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Convex vNM–Stable Sets for a Semi Orthogonal Game

Part II:

Rich Central Commodity

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Abstract

This paper constitutes the second part in a series dealing with vNM–Stable sets for (cooperative) linear production games with a continuum of players, see [2]. The coalitional function is generated by $r+1$ "production factors" (non atomic measures). r factors are given by orthogonal probabilities ("cornered" production factors) while factor $r+1^{th}$ is provided "across the corners" of the market.

We consider convex vNM–Stable Sets of this game.

Within this second part we present an economy reflecting relative abundance of the "central" commodity. In this situation the model allows for a vNM–Stable Set including but not equal to the core of the game. Rather there is an additional imputation such that the vNM–Stable Set is the convex hull of this imputation and the core. This imputation is essentially described by the density of the $r+1^{st}$ production factor - *mutatis mutandis.*

1 Introduction

Within this second part we discuss a market notably different from the original version that has been discussed in [2]. Here, the third production factor appears in quantities such that the core fails to dominate all imputations outside of itself. We observe the existence of a vNM–Stable set containing but not equal to the core.

In the present model, the non–cornered production factor is available in relatively rich quantities. Nevertheless, as all factors are necessary for to achieve production in a coalition, this distribution provides an additional obstacle for the formation of effective coalitions when compared to the "purely" orthogonal situation. This is reflected by the additional extremal point of the vNM–Stable Set.

At this stage let us enter a discussion regarding the concept of the vNM–Stable Set. This concept is not too popular in economic circles mainly, as it would seem, by two reasons: the first one being its awkward tractability. It is mathematically complicated, existence theorems are not obtained by some standard or prefabricated set of procedures and results may have a strange shape, difficult to justify or to interpret in economic terms. The second reason is its "*ex ante*" interpretation: what does the concept yield in view of its very definition. With both respects the vNM–Stable set is usually considered to be inferior to the core or the competitive equilibrium. Note however, that the concept was the first one developed by von Neumann and Morgenstern in [6] while the core (not mentioning Edgeworth) appears later in the work of Gillies and Shapley.

Since the concept yields set–valued solutions which - in general - are not even unique, there have been quite a few attempts to justify this richness/abundance/chaotic appearance. E.g., it has been said (by the creators in [6]), that the preselection of a particular Stable Set reflects a "standard of behavior“. In the 3–person weighted majority game (the paradigm for a successful evaluation of vNM–Stable Sets) the results illustrate this idea rather impressively: either there is an implicit agreement that minimal winning 2–person coalitions should be taken into considerations only (resulting in the main simple solution) or else there is a (social?) convention according to which one party (holding extreme political views?) is excluded from the bargaining process *a priori* in which case the remaining two have to discuss the "discriminating solution" (see [6]).

We would like to just scale down this cloudy idea, thus obtaining an interpretation referring to the bargaining process around the game only. Vaguely speaking, we want to replace the establishment of a "social standard of behavior" by a first round of bargaining - not exactly between the players involved but between certain coalitions. More precisely, there are now *two rounds* of bargaining. The first one is being performed by "socially relevant groups", i.e., by (representatives of) certain *ex ante* influential coalitions. The second round then takes place *inside* those “influential” coalitions.

Now, for the 3–person majority game this idea may appear in just a rudimentary version, as the “influential” coalitions will have to be the single players (the political parties in a parliament in the most common interpretation of that model). Thus, in the first round of the 3–person majority game players (political parties) will discuss whether to adopt the main simple solution (meaning: all players/parties are on equal footing, everyone is willing to communicate with everybody else; all the minimal winning coalitions are considered to be feasible). Or else it might turn out that one player/party is being shunned, the other two do not want to be seen talking with the political extremists, so there is communication only between the remaining two players. This result of the first round of bargaining would then lead to the discriminating solution (meaning that, because of the large political gap, one party is excluded from the discussion about forming a government).

In the second round then coalitions will internally agree about distributing the worth of the game. That is, in the main simple solution (because of the inherent pressure of a third eligible party being present) there will be a fifty–fifty split in a minimal winning coalition. And in the discriminating case – as the third party is ruled out – there is ample room for distribution the wealth of this coalition between the two remaining parties at will.

In the light of this slight re–interpretation of the “classical” case let us focus shortly on the result in the orthogonal market game discussed in [3] and [4]. In this context, there is a great appeal in restricting oneself to convex Stable Sets. While this does not seem to be appropriate for the 3 person majority game (as it is “zero-sum”) or more generally the class of homogenous weighted majority games, it is well behaved in the context of the market game discussed in those papers as well as in our present context. The (“purely”) orthogonal game allows for an obvious interpretation of the various corners of the market (the carriers of the orthogonal measures establishing production factors): these are cartels guarding their commodity; all commodities are indispensable for production and that establishes a considerable bargaining power of those cartels. Specifically, cartels commanding a commodity distribution of total wealth $1 = \boldsymbol{v}(\boldsymbol{I})$ which is a probability, form the “short side” of the market, while cartels possessing anabundance of their commodity (a measure with high total mass) belong to the long side. (We use term “probability” only to indicate a normalized measure).

The results of [3] and [4] characterize (!) all such solutions and they strongly support our view of a two round bargaining process. Indeed, all vNM–Stable Sets are being obtained in two stages: there is a probability distribution selected in each cartel and the convex hull of these distributions establishes a vNM–Stable Set.

We suggest that this feature corresponds to two rounds of the bargaining process: in one round (“external” from the view of the members of a cartel) (representatives of) the cartels (the corners of the market) establish a convex combination of local distributions of the various cartels – i.e. the share of each cartel in the distribution of wealth. Another round of discussions ("internal") is then engineered within a cartel in order to establish a ("local") distribution of the wealth inside each cartel (normalized, i.e., a probability).

There is no harm in regarding the order of organization of these two rounds as a small matter. Yet, it is clear that for cartels on the short side of the market, the internal distribution - a probability - is dictated by just this probability. For cartels on the long side, players have to agree about some probability inside this cartel which, however, (a result of the Characterization Theorem [3],[4]) is absolutely continuous w.r.t. the commodity distribution of this cartel and yields a density not exceeding one.

Now, an important point is that, as we have a large ("continuous") game, the core assigns no wealth to cartels on the long side of the market. In view of the convergence or "equivalence" theorems, so do the competitive equilibrium and the Shapley value. Of course, in the context of these concepts, the interpretation of this fact is at hand: exploitation of the long side of the market will take place as there is so much competition to rule out any payment to members of the long side.

To which it can be said that, while this reflects the competitive notion of the market, it fails to amply reflect the idea of cooperation: one *does* need the longer side of the market for a coalition to yield the total worth $\boldsymbol{v}(\boldsymbol{I})$. Why the members of any coalition that belong to the long side should agree to obtain just nothing is – from the view of cooperative theory – a mystery. Of course it is the view based on the Equivalence Theorem that competition forces out cooperation in a large market. This author's view is rather that competitive concepts are not appropriate for reflecting the results of cooperation – not even in a large game.

At this stage we would like to shortly hint at some other solution concept that also respects the power of cartels and does not assign zero worth to the long side of the market. The reader is directed to take notice of a similar discussion in [5] regarding the modified nucleolus or modiclus. This concept shows quite an analogous development and reflect the results of cooperation in a striking way: the long side of the market is obtaining value according to its relevance and impact. On the other hand, it clearly suffers from the same backlash as the vNM–Stable Set: it is mathematically difficult to handle and requires some effort to interpret it "ex ante". This effort is rewarded: in the case of the modiclus the introduction of the dual game bears consequences that definitely shed light on a feature neglected by some cooperative concepts: the power to prevent as opposed to the power to achieve.

In view of these two concepts we argue that the class of concepts of the "equivalence theorems" though quite powerful within contexts of competition may not be exclusively appropriate for a class of cooperative market games like linear production games. Hence, the concepts we have in mind are still worth a lengthy strain of research – no matter what happens within that realm of the 70-ies till 90-ies.

Let us now turn to our present framework of a quasi orthogonal game and its vNM–Stable sets. We consider a distribution $\boldsymbol{\lambda}^0$ of a "central commodity" that exhibits mass across the cartels, hence is counteracting the cartelization tendencies. As we have a "min game", the central commodity is more important the smaller the contribution of $\boldsymbol{\lambda}^0$. Now the central commodity is, other then the goods in corners, *not* attached to any cartel like coalition. We suggest that the role of the various cartels is augmented by a similar role of the grand coalition. It may act in the external round by participating in the discussion concerning the share of the cartels. And it may then engineer an internal round of discussion like the cartels do – internal meaning in this case on the global scale of the grand coalition.

Hence, when discussing a distribution within the grand coalition via some imputation the "internal round" of discussions (the one that took place within the cartels previously) will have to include some discussion within the grand coalition that hinges on $\boldsymbol{\lambda}^0$. But $\boldsymbol{\lambda}^0$ is decidedly *not* an imputation, we assume $\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1$. Our results suggest an outcome of this "internal" discussion within the grand coalition: this consist of a pre–imputation $\bar{\boldsymbol{x}}$ which equals the density of $\boldsymbol{\lambda}^0$ up to a few corrections. These corrections take place on sectors $\boldsymbol{D}^\tau$ on which $\boldsymbol{\lambda}^0$ is rather large (hence less influential) and on sectors on which it is rather small (hence overly influential).

Consider Figure 3.2. Here $\bar{\boldsymbol{x}}$ is obtained by decreasing the density $\boldsymbol{h}$ of $\boldsymbol{\lambda}^0$ in sectors with values 1 and increasing it in sectors with small values.

Now, the vNM–Stable set obtained is given (in terms of pre– imputations) by the convex hull of the core and $\bar{\boldsymbol{x}}$. Thus, we see that again there are two rounds of bargaining suggested. The one ("internal") involves a discussion in each cartel *and* a discussion in the grand coalition about which distribution to choose "internally". The external round consists of some discussion among (representatives of) cartels plus the grand coalition as to the coefficients of the convex combination of the "internal" imputations applied in order to result in an element of the vNM–Stable Set.

Thus, the vNM–Stable Set subtly reacts to the quasi orthogonal setup, that is, the existence of a non cornered commodity. The procedure from the orthogonal setup is copied up to the fact that the cartels are augmented by the grand coalition. The internal procedure is, therefore, augmented by the result of some global discussion in view of the central commodity. The external procedure is also augmented by asking for some regard to the result of the internal discussion of the grand coalition. The grand coalition kind of plays the same role as the cartels – in all details.

In models to be discussed later on, vNM–Stable Sets may appear in abundance – as there is no uniqueness when certain data of $\boldsymbol{\lambda}^0$ are being relaxed. In this case it is even more striking that the grand coalition is taking place as a partner in the discussion of the cartels - just that the result of the internal discussion of the grand coalition is not unique.

Next want to stress a second feature of our concept which we believe to be responsible for the power of cartels, that feature being neglected in the concepts of the "equivalence theorem class".

The essential argument appears in [5] regarding the modiclus: Cartels cannot achieve any worth ($\boldsymbol{v}(\boldsymbol{C}^{\rho}) = 0$) but they can prevent their opponents from getting any worth as well ($\boldsymbol{v}(\boldsymbol{I} \setminus \boldsymbol{C}^{\rho}) = 0$). The "power to prevent" is incorporated within the modiclus concept as it involves the dual game – the game representing the preventing power of coalitions *ipso forte*. This feature is completely missing regarding the core: the core is indicating what coalitions can achieve and nothing else.

Now the core is simultaneously the set of undominated imputations. It is always internally stable. That is, internal stability and the power to achieve are closely connected.

Outside the core one may find imputations that cannot be dominated by elements of the core, i.e., certain coalitions cannot be prevented from striving for these imputations. Obviously the power to prevent is caught by the concept of external stability. Thus, vNM–Stable Sets result in reflecting the "preventive power" of cartels because they involve arguments of prevention via external stability. This property, though by a different reason, is quite similar to the one exhibited by the modiclus.

Finally let us comment on the subtlety of the nature of the vNM–Stable Set as seen from the Mathematical or Computational viewpoint. It has been argued (Y. Kannai, verbal communication) that it is a main disadvantage of such a concept to be undecidable *ex ante*. I.e., given an imputation one can at once (in principle) decide upon whether it is an element of the core (say, by verifying a system of inequalities). There is no such decisive – not to speak of constructive – procedure regarding vNM–Stable Sets. Frequently they are not unique and any attempt to somehow find a computational idea for exhibiting them is perturbed by the lack of transitivity of the domination relation. Thus, there seems to be no way to *ex ante* decide about some imputation being a member of a vNM–Stable Set, and of which one. Hence one may feel uncomfortable with such a concept.

To which we would like to reply that indeed, the core in a sense is a set of "optimal elements" – hence "approachable". As such it is also in a certain sense naive. vNM on the other hand results from a more subtle consideration, taking into account not just what one can do but also what one can prevent others from doing. In what sense soever – optimality is just the naive approach of Optimization – the one player game. Reflecting about the influence of opposing coalitions is much more adapted to the real world situation.

There are many versions of stability (of course the Nash equilibrium in a non–cooperative game comes to ones mind) that are non unique, resistant to computational methods, but intuitively closer to representing a situation of the real world. Frequently we do not know in advance which result to expect from conflict and cooperation – this is all what Game Theory is about. To revert the 11^{th} thesis of a 19^{th} century economist–philosopher: The Politicians–Economists have but changed the world in various ways – it is however essential to interpret it.

2 Notations and Definitions

We use definitions and notations as provided in [2] and previously in [3] and [4]. Thus, we consider a (cooperative) ***game*** with a continuum of players, i.e., a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ where $\boldsymbol{I}$ is some interval in the reals (the ***players***), $\underline{\underline{\mathbf{F}}}$ is the $\sigma-$field of (Borel) measurable sets (the ***coalitions***) and $\boldsymbol{v}$ (the ***coalitional function***) is a mapping $\boldsymbol{v} : \underline{\underline{\mathbf{F}}} \to \mathbb{R}_+$ which is absolutely continuous w.r.t. the Lebesgue measure $\boldsymbol{\lambda}$. We focus on "linear production games", that is, $\boldsymbol{v}$ is described by finitely many measures $\boldsymbol{\lambda}^\rho, (\rho \in \{0, 1, \ldots, r\})$ via

$$\boldsymbol{v}(S) := \min \{\boldsymbol{\lambda}^\rho(S) \mid \rho \in \{0, 1, \ldots, r\} \quad (S \in \underline{\underline{\mathbf{F}}}). \tag{2.1}$$

or

$$\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} = \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho , \tag{2.2}$$

(as previously, we use $\mathbf{R} = \{1, \ldots, r\}$ and $\mathbf{R}_0 = \mathbf{R} \cup \{0\}$). All measures are absolutely continuous w.r.t to Lebesgue measure $\boldsymbol{\lambda}$. The measures $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r$ are orthogonal copies of Lebesgue measure. Thus we choose the player set to be $\boldsymbol{I} := [0, r)$. The carriers $\boldsymbol{C}^\rho = (\rho - 1, \rho]$ $(\rho = 0, \ldots, r)$ of the measures $\boldsymbol{\lambda}^\rho$ are the "cartels" commanding commodity ρ.

The measure $\boldsymbol{\lambda}^0$ is assumed to have a piecewise constant density $\dot{\boldsymbol{\lambda}}^0$ w.r.t $\boldsymbol{\lambda}$ given by

$$\dot{\boldsymbol{\lambda}}^0 = h_\tau \text{ on } \boldsymbol{D}^\tau, \quad (\tau \in \mathbf{T}) \tag{2.3}$$

where $\{\boldsymbol{D}^\tau\}_{\tau \in \mathbf{T}^\rho}$ constitutes a partition of the carrier $\boldsymbol{C}^\rho$ of $\boldsymbol{\lambda}^\rho$ such that $\bigcup_{\tau \in \mathbf{T}^\rho} \boldsymbol{D}^\tau = \boldsymbol{C}^\rho$. Further details of our notation are exactly those presented in [2].

In particular, we shall discuss the 2×2– example that goes back to EINY ET AL. [1] but with initial data changed according to our present model. Thus, we assume the density of $\boldsymbol{\lambda}^0$ (i.e., the levels h_τ) to be "small" on most blocks $\boldsymbol{D}^\tau$, meaning that the non–cornered factor is scarce. Within the example (see Figure 2.1) this results in a requirement $h_2 + h_3 < 1$. More precisely:

Example 2.1. Figure 2.1 illustrates a situation for $r = t = 2$. We assume

$$\begin{aligned} &\lambda_1 + \lambda_3 < 1 \\ &h_1 = 0,\ h_2, h_3 < 1, \quad h_4 = 1 \\ &h_2\lambda_2 + h_3\lambda_3 + \lambda_4 > 1 \end{aligned} \tag{2.4}$$

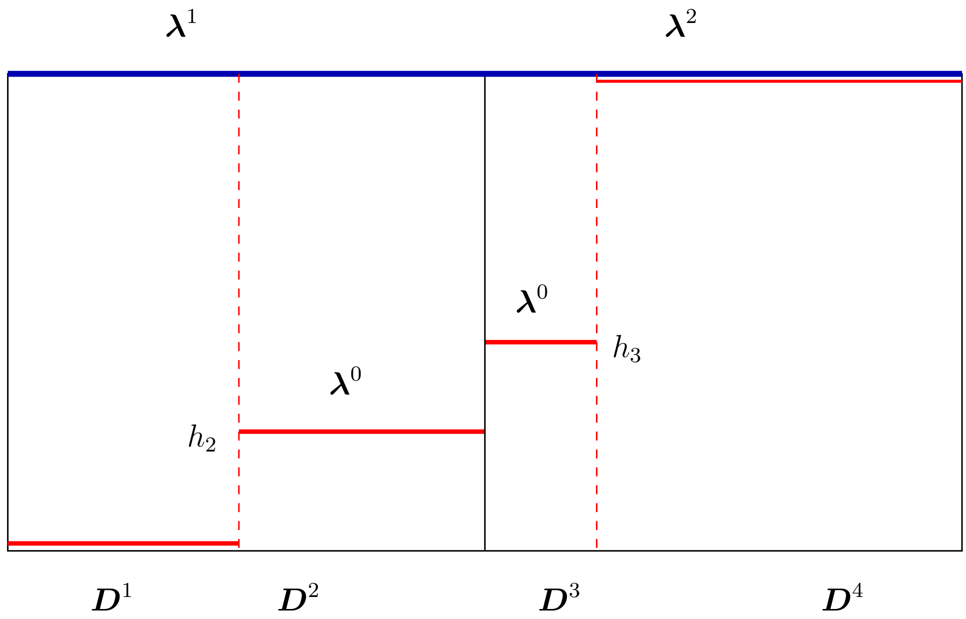

Figure 2.1: The case of 4 steps: the density of $\boldsymbol{\lambda}^0$

Our solution concept is given by the ***vNM–Stable Set*** (VON NEUMANN-MORGENSTERN [6]). We repeat the definition for a continuum of players, see also Part 1 [2].

Definition 2.2. *Let $(I, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ be a game. An **imputation** is a measure $\boldsymbol{\xi}$ with $\boldsymbol{\xi}(I) = \boldsymbol{v}(I)$. An imputation $\boldsymbol{\xi}$ **dominates** an imputation $\boldsymbol{\eta}$ w.r.t $S \in \underline{\underline{\mathbf{F}}}$ if $\boldsymbol{\xi}$ is **effective** for S, i.e.,*

$$\boldsymbol{\lambda}(S) > 0 \quad \textit{and} \quad \boldsymbol{\xi}(S) \leq \boldsymbol{v}(S) \tag{2.5}$$

and if

$$\boldsymbol{\xi}(T) > \boldsymbol{\eta}(T) \quad (T \in \underline{\underline{\mathbf{F}}},\ T \subseteq S, \boldsymbol{\lambda}(T) > 0) \tag{2.6}$$

holds true. That is, every subcoalition of S (almost every player in S) strictly improves its payoff at $\boldsymbol{\xi}$ versus $\boldsymbol{\eta}$. We write $\boldsymbol{\xi} \operatorname{dom}_S \boldsymbol{\eta}$ to indicate domination. It is standard to use $\boldsymbol{\xi} \operatorname{dom} \boldsymbol{\eta}$ whenever $\boldsymbol{\xi} \operatorname{dom}_S \boldsymbol{\eta}$ holds true for some *coalition $S \in \underline{\underline{\mathbf{F}}}$.*

Definition 2.3. *Let $\boldsymbol{v}$ be a game. A set $\mathcal{S}$ of imputations is called a **vNM–Stable Set** if*

- *there is no pair $\boldsymbol{\xi}, \boldsymbol{\mu} \in \mathcal{S}$ such that $\boldsymbol{\xi} \operatorname{dom} \boldsymbol{\mu}$ holds true (*"internal stability"*).*
- *for every imputation $\boldsymbol{\eta} \notin \mathcal{S}$ there exists $\boldsymbol{\xi} \in \mathcal{S}$ such that $\boldsymbol{\xi} \operatorname{dom} \boldsymbol{\eta}$ is satisfied (*"external stability"*).*

The discrete nature of the density of $\boldsymbol{\lambda}^0$ carries some implications for the establishment of dominance based on discrete analogues of concepts like imputations, coalitions etc. We refer to these analogues as "pre–concepts". Again see Part 1 [2] for the details.

We now specify the basic assumptions for the model under considerations within this second part of our presentation. Within the following definition it is understood that we pick r as a distinguished element of $\mathbf{R}$.

Definition 2.4. *We call*

$$\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} \quad = \quad \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho .$$

a game with ***Rich Central Commodity*** *if the following is satisfied.*

There is a unique sequence $\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r) \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$ *such that*

$$\sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} \; < \; 1 \tag{2.7}$$

holds true. Thus, we have

$$\sum_{\rho \in \mathbf{R}} h_{\tau_\rho} \; \geq \; 1 \tag{2.8}$$

for any sequence $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_r) \in \mathbf{T}^1$ *that does not contain at least one* $\widehat{\tau}_\rho$.

The 2×2 Example 2.1 is captured by this definition if we assume $h_2 + h_3 \geq 1$. For in this case the sequence $(h_1, h_3) = (0, h_3)$ is the only one that satisfies (2.7).

Pre–coalitions along the sequence $\widehat{\boldsymbol{\tau}}$ play an essential role in order to provide domination certain imputations outside the core that cannot be dominated from inside the core. Along this sequence the central commodity is more scarce than any cornered commodity that is, any extremal of the core. Hence, the distribution of $\boldsymbol{\lambda}^0$ along this sequence is decisive for external dominance. Consequently, the core is not sufficient in order to reject certain imputations with a small outfit of the central commodity: in such a coalition the value of the game is dictated by the central commodity and hence domination requires some additional imputation in the vNM–Stable Set.

It should be noted that the central commodity would be even richer in case that there is *no* sequence at all satisfying (2.7). In this case however, it is seen that the central commodity is of no influence to the game: any relevant pre–coalition involves the short commodities of the corners and the long commodity of the center is always available in abundance. Hence, the game is essentially the "pure" orthogonal game. For this the vNM–Stable Sets are well known ([3],[4]).

Consider the pre–coalition $\boldsymbol{a}^\oplus$ with coordinates 1 along the sequence $\widehat{\boldsymbol{\tau}}$, i.e.,

$$\boldsymbol{a}^\oplus \;=\; (0, \ldots, 0, 1, 0 \ldots, 0, 1, 0, , \ldots, 0, \frac{1 - (h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}})}{h_{\widehat{\tau}_r}}, 0 \ldots, 0) \,.$$

and similarly

$$\boldsymbol{a}^\ominus \;=\; \left(1, \ldots, 1, \frac{(h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}} \,,\, \frac{1 - (h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}} ; \right)$$

we have $v(\boldsymbol{a}^{\oplus}) = 1$ and $v(\boldsymbol{a}^{\oplus}) = 1$. Now we will construct the extremal $\bar{\boldsymbol{x}}$ of the hypothetical vNM–Stable Set such that $\bar{\boldsymbol{x}}\boldsymbol{a}^{\oplus} = 1 = v(\boldsymbol{a}^{\oplus})$ and – even more significant –

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} < 1 = v(\boldsymbol{a}^{\oplus}) .$$

This construction will permit the (pre–)domination of certain imputations holding small amounts of the central commodity.

Recall that the pre–coalitions mentioned above are relevant vectors in the sense of Definition 3.1 of [2]. The crucial result presented in [2] is offered by the "Inheritance Theorem" (Theorem 3.3) according to which it is sufficient and necessary to consider the relevant vectors in order to study dominance relations for the Semi Orthogonal Game in our sense. More precisely, recalling the notation of the vector valued measure $\vec{\boldsymbol{\lambda}}$ via

$$\vec{\boldsymbol{\lambda}}(\star) := \left\{\boldsymbol{\lambda}\left(\star \cap \boldsymbol{D}^{\tau}\right)_{\tau\in\mathbf{T}}\right\}$$

it is sufficient and necessary to study ε–relevant coalitions, i.e., coalitions $T = T^{\varepsilon\boldsymbol{a}}$ that, for some small $\varepsilon > 0$ and some relevant vector $\boldsymbol{a}$, yield

$$\vec{\boldsymbol{\lambda}}(T) = \varepsilon\boldsymbol{a} \text{ and } \boldsymbol{v}(T) = 1 .$$

3 Rich Central Commodity

Recall the set of pre–imputations

$$\boldsymbol{H} \;=\; \{\boldsymbol{x} \in \boldsymbol{J} \mid \boldsymbol{x}\boldsymbol{a} \geq v(\boldsymbol{a}) \;=\; 1 \;\; (\boldsymbol{a} \in \boldsymbol{A}^s)\} \tag{3.1}$$

that was introduced in in (4.7) of Part I of [2] ($\boldsymbol{J} = \boldsymbol{J}(v)$ denotes the pre–imputations of the pre–game v). Within this section we exhibit a pre–imputation $\bar{\boldsymbol{x}}$ that, given appropriate requirements regarding $\boldsymbol{\lambda}^0$, renders

$$\boldsymbol{H} \;=\; \mathbf{ConvH}\left\{\bar{\boldsymbol{x}},\; \boldsymbol{e}^{T^\rho} \;\; (\rho \in \mathbf{R})\right\} \;.$$

Then we show that $\boldsymbol{H}$ provides the basis for a vNM–Stable Set

$$\mathcal{H} \;=\; \mathbf{ConvH}\left\{\boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}},\; \boldsymbol{\lambda}^\rho \;\; (\rho \in \mathbf{R})\right\} \;.$$

To this end we pick r as the distinguished element of $\mathbf{R}$. We assume that our conditions regarding a rich central commodity (Definition 2.4) are satisfied.

As a prerequisite we consider a linear system of equations that eventually will serve to define the pre–imputation $\bar{\boldsymbol{x}}$.

Definition 3.1. *Let* $\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r) \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$ *and let* $(\widehat{x}_{\widehat{\tau}_1}, \ldots, \widehat{x}_{\widehat{\tau}_r}) \in \mathbb{R}^r_+$. *We shall say that* $\widehat{\boldsymbol{x}} \in \mathbb{R}^{rt}_+$ *obeys the* ***defining system*** *with boundary conditions* $(\widehat{x}_{\widehat{\tau}_1}, \ldots, \widehat{x}_{\widehat{\tau}_r})$ *if the coordinates* $\widehat{x}_\tau$ $(\tau \in \mathbf{T})$ *are determined by the following linear system of equations in variables* x_τ $(\tau \in \mathbf{T})$*:*

$$x_\tau \;=\; \widehat{x}_\tau \;\; (\tau \in \{\widehat{\tau}_1, \ldots, \widehat{\tau}_r\}) \;, \tag{3.2}$$

$$x_\tau \;=\; x_{\tau'} \;\; (\tau, \tau\prime \in \mathbf{T}^\rho \setminus \{\widehat{\tau}_\rho\},\; \rho \in \mathbf{R}) \;, \tag{3.3}$$

$$x_{\tau_\sigma} + \sum_{\rho \in \mathbf{R} \setminus \{\sigma, r\}} \widehat{x}_{\widehat{\tau}_\rho} + x_{\tau_r} \;=\; 1 \quad (\sigma \in \mathbf{R} \setminus \{r\}) \;, \tag{3.4}$$

$$\sum_{\tau \in \mathbf{T}} \lambda_\tau x_\tau \;=\; 1 \;. \tag{3.5}$$

Remark 3.2. As r plays a distinguished role, equation (3.4) is not required for $\rho = r$, instead the pre–imputation requirement (3.5) is supplied. As (3.2) shows, there is essentially one value x_τ for $\tau \in \mathbf{T}^\rho, \tau \neq \widehat{\tau}_\rho$ to be determined, let the generic element be denoted by $x_{\tau_\rho} \in \mathbf{T}^\rho$. Then the defining system is essentially equivalent to a linear system in r variables x_{τ_ρ} with r equations

$$\begin{aligned} x_{\tau_\sigma} + x_{\tau_r} \;&=\; 1 - \sum_{\rho \in \mathbf{R} \setminus \{\sigma, r\}} x_{\widehat{\tau}_\rho} \quad (\sigma \in \mathbf{R} \setminus \{r\}) \\ \sum_{\rho \in \mathbf{R}} \Lambda_\rho x_{\tau_\rho} \;&=\; 1 \;-\; \sum_{\rho \in \mathbf{R}} \lambda_{\widehat{\tau}_\rho} x_{\widehat{\tau}_\rho} \;\;. \end{aligned} \tag{3.6}$$

with

$$\Lambda_\rho \;:=\; \sum_{\tau \in \mathbf{T}^\rho \setminus \{\widehat{\tau}_\rho\}} \lambda_\tau \;=\; 1 - \lambda_{\widehat{\tau}_\rho} \tag{3.7}$$

In order to exhibit the solution of (3.6) (and hence of the defining system), we attempt to eliminate the variables $x_{\tau_\sigma} \quad (\sigma \neq r)$. We introduce

$$H_\sigma \;:=\; \sum_{\rho \in \mathbf{R} \setminus \{\sigma, r\}} x_{\widehat{\tau}_\rho} \;, \tag{3.8}$$

then the first line of (3.6) reads

$$x_{\tau_\sigma} \;=\; 1 - H_\sigma - x_{\tau_r} \qquad (\sigma \in \mathbf{R} \setminus \{r\}) \tag{3.9}$$

The second line of (3.6) is then transformed to

$$\sum_{\rho \in \mathbf{R} \setminus \{r\}} \Lambda_\rho (1 - H_\rho - x_{\tau_r}) + \Lambda_r x_{\tau_r} \;=\; 1 \;-\; \sum_{\rho \in \mathbf{R}} \lambda_{\widehat{\tau}_\rho} x_{\widehat{\tau}_\rho} \tag{3.10}$$

which permits to come up with a closed expression for x_{τ_r} in terms of $x_{\widehat{\tau}_\rho}$ and λ_ρ as follows:

(3.11)

$$\begin{aligned} x_{\tau_r} \;&=\; \frac{1 - \sum_{\rho \in \mathbf{R}} \lambda_{\widehat{\tau}_\rho} x_{\widehat{\tau}_\rho} - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \Lambda_\rho (1 - H_\rho)}{\Lambda_r - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \Lambda_\rho} \\ &=\; \frac{1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \Lambda_\rho - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \left\{ \lambda_{\widehat{\tau}_\rho} - \sum_{\sigma \in \mathbf{R} \setminus \{\rho, r\}} \Lambda_\sigma \right\} x_{\widehat{\tau}_\rho} - \lambda_{\widehat{\tau}_r} x_{\widehat{\tau}_r}}{\Lambda_r - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \Lambda_\rho} \end{aligned}$$

Combining our results, we come up with the following lemma.

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Lemma 3.3. *Given the data* $x_{\widehat{\tau}_\rho}$ *and* λ_ρ, *a solution of the defining system necessarily satisfies equations* (3.9) *and* (3.11).

On the other hand, let $x_{\widehat{\tau}_\rho}$ *and* λ_ρ *satisfy the following conditions.*

$$\begin{aligned} \Lambda_r \;&>\; \sum_{\rho \in \mathbf{R} \setminus \{r\}} \Lambda_\rho \;, \\ \lambda_{\widehat{\tau}_\rho} \;&>\; \sum_{\sigma \in \mathbf{R} \setminus \{\rho, r\}} \Lambda_\sigma \quad (\rho \in \mathbf{R} \setminus \{r\}) \;, \\ 1 \;&>\; \sum_{\rho \in \mathbf{R} \setminus \{r\}} \Lambda_\rho - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \left\{ \lambda_{\widehat{\tau}_\rho} - \sum_{\sigma \in \mathbf{R} \setminus \{\rho, r\}} \Lambda_\sigma \right\} x_{\widehat{\tau}_\rho} - \lambda_{\widehat{\tau}_r} x_{\widehat{\tau}_r} \end{aligned} \tag{3.12}$$

Then equations (3.9) *and* (3.11) *define a unique pre–imputation that satisfies the defining system.*

Remark 3.4. The conditions of Lemma 3.3 define a nonempty set of data λ_ρ such that the conclusions of the lemma hold true. For, rewriting these conditions in view of (3.7), we obtain

$$\begin{aligned} \sum_{\rho \in \mathbf{R}\setminus\{r\}} \lambda_{\widehat{\tau}_\rho} &> (r-2) + \lambda_{\widehat{\tau}_r} \ , \\ \sum_{\rho \in \mathbf{R}\setminus\{r\}} \lambda_{\widehat{\tau}_\rho} &> r-2 \end{aligned} \tag{3.13}$$

for the first two inequalities, while the third one follows immediately if the first two are satisfied. However, (3.13) shows, that the second one follow also once the first one is satisfied, hence a sufficient condition is actually provided by the first inequality in (3.13). For example, assume that for some $\varepsilon > 0$ we have

$$\lambda_{\widehat{\tau}_\rho} \geq 1 - \varepsilon \quad (\rho \in \mathbf{R} \setminus \{r\}), \qquad \lambda_{\widehat{\tau}_r} \leq \varepsilon \ , \tag{3.14}$$

then it is sufficient to assume that

$$(r-1)(1-\varepsilon) > (r-2) + \varepsilon \text{ i.e. } \varepsilon < \frac{1}{2(r-1)}$$

holds true. Thus we observe that the regions $\boldsymbol{D}^{\widehat{\tau}_\rho}$ with small density factor, say $\widehat{x}_{\widehat{\tau}_\rho} = k_{\widehat{\tau}_\rho}$ have relatively large masses $\lambda_{\widehat{\tau}_r}$ while the region $\boldsymbol{D}^{\widehat{\tau}_r}$ has small mass.

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Now define a pre–imputation $\bar{\boldsymbol{x}} \in \boldsymbol{J}$ as follows:

Definition 3.5. *Let* $\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r) \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$. *Assume that* (3.12) *is satisfied for* $\widehat{x}_\tau = h_\tau \quad (\tau \in \{\widehat{\tau}_1, \ldots, \widehat{\tau}_r\})$. *Then* $\bar{\boldsymbol{x}}$ *is the unique pre–imputation resulting from Lemma 3.3.*

That is, $\bar{\boldsymbol{x}}$ coincides with the density of $\boldsymbol{\lambda}^0$ along the specified sequence $\widehat{\boldsymbol{\tau}}$ and all values $\widehat{x}_\tau$ within some $\mathbf{T}^\rho$ are equal for $\tau \neq \widehat{\tau}_r$ and determined by Definition 3.1 or (3.6) respectively.

Figures 3.1 and 3.2 provide a visual presentation of $\dot{\boldsymbol{\lambda}}^0$ and $\bar{\boldsymbol{x}}$. The sketch reflects the assumption that $\widehat{\tau}_\rho$ is the first element in $\mathbf{T}^\rho$, that is $\widehat{\tau}_\rho = (\rho - 1)t + 1$.

We write

$$\begin{aligned} \Lambda^r &:= \sum_{\rho \in \mathbf{R}\setminus\{r\}} \Lambda_\rho \ , \\ c_\rho &:= \lambda_{\widehat{\tau}_\rho} - \sum_{\sigma \in \mathbf{R}\setminus\{\rho, r\}} \Lambda_\sigma \\ Q &:= \frac{\Lambda_r - \Lambda^r}{1 - \Lambda^r} \ . \end{aligned} \tag{3.15}$$

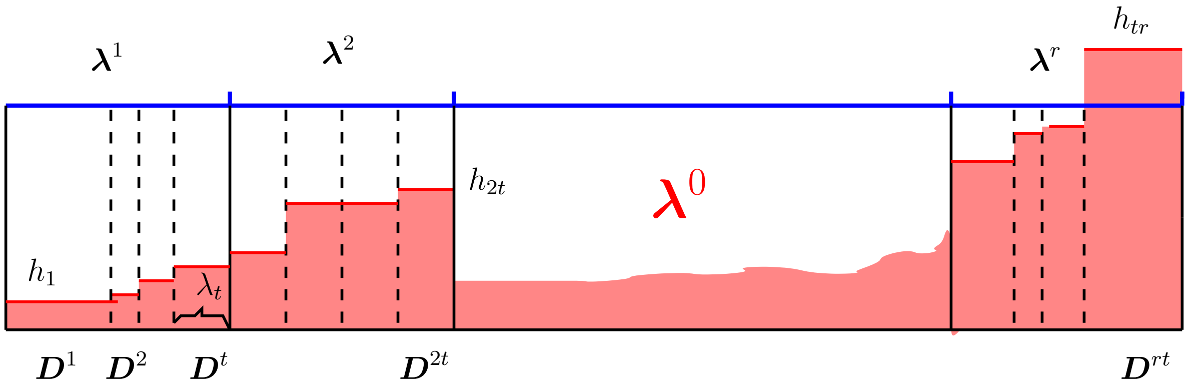

Figure 3.1: The density of $\boldsymbol{\lambda}^0$

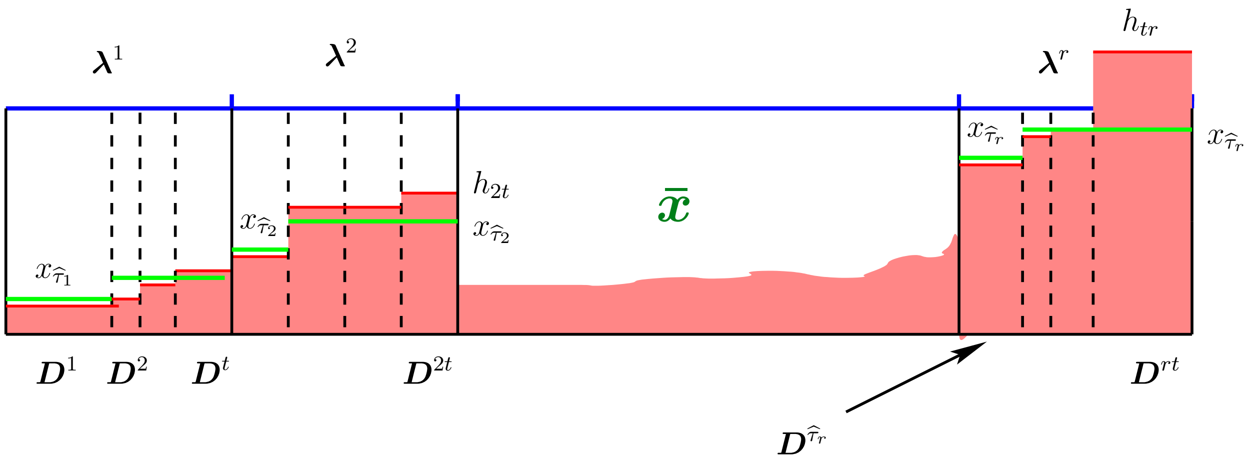

Figure 3.2: The pre–imputation $\bar{\boldsymbol{x}}$

Note that these quantities depend on the λ_τ only. The following Lemma capitalizes essentially on this fact.

Lemma 3.6. *Let* $(x_{\widehat{\tau}_1}, \dots, x_{\widehat{\tau}_r}) \in \mathbb{R}^r_+$ *(a vector of "boundary conditions") and let* $\boldsymbol{x} \in \boldsymbol{J}$ *be the corresponding solution of the defining system. Assume that the data* λ_ρ $(\rho \in \mathbf{R})$ *satisfy the conditions* (3.12). *Then*

$$x_{\tau_r} - x_{\widehat{\tau}_r} \;=\; \frac{1 - \sum_{\rho\in\mathbb{R}} x_{\widehat{\tau}_\rho}}{Q} \;. \tag{3.16}$$

Proof:

Using the abbreviations of (3.15) we find that (3.11) is shortened to

$$x_{\tau_r} \;=\; \frac{1 - \Lambda^r - \sum_{\rho\in\mathbf{R}\setminus\{r\}} c_\rho x_{\widehat{\tau}_\rho} - \lambda_{\widehat{\tau}_r} x_{\widehat{\tau}_r}}{\Lambda_r - \Lambda^r} \;. \tag{3.17}$$

Thus

$$\begin{aligned} x_{\tau_r} - x_{\widehat{\tau}_r} \;&=\; \frac{1 - \Lambda^r - \sum_{\rho\in\mathbf{R}\setminus\{r\}} c_\rho x_{\widehat{\tau}_\rho} - \left\{\lambda_{\widehat{\tau}_r} + \Lambda_r - \Lambda^r\right\} x_{\widehat{\tau}_r}}{\Lambda_r - \Lambda^r} \\ &=\; \frac{1 - \sum_{\rho\in\mathbf{R}\setminus\{r\}} \frac{c_\rho}{1-\Lambda^r} x_{\widehat{\tau}_\rho} - \frac{\left\{\lambda_{\widehat{\tau}_r} + \Lambda_r - \Lambda^r\right\}}{1-\Lambda^r} x_{\widehat{\tau}_r}}{\frac{\Lambda_r - \Lambda^r}{1-\Lambda^r}} \;. \end{aligned} \tag{3.18}$$

Hence we are done if all coefficients to the $x_{\widehat{\tau}_\rho}$ equal 1. Now for $\rho \in \mathbf{R} \setminus \{r\}$

$$\begin{aligned} c_\rho \;&=\; \lambda_{\widehat{\tau}_\rho} - \sum_{\sigma\in\mathbf{R}\setminus\{\rho,r\}} \Lambda_\sigma \\ &=\; 1 - \Lambda_\rho - \sum_{\sigma\in\mathbf{R}\setminus\{\rho,r\}} \Lambda_\sigma \\ &=\; 1 - \sum_{\sigma\in\mathbf{R}\setminus\{r\}} \Lambda_\sigma \;, \end{aligned} \tag{3.19}$$

that is the coefficient of $x_{\widehat{\tau}_\rho}$ equals

$$\frac{c_\rho}{1 - \sum_{\sigma\in\mathbf{R}\setminus\{r\}} \Lambda_\sigma \;,} \;=\; 1.$$

Next

$$\lambda_{\widehat{\tau}_r} + \Lambda_r - \Lambda^r \;=\; 1 - \Lambda^r \;, \tag{3.20}$$

thus is the coefficient of $x_{\widehat{\tau}_r}$ also equals 1. This proves the lemma.

q.e.d.

Remark 3.7. Lemma 3.6 explains the complicated choice of some pre–imputation $\widehat{x}$ via the defining system (Definition 3.1 and Remark 3.2). For, the coordinates $\widehat{x}_\tau$ resulting from the choice of the $\widehat{x}_{\widehat{\tau}_\rho}$ (as given by (3.9) and (3.11)) ensure that the coefficients to these coordinates $\widehat{x}_\tau$ are linear in the boundary coordinates $\widehat{x}_{\widehat{\tau}_\rho}$ in such a way, that they allow for a simple form of the difference (3.16). This difference turns out to be crucial when dominance w.r.t the pre–coalitions of type $\boldsymbol{a}^{\ominus}$ is at stake.

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Corollary 3.8. *Let $\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r) \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$ and let $(\widehat{x}_{\widehat{\tau}_1}, \ldots, \widehat{x}_{\widehat{\tau}_r}) \in \mathbb{R}^r_+$. Let $\widehat{\boldsymbol{x}} \in \boldsymbol{J}$ be the corresponding solution of the defining system. Then*

$$\widehat{x}_\tau \geq \widehat{x}_{\widehat{\tau}_\rho} \quad (\tau \in T^\rho \setminus \{\widehat{\tau}_\rho\},\ \rho \in \mathbf{R}). \tag{3.21}$$

In particular, whenever $\bar{\boldsymbol{x}}$ is determined via Definition 3.5, then

$$\overline{x}_\tau \geq \overline{x}_{\widehat{\tau}_\rho} \;=\; h_{\widehat{\tau}_\rho} \quad (\tau \in T^\rho \setminus \{\widehat{\tau}_\rho\},\ \rho \in \mathbf{R}). \tag{3.22}$$

Proof:

For $\rho = r$ this follows from Lemma 3.6. For $\rho < r$ we have in view of (3.6)

$$\begin{aligned}
\widehat{x}_{\tau_\sigma} - \widehat{x}_{\widehat{\tau}_\sigma} &= 1 - \sum_{\rho \in \mathbf{R}\setminus\{\sigma, r\}} \widehat{x}_{\widehat{\tau}_\rho} - \widehat{x}_{\tau_r} - \widehat{x}_{\widehat{\tau}_\sigma} \\
&= 1 - \sum_{\rho \in \mathbf{R}} \widehat{x}_{\widehat{\tau}_\rho} - \widehat{x}_{\tau_r} + \widehat{x}_{\widehat{\tau}_r} \\
&= 1 - \sum_{\rho \in \mathbf{R}} \widehat{x}_{\widehat{\tau}_\rho} - (\widehat{x}_{\tau_r} - \widehat{x}_{\widehat{\tau}_r}) \\
&= \left(1 - \sum_{\rho \in \mathbf{R}} \widehat{x}_{\widehat{\tau}_\rho}\right)\left(1 - \frac{1}{Q}\right) \\
&\geq 0
\end{aligned}$$

as $Q > 1$,

q.e.d.

Definition 3.9. *We denote by*

$$\bar{\boldsymbol{a}}^{\oplus} \;=\; (0, \ldots, 0, 1, 0 \ldots, 0, 1, 0, , \ldots, 0, \frac{1 - (h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}})}{h_{\widehat{\tau}_r}}, 0 \ldots, 0)\;.$$

the pre–coalition where the non–zero coordinates appear exactly along $\widehat{\boldsymbol{\tau}} \;=\; (\widehat{\tau}_1, \widehat{\tau}_2, \ldots, \widehat{\tau}_r)$.

Also,

$$\bar{\boldsymbol{a}}^{\ominus} \;:=\; \left(1, \ldots, 1, \frac{(h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}}\;, \frac{1 - (h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}}\right) \tag{3.23}$$

denotes a/the pre–coalition with non vanishing coordinates along the sequence $(\boldsymbol{\tau}, \tau_r) = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \tau_r)$ for some $\tau_r \neq \widehat{\tau}_r$.

Moreover,

$$\bar{\boldsymbol{a}}^{\odot} \;=\; (0,\ldots,0,1,0\ldots,0,1,0,,\ldots,,\ldots,0,1,0\ldots,0)\ .$$

denotes a pre–coalition the coordinates 1 *of which appear exactly along a sequence* $\boldsymbol{\tau}$ *that differs from* $\widehat{\boldsymbol{\tau}}$ at least *at one coordinate* ρ*, e.g.*
$\boldsymbol{\tau} = (\widehat{\tau}_1,\ldots,\widehat{\tau}_{\rho-1},\tau_\rho,\widehat{\tau}_{\rho+1},\ldots,\widehat{\tau}_r)$.

Theorem 3.10. *1. The pre–coalitions* $\bar{\boldsymbol{a}}^{\oplus}$ *and* $\bar{\boldsymbol{a}}^{\ominus}$ *are efficient. More precisely,*

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\oplus} = 1\ ,\quad \bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} < 1\ . \tag{3.24}$$

2. $\bar{\boldsymbol{x}}\boldsymbol{a}^{\odot} \geq 1 \;=\; v(\boldsymbol{a}^{\odot})$.

3. $\bar{\boldsymbol{x}}$ *is an extremal point of*

$$\boldsymbol{H} \;=\; \{\boldsymbol{x}\in\boldsymbol{J}\mid \boldsymbol{x}\boldsymbol{a}\geq v(\boldsymbol{a}) = 1 \quad (\boldsymbol{a}\in\boldsymbol{A}^s)\}$$

4. $\bar{\boldsymbol{x}}$ *is the only extremal of* $\boldsymbol{H}$*, i.e.,*

$$\boldsymbol{H} \;=\; \boldsymbol{ConvH}\big\{\bar{\boldsymbol{x}},\ \boldsymbol{e}^{T^\rho}\quad(\rho\in\mathbf{R})\big\}\ .$$

Proof:

1^st^STEP :

The pre–imputation $\bar{\boldsymbol{x}}$ is the solution of the defining system with boundary conditions given by $\overline{x}_\tau = h_t$ $(\tau = \widehat{\tau}_1,\widehat{\tau}_2,\ldots,\widehat{\tau}_r)$. Hence, along $\widehat{\tau} \;=\; (\widehat{\tau}_1,\widehat{\tau}_2,\ldots,\widehat{\tau}_r)$ all the coordinates of $\bar{\boldsymbol{x}}$ equal those of $\boldsymbol{h}$, and consequently $\bar{\boldsymbol{x}}$ obviously satisfies all equations

$$\begin{aligned}\bar{\boldsymbol{x}}\bar{\boldsymbol{a}}^{\oplus} &= \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} x_{\widehat{\tau}_\rho} + \frac{1-(\sum_{\mathbf{R}\setminus\{\sigma\}} h_{\widehat{\tau}_\rho})}{h_{\widehat{\tau}_\sigma}} h_{\widehat{\tau}_\sigma}\\ &= \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} h_{\widehat{\tau}_\rho} + \frac{1-(\sum_{\mathbf{R}\setminus\{\sigma\}} h_{\widehat{\tau}_\rho})}{h_{\widehat{\tau}_\sigma}} h_{\widehat{\tau}_\sigma} \;=\; 1\quad .\end{aligned} \tag{3.25}$$

Next,

(3.26)

$$\begin{aligned}\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} &= (h_{\widehat{\tau}_1},\ldots,h_{\widehat{\tau}_r},\overline{x}_{\overline{\overline{\tau}}_r})\left(1,\ldots,1,\frac{(h_{\widehat{\tau}_1}+\ldots+h_{\widehat{\tau}_{r-1}}+h_{\overline{\overline{\tau}}_r})-1}{h_{\overline{\overline{\tau}}_r}-h_{\widehat{\tau}_r}}\ ,\frac{1-(h_{\widehat{\tau}_1}+\ldots+h_{\widehat{\tau}_r})}{h_{\overline{\overline{\tau}}_r}-h_{\widehat{\tau}_r}},\right)\\ &= h_{\widehat{\tau}_1}+\ldots+h_{\widehat{\tau}_{r-1}}+\alpha h_{\widehat{\tau}_r}+\beta\overline{x}_{\overline{\overline{\tau}}_r}\ ,\end{aligned}$$

where α,β are the last two coordinates of $\boldsymbol{a}^{\ominus}$ which are positive and sum up to 1. Hence, if $h_{\widehat{\tau}_r} \geq \overline{x}_{\tau_r}$, then

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} \leq \sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho} < 1\ .$$

On the other hand, if $h_{\widehat{\tau}_r} < \overline{x}_{\tau_r}$, then

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} \leq \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\widehat{\tau}_\rho} + \overline{x}_{\tau_r} < h_{\tau_1} + \sum_{\rho \in \mathbf{R} \setminus \{1,r\}} h_{\widehat{\tau}_\rho} + \overline{x}_{\tau_r} = 1 \ ,$$

using equations (3.4).

2$^{\text{nd}}$STEP : For any pre–coalition $\boldsymbol{a}^{\odot}$ as above we have

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\odot} = \sum_{\rho \in \mathbf{R}} \overline{x}_{\tau_\rho} \geq \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} = 1 \tag{3.27}$$

by (3.22).

3$^{\text{rd}}$STEP : Obviously $\bar{\boldsymbol{x}}$ satisfies a system of equations that are among the inequalities defining $\boldsymbol{H}$. As $\bar{\boldsymbol{x}} \in \boldsymbol{H}$ by the first two steps, clearly, $\bar{\boldsymbol{x}}$ is extremal in $\boldsymbol{H}$.

4$^{\text{rd}}$STEP : Finally, let $\widetilde{\boldsymbol{x}}$ be a further extremal point of $\boldsymbol{H}$. Then $\widetilde{\boldsymbol{x}}$ has to satisfy all the equations (3.4) at least with an inequality, say

$$x_{\tau_\sigma} + \sum_{\rho \in \mathbf{R} \setminus \{\sigma, r\}} \widehat{x}_{\widehat{\tau}_\rho} + x_{\tau_r} \ \geq \ 1 \quad (\sigma \in \mathbf{R} \setminus \{r\}) \ , \tag{3.28}$$

Moreover, there will be further inequalities involving more than one coordinate $\tau \notin \{\widehat{\tau}_1, \ldots, \widehat{\tau}_r\}$, say

$$x_{\tau_\sigma} + x_{\tau_{\sigma'}} \sum_{\rho \in \mathbf{R} \setminus \{\sigma, \sigma', r\}} \widehat{x}_{\widehat{\tau}_\rho} + x_{\tau_r} \ \geq \ 1 \quad (\sigma \in \mathbf{R} \setminus \{r\}) \ . \tag{3.29}$$

Suppose that we have an equation in one of the inequalities (3.29). then there are two corresponding equations of type (3.28) and it is seen that at least one of the terms $x_{\tau_\sigma}, x_{\tau_{\sigma'}}$ has to be zero as all terms involved are non negative. We would then proceed by induction (neglecting for the moment our requirement that all $\boldsymbol{T}^{\tau}$ are of equal size). Thus, let us assume that all equations determining $\widetilde{\boldsymbol{x}}$ are of the type (3.28).

Next, assume that two of the terms $x_{\tau_s}, x_{\tau_{\sigma'}}$ are different, say $x_{\tau_s} > x_{\tau_{\sigma'}}$ for some $\tau_s, \tau_{\sigma'} \in \boldsymbol{T}^{\sigma}$. Both terms would have to appear in certain inequalities of type (3.28). Regarding both these equations exchanging x_{τ_σ} and $x_{\tau_{\sigma'}}$ would necessarily result in an inequality of at least one of these equations, a contradiction. Hence all terms $x_{\tau_s}, x_{\tau_{\sigma'}}$ for $\tau_s, t_{\sigma'} \in \boldsymbol{T}^{\sigma}$ have to be equal. Consequently, $\widetilde{x}$ satisfies equations (3.3) as well.

Finally, the variables $\widetilde{x}_{\widehat{\tau}_\rho}$ can appear only in equations of the type dictated by $\boldsymbol{a}^{\ominus}$ from which it follows that $\widetilde{\boldsymbol{x}}$ satisfies all equations determining $\bar{\boldsymbol{x}}$, hence $\widetilde{\boldsymbol{x}} = \bar{\boldsymbol{x}}$.

q.e.d.

Now we have

Theorem 3.11. *Assume the conditions for rich central commodity to be satisfied. Then*

$$\mathcal{H} \;=\; \boldsymbol{ConvH}\left\{\boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}, \boldsymbol{\lambda}^{\rho}(\rho \in \mathbf{R})\right\}$$

i.e., the set of imputations induced by

$$\boldsymbol{H} \;=\; \boldsymbol{ConvH}\left\{\bar{\boldsymbol{x}}, \boldsymbol{e}^{\mathbf{T}^{\rho}} \quad (\rho \in \mathbf{R})\right\} \;, \tag{3.30}$$

is internally stable.

Proof:

1^st STEP : By definition, all vectors $\boldsymbol{x} \in \boldsymbol{H}$ satisfy $\boldsymbol{x}\boldsymbol{a} = v(\boldsymbol{a}) = 1$ for all separating pre–coalitions $\boldsymbol{a} \in \boldsymbol{A}^s$. Hence, no separating relevant vector induces a coalition that yields a domination. Therefore, we can restrict ourselves to domination via the non–separating relevant vectors of the type $\boldsymbol{a}^{\ominus}$ described by item 3 of Theorem 3.4 of [2] (relevant vectors of the third type), also described in Definition 3.9. These vectors $\bar{\boldsymbol{a}}^{\ominus}$ are given by a sequence $(\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}}_r)$ by

$$\begin{aligned}
\overline{a}_{\widehat{\tau}_\rho} &= 1 \quad (\rho \in \mathbf{R} \setminus \{r\}) \\
\overline{a}_{\widehat{\tau}_r} &= \frac{(h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}} \\
\overline{a}_{\overline{\overline{\tau}}_r} &= \frac{1 - (h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}}, \\
\overline{a}_{\tau} &= 0 \quad \text{otherwise}
\end{aligned} \tag{3.31}$$

with

$$h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_r} < \; 1 \; < h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r} \;.$$

Now according to (3.28) and (3.29) we have for the extremal $\bar{\boldsymbol{x}}$ of $\boldsymbol{H}$

$$\sum_{\rho \in \mathbf{R}\setminus\{r\}} \overline{x}_{\widehat{\tau}_\rho} + \overline{x}_{\overline{\overline{\tau}}_r} \geq 1 \quad . \tag{3.32}$$

for any separating sequence $\overline{\tau}$ including $\overline{\overline{\tau}}_r$. On the other hand, for any separating sequence *not* including $\overline{\overline{\tau}}_r$ we have

$$\sum_{\rho \in \mathbf{R}} \overline{x}_{\widehat{\tau}_\rho} < 1 \quad . \tag{3.33}$$

Finally, any $\boldsymbol{e}^{T^{\rho}}$ $(\rho \in \mathbf{R})$ and hence any vector $\boldsymbol{e}$ of the pre–core satisfies

$$\sum_{\rho \in \mathbf{R}} e_{\widehat{\tau}_\rho} = 1 \quad , \tag{3.34}$$

no matter whether the separating sequence ends up with or without $\overline{\overline{\tau}}_r$.

2^nd STEP :

The vectors of $\boldsymbol{H}$ are of the form

$$\boldsymbol{x} = \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{e}^{T^\rho} + \overline{\alpha}\bar{\boldsymbol{x}} \tag{3.35}$$

with a convex coefficients $(\alpha_1, \ldots, \alpha_r, \overline{\alpha})$. We write this

$$\boldsymbol{x} = \left(\sum_{\sigma\in\mathbf{R}} \alpha_\sigma\right) \sum_{\rho\in\mathbf{R}} \frac{\alpha_\rho}{\sum_{\sigma\in\mathbf{R}} \alpha_\sigma} \boldsymbol{e}^{T^\rho} + \overline{\alpha}\bar{\boldsymbol{x}} = (1-\overline{\alpha})\boldsymbol{e} + \overline{\alpha}\bar{\boldsymbol{x}} \ ; \tag{3.36}$$

in other words, any $\boldsymbol{x} \in \boldsymbol{H}$ is a convex combination of a core element and $\bar{\boldsymbol{x}}$. Suppose now, that domination takes place between two elements of $\mathcal{H}$ via some relevant vector as described by (3.31). Then necessarily we have vectors $\boldsymbol{e}$ and $\boldsymbol{e}'$ in the pre–core satisfying

$$(1-\overline{\alpha})\boldsymbol{e} + \overline{\alpha}\bar{\boldsymbol{x}} > (1-\overline{\beta})\boldsymbol{e}' + \overline{\beta}\bar{\boldsymbol{x}} \ . \tag{3.37}$$

with suitable coefficients $\overline{\alpha}$, $\overline{\beta}$ and for the coordinates $(\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}}_r)$. First, consider the separating sequence obtained by omitting $\widehat{\tau}_r$, i.e., $(\widehat{\tau}_1, \ldots, \overline{\overline{\tau}}_r)$. Then, according to (3.32) and (3.34) we find by taking the sum $\sum_{\rho\in\mathbf{R}\setminus\{r\}} x_{\widehat{\tau}_\rho} + x_{\overline{\overline{\tau}}_r} > 1$ on both sides and writing $\xi := \sum_{\rho\in\mathbf{R}\setminus\{r\}} x_{\widehat{\tau}_\rho} + x_{\overline{\overline{\tau}}_r}$

$$\begin{gathered}(1-\overline{\alpha}) + \overline{\alpha}\xi > (1-\overline{\beta}) + \overline{\beta}\xi \\ \text{i.e.} \\ \overline{\alpha}(\xi-1) > \overline{\beta}(\xi-1) \\ \overline{\alpha} > \overline{\beta} \ .\end{gathered}$$

Now we perform the same operation along the sequence $(\widehat{\tau}_1, \ldots, \widehat{\tau}_r)$ *not* including $\overline{\overline{\tau}}_\rho$. Then the sum $\eta := \sum_{\rho\in\mathbf{R}} x_{\widehat{\tau}_\rho} < 1$ can be employed so that summation along the sequence now produces

$$\begin{gathered}(1-\overline{\alpha}) + \overline{\alpha}\eta > (1-\overline{\beta}) + \overline{\beta}\eta \\ \text{i.e.} \\ \overline{\alpha}(\eta-1) > \overline{\beta}(\eta-1) \\ \overline{\alpha}(1-\eta) < \overline{\beta}(1-\eta) \\ \overline{\alpha} < \overline{\beta} \ .\end{gathered}$$

This contradiction proves that domination cannot take place inside $\boldsymbol{H}$ via a non–separating sequence resulting from a relevant vector described by (3.31).

q.e.d.

Remark 3.12. Note that internal stability as treated above does not make any specific use of the particular shape of the pre–imputation $\bar{\boldsymbol{x}}$. We just use the inequalities $\bar{\boldsymbol{x}}\boldsymbol{a} \geq v(\boldsymbol{a}) = 1$ for all separating relevant vectors. Hence the theorem rests essentially on the shape of the relevant vectors only, which was established in Part I (see [2]). Consequently we can say that, whenever some $\widetilde{\boldsymbol{x}} \in \boldsymbol{J}(v)$ satisfies $\widetilde{\boldsymbol{x}}\boldsymbol{a} \geq v(\boldsymbol{a}) = 1$ for all separating relevant vectors, then the convex hull

$$\mathbf{ConvH}\left\{\widetilde{\boldsymbol{x}}, \boldsymbol{e}^{\mathbf{T}^{\rho}} \quad (\rho \in \mathbf{R})\right\} \ , \tag{3.38}$$

is internally stable.

This observation will be useful later on when other versions of the pre–imputation $\bar{\boldsymbol{x}}$ may occur. The above proof will not work for for the convex hull of the $\boldsymbol{e}^{T^{\rho}}$ and *more* than one additional pre–imputation.

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Eventually we shall verify that $\boldsymbol{H}$ or $\mathcal{H}$ is externally stable as well.

Theorem 3.13. *1. Let $\bar{\boldsymbol{a}}^{\oplus}$ be the separating vector of the second type given by Definition 3.9. Let $\mathring{\boldsymbol{x}}$ be an imputation such that*

$$\sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_{\rho}} < \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_{\rho}} \ , \tag{3.39}$$

Then

$$\widehat{\boldsymbol{x}} \operatorname{dom}_{\bar{\boldsymbol{a}}^{\oplus}} \mathring{\boldsymbol{x}} \ . \tag{3.40}$$

2. Let $\boldsymbol{\vartheta}$ be an imputation with minima vector $\boldsymbol{m}$. If

$$\sum_{\rho \in \mathbf{R}} m_{\widehat{\tau}_{\rho}} < \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_{\rho}} \ , \tag{3.41}$$

Then, for sufficiently small $\varepsilon > 0$, there exists an ε–$\bar{\boldsymbol{a}}^{\oplus}$ relevant coalition $T^{\varepsilon} = T^{\varepsilon\bar{\boldsymbol{a}}^{\oplus}}$ and $\widehat{\boldsymbol{x}} \in \boldsymbol{H}$ such that

$$\boldsymbol{\vartheta}^{\widehat{x}} \operatorname{dom}_{T^{\varepsilon}} \boldsymbol{\vartheta} \ . \tag{3.42}$$

Proof:

1$^{\text{st}}$STEP : Assume w.l.g. that r minimizes the quotients $\frac{\mathring{x}_{\widehat{\tau}_{\rho}}}{h_{\widehat{\tau}_{\rho}}}$, i.e, $\frac{\mathring{x}_{\widehat{\tau}_{r}}}{h_{\widehat{\tau}_{r}}} \leq \frac{\mathring{x}_{\widehat{\tau}_{\rho}}}{h_{\widehat{\tau}_{\rho}}}$, or

$$\frac{\mathring{x}_{\widehat{\tau}_{r}}}{h_{\widehat{\tau}_{r}}} h_{\widehat{\tau}_{\rho}} \leq \mathring{x}_{\widehat{\tau}_{\rho}} \quad (\rho \in \mathbf{R}) \ . \tag{3.43}$$

Define $\overline{\alpha} := \frac{\mathring{x}_{\widehat{\tau}_{r}}}{h_{\widehat{\tau}_{r}}} < 1$. Now because of

$$1 - \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_{\rho}} > 1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_{\rho}}$$

it follows that

$$\frac{\left(1 - \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_{\rho}}\right) + \mathring{x}_{\widehat{\tau}_{r}}}{\left(1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_{\rho}}\right) + h_{\widehat{\tau}_{r}}} > \frac{\mathring{x}_{\widehat{\tau}_{r}}}{h_{\widehat{\tau}_{r}}} \ = \ \overline{\alpha} \ ,$$

or, equivalently

$$\frac{1-\sum_{\rho\in\mathbf{R}\setminus\{r\}}\overset{\circ}{x}_{\widehat{\tau}_\rho}}{1-\sum_{\rho\in\mathbf{R}\setminus\{r\}}h_{\widehat{\tau}_\rho}} \;>\; \overline{\alpha}\;,$$

$$1-\sum_{\rho\in\mathbf{R}\setminus\{r\}}\overset{\circ}{x}_{\widehat{\tau}_\rho} \;>\; \overline{\alpha}\left(1-\sum_{\rho\in\mathbf{R}\setminus\{r\}}h_{\widehat{\tau}_\rho}\right)$$

which is

$$1-\overline{\alpha} \;>\; \sum_{\rho\in\mathbf{R}\setminus\{r\}}\left(\overset{\circ}{x}_{\widehat{\tau}_\rho}-\overline{\alpha}h_{\widehat{\tau}_\rho}\;.\right) \tag{3.44}$$

Because of (3.43) the terms under sum in (3.44) are all non negative. Therefore, (3.44) permits to choose positive reals $\alpha_1,\ldots,\alpha_r$ such that

$$1-\overline{\alpha} \;>\; 1-\alpha_r \;>\; \sum_{\rho\in\mathbf{R}\setminus\{r\}}\left(\overset{\circ}{x}_{\widehat{\tau}_\rho}-\overline{\alpha}h_{\widehat{\tau}_\rho}\;.\right) \tag{3.45}$$

$$\alpha_\rho > \overset{\circ}{x}_{\widehat{\tau}_\rho}-\alpha_r h_{\widehat{\tau}_\rho} \quad (\rho\in\mathbf{R}\setminus\{r\})\;, \tag{3.46}$$

and

$$1-\alpha_r \;=\; \sum_{\rho\in\mathbf{R}\setminus\{r\}}\alpha_\rho \tag{3.47}$$

holds true. In other words, the α_ρ are positive convex coefficients,

$$\sum_{\rho\in\mathbf{R}}\alpha_\rho \;=\; 1\;. \tag{3.48}$$

Also, we have

$$\alpha_r > \overline{\alpha} = \frac{\overset{\circ}{x}_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}}\;. \tag{3.49}$$

Now consider the vector

$$\widehat{\boldsymbol{x}} \;:=\; \sum_{\rho\in\mathbf{R}\setminus\{r\}}\alpha_\rho\boldsymbol{e}^{\boldsymbol{T}^\rho}+\alpha_r\overline{\boldsymbol{x}}\;. \tag{3.50}$$

Then clearly for $\rho\in\mathbf{R}\setminus\{r\}$ we have

$$\widehat{x}_{\widehat{\tau}_\rho} \;=\; \alpha_\rho+\alpha_r h_{\widehat{\tau}_\rho} \;>\; \overset{\circ}{x}_{\widehat{\tau}_\rho} \tag{3.51}$$

(in view of (3.46)), and for $\rho=r$

$$\widehat{x}_{\widehat{\tau}_r} \;=\; \alpha_r h_{\widehat{\tau}_r} > \overline{\alpha}h_{\widehat{\tau}_r} \;=\; \overset{\circ}{x}_{\widehat{\tau}_r} \tag{3.52}$$

(in view of (3.49)). Next

$$\widehat{\boldsymbol{x}}\boldsymbol{a}^{\oplus} \;=\; \sum_{\rho\in\mathbf{R}\setminus\{r\}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{T}^\rho}\boldsymbol{a}^{\oplus} + \alpha_r \bar{\boldsymbol{x}}\boldsymbol{a}^{\oplus} \;=\; \sum_{\rho\in\mathbf{R}} \alpha_r = 1 \ . \tag{3.53}$$

Now (3.51),(3.52) and (3.53) imply

$$\widehat{\boldsymbol{x}} \operatorname{dom}_{\boldsymbol{a}^{\oplus}} \mathring{\boldsymbol{x}} \ .$$

2^nd STEP : If $\boldsymbol{\vartheta}$ is an imputation satisfying the condition specified for $\boldsymbol{m}$, then $\boldsymbol{m}$ can play the role of $\mathring{\boldsymbol{x}}$. Hence by Theorem 4.5. of [2] we find, for $\varepsilon > 0$ sufficiently small, an ε–relevant coalition $T^\varepsilon = T^{\varepsilon \boldsymbol{a}^{\oplus}}$ such that

$$\boldsymbol{\vartheta}^{\widehat{x}} \operatorname{dom}_{T^\varepsilon} \boldsymbol{\vartheta} \ ,$$

q.e.d.

Theorem 3.14. *Let $\mathring{\boldsymbol{x}} \in \mathbb{R}^{rt}$ satisfy*

1.

$$1 > \sum_{\rho\in\mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho} > \sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho} \ , \tag{3.54}$$

2. *for all $\tau \in \mathbf{T}^r \setminus \{\widehat{\tau}_\rho\}$*

$$\mathring{x}_{\widehat{\tau}_1} + \mathring{x}_{\widehat{\tau}_2} +, \ldots, + \mathring{x}_{\widehat{\tau}_{r-1}} + \mathring{x}_\tau \;\geq\; 1 \ , \tag{3.55}$$

3.

$$\sum_{\tau\in\mathbf{T}} \lambda_\tau \mathring{x}_\tau \;\leq\; 1 \tag{3.56}$$

that is, $\mathring{\boldsymbol{x}}$ is a "pre–subimputation".

Then there exists $\overset{*}{\boldsymbol{x}} \in \boldsymbol{H}$ and $\overset{*}{\tau} \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\}$ as well as $\boldsymbol{a}^{\ominus}$ with values at coordinates $\widehat{\tau}_1, \widehat{\tau}_2, \ldots, \widehat{\tau}_r, \overset{*}{\tau}$ provided for by Definition 3.9 such that

$$\overset{*}{\boldsymbol{x}} \operatorname{dom}_{\boldsymbol{a}^{\ominus}} \mathring{\boldsymbol{x}} \tag{3.57}$$

holds true.

Proof:

1^st STEP :

Define

$$\widehat{\alpha} \;:=\; \frac{1 - \sum_{\rho\in\mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho}}{1 - \sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho}} \ , \quad 0 < \widehat{\alpha} < 1 \ , \tag{3.58}$$

and

$$\widehat{\alpha}_\rho \ := \ \overset{\circ}{x}_{\widehat{\tau}_\rho} - \widehat{\alpha} h_{\widehat{\tau}_\rho} \quad \rho \in \mathbf{R} \ . \tag{3.59}$$

If it so happens that $\widehat{\alpha}_\rho \geq 0$ for $\rho \in \mathbf{R}$, then $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r, \widehat{\alpha}$ constitutes a set of "convex coefficients". If this is not so, then a slight digression is necessary in order to adjust these coefficients as follows. Because of

$$\widehat{\alpha}(1 - \sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho}) \ = \ (1 - \sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho})$$

i.e .

$$\sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} - \widehat{\alpha} \sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho} = 1 - \widehat{\alpha}$$

i.e.

$$\sum_{\rho\in\mathbf{R}} (\overset{\circ}{x}_{\widehat{\tau}_\rho} - \widehat{\alpha} h_{\widehat{\tau}_\rho}) = 1 - \widehat{\alpha}$$

we have

$$\sum_{\rho\in\mathbf{R}} \widehat{\alpha}_\rho = 1 - \widehat{\alpha} \ .$$

We write $\alpha^+ \ := \ \max\{0, \alpha\}$ for real α. Now consider

$$L(\bullet) : [0,1] \to [0,1]$$

given by

$$L(\alpha) \ := \ \sum_{\rho\in\mathbf{R}} (\overset{\circ}{x}_{\widehat{\tau}_\rho} - \alpha h_{\widehat{\tau}_\rho})^+ \quad (\alpha \in [0,1])$$

which is continuous and decreasing in α. We have

$$L(0) = \sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} \ < \ 1$$

$$L(1) \geq \sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} - \sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho} \ > 0$$

Compare this with the decreasing function $\alpha \to 1 - \alpha$ on $[0,1]$ which has values 1 and 0 at arguments 0 and 1. Clearly we can find some $\overline{\alpha} \in [0,1]$, $\overline{\alpha} \leq \widehat{\alpha}$, such that both functions are equal, that is

$$1 - \overline{\alpha} \ = \ \sum_{\rho\in\mathbf{R}} (\overset{\circ}{x}_{\widehat{\tau}_\rho} - h_{\widehat{\tau}_\rho})^+ \geq \sum_{\rho\in\mathbf{R}} (\overset{\circ}{x}_{\widehat{\tau}_\rho} - h_{\widehat{\tau}_\rho}) \ > 0 \ . \tag{3.60}$$

Define $\overline{\alpha}_1, \ldots, \overline{\alpha}_r, \geq 0$ by

$$\overline{\alpha}_\rho \ := \ (\overset{\circ}{x}_{\widehat{\tau}_\rho} - \alpha h_{\widehat{\tau}_\rho})^+ \geq (\overset{\circ}{x}_{\widehat{\tau}_\rho} - \alpha h_{\widehat{\tau}_\rho}) \tag{3.61}$$

then

$$\sum_{\rho\in\mathbf{R}} \overline{\alpha}_\rho = 1 - \overline{\alpha} \ , \qquad \overline{\alpha} < \widehat{\alpha} \ .$$

Thus, the set of coefficients $\overline{\alpha}_1, \dots, \overline{\alpha}_r, \overline{\alpha}$ replaces the initial set $\widehat{\alpha}_1, \dots, \widehat{\alpha}_r, \widehat{\alpha}$ as constructed above.

2$^{\text{nd}}$STEP :

Now we put

$$\overset{*}{\boldsymbol{x}} := \sum_{\rho \in \mathbf{R}} \overline{\alpha}_\rho \boldsymbol{e}^{T^\rho} + \bar{\alpha} \bar{\boldsymbol{x}} \in \boldsymbol{H} .$$

Then clearly

$$\overset{*}{x}_{\widehat{\tau}_\rho} = \overline{\alpha}_\rho + \overline{\alpha} h_{\widehat{\tau}_\rho} \geq \overset{\circ}{x}_{\widehat{\tau}_\rho}$$

by just rewriting (3.61).

Now by (3.24) we know that $\bar{\boldsymbol{x}} \boldsymbol{a}^\ominus < 1$ and as $\overline{\alpha} > 0$ it follows that

$$\overset{*}{\boldsymbol{x}} \boldsymbol{a}^\ominus < 1 . \tag{3.62}$$

Hence, if we can show that

$$\overset{*}{x}_{\overset{*}{\tau}} > \overset{\circ}{x}_{\overset{*}{\tau}} \text{ for some } \overset{*}{\tau} \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\} , \tag{3.63}$$

then by a slight change in the coordinates of $\overset{*}{\boldsymbol{x}}$ we obtain a strict inequality for coordinates along $\boldsymbol{a}^\ominus$, hence

$$\overset{*}{\boldsymbol{x}} \operatorname{dom}_{\boldsymbol{a}^\ominus} \overset{\circ}{\boldsymbol{x}} , \tag{3.64}$$

as we have claimed.

Indeed, all we have to do is replacing $\overset{*}{\boldsymbol{x}}$ by a suitable convex combination of $\overset{*}{\boldsymbol{x}}$ and the $\boldsymbol{e}^{T^\rho}$. E.g., as $\boldsymbol{e}^{T^1}_{\widehat{\tau}_1} = 1$ and $\overset{*}{x}_1 < 1$ we find that, for sufficiently small $\varepsilon_1 > 0$, the vector

$$\overset{\star}{\boldsymbol{x}}^1 := (1 - \varepsilon_1) \overset{*}{\boldsymbol{x}} + \varepsilon_1 \boldsymbol{e}^{T^1}$$

exceeds $\overset{\circ}{\boldsymbol{x}}$ on the first coordinate without disturbing inequalities (3.62) and (3.63). Next, for sufficiently small $\varepsilon_2 > 0$ the vector

$$\overset{\star}{\boldsymbol{x}}^2 := (1 - \varepsilon_2) \overset{*}{\boldsymbol{x}}^1 + \varepsilon_2 \boldsymbol{e}^{T^2}$$

exceeds $\overset{\circ}{\boldsymbol{x}}$ on the first and second coordinate without disturbing inequalities (3.62) and (3.63); etc.

Thus, it remains to prove (3.63).

3$^{\text{rd}}$STEP :

We shall treat the case that

$$\overset{\circ}{x}_{\widehat{\tau}_\rho} \geq \widehat{\alpha} h_{\widehat{\tau}_\rho} \quad \rho \in \mathbf{R}$$

only, this permits a simplification of the notation. The remaining cases would be treated quite analogously.

Thus we have

$$\overline{\alpha} \;=\; \widehat{\alpha} \;=\; \frac{1-\sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho}}{1-\sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho}} \;<1 \tag{3.65}$$

with $0<\overline{\alpha}<1$. Also

$$\overline{\alpha}_\rho \;:=\; \overset{\circ}{x}_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho} \quad \rho\in\mathbf{R}\;; \tag{3.66}$$

and these quantities are assumed to be non–negative and bounded by 1.

Then, we recall

$$\begin{aligned}
&\overline{\alpha}(1-\sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho}) \;=\; (1-\sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho})\\
&\text{i.e .}\\
&\sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} - \overline{\alpha}\sum_{\rho\in\mathbf{R}} h_{\widehat{\tau}_\rho} = 1-\widehat{\alpha}\\
&\text{i.e.}\\
&\sum_{\rho\in\mathbf{R}} (\overset{\circ}{x}_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho}) = 1-\overline{\alpha}
\end{aligned}$$

and

$$\sum_{\rho\in\mathbf{R}} \overline{\alpha}_\rho = 1-\overline{\alpha}\;,\ \text{i.e.}\ \sum_{\rho\in\mathbf{R}} \overline{\alpha}_\rho + \overline{\alpha} = 1\;.$$

Hence

$$\overset{*}{\boldsymbol{x}} \;:=\; \sum_{\rho\in\mathbf{R}} \overline{\alpha}_\rho \boldsymbol{e}^{T^\rho} + \bar{\alpha}\bar{\boldsymbol{x}} \in \boldsymbol{H}\;,$$

and

$$\overset{*}{x}_{\widehat{\tau}_\rho} = \overline{\alpha}_\rho + \overline{\alpha} h_{\widehat{\tau}_\rho} \;=\; \overset{\circ}{x}_{\widehat{\tau}_\rho}\;.$$

5

5^{th}STEP :

Now, in order to show that

$$\overset{*}{x}_\tau \;=\; \alpha_r + \overline{\alpha x}_\tau > \overset{\circ}{x}_\tau \tag{3.67}$$

holds true for some $\tau\in\mathbf{T}^r\setminus\{\widehat{\tau}_r\}$ we insert $\alpha_r \;=\; \overset{\circ}{x}_{\widehat{\tau}_r} - \overline{\alpha} h_{\widehat{\tau}_r}$ so that we need to show

$$\overset{\circ}{x}_{\widehat{\tau}_r} - \overline{\alpha} h_{\widehat{\tau}_r} + \overline{\alpha x}_\tau > \overset{\circ}{x}_\tau\;,\ \text{i.e.}\ \overset{\circ}{x}_{\widehat{\tau}_r} - \overset{\circ}{x}_\tau > \overline{\alpha}(\overline{x}_{\widehat{\tau}_r} - \overline{x}_\tau)$$

for some $\tau\in\mathbf{T}^r\setminus\{\widehat{\tau}_r\}$. The coefficient of $\overline{\alpha}$ on the right side will turn out to be positive ($5^{th}STEP$, formula (3.73)); hence we have to show

$$\frac{\overset{\circ}{x}_{\widehat{\tau}_r} - \overset{\circ}{x}_\tau}{\overline{x}_{\widehat{\tau}_r} - \overline{x}_\tau} \;>\; \overline{\alpha} \tag{3.68}$$

for some $\tau \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\}$.

4thSTEP :

We start out with

$$\mathring{x}_\tau - \mathring{x}_{\widehat{\tau}_r} \;>\; \frac{1 - \sum_{\rho\in\mathbb{R}} x_{\widehat{\tau}_\rho}}{Q} \tag{3.69}$$

for some $\tau \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\}$. Observe that the inequality looks similar to equation (3.16) in Lemma 3.6 up to the equation being replaced by an inequality.

To this end we first of all argue that w.l.g. we may assume that

$$\mathring{x}_\tau \;=\; \mathring{x}_{\tau'} \quad (\tau, \tau\prime \in \mathbf{T}^\rho \setminus \{\widehat{\tau}_\rho\},\; \rho \in \mathbf{R}) \;, \tag{3.70}$$

that is, equations (3.3) are satisfied for $\boldsymbol{x} = \mathring{\boldsymbol{x}}$. Indeed, otherwise replace $\mathring{x}_\tau$ by

$$\dot{x}_\tau \;:=\; \frac{\sum_{\tau\in\mathbf{T}^\rho\setminus\{\widehat{\tau}_\rho\}} \lambda_t \mathring{x}_\tau}{\sum_{\tau\in\mathbf{T}^\rho\setminus\{\widehat{\tau}_\rho\}} \lambda_t} \;.$$

This leads to an imputation $\dot{\boldsymbol{x}}$ which is the same as $\mathring{\boldsymbol{x}}$ on coordinates $\widehat{\tau}_1, , \ldots, \widehat{\tau}_{r-1}$ and smaller on at least *some* coordinate $\overset{*}{\tau} \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\}$. Then a domination of $\dot{\boldsymbol{x}}$ along coordinates $\widehat{\tau}_1, , \ldots, \widehat{\tau}_r, \tau_r$ can at once be seen to induce a domination of $\mathring{\boldsymbol{x}}$ along coordinates $\widehat{\tau}_1, , \ldots, \widehat{\tau}_r, \overset{*}{\tau}$ for some $\overset{*}{\tau} \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\}$.

Next, we observe that instead of equations (3.3) and (3.4) we now have inequalities (3.55) and (3.56). Therefore, returning to Remark 3.2, we obtain

$$\mathring{x}_{\tau_\sigma} \;>\; 1 - H_\sigma - \mathring{x}_{\tau_r} \;. \tag{3.71}$$

replacing equation (3.9) and

$$\sum_{\rho\in\mathbf{R}\setminus\{r\}} \Lambda_\rho(1 - H_\rho - \mathring{x}_{\tau_r}) + \Lambda_r \mathring{x}_{\tau_r} \;\leq\; 1 \;-\; \sum_{\rho\in\mathbf{R}} \lambda_{\widehat{\tau}_\rho} \mathring{x}_{\widehat{\tau}_\rho} \tag{3.72}$$

replacing equation (3.10). We may then follow up the development in Lemma 3.6 to exactly obtain the desired inequality (3.69).

5thSTEP :

Now in order to verify the inequality (3.68) we use inequality (3.69) and equation (3.16) in Lemma 3.6, again noting that the denominator Q in both terms is independent of the initial data $\overline{x}_{\widehat{\tau}_\rho}$ or $\mathring{x}_{\widehat{\tau}_\rho}$ respectively. It follows then that

$$\begin{gathered} \mathring{x}_{\widehat{\tau}_r} - \mathring{x}_\tau \;\geq\; \frac{1 - \sum_{\rho\in\mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho}}{Q} \\ \text{and} \\ \overline{x}_{\widehat{\tau}_r} - \overline{x}_\tau \;=\; \frac{1 - \sum_{\rho\in\mathbf{R}} \overline{x}_{\widehat{\tau}_\rho}}{Q} \;, \end{gathered} \tag{3.73}$$

consequently

$$\frac{\overset{\circ}{x}_{\widehat{\tau}_r} - \overset{\circ}{x}_{\tau}}{\overline{x}_{\widehat{\tau}_r} - \overline{x}_{\tau}} \geq \frac{1 - \sum_{\rho\in\mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho}}{1 - \sum_{\rho\in\mathbf{R}} \overline{x}_{\widehat{\tau}_\rho}} = \overline{\alpha} \tag{3.74}$$

where the quantifier is now meant for all $\tau \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\}$ as all $\overset{\circ}{x}_\tau$ and $\overline{x}_\tau$ are all equal for $\tau \in \mathbf{T}^r \setminus \{\widehat{\tau}_r\}$. Thus, the quotient is which proves (3.68).

6$^{\text{th}}$STEP :

Now, we have

$$\overset{*}{x}_{\overset{*}{\tau}} = \overline{\alpha}_r + \overline{\alpha} h_\tau > \overset{\circ}{x}_\tau \tag{3.75}$$

we know that $\widehat{\boldsymbol{x}}$ exceeds $\overset{\circ}{\boldsymbol{x}}$ strictly along all the coordinates of $\boldsymbol{a}^\ominus$. Moreover

$$\widehat{\boldsymbol{x}}\boldsymbol{a}^\ominus = \sum_{\rho\in\mathbf{R}} \overline{\alpha}_\rho \boldsymbol{e}^{T^\rho}\boldsymbol{a}^\ominus + \overline{\alpha}\bar{\boldsymbol{x}}\boldsymbol{a}^\ominus < \sum_{\rho\in\mathbf{R}} \overline{\alpha}_\rho + \overline{\alpha} = 1 \tag{3.76}$$

using (3.24). Hence we have $\widehat{\boldsymbol{x}}\,\mathrm{dom}_{\boldsymbol{a}^\ominus}\,\overset{\circ}{\boldsymbol{x}}$.

7$^{\text{th}}$STEP : Given $\boldsymbol{\vartheta}$, the minima vector $\boldsymbol{m}$ can play the role of $\overset{\circ}{\boldsymbol{x}}$. By Theorem 4.5. of [2] we find, for $\varepsilon > 0$ sufficiently small, an ε–relevant coalition $T^\varepsilon = T^{\varepsilon\boldsymbol{a}^\ominus}$ such that

$$\boldsymbol{\vartheta}^{\widehat{x}}\,\mathrm{dom}_{T^\varepsilon}\,\boldsymbol{\vartheta} \ ,$$

q.e.d.

Remark 3.15. If $\overset{\circ}{\boldsymbol{x}}$ satisfies

$$\overset{\circ}{x}_{\tau_1} + \overset{\circ}{x}_{\tau_2} +, \ldots, + \overset{\circ}{x}_{\tau_{r-1}} + \overset{\circ}{x}_{\tau_r} \geq 1 \ ,$$

for *all* sequences $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r)$, then $\overset{\circ}{\boldsymbol{x}}$ equals some $\boldsymbol{e}^{\mathbf{T}^\rho}$. This follows from Lemma 4.8 and Theorem 4.9 of [2]. Therefore, if $\boldsymbol{\vartheta}$ is an imputation such that the minima vector $\boldsymbol{m}$ satisfies

$$\overset{\circ}{m}_{\tau_1} + \overset{\circ}{m}_{\tau_2} +, \ldots, + \overset{\circ}{m}_{\tau_{r-1}} + \overset{\circ}{m}_{\tau_r} \geq 1 \ ,$$

for *all* sequences $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r)$, then $\boldsymbol{m} = \boldsymbol{e}^{\mathbf{T}^\rho}$ for some $\rho \in \mathbf{R}$. Hence the minima vector is a pre–imputation from which it follows at once that $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^{\boldsymbol{m}} = \boldsymbol{\vartheta}^{\boldsymbol{e}^{\mathbf{T}^\rho}} = \boldsymbol{\lambda}^\rho$.

◦ ∼∼∼∼∼∼∼ ◦

Theorem 3.16. *Let $\boldsymbol{v}$ be a game with rich central commodity. Let $\boldsymbol{H}$ be the set of preimputations*

$$\boldsymbol{H} = \{\boldsymbol{x} \in \boldsymbol{J} \,\big|\, \boldsymbol{x}\boldsymbol{a} \geq 1 \quad (\boldsymbol{a} \in \boldsymbol{A}^\sigma)\} = \boldsymbol{ConvH}\big\{\bar{\boldsymbol{x}}, \boldsymbol{e}^{\mathbf{T}^\rho} \quad (\rho \in \mathbf{R})\big\} \tag{3.77}$$

Then

$$\mathcal{H} = \boldsymbol{ConvH}\big\{\boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}, \boldsymbol{\lambda}^\rho(\rho \in \mathbf{R})\big\}$$

is externally stable, hence a vNM–Stable Set.

Proof: Extern stability follows from Theorems 3.13, 3.14 and the above Remark 3.15. Intern stability has been proven in Theorem 3.11

q.e.d.

Remark 3.17. As we have seen in Remark 3.13, the conditions to the quantities $\lambda_\bullet$ determining the area of the domains of the density amount to small density factors implying large areas. Now, the result established with regard to the vNM–Stable Set we have exhibited is quite fine–tuned:

Other than the case studied in Part I, the core is no longer stable. Instead, there appears a unique further pre–imputation $\bar{\boldsymbol{x}}$ which is obtained by correcting the density of $\boldsymbol{\lambda}^0$ outside the areas of small values in a way such that a pre–imputation results. The resulting imputation together with the elements of the core now establishes the vNM–Stable Set. Clearly, as we have a min game, a rich central commodity along some sequence $h_{\widehat{\tau}_\rho}$ increases the importance of that commodity for the formation of dominating imputations and it is this importance that is exactly reflected by $\boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}}$.

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