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# Weak approximation of $G$-expectation with discrete state space*

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**Abstract**

We refine the discretization of $G$-expectation by Y. Dolinsky, M. Nutz, and M. Soner (*Stochastic Processes and their Applications, 122 (2012), 664–675*), in order to obtain a discretization of the sublinear expectation where the martingale laws are defined on a finite lattice rather than the whole set of reals.

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## 1 Introduction

Dolinsky et al. [4] showed a Donsker-type result for $G$-Brownian motion, henceforth referred to as *G-Donsker,* by introducing a notion of volatility uncertainty in discrete time and defined a discrete version of *Peng's* $G$-expectation. In the continuous-time limit, the resulting sublinear expectation converges weakly to $G$-expectation. In their discretization, Dolinsky et al. [4] allow for martingale laws whose support is the whole set of reals. In other words, they only discretize the time line, but not the state space of the canonical process. Now for certain applications, for example a hyperfinite construction of $G$-expectation in the sense of Robinsonian nonstandard analysis, a discretization of the state space would be necessary. Thus, we

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develop a modification of the construction by Dolinsky et al. [4] which even ensures that the sublinear expectation operator for the discrete-time canonical process corresponding to this discretization of the state space (whence the martingale laws are supported by a finite lattice only) converges to the $G$-expectation. The proof is based on technique from (linear) probability theory. Ruan [9] constructed the $G$-Brownian motion via the weak limit of a sequence of *$G$-random walks* which can be seen as the invariance principle of $G$-Brownian motion. The proof relies heavily on the theory of sublinear expectation.

This paper is organised as follows: in Section 2, we introduce the $G$-expectation, and the discrete-time and continuous-time version of the sublinear expectation in the spirit of Dolinsky et al. [4]. Unlike in [4], we require the discretization of the martingale laws to be defined on a finite lattice rather than the whole set of reals. We also introduce the strong formulation of volatility uncertainty. In Section 3, we show that a natural push forward of our discretize sublinear expectation converges weakly to $G$-expectation as $n \to \infty$ provided the domain of volatility uncertainty $\mathbf{D}$ is scaled by $1/n$. Finally, we prove that

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] = \lim_{n \to \infty} \max_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)].$$

# 2 Framework

## 2.1 $G$-expectation via volatility uncertainty

Peng [8] introduced a sublinear expectation on a well-defined space $\mathbb{L}^1_G$, the completion of $\mathrm{Lip}_{b.cyl}(\Omega)$ (bounded and Lipschitz cylinder function) under the norm $\|\cdot\|_{\mathbb{L}^1_G}$, under which the increments of the canonical process $(B_t)_{t>0}$ are zero-mean, independent and stationary and can be proved to be $(G)$-normally distributed. This type of process is called *$G$-Brownian motion* and the corresponding sublinear expectation is called *$G$-expectation.* We fix a constant $T > 0$ and replace the $d$-dimensional setting by Dolinsky et al. [4] with $d = 1$. We also fix a nonempty, compact and convex set $\mathbf{D} \subseteq \mathbb{R}_+$ such that the volatility processes take values in $\mathbf{D}$.

The $G$-expectation $\xi \mapsto \mathcal{E}^G(\xi)$ is a sublinear operator defined on a class of random variables on $\Omega$. The symbol $G$ refers to a given function

$$G(\gamma) := \frac{1}{2} \sup_{c \in \mathbf{D}} c\gamma : \mathbb{R} \to \mathbb{R} \tag{1}$$

where $\mathbf{D} = [r_{\mathbf{D}}, R_{\mathbf{D}}]$ and $0 \leq r_{\mathbf{D}} \leq R_{\mathbf{D}} < \infty$ are fixed numbers. The construction of the $G$-expectation is as follows. Let $\xi = f(B_T)$, where $B_T$ is the $G$-Brownian motion and $f$ a sufficiently regular function. Then $\mathcal{E}^G(\xi)$ is defined to be the initial value $u(0, 0)$ of the solution of the nonlinear backward heat equation,

$$\partial_t u - G(\partial^2_{xx} u) = 0,$$

with terminal condition $u(\cdot, T) = f$, Pardoux and Peng [7]. The mapping $\mathcal{E}^G$ can be extended to random variables of the form $\xi = f(B_{t_1}, \cdots, B_{t_n})$ by a stepwise evaluation of the PDE and then to the completion $\mathbb{L}^1_G$ of the space of all such random variables. Denis et al. [3] showed that $\mathbb{L}^1_G$ is the completion of $\mathcal{C}_b(\Omega)$ and $\text{Lip}_{b.cyl}(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^1_G}$, and that $\mathbb{L}^1_G$ is the space of the so-called quasi-continuous function and contains all bounded continuous functions on the canonical space $\Omega$, but not all bounded measurable functions are included. Theorem 6 (our main result in this paper) cannot be extended to the case where $\xi$ is defined on $\mathbb{L}^1_G$ under the norm $\|\cdot\|_{\mathbb{L}^1_G}$ (see below), thus, we work in a smaller space $\mathbb{L}^1_*$ defined as the completion of $\mathcal{C}_b(\Omega;\mathbb{R})$ under the norm $\|\cdot\|_*$. Our setting is based on a set of martingale laws not a single probability measure. However, when $r_{\mathbf{D}} = R_{\mathbf{D}} = 1$, the canonical process under $\mathcal{E}^G(\xi)$, $G$-Brownian motion, becomes the (standard) Brownian motion since $\mathcal{E}^G(\xi)$ will be a (linear) expectation under the Wiener measure.

There also exists an alternative representation of the $G$-expectation known as the dual view on $G$-expectation via volatility uncertainty, see Denis et al. [3]: One can show that the $G$-expectation can be expressed as the upper expectation

$$\mathcal{E}^G(\xi) = \sup_{P \in \mathcal{P}^G} \mathbb{E}^P[\xi], \quad \xi = f(B_T), \tag{2}$$

where $\mathcal{P}^G$ is defined as the set of probability measures on $\Omega$ such that, for any $P \in \mathcal{P}^G$, $B$ is a martingale with the volatility $d\langle B\rangle_t/dt \in \mathbf{D} \quad P \otimes dt$ a.e, and $\mathbf{D} = [r_{\mathbf{D}}, R_{\mathbf{D}}]$, for $0 \le r_{\mathbf{D}} \le R_{\mathbf{D}} < \infty$.

**Remark 1.** (2) *can be seen as the cheapest super-hedging price of a European contingent claim where $\xi$ can be regarded as the discounted payoff.*

## 2.2 Continuous-time construction of sublinear expectation

Let $\Omega = \{\omega \in \mathcal{C}([0,T];\mathbb{R}) : \omega_0 = 0\}$ be the canonical space of continuous paths with time horizon $T \in (0, \infty)$, endowed with uniform norm $\|\omega\|_\infty = \sup_{0 \le t \le T} |\omega_t|$, where the Euclidean norm on $\mathbb{R}$ is given by $|\cdot|$. Let $B$ be the canonical process $B_t(\omega) = \omega_t$, and $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$ is the filtration generated by $B$. A probability measure $P$ on $\Omega$ is called a martingale law provided $B$ is a $P$-martingale and $B_0 = 0$ $P$ a.s. Then, $\mathcal{P}_{\mathbf{D}}$ is the set of martingale laws on $\Omega$ and the volatility takes values in $\mathbf{D}$, $P \otimes dt$ a.e;

$$\mathcal{P}_{\mathbf{D}} = \{P \text{ martingale law on } \Omega\text{: } d\langle B\rangle_t/dt \in \mathbf{D},\ P \otimes dt \text{ a.e.}\}.$$

Thus, the sublinear expectation is given by

$$\mathcal{E}_{\mathbf{D}}(\xi) = \sup_{P \in \mathcal{P}_{\mathbf{D}}} \mathbb{E}^P[\xi], \tag{3}$$

such that, for any $\xi : \Omega \to \mathbb{R}$, $\xi$ is $\mathcal{F}_T$-measurable and integrable for all $P \in \mathcal{P}_{\mathbf{D}}$. $\mathbb{E}^P$ denotes the expectation under $P$. It is important to note that the continuous-time sublinear expectation (3) can be considered as the $G$-expectation (for every $\xi \in \mathbb{L}^1_G$ where $\mathbb{L}^1_G$ is defined as the $\mathbb{E}[|\cdot|]$−norm completion of $\mathcal{C}_b(\Omega; \mathbb{R})$) provided (1) is satisfied (cf. Dolinsky et al. [4]).

### 2.3 Discrete-time construction of sublinear expectation

Here we introduce the setting of the discrete-time sublinear expectation. We denote

$$\mathcal{L}_n = \left\{ \frac{j}{n\sqrt{n}}, \quad -n^2\sqrt{R_{\mathbf{D}}} \le j \le n^2\sqrt{R_{\mathbf{D}}}, \quad \text{for } j \in \mathbb{Z} \right\},$$

and $\mathcal{L}_n^{n+1} = \mathcal{L}_n \times \cdots \times \mathcal{L}_n (n+1 \text{ times})$, for $n \in \mathbb{N}$. Let $X^n = (X^n_k)^n_{k=0}$ be the canonical process $X^n_k(x) = x_k$ defined on $\mathcal{L}_n^{n+1}$ and $(\mathcal{F}^n_k)^n_{k=0} = \sigma(X^n_l, l = 0, \ldots, k)$ be the filtration generated by $X^n$. Let

$$\mathbf{D}'_n = \mathbf{D} \cap \left(\frac{1}{n}\mathbb{N}\right)^2$$

be a nonempty bounded set of volatilities. Recall $\mathbf{D} = [r_{\mathbf{D}}, R_{\mathbf{D}}]$, for $0 \le r_{\mathbf{D}} \le R_{\mathbf{D}} < \infty$. We note that $R_{\mathbf{D}} = \sup_{\alpha \in \mathbf{D}} |\alpha|$, where $|\cdot|$ denotes the absolute value. A probability measure $P$ on $\mathcal{L}_n^{n+1}$ is called a martingale law provided $X^n$ is a $P$-martingale and $X^n_0 = 0$ $P$ a.s. The increment $\Delta X^n$ denotes the difference by $\Delta X^n_k = X^n_k - X^n_{k-1}$. Let $\mathcal{P}^n_{\mathbf{D}}$ be the set of martingale laws of $X^n$ on $\mathbb{R}^{n+1}$, i.e.,

$$\mathcal{P}^n_{\mathbf{D}} = \left\{ P \text{ martingale law on } \mathbb{R}^{n+1}\text{: } r_{\mathbf{D}} \le |\Delta X^n_k|^2 \le R_{\mathbf{D}},\ P \text{ a.s.} \right\},$$

such that for all $n$, $\mathcal{L}_n^{n+1} \subseteq \mathbb{R}^{n+1}$.

In order to establish a relation between the continuous-time and discrete-time settings, we obtained a continuous-time process $\widehat{x}_t \in \Omega$ from any discrete path $x \in \mathcal{L}_n^{n+1}$ by linear interpolation. i.e.,

$$\widehat{x}_t := (\lfloor nt/T \rfloor + 1 - nt/T)x_{\lfloor nt/T \rfloor} + (nt/T - \lfloor nt/T \rfloor)x_{\lfloor nt/T \rfloor + 1}$$

where $\widehat{\cdot}\colon \mathcal{L}_n^{n+1} \to \Omega$ is the linear interpolation operator, $x = (x_0, \ldots, x_n) \mapsto \widehat{x} = \{(\widehat{x})_{0 \le t \le T}\}$, and $\lfloor y \rfloor$ denotes the greatest integer less than or equal to $y$. If $X^n$ is the canonical process on $\mathcal{L}_n^{n+1}$ and $\xi$ is a random variable on $\Omega$, then $\xi(\widehat{X}^n)$ defines a random variable on $\mathcal{L}_n^{n+1}$.

### 2.4 Strong formulation of volatility uncertainty

We introduce the so-called strong formulation of volatility uncertainty for the continuous-time construction, as in Dolinsky et al. [4], Nutz [6], Soner et al. [10, 11], and for the discrete-time construction, as in Dolinsky et al. [4]; i.e., we consider martingale laws generated by stochastic integrals with respect to a fixed Brownian motion and a fixed random walk.
For the continuous-time construction; let $\mathcal{Q}_{\mathbf{D}}$ be the set of martingale laws of the form:

$$\mathcal{Q}_{\mathbf{D}} = \left\{ P_0 \circ (M)^{-1};\ M = \int f(t, B) dB_t, \text{ and } f \in \mathcal{C}([0, T] \times \Omega; \sqrt{\mathbf{D}}) \text{ is adapted} \right\}.$$

$B$ is the canonical process under the Wiener measure $P_0$, and $\mathbf{D}$ is a convex set.

**Remark 2.** *The elements of $\mathcal{Q}_{\mathbf{D}}$, in particular $M$, with nondegenerate $f$ which satisfies the predictable representation condition, correspond to the analogy of market completeness in finance (martingale representation theorem).*

For the discrete-time construction; we fix $n \in \mathbb{N}$, $\Omega_n = \{\omega = (\omega_1, \ldots, \omega_n) : \omega_i \in \{\pm 1\}, \quad i = 1, \ldots, n\}$ equipped with the power set and let

$$P_n = \underbrace{\frac{\delta_{-1} + \delta_{+1}}{2} \otimes \cdots \otimes \frac{\delta_{-1} + \delta_{+1}}{2}}_{\text{n times}}$$

where for all $A \subseteq \mathbb{R}$,

$$\delta_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

be the product probability associated with the uniform distribution. Let $\xi_1, \ldots, \xi_n$ be an i.i.d sequence of $\{\pm 1\}$ -valued random variables. The components of $\xi_k$ are orthonormal in $L^2(P_n)$. We denote the associated random walk by

$$Z_k^n = \sum_{l=1}^{k} \xi_l,$$

then, we can view

$$\sum_{l=1}^{k} f(l-1, \mathbb{X}) \Delta \mathbb{X}_l$$

as the discrete-time stochastic integrals of $\mathbb{X}$, where $f$ is $\mathcal{F}^n$-adapted and

$$\mathbb{X} = \frac{1}{\sqrt{n}} Z^n$$

is the scaled random walk. We denote by $\mathcal{Q}^n_{\mathbf{D}'_n}$ the set of martingale laws of the form:

$$\mathcal{Q}^n_{\mathbf{D}'_n} = \Big\{ P_n \circ (M^{f,\mathbb{X}})^{-1};\ f : \{0,\dots,n\} \times \mathcal{L}_n^{n+1} \to \sqrt{\mathbf{D}'_n} \text{ is } \mathcal{F}^n\text{-adapted.} \Big\} \tag{4}$$

where

$$M^{f,\mathbb{X}} = \left( \sum_{l=1}^{k} f(l-1, \mathbb{X}) \Delta \mathbb{X}_l \right)_{k=0}^{n}.$$

## 3 Results and proofs

Proposition 4 states that a sublinear expectation with discrete-time volatility uncertainty on our finite lattice converges to the $G$-expectation.

**Lemma 3.** *Let*

$$\mathcal{Q}^n_{\mathbf{D}} = \left\{ P_n \circ \left( M^{f,\mathbb{X}} \right)^{-1};\ f : \{0,\dots,n\} \times \mathbb{R}^{n+1} \to \sqrt{\mathbf{D}} \textit{ is adapted.} \right\}$$

*where*

$$M^{f,\mathbb{X}} = \left( \sum_{l=1}^{k} f(l-1, \mathbb{X}) \Delta \mathbb{X}_l \right)_{k=0}^{n}.$$

*Then* $\mathcal{Q}^n_{\mathbf{D}} \subseteq \mathcal{P}^n_{\mathbf{D}}$.

*Proof.* From the above equation, we can say that $\Delta M^f_k = f(k, \mathbb{X})\xi_k$. And by the orthonormality property of $\xi_k$, we have

$$\mathbb{E}^{P_n}[f(k,\mathbb{X})^2 \xi_k^2 | \mathcal{F}^n_k] = \mathbb{E}^{P_n}[f(k,\mathbb{X})^2 | \mathcal{F}^n_k] \le \mathbb{E}^{P_n}[(\sqrt{R_{\mathbf{D}}})^2 | \mathcal{F}^n_k] = R_{\mathbf{D}} \quad P_n \text{ a.s.},$$

as $|\xi_k| = 1,\ f(\cdots)^2 \in \mathbf{D}$ implies

$$|(\Delta M^f_k)^2| = |f(k,\mathbb{X})|^2 \in [r_{\mathbf{D}}, R_{\mathbf{D}}] \quad P_n \text{ a.s.}$$

□

**Proposition 4.** *Let* $\xi : \Omega \to \mathbb{R}$ *be a continuous function satisfying* $|\xi(\omega)| \le a(1+ \parallel \omega \parallel_\infty)^b$ *for some constants* $a, b > 0$. *Then,*

$$\lim_{n\to\infty} \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] = \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]. \tag{5}$$

*Proof.* To prove (5), we prove two separate inequalities together with a density argument which imply (5). Before then, we introduce a smaller space $\mathbb{L}^1_*$ that is defined as the completion of $\mathcal{C}_b(\Omega;\mathbb{R})$ under the norm

$$\parallel \xi \parallel_* := \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q |\xi|, \quad \mathcal{Q} := \mathcal{P}_{\mathbf{D}} \cup \{ P \circ (\widehat{X}^n)^{-1}; P \in \mathcal{P}^n_{\mathbf{D}/n}, n \in \mathbb{N}. \}.$$

This is because Proposition 4 will not hold if $\xi$ just belong to $\mathbb{L}^1_G$, where $\mathbb{L}^1_G$ is the completion of $\mathcal{C}_b(\Omega;\mathbb{R})$ under the norm

$$\| \xi \|_G := \sup_{P\in\mathcal{P}_{\mathbf{D}}} \mathbb{E}^P[|\xi|]. \tag{6}$$

In fact, a random variable which is defined on a set of paths of finite variation will have zero expectation under any martingale law $P \in \mathcal{P}_{\mathbf{D}}$ because the support of the martingale laws is disjoint to a set of paths of finite variation whereas it will have non zero expectation under an element of $\mathcal{Q}$.
Dolinsky et al. [4, Lemma 3.4] show that if $\xi : \Omega \to \mathbb{R}$ satisfies the condition of Proposition 4, then $\xi \in \mathbb{L}^1_*$.

**First inequality** (for $\leq$ in (5))**:**

$$\limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \leq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]. \tag{7}$$

For all $n$, trivially $\sqrt{\mathbf{D}'_n/n} \subseteq \sqrt{\mathbf{D}/n}$ and $\mathcal{L}^{n+1}_n \subseteq \mathbb{R}^{n+1}$. Thus, $\mathcal{Q}^n_{\mathbf{D}'_n/n} \subseteq \mathcal{Q}^n_{\mathbf{D}/n}$. Therefore,

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \leq \sup_{P\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^P[\xi(\widehat{X}^n)],$$

and for all $n$, we have

$$\limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \leq \limsup_{n\to\infty} \sup_{P\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^P[\xi(\widehat{X}^n)]. \tag{8}$$

In Dolinsky et al. [4], it was shown that

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}^n_{\mathbf{D}/n}} \mathbb{E}^P[\xi(\widehat{X}^n)] \leq \sup_{P\in\mathcal{P}_{\mathbf{D}}} \mathbb{E}^P[\xi].$$

Since Lemma 3 shows that $\mathcal{Q}^n_{\mathbf{D}} \subseteq \mathcal{P}^n_{\mathbf{D}}$, and the convex hull of $\mathcal{Q}_{\mathbf{D}}$ is a weakly dense subset of $\mathcal{P}_{\mathbf{D}}$, see Dolinsky et al. [4, Proposition 3.5], then,

$$\limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \leq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi].$$

Hence, (7) follows.

**Second inequality** (for $\geq$ in (5))**:**

It remains to show that

$$\liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \geq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi].$$

For arbitrary $P \in \mathcal{Q}_{\mathbf{D}}$, we construct a sequence $(P^n)_n$ such that for all $n$,

$$P^n \in \mathcal{Q}^n_{\mathbf{D}'_n/n}, \tag{9}$$

and

$$\mathbb{E}^P[\xi] \leq \liminf_{n\to\infty} \ \mathbb{E}^{P^n}[\xi(\widehat{X}^n)]. \tag{10}$$

Fix $n$ and let $\xi_1, \ldots, \xi_n$ be some i.i.d sequence of random variables on $\Omega_n$ as defined in Section 2, i.e., $\xi_i : \Omega_n \to \{\pm 1\}$, for $i = 1, \ldots, n$. Now, we want to construct martingales $M^n$ whose laws are in $\mathcal{Q}^n_{\mathbf{D}'_n/n}$ and the laws of their interpolations tend to $P$. To achieve the above task, we introduce a scaled random walk with the piecewise constant càdlàg property (right continuity with left limits),

$$W^n_t := \frac{1}{\sqrt{n}} \sum_{l=1}^{\lfloor nt/T \rfloor} \xi_l = \frac{1}{\sqrt{n}} Z^n_{\lfloor nt/T \rfloor}, \quad 0 \leq t \leq T, \tag{11}$$

and we denote the continuous version of (11) obtained by linear interpolation by

$$\widehat{W}^n_t := \frac{1}{\sqrt{n}} \widehat{Z}^n_{\lfloor nt/T \rfloor}, \quad 0 \leq t \leq T. \tag{12}$$

By the central limit theorem;

$$(W^n, \widehat{W}^n) \Rightarrow (W, W)$$

as $n \to \infty$ on $D([0,T]; \mathbb{R}^2)$ ($\Rightarrow$ implies convergence in distribution). i.e., the law $(P_n)$ converges to the law $P_0$ on the Skorohod space $D([0,T]; \mathbb{R}^2)$ Billingsley [1, Theorem 27.1]. Let $g \in \mathcal{C}([0,T] \times \Omega, \sqrt{\mathbf{D}})$, such that

$$P = P_0 \circ \left( \underbrace{\int g(t, W) dW_t}_{M} \right)^{-1}.$$

Since $g$ is continuous and $\widehat{W}^n_t$ is the interpolated version of (11), it turns out that

$$\left( W^n, \left( g\left( \lfloor nt/T \rfloor T/n, \widehat{W}^n_t \right) \right)_{t\in[0,T]} \right) \Rightarrow \left( W, (g(t, W_t))_{t\in[0,T]} \right)$$

as $n \to \infty$ on $D([0,T]; \mathbb{R}^2)$. We introduce martingales with discrete-time integrals,

$$M^n_k := \sum_{l=1}^{k} g\left( (l-1)T/n, \widehat{W}^n \right) \widehat{W}^n_{lT/n} - \widehat{W}^n_{(l-1)T/n}. \tag{13}$$

In order to construct a discretize martingale $M^n$ which is "close" to $M$ and also is such that $P_n \circ (M^n)^{-1} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}$. We shall choose some

$$g_n : \{0, \dots, n\} \times \mathcal{L}_n^{n+1} \to \sqrt{\mathbf{D}'_n/n},$$

such that,

$$M_k^n = \sum_{l=1}^{k} g_n \left( l-1, \frac{1}{\sqrt{n}} Z^n \right) \frac{1}{\sqrt{n}} \Delta Z_l^n.$$

Let $d_{J_1}$ be the Kolmogorov metric for the Skorohod $J_1$ topology. We choose $\widetilde{h}_n : \{0, \cdots, n\} \times \Omega \to \sqrt{\mathbf{D}'_n/n}$ such that

$$d_{J_1} \left( \left( \widetilde{h}_n(\lfloor nt/T \rfloor T/n, \widehat{W}_t^n) \right)_{t\in[0,T]}, \left( g(\lfloor nt/T \rfloor T/n, \widehat{W}_t^n) \right)_{t\in[0,T]} \right)$$

is minimal (this is possible because there are only finitely many choices for $\left( \widetilde{h}_n(\lfloor nt/T \rfloor T/n, \widehat{W}_t^n) \right)_{t\in[0,T]}$). This implies, due to the construction of $\mathbf{D}'_n$ as a discretization of $\mathbf{D}$ that

$$d_{J_1} \left( \left( \widetilde{h}_n(\lfloor nt/T \rfloor T/n, \widehat{W}_t^n) \right)_{t\in[0,T]}, \left( g(\lfloor nt/T \rfloor T/n, \widehat{W}_t^n) \right)_{t\in[0,T]} \right) \to 0$$

as $n \to \infty$ on $D([0,T];\mathbb{R})$. From Billingsley [2, Theorem 3.1 and Theorem 14.1], it follows that

$$\left( W^n, \left( \widetilde{h}_n \left( \lfloor nt/T \rfloor T/n, \widehat{W}_t^n \right) \right)_{t\in[0,T]} \right) \Rightarrow \left( W, g(t, W_t)_{t\in[0,T]} \right)$$

as $n \to \infty$ on $D([0,T];\mathbb{R}^2)$. We then define $g_n : \{0, \dots, n\} \times \mathcal{L}_n^{n+1} \to \sqrt{\mathbf{D}'_n/n}$ by

$$g_n : (\ell, \vec{\mathbb{X}}) \mapsto \widetilde{h}_n(\ell, \widehat{\vec{X}}).$$

Let $M^n$ be defined by

$$M_k^n = \sum_{l=1}^{k} g_n \left( l-1, \frac{1}{\sqrt{n}} Z^n \right) \frac{1}{\sqrt{n}} \Delta Z_l^n, \quad \forall k \in \{0, \cdots, n\}.$$

By stability of stochastic integral (see Duffie and Protter [5, Theorem 4.3 and Definition 4.1]),

$$\left( M^n_{\lfloor nt/T \rfloor} \right)_{t\in[0,T]} \Rightarrow M \quad \text{as } n \to \infty \text{ on } D([0,T];\mathbb{R})$$

because

$$M^n_{\lfloor nt/T \rfloor} = \sum_{l=1}^{\lfloor nt/T \rfloor} \widetilde{h}_n \left( (l-1)T/n, \left( \widehat{W}_{kT/n} \right)_{k=0}^{n} \right) \Delta \widehat{W}_{lT/n}.$$

By Dolinsky et al. [4], the continuous version of (13) obtained by linear interpolation $\widehat{M}^n$ converges in distribution to $M$ on $\Omega$ endowed with the uniform metric on the Skorohod space, i.e., $\widehat{M}^n \Rightarrow M$ on $\Omega$. Since $\xi$ is bounded and continuous,

$$\lim_{n\to\infty} \mathbb{E}^{P_n\circ(M^n)^{-1}}[\xi(\widehat{X}^n)] = \mathbb{E}^{P_0\circ M^{-1}}[\xi]. \tag{14}$$

Therefore, (9) is satisfied for $P^n = P_n \circ (M^n)^{-1} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}$. Trivially, (9) implies

$$\mathbb{E}^{P^n}[\xi(\widehat{X}^n)] \leq \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{15}$$

Combining (14) and (15), and taking the lim inf as $n$ tends to $\infty$, gives

$$\mathbb{E}^{P}[\xi] \leq \liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{16}$$

Taking the supremum of (16) over $P \in \mathcal{Q}_{\mathbf{D}}$, the equation becomes

$$\sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^{P}[\xi] \leq \liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{17}$$

Combining (7) and (17),

$$\begin{aligned}
\sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^{P}[\xi] &\geq \limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \\
&\geq \liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \\
&\geq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^{P}[\xi].
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] = \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^{P}[\xi]. \tag{18}$$

Density argument: Hence (5) is established for all $\xi \in \mathcal{C}_b(\Omega, \mathbb{R})$. Since $\mathcal{Q}^n_{\mathbf{D}'_n} \subseteq \mathcal{Q}$ and $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{Q}$, this implies that $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{P}_{\mathbf{D}}$ Dolinsky et al. [4, Propostion 3.5] that is (5) holds for all $\xi \in \mathbb{L}^1_*$, and hence Dolinsky et al. [4, Lemma 3.4] holds for all $\xi$ that satisfy condition of Proposition 4. □

**Proposition 5.** *Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq a(1 + \|\omega\|_\infty)^b$ for some constants $a, b > 0$ and $\mathcal{Q}^n_{\mathbf{D}'_n}$ be the set of probability measures as defined in (4), then*

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] = \max_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{19}$$

*Proof.* The LHS of (19) can be written as

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] = \sup_{f\in\mathcal{A}} \mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)],$$

where $\mathcal{A} = \left\{f : \{0,\dots,n\}\times\mathcal{L}_n^{n+1} \to \sqrt{\mathbf{D}'_n}\right\}$ such that $f$ is $\mathcal{F}^n$-adapted. We shall prove that $\mathcal{A}$ is a compact subset of a finite-dimensional vector space, and that $f \mapsto \mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)]$ is continuous.

**First part**

Recall that for fixed $n\in\mathbb{N}$, $X^n = (X^n_k)^n_{k=0}$ is the canonical process defined by $X^n_k(x) = x_k$ for $x = (x_0,\dots,x_n)\in\mathcal{L}_n^{n+1}$, and $(\mathcal{F}^n_k)^n_{k=0} = \sigma(X^n_l, l = 0,\dots,k)$ is the filtration generated by $X^n$. We consider $\Omega_n = \{\omega = (\omega_1,\dots,\omega_n) : \omega_i\in\{\pm1\},\ i = 1,\dots,n\}$ equipped with the power set. Let

$$P_n = \underbrace{\frac{\delta_{-1}+\delta_{+1}}{2}\otimes\cdots\otimes\frac{\delta_{-1}+\delta_{+1}}{2}}_{\text{n times}}$$

where for all $A\subseteq\mathbb{R}$,

$$\delta_x(A) = \begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases},$$

be the product probability associated with the uniform distribution. $\xi_1,\dots,\xi_n$ is the i.i.d sequence of real-valued random variables such that $\xi_k$ belongs to $\{\pm1\}$ and the components of $\xi_k$ are orthonormal in $L^2(P_n)$. We denote the associated random walk by $Z^n_k = \sum_{l=1}^k \xi_l$.
$\mathcal{A}$ is closed[1] and obviously bounded with respect to the norm $\|\cdot\|_\infty$ as $\mathbf{D}'_n$ is bounded[2]. By Heine-Borel theorem, $\mathcal{A}$ is a compact subset of a $N(n,n)$-dimensional vector space equipped with the norm $\|\cdot\|_\infty$.

**Second part**

Here, we want to show that $F : f\mapsto \mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)]$ is continuous.

$$\mathcal{Q}^n_{\mathbf{D}'_n} = \left\{P_n\circ(M^{f,\mathbb{X}})^{-1};\ f:\{0,\dots,n\}\times\mathcal{L}_n^{n+1}\to\sqrt{\mathbf{D}'_n} \text{ is } \mathcal{F}^n\text{-adapted.}\right\}$$

[1]The cardinality of $\mathcal{L}_n$, $\#\mathcal{L}_n = 2n+1$, $\#\mathcal{L}_n^{n+1} = (2n+1)^{n+1}$, and $\#(\{0,\dots,n\}\times\mathcal{L}_n^{n+1}) = (n+1)(2n+1)^{n+1} = N(n,n)$. Let $(f^m)_m\in\mathcal{A}^{N(n,n)}$ and $f:\{0,\dots,n\}\times\mathcal{L}_n^{n+1}\to\mathbb{R}$, such that $f^m\to f$, as $m\to\infty$, with respect to the maximum norm $\|\cdot\|_\infty$ (or any norm as a result of norm equivalency) on $\mathbb{R}^{N(n,n)}$. We have to prove that $f$ is adapted and $\sqrt{\mathbf{D}'_n}$-valued (is obvious, $\sqrt{\mathbf{D}'_n}$ is closed). For the first part, let $j\in\{0,\dots,n\}$. We want to show that $f(j,\cdot)$ is $\mathcal{F}^n_j$-measurable. This, however, follows from Billingsley [1, Theorem 13.4$(ii)$].

[2]If $V\in\mathbb{R}_{>0}$ such that $\mathbf{D}'_n\subseteq[0,V]$, then obviously $\|f\|_\infty = \max_{\substack{j\in\{0,\dots,n\}\\ \omega\in\mathcal{L}_n^{n+1}}}|f(j,\omega)|\le\sqrt{V}$.

where

$$M^{f,\mathbb{X}} = \left( \sum_{l=1}^{k} f(l-1,\mathbb{X})\Delta\mathbb{X}_l \right)_{k=0}^{n}.$$

$$\begin{aligned}
\mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)] &= \int_{\mathcal{L}_n^{n+1}} \xi(\widehat{X}^n) dP_n \circ (M^{f,\mathbb{X}})^{-1}, \\
&= \int_{\Omega_n} \xi(\widehat{X}^n(M^{f,\mathbb{X}})) dP_n, \quad \text{(transforming measure)} \\
&= \sum_{\omega_n\in\Omega_n} P_n\{\omega_n\}\xi \circ (\widehat{X}^n) \circ M^{f,\mathbb{X}}(\omega_n).
\end{aligned}$$

From Proposition 4 we know that $\xi$ is continuous, $\widehat{X}^n$ is the interpolated canonical process, i.e., $\widehat{X} : \mathcal{L}_n^{n+1} \to \Omega$, thus $\widehat{X}^n$ is continuous and $P_n$ takes it values from the set of real numbers. For $F : f \mapsto \mathbb{E}^{P_n\circ(M^{f,\mathbb{X}})^{-1}}[\xi(\widehat{X}^n)]$ to be continuous, $\psi : f \mapsto M^{f,\mathbb{X}}$ has to be continuous. Since $\mathcal{A} = \{f : \{0,\ldots,n\} \times \mathcal{L}_n^{n+1} \to \sqrt{\mathbf{D}'_n}$, where $f$ is adapted with respect to the filtration generated by $\mathbb{X}\}$ is a compact subset of a $N(n,n)$-dimensional vector space for fixed $n \in \mathbb{N}$ and $M^{f,\mathbb{X}} : \Omega_n \to \mathcal{L}_n^{n+1}$, for all $f, g \in \mathcal{A}$,

$$|M^{f,\mathbb{X}} - M^{g,\mathbb{X}}| = |\|f\|_\infty - \|g\|_\infty| \le \|f-g\|_\infty.$$

Thus, $\psi$ is continuous with respect to the norm $\|\cdot\|_\infty$. Hence $F$ is continuous with respect to any norm[3] on $\mathbb{R}^{N(n,n)}$.

□

**Theorem 6.** *Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \le a(1+\|\omega\|_\infty)^b$ for some constants $a, b > 0$. Then,*

$$\sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] = \lim_{n\to\infty} \max_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)]. \tag{20}$$

*Proof.* The proof follows directly from Proposition 4 and Proposition 5. □

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[3]For any two vector norms $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, and $C_1, C_2 > 0$, we have $C_1\|A\|_\alpha \le \|A\|_\beta \le C_2\|A\|_\alpha$, for all matrices $A \in \mathbb{R}^{N(n,n)}$. i.e., all norms on $\mathbb{R}^{N(n,n)}$ are equivalent because $\mathbb{R}^{N(n,n)}$ has $N(n,n)$-dimension for fixed $n \in \mathbb{N}$.

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# Appendix

## Density argument verification

Let

$$f : \xi \mapsto \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]$$

and

$$g : \xi \mapsto \lim_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)].$$

From (18), we know that for all $\xi \in \mathcal{C}_b(\Omega, \mathbb{R})$, $f(\xi) = g(\xi)$. Since $\mathbb{L}_*^1$ is the completion of $\mathcal{C}_b(\Omega, \mathbb{R})$ under the norm $\|\cdot\|_*$, $\mathcal{C}_b(\Omega, \mathbb{R})$ is dense in $\mathbb{L}_*^1$; and we want to prove for all $\xi \in \mathbb{L}_*^1$, $f(\xi) = g(\xi)$. To prove this, it is sufficient to show that $f$ and $g$ are continuous with respect to the norm $\|\cdot\|_*$.

**For continuity of $f$:**

For all $P \in \mathcal{Q}_{\mathbf{D}}$ and $\xi, \xi' \in \mathbb{L}_*^1$,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi'] \leq \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi - \xi']$$

and

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi - \xi'] \leq \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[|\xi - \xi'|].$$

Since, $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{Q}$,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[|\xi - \xi'|] \leq \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[|\xi - \xi'|] = \|\xi - \xi'\|_*.$$

Then,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi'] \leq \|\xi - \xi'\|_*. \tag{21}$$

Interchanging $\xi$ and $\xi'$,

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi'] - \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] \leq \|\xi' - \xi\|_*. \tag{22}$$

Adding (21) and (22), we have

$$\left| \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi'] \right| \leq \|\xi - \xi'\|_*. \tag{23}$$

Hence,

$$|f(\xi) - f(\xi')| \leq \|\xi - \xi'\|_*.$$

**For continuity of $g$:**

We can follow the same argument as above; for all $\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}$, $\xi, \xi' \in \mathbb{L}_*^1$ and for all $n$,

$$\begin{aligned}
\sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] - \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi'(\widehat{X}^n)] \\
\leq \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n) - \xi'(\widehat{X}^n)] \\
\leq \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}'_n/n}} \mathbb{E}^{\mathbb{Q}}[|\xi(\widehat{X}^n) - \xi'(\widehat{X}^n)|].
\end{aligned}$$

Since, $\mathcal{Q}^n_{\mathbf{D'}_n/n} \subseteq \mathcal{Q}^n_{\mathbf{D}/n}$ and $\mathcal{Q}^n_{\mathbf{D}/n} \subseteq \mathcal{Q}$, we can say that

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[|\xi(\widehat{X}^n) - \xi^{'}(\widehat{X}^n)|] \leq \sup_{Q\in\mathcal{Q}} \mathbb{E}^{Q}[|\xi - \xi^{'}|] = \|\xi - \xi^{'}\|_*,$$

then,

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] - \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X}^n)] \leq \|\xi - \xi^{'}\|_*. \tag{24}$$

Taking the limit when $n$ goes to $\infty$, (24) becomes,

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] - \lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X}^n)] \leq \|\xi - \xi^{'}\|_*. \tag{25}$$

Interchanging $\xi$ and $\xi^{'}$,

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X}^n)] - \lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] \leq \|\xi^{'} - \xi\|_*. \tag{26}$$

Adding (25) and (26), we have

$$\left| \lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X}^n)] - \lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D'}_n/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X}^n)] \right| \leq \|\xi - \xi^{'}\|_*.$$

Hence,

$$\left| g(\xi) - g(\xi^{'}) \right| \leq \|\xi - \xi^{'}\|_*.$$