (H^T,d)$ can serve as such an $(A,d) \in \sum$, which simplifies SD.

A strictly weaker axiom than this simplified version of SD has been used by Trockel (2009) together with MD restricted to hyperplane games for an axiomatization of DR.

2 A Counterexample to Proposition 3

Let $n := 2$. The mapping $m : \sum \longrightarrow \mathbb{R}^2 : (S,d) \mapsto m(S,d) \in S$ is a solution on $\sum$. It provides the counterexample.

I have to verify that m satisfies each of the three axioms of Proposition 3 of Anbarci and Sun (2013), namely $MD, INMD^*$ and RIA-Box.

2.1 MD

The solution m satisfies the required weak inequality in MD as equality.

2.2 INMD*

It has already been remarked before and in fact, also by Anbarci and Sun, that m satisfies $INMD^*$.

2.3 RIA - Box

In order to establish RIA-Box for m we need to verify that:

$(D(S,d,m) \cup \{m(S,d)\}) \cap (D(T,d,m) \cup \{m(T,d)\}) \backslash \{d\} \neq \emptyset$

for all $(S,d),(T,d) \in \sum$ with $S \subset T$ and $b(S,d) = b(T,d)$.

By $b(S,d) = b(T,d)$ we get $m(S,d) = m(T,d) > d$, and are finished.

This proves the correctness of the counterexample.

3 The proof of Anbarci and Sun

I will follow the proof step by step.

First, it is true and easily established that DR satisfies the three axioms $MD, INMD^*$ and RIA-Box. Next these three axioms are assumed for a solution f with the goal to establish that $f = DR$. As we have shown also m satisfies these three axioms. So we continue the proof of Anbarci and Sun with a simplified but more restrictive version of RIA-Box.

RIA-Box can be equivalently formulated as follows:

$\forall (S,d),(T,d) \in \sum$ with $S \subset T, b(S,d) = b(T,d)$.

$A : (D(S,d,f) \cap D(T,d,f)) \setminus \{d\} \neq \emptyset$ or

$B : (D(S,d,f) \cap \{f(T,d)\}) \setminus \{d\} \neq \emptyset$ or

$C : (f\{(S,d)\}) \cap D(T,d,f)) \setminus \{d\} \neq \emptyset$ or

$D : \{f(S,d)\} \cap \{f(T,d)\} \setminus \{d\} \neq \emptyset$

We skip D in order to prevent m as a solution, and B as it is always violated for DR.

Then we get RIA^*-Box: $\forall (S,d),(T,d) \in \sum$ with $S \subset T, b(S,d) = b(T,d)$:

$(D(S,d,f) \cup \{f(S,d)\}) \cap D(T,d,f) \setminus \{d\} \neq \emptyset$.

It is immediate that DR still satisfies the stronger RIA^*-Box because $(H^{S,d},d) = (H^{T,d},d)$ and $d \neq DR(H^{S,d},d) = m(H^{S,d},d) \in D(S,d,f) \cap D(T,d,f)$.

On their way to establish $f = DR$ Anbarci and Sun first consider $(S,d) = (H^{S,d},d)$ and observe correctly that in this case $DR(S,d) = m(S,d) = f(S,d)$.

Next they consider $(S,d) \neq (H^{S,d},d)$ and claim, that in this case $f(S,d) = f(S,m(S,d))$ suffices to establish $f(S,d) = DR(S,d)$. That statement is correct.

In order to prove that they consider $T := H^{S,d} \subset S$ and conclude, again correctly, that $f(T,d)(= m(T,d)) = m(S,d)$ and that $D(T,d,f) = [d, m(S,d)) := co\{d, m(S,d)\} \setminus \{m(S,d)\}$.

Their next statement is again correct: "By RIA-Box, there exists a common intermediate agreement $a \in [d, m(S,d)) \ \cup \ \{m(S,d)\}$ such that $f(S,d) = f(S,a)$".

Indeed, one has to use RIA^*-Box, which applied to (S,d) and $(T,d) = (S^{H,d},d)$ states

$(D(T,d,f) \cup \{f(T,d)\}) \ \cap \ D(S,d,f) \setminus \{d\} \neq \emptyset$

$\iff \ (D(T,d,f) \setminus \{d\} \ \cup \ \{f(T,d)\}) \cap D(S,d,f) \setminus \{d\} \neq \emptyset$

$\iff \ ((d, m(S,d) \cup \{m(S,d)\}) \ \cap \ D(S,d,f) \setminus \{d\} \neq \emptyset$

Therefore there exists an $a \in D(S,d,f)$, i.e. satisfying $f(S,a) = f(S,d)$, such that $a = m(S,d)$ or $a \in (d, m(S,d))$.

The following last two sentences of the proof of Anbarci and Sun are, however, not conclusive! They do **not** prove $INMD^*$ to "exclude all points in $(d, m(S,d))$ from being a common intermediate agreement!" Hence one **cannot** conclude that $a = m(S,d)$!

This is illustrated by applying the solution DR that does satisfy $INMD^*$, to the games $(S,d),(T,d)$ with $S := [0,1]^2, T = S^H$ and $d = 0$. Here obviously the line segment $(0, m(T,0)) = (0, m(S,0))$ satisfies $(0, m(T,0)) \subset D(T,0,DR) \cap D(S,0,DR)$

So **no** point a "below" $m(S,0)$ is excluded from being a common intermediate agreement.

In fact, with RIA-Box instead of RIA^*-Box we could have possibly $f(S,d) = m(S,d) = m(S,a)$ for $a \in (d, m(S,d))$, and f could be the mid-point solution m. The same is true for any solution defined by $f^\alpha := \alpha DR + (1-\alpha)m$ for $\alpha \in [0,1]$.

f^α obviously satisfies MD. As DR and m both satisfy $INMD^*$ and RIA-Box so does f^α for any $\alpha \in (0,1)$.

But even RIA^*-Box is not strong enough together with MD and $INMD^*$ to uniquely determine the discrete Raiffa solution DR. In fact, a comparison with Salonen's axiom SD that is effective in characterizing together with symmetry and covariance the solution DR shows that SD is a considerable strengthening of RIA*-Box.

This analysis has shown where the proof of Anbarci and Sun goes wrong. Even though by modifying RIA-Box to RIA^*-Box we excluded m as a solution, that does not mean that uniqueness of DR as satisfying $MD, INMD^*$ and RIA^*-Box has been proven.

4 Axiomatization of DR

To simplify the following considerations I introduce some new notation.

For any $(S,z) \in \sum$ define $(S_z, z) := (IR(S,z), z)$. Also notice that

for any arbitrary given $(S,d) \in \sum$ we have $m(S_{m_0}, m_0) = m(S, m_0)$, hence

$m_0 = m(S,d), m_1 = m(S_{m_0}, m_0)$ and $\forall k \in \mathbb{N}\ m_k = m(S_{mk-1}, m_{k-1})$.

Define $\hat{\sum} := \sum \ \cup \ \{(\{d\}, d) \mid d \in \mathbb{R}^2\}$

Proposition: A solution f on $\hat{\sum}$ satisfies $INMD$ if and only if $f = DR$.

Proof:

$INMD \Longleftrightarrow \forall (S,d) \in \sum : f(S,d) = f(S_{m_0}, m_0)$.

$\Longrightarrow \forall (S,d) \in \sum \ \forall\ k \in \mathbb{N} : f(S,d) = f(S_{m_k}, m_k)$.

By $INMD$ now $\forall\ k \in \mathbb{N} : f(S, m_k) = f(S_{m_{k+1}}, m_{k+1}) = f(S,d)$.

Therefore $\lim_{k \longrightarrow \infty}\ f(S, m_k) = f(S,d)$.

As the sequence $(S_{m_k})_{k\in\mathbb{N}}$ converges to $\{f(S,d)\}$ in the Hausdorff distance, we get convergence of the sequence $(\| m_{k+1} - f(S,d) \|_2)_{k\in\mathbb{N}}$ to zero.

As by definition $\lim_{k\in\mathbb{N}_0}\ m_{k+1} = DR(S,d)$ the triangle inequality yields $f(S,d) = DR(S,d)$. This proves that f equals DR if it satisfies $INMD$. That DR satisfies $INMD$ is obvious (hence the assumption "f satisfies $INMD$" not void!) as

$DR(S,d) = \lim_{k\in\mathbb{N}_0}\ m_k = \lim_{k\in\mathbb{N}}\ m_k = DR(S_{m_0}, m_0)$.

$\square$

The omission of MD in the Proposition is enabled by the extension of $\sum$ via admitting singleton bargaining games. That implies that $INMD$ is well-defined even for hyperplane games $(S,d) = (H^{S,d}, d)$. For them we have $(H^{(S,d)}_{m_0}, m_0) = (\{m_0\}, m_0)$ with $m_0 = DR(\{m_0\}, m_0) = DR(H^{S,d}, d)$.

Denote by $\sum^H$ the set of hyperplane games in $\sum$. For $\sum$ instead of $\hat{\sum}$ we get the following modified version of Proposition 3 in Anbarci and Sun (2013).

Corollary: A solution f on $\sum$ equals DR if and only if it satisfies

a) MD restricted to $\sum^H$

b) $INMD$ restricted to $\sum \setminus \sum^H$.

On the larger class $\mathcal{B}$ of compact, convex two person bargaining games that contains $\sum, \tilde{\sum}, \hat{\sum}$ as proper subsets axiom $INMD$ implies MD and imposed on a solution $f : \mathcal{B} \longrightarrow \mathbb{R}^2$ it is equivalent to $f = DR$ (cf. Trockel (2009), Corollary 2).

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