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Convex vNM–Stable Sets for a Semi Orthogonal Game

Part III:

A Small Economy - Uniqueness and Multiple Solutions

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Abstract

This paper constitutes the third part in a series dealing with vNM–Stable Sets, see [2], [3]. We consider (cooperative) linear production games with a continuum of players. The coalitional function is generated by $r + 1$ "production factors" (non atomic measures). r factors are given by orthogonal probabilities ("cornered" production factors) while factor $r+1^{th}$ is provided "across the corners" of the market.

We consider convex vNM–Stable Sets of this game.

Within this third part we exhaustively discuss the situation in a small but very significant economy or game. In this situation, there are two corners of the market (factors represented by orthogonal probabilities), each of which being divided into two sectors of constant density of the non cornered commodity (a measure exhibiting mass across all corners of the market). For short, this is the 2×2–case, the foundations of which have been laid in Example 2.1 of Part I (cf. [2]).

It turns out that, depending on the boundary conditions, we obtain two different scenarios. The first one reflects a situation that exhibits a unique vNM–Stable Set. The second scenario allows for a variety of vNM–Stable Sets including but not equal to the core of the game.

1 Introduction, Notation

Within this third part of our series about vNM–Stable Sets of Semi Orthogonal Games we embark on a detailed discussion of Example of a "small economy", more precisely the 2×2 case, that has been treated in some version already in Example 2.1 of Part I ([2]) (for short Example 2.1*I). We are going to vary the boundary conditions for h_τ, λ_τ $(\tau = 1, 2, 3, 4)$. That is, we fix the assumption that $h_1 = 0, h_4 = 1$ but otherwise attempt to do an exhaustive presentation.

Within this present paper it turns out that vNM–Stable Sets may exhibit various forms of appearance.

In the first section the vNM–Stable Set presented is unique. The example is *not* exhibiting a rich central commodity in the technical sense of Definition 3.1 of Part II, i.e. [2] (for short Definition 3.1*II). Yet the situation allows for a similar treatment as in SECTION 3*II.

The second section then exhibits quite the opposite behavior of vNM–Stable sets. In this case there is an abundance of such stable sets within certain restrictions. While there is no uniqueness, we can again come up with a full characterization: the family of vNM–Stable Sets described is exhaustive, all such stable sets will be of the nature exhibited.

We use definitions and notations as provided in [2], [3] and previously in [4] and [5]. Thus, we consider a (cooperative) ***game*** with a continuum of players, i.e., a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ where $\boldsymbol{I}$ is some interval in the reals (the ***players***), $\underline{\underline{\mathbf{F}}}$ is the $\sigma-$field of (Borel) measurable sets (the ***coalitions***) and $\boldsymbol{v}$ (the ***coalitional function***) is a mapping $\boldsymbol{v} : \underline{\underline{\mathbf{F}}} \to \mathbb{R}_+$ which is absolutely continuous w.r.t. the Lebesgue measure $\boldsymbol{\lambda}$. We focus on "linear production games", that is, $\boldsymbol{v}$ is described by finitely many measures $\boldsymbol{\lambda}^\rho, (\rho \in \{0, 1, \ldots, r\})$ via

$$(1.1) \qquad \boldsymbol{v}(S) := \min\{\boldsymbol{\lambda}^\rho(S) \mid \rho \in \{0, 1, \ldots, r\}\} \quad (S \in \underline{\underline{\mathbf{F}}}).$$

or

$$(1.2) \qquad \boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} = \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho ,$$

(as previously, we use $\mathbf{R} = \{1, \ldots, r\}$ and $\mathbf{R}_0 = \mathbf{R} \cup \{0\}$). All measures are absolutely continuous w.r.t to Lebesgue measure $\boldsymbol{\lambda}$. The measures $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r$ are orthogonal copies of Lebesgue measure. Thus we choose the player set to be $\boldsymbol{I} := [0, r)$. The carriers $\boldsymbol{C}^\rho = (\rho - 1, \rho]$ $(\rho = 0, \ldots, r)$ of the measures $\boldsymbol{\lambda}^\rho$ are the "cartels" commanding commodity ρ.

The measure $\boldsymbol{\lambda}^0$ is assumed to have a piecewise constant density $\dot{\boldsymbol{\lambda}}^0$ w.r.t $\boldsymbol{\lambda}$ given by

$$(1.3) \qquad \dot{\boldsymbol{\lambda}}^0 = h_\tau \text{ on } \boldsymbol{D}^\tau, \quad (\tau \in \mathbf{T})$$

where $\{\boldsymbol{D}^\tau\}_{\tau\in\mathbf{T}^\rho}$ constitutes a partition of the carrier $\boldsymbol{C}^\rho$ of $\boldsymbol{\lambda}^\rho$ such that $\bigcup_{\tau\in\mathbf{T}^\rho}\boldsymbol{D}^\tau = \boldsymbol{C}^\rho$. Further details of our notation are exactly those presented in [2], [3].

Here we shall discuss the 2×2–example, the first rudimentary version of which appears in EINY ET AL. [1]. however, we change the boundary data so as to obtain an exhaustive treatment. Indeed, in what follows we explain why the 2×2 case is completely treated within the two sections of this paper. As previously we write

$$\begin{aligned}
\lambda_\tau &:= \boldsymbol{\lambda}(\boldsymbol{D}^\tau) \quad (\tau = 1,\ldots,4),\\
&\text{and also}\\
\lambda^0_{23} &:= \boldsymbol{\lambda}^0(\boldsymbol{D}^2\cup\boldsymbol{D}^3)\\
&\text{such that}\\
\lambda_1+\lambda_2 &= 1, \quad \lambda_3+\lambda_4 = 1 \quad .
\end{aligned} \tag{1.4}$$

Now we distinguish alternatives according whether

$$\begin{gathered}
\lambda_2+\lambda_3 \geq 1 \text{ or } \lambda_2+\lambda_3 \leq 1\\
h_2+h_3 \geq 1 \text{ or } h_2+h_3 \leq 1
\end{gathered}$$

holds true. Observe that

$$\lambda_1+\lambda_3 \geq 1 \quad \text{and } h_2+h_3 \leq 1$$

imply $\lambda_3 \geq 1-\lambda_1 \ = \ \lambda_2$, hence

$$\lambda_3 \geq \lambda_3 h_2 + \lambda_3 h_3 \geq \lambda_3 h_2 + \lambda_2 h_2 = \lambda^0_{23}\ ,\ \text{i.e. } 1-\lambda_4 \geq \lambda^0_{23}$$

hence

$$1 > \lambda^0_{23} + \lambda_4 \ = \ \boldsymbol{\lambda}^0(\boldsymbol{I})\ , \tag{1.5}$$

contradicting our basic assumption $\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1$. Hence, there remain the following 3 cases to be distinguished. Note that in all these cases we then assume that

$$\frac{1-\lambda^0_{23}}{\lambda_4} < 1 \quad . \tag{1.6}$$

is satisfied. Note that $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ represent the corners of the market while $\boldsymbol{\lambda}^0$ reflects the commodities in the center of the market.

First Case: The EHMS example.

$$\lambda_1+\lambda_3 \leq 1, \quad \text{i.e.} \quad \lambda_1 \leq \lambda_4, \quad \lambda_3 \leq \lambda_2 \quad , \tag{1.7}$$

$$h_2+h_3 \geq 1 \quad . \tag{1.8}$$

this case has been treated in Part I; the result shows that the core is a vNM–Stable Set.

Second Case:

$$(1.9)\qquad \lambda_1 + \lambda_3 \leq 1, \quad \text{i.e.} \quad \lambda_1 \leq \lambda_4, \quad \lambda_3 \leq \lambda_2 \quad ,$$

$$(1.10)\qquad h_2 + h_3 < 1 \quad .$$

This case is treated in SECTION 2 below.

Third Case:

$$(1.11)\qquad \lambda_1 + \lambda_3 \geq 1, \quad \text{i.e.} \quad \lambda_1 \geq \lambda_4, \quad \lambda_3 \geq \lambda_2 \quad ,$$

$$(1.12)\qquad h_2 + h_3 \geq 1 \quad .$$

This case is treated in SECTION 3 below.

2 A Characterization: The 2×2 Example with a unique vNM–Stable Set

Throughout this section we assume that the density $\dot{\boldsymbol{\lambda}}^0$ satisfies

$$(2.1)\qquad h_1 = 0,\ h_2 + h_3 < 1,\ h_4 = 1 \ .$$

and

$$(2.2)\qquad \begin{array}{l} \lambda_1 + \lambda_3 \ < \ 1 \ , \\ \text{equivalently} \\ \lambda_2 + \lambda_4 \ > \ 1, \ \ \lambda_1 < \lambda_4 \ \ , \lambda_3 < \lambda_2 \ \ , \lambda_2 + \lambda_4 > 1 \ . \end{array}$$

This is not the case of a rich central commodity as we have two sequences (h_1, h_3) and (h_2, h_3) satisfying 2.7.*II. On the other hand, (2.2) is formally the same as 3.12 in Lemma 3.3*II for the sequence $(\widehat{\tau}_1, \widehat{\tau}_2) = (2, 3)$. The example is treated with this sequence in mind. Then conditions 3.12*II or 3.13*II boil down to (2.2).

Observe also, that the Example assumes $h_1 = 0$; hence we cannot compute the relevant vectors (pre–coalitions) just by referring to Theorem 3.5*I, i.e., of [2]. Hence, we cannot call on Theorem 3.15 Part II, but instead have to go through detailed proofs for existence and uniqueness. The proof for the existence theorem nevertheless proceeds along the same path as exhibited in SECTION 3*II.

In order to have integer expressions in the η_τ, we prefer to normalize the relevant vectors differently (see Remark 3.6*I). Accordingly, we list the relevant vectors in a way that avoids rational expressions.

Lemma 2.1. *With assumptions* (2.1) *and* (2.2), *the relevant vectors and their values are given as follows:*

$$
\begin{aligned}
\boldsymbol{a}^{14} :=\ & (1,0,0,1)\ ; \\
& v(\boldsymbol{a}^{14}) = 1 = \boldsymbol{e}^{12}\boldsymbol{a}^{14} = \boldsymbol{e}^{34}\boldsymbol{a}^{14} = \boldsymbol{c}^{0}\boldsymbol{a}^{14} \\
\boldsymbol{a}^{24} :=\ & (0,1,0,1)\ ; \\
& v(\boldsymbol{a}^{24}) = 1 = \boldsymbol{e}^{12}\boldsymbol{a}^{24} = \boldsymbol{e}^{34}\boldsymbol{a}^{24} < \boldsymbol{c}^{0}\boldsymbol{a}^{24} = h_2 + 1 \\
\boldsymbol{a}^{13} :=\ & (h_3,0,1,0)\ ; \\
& v(\boldsymbol{a}^{13}) = h_3 = \boldsymbol{e}^{12}\boldsymbol{a}^{13} = \boldsymbol{c}^{0}\boldsymbol{a}^{13} < \boldsymbol{e}^{34}\boldsymbol{a}^{13} = 1 \\
\boldsymbol{a}^{234} :=\ & (0,1-h_3,h_2,1-(h_2+h_3))\ ; \\
& v(\boldsymbol{a}^{234}) = 1-h_3 = \boldsymbol{e}^{12}\boldsymbol{a}^{234} = \boldsymbol{c}^{0}\boldsymbol{a}^{234} = \boldsymbol{e}^{34}\boldsymbol{a}^{234} \\
\boldsymbol{a}^{23} :=\ & (0,1-h_3,h_2,0)\ ; \\
& v(\boldsymbol{a}^{23}) = h_2 = \boldsymbol{e}^{34}\boldsymbol{a}^{23} = \boldsymbol{c}^{0}\boldsymbol{a}^{23} < \boldsymbol{e}^{12}\boldsymbol{a}^{23} \\
\boldsymbol{a}^{32} :=\ & (0,h_3,1-h_2,0) \\
& v(\boldsymbol{a}^{32}) = h_3 = \boldsymbol{e}^{12}\boldsymbol{a}^{32} = \boldsymbol{c}^{0}\boldsymbol{a}^{32} < \boldsymbol{e}^{34}\boldsymbol{a}^{32}
\end{aligned}
\tag{2.3}
$$

The lemma is verified by some standard procedure for the computation of the extremals of a compact convex polyhedron – but also in view of Theorem 3.5.,[2]. Note that pre–coalitions $\boldsymbol{a}^{14}$ and $\boldsymbol{a}^{24}$ are of the first type $\boldsymbol{a}^{\odot}$ of Theorem 3.5 of [2]. Precoalitions $\boldsymbol{a}^{13}, \boldsymbol{a}^{23}, \boldsymbol{a}^{32}$ are of the second type $\boldsymbol{a}^{\oplus}$ and pre–coalition $\boldsymbol{a}^{234}$ is of the third type $\boldsymbol{a}^{\ominus}$.

Accordingly, we adjust our notation of ε–relevant coalitions. Eg. whenever for some $\varepsilon > 0$ a coalition $\boldsymbol{T}^{13} \subseteq \boldsymbol{D}^{13}$ satisfies $\overrightarrow{\boldsymbol{\lambda}}(T^{13}) = \varepsilon\boldsymbol{a}^{13}$, then we call T^{13} an $\boldsymbol{\varepsilon}$–**13**–***relevant*** coalition. Then

$$
\boldsymbol{v}(\boldsymbol{T}^{13}) \ = \ v(\varepsilon\overrightarrow{\boldsymbol{\lambda}}(T^{13})) \ = \ v(\varepsilon\boldsymbol{a}^{13}) \ = \ \varepsilon h_3 \tag{2.4}
$$

holds true. A similar notation is used for other relevant vectors $\boldsymbol{a}^{\bullet}$. Accordingly, we slightly change the notation for

$$
\boldsymbol{H} \ = \ \{\boldsymbol{x} \in \boldsymbol{J} \,|\, \boldsymbol{x}\boldsymbol{a} \geq v(\boldsymbol{a}) \ \ (\boldsymbol{a} \in \boldsymbol{A}^{s})\} \tag{2.5}
$$

such that in the present situation we have

$$
\begin{aligned}
\boldsymbol{H} \ = \ \{\boldsymbol{x} \in \boldsymbol{J} \,|\, & x_1 + x_4 \geq 1,\ x_2 + x_2 \geq 1, \\
& \boldsymbol{a}^{23}\boldsymbol{x} \geq h_2,\ \boldsymbol{a}^{32}\boldsymbol{x} \geq h_3,\ \boldsymbol{a}^{13}\boldsymbol{x} \geq h_3\}.
\end{aligned}
\tag{2.6}
$$

Finally we note that the vector $\bar{\boldsymbol{x}}$, i.e., the unique element of $\boldsymbol{J}$ satisfying $\boldsymbol{a}^{23}\boldsymbol{x} = h_2,\ \boldsymbol{a}^{32}\boldsymbol{x} = h_3$ and $x_1 + x_4 = 1$, is given by

$$
\bar{\boldsymbol{x}} \ := \ \left(\frac{\lambda_3 + \lambda^0_{23}}{\lambda_4 - \lambda_1}, h_2, h_3, \frac{\lambda_2 - \lambda^0_{23}}{\lambda_4 - \lambda_1}\right) \ , \tag{2.7}
$$

with $\lambda_{23}^0 = h_2\lambda_2^0 + h_3\lambda_3^0 < \lambda_2$.

This vector corresponds exactly to the one exhibited in Section 3*II if we choose the sequence $(\widehat{\tau}_1, \widehat{\tau}_2) = (2,3)$. The following sketch (Figure 2.1) represents $\bar{\boldsymbol{x}}$ as a density in $\boldsymbol{I}$.

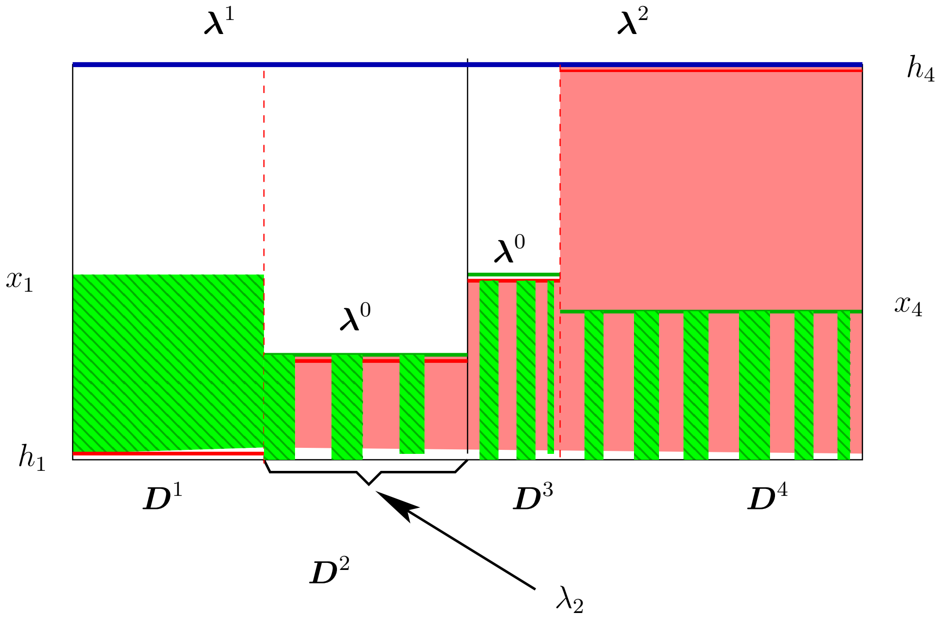

Figure 2.1: The density suggested by $\bar{\boldsymbol{x}}$

Lemma 2.2. *For $\lambda_1 + \lambda_3 \leq 1$ the extremals of $\boldsymbol{H}$ are*

$$\boldsymbol{e}^{12},\ \boldsymbol{e}^{34},\ \textit{and}\ \bar{\boldsymbol{x}}\ .$$

Hence

$$\boldsymbol{H} \quad = \quad \boldsymbol{ConvH}\{\boldsymbol{e}^{12},\ \boldsymbol{e}^{34}, \bar{\boldsymbol{x}}\}$$

The proof can be done by a routine computation of extremals. We cannot rely on Theorem 3.10*II – again the *caveat* concerning the assumption $h_1 = 0$ has to be observed. Yet, that theorem yields relevant vectors though not all of them.

The following Lemma is verified by a standard argument.

Lemma 2.3. *Let $\boldsymbol{\eta}$ be an imputation with minima vector $\boldsymbol{m}$. If*

$$m_1 + m_4 < 1 \tag{2.8}$$

holds true, then an element of the core dominates $\boldsymbol{\eta}$ via an ε–14–relevant coalition. The same holds true if $\{14\}$ is replaced by $\{24\}$.

Corollary 2.4. *1. The set*

$$\boldsymbol{F} \ := \ \{\boldsymbol{x} \in \boldsymbol{J} \mid x_1 + x_4 \geq 1\ ,\ x_2 + x_4 \geq 1,\ x_2 + x_3 \geq 1\} \tag{2.9}$$

has extremals $\boldsymbol{e}^{12}$ and $\boldsymbol{e}^{34}$ only, hence $\boldsymbol{F} \ = \ \boldsymbol{C}(v)$.

2. *Let $\boldsymbol{\eta} \notin \mathcal{C}(\boldsymbol{v})$ be an imputation with minima vector $\boldsymbol{m}$. Suppose*

$$m_1 + m_4 \geq 1 \text{ and } m_2 + m_4 \geq 1 \tag{2.10}$$

holds true. Then

$$m_2 + m_3 < 1 \ . \tag{2.11}$$

Proof:

The first statement is verified by a standard computation, it is essential that $\frac{\lambda_3}{\lambda_2} < 1$ holds true. The second statement follows then immediately, if (2.11) is violated, then $\boldsymbol{m} \in \boldsymbol{C}(\boldsymbol{v})$ and hence necessarily $\boldsymbol{\eta} = \boldsymbol{\vartheta}^{\boldsymbol{m}} \in \mathcal{C}(\boldsymbol{v})$.

q.e.d.

Note that this Corollary is close to Lemma 4.8 and Theorem 4.9 of [2], it serves the same purpose.

Lemma 2.5. *Let $\boldsymbol{\vartheta}$ be an imputation with minima vector $\boldsymbol{m}$. If $\boldsymbol{a}^{23}\boldsymbol{m} \leq h_2$, then $\boldsymbol{\vartheta}$ is dominated by means of the core and itself. The same holds true for $\boldsymbol{a}^{32}$ and $\boldsymbol{a}^{13}$.*

Proof:

Follows from our general theory, i.e., Theorem 4.4*I.

q.e.d.

Theorem 2.6. *$\mathcal{H}$ is internally stable.*

Proof:

This actually does follow from Theorem 3.11*II. In the 2×2–case we can, however, provide a second argument as follows.

We know that for internal domination only $\varepsilon - 234$–relevant coalitions have to be taken into account as all pre–imputations $\boldsymbol{x} \in \boldsymbol{H}$ induce the same value $\boldsymbol{v}(\boldsymbol{a}) = \boldsymbol{x}\boldsymbol{a}$ for the pre–coalitions of the first and second type.

Now all the extremals of $\boldsymbol{H}$, i.e., the vectors $\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \bar{\boldsymbol{x}}$ satisfy $x_1 + x_4 = 1$ as well as the imputation equation, i.e., $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 1$. Subtracting both we obtain for any $\boldsymbol{x}$ satisfying both equations

$$\lambda_2 x_2 + \lambda_3 x_3 + (\lambda_4 - \lambda_1) x_4 = 1 - \lambda_1 = \lambda_2 \ .$$

Hence the vector $(\lambda_2, \lambda_3, \lambda_4 - \lambda_1) > \mathbf{0}$ serves as a gradient to the hyperplane in $\mathbb{R}^4_{234}$ containing the projections of $\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \bar{\boldsymbol{x}}$. Clearly, for any two vectors $\boldsymbol{x}, \boldsymbol{y}$ located within this hyperplane a relation $x_\tau > y_\tau \quad (\tau = 2, 3, 4)$ cannot occur.

q.e.d.

We list the following standard arguments just for comprehensiveness.

Corollary 2.7. *Let $\boldsymbol{\eta} \in \mathcal{I} \setminus \mathcal{H}$ with minima vector $\boldsymbol{m}$.*

1. *If, for some linear combination $\boldsymbol{x}$ of $\bar{\boldsymbol{x}}$ and $\mathbf{e}^{12}$ we have $x_2 > m_2$ and $x_3 > m_3$, then, for sufficiently small $\varepsilon > 0$ there is an ε–32–relevant coalition S^{32} such that*
$$\boldsymbol{\vartheta}^{\boldsymbol{x}} \operatorname{dom}_{S^{32}} \boldsymbol{\eta} \ .$$
2. *If, for some linear combination $\boldsymbol{x}$ of $\bar{\boldsymbol{x}}$ and $\mathbf{e}^{34}$ we have $x_2 > m_2$ and $x_3 > m_3$, then, for sufficiently small $\varepsilon > 0$ there is an ε–23–relevant coalition S^{23} such that*
$$\boldsymbol{\vartheta}^{\boldsymbol{x}} \operatorname{dom}_{S^{23}} \boldsymbol{\eta} \ .$$
3. *If, for some linear combination $\boldsymbol{x}$ of $\bar{\boldsymbol{x}}, \mathbf{e}^{12}$, and $\mathbf{e}^{34}$ we have $x_2 > m_2, x_3 > m_3$ and $x_4 > m_4$, then, for sufficiently small $\varepsilon > 0$ there is an ε–23–relevant coalition S^{23} such that*
$$\boldsymbol{\vartheta}^{\boldsymbol{x}} \operatorname{dom}_{S^{23}} \boldsymbol{\eta} \ .$$

Lemma 2.8. *Let $(x_2, x_3) \geq 0$ be such that $x_2 + x_3 < h_2 + h_3$. Then the following holds true.*

1. *If $x_2 \leq h_2$ then the 23–inequality*
$$\boldsymbol{a}^{23}\boldsymbol{x} \ = \ (1 - h_3)x_2 + h_2 x_3 \leq h_2, \tag{2.12}$$
is satisfied with a strict inequality.
2. *If $x_3 \leq h_3$ then the 32–inequality*
$$\boldsymbol{a}^{32}\boldsymbol{x} \ = \ h_3 x_2 + (1 - h_2)x_3 \leq h_3 \tag{2.13}$$
is satisfied with a strict inequality.

Proof: We check the first item.

As $x_2 + x_3 < 1$ we have $t := \frac{1-x_3}{x_2} > 1$. Now, adding $x_2 + x_3 < h_2 + h_3$ and $(t-1)x_2 \leq (t-1)h_2$ we obtain
$$1 = t x_2 + x_3 < t h_2 + h_3,$$
i.e.,
$$1 < \frac{1 - x_3}{x_2} h_2 + h_3,$$
i.e.,
$$x_2 < (1 - x_3)h_2 + x_2 h_3,$$
which is indeed the 23–inequality (2.12),

Lemma 2.9. *Let $\boldsymbol{\eta} \in \mathcal{I}$ be an imputation and let let T be a 23–coalition. Denote*

$$m_\tau \;:=\; \text{ess inf}_{T^\tau}\, \boldsymbol{\eta} \quad (\tau \;=\; 2,3). \tag{2.14}$$

If

$$m_2 + m_3 < h_2 + h_3 \ , \tag{2.15}$$

then, for sufficiently small $\varepsilon > 0$, there is an ε–23–coalition $S \subseteq T$ and a linear combination $\widehat{\boldsymbol{x}}$ of $\bar{\boldsymbol{x}}$ and $\boldsymbol{e}^{34}$ satisfying $\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}}\, \text{dom}_S\, \boldsymbol{\eta}$.

q.e.d.

Proof:

Assume that $m_2 < h_2$ is true (either this or $m_3 < h_3$ has to be the case, the second case is treated analogously). Because of Lemma 2.8 we can assume that the 23–inequality (2.12) holds true for $\boldsymbol{m}$, i.e., we have

$$(1 - h_3)m_2 + h_2 m_3 < h_2 \ . \tag{2.16}$$

Consider the vector

$$x \;:=\; \frac{m_2}{h_2}\bar{\boldsymbol{x}} + (1 - \frac{m_2}{h_2})\boldsymbol{e}^{34} \in \mathbf{J} \ . \tag{2.17}$$

Now, because of

$$h_2(1 - m_3) > m_2(1 - h_3), \text{ i.e., } m_2 h_3 + h_2 - m_2 > h_2 m_3$$

and $\overline{x}_3 = h_3$ we have

$$m_2\overline{x}_3 + (h_2 - m_2) > h_2 m_3$$

i.e.,

$$\frac{m_2}{h_2}\overline{x}_3 + (1 - \frac{m_2}{h_2}) > m_3 \ .$$

that is,

$$x_2 = m_2, x_3 > m_3 \tag{2.18}$$

Then, for sufficiently small $\delta > 0$ the vector

$$x^\delta \;:=\; (\frac{m_2}{h} + \delta)\bar{\boldsymbol{x}} + (1 - \frac{m_2}{h} - \delta)\boldsymbol{e}^{22} \in \mathbf{J} \tag{2.19}$$

yields

$$x_2^\delta > m_2, x_3^\delta > m_3 \ . \tag{2.20}$$

Now we can choose an ε–23–coalition $S \subseteq T$ satisfying

$$(\boldsymbol{\lambda}(S^2), \boldsymbol{\lambda}(S^3)) = \varepsilon\,[(1-h_3), h_2]$$

such that

$$\boldsymbol{\vartheta}^{\boldsymbol{x}^\delta} > \boldsymbol{\eta} \text{ on } S \tag{2.21}$$

holds true. For this coalition we have

$$\boldsymbol{\lambda}^{3\boldsymbol{h}}(S) \;=\; \varepsilon\,(h_2(1-h_3) + h_3 h_2) = \varepsilon h_2 = \boldsymbol{\lambda}^2(S). \tag{2.22}$$

As $\boldsymbol{x}^\delta$ is a convex combination of $\bar{\boldsymbol{x}}$ and $\boldsymbol{e}^{34}$, we can apply Lemma 2.2 and obtain

$$\boldsymbol{\vartheta}^{\boldsymbol{x}^\delta}(S) = \varepsilon h_2 \tag{2.23}$$

By (2.21) and (2.23) $\boldsymbol{\vartheta}^{x^\delta}$ dominates $\boldsymbol{\eta}$ on S for sufficiently small $\varepsilon > 0$.

q.e.d.

Now consider the $\varepsilon - 234$–relevant coalitions corresponding to $\boldsymbol{a}^{234}$. The following Lemma brakes the path.

Lemma 2.10. *Let $\boldsymbol{x}^0 \in \mathbb{R}^4_+$, $\boldsymbol{x}^0 \neq \bar{\boldsymbol{x}}$, satisfy*

$$\lambda_1 x_1^0 + \lambda_2 x_2^0 + \lambda_3 x_3^0 + \lambda_4 x_4^0 \leq 1, \quad x_1^0 + x_4^0 \geq 1, \quad \text{and} \quad x_2^0 + x_3^0 \leq 1$$

Then there is a set of convex (i.e., nonnegative and summing to 1) coefficients $\alpha_2, \alpha_3, \overline{\alpha}$ such that

$$\boldsymbol{x}^\star \;:=\; \alpha_2 \boldsymbol{e}^{12} + \alpha_3 \boldsymbol{e}^{34} + \overline{\alpha}\bar{\boldsymbol{x}} \in \boldsymbol{H} \tag{2.24}$$

satisfies

$$x_2^\star = x_2^0, \quad x_3^\star = x_3^0, \quad x_4^\star > x_4^0 \;\; . \tag{2.25}$$

Consequently, for sufficiently small $\varepsilon > 0$, there is a relevant ε–234–coalition $S \subseteq T$ and a convex combination $\widehat{\boldsymbol{x}}$ of $\bar{\boldsymbol{x}}$, $\boldsymbol{e}^{12}$, and $\boldsymbol{e}^{34}$ satisfying $\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}}\,\mathrm{dom}_S\,\boldsymbol{\eta}$.

Proof:

1$^{\text{st}}$STEP : Let

$$\overline{\alpha} \;:=\; \frac{1-(x_2^0+x_3^0)}{1-(h_2+h_3)} \;<\; 1 \tag{2.26}$$

First of all consider the case that

$$x_\tau^0 \geq h_\tau \overline{\alpha} \quad (\tau = 2,3)\;. \tag{2.27}$$

Then let

$$0 \leq \alpha_2 = x_2^0 - h_2\overline{\alpha} \;<\; 1 \tag{2.28}$$

$$0 \le \alpha_3 = x_3^0 - h_3\overline{\alpha} \ < \ 1 \ . \tag{2.29}$$

Because of

$$\alpha_2 + \overline{\alpha} h_2 = x_2^0, \quad \alpha_3 + \overline{\alpha} h_3 = x_3^0, \tag{2.30}$$

and

$$\alpha_2 + \alpha_3 \ = \ (x_2^0 + x_3^0) - \overline{\alpha}(h_2 + h_3) \ = \ 1 - \overline{\alpha} \tag{2.31}$$

we have $\alpha_1 + \alpha_2 + \overline{\alpha} = 1$, that is the coefficients define a convex combination. Thus we have determined the coefficients $\overline{\alpha}, \alpha_2, \alpha_2$ of $\bar{\boldsymbol{x}}$,$\boldsymbol{e}^{12}$, and $\boldsymbol{e}^{34}$ such that (2.25) is satisfied.

2^nd STEP :

We wish to verify

$$\alpha_3 + \overline{\alpha x}_4 > x_4^0 \tag{2.32}$$

or equivalently

$$\overline{\alpha}(\overline{x}_4 - h_3) > x_4^0 - x_3^0 \tag{2.33}$$

As we have

$$\lambda_1 x_1^0 + \lambda_2 x_2^0 + \lambda_3 x_3^0 + \lambda_4 x_4^0 \le 1$$

and $x_1^0 + x_4^0 \ge 1$, we have also

$$\lambda_1 x_1^0 + \lambda_1 x_4^0 \ge \lambda_1$$

subtracting the second from first we obtain

$$\lambda_2 x_2^0 + \lambda_3 x_3^0 + (\lambda_4 - \lambda_1) x_4^0 \le 1 - \lambda_1 = \lambda_2$$

that is,

$$x_4^0 \le \frac{\lambda_2 - (\lambda_2 x_2^0 + \lambda_3 x_3^0)}{\lambda_4 - \lambda_1} \tag{2.34}$$

with an equation if and only if $x_1^0 + x_4^0 = 1$ and $\lambda_1 x_1^0 + \lambda_2 x_2^0 + \lambda_3 x_3^0 + \lambda_4 x_4^0 \ = \ 1$. However, whenever both of these equations are satisfied, then necessarily all inequalities involved are equations which means that $\mathring{\boldsymbol{x}} \ = \ \bar{\boldsymbol{x}}$. Thus, we may assume that there is a strict inequality in (2.34). This implies

$$\begin{aligned} x_4^0 - x_3^0 \ &< \ \frac{\lambda_2 - (\lambda_2 x_2^0 + \lambda_3 x_3^0) - x_3^0(\lambda_4 - \lambda_1)}{\lambda_4 - \lambda_1} \\ &= \ \frac{\lambda_2 - \lambda_2 x_2^0 - x_3^0(\lambda_3 + \lambda_4 - \lambda_1)}{\lambda_4 - \lambda_1} \\ &= \ \frac{\lambda_2(1 - (x_2^0 + x_3^0))}{\lambda_4 - \lambda_1} \end{aligned} \tag{2.35}$$

On the other hand, using the definition of $\bar{\boldsymbol{x}}$ as given in (2.7), we obtain

$$
\begin{aligned}
\overline{x}_4 - h_3 &= \frac{\lambda_2 - h_2\lambda_2 - h_3\lambda_3}{\lambda_4 - \lambda_1} - h_3 \\
&= \frac{\lambda_2 - h_2\lambda_2 - h_3(\lambda_3 + \lambda_4 - \lambda_1)}{\lambda_4 - \lambda_1} \\
&= \frac{\lambda_2(1 - (h_2 + h_3))}{\lambda_4 - \lambda_1} ,
\end{aligned}
\tag{2.36}
$$

hence

$$
\bar{\boldsymbol{\alpha}}(\overline{x}_4 - h_3) = \frac{\lambda_2(1 - (x_2^0 + x_3^0))}{\lambda_4 - \lambda_1} ,
\tag{2.37}
$$

Now comparing (2.37) and (2.35)we obtain directly the desired inequality (2.33) and hence (2.32).

3rdSTEP : In the case that (2.27) is not satisfied, we can proceed quite analogously. Eg., if $x_2^0 < h_2\overline{\alpha}$, the put $\alpha_2 = 0$ and determine α_3 by (2.29), the remaining computations then run exactly along the same line as above.

4thSTEP : Now consider the vector

$$
\boldsymbol{x} \ := \ \alpha_2\boldsymbol{e}^{12} + \alpha_3\boldsymbol{e}^{34} + \overline{\alpha}\bar{\boldsymbol{x}}.
$$

Clearly, by the construction of α_2, α_2 and $\overline{\alpha}$ this vector equals $\boldsymbol{m}$ on coordinates $2, 3$ (by (2.30)) and exceeds $\boldsymbol{m}$ on coordinate 4 (by (2.32). Thus, we can increase $\boldsymbol{x}$ slightly on coordinates $2, 3$. I.e., for sufficiently small $\delta < 0$

$$
\boldsymbol{x}^\delta \ := \ (\alpha_2 + \delta)\boldsymbol{e}^{12} + (\alpha_3 + \delta)\boldsymbol{e}^{34} + (\overline{\alpha} - 2\delta)\bar{\boldsymbol{x}}
\tag{2.38}
$$

exceeds $\boldsymbol{m}$ on coordinates $2, 3, 4$. Now effectiveness results from Corollary 2.14. Consequently $\widehat{\boldsymbol{x}} = \boldsymbol{x}^\delta$ serves to fullfill the claim of the present lemma.

q.e.d.

Lemma 2.11. *Let T be a 234–coalition and let $\boldsymbol{\eta} \in \mathfrak{I}$ be an imputation. Let $\boldsymbol{m}$ be the minima vector of $\boldsymbol{\eta}$. If*

$$
h_2 + h_3 \leq m_2 + m_3, \ \textit{and}\ m_2 < h_2, \ \ h_3 < m_3,
\tag{2.39}
$$

then, for sufficiently small $\varepsilon > 0$, there is a relevant ε–234–coalition $S \subseteq T$ or a relevant ε–23–coalition $S \subseteq T$ as well as a linear combination $\widehat{\boldsymbol{x}}$ of $\bar{\boldsymbol{x}}$,$\boldsymbol{e}^{12}$, and $\boldsymbol{e}^{34}$ satisfying $\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}}\ \mathrm{dom}_S\, \boldsymbol{\eta}$. (The symmetric case with $m_2 > h_2, \ h_3 > m_3$ is treated analogously).

Proof:

Either the 23–equation (2.12) is satisfied, in which case we can just repeat the construction of the Lemma 2.9. Or else, the reverse inequality holds true, i.e., we have

$$
m_2(1 - h_3) \geq h_2(1 - m_3).
\tag{2.40}
$$

Then
$$m_2[1-(h_3+h_3)] \geq h_2[1-(m_3+m_3)],$$
that is
$$\alpha_2 \; := \; m_2 - h_2\frac{1-(m_3+m_3)}{1-(h_3+h_3)} \; \geq 0$$
Hence we can define the quantities $\alpha_2, \alpha_3, \overline{\alpha}$ as in the proof of Lemma 2.10 and the argument of that proof is repeated.

q.e.d.

Theorem 2.12. *Let $\lambda_1 + \lambda_3 < 1$ and $h_2 + h_3 < 1$. Then $\mathcal{H}$ is externally stable, hence a vNM–Stable Set.*

Proof: 1$^{\text{st}}$STEP :

Let $\boldsymbol{\eta} \in \mathcal{I} \setminus \mathcal{H}$ and let $\boldsymbol{m}$ be the minima vector of $\boldsymbol{\eta}$.

Now, if $\boldsymbol{m}$ violates one of the inequalities regarding relevant 14 and 24 coalitions, then we can immediately dominate $\boldsymbol{\eta}$ (actually via the core) in view of Corollary 2.3. Thus we have $m_1 + m_4 \geq 1, \; m_2 + m_4 \geq 1$.

Moreover, we can assume

$$m_2 + m_3 \leq 1 \tag{2.41}$$

holds true in view of Corollary 2.4.

3$^{\text{rd}}$STEP : Assume now

$$m_2 + m_3 < h_2 + h_3 \; . \tag{2.42}$$

Then by Lemma 2.9 we obtain some linear combination $\widehat{\boldsymbol{x}}$ of $\bar{\boldsymbol{x}}$ and $\boldsymbol{e}^{12}$ such that for sufficiently small $\varepsilon > 0$, the imputation $\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}}$ dominates $\boldsymbol{\eta}$ w.r.t. some relevant ε–23–set S.

4$^{\text{th}}$STEP : Consider now the situation in which

$$h_\tau < m_\tau \; (\tau = 2, 3) \tag{2.43}$$

holds true.

Then by Lemma 2.10 we obtain some linear combination $\widehat{\boldsymbol{x}}$ of $\bar{\boldsymbol{x}}$, $\boldsymbol{e}^{12}$ and $\boldsymbol{e}^{34}$ such that, for sufficiently small $\varepsilon > 0$, the imputation $\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}}$ dominates $\boldsymbol{\eta}$ w.r.t some relevant e–234–set S

5$^{\text{th}}$STEP : Finally, consider the situation in which

$$h_2 + h_3 \leq m_2 + m_3 \leq 1, \text{ and } m_2 < h_2, \; h_3 < m_3 \; . \tag{2.44}$$

is true

Here we can either construct a relevant 23 or a relevant 234 coalition. The details follow immediately from Lemma 2.10. The symmetric case with $m_2 > h_2, \; h_3 > m_3$ is treated analogously.

q.e.d.

For the present setup at which there is just one additional extremal point in $\boldsymbol{H}$ we can actually prove uniqueness of our vNM–Stable Set.

Theorem 2.13. *Let $\widehat{\mathcal{H}}$ be a vNM–Stable Set. Then $\bar{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}} \in \widehat{\mathcal{H}}$.*

Proof:

If $\bar{\boldsymbol{\vartheta}} \notin \widehat{\mathcal{H}}$, then there has to be an imputation $\widehat{\boldsymbol{\vartheta}} \in \widehat{\mathcal{H}}$ (*in particular not dominated by the core !*) as well as a coalition $\widehat{S}$ such that

$$\widehat{\boldsymbol{\vartheta}} \operatorname{dom}_{\widehat{S}} \bar{\boldsymbol{\vartheta}} \tag{2.45}$$

holds true. Let the essential minima of $\widehat{\boldsymbol{\vartheta}}$ be denoted by $\widehat{\boldsymbol{m}} = (\widehat{m}_1, \ldots, \widehat{m}_4)$ with

$$\widehat{m}_\tau \ := \ \text{ess inf}_{\boldsymbol{D}^\tau} \widehat{\boldsymbol{\vartheta}} \ \ (\tau \ = \ 1,2,3,4). \tag{2.46}$$

1$^{\text{st}}$STEP :

$$\text{We claim that w.l.o.g } (\widehat{m}_2, \widehat{m}_3) \geq (h_2, h_3). \tag{2.47}$$

Indeed, because $\widehat{\boldsymbol{\vartheta}}$ is not dominated by the core (Lemma 2.5), we have

$$\boldsymbol{a}^{23}\widehat{\boldsymbol{m}} \geq h_2, \quad \boldsymbol{a}^{32}\widehat{\boldsymbol{m}} \geq h_3.$$

If we focus in coordinates 2 and 3 of $\widehat{\boldsymbol{m}}$, then the situation is represented in Figure 2.2. Obviously at most one of the coordinates $2, 3$ can be dominated by a coordinate of $\bar{\boldsymbol{x}}$, i.e., of (h_2, h_3). Assume $\widehat{m}_3 \leq h_3$ as indicated by Figure 2.2.

Then consider the imputation $\boldsymbol{\vartheta}^0 \ := \ (1-t)\widehat{\boldsymbol{\vartheta}} + t\boldsymbol{e}^{23} \in \widehat{\mathcal{H}}$ for $t \ := \ \bar{x}_3 - \widehat{m}_3$. Let the minima of $\boldsymbol{\vartheta}^0$ be given by $\boldsymbol{m}^0$, then

$$(m_2^0, m_3^0) \geq (\bar{x}_2, \bar{x}_3). \tag{2.48}$$

Therefore, the claim formulated in 2.47 is proved.

2$^{\text{nd}}$STEP : Our next claim is

$$\bar{x}_4 \geq \widehat{m}_4 \ \ . \tag{2.49}$$

Because of (2.48) and

$$\begin{aligned} &\bar{x}_1\lambda_1 + \bar{x}_2\lambda_2 + \bar{x}_3\lambda_3 + \bar{x}_4\lambda_4 = 1, \\ &\widehat{m}_1\lambda_1 + \widehat{m}_2\lambda_2 + \widehat{m}_3\lambda_3 + \widehat{m}_4\lambda_4 \leq 1 \end{aligned}$$

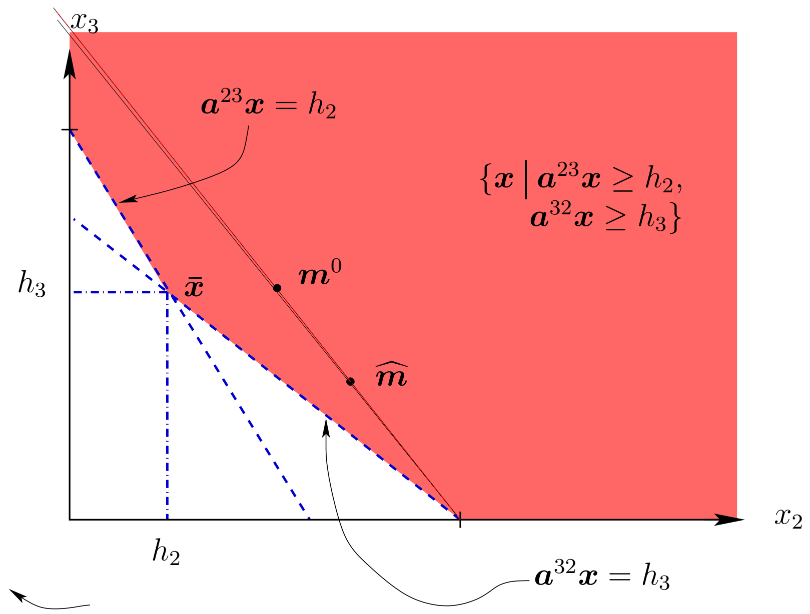

Figure 2.2: Coordinates 2 and 3 of $\widehat{\boldsymbol{m}}$ and $\bar{\boldsymbol{x}}$

we have

$$\begin{aligned}\widehat{m}_1\lambda_1 + \widehat{m}_4\lambda_4 \leq \bar{x}_1\lambda_1 + \bar{x}_4\lambda_4 &= 1 - (\bar{x}_2\lambda_2 + \bar{x}_3\lambda_3) \\ &= 1 - (h_2\lambda_2 + h_3\lambda_3) = 1 - \boldsymbol{\lambda}^0{}_{23}.\end{aligned}$$

On the other hand, $\boldsymbol{\vartheta}^0$ cannot be dominated by the core, hence we have necessarily

$$\widehat{m}_1 + \widehat{m}_4 \geq 1 = \bar{x}_1 + \bar{x}_4 \ .$$

Adding

$$\begin{array}{l} -\widehat{m}_1\lambda_1 - \widehat{m}_4\lambda_1 \leq -\bar{x}_1\lambda_1 - \bar{x}_4\lambda_4 \\ \text{and} \\ \widehat{m}_1\lambda_1 + \widehat{m}_4\lambda_4 \leq \bar{x}_1\lambda_1 + \bar{x}_4\lambda_4 \end{array}$$

we come up with $(\lambda_4 - \lambda_1)m_4^0 \leq (\lambda_4 - \lambda_1)\bar{x}_4$. Hence (2.49) holds true indeed..

Similarly, one finds $\bar{x}_1 \leq \widehat{m}_1$. Alternatively, one may at once focus on coordinates $1, 4$ of $\bar{\boldsymbol{x}}$ and $\boldsymbol{m}^0$ and inspect Figure 2.3.

3rdSTEP :

We now claim that w.l.o.g

$$\widehat{m}_1 + \widehat{m}_4 = 1 \quad . \tag{2.50}$$

Indeed, consider the measure

$$\boldsymbol{\vartheta}^1 \ := \ \widehat{\boldsymbol{\vartheta}} - t\lambda_4\boldsymbol{\lambda}_{\,|\,\boldsymbol{D}^1} + t\lambda_1\boldsymbol{\lambda}_{\,|\,\boldsymbol{D}^4}$$

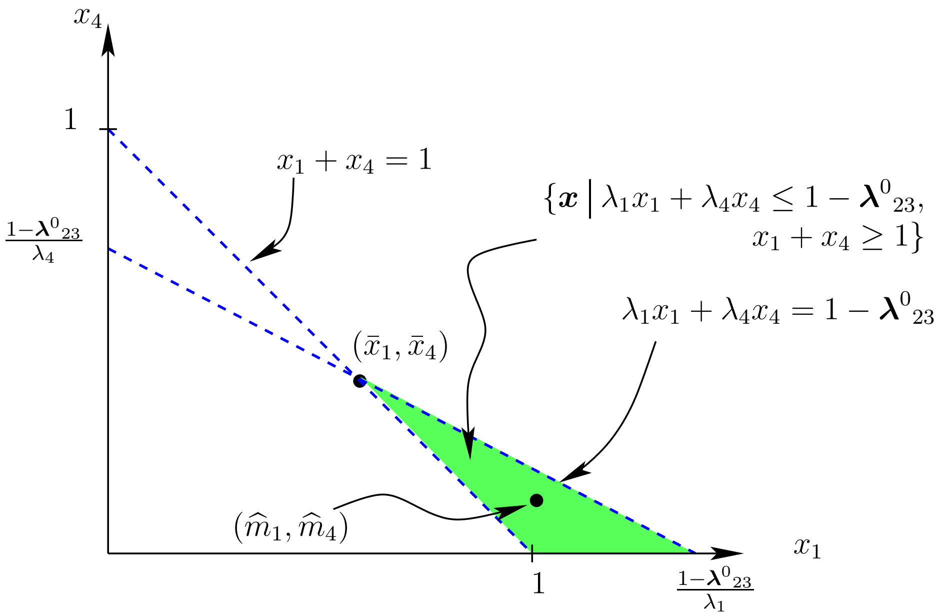

Figure 2.3: Coordinates 1 and 4 of $\widehat{\boldsymbol{m}}$ and $\bar{\boldsymbol{x}}$

for

$$t \;:=\; \frac{(\widehat{m}_1 + \widehat{m}_4) - 1}{\lambda_4 - \lambda_1}$$

Note that the minima in the second and third coordinate are not changed, so the vector of minima $\boldsymbol{m}^1$ of $\boldsymbol{\vartheta}^1$ inherits (2.48), that is

$$(m_2^1, m_3^1) \geq (\bar{x}_2, \bar{x}_3). \tag{2.51}$$

Moreover by the choice of t we observe that

$$(m_1^1 + m_4^1) = 1 \tag{2.52}$$

But (m_1^1, m_4^1) also inherits the inequality

$$\lambda_1 x_1 + \lambda_4 x_4 \leq 1 - \boldsymbol{\lambda}^0{}_{23} \tag{2.53}$$

from $(\widehat{m}_1, \widehat{m}_4)$. Now, $\widehat{m}_4^1 \geq \widehat{m}_4$ is obvious and $\widehat{m}_1^1 \geq \widehat{m}_1$ follows by an inspection of Figure 2.3 which shows that (2.52) and (2.53) imply that any solution $\boldsymbol{x}$ of these two inequalities necessarily yields $x_1 \geq \bar{x}_1$. Thus $\boldsymbol{\vartheta}^1$ is nonnegative, hence an imputation. But any domination of $\boldsymbol{\vartheta}^1$ via some ε–234–relevant coalition implies *a fortiori* a domination of $\widehat{\boldsymbol{\vartheta}}$, as both are equal in in $\boldsymbol{D}^2 \cup \boldsymbol{D}^3$ and $\boldsymbol{\vartheta}^1$ exceeds $\widehat{\boldsymbol{\vartheta}}$ on $\boldsymbol{D}^4$. Consequently, $\boldsymbol{\vartheta}^1 \in \widehat{\mathcal{H}}$ and we may as well replace $\widehat{\boldsymbol{\vartheta}}$ by $\boldsymbol{\vartheta}^1$, in other words, we have verified (2.50).

4thSTEP :

Next we reduce the minima of $\widehat{\boldsymbol{\vartheta}}$ in $\boldsymbol{D}^2 \cup \boldsymbol{D}^3$ to (h_2, h_3). That is our next claim is that w.l.o.g

$$(\widehat{m}_2, \widehat{m}_3) = (h_2, h_3) \tag{2.54}$$

holds true.

To this end we recall Lemma 2.10 and choose – using our present $\widehat{\boldsymbol{m}}$ – the coefficients $\alpha_1, \alpha_2, \bar{\alpha}$ as given by that lemma. Then we know that

$$\boldsymbol{x}^\star \;:=\; \alpha_1 \boldsymbol{e}^{12} + \alpha_2 \boldsymbol{e}^{34} + \bar{\alpha}\overline{\boldsymbol{x}}$$

satisfies

$$(x_2^\star, x_3^\star) = (\widehat{x}_2, \widehat{x}_3) = (\alpha_1, \alpha_2) + \bar{\alpha}(h_2, h_3) \ .$$

Hence

$$(x_2^\star, x_3^\star) - (\alpha_1, \alpha_2) = \bar{\alpha}(h_2, h_3) \ .$$

Consequently, the vector

$$\boldsymbol{x}^{\star\star} \;:=\; \frac{\widehat{\boldsymbol{x}} - (\alpha_1 \boldsymbol{e}^{12} + \alpha_2 \boldsymbol{e}^{34})}{\bar{\alpha}}$$

$$\boldsymbol{x}^{\star\star} \;:=\; \frac{(x_2^\star, x_3^\star) - (\alpha_1, \alpha_2)}{\bar{\alpha}}$$

satisfies

$$(\boldsymbol{x}_2^{\star,\star}, \boldsymbol{x}_3^{\star,\star}) = (h_2, h_3) \ .$$

We claim that $\boldsymbol{x}^{\star\star} \geq 0$ holds true. To this end, consider, for $0 \leq t \leq 1$ the vector

$$\widehat{\boldsymbol{x}} - t(\alpha_1 \boldsymbol{e}^{12} + \alpha_2 \boldsymbol{e}^{34})$$

this vector inherits the inequalities $x_1 + x_4 = 1$ and $x_2 + x_4 \geq 1$ from $\widehat{\boldsymbol{x}}$ as the $\boldsymbol{e}^{12}$ and $\boldsymbol{e}^{34}$ yield an equation in both inequalities. Now, if for some $t < 1$ we would have $x_4^t = 0$, then $x_1^t = 1$ and $x_2^t \geq 1$ hence $\lambda_1 x_1^t + \ldots + \lambda_4 x_4^t > \lambda_1 + \lambda_2 = 1$, a contradiction. Consequently $x_4^t = x_4^{\star\star} \geq 0$. Next, $x_1^{\star\star} > 0$ is immediately seen by inspecting Figure 2.3 as the inequalities $x_1 + x_4 = 1, \lambda_1 x_1^t + \lambda_4 x_4^t \leq 1 - \lambda_{23}^0$ admit only solutions $\boldsymbol{x}$ with $x_1 \geq \overline{x}_1$.

This implies that

$$\boldsymbol{\vartheta}^{\star\star} \;:=\; \frac{\widehat{\boldsymbol{\vartheta}} - (\alpha_1 \boldsymbol{e}^{12} + \alpha_2 \boldsymbol{e}^{34})}{\bar{\alpha}}$$

is indeed an imputation with minima (h_2, h_3) in $\boldsymbol{D}^2$ and $\boldsymbol{D}^3$. However, as

$$\widehat{\boldsymbol{\vartheta}} \;=\; \alpha \boldsymbol{\vartheta}^{\star\star} + (\alpha_1 \boldsymbol{e}^{12} + \alpha_2 \boldsymbol{e}^{34}) \ ,$$

we observe that any domination of $\boldsymbol{\vartheta}^{\star\star}$ by some $\widetilde{\boldsymbol{\vartheta}}$ via some $\varepsilon - 234$–relevant coalition induces a domination of $\widehat{\boldsymbol{\vartheta}}$ by $\alpha\widetilde{\boldsymbol{\vartheta}} + (\alpha_1 \boldsymbol{e}^{12} + \alpha_2 \boldsymbol{e}^{34})$ via the same $\varepsilon -$ 234–relevant coalition, a contradiction. Hence we have $\boldsymbol{\vartheta}^{\star\star} \in \widehat{\mathcal{H}}$. Therefore, we may, if necessary, replace $\widehat{\boldsymbol{\vartheta}}$ by $\boldsymbol{\vartheta}^{\star,\star}$, that is we have verified our claim (2.54).

5$^{\text{th}}$STEP :

Within this step we now show that $\widehat{\boldsymbol{\vartheta}}$ can actually be assumed to be constant in $\boldsymbol{D}^2 \cup \boldsymbol{D}^3$. That is we are going to prove that w.l.o.g

$$\widehat{\boldsymbol{\vartheta}} = h_2 \text{ on } \boldsymbol{D}^2, \quad \widehat{\boldsymbol{\vartheta}} = h_3 \text{ on } \boldsymbol{D}^3, \tag{2.55}$$

To this end let $\boldsymbol{\vartheta}^\star$ be obtained by shifting all mass of $\widehat{\boldsymbol{\vartheta}}$ above (h_2, h_3) from $\boldsymbol{D}^2 \cup \boldsymbol{D}^3$ to $\boldsymbol{D}^4$, that is we define

$$\boldsymbol{\vartheta}^\star \;:=\; \widehat{\boldsymbol{\vartheta}}_{\mid \boldsymbol{D}^1} + h_2 \mathbb{1}_{\boldsymbol{D}^2} + h_3 \mathbb{1}_{\boldsymbol{D}^3} + \widehat{\boldsymbol{\vartheta}}_{\mid \boldsymbol{D}^4} + \Big(\int\limits_{\boldsymbol{D}^2 \cup \boldsymbol{D}^3} \widehat{\boldsymbol{\vartheta}} d\boldsymbol{\lambda} - \boldsymbol{\lambda}^0{}_{23} \Big) \mathbb{1}_{\boldsymbol{D}^4} \quad .$$

Suppose now that $\boldsymbol{\vartheta}^\star$ is dominated by some $\widetilde{\boldsymbol{\vartheta}}$ via some $\varepsilon - 234$–relevant coalition $\overset{\circ}{T}{}^{234}$ for some $\varepsilon > 0$. The measure

$$\boldsymbol{\lambda}^{\boldsymbol{h}} \;:=\; \frac{h_2}{h_2 + h_3} \boldsymbol{\lambda}^1 + \frac{h_3}{h_2 + h_3} \boldsymbol{\lambda}^2$$

exceeds $\widehat{\boldsymbol{\vartheta}}$ on some set $\widehat{T}^{23} \subseteq \boldsymbol{D}^{23}$ of sufficiently small measure, as the minima of this measure exceed (h_2, h_3).

Moreover, $\widetilde{\boldsymbol{\vartheta}}$ exceeds $\widehat{\boldsymbol{\vartheta}}$ on $\overset{\circ}{T}{}^4$ as $\boldsymbol{\vartheta}^\star$ exceeds $\widehat{\boldsymbol{\vartheta}}$ on $\overset{\circ}{T}{}^4$.

Hence, for sufficiently small $\widetilde{\varepsilon} > 0$ we can find an $\widetilde{\varepsilon} - 123$–relevant coalition $\widetilde{T}^{234} \subseteq \widehat{T}^{23} \cup \overset{\circ}{T}{}^4$ such that, for sufficiently small $\widetilde{\delta} > 0$ the measure

$$\widetilde{\boldsymbol{\vartheta}} \;:=\; (1 - \widetilde{\delta}) \widehat{\boldsymbol{\vartheta}} + \widetilde{\delta} \frac{\boldsymbol{\lambda}^{\boldsymbol{h}} + \widetilde{\boldsymbol{\vartheta}}}{2}$$

exceeds $\widehat{\boldsymbol{\vartheta}}$ on $\widetilde{T}^{234}$. Now, as $\widehat{\boldsymbol{\vartheta}}(\widetilde{T}^{234}) < \widetilde{\varepsilon}(1 - h_3) = \boldsymbol{v}(\widetilde{T}^{234})$ and $\boldsymbol{\lambda}^{\boldsymbol{h}}(\widetilde{T}^{234}) \leq \widetilde{\varepsilon}(1 - h_3) = \boldsymbol{v}(\widetilde{T}^{234})$ we can choose, if necessary, $\widetilde{\delta}$ even smaller such that $\widetilde{\boldsymbol{\vartheta}}(\widetilde{T}^{234}) < \widetilde{\varepsilon}(1 - h_3) = \boldsymbol{v}(\widetilde{T}^{234})$ is true. Hence we have indeed

$$\widetilde{\boldsymbol{\vartheta}} \operatorname{dom}_{\widetilde{T}^{234}} \widehat{\boldsymbol{\vartheta}}$$

a contradiction. Consequently, $\boldsymbol{\vartheta}^\star \in \mathcal{H}$ and we may replace $\widehat{\boldsymbol{\vartheta}}$ by $\boldsymbol{\vartheta}^\star$. That is, we have indeed verified our claim (2.55).

6^th^STEP :

Altogether we have found $\widehat{\boldsymbol{\vartheta}} \in \widehat{\mathcal{H}}$ such that $\widehat{\boldsymbol{\vartheta}} = \widehat{x}_1 \geq \overline{x}_1$ on $\boldsymbol{D}^1$ and $\widehat{\boldsymbol{\vartheta}} = h_\tau$ on $\boldsymbol{D}^\tau$ $(\tau = 2, 3)$ is the case. Moreover, $\widehat{x}_1 + \widehat{x}_4 = 1$ Now consider the average over $\boldsymbol{D}^4$ given by

$$\widehat{\widehat{x}}_4 \;:=\; \frac{1}{\boldsymbol{\lambda}(\boldsymbol{D}^4)} \int\limits_{\boldsymbol{D}^4} \widehat{\boldsymbol{\vartheta}} d\boldsymbol{\lambda} \quad .$$

Then clearly

$$\lambda_1 \widehat{x}_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 \widehat{\widehat{x}}_4 = 1$$

thus

$$\lambda_1 \widehat{x}_1 + \lambda_4 \widehat{\widehat{x}}_4 = 1 - \lambda_2^0 3$$

and

$$\widehat{x}_1 + \widehat{\overline{x}}_4 \geq 1 \quad .$$

However, inspection of Figure 2.3 reveals that $(\overline{x}_1, \overline{x}_4)$ is the only solution to this pair of inequality/equation. Hence $\widehat{x}_1 = \overline{x}_1$ and because of $\widehat{x}_1 + \widehat{x}_4 = 1$ it follows that $\widehat{x}_4 = \overline{x}_4$. Consequently, $\widehat{\boldsymbol{\vartheta}} \geq \boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}$ which implies $\widehat{\boldsymbol{\vartheta}} \; = \; \boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}$ as both are imputations. This finally proves our Theorem.

q.e.d.

Theorem 2.14. $\mathcal{H}$ *is the unique vNM–Stable Set.*

Proof:

Let $\widehat{\mathcal{H}}$ be a second vNM–Stable Set. First of all $\widehat{\mathcal{H}}$ must contain the core as the core is always undominated. By Theorem 2.13, we have $\overline{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}^{\widehat{x}} \in \widehat{\mathcal{H}}$. Consequently $\widehat{\mathcal{H}} \subseteq \mathcal{H}$ and hence $\widehat{\mathcal{H}} = \mathcal{H}$,

q.e.d.

3 Multiplicity of vNM–Stable Sets

Recall the standard 2×2 Example (see [2], [3]); this we will now extensively discuss under the assumption

$$h_1 = 0, \quad h_2 + h_3 \geq 1 \quad h_4 = 1 \tag{3.1}$$

and

$$\lambda_1 + \lambda_3 \geq 1 \ . \tag{3.2}$$

Then $(\widehat{\tau}_1, \widehat{\tau}_2) = (1, 3)$ is the only sequence that yields $h_{\widehat{\tau}_1} + h_{\widehat{\tau}_2} = 0 + h_3 < 1$. Thus, this is the case of a long central distribution but without the conditions regarding $\boldsymbol{\lambda}^0$ that ensure the existence in Part II ([3]).

In order to avoid rational expressions in terms h_1, h_2, we adjust our notation of ε–relevant coalitions in accordance with Remark 3.6 and Example 3.7 of [2]. Thus the following is a complete description of the relevant vectors.

Lemma 3.1. *Given the assumptions* (3.1),(3.2), *the relevant vectors and their values are given as follows:*

$$\begin{array}{rll}
\boldsymbol{a}^{14} := & (1,0,0,1) \ ; \\
& v(\boldsymbol{a}^{14}) = 1 = \boldsymbol{e}^{12}\boldsymbol{a}^{14} = \boldsymbol{e}^{34}\boldsymbol{a}^{14} = \boldsymbol{c}^{0}\boldsymbol{a}^{14} \\
\boldsymbol{a}^{23} := & (0,1,1,0) \\
& v(\boldsymbol{a}^{23}) = 1 = \boldsymbol{e}^{12}\boldsymbol{a}^{23} = \boldsymbol{e}^{34}\boldsymbol{a}^{23} < \boldsymbol{c}^{0}\boldsymbol{a}^{23} = h_2 + h_3 \\
\boldsymbol{a}^{24} := & (0,1,0,1) \ ; \\
& v(\boldsymbol{a}^{24}) = 1 = \boldsymbol{e}^{12}\boldsymbol{a}^{24} = \boldsymbol{e}^{34}\boldsymbol{a}^{24} < \boldsymbol{c}^{0}\boldsymbol{a}^{24} = h_2 + 1 \\
\boldsymbol{a}^{13} := & (h_3,0,1,0) \\
& v(\boldsymbol{a}^{13}) = h_3 = \boldsymbol{e}^{12}\boldsymbol{a}^{23} = \boldsymbol{c}^{0}\boldsymbol{a}^{23} < \boldsymbol{e}^{34}\boldsymbol{a}^{23} = 1 \\
\boldsymbol{a}^{123} := & ((h_2+h_3)-1, 1-h_3, h_2, 0) \\
& v(\boldsymbol{a}^{123}) = 1 - h_3 = \boldsymbol{e}^{12}\boldsymbol{a}^{123} = \boldsymbol{e}^{34}\boldsymbol{a}^{123} = \boldsymbol{c}^{0}\boldsymbol{a}^{123}
\end{array} \tag{3.3}$$

Remark 3.2. Accordingly, we reformulate the notion for ε–relevant coalitions under the above two assumptions regarding a "long central commodity". Thus, whenever for some $\varepsilon > 0$ a coalition satisfies

$$\begin{array}{l}
\vec{\boldsymbol{\lambda}}(T^{13}) \ = \ \varepsilon \boldsymbol{a}^{13} \ = \ \varepsilon(h_3, 0, 1, 0) \\
\text{and hence} \\
\boldsymbol{v}(T^{13}) \ = \ \boldsymbol{\lambda}^1(T) = \boldsymbol{\lambda}^3(T) = \varepsilon h_3 < \boldsymbol{\lambda}^2(T) = \varepsilon,
\end{array} \tag{3.4}$$

then we call T^{13} an ***ε*–13–*relevant*** coalition. Similarly, if $\boldsymbol{T}^{123} \subseteq \boldsymbol{D}^{123}$ satisfies

$$\begin{array}{l}
\vec{\boldsymbol{\lambda}}(T^{123}) \ = \ \varepsilon \boldsymbol{a}^{123} \ = \ \varepsilon((h_2+h_3)-1, 1-h_3, h_2, 0), \\
\text{and hence} \\
\boldsymbol{v}(T^{123}) \ = \ \boldsymbol{\lambda}^1(T) = \boldsymbol{\lambda}^2(T) = \boldsymbol{\lambda}^3(T) = \varepsilon(1-h_3) \ ,
\end{array} \tag{3.5}$$

then we call T^{123} an $\boldsymbol{\varepsilon}$–**123**–***relevant*** coalition, etc.

◦ ~~~~~~~ ◦

Next we turn to the set of pre–imputations

$$\boldsymbol{H} := \{\boldsymbol{x} \in \boldsymbol{J} \mid \boldsymbol{a}\boldsymbol{x} \geq v(\boldsymbol{a}) \quad (\boldsymbol{a} \in \boldsymbol{A}^s)\} \tag{3.6}$$

which in the present 2×2 case is

$$\begin{aligned} \boldsymbol{H} = \{\boldsymbol{x} \in \boldsymbol{J} \mid x_1 + x_4 \geq 1, x_2 + x_4 \geq 1, x_2 + x_3 \geq 1, \\ h_3 x_1 + x_3 \geq h_3\}\,. \end{aligned} \tag{3.7}$$

In the context of our previous presentations (cf. [2], [3]), this set was eventually discovered to yield a vNM–Stable set – unique in the context of a large central commodity. In the present context this will not do. Rather we have to identify a proper subset of $\boldsymbol{H}$ for our purpose.

We shall represent $\boldsymbol{H}$ in a suitable geometrical sketch (Figure 3.1). To this end, we introduce the set of pre–imputations

$$\boldsymbol{F} := \{\boldsymbol{x} \in \boldsymbol{J} \mid x_1 + x_4 \geq 1,\ x_2 + x_4 \geq 1, x_2 + x_3 \geq 1\}\ . \tag{3.8}$$

In the present context, if one of the inequalities of $\boldsymbol{F}$ is violated, then we can immediately set up a domination by the core. Thus, $\boldsymbol{F}$ is the set of all pre–imputations that cannot be dominated by the core.

Clearly, $\boldsymbol{H}$ is the set of all such pre–imputations that, in addition, cannot be (“internally”) dominated by some other element of $\boldsymbol{H}$ via some ε–23–relevant coalition.

Now we compute the extremals of $\boldsymbol{H}$.

Lemma 3.3. *Let*

$$\begin{aligned} \boldsymbol{x}^{(3424)} &:= h_3\boldsymbol{e}^{34} + (1-h_3)\boldsymbol{x}^{(24)} = \left(0, (1-h_3)\frac{\lambda_3}{\lambda_2}, h_3, 1\right) \\ \boldsymbol{x}^{(3442)} &:= h_3\boldsymbol{e}^{34} + (1-h_3)\boldsymbol{x}^{(42)} \\ &\qquad = \left(0, (1-h_3), h_3, (1-h_3)\frac{\lambda_1}{\lambda_4} + h_3\right), \end{aligned} \tag{3.9}$$

then

$$\boldsymbol{H} = \boldsymbol{ConvH}\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \boldsymbol{x}^{(3424)}, \boldsymbol{x}^{(3442)}\}\ . \tag{3.10}$$

Proof: We start with the set $\boldsymbol{F}$ as defined in (3.8). The extremals of $\boldsymbol{F}$ are given by $\boldsymbol{e}^{12}, \boldsymbol{e}^{34}$ and

$$\boldsymbol{x}^{(24)} := \left(0, \frac{\lambda_3}{\lambda_2}, 0, 1\right), \text{ and } \boldsymbol{x}^{(42)} := \left(0, 1, 0, \frac{\lambda_1}{\lambda_4}\right)\ . \tag{3.11}$$

Hence, $\boldsymbol{F}$ is a simplex in $\mathbb{R}^4_+$ as depicted in Figure 3.1. Now observe that the following inequalities and equations hold true:

$$
\begin{aligned}
\boldsymbol{a}^{13}\boldsymbol{e}^{12} &= \boldsymbol{a}^{13}\boldsymbol{x}^{(3424)} = \boldsymbol{a}^{13}\boldsymbol{x}^{(3442)} = h_3 \\
\boldsymbol{a}^{13}\boldsymbol{e}^{34} &= 1 > h_3 \\
\boldsymbol{a}^{13}\boldsymbol{x}^{(24)} &= \boldsymbol{a}^{13}\boldsymbol{x}^{(42)} = 0 < h_3 \ .
\end{aligned}
\tag{3.12}
$$

Thus the subsimplex

$$
\boldsymbol{H}^0 \ := \ \{\boldsymbol{x} \in \boldsymbol{F} \,|\, \boldsymbol{a}^{13}\boldsymbol{x} = h_3\}
\tag{3.13}
$$

has the extremals

$$
\boldsymbol{e}^{12}, \quad \boldsymbol{x}^{(3424)}, \quad \boldsymbol{x}^{(3442)} \ .
\tag{3.14}
$$

This subsimplex divides the simplex $\boldsymbol{F}$ into a simplex and a tetrahedron; clearly $\boldsymbol{H}$ is the simplex spanned by $\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \boldsymbol{x}^{(3424)}, \boldsymbol{x}^{(3442)}$,

q.e.d.

Figure 3.1 shows the details representing the simplex $\boldsymbol{H}$ within the simplex $\boldsymbol{F}$ spanned by $\boldsymbol{e}^{12}$, $\boldsymbol{e}^{34}$ and $\boldsymbol{x}^{(24)}$ and $\boldsymbol{x}^{(42)}$.

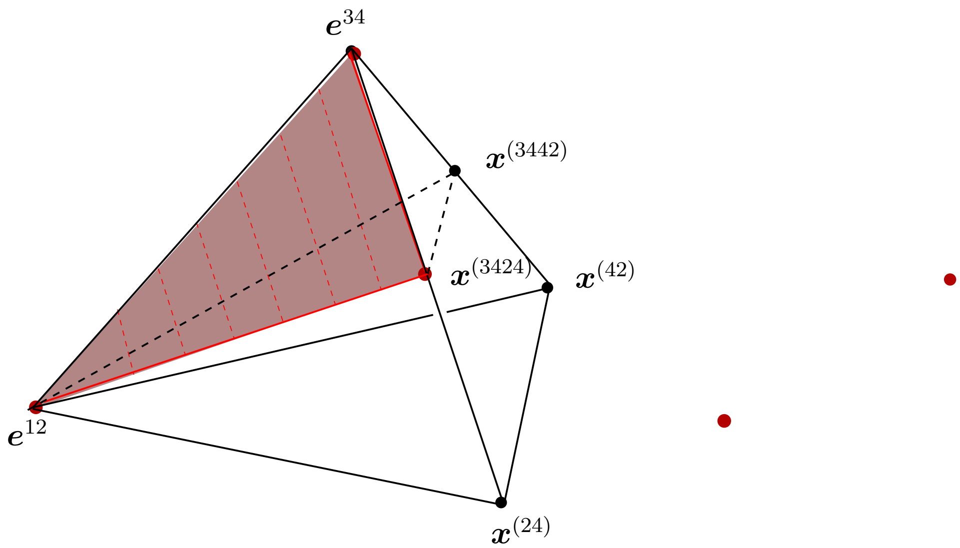

Figure 3.1: The set $\boldsymbol{H}$ viewed within the simplex $\boldsymbol{F}$

Lemma 3.4. *Let $\boldsymbol{\vartheta}$ be an imputation with minimum vector $\boldsymbol{m}$. If $\boldsymbol{a}^{13}\boldsymbol{m} \geq h_3$, then $\boldsymbol{\vartheta}$ cannot be dominated via some $\varepsilon - 13$–relevant coalition.*

The **Proof** is obvious.

We continue with similar considerations concerning $\varepsilon - 123$–relevant coalitions.

Lemma 3.5. *The following inequalities and equations hold true:*

$$
\begin{array}{rcl}
\boldsymbol{a}^{123}\boldsymbol{e}^{12} & = & \boldsymbol{a}^{123}\boldsymbol{e}^{34} \;=\; h_2 \\
\boldsymbol{a}^{123}\boldsymbol{x}^{(42)} & = & 1 - h_3 < h_2 \\
\boldsymbol{a}^{123}\boldsymbol{x}^{(24)} & < & h_2 \\
 & & \textit{which implies} \\
\boldsymbol{a}^{123}\bar{\boldsymbol{x}}^{3442} & < & h_2 \\
\boldsymbol{a}^{123}\bar{\boldsymbol{x}}^{3424} & < & h_2 \quad .
\end{array}
\tag{3.15}
$$

Proof: The first three lines are directly computed and the last two lines follow from the fact that $\boldsymbol{x}^{3442}$ is a "convex" combination of $\boldsymbol{e}^{34}$ and $\boldsymbol{x}^{(42)}$.

q.e.d.

The following Lemma refers to the projection of $\boldsymbol{F}$ on $\mathbb{R}_{123}$.

Lemma 3.6. *Let* $1 - h_3 \leq \xi \leq h_2$ *and define*
(3.16)
$$
\Delta_0{}^{\xi} \;:=\; \boldsymbol{ConvH}\{(1,1,0),(0,0,1),(0,\xi,h_3)\} \;=\; \boldsymbol{ConvH}\left\{\boldsymbol{e}^{12}_{\mathbb{R}_{123}},\boldsymbol{e}^{34}_{\mathbb{R}_{123}},(0,\xi,h_3)\right\}
$$

There is no pair $\boldsymbol{x},\boldsymbol{y} \in \Delta_0{}^{\xi}$ *satisfying*

$$
x_\tau > y_\tau \quad (\tau = 1,2,3) \quad . \tag{3.17}
$$

Proof: The normal $\mathfrak{n}$ to the hyperplane spanned by the vectors $\boldsymbol{e}^{12}$,$(0,0,1)$, and $(0,\xi,h_3)$ is positive. Indeed, up to normalization it is given by $\mathfrak{n} = (h_3 + \xi - 1, 1 - h_3, \xi)$ such that $\mathfrak{n}\boldsymbol{e}^{12} \;=\; \mathfrak{n}(0,0,1) \;=\; \mathfrak{n}(0,\xi,h_3) \;=\; \xi$ holds true. Hence, for any two vectors $\boldsymbol{x},\boldsymbol{y} \in \Delta_0{}^{\xi}$ we have $\mathfrak{n}\boldsymbol{x} = \mathfrak{n}\boldsymbol{y} \;=\; \xi$, that is, $\mathfrak{n}\boldsymbol{x} - \mathfrak{n}\boldsymbol{y} \;=\; \boldsymbol{0}$. Therefore, whenever $\boldsymbol{x} - \boldsymbol{y} \neq 0$, then $\boldsymbol{x} - \boldsymbol{y}$ must have positive and negative coordinates,

q.e.d.

With due caution the shape of $\Delta_0{}^{\xi}$ can be visualized by Figure 3.7 for $\xi = h_2$ and $\xi = 1-h_3$. The intermediate values of ξ can be imagined to yield similar versions of Δ_0^{ξ} Observe that the normal to these simplices is always positive.

Definition 3.7. *An imputation* $\boldsymbol{\vartheta}$ *is said to be* ***vNM–extremal*** *if*

$$
\begin{array}{rcll}
\boldsymbol{\vartheta} & = & 0 & on \boldsymbol{D}^1 \\
(1-h_3) \leq \boldsymbol{\vartheta} & \leq & h_2 & on \boldsymbol{D}^2 \\
\boldsymbol{\vartheta} & = & h_3 & on \boldsymbol{D}^3 \\
\boldsymbol{\vartheta} & = & 1 & on \boldsymbol{D}^4
\end{array}
\tag{3.18}
$$

That is, $\boldsymbol{\vartheta}$ coincides with $\boldsymbol{x}^{(3424)}$ on $\boldsymbol{D}^1, \boldsymbol{D}^3, \boldsymbol{D}^4$ and the remaining mass is distributed in $\boldsymbol{D}^2$ such that the values are located in Δ_0. In particular we

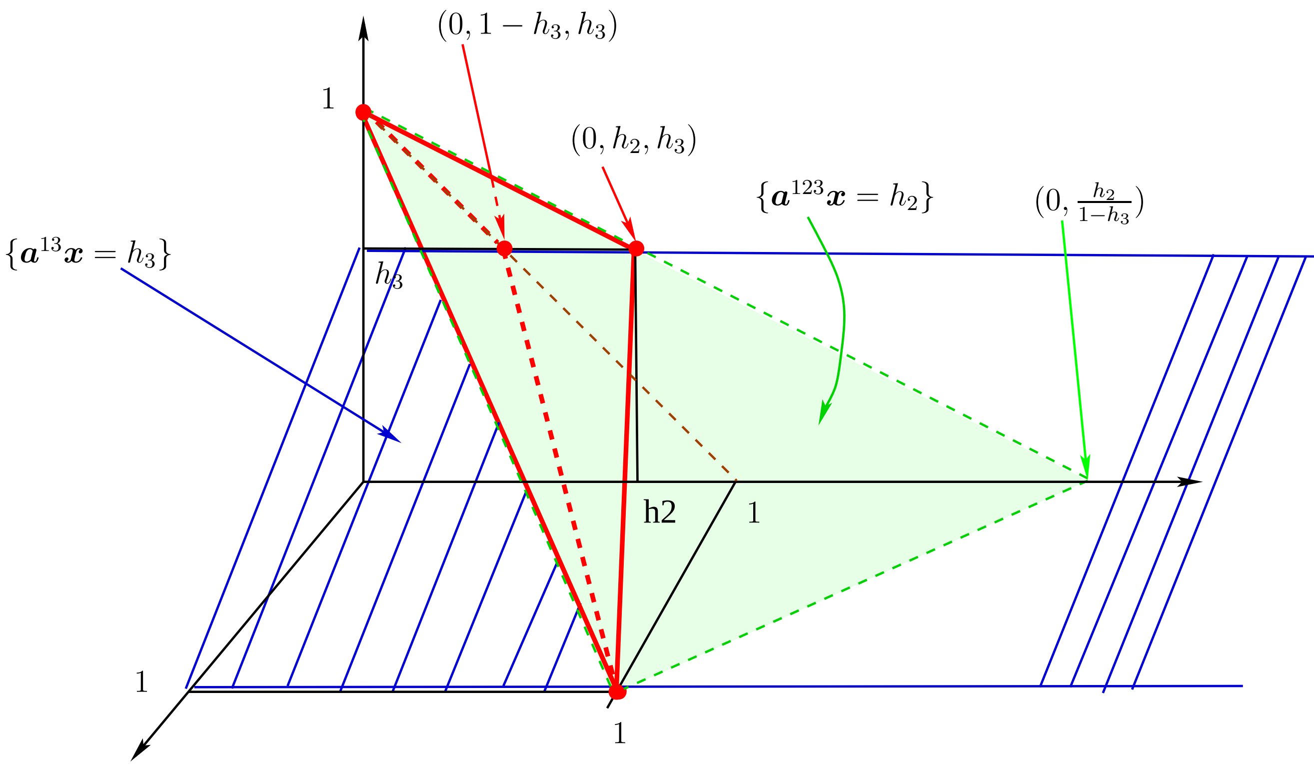

Figure 3.2: The Simplex ${\Delta_0}^{\xi}$

have

$$\int\limits_{\boldsymbol{D}^2} \boldsymbol{\vartheta} d\boldsymbol{\lambda} = \boldsymbol{x}_2^{(3424)} \boldsymbol{\lambda}(\boldsymbol{D}^2) = (1-h_3)\lambda_3$$

cf. Figure 3.3

We are now in the position to specify our candidates for vNM–Stable Sets. The typical representative is provided via some vNM–extremal imputation $\overline{\boldsymbol{\vartheta}}$ via $\overline{\mathcal{G}} := \mathbf{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \overline{\boldsymbol{\vartheta}}\right\}$. That is, we choose a candidate for a vNM–Stable Set by specifying a third extremal apart from the two extremals of the core.

Example 3.8. We can represent this candidate for the particular case that we have the (unique) vNM–extremal imputation that is constant on $\boldsymbol{D}^2$. This is the imputation $\boldsymbol{\vartheta}^{(3424)} := \boldsymbol{\vartheta}^{\boldsymbol{x}^{(3424)}}$. In this case we have a candidate for a vNM–Stable Set given by

(3.19)

$$\begin{aligned} \boldsymbol{G}^{(3424)} &:= \mathbf{ConvH}\left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \boldsymbol{x}^{(3424)}\right\} &= \boldsymbol{G}^0 \\ \mathcal{G}^{(2434)} &:= \left\{\boldsymbol{\vartheta}^{\boldsymbol{x}} \,\middle|\, \boldsymbol{x} \in \boldsymbol{G}^{(3424)}\right\} = \mathbf{ConvH}\left\{\boldsymbol{\lambda}^1,\ \boldsymbol{\lambda}^2, \boldsymbol{\vartheta}^{(3424)}\right\} &= \mathcal{G}^0\ ; \end{aligned}$$

the first set is sketched in Figure 3.4.

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For the sake of lucidity we shall treat the (vNM–Stable) set provided by this example separately in a first approach. The general version will then

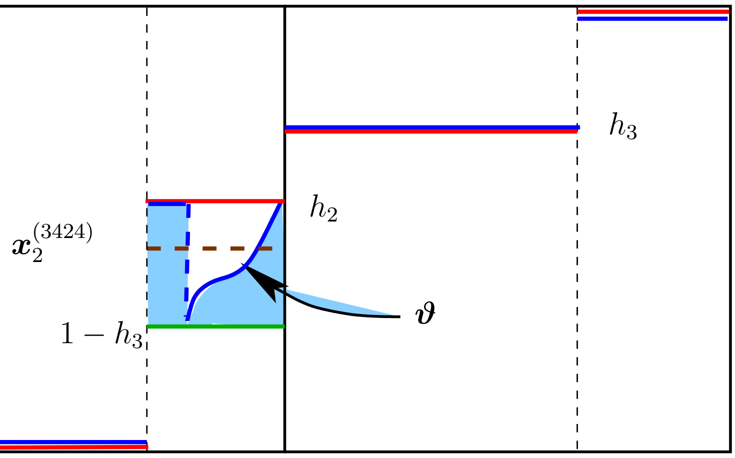

Figure 3.3: A vNM–extreme imputation

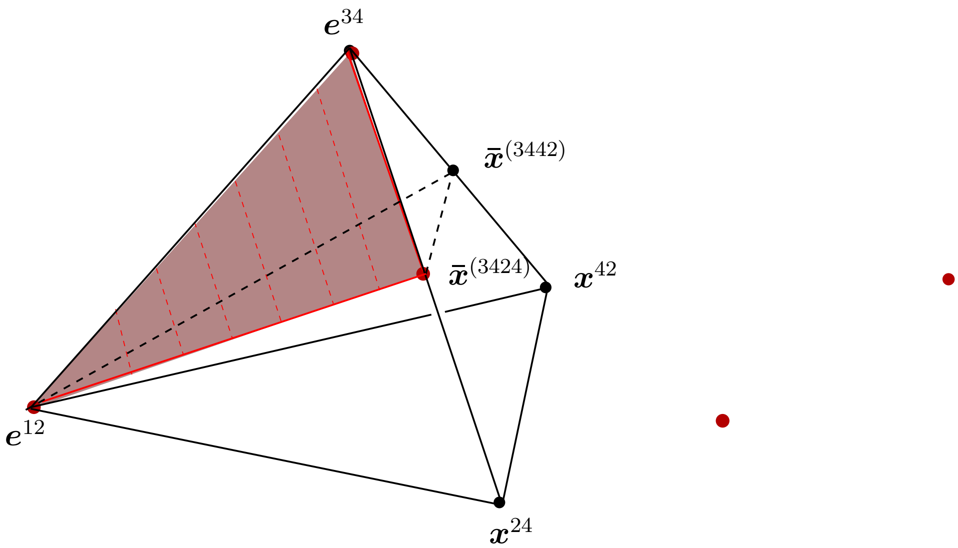

Figure 3.4: The set $\boldsymbol{G}^{(3424)}$

be easier understood and is completely treated in the subsequently following theorems.

Thus we start out with (3.19) using abbreviations $\boldsymbol{G}^0$ and $\mathcal{G}^0$.

Theorem 3.9.

$$\boldsymbol{G}^0 = \boldsymbol{ConvH}\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \boldsymbol{x}^{(3424)}\} \tag{3.20}$$

is internally pre–stable, hence

$$\mathcal{G}^0 \ := \ \boldsymbol{ConvH}\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\vartheta}^{3424}\} \tag{3.21}$$

is internally stable.

Proof:

We have to check with respect to $\varepsilon - 13$– and $\varepsilon - 123$–relevant coalitions only. Now Lemma 3.4 shows immediately that internal domination cannot take place with respect to $\varepsilon - 13$–relevant coalitions. Geometrically, $\boldsymbol{G}^0$ is a subset of $\boldsymbol{H}$ as discussed in the context of that Lemma. In other words, all elements of $\boldsymbol{G}^0$ are located above the hyperplane $\boldsymbol{H}_0$, hence satisfy $\boldsymbol{a}^{13}\boldsymbol{x} \geq h_3$ which implies for the corresponding imputation $\boldsymbol{\vartheta}^{\boldsymbol{x}}(\boldsymbol{T}^{13}) \geq \varepsilon h_3 = \boldsymbol{\lambda}(\boldsymbol{T}^{13})$ whenever $\boldsymbol{T}^{13}$ is an $\varepsilon - 13$–coalition.

It remains to discuss $\varepsilon - 123$–relevant coalitions. As $\boldsymbol{G}^0$ has the shape indicated in Figure 3.4, we have $\boldsymbol{a}^{123}\boldsymbol{x} < h_2$ for all $\boldsymbol{x} \in \boldsymbol{G}^0$ outside the core. However, for all $\boldsymbol{x} \in \boldsymbol{G}^0$ we have in this case $x_4 = 0$. Therefore, if for some $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{G}^0$ and some $\varepsilon - 123$–coalition $\boldsymbol{T}^{123}$ we have $\boldsymbol{\vartheta}^{\boldsymbol{x}} \operatorname{dom}_{\boldsymbol{T}^{123}} \boldsymbol{\vartheta}^{\boldsymbol{y}}$, then necessessarily $x_\tau > y_\tau$ for $\tau = 1, 2, 3$ which, by Corollary 3.6 is not possible.

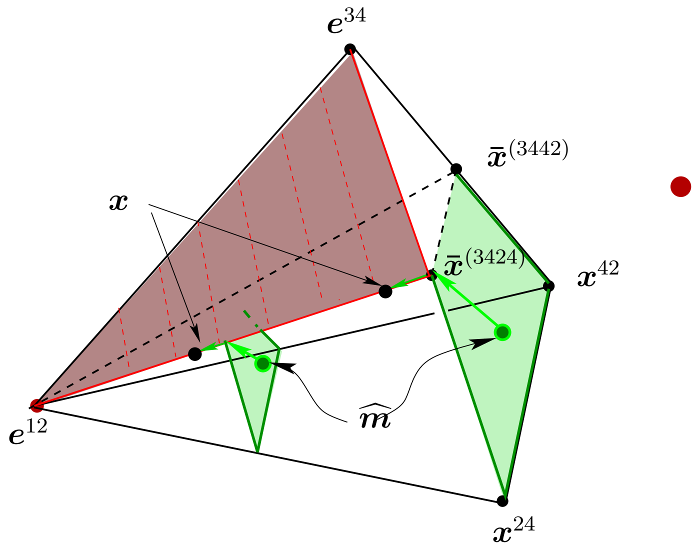

Figure 3.5: External dominance by $\boldsymbol{G}^{(3424)}$ via $\varepsilon - 13$–coalitions

q.e.d.

Thus internal stability is being delt with. Now we follow up with

Theorem 3.10.

$$G^0 = \boldsymbol{ConvH}\left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \boldsymbol{x}^{(3424)}\right\} \tag{3.22}$$

is externally pre–stable, hence

$$\mathcal{G}^0 := \boldsymbol{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\vartheta}^{3424}\right\} \tag{3.23}$$

is externally stable.

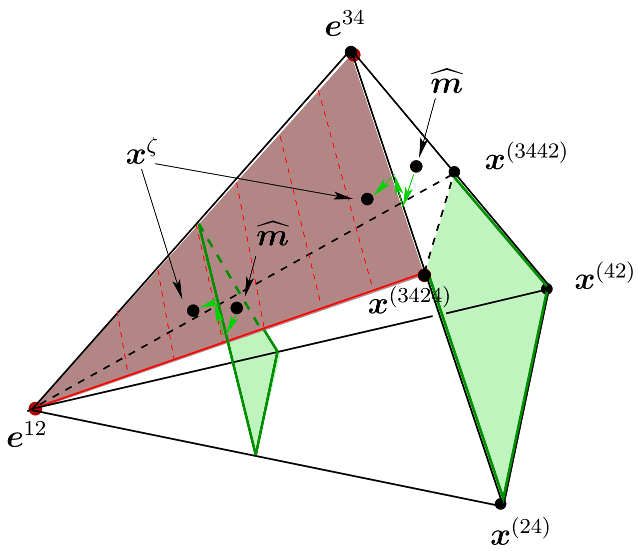

Figure 3.6: External dominance by $\boldsymbol{G}^{(3424)}$ via $\varepsilon - 123$–coalitions

Proof:

Let $\widehat{\boldsymbol{\vartheta}} \notin \overline{\mathcal{G}}^0$ be an imputation and let $\widehat{\boldsymbol{m}} = (\widehat{m}_1, \ldots, \widehat{m}_4)$ be it minima vector. W.l.o.g. we may assume $\widehat{\boldsymbol{m}} \in \boldsymbol{F}$ holds true, otherwise there is a vector in $\boldsymbol{F}$ exceeding $\widehat{\boldsymbol{m}}$ coordinatewise which we can argue about. Then necessarily $\widehat{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}^{\widehat{\boldsymbol{m}}}$. Clearly $\widehat{\boldsymbol{m}} \in \boldsymbol{H} = \mathbf{ConvH}\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \boldsymbol{x}^{(3424)}, \boldsymbol{x}^{(3442)}\}$ for otherwise $\widehat{\boldsymbol{\vartheta}}$ is dominated by the core. Thus $\widehat{\boldsymbol{m}} \in \boldsymbol{H} \setminus \boldsymbol{G}^0$.

The procedure is depicted in Figure 3.6. Technically we deal only with the case that $\widehat{\boldsymbol{m}}$ has a zero first coordinate,the general case can obviously be discussed similarly (just move $\widehat{\boldsymbol{m}}$ towards $\boldsymbol{e}^{14}$ - there is another vector denoted by $\widehat{\boldsymbol{m}}$ depicted in the sketch). Within this context we now construct a dominance via some $\varepsilon - 123$ – coalition. Compare Figure 3.6 for the details.

Observing

$$\widehat{\boldsymbol{m}} \in \{\boldsymbol{x} \in \boldsymbol{H} \,\big|\, x_1 = 0\} = \mathbf{ConvH}\left\{\boldsymbol{e}^{34}, \boldsymbol{x}^{(3424)}, \boldsymbol{x}^{(3442)}\right\} .$$

we move $\widehat{\boldsymbol{m}}$ parallel to the line $[\boldsymbol{x}^{3442}, \boldsymbol{x}^{3424}]$ towards the line $[\boldsymbol{e}^{12}, \boldsymbol{x}^{3424}]$. Then the coordinate x_1 is unchanged $(= 0)$ and so is coordinate x_3 as $\boldsymbol{x}^{3442}$ and $\boldsymbol{x}^{3424}$ have the same 3^{rd} coordinate.

Thus, necessarily, the second coordinate is increased (again compare the situation for $\boldsymbol{x}^{3442}$ and $\boldsymbol{x}^{3424}$). Let the resulting vector be $\bar{\boldsymbol{x}}$. Now, we move in direction of $\boldsymbol{e}^{34}$ for some small $\delta > 0$ to some vector called $\boldsymbol{x}^{\delta}$ such that the second coordinate of $\boldsymbol{x}^{\delta}$ is still exceeding the second coordinate of $\bar{\boldsymbol{x}}$ and $\widehat{\boldsymbol{m}}$ and, in addition, the 3^{rd} coordinate has been increased. Finally we move from $\boldsymbol{x}^{\delta}$ in direction of $\boldsymbol{e}^{12}$ for some small $\zeta > 0$ to some vector called $\boldsymbol{x}^{\zeta}$ such that the second and third coordinate still exceed the ones of $\widehat{\boldsymbol{m}}$ and the first one has been increased in addition. Thus, $\boldsymbol{x}^{\zeta}$ exceeds $\widehat{\boldsymbol{m}}$ with respect to the coordinates $1, 2, 3$ and still – as we have an element of $\boldsymbol{G}$ – yields $\boldsymbol{a}^{123}\boldsymbol{y}^{\zeta} < h_2$ in view of Lemma 3.5. Consequently

$$\boldsymbol{x}^{\zeta} \operatorname{dom}_{\boldsymbol{a}^{123}} \widehat{\boldsymbol{m}} \ ,$$

q.e.d.

So far we have completed the treatment of

$$\mathcal{G}^0 \ = \ \textbf{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\vartheta}^{3424}\right\} \ = \ \mathcal{G}^{3424} \ . \tag{3.24}$$

Let us now turn to the general case of a vNM–extremal imputation generating a vNM–Stable Set.

Theorem 3.11. *Let $\bar{\boldsymbol{\vartheta}}$ be a vNM–extremal imputation. Then*

$$\overline{\mathcal{G}} \ := \ \boldsymbol{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \bar{\boldsymbol{\vartheta}}\right\} \tag{3.25}$$

is internally stable.

Proof: We have to check for $\varepsilon - 13$– and $\varepsilon - 123$–relevant coalitions only. Now, internal stability against domination by $\varepsilon - 13$–coalitions follows at once from Lemma 3.4. For $\overline{\boldsymbol{\vartheta}}$ has constant densities 0 and h_3 on $\boldsymbol{D}^1$ and $\boldsymbol{D}^3$ respectively, hence satisfies the conditions of that lemma. The same is true for $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$, hence for any convex combination $\boldsymbol{\vartheta} \in \overline{\mathcal{G}}$

Hence we have to show that $\overline{\mathcal{G}}$ is internally stable against domination via some $\varepsilon - 123$–coalition.

Now,for any $\boldsymbol{\vartheta} \in \overline{\mathcal{G}}$ and $t \in \boldsymbol{I}$ we have

$$\dot{\boldsymbol{\vartheta}}(t) \in \textbf{ConvH}\left\{\dot{\boldsymbol{\lambda}}^1(t),\ \dot{\boldsymbol{\lambda}}^2(t),\ \dot{\overline{\boldsymbol{\vartheta}}}(t)\right\} \quad .$$

Thus, in particular for any $\varepsilon - 123$–relevant coalition T^{123} and for any $t_\tau \ \in T^\tau (\tau = 1, 2, 3)$ we have

$$\begin{aligned}
\boldsymbol{y} \ := \ (\dot{\boldsymbol{\vartheta}}(t_1), \dot{\boldsymbol{\vartheta}}(t_2), \dot{\boldsymbol{\vartheta}}(t_3)) \in \textbf{ConvH}&\left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \left(\dot{\overline{\boldsymbol{\vartheta}}}(t_1), \dot{\overline{\boldsymbol{\vartheta}}}(t_2), \dot{\overline{\boldsymbol{\vartheta}}}(t_3)\right)\right\} \\
= \ \textbf{ConvH}&\left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \left(0, \dot{\overline{\boldsymbol{\vartheta}}}(t_2), h_3\right)\right\} \\
= \ \textbf{ConvH}&\left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, (0, \xi, h_3)\right\}
\end{aligned}$$

with $1-h_3 \leq \xi \leq h_2$. Now, whenever we have some domination $\widetilde{\boldsymbol{\vartheta}}\,\mathrm{dom}_{\{T^{123}\}}\,\boldsymbol{\vartheta}$ for some $\widetilde{\boldsymbol{\vartheta}}$, then for the same $t_\tau \in T^\tau (\tau = 1,2,3)$ the vector

$$\boldsymbol{x} \ := \ (\dot{\widetilde{\boldsymbol{\vartheta}}}(t_1), \dot{\widetilde{\boldsymbol{\vartheta}}}(t_2), \dot{\widetilde{\boldsymbol{\vartheta}}}(t_3))$$

satisfies $\boldsymbol{x} > \boldsymbol{y}$ and $\boldsymbol{x} \in \mathbf{ConvH}\left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, (0,\xi,h_3)\right\}$, contradicting Lemma 3.6.

q.e.d.

Theorem 3.12. *Let $\bar{\boldsymbol{\vartheta}}$ be a vNM–extremal imputation. Then*

$$\overline{\mathcal{G}} \ := \ \boldsymbol{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \bar{\boldsymbol{\vartheta}}\right\} \tag{3.26}$$

is externally stable, hence a vNM–Stable Set.

Proof:

Let $\boldsymbol{\vartheta}^0 \notin \overline{\mathcal{G}}$ be an imputation. We have to show that it is dominated by an element from $\overline{\mathcal{G}}$.

1$^{\text{st}}$STEP :

We prove the following statement:

(3.27) Without loss of generality $\boldsymbol{\vartheta}^0$ is constant on $\boldsymbol{D}^1 \cup \boldsymbol{D}^3 \cup \boldsymbol{D}^4$.

Indeed , let $\underline{\boldsymbol{m}}^0 = (\underline{m}_1^0, \underline{m}_1^0, \underline{m}_2^0, \underline{m}_3^0, \underline{m}_4^0)$ denote the minima vector of $\boldsymbol{\vartheta}^0$. Consider the imputation $\boldsymbol{\vartheta}^1$ that is obtained from $\boldsymbol{\vartheta}^0$ by collecting all mass above the minima on $\boldsymbol{D}^\tau$ $(\tau = 1,3,4)$ on $\boldsymbol{D}^2$. That is, consider

$$\boldsymbol{\vartheta}^1 \ := \ \underline{m}_1^0 \mathbb{1}_{\boldsymbol{D}^1} + (\boldsymbol{\vartheta}^0 + c)\mathbb{1}_{\boldsymbol{D}^2} + \underline{m}_3^0 \mathbb{1}_{\boldsymbol{D}^3} + \underline{m}_4^0 \mathbb{1}_{\boldsymbol{D}^4}$$

with

$$c \ := \int\limits_{\boldsymbol{D}^1 \cup \boldsymbol{D}^3 \cup \boldsymbol{D}^4} \boldsymbol{\vartheta}^0 d\boldsymbol{\lambda} - \sum_{\tau=1,3,4} \underline{m}_\tau^0 \lambda_\tau \quad .$$

If $\boldsymbol{\vartheta}^0$ does not satisfy 3.27, then $c > 0$ and $\boldsymbol{\vartheta}^1$ exceeds $\boldsymbol{\vartheta}^0$ on $\boldsymbol{D}^2$. Now, if some $\overline{\boldsymbol{\vartheta}}$ dominates $\boldsymbol{\vartheta}^1$ via some ε–relevant coalition – for example some $\varepsilon - 123$–relevant coalition T^{123} – then cleary $\overline{\boldsymbol{\vartheta}}$ exceeds $\boldsymbol{\vartheta}^0$ on $T^2 \subseteq \boldsymbol{D}^2$. Moreover, as both, $\overline{\boldsymbol{\vartheta}}$ and $\boldsymbol{\vartheta}^1$ are constant outside $\boldsymbol{D}^2$, it follows that $\overline{\boldsymbol{\vartheta}}$ exceeds $\boldsymbol{\vartheta}^1$ everywhere on $\boldsymbol{D}^1 \cup \boldsymbol{D}^3$. Therefore, by definition of the essential minima, $\overline{\boldsymbol{\vartheta}}$ exceeds $\boldsymbol{\vartheta}^0$ on sets $\widetilde{\boldsymbol{D}}^\tau$ $(\tau = 1,3)$ of positive measure. Choosing an $\varepsilon - 123$–coalition $\overline{T}^{123}$ with $\overline{T}^2 \subseteq T^2$ and $\overline{T}^\tau \subseteq \widetilde{T}^\tau$ $(\tau = 1,3)$ yields a domination $\overline{\boldsymbol{\vartheta}}\,\mathrm{dom}_{\overline{T}^{123}}\,\boldsymbol{\vartheta}^0$.

The same consideration holds true *a fortiori* for domination via some $\varepsilon - 13$–relevant coalition.

Henceforth we assume (3.27) and denote the constant values of $\boldsymbol{\vartheta}^0$ on $\boldsymbol{D}^\tau$ by x_τ^0 for $\tau = 1,3,4$.

2ndSTEP :

Our next claim is

$$\text{Without loss of generality } x^0_1 + x^0_4 = 1 \quad . \tag{3.28}$$

Indeed, first of all we have necessarily $x^0_1 \leq 1$.

For, consider the vector $\underline{\boldsymbol{m}} = (x^0_1, \underline{m}^0_2, x^0_3, x^0_4)$. If it is not a pre–imputation, construct a pre–imputation $\widetilde{\underline{\boldsymbol{m}}}$ exceeding $\underline{\boldsymbol{m}}$ coordinatewise. If one of the inequalities defining $\boldsymbol{F}$ is violated, then we have immediately a dominance by the core (implying at once a domination of $\boldsymbol{\vartheta}^0$ by the core). Hence $\widetilde{\underline{\boldsymbol{m}}} \in \boldsymbol{F}$. But for all the extremals of $\boldsymbol{F}$ we have a first coordinate not exceeding one. Hence $\widetilde{\underline{m}}_1 \leq 1$ and *a fortiori* $\underline{m}_1 = x^0_1 \leq 1$.

Next, assume that $x^0_1 + x^0_4 > 1$ is true. ($<$ would constitute a domination by the core). Let

$$t \;:=\; \frac{(x^0_1 + x^0_4) - 1}{\frac{\lambda_1}{\lambda_4} - 1} \;,$$

then

$$x^0_4 - t(\frac{\lambda_1}{\lambda_4} - 1) \;=\; x^0_4 - ((x^0_1 + x^0_4) - 1) \;=\; 1 - x_0 \leq 1 \;.$$

Consider the (signed) measure $\boldsymbol{\eta}$ given by the density

$$\overset{\bullet}{\boldsymbol{\eta}} \;:=\; \left(\frac{\lambda_3}{\lambda_2} - 1\right) \mathbb{1}_{\boldsymbol{D}^2} - \left(\frac{\lambda_1}{\lambda_4} - 1\right) \mathbb{1}_{\boldsymbol{D}^4} \quad .$$

with $\int_{\boldsymbol{I}} \boldsymbol{\eta} d\boldsymbol{\lambda} = 0$ as well as

$$\widehat{\boldsymbol{\vartheta}} \;:=\; \boldsymbol{\vartheta}^0 + t\boldsymbol{\eta} \quad .$$

Then $\widehat{\boldsymbol{\vartheta}}$ is a preimputation as the last coordinate exceeds 0. Also, $\widehat{\boldsymbol{\vartheta}}$ is constant on $\boldsymbol{D}^1 \cup \boldsymbol{D}^4$ with values summing up to one:

$$x^0_1 + x^0_4 - t(\frac{\lambda_1}{\lambda_4} - 1) \;=\; x^0_1 + x^0_4 - ((x^0_1 + x^0_4) - 1) \;=\; 1 \;.$$

Now, domination of either $\boldsymbol{\vartheta}^0$ or $\widehat{\boldsymbol{\vartheta}}$ via some $\varepsilon - 14$–relevant coalition is impossible. With respect to any other form of domination – e.g. domination via some ε–123–relevant coalition – clearly $\widetilde{\boldsymbol{\vartheta}}\,\mathrm{dom}_{T^{123}}\,\widehat{\boldsymbol{\vartheta}}$ implies $\widetilde{\boldsymbol{\vartheta}}\,\mathrm{dom}_{\overline{T}^{123}}\,\boldsymbol{\vartheta}^0$. That is, we can as well replace $\boldsymbol{\vartheta}^0$ by $\widehat{\boldsymbol{\vartheta}}$ or just assume (3.28) in the first place.

3rdSTEP : Our next claim is

$$\text{Without loss of generality } x^0_1 = 0 \quad . \tag{3.29}$$

Assume $x^0_1 > 0$ holds true. If $x^0_2 + x^0_4 < 1$, then $\boldsymbol{\vartheta}^0$ is dominated by the core. Hence (using (3.28)) $x^0_2 \geq 1 - x^0_4 = x^0_1$. Therefore

$$\boldsymbol{\vartheta}' \;:=\; \frac{\boldsymbol{\vartheta}^0}{1 - x_1} - \frac{x^0_1}{1 - x^0_1}\boldsymbol{\lambda}^1$$

is nonnegative, hence an imputation which equals 0 on $\boldsymbol{D}^1$. As

$$\boldsymbol{\vartheta}^0 = (1 - x_1^0)\boldsymbol{\vartheta}' + x_1^0 \boldsymbol{\lambda}^1$$

we observe that any domination of $\boldsymbol{\vartheta}'$ by some $\overline{\boldsymbol{\vartheta}}$ induces a domination of $\boldsymbol{\vartheta}^0$ by $(1 - x_1^0)\overline{\boldsymbol{\vartheta}} + x_1^0 \boldsymbol{\lambda}^1$. Hence we may assume (3.29) at once.

4$^{\text{th}}$STEP :

Similarly we prove

(3.30) Without loss of generality $x_3^0 = h_3$.

For in view of (3.28) and (3.29) we have now $x_4^0 = 1$. Now, an inequality $x_3^0 > 1$ would imply $\boldsymbol{\vartheta}^0(\boldsymbol{I}) \geq x_3^0 \lambda_3 + x_4^0 \lambda_4 > \lambda_3 + \lambda_4 = 1$, hence $x_3^0 \leq x_4^0$. Consequently

$$\boldsymbol{\vartheta}'' := \frac{\boldsymbol{\vartheta}^0}{1 - (x_3 - h_3)} - \frac{x_1^0 - h_3}{1 - (x_3^0 - h_3)} \boldsymbol{\lambda}^1$$

is an imputation which equals h_3 on $\boldsymbol{D}^3$. As

$$\boldsymbol{\vartheta}^{00} = (1 - (x_3^0 - h_3))\boldsymbol{\vartheta}'' + (x_1^0 - h_3)\boldsymbol{\lambda}^1$$

we observe that any domination of $\boldsymbol{\vartheta}''$ by some $\overline{\boldsymbol{\vartheta}}$ induces a domination of $\boldsymbol{\vartheta}^0$ by $(1 - (x_3^0 - h_3))\overline{\boldsymbol{\vartheta}} + (x_3^0 - h_3)\boldsymbol{\lambda}^1$.

5$^{\text{th}}$STEP : By the first 4 *STEPs* we can now assume that $\boldsymbol{\vartheta}^0$ is vNM extremal. Clearly, both $\bar{\boldsymbol{\vartheta}}$ and$\boldsymbol{\vartheta}^0$ differ only on $\boldsymbol{D}^2$. Choose $T^2 \subsetneq \boldsymbol{D}^2$ of positive measure such that

$$\bar{\boldsymbol{\vartheta}} > \boldsymbol{\vartheta}^0 \text{ on } T^2.$$

Then, for $\delta < 1 - h_3$ the we have

$$\boldsymbol{e}^\delta := \delta \boldsymbol{e}^{12} + (1 - \delta)\boldsymbol{e}^{34} > \boldsymbol{\vartheta}^0 \text{ on } \boldsymbol{D}^1 \cup \boldsymbol{D}^3 .$$

Hence, for sufficiently small $\varepsilon > 0$ we have necessarily

$$\boldsymbol{\vartheta}^{\delta\varepsilon} := (1 - \varepsilon)\overline{\boldsymbol{\vartheta}} + \varepsilon \boldsymbol{e}^\delta > \boldsymbol{\vartheta}^0 \text{ on } \boldsymbol{D}^1 \cup T^2 \cup \boldsymbol{D}^3$$

Choosing an $\varepsilon - 123$–relevant coalition $\overline{T}^{123} \subseteq \boldsymbol{D}^1 \cup T^2 \cup \boldsymbol{D}^3$ obviously constitutes a domination $\boldsymbol{\vartheta}^{\delta\varepsilon} \operatorname{dom}_{T^{123}} \boldsymbol{\vartheta}^0$. This completes the proof of our theorem.

q.e.d.

Thus we have pointed out a class of vNM–Stable sets, each of them being constructed by picking exactly one vNM–extremal imputation in the sense of Definition 3.7 (Figure 3.3) and constructing the convex hull with the core. We now set out to show that Theorem 3.12 provides a complete characterization of all vNM–Stable Sets.

To this end let us define

$$\Delta_0 \ := \ \boldsymbol{J}_0 \mid_{\mathbb{R}^4_{\{123\}}} \bigcap \{\boldsymbol{x} \in \mathbb{R}^3_+ \mid \boldsymbol{a}^{13}\boldsymbol{x} \geq h_3. \ \ \boldsymbol{a}^{123}\boldsymbol{x} \leq h_2\}$$

The set Δ_0 constitutes a simplex in $\mathbb{R}^3_+$ the extremals of which are given by

$$(1,1,0), \ \ (0,0,1), \ \ (0,1-h_3,h_3), \text{ and } (0,h_2,h_3) \ \ ,$$

see Figure 3.7. Note that the set $\Delta_0{}^{\xi}$ as considered in Lemma 3.6 is a lower dimensional subsimplex of Δ_0.

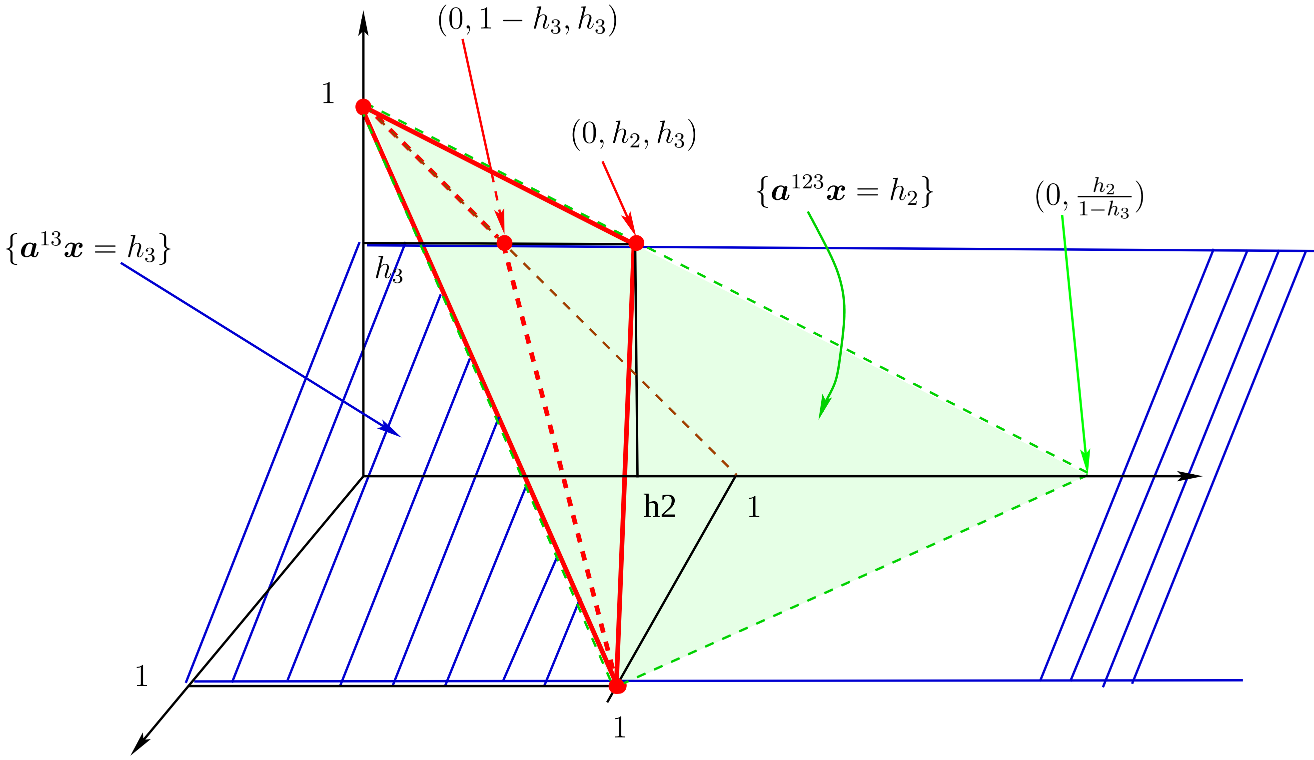

Figure 3.7: The Simplex Δ_0

Lemma 3.13. *Let $\widehat{\mathfrak{G}}$ be a convex vNM–Stable set and let $\overline{\boldsymbol{\vartheta}} \in \widehat{\mathfrak{G}}$. Let $\underline{\boldsymbol{m}} = (\underline{m}_1, \underline{m}_2, \underline{m}_3, \underline{m}_4)$ denote the essential minima of $\overline{\boldsymbol{\vartheta}}$. Also, assume that $\overline{\boldsymbol{\vartheta}}$ dominates some imputation via some $\varepsilon - 123$–relevant coalition T^{123}.*

1. *Then*

$$\begin{array}{c} \underline{m}_1 + \underline{m}_4 \geq 1, \ \ \underline{m}_2 + \underline{m}_4 \geq 1, \ \ \underline{m}_2 + \underline{m}_3 \geq 1, \\ \boldsymbol{a}^{13}\underline{\boldsymbol{m}} \geq h_3, \quad \boldsymbol{a}^{123}\underline{\boldsymbol{m}} \leq h_2, \quad \underline{m}_1 \leq \underline{m}_2. \end{array} \tag{3.31}$$

   In addition, the projection $\underline{\boldsymbol{m}}^{123}$ of $\underline{\boldsymbol{m}}$ onto $\mathbb{R}_{123}$ satisfies

$$\underline{m} \in \Delta_0. \tag{3.32}$$

2. *For sufficiently small* $0 < \hat{\varepsilon} < \varepsilon$ *there exists an* $\hat{\varepsilon}-123$*–relevant coalition* $\widehat{T}^{123} \subseteq T^{123}$ *such that for all* $t_\tau \in T^\tau \quad (\tau = 1, 2, 3)$

$$(\overset{\bullet}{\boldsymbol{\vartheta}}(t_1), \overset{\bullet}{\boldsymbol{\vartheta}}(t_2), \overset{\bullet}{\boldsymbol{\vartheta}}(t_3)) \in \Delta_0 \quad , \overset{\bullet}{\boldsymbol{\vartheta}}(t_1) \leq \overset{\bullet}{\boldsymbol{\vartheta}}(t_2) \tag{3.33}$$

holds true.

3. *If, in addition,* $\underline{m}_1 = 0$ *holds true, then*

$$\underline{m}_4 \geq 1, \quad m_3 = \boldsymbol{a}^{13}\underline{\boldsymbol{m}} \geq h_3 \ . \tag{3.34}$$

4. *Next, if* $\underline{m}_3 \geq h_3$*, then, for* $\widehat{T}^{1231}$ *chosen by* item2,

$$\overline{\overset{\bullet}{\boldsymbol{\vartheta}}} \leq h_2 \ \text{on} \ \widehat{T}^2 \ . \tag{3.35}$$

5. *Finally, if* $\underline{m}_1 = 0, \underline{m}_3 = h_3$*, then*

$$\underline{m}_2 \geq 1 - h_3, \quad \text{that is } 1 - h_3 \leq \overline{\overset{\bullet}{\boldsymbol{\vartheta}}} \leq h_2 \ \text{on} \ \hat{T}^2 \quad . \tag{3.36}$$

holds true.

Proof:

1$^{\text{st}}$STEP :

The vector $\underline{\boldsymbol{m}}$ has to satisfy the inequalities defining $\boldsymbol{F}$ as listed in (3.7). Otherwise a domination via the core would occur. Also, it has to satisfy the inequality $\boldsymbol{a}^{13}\underline{\boldsymbol{m}} \geq h_3$, otherwise it would be pre–dominated by itself and the core (Theorem 4.4. of [2]). Moreover we have $\boldsymbol{a}^{123}\underline{\boldsymbol{m}} \leq h_2$ in view of Lemma 3.5. The inequalities are thus

$$\begin{gathered} \underline{m}_1 + \underline{m}_4 \geq 1, \ \underline{m}_2 + \underline{m}_4 \geq 1, \ \underline{m}_2 + \underline{m}_3 \geq 1, \\ \boldsymbol{a}^{13}\underline{\boldsymbol{m}} \geq h_3, \quad \boldsymbol{a}^{123}\underline{\boldsymbol{m}} \leq h_2. \end{gathered} \tag{3.37}$$

That is, the ones required in (3.31) apart from the last one.

The first four of these are the inequalitis defining $\boldsymbol{H}$, thus the projection into $\mathbb{R}_{123}$ has to be located with the convex hull of the projection of the extremals of $\boldsymbol{H}$, that is

$$(1,1,0), (0,0,1), (0, 1 - h_3, h_2), (0, (1 - h_3)\frac{\lambda_3}{\lambda_2}, h_3). \tag{3.38}$$

Moreover, the projection of $\underline{\boldsymbol{m}}$ has to be located below the hyperplane $\{\boldsymbol{x} \,|\, \boldsymbol{a}^{123}\boldsymbol{x} \leq h_2$, which cuts the line $\{\boldsymbol{x} \in \mathbb{R}_{123} \,|\, x_1 = 0, x_3 = h\}$ at $(0, h_2, h_3)$. Thus, the fourth extremal as listed in has second coordinate at most h_2, that is we have

$$\underline{\boldsymbol{m}}_{123} \in \Delta_0. \tag{3.39}$$

As a consequence, the last inequality in (3.31) namely

$$(3.40) \qquad \underline{m}_1 \leq \underline{m}_2$$

follows from the fact that all extremals of Δ_0 as listed above satisfy these inequalities.

2ndSTEP : The $\geq$–inequalities verified for $\underline{\boldsymbol{m}}$ are also satisfed by any tripel

$$(\overset{\bullet}{\boldsymbol{\vartheta}}(t_1), \overset{\bullet}{\boldsymbol{\vartheta}}(t_2), \overset{\bullet}{\boldsymbol{\vartheta}}(t_3))(t_\tau \in \boldsymbol{D}^\tau, \quad \tau = 1, 2, 3).$$

For if one of these inequalities is violated on a set of positive measure, we would again be able to construct an internal domination.

On the other hand, the inequality involving $\boldsymbol{a}^{123}$ is satisfied on a set of positive measure. To see this, denote the averages of $\bar{\boldsymbol{\vartheta}}$

$$\widetilde{m}_\tau \ := \ \frac{\int_{T^\tau} \overset{\bullet}{\bar{\boldsymbol{\vartheta}}} d\boldsymbol{\lambda}}{\boldsymbol{\lambda}(T\tau)} \ = \ \frac{\bar{\boldsymbol{\vartheta}}(T^\tau) d\boldsymbol{\lambda}}{\boldsymbol{\lambda}(T\tau)} \ \ (\tau = 1, 2, 3) \ \ , \widetilde{\boldsymbol{m}} \ := \ (\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3).$$

As $\bar{\boldsymbol{\vartheta}}$ dominates some imputation via some $\varepsilon - 123$–relevant coalition T^{123} we have

$$(3.41) \qquad \bar{\boldsymbol{\vartheta}}(T^{123}) = \int\limits_{T^{123}} \overset{\bullet}{\bar{\boldsymbol{\vartheta}}} d\boldsymbol{\lambda} \leq \boldsymbol{v}(\boldsymbol{I}) \ = \ \varepsilon h_2 \ \ .$$

Consequently

$$(3.42) \qquad \begin{aligned} \varepsilon \boldsymbol{a}^{123} \widetilde{\boldsymbol{m}} \ &= \ \sum_{\tau=1}^{3} \int\limits_{T^\tau} \widetilde{m}_\tau d\boldsymbol{\lambda} \ = \ \sum_{\tau=1}^{3} \int\limits_{T^\tau} \overset{\bullet}{\bar{\boldsymbol{\vartheta}}} d\boldsymbol{\lambda} \\ &= \ \int\limits_{T^{123}} \overset{\bullet}{\bar{\boldsymbol{\vartheta}}} d\boldsymbol{\lambda} \ \leq \ \varepsilon h_2 \ \ , \end{aligned}$$

that is,

$$(3.43) \qquad \boldsymbol{a}^{123} \widetilde{\boldsymbol{m}} \leq h_2 \ \ .$$

Now let $\widetilde{T}^\tau \ := \ \{t \in T\tau \,\big|\, \overset{\bullet}{\bar{\boldsymbol{\vartheta}}} \leq \widetilde{m}_\tau\} \ \ (\tau = 1, 2, 3)$. Then $\boldsymbol{\lambda}(\widetilde{\boldsymbol{D}}\tau) > 0$ and for any $t_\tau \in \widetilde{T}^\tau \ \ (\tau = 1, 2, 3)$ we have

$$(3.44) \qquad \boldsymbol{a}^{123} \underline{\boldsymbol{m}} \leq \boldsymbol{a}^{123} (\overset{\bullet}{\boldsymbol{\vartheta}}(t_1), \overset{\bullet}{\boldsymbol{\vartheta}}(t_2), \overset{\bullet}{\boldsymbol{\vartheta}}(t_3)) \leq \boldsymbol{a}^{123} \widetilde{\boldsymbol{m}} \leq h_2 \ \ .$$

This shows that indeed $(\overset{\bullet}{\boldsymbol{\vartheta}}(t_1), \overset{\bullet}{\boldsymbol{\vartheta}}(t_2), \overset{\bullet}{\boldsymbol{\vartheta}}(t_3)) \in \Delta_0$ holds true for any $t_\tau \in \widetilde{T}^\tau \ \ (\tau = 1, 2, 3)$. Now we can choose an $\hat{\varepsilon}-123$–relevant coalition $\widehat{\boldsymbol{T}}^{123} \subseteq \widetilde{T}^{123}$ in order to establish (3.33) (the inequality $\overset{\bullet}{\boldsymbol{\vartheta}}(t_1) \leq \overset{\bullet}{\boldsymbol{\vartheta}}(t_2)$ follows as above).

3rdSTEP :

The inequalities listen in (3.34) follow immediately by specifying $\underline{m}_1 = 0$ in (3.31).

4thSTEP :

Assume now $\underline{m}_3 \geq h_3$. Choose $\widehat{T}$ as in *item 2* of the Lemma, i.e., as in the $2^{nd}STEP$ above.

By (3.44) we have $\boldsymbol{a}^{123}(\dot{\boldsymbol{\vartheta}}(t_1), \dot{\boldsymbol{\vartheta}}(t_2), \dot{\boldsymbol{\vartheta}}(t_3)) \leq h_2$ for any $t_\tau \in \widehat{T}^\tau$ $(\tau = 1, 2, 3)$. That is

$$((h_2 + h_3) - 1)\dot{\bar{\boldsymbol{\vartheta}}}(t_1) + (1 - h_3)\dot{\bar{\boldsymbol{\vartheta}}}(t_2) + h_2\dot{\bar{\boldsymbol{\vartheta}}}(t_3) \leq h_2$$

so *a fortiori*

$$(1 - h_3)\dot{\bar{\boldsymbol{\vartheta}}}(t_2) + h_2\dot{\bar{\boldsymbol{\vartheta}}}(t_3) \leq h_2$$

and hence, using $\dot{\bar{\boldsymbol{\vartheta}}}(t_3) \geq \underline{m}_3 \geq h_3$

$$(1 - h_3)\dot{\bar{\boldsymbol{\vartheta}}}(t_2) \leq h_2 - h_2\dot{\bar{\boldsymbol{\vartheta}}}(t_3) \leq h_2 - h_3h_2 = (1 - h_3)h_2 \quad . \tag{3.45}$$

This proves (3.35).

5thSTEP :

The remaining inequalities (3.36) follow again from inspecting the extremal points of Δ_0 (see Figure 3.7).

q.e.d.

Theorem 3.14. *Let $\widehat{\mathcal{G}}$ be a convex vNM–Stable Set. Then there exists a vNM–extremal imputation $\boldsymbol{\vartheta} \in \widehat{\mathcal{G}}$.*

Proof:

1stSTEP :

Let $\widehat{\mathcal{G}}$ be a vNM–Solution. If $\boldsymbol{\vartheta}^{(3424)} \in \widehat{\mathcal{G}}$, then we are done. Otherwise, let $\widehat{\boldsymbol{\vartheta}} \in \widehat{\mathcal{G}}$ be such that $\widehat{\boldsymbol{\vartheta}}\,\mathrm{dom}_{T^{123}}\,\boldsymbol{\vartheta}^{(3424)}$ holds true with some $\varepsilon - 123$–relevant coalition (only this kind of domination can occur). Let $\widehat{\underline{\boldsymbol{m}}}$ denote the minima vector of $\widehat{\boldsymbol{\vartheta}}$. A similar notion ($\underline{\boldsymbol{m}}^0$ for the minima vector of $\boldsymbol{\vartheta}^0$ etc) will be used in what follows.

Within the next two steps we are going to show that

$$\begin{gathered}\text{There exists an imputation } \boldsymbol{\vartheta}^0 \in \widehat{\mathcal{G}} \text{ such} \\ \text{that } \boldsymbol{\vartheta}^0 \in \widehat{\mathcal{G}}\ \ , \underline{m}_1^0 = 0 \text{ and } \underline{m}_3^0 = h_3.\end{gathered} \tag{3.46}$$

2ndSTEP : First, if $\widehat{\underline{m}}_1 = 0$ is not the case, then consider the measure

$$\boldsymbol{\vartheta}' := \frac{\widehat{\boldsymbol{\vartheta}}}{1 - \widehat{\underline{m}}_1} - \widehat{\underline{m}}_1 \frac{\boldsymbol{\lambda}^1}{1 - \widehat{\underline{m}}_1} \tag{3.47}$$

which is nonnegative (as $\underline{m}_1 \leq \underline{m}_2$, cf. (3.31)) and has total mass 1, hence constitutes an imputation. As

$$\widehat{\boldsymbol{\vartheta}} \ := \ (1 - \widehat{\underline{m}}_1)\boldsymbol{\vartheta}' + \widehat{\underline{m}}_1 \boldsymbol{\lambda}^1$$

it is seen that domination of $\boldsymbol{\vartheta}'$ by some $\widetilde{\boldsymbol{\vartheta}} \in \widehat{\mathcal{G}}$ induces domination of $\widehat{\boldsymbol{\vartheta}}$ by $(1 - \widehat{\underline{m}}_1)\widetilde{\boldsymbol{\vartheta}} + \widehat{\underline{m}}_1 \boldsymbol{e}^{12}$ which would constitute an internal dominance within $\widehat{\mathcal{G}}$. Hence $\boldsymbol{\vartheta}' \in \widehat{\mathcal{G}}$ follows at once. $\boldsymbol{\vartheta}'$ has zero minimum on $\boldsymbol{D}^1$, hence we can henceforth assume that $\boldsymbol{\vartheta}^0$ yields $\underline{m}_1 = 0$, i.e., the first part of (3.46).

3rdSTEP : Now we proceed in a similar way in order to generate an element of $\widehat{\mathcal{G}}$ that has a minimal value h_3 in the third coordiate.

First of all the essential minimum m_1' of $\boldsymbol{\vartheta}'$ over $\boldsymbol{D}^1$ satisfies $m_1' = 0$. Hence the essential minimum over $\boldsymbol{D}^4$ satisfies $m_4' \geq 1$, for otherwise $\boldsymbol{\vartheta}'$ would be dominated by the core via some $\varepsilon-14$–relevant coalition. Now, if $m_3' + m_4' \geq 1$ would be true, then a contradiction would follow from

$$\boldsymbol{\vartheta}(\boldsymbol{I}) > m_1'\lambda_1 + \underline{m}_2'\lambda_2 + \lambda_3 + \lambda_4 > \lambda_3 + \lambda_4 = 1.$$

Hence we know $m_3' \leq m_4'$. Next, if we have

$$m_3' = \frac{\widehat{\underline{m}}_3}{1 - \widehat{\underline{m}}_1} > h_3 \ ,$$

then we consider the imputation

$$\boldsymbol{\vartheta}^0 \ := \ \frac{\boldsymbol{\vartheta}'}{1 - (\underline{m}_3' - h_3)} - (\underline{m}_3' - h_3)\frac{\boldsymbol{\lambda}^2}{1 - (\underline{m}_3' - h_3)}. \tag{3.48}$$

As in the $2^{nd}STEP$ this imputation inherits all domination inside $\widehat{\mathcal{G}}$ from $\boldsymbol{\vartheta}'$ (and $\widehat{\boldsymbol{\vartheta}}$), hence it has to be an element of $\widehat{\mathcal{G}}$.

Thus we have found $\boldsymbol{\vartheta}^0 \in \widehat{\mathcal{G}}$ satisfying our statement (3.46).

4thSTEP : Let us complete the proof of our theorem. To this end, consider $\boldsymbol{\vartheta}^0$ as constructed in the previous steps satisfying (3.46). Now, if $\boldsymbol{\vartheta}^0$ is vNM–extremal, then we are done. Otherwise, there is some positive mass of $\boldsymbol{\vartheta}^0$ on

$$\{\boldsymbol{D}^1 \cap \{\dot{\boldsymbol{\vartheta}}^0 > 0\}\} \cup \{\boldsymbol{D}^2 \cap \{\dot{\boldsymbol{\vartheta}}^0 > h_2\}\} \cup \{\boldsymbol{D}^3 \cap \{\dot{\boldsymbol{\vartheta}}^0 > h_3\}\} \cup \{\boldsymbol{D}^4 \cap \{\dot{\boldsymbol{\vartheta}}^0 > 1\}\}$$

Now we are going to throw all of this mass onto $\{\boldsymbol{D}^2 \cap \{\dot{\boldsymbol{\vartheta}} < h_2\}\}$. Note that the set $\{\boldsymbol{D}^2 \cap \{\dot{\boldsymbol{\vartheta}} < h_2\}\}$ has positive $\boldsymbol{\vartheta}^0$ measure for otherwise we would obtain

$$\boldsymbol{\vartheta}^0\{\boldsymbol{I}\} \geq h_2\lambda_2 + h_3\lambda_3 + \lambda_4 = \boldsymbol{\lambda}^3(I) > 1 \ \ .$$

That is, for sufficiently small $\varepsilon > 0$, we construct an imputation $\boldsymbol{\vartheta}^1$ such that

$$\begin{aligned}
&\boldsymbol{\vartheta}^1 = 0 \text{ on } \boldsymbol{D}^1, \quad \boldsymbol{\vartheta}^1 = h_3 \text{ on } \boldsymbol{D}^3, \quad \boldsymbol{\vartheta}^1 = 1 \text{ on } \boldsymbol{D}^4, \\
&\boldsymbol{\vartheta}^1 \geq 1 - h_3 \text{ on } \boldsymbol{D}^2 \\
&\dot{\boldsymbol{\vartheta}}^1 = h_2 \text{ on } \{\boldsymbol{D}^2 \cap \{\dot{\boldsymbol{\vartheta}}^0 \geq h_2\}\}, \\
&\dot{\boldsymbol{\vartheta}}^1 > \dot{\boldsymbol{\vartheta}}^0 \text{ on } \{\boldsymbol{D}^2 \cap \{\dot{\boldsymbol{\vartheta}}^0 < \ h_2\}\}
\end{aligned} \tag{3.49}$$

Clearly, $\boldsymbol{\vartheta}^1$ is vNM–extreme. Therefore, if $\boldsymbol{\vartheta}^1 \in \widehat{\mathfrak{G}}$, then we are done.

Assume therefore that $\boldsymbol{\vartheta}^1 \notin \widehat{\mathfrak{G}}$ holds true. Then there is $\boldsymbol{\vartheta}^2 \in \widehat{\mathfrak{G}}$ dominating $\boldsymbol{\vartheta}^1$. This domination can take place only with respect to some $\varepsilon - 123$–relevant coalition T^{123} as the density $\dot{\boldsymbol{\vartheta}}^1$ respects all inequalities prohibiting domination by the core or via some $\varepsilon - 13$–coalition.

We have $\boldsymbol{\vartheta}^2 > \boldsymbol{\vartheta}^1$ on T^2. However, according to (3.33) there is some coalition $\widetilde{T}^2 \in T^2$ such that $\dot{\boldsymbol{\vartheta}}^2 \leq h_2$ holds true on $\widetilde{T}^2$. Consequently, $\widetilde{T}^2 \subseteq T^2 \cap \{\boldsymbol{\vartheta}^0 < h_2\}$ and hence

(3.50) $$h_2 \geq \dot{\boldsymbol{\vartheta}}^2 > \dot{\boldsymbol{\vartheta}}^1 > \dot{\boldsymbol{\vartheta}}^0 \text{ on } \widetilde{T}^2 \quad .$$

This implies

(3.51) $$\boldsymbol{\vartheta}^2(\overline{T}^{123}) \leq \varepsilon(h_2(1-h_3) + h_3 h_2) = \varepsilon h_2$$

for *any* $\varepsilon - 123$–relevant coalition $\overline{T}^{123} = \overline{T}^1 \cup \overline{T}^2 \cup \overline{T}^3$ with $\overline{T}^2 \subseteq \widetilde{T}^2$.

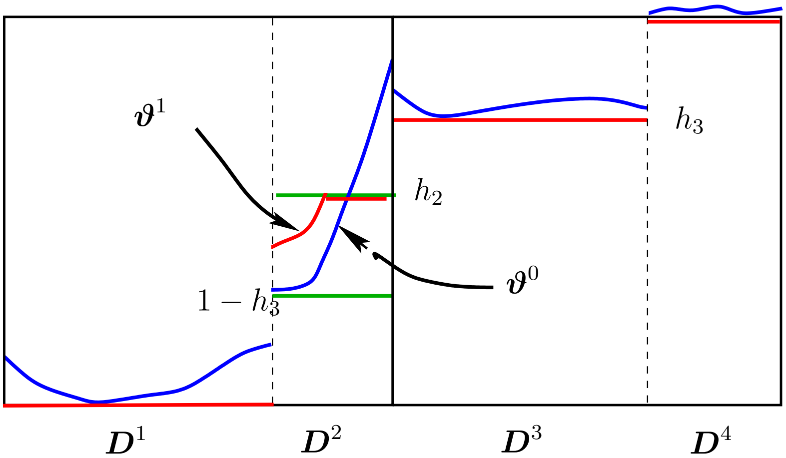

Figure 3.8: The situation of $\boldsymbol{\vartheta}^0$ vs. $\boldsymbol{\vartheta}^1$

Now choose positive $\delta < 1 - h_3$ and some small $\zeta > 0$ as well as coalitions $\widehat{T}^\tau \subseteq T^\tau \quad (\tau = 1, 3), \quad \widehat{T}^2 \subseteq \widetilde{T}^2$ such that

(3.52) $$(1-\zeta)\boldsymbol{\vartheta}^2 > \boldsymbol{\vartheta}^0 \text{ on } \widehat{T}^2, \quad \zeta\delta > \boldsymbol{\vartheta}^0 \text{ on } \widehat{T}^1, \quad (1-\zeta)h_3 + \zeta(1-\delta) > \boldsymbol{\vartheta}^0 \text{ on } \widehat{T}^3 \quad .$$

This is clearly possible by (3.46) and (3.52). Therefore

$$\boldsymbol{\vartheta}^\zeta \;:=\; (1-\zeta)\boldsymbol{\vartheta}^2 + \zeta(\delta \boldsymbol{e}^{12} + (1-\delta)\boldsymbol{e}^{34})$$

exceeds $\boldsymbol{\vartheta}^0$ on $\widehat{T}^{123} = \widehat{T}^1 \cup \widehat{T}^2 \cup \widehat{T}^3$. Hence, if we choose an $\varepsilon - 123$–coalition $\overline{T}^{123} \subseteq \widehat{T}^{123}$, then clearly (use (3.51))

$$\boldsymbol{\vartheta}^{\varepsilon'} \operatorname{dom}_{\overline{T}^{123}} \boldsymbol{\vartheta}^0,$$

contradicting internal stability. Hence, $\boldsymbol{\vartheta}^0 \in \widehat{\mathfrak{G}}$ is necessarily vNM–extreme.

q.e.d.

finally we have

Theorem 3.15. *Let $\widehat{\mathcal{G}}$ be a convex vNM–Stable Set. Then there exists a vNM–extremal imputation $\widehat{\boldsymbol{\vartheta}} \in \widehat{\mathcal{G}}$ such that*

$$\widehat{\mathcal{G}} \;=\; \boldsymbol{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \widehat{\boldsymbol{\vartheta}}\right\} \tag{3.53}$$

Proof: By Theorem 3.14 there is at least one vNM–extremal imputation $\widehat{\boldsymbol{\vartheta}} \in \widehat{\mathcal{G}}$. By Theorem 3.12 the convex hull

$$\widehat{\widehat{\mathcal{G}}} \;:=\; \mathbf{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \widehat{\boldsymbol{\vartheta}}\right\}$$

is a vNM–Stable Set. As $\widehat{\widehat{\mathcal{G}}} \subseteq \widehat{\mathcal{G}}$ we have necessarily $\widehat{\widehat{\mathcal{G}}} \;=\; \widehat{\mathcal{G}}$.

q.e.d.

Corollary 3.16. *Let* (3.1) *and* (3.2) *be satisfied. Then the vNM–extremal imputations supply a complete characterization of all vNM–Stable Sets. That is, any vNM–extremal imputation $\widehat{\boldsymbol{\vartheta}}$ provides a vNM–stable set via* (3.53) *and every vNM–stable set is of this shape.*

In particular, if $h_2 + h_3 = 1$ holds true, the only vNM–Stable Set is given by the only vNM–extremal imputation, which is the constant imputation $\boldsymbol{\vartheta}^{(3424)} \;=\; \boldsymbol{\vartheta}^{\boldsymbol{x}^{(3424)}}$. That is, in this case

$$\mathcal{G}^{(3424)} \;=\; \boldsymbol{ConvH}\left\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\vartheta}^{(2434)}\right\}$$

is the unique vNM–Stable Set.

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