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Convex vNM–Stable Sets for a Semi Orthogonal Game

Part I:

ε-Relevant Coalitions

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Abstract

We consider (cooperative) linear production games with a continuum of players. The coalitional function is generated by $r+1$ "production factors" that is, non atomic measures defined on an interval. r of these are orthogonal probabilities which, economically, can be considered as "cornered" production factors. The $r+1^{th}$ measure involved has positive mass "across the carriers" of the orthogonal probabilities. That is, there is a "non–cornered" (or "central") production factor available throughout the market. We consider convex vNM–Stable Sets of this game. Depending on the size of the central measure, we observe cases in which a vNM–Stable Set is uniquely defined to be either the core or the convex hull of the core plus a unique additional imputation. We observe other situations in which a variety of vNM–Stable Sets exists.

Within this first part we will present the coalitions that are necessary and sufficient for dominance relations between imputations. In the context of the "purely orthogonal" production game this question is answered in a rather straightforward way by the "Inheritance Theorem" established in [3]. However, once orthogonality is abandoned one has to establish prerequisites about ε–relevant coalitions. Thus, this first part centers around the formulation of a generalized "Inheritance Theorem".

As a consequence, based on the Inheritance Theorem, we provide conditions for the core to be a vNM–Stable Set whenever the central commodity is available in abundance.

1 Introduction

The main theorem in [3]and [4] is the Characterization Theorem for convex vNM–Stable Sets of non–atomic orthogonal linear production games with a continuum of players. This theorem states that every vNM–Stable Set is *standard*, that is, generated as the convex hull of finitely many probabilities that are absolutely continuous to the factor distributions of the market respectively.

Within this paper we start out to discuss a non–orthogonal linear production game. We describe vNM–stable sets for this game that – in addition to a set of orthogonal measures – admits of a further measure "across" the carriers. Thus, apart from several separate sectors of a market ("corners") providing a unique factor, production is also enabled by a distribution that is available in all sectors.

Rudimentary, this kind of game appears in HOLZMAN, MONDERER, EINY, AND SHITOVITZ, see [2]. They provide an example with a measure which has mass accumulated across the carriers of two orthogonal probabilities. Here, the third factor appears in such quantities that the core still provides the unique vNM–Stable Set.

After a change of some quantities the core fails to dominate all imputations outside. Then there appears a (sometimes unique) vNM–Stable set containing the core. We provide a description of the EHMS–example in SECTION 3 as well as in a subsequent paper regarding this topic.

The aim of the present paper is, however, the discussion of the general "Semi Orthogonal Game" and in particular, its "ε–relevant coalitions". By contrast we refer to the model discussed in [3] and [4] as to the "Purely Orthogonal Game". Within this paper we provide the appropriate generalization of the Inheritance Theorem of [3].

2 Notations and Definitions

We start by collecting some definitions and notations necessary for our presentation. We follow the conventions used in [3] and [4].

The general background is given by a (cooperative) ***game*** with a continuum of players, i.e., is a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ such that $\boldsymbol{I}$ is some interval in the reals (the ***players***), $\underline{\underline{\mathbf{F}}}$ is the σ−field of (Borel) measurable sets (the ***coalitions***) and $\boldsymbol{v}$ (the ***coalitional function***) is a mapping $\boldsymbol{v} : \underline{\underline{\mathbf{F}}} \to \mathbb{R}_+$ which is absolutely continuous w.r.t. the Lebesgue measure $\boldsymbol{\lambda}$. We focus on "linear production games", that is, $\boldsymbol{v}$ is described by finitely many measures $\boldsymbol{\lambda}^\rho$, $(\rho \in \{0, 1, \ldots, r\})$ via

$$\boldsymbol{v}(S) := \min\{\boldsymbol{\lambda}^\rho(S) \mid \rho \in \{0, 1, \ldots, r\}\} \quad (S \in \underline{\underline{\mathbf{F}}}). \tag{2.1}$$

We write $\mathbf{R} := \{1, \ldots, r\}$ and $\mathbf{R}_0 := \{0, 1, \ldots, r\} = \mathbf{R} \cup \{0\}$. Also, we use $\bigwedge$ to denote the **min *operation*** implied by (2.1) in the lattice of set functions on $\underline{\underline{\mathbf{F}}}$. Then (2.1) is equivalent to

$$v = \bigwedge \{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} \;=\; \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho. \tag{2.2}$$

The ***carrier*** of a measure $\boldsymbol{\mu}$ is denoted by $\boldsymbol{C}(\boldsymbol{\mu})$. As we consider measures that are absolutely continuous w.r.t to Lebesgue measure $\boldsymbol{\lambda}$, we use $\dot{\boldsymbol{\mu}}$ to denote the Radon–Nikodym density of $\boldsymbol{\mu}$ w.r.t. $\boldsymbol{\lambda}$. The assumption of an underlying "reference measure" and existing densities for the members of a vNM–Stable Set is justified in [3],[4]. As in these previous papers statements referring to the reference measure are meant to hold true almost surely – abbreviations like "a.s" or "a.e." will generally be omitted.

We assume that the measures $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r$ are orthogonal probabilities. There is no loss of generalization in assuming that they are copies of Lebesgue measure. Thus we choose the player set to be $\boldsymbol{I} := [0, r)$. The carrier of $\boldsymbol{\lambda}^\rho$ is $\boldsymbol{C}^\rho := [\rho - 1, \rho)$ $(\rho \in \mathbf{R})$ such that $\boldsymbol{\lambda}^\rho := \boldsymbol{\lambda}_{\boldsymbol{C}^\rho}$. This way the $\boldsymbol{\lambda}^\rho$ are orthogonal probabilities defined on $\boldsymbol{I}$ satisfying $\bigcup_{\rho \in \mathbf{R}} \boldsymbol{C}^\rho = \boldsymbol{I}$.

The measure $\boldsymbol{\lambda}^0$ is assumed to have a piecewise constant density $\dot{\boldsymbol{\lambda}}^0$ w.r.t $\boldsymbol{\lambda}$. To describe this density, we choose $t \in \mathbb{N}$ and, for $\rho \in \mathbf{R}$, define $\mathbf{T}^\rho := \{(\rho - 1)t + 1, \ldots, \rho t\}$ as well as $\mathbf{T} := \cup_{\rho \in \mathbf{R}} \mathbf{T}^\rho)$. Next we choose a partition $\{\boldsymbol{D}^\tau\}_{\tau \in \mathbf{T}^\rho}$ of $\boldsymbol{C}^\rho$ such that $\bigcup_{\tau \in \mathbf{T}^\rho} \boldsymbol{D}^\tau = \boldsymbol{C}^\rho$ and $\boldsymbol{D}^\tau \cap \boldsymbol{D}^{\tau'} = \emptyset$ $(\tau, \tau' \in \mathbf{T}^\rho)$. Finally we choose constants $h_\tau \geq 0$ $(\tau \in \mathbf{T})$ such that the density $\dot{\boldsymbol{\lambda}}^0$ is given by

$$\dot{\boldsymbol{\lambda}}^0 = h_\tau \text{ on } \boldsymbol{D}^\tau, \quad (\tau \in \mathbf{T}) \tag{2.3}$$

Introducing the indicator function $\mathbb{1}_S$of a set S we write (2.3) also as

$$\dot{\boldsymbol{\lambda}}^0 \;=\; \sum_{\tau \in \mathbf{T}} h_\tau \, \mathbb{1}_{\boldsymbol{D}^\tau} \;; \tag{2.4}$$

We write

$$\lambda_\tau \;:=\; \boldsymbol{\lambda}(\boldsymbol{D}^\tau) \;\; (\tau \in \mathbf{T}), \;\; \lambda^0_\tau \;:=\; \boldsymbol{\lambda}^0(\boldsymbol{D}^\tau) \;\; (\tau \in \mathbf{T}), \tag{2.5}$$

and for any $\mathbf{T}' \subseteq \mathbf{T}$

$$\lambda_{\mathbf{T}'} \;=\; \boldsymbol{\lambda}(\cup_{\tau \in \mathbf{T}'} \boldsymbol{D}_\tau), \;\; \lambda^0_{\mathbf{T}'} \;=\; \boldsymbol{\lambda}^0(\cup_{\tau \in \mathbf{T}'} \boldsymbol{D}_\tau) \tag{2.6}$$

such that $\sum_{\tau \in \mathbf{T}^\rho} \lambda^\rho_\tau = 1$ for $\rho \in \mathbf{R}$ holds true. The measure $\boldsymbol{\lambda}^0$ is *not* assumed to be a probability and it is carrying mass across the carriers of the $\boldsymbol{\lambda}^\rho (\rho \in \mathbf{R})$. More precisely, we assume

$$\boldsymbol{\lambda}^0(\boldsymbol{I}) \;=\; \sum_{\tau \in \mathbf{T}} h_\tau \lambda_t > 1 \;, \tag{2.7}$$

which implies that the core of $\boldsymbol{v}$ is given by

$$\mathcal{C}(\boldsymbol{v}) \;=\; \{\boldsymbol{\lambda}^{\rho} \,|\, \rho \in \mathbf{R}\} \quad ,$$

see (BILLERA–RANAAN [1]).

Example 2.1. Figure 2.1 illustrates a situation for $r = t = 2$. We assume

(2.8) $$h_1 = 0,\; h_2, h_3 < 1, \quad h_4 = 1 \text{ and } h_2\lambda_2 + h_3\lambda_3 + \lambda_4 > 1$$

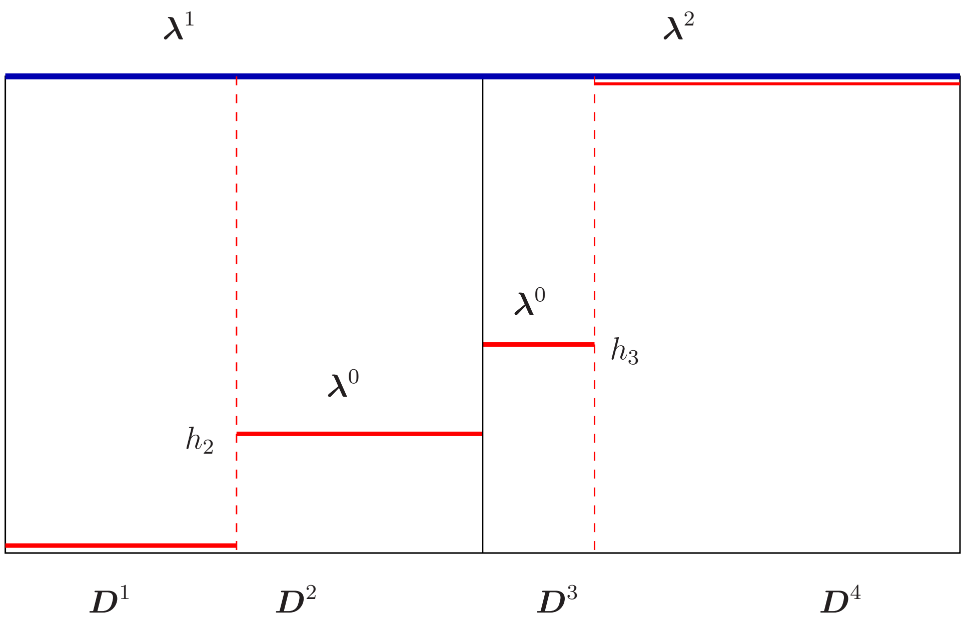

Figure 2.1: The case of 4 steps: the density of $\boldsymbol{\lambda}^0$

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Our solution concept is given by the ***vNM–Stable Set*** . The version for a finite player set has been introduced by VON NEUMANN-MORGENSTERN [5]. Let us repeat the definitions for a continuum of players which is the appropriate one for the present context.

Definition 2.2. *Let* $(I, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ *be a game. An* ***imputation*** *is a measure* $\boldsymbol{\xi}$ *with* $\boldsymbol{\xi}(I) = \boldsymbol{v}(I)$. *An imputation* $\boldsymbol{\xi}$ ***dominates*** *an imputation* $\boldsymbol{\eta}$ *w.r.t* $S \in \underline{\underline{\mathbf{F}}}$ *if* $\boldsymbol{\xi}$ *is* ***effective*** *for* S, *i.e.,*

(2.9) $$\boldsymbol{\lambda}(S) > 0 \quad \textit{and} \quad \boldsymbol{\xi}(S) \le \boldsymbol{v}(S)$$

and if

(2.10) $$\boldsymbol{\xi}(T) > \boldsymbol{\eta}(T) \quad (T \in \underline{\underline{\mathbf{F}}},\; T \subseteq S, \boldsymbol{\lambda}(T) > 0)$$

holds true. That is, every subcoalition of S *(almost every player in* S *) strictly improves its payoff at* $\boldsymbol{\xi}$ *versus* $\boldsymbol{\eta}$. *We write* $\boldsymbol{\xi}\,\mathrm{dom}_S\,\boldsymbol{\eta}$ *to indicate domination. It is standard to use* $\boldsymbol{\xi}\,\mathrm{dom}\,\boldsymbol{\eta}$ *whenever* $\boldsymbol{\xi}\,\mathrm{dom}_S\,\boldsymbol{\eta}$ *holds true for* some *coalition* $S \in \underline{\underline{\mathbf{F}}}$.

We allow domination also to take place between "subimputations", i.e., measures with total mass less than $\boldsymbol{v}(I)$.

Definition 2.3. *Let* $\boldsymbol{v}$ *be a game. A set* $\mathcal{S}$ *of imputations is called a* ***vNM–Stable Set*** *if*

- *there is no pair* $\boldsymbol{\xi}, \boldsymbol{\mu} \in \mathcal{S}$ *such that* $\boldsymbol{\xi} \operatorname{dom} \boldsymbol{\mu}$ *holds true (*"internal stability"*).*
- *for every imputation* $\boldsymbol{\eta} \notin \mathcal{S}$ *there exists* $\boldsymbol{\xi} \in \mathcal{S}$ *such that* $\boldsymbol{\xi} \operatorname{dom} \boldsymbol{\eta}$ *is satisfied (*"external stability"*).*

The discrete nature of the density of $\boldsymbol{\lambda}^0$ carries some implications for the establishment of dominance based on discrete analogues of concepts like imputations, coalitions etc. We refer to these analogues as "pre–concepts". E.g., the "discrete" analogues to imputations are vectors $\boldsymbol{x} \in \mathbb{R}_+^{|\mathbf{T}|} = \mathbb{R}_+^{rt}$ as follows.

Definition 2.4. *The set*

$$\mathbf{J} := \left\{ \boldsymbol{x} = (x_\tau)_{\tau \in \mathbf{T}} \in \mathbb{R}_+^{rt} \;\middle|\; \sum_{\tau \in \mathbf{T}} \lambda_\tau x_\tau = 1 \right\} \tag{2.11}$$

is called the set of ***pre–imputations****. For any* $\boldsymbol{x} \in \boldsymbol{J}$ *the measure* $\boldsymbol{\vartheta}^{\boldsymbol{x}}$ *defined by the density*

$$\dot{\boldsymbol{\vartheta}}^{\boldsymbol{x}} := \sum_{\tau \in \mathbf{T}} x_\tau \mathbb{1}_{\boldsymbol{D}^\tau} , \tag{2.12}$$

clearly constitutes a ("piecewise constant", "step function") imputation $\boldsymbol{\vartheta}^{\boldsymbol{x}} \in \mathcal{I}(\boldsymbol{v})$*.*

Next, a ***pre–coalition*** is a nonnegative vector $\boldsymbol{a} = (a_\tau)_{\tau \in \mathbf{T}} \in \mathbb{R}_+^{rt}$. Such a vector may serve as a "discrete" analogue to coalitions (or rather to the Lebesgue measure of coalitions). More precisely, let $\vec{\boldsymbol{\lambda}}$ be the vector–valued measure defined by

$$\vec{\boldsymbol{\lambda}}(\star) = \{\boldsymbol{\lambda}(\star \cap \boldsymbol{D}^\tau)\}_{\tau \in \mathbf{T}} \tag{2.13}$$

Then for some $T \in \underline{\underline{\mathbf{F}}}$ the vector

$$\boldsymbol{a} = (a_\tau)_{\tau \in \mathbf{T}} := \{\boldsymbol{\lambda}(T \cap \boldsymbol{D}^\tau)\}_{\tau \in \mathbf{T}} = \vec{\boldsymbol{\lambda}}(T) \tag{2.14}$$

reflects the coalition T properly. In particular, for $\varepsilon > 0$ and some vector $\boldsymbol{a} \in \mathbb{R}_+^{rt}$, a coalition $T^{\varepsilon \boldsymbol{a}}$ satisfying $\boldsymbol{\lambda}(T^{\varepsilon \boldsymbol{a}} \cap \boldsymbol{D}^\tau) = \varepsilon a_\tau \quad (\tau \in \mathbf{T})$ yields the vector valued measure $\vec{\boldsymbol{\lambda}}$ evaluated at $T^{\varepsilon \boldsymbol{h}}$ via

$$\vec{\boldsymbol{\lambda}}(T^{\varepsilon \boldsymbol{h}}) = \varepsilon \boldsymbol{a} \quad . \tag{2.15}$$

Consequently, we also introduce the concept of ***pre–measures***,which are functionals on pre–coalitions, hence in the discrete context, vectors as well.

Generally, a vector $\boldsymbol{c} \in \mathbb{R}^{rt}_{+}$ when seen as a density constant on the $\boldsymbol{D}^{\tau}$, gives rise to an (absolutely continuous) measure $\boldsymbol{\mu}$ on $\underline{\underline{\mathbf{F}}}$ via

$$(2.16) \qquad \dot{\boldsymbol{\mu}} := \sum_{\tau \in \mathbf{T}} c_{\tau} \mathbb{1}_{\boldsymbol{D}^{\tau}} \quad \text{or} \quad \boldsymbol{\mu}(\star) := \sum_{\tau \in \mathbf{T}} c_{\tau} \boldsymbol{\lambda}(\star \cap \boldsymbol{D}^{\tau}) \ .$$

Then, for some pre–coalition $\boldsymbol{a}$ and a coalition $T = T^{\boldsymbol{a}}$ as in (2.14) above we have

$$\boldsymbol{\mu}(T^{\boldsymbol{a}}) = \sum_{\tau \in \mathbf{T}} c_{\tau} \boldsymbol{\lambda}(T^{\boldsymbol{a}} \cap \boldsymbol{D}^{\tau}) = \sum_{\tau \in \mathbf{T}} c_{\tau} a_{\tau} = \boldsymbol{c}\boldsymbol{a} = \boldsymbol{c}(\boldsymbol{a}) \ ;$$

thus the linear functional on pre–coalitions

$$(2.17) \qquad \boldsymbol{c} : \mathbb{R}^{rt} \to \mathbb{R} \ , \ \ \boldsymbol{c}(\boldsymbol{a}) := \sum_{\tau \in \mathbf{T}} c_{\tau} a_{\tau} \ .$$

reflects the action of the measure $\boldsymbol{\mu}$ on coalitions.

Specifically the vector

$$\boldsymbol{e}^{0} := (h_t)_{\tau \in \mathbf{T}}$$

reflecting the density of $\boldsymbol{\lambda}^{0}$, corresponds to the functional

$$(2.18) \qquad \boldsymbol{c}^{0} : \mathbb{R}^{rt} \to \mathbb{R} \ , \ \ \boldsymbol{c}^{\rho}(\boldsymbol{a}) := \sum_{\tau \in \mathbf{T}} h_{\tau} a_{\tau}$$

such that for some $T^{\boldsymbol{a}}$ as above we have

$$\boldsymbol{\lambda}^{0}(T^{\boldsymbol{a}}) = \boldsymbol{c}^{0}(\boldsymbol{a}) = \boldsymbol{e}^{0}\boldsymbol{a} \ .$$

Also, we introduce the vector

$$\boldsymbol{e}^{\mathbf{T}^{\rho}} = \mathbb{1}_{\mathbf{T}^{\rho}}(\bullet) \in \mathbb{R}^{rt}_{+} \quad (\rho \in \mathbf{R}) \ .$$

by

$$(2.19) \qquad e^{\mathbf{T}^{\rho}}_{\tau} = \left\{ \begin{array}{ll} 1 & \tau \in \mathbf{T}^{\rho} \\ 0 & \text{otherwise} \end{array} \right\} \ .$$

This vector can be interpreted in three ways:

First of all $\boldsymbol{e}^{\mathbf{T}^{\rho}}$ reflects the density of $\boldsymbol{\lambda}^{\rho}$ w.r.t. to Lebesgue measure, i.e.,

$$(2.20) \qquad \dot{\boldsymbol{\lambda}}^{\rho}(\star) = \mathbb{1}_{\boldsymbol{C}^{\rho}}(\star) = \sum_{\tau \in \mathbf{T}^{\rho}} \mathbb{1}_{\boldsymbol{D}^{\tau}}(\star) = \sum_{\tau \in \mathbf{T}} \boldsymbol{e}^{\mathbf{T}^{\rho}}_{\tau} \mathbb{1}_{\boldsymbol{D}^{\tau}}(\star)$$

and for $T = T^{\boldsymbol{a}}$ as above

$$(2.21) \quad \boldsymbol{\lambda}^{\rho}(T) = \boldsymbol{\lambda}(\boldsymbol{C}^{\rho} \cap T) = \sum_{\tau \in \mathbf{T}^{\rho}} \boldsymbol{\lambda}(\boldsymbol{D}^{\tau} \cap T) = \sum_{\tau \in \mathbf{T}} \boldsymbol{e}^{\mathbf{T}^{\rho}}_{\tau} \boldsymbol{\lambda}(\boldsymbol{D}^{\tau} \cap T) = \boldsymbol{e}^{\mathbf{T}^{\rho}} \boldsymbol{a}$$

Thus, if $\boldsymbol{e}^{\mathsf{T}^\rho}$ is seen as the linear functional

$$\text{(2.22)} \qquad \boldsymbol{c}^\rho \ : \ \mathbb{R}^{rt} \to \mathbb{R} \ , \quad \boldsymbol{c}^\rho(\boldsymbol{a}) \ := \ \sum_{\tau \in \mathsf{T}} e_\tau^{\mathsf{T}^\rho} a_\tau \ = \ \sum_{\tau \in \mathsf{T}^\rho} a_\tau \ .$$

Then, for a pre–coalition $\boldsymbol{a}$, and $T^{\boldsymbol{a}}$ as above, we have

$$\boldsymbol{\lambda}^\rho(T^{\boldsymbol{a}}) \ = \ \sum_{\tau \in \mathsf{T}^\rho} \boldsymbol{\lambda}(T^{\boldsymbol{a}} \cap \boldsymbol{D}^\tau) \ = \ \sum_{\tau \in \mathsf{T}^\rho} a_\tau \ = \ \boldsymbol{c}^\rho(\boldsymbol{a}) \ = \ \boldsymbol{e}^{\mathsf{T}^\rho} \boldsymbol{a} \ ;$$

that is, $\boldsymbol{e}^{\mathsf{T}^\rho}$ or the pre–measure $\boldsymbol{c}^\rho$ correspond to $\boldsymbol{\lambda}^\rho$.

Secondly note that $\boldsymbol{e}^{\mathsf{T}^\rho} \in \boldsymbol{J}$ constitutes a pre–imputation. The imputation generated by $\boldsymbol{x} = \boldsymbol{e}^{T^\rho}$ is of course $\boldsymbol{\vartheta}^{\boldsymbol{x}} = \boldsymbol{\lambda}^\rho$ which plays a double role as an imputation as well as a measure or linear function on coalitions.

Finally, the vector $\boldsymbol{a} = \mathbb{1}_{\mathsf{T}^\rho}(\bullet)$ corresponds to the coalition $T^{\boldsymbol{a}} = \boldsymbol{C}^\rho$.

Next we introduce the discrete version of the coalitional function $\boldsymbol{v}$ which is the ***pre–game*** v. For any pre–coalition $\boldsymbol{a} \in \mathbb{R}^{rt}$ we define

$$\text{(2.23)} \qquad v(\boldsymbol{a}) \ := \ \min\{\boldsymbol{c}^\rho \mid \quad (\rho \in \mathbf{R}_0) \ \}$$

Note that v is a positively homogenous function, i.e., $v(t\boldsymbol{a}) = tv(\boldsymbol{a})$ for any positive real t. The connection to $\boldsymbol{v}$ is of course

$$\varepsilon v(\boldsymbol{a}) = v(\varepsilon \boldsymbol{a}) = \boldsymbol{v}(T^{\varepsilon \boldsymbol{a}})$$

for $\varepsilon > 0, \boldsymbol{a} \in \mathbb{R}_+^{rt}$ and $T^{\varepsilon \boldsymbol{a}}$ as above.

Finally, the convex hull

$$\boldsymbol{C}(v) \ := \ \mathbf{ConvH}\left\{\boldsymbol{e}^{\mathsf{T}^\rho}\right\}_{\rho \in \mathbf{R}} \subseteq \boldsymbol{J}$$

is called the ***pre–core*** and the core of the game satisfies

$$\mathcal{C}(\boldsymbol{v}) \ = \ \mathbf{ConvH}\{\boldsymbol{\lambda}^\rho\}_{\rho \in \mathbf{R}} \ = \ \{\boldsymbol{\vartheta}^{\boldsymbol{x}} \mid \boldsymbol{x} \in \boldsymbol{C}(v)\} \ .$$

3 ε–Relevant Coalitions

We shall now exhibit a set of coalitions that will predominantly be effective for domination. In the "purely" orthogonal case as treated in [3]and [4] (i.e., when there is no measure $\boldsymbol{\lambda}^0$ with carrier across the carriers $\boldsymbol{C}^\rho$) the "Inheritance Theorem" states that it is necessary and sufficient to restrict all considerations regarding domination on coalitions that yield a vector–valued measure $\vec{\boldsymbol{\lambda}}(T) = (\varepsilon, \ldots, \varepsilon)$. In the present semi–orthogonal case the situation is more involved. Yet, we can demonstrate that domination is governed by an essentially finite family of coalitions. As previously we write $\mathbf{R}_0 := \mathbf{R} \cup \{0\} = \{0, \ldots, r\}$.

Definition 3.1. *1. Let*

$$\boldsymbol{A} \; : \; \left\{ \boldsymbol{a} \in \mathbb{R}_+^{rt} \,\middle|\, \boldsymbol{c}^\rho(\boldsymbol{a}) \geq 1 \quad (\rho \in \mathbf{R}_0) \right\} \tag{3.1}$$

*Then $\boldsymbol{A}$ is a convex set. The extremal points of $\boldsymbol{A}$ are called the **relevant vectors**; $\boldsymbol{A}^e$ denotes the set of these extremal points.*

*2. For $\varepsilon > 0$ a coalition T is called **$\boldsymbol{\varepsilon}$–relevant** if there is a relevant vector $\boldsymbol{a} \in \boldsymbol{A}^e$ such that $\vec{\boldsymbol{\lambda}}(T) = \varepsilon \boldsymbol{a}$. We shall also use the term **$\boldsymbol{\varepsilon}$–$\boldsymbol{a}$–relevant** coalition if $\boldsymbol{a}$ has been exhibited.*

Recall the vector valued measure $\vec{\boldsymbol{\lambda}}$ for some coalition $T^{\boldsymbol{a}}$ such that

$$\boldsymbol{a} \;=\; \vec{\boldsymbol{\lambda}}(T^{\boldsymbol{a}}) \;:=\; (\boldsymbol{\lambda}(T^{\boldsymbol{a}} \cap \boldsymbol{D}^\tau))_{\tau \in \mathbf{T}} \;. \tag{3.2}$$

Now, for some coalition $T = T^{\boldsymbol{a}}$ with

$$\boldsymbol{v}(T) = \min\{\boldsymbol{\lambda}^\rho(T) \,|\, \rho \in \mathbf{R}_0\} \;=\; \min\{\boldsymbol{c}^\rho(\boldsymbol{a}) \,|\, \rho \in \mathbf{R}_0\} \;=\; v(\boldsymbol{a}),$$

we obtain that

$$\boldsymbol{a}^0 \;:=\; \frac{\boldsymbol{a}}{v(\boldsymbol{a})} \;=\; \frac{\vec{\boldsymbol{\lambda}}(T)}{\boldsymbol{v}(T)} \in \boldsymbol{A}, \quad v(\boldsymbol{a}^0) = \frac{1}{v(\boldsymbol{a})} v(\boldsymbol{a}) = 1,$$

thus, $\boldsymbol{a}^0$ is located in the boundary of $\boldsymbol{A}$.

We may emulate the "purely" orthogonal case (i.e., when $\boldsymbol{\lambda}^0$ is missing) by choosing all $\mathbf{T}^\rho = \{r\}$ to be single valued in which case it is easy to see that $(1, \ldots, 1)$ is the only extremal of $\boldsymbol{A}$. Then the Inheritance Theorem mentioned above shows any coalition S with positive value contains an ε–relevant coalition T with $\vec{\boldsymbol{\lambda}}(T) = (\varepsilon, \ldots, \varepsilon)$ such that dominance is inherited.

The following Lemma, while of a purely geometric nature, indicates that eventually we will be able to imitate the above consideration within our present context.

Lemma 3.2. *For any $\widehat{\boldsymbol{a}} \in \boldsymbol{A}$ there exists $\bar{\boldsymbol{a}} \in \boldsymbol{A}$ such that*

1. $\bar{\boldsymbol{a}} \leq \widehat{\boldsymbol{a}}$

2. *There is a set* $\boldsymbol{E} \subseteq \boldsymbol{A}^e$, *say*

$$\boldsymbol{E} \ = \ \left\{\boldsymbol{a}^{(k)} \,\middle|\, k \in \mathbf{K}\right\} \subseteq \boldsymbol{A}^e \quad \textit{with} \quad \mathbf{K} \ := \ \{1, \ldots, K\}$$

of relevant vectors as well as a set of "convex" (i.e., nonnegative and summing up to one) coefficients $\{\gamma_k\}_{k\in\mathbf{K}}$ *such that*

$$\bar{\boldsymbol{a}} \ = \ \sum_{k\in\mathbf{K}} \gamma_k \boldsymbol{a}^{(k)} \tag{3.3}$$

holds true.

3. *If* $\min\{\boldsymbol{c}^{\rho}\widehat{\boldsymbol{a}} \mid \rho \in \mathbf{R}_0\} \ = \ 1$, *then there is some* $\overline{\rho} \in \mathbf{R}_0$ *such that for* $k \in \boldsymbol{K}$

$$\boldsymbol{c}^{\overline{\rho}}\widehat{\boldsymbol{a}} \ = \ \boldsymbol{c}^{\overline{\rho}}\bar{\boldsymbol{a}} \ = \ \boldsymbol{c}^{\overline{\rho}}\boldsymbol{a}^{(k)} \ = \ 1$$

holds true.

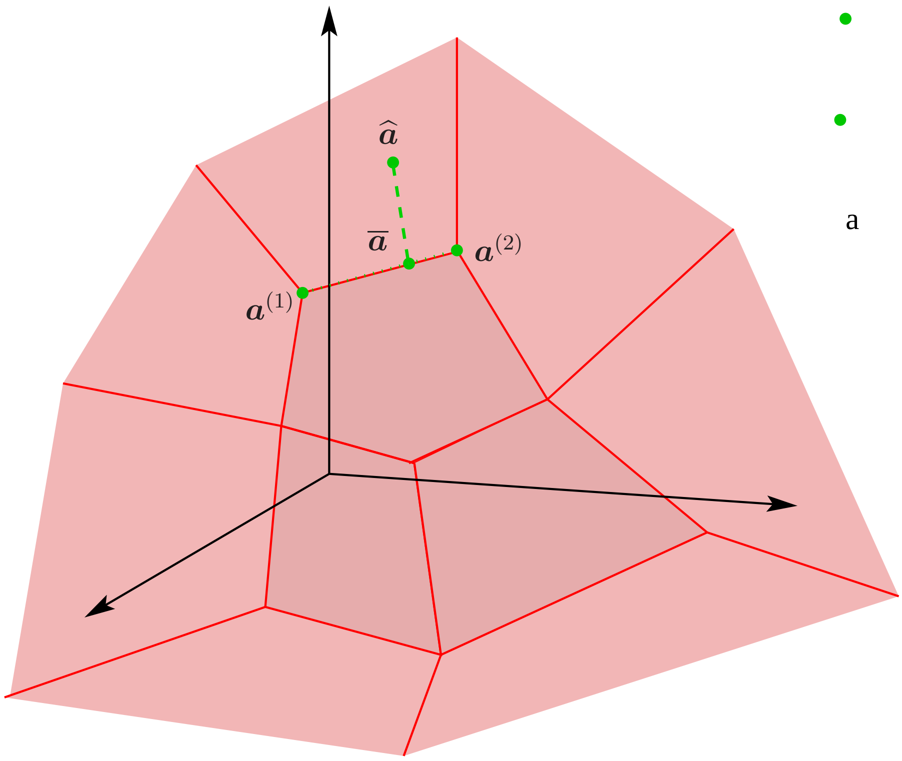

Figure 3.1: The shape of $\boldsymbol{A}$

Proof: Essentially one has to decrease coordinates of $\widehat{\boldsymbol{a}}$ until a compact boundary facet of $\boldsymbol{A}$ is reached. Then the vector obtained is a "convex combination" of vectors in $\boldsymbol{A}^{\boldsymbol{e}}$ by the Krein–Milman Theorem.

1stSTEP :

Given $\widehat{\boldsymbol{a}}$, let $\mathbf{R}_1 := \ \{\rho \in \mathbf{R}_0 \,|\, \boldsymbol{c}^{\rho}\widehat{\boldsymbol{a}} = 1\}$ and put

$$\boldsymbol{I}_0 \ := \ \{\tau \,|\, \widehat{\boldsymbol{a}}_{\tau} > 0\}, \quad \boldsymbol{I}_+ \ := \ \{\tau \,|\, c^{\rho}_{\tau} > 0 \text{ for at least one } \rho \in \mathbf{R}_1\}$$

($\boldsymbol{I}_+ = \emptyset$ for $\mathbf{R}_1 = \emptyset$). Assume that $\boldsymbol{I}_0 \cap \boldsymbol{I}_+^c \neq \emptyset$ is the case. Pick some $\tau \in \boldsymbol{I}_0 \cap \boldsymbol{I}_+^c$. Then $\widehat{a}_\tau > 0$ and $c_\tau^\rho = 0$ for all $\rho \in \mathbf{R}_1$. Consider for $t \geq 0$

$$\widehat{\boldsymbol{a}}^t \;:=\; \widehat{\boldsymbol{a}} - t\boldsymbol{e}^\tau .$$

Clearly, $\widehat{\boldsymbol{a}}^t c^\rho \geq 1$ for all $t \geq 0$ and $\rho \in \mathbf{R}_1$. Therefore there is $t_0 > 0$ such that $\boldsymbol{a}^0 \;:=\; \boldsymbol{a}^{t_0} \in \boldsymbol{A}$ and either $\boldsymbol{a}_\tau^0 = 0$ for some $\tau \in \widehat{\boldsymbol{I}}$ or else $c^\rho \boldsymbol{a}^0 = 1$ for some $\rho \notin \mathbf{R}_1$. In other words, t_0 is the first t such that an additional equation from the inequalities defining $\boldsymbol{A}$ prevails. Replacing $\widehat{\boldsymbol{a}}$ by $\boldsymbol{a}^0$ and defining $\widehat{\boldsymbol{I}}$ and $\boldsymbol{I}_+$ accordingly, we have diminished $\boldsymbol{I}_0 \cap \boldsymbol{I}_+^c$ by at least one element. Clearly $\boldsymbol{a}^0 \leq \widehat{\boldsymbol{a}}$. Proceeding this way, we find, therefore, $\bar{\boldsymbol{a}} \leq \widehat{\boldsymbol{a}}$ such that (with appropriate redefinitions) $\boldsymbol{I}_0 \cap \boldsymbol{I}_+^c = \emptyset$ holds true, that is $\boldsymbol{I}_0 \subseteq \boldsymbol{I}_+$ is the case.

2$^{\text{nd}}$STEP : Thus we have found $\bar{\boldsymbol{a}} \leq \widehat{\boldsymbol{a}}$ such that, for any τ with $\overline{a}_t > 0$ there is some ρ satisfying $c_\tau^\rho > 0$. Let

$$\widehat{\boldsymbol{A}} \;:=\; \boldsymbol{A}_{\big|\mathbb{R}^n_{\boldsymbol{I}_0}} \;=\; \boldsymbol{A} \cap \left\{\boldsymbol{a} \,\middle|\, a_\tau = 0 \quad (\tau \notin \boldsymbol{I}_0)\right\}$$

For all $\boldsymbol{a} \in \widehat{\boldsymbol{A}}$ and for all τ with $x_\tau > 0$ there is ρ such that $c_t^\rho > 0$ and $c^\rho \boldsymbol{a} = 1$. hence, $\widehat{\boldsymbol{A}}$ is a compact convex polyhedron containing $\bar{\boldsymbol{a}}$. Let $\left\{\boldsymbol{a}^{(k)}\right\}_{k \in \mathbf{K}}$ denote the extremal points of this set. Then there exists a set of "convex" coefficients $\{\gamma_k\}_{k \in \mathbf{K}}$ such that

$$\bar{\boldsymbol{a}} \;=\; \sum\nolimits_{k \in \mathbf{K}} \gamma_k \boldsymbol{a}^{(k)} \quad .$$

It is not hard to see that the vectors $\boldsymbol{a}^{(k)}$ $(k \in \mathbf{K})$ are extremals of $\boldsymbol{A}$ as well as all coordinates vanish outside of $\boldsymbol{I}_0$. Hence our Lemma is verified. **q.e.d.**

We are now in the position to formulate the main result of this paper.

Theorem 3.3 **(The Inheritance Theorem).** *Let $\boldsymbol{\vartheta}, \boldsymbol{\eta}$ be imputations and let S be a coalition such that $\boldsymbol{\vartheta}\,\mathrm{dom}_S\,\boldsymbol{\eta}$. Then there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there is a relevant vector $\boldsymbol{a} \in \boldsymbol{A}^e$ and a coalition $T \;=\; T^{\boldsymbol{a},\varepsilon} \subseteq S$ satisfying*

$$\vec{\boldsymbol{\lambda}}(T) = \varepsilon \boldsymbol{a} \qquad \textit{and} \qquad \boldsymbol{\vartheta}\,\mathrm{dom}_T\,\boldsymbol{\eta} \; .$$

In other words, with respect to domination it is sufficient and necessary to consider ε–relevant coalitions only.

Proof:

1$^{\text{st}}$STEP :

Let $\widetilde{\boldsymbol{a}} := \vec{\boldsymbol{\lambda}}(S)$ and $\widehat{\boldsymbol{a}} := \frac{\vec{\boldsymbol{\lambda}}(S)}{\boldsymbol{v}(S)} = \frac{\widetilde{\boldsymbol{a}}}{v(\widetilde{\boldsymbol{a}})}$ such that $v(\widehat{\boldsymbol{a}}) = \min\{\boldsymbol{e}^\rho \widehat{\boldsymbol{a}} \mid \rho \in \mathbf{R}_0\}\} = 1$. Choose $\bar{\boldsymbol{a}} \leq \widehat{\boldsymbol{a}}$ according to Lemma 3.2, *item 1* and $\overline{\rho}$ according to *item 3.* Next choose a coalition $\bar{S} \subseteq S$ such that

$$\vec{\boldsymbol{\lambda}}(\bar{S}) = \boldsymbol{v}(S)\bar{\boldsymbol{a}} \leq \boldsymbol{v}(S)\widehat{\boldsymbol{a}} \quad .$$

Then clearly

$$
(3.4)\qquad
\begin{aligned}
\boldsymbol{v}(\bar{S}) &= \min\{\boldsymbol{\lambda}^{\rho}(\bar{S}) \mid \rho \in \mathbf{R}_0\} \\
&= \min\{\boldsymbol{c}^{\rho}\vec{\boldsymbol{\lambda}}(\bar{S}) \big| \rho \in \mathbf{R}_0\} \\
&= \boldsymbol{c}^{\overline{\rho}}\vec{\boldsymbol{\lambda}}(\bar{S}) \;=\; \boldsymbol{v}(S)\boldsymbol{c}^{\overline{\rho}}\bar{\boldsymbol{a}} \;=\; \boldsymbol{v}(S) \\
&= \boldsymbol{v}(S)\boldsymbol{c}^{\overline{\rho}}\widehat{\boldsymbol{a}} \;=\; \boldsymbol{c}^{\overline{\rho}}\vec{\boldsymbol{\lambda}}(S) \\
&= \min\{\boldsymbol{c}^{\rho}\vec{\boldsymbol{\lambda}}(S) \big| \rho \in \mathbf{R}_0\} \\
&= \boldsymbol{v}(S)\ .
\end{aligned}
$$

Now, as $\boldsymbol{\vartheta}$ exceeds $\boldsymbol{\eta}$ strictly on S, it dominates $\boldsymbol{\eta}$ a fortiori on $\bar{S}\subseteq S$, hence we have

$$(3.5)\qquad \boldsymbol{\vartheta}\,\mathrm{dom}_{\bar{S}}\,\boldsymbol{\eta}$$

2

2^{nd}STEP :

Let $\overline{m}$ be the ess inf of $\boldsymbol{\vartheta}$ over $\overline{S}$. Then

$$(3.6)\qquad \overline{m}\boldsymbol{\lambda}(\overline{S}) \leq \boldsymbol{v}(\bar{S}) = \boldsymbol{\lambda}^{\overline{\rho}}(\bar{S})$$

Now, if we have an equation in (3.6), then necessarily $\boldsymbol{\vartheta} = \boldsymbol{\lambda}^{\overline{\rho}}$ on $\bar{S}$. Then $\boldsymbol{\vartheta}$ cannot dominate as $\boldsymbol{\lambda}^{\overline{\rho}}$ equals 0 outside $\boldsymbol{C}^{\overline{\rho}}$.

Hence we can find $\bar{T}\subseteq\bar{S}$ with positive measure such that, for all $T^0\subseteq\bar{T}$

$$(3.7)\qquad \boldsymbol{\vartheta}(T^0) < \boldsymbol{\lambda}^{\overline{\rho}}(T^0)$$

holds true. In particular, we can choose $T^0\subseteq\bar{T}$ and $\delta^0 > 0$ such that for all $T'\subseteq T^0$

$$(3.8)\qquad \boldsymbol{\vartheta}(T') < \boldsymbol{\lambda}^{\overline{\rho}}(T')$$

and

$$(3.9)\qquad \vec{\boldsymbol{\lambda}}(T^0) \;=\; \delta^0\bar{\boldsymbol{a}}$$

holds true. This implies

$$(3.10)\qquad \boldsymbol{v}(T^0) = \boldsymbol{\lambda}^{\overline{\rho}}(T^0) \;=\; \vec{\boldsymbol{\lambda}}(T^0)\boldsymbol{c}^{\overline{\rho}} \;=\; \delta^0\boldsymbol{c}^{\overline{\rho}}\bar{\boldsymbol{a}}$$

and

$$(3.11)\qquad \boldsymbol{\vartheta}(T') < \boldsymbol{\lambda}^{\overline{\rho}}(T') \text{ for all } T'\subseteq T^0$$

hence

$$(3.12)\qquad \boldsymbol{\vartheta}(T^0) < \boldsymbol{\lambda}^{\overline{\rho}}(T^0) \;=\; \boldsymbol{v}(T^0)\ .$$

Moreover, according to the *second item* of Lemma 3.2, we can find a set of extremals

$$\boldsymbol{E} \;=\; \left\{\boldsymbol{a}^{(k)} \,\middle|\, k \in \mathbf{K}\right\}$$

and coefficients $\{\gamma_k\}_{k\in\mathbf{K}}$ satisfying

$$\bar{\boldsymbol{a}} \;=\; \sum\nolimits_{k\in\mathbf{K}} \gamma_k \boldsymbol{a}^{(k)} \;. \tag{3.13}$$

3^rd STEP : Now choose $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $k \in \mathbf{K}$ there exists a coalition $T^{\star(k)} \subseteq T^0$ such that

$$\vec{\boldsymbol{\lambda}}(T^{\star(k)}) = \varepsilon \boldsymbol{a}^{(k)} \tag{3.14}$$

and all the $T^{\star(k)}$ are mutually disjoint. Then, considering the $(|\mathbf{T}|+1) = (rt+1)$ dimensional vector valued measure $(\vec{\boldsymbol{\lambda}}, \boldsymbol{\vartheta})$, choose for every $k \in \mathbf{K}$ a coalition $T^{(k)} \subseteq T^{\star(k)}$ satisfying

(3.15)

$$(\vec{\boldsymbol{\lambda}}(T^{(k)})), \boldsymbol{\vartheta}(T^{(k)}) = \gamma_k (\vec{\boldsymbol{\lambda}}(T^{\star(k)}), \boldsymbol{\vartheta}(T^{\star(k)})) \;=\; (\varepsilon\gamma_k \boldsymbol{a}^{(k)}, \varepsilon\gamma_k \boldsymbol{\vartheta}(T^{\star(k)})) \;.$$

Then

$$\boldsymbol{v}(T^{(k)}) = \boldsymbol{c}^{\bar{\rho}} \vec{\boldsymbol{\lambda}}(T^{(k)}) = \varepsilon\gamma_k \boldsymbol{c}^{\bar{\rho}} \boldsymbol{a}^{(k)} \;.$$

Define $T^\star \;:=\; \bigcup_{k=1,\ldots,K} T^{\star(k)}$. Because of $T^\star \subseteq T^0$ we have

$$\boldsymbol{\vartheta}(T^\star) \;<\; \boldsymbol{v}(T^\star) \;.$$

Moreover,

$$\begin{aligned}
\sum\nolimits_{k\in\mathbf{K}} \gamma_k \boldsymbol{\vartheta}(T^{(k)}) \;&=\; \sum\nolimits_{k\in\mathbf{K}} \boldsymbol{\vartheta}(T^{\star(k)}) \;=\; \boldsymbol{\vartheta}(T^\star) \;<\; \boldsymbol{v}(T^\star) \\
&=\; \min\{\boldsymbol{c}^{\rho} \vec{\boldsymbol{\lambda}}(T^\star) \,\big|\, \rho \in \mathbf{R_1}\} \;\leq\; \boldsymbol{c}^{\bar{\rho}} \vec{\boldsymbol{\lambda}}(T^\star) \\
&=\; \boldsymbol{c}^{\bar{\rho}} \sum\nolimits_{k\in\mathbf{K}} \vec{\boldsymbol{\lambda}}(T^{\star k}) \;=\; \sum\nolimits_{k\in\mathbf{K}} \boldsymbol{c}^{\bar{\rho}} \vec{\boldsymbol{\lambda}}(T^{\star k}) \\
&=\; \sum\nolimits_{k\in\mathbf{K}} \boldsymbol{c}^{\bar{\rho}} \gamma_k \vec{\boldsymbol{\lambda}}(T^{k}) \;=\; \sum\nolimits_{k\in\mathbf{K}} \gamma_k \boldsymbol{c}^{\bar{\rho}} \vec{\boldsymbol{\lambda}}(T^{k}) \\
&=\; \sum\nolimits_{k\in\mathbf{K}} \gamma_k \boldsymbol{v}(T^k) \;.
\end{aligned} \tag{3.16}$$

Hence $\boldsymbol{\vartheta}(T^{(k)}) < \boldsymbol{v}(T^k)$ for at least one $k \in \boldsymbol{K}$. However, $\boldsymbol{\vartheta}$ strictly exceeds $\boldsymbol{\eta}$ on $T^{(k)} \subseteq S$, hence we have $\boldsymbol{\vartheta}\, \mathrm{dom}_{T^{(k)}}\, \boldsymbol{\eta}$.

In view of (3.14) and (3.15) we see that $T^{(k)}$ is an $\varepsilon\gamma_k$ relevant coalition and because of

$$T^{(k)} \subseteq T^{\star(k)} \subseteq T^0 \subseteq \overline{T} \subseteq \overline{S} \subseteq S$$

the proof is complete. **q.e.d.**

We proceed by describing the generic shape of the relevant vectors.

Lemma 3.4. *Let $\bar{\boldsymbol{a}} \in \boldsymbol{A}^e$ be a relevant vector, i.e., an extremal vector of $\boldsymbol{A}$. Then $\bar{\boldsymbol{a}}$ has at most $r+1$ and at least r positive coordinates.*

Proof: If $\bar{\boldsymbol{a}}$ is extremal, then there have to be rt equations among the defining inequalities of $\boldsymbol{A}$ that $\bar{\boldsymbol{a}}$ has to satisfy. As there are $r+1$ linear functionals only, there have to be at least $rt-(r+1)$ equations of the form $x_\tau = 0$ that $\bar{\boldsymbol{a}}$ satisfies as well. Hence there are at most $r+1$ positive coordinates of $\bar{\boldsymbol{a}}$. Now, if there are less than r positive coordinates of $\bar{\boldsymbol{a}}$, then in at least one T^ρ there is no positive coordinate, hence $\boldsymbol{c}^\rho(\bar{\boldsymbol{a}}) = \boldsymbol{e}^{T^\rho}\bar{\boldsymbol{a}} = 0$, contradicting $\bar{\boldsymbol{a}} \in \boldsymbol{A}$.

q.e.d.

Theorem 3.5. *Let $\boldsymbol{a} \in \mathbb{R}^{rt}_{+}$ and let $\bar{\boldsymbol{\tau}} \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$ be a sequence such that $\boldsymbol{a}$ is described by one of the following alternatives. Then $\boldsymbol{a}$ is relevant. If $\boldsymbol{h} > \boldsymbol{0}$, then all relevant vectors are obtained this way.*

1. *Let*

$$\sum_{\rho \in \mathbf{R}} h_{\overline{\tau}_\rho} \geq 1 \ . \tag{3.17}$$

Then $\boldsymbol{a}^{\odot} \in \boldsymbol{A}$ defined via

$$a^{\odot}{}_{\bar{\tau}_\rho} = 1 \quad (\rho \in \mathbf{R}) \text{ and } a^{\odot}_\tau = 0 \text{ whenever } \tau \notin \{\overline{\tau}_1, \ldots, \overline{\tau}_r\} \ , \tag{3.18}$$

i.e.,

$$\boldsymbol{a}^{\odot} \ = \ (0, \ldots, 0, 1, 0 \ldots, 0, 1, 0, , \ldots, , \ldots, 0, 1, 0 \ldots, 0) \ .$$

constitutes a relevant vector. Clearly $\boldsymbol{a}^{\odot}$ satisfies

$$1 = v(\boldsymbol{a}^{\odot}) = \boldsymbol{e}^{T^\rho}\boldsymbol{a}^{\odot} \leq \boldsymbol{c}^0(\boldsymbol{a}^{\odot}) \quad (\rho \in \mathbf{R}) \ .$$

2. *Let $h_{\overline{\tau}_\rho} > 0$ $(\rho \in \mathbf{R})$ and let*

$$\sum_{\rho \in \mathbf{R}} h_{\overline{\tau}_\rho} < 1 \ , \tag{3.19}$$

then the vector $\boldsymbol{a}^{\oplus}$ given by

$$\begin{aligned} a^{\oplus}{}_{\overline{\tau}_\rho} &= 1 \quad (\rho \in \mathbf{R} \setminus \{r\}) \\ a^{\oplus}{}_{\overline{\tau}_r} &= \frac{1-(h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}})}{h_{\overline{\tau}_r}} > 1, \\ a^{\oplus}{}_\tau &= 0 \quad \textit{otherwise} \ , \end{aligned} \tag{3.20}$$

i.e.,

$$\boldsymbol{a}^{\oplus} \ = \ (0, \ldots, 0, 1, 0 \ldots, 0, 1, 0, , \ldots, 0, \frac{1-(h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}})}{h_{\overline{\tau}_r}}, 0, \ldots, 0) \ .$$

yields a relevant vector. Clearly $\boldsymbol{a}^{\oplus}$ *satisfies*

$$1 = v(\boldsymbol{a}^{\oplus}) = \boldsymbol{e}^{T^\rho}\boldsymbol{a}^{\oplus} = \boldsymbol{c}^0(\boldsymbol{a}^{\oplus}) \leq \boldsymbol{e}^{T^r}\boldsymbol{a}^{\oplus} \quad (\rho = 1, \ldots, r-1) \ .$$

An analogous constructions is obtained by replacing r *by any* $\sigma \in \{1, \ldots, r\}$*. This yields a relevant vector of the shape*

$$\begin{aligned} &{a^{\oplus}}_{\overline{\tau}_\rho} = 1 \quad (\rho \in \mathbf{R} \setminus \{\sigma\}) \\ &{a^{\oplus}}_{\overline{\tau}_\sigma} = \frac{1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{\sigma-1}} + h_{\overline{\tau}_{\sigma+1}} + \ldots + h_{\overline{\tau}_r})}{h_{\overline{\tau}_\sigma}} > 1, \\ &{a^{\oplus}}_{\tau} = 0 \quad \textit{otherwise} \ , \end{aligned} \tag{3.21}$$

or

$$\boldsymbol{a}^{\oplus} \ = \ (0, \ldots, 0, 1, 0 \ldots, 0, \frac{1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{\sigma-1}} + h_{\overline{\tau}_{\sigma+1}} + \ldots + h_{\overline{\tau}_r})}{h_{\overline{\tau}_\sigma}}, 0, \ldots, 0, 1, 0 \ldots, 0) \ .$$

3. *Finally, as a third alternative assume that there exists* $\overline{\overline{\tau}}_r \in \mathbf{T}^r$ *in addition to the sequence* $\overline{\boldsymbol{\tau}}$*, such that and*

$$\sum_{\rho \in \mathbf{R}} h_{\overline{\tau}_\rho} < 1 < \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\overline{\tau}_\rho} + h_{\overline{\overline{\tau}}_r} \tag{3.22}$$

(implying $h_{\overline{\overline{\tau}}_r} > h_{\overline{\tau}_r}$*). Then the vector* $\boldsymbol{a}^{\ominus}$ *defined by*

$$\begin{aligned} &{a^{\ominus}}_{\overline{\tau}_\rho} = 1 \quad (\rho \in \mathbf{R} \setminus \{r\}) \\ &{a^{\ominus}}_{\overline{\tau}_r} = \frac{1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\overline{\tau}_r}}, \\ &{a^{\ominus}}_{\overline{\overline{\tau}}_r} = \frac{(h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\overline{\tau}_r}}, \\ &{a^{\ominus}}_{\tau} = 0 \quad \textit{otherwise} \ . \end{aligned} \tag{3.23}$$

with

$$v(\boldsymbol{a}^{\ominus}) = \boldsymbol{e}^{T^\rho}\boldsymbol{a}^{\ominus} = \boldsymbol{c}^0(\boldsymbol{a}^{\ominus}) = 1 \quad (\rho \in \mathbf{R})$$

There is also the vector obtained via exchanging r *by some* σ*:*

$$\begin{aligned} &{a^{\ominus}}_{\overline{\tau}_\rho} = 1 \quad (\rho \in \mathbf{R} \setminus \{\sigma\}) \\ &{a^{\ominus}}_{\overline{\tau}_\sigma} = \frac{1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_r})}{h_{\overline{\overline{\tau}}_\sigma} - h_{\overline{\tau}_\sigma}}, \\ &{a^{\ominus}}_{\overline{\overline{\tau}}_\sigma} = \frac{(h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_\sigma} - h_{\overline{\tau}_\sigma}) - 1}{h_{\overline{\overline{\tau}}_\sigma} - h_{\overline{\tau}_\sigma}}, \\ &{a^{\ominus}}_{\tau} = 0 \quad \textit{otherwise} \ . \end{aligned} \tag{3.24}$$

Proof:

1^st^STEP : The first situation occurs exactly whenever $\boldsymbol{a}^{\odot}$ satisfies the r equations

$$\boldsymbol{e}^{T^\rho}\boldsymbol{a}^{\odot} = 1 \quad (\rho = 1, \ldots, r)$$

together with suitably many equations of the type $a_\tau = 0$. Also, $\boldsymbol{c}^0(\boldsymbol{a}^{\odot}) = \sum_{\rho\in\mathbf{R}} h_{\overline{\tau}_\rho} \geq 1$ shows that $\boldsymbol{a}^{\odot}$ is indeed extremal in $\boldsymbol{A}$, that is, a relevant vector.

2^nd^STEP : In the second case the vector $\boldsymbol{a}^{\oplus}$ is determined by $r-1$ equations $\boldsymbol{e}^{T^\rho}\boldsymbol{a} = 1 \quad (\rho \in \mathbf{R} \setminus \{r\})$ and the equation $\boldsymbol{c}^0\boldsymbol{a} = 1$. Moreover,

$$\frac{1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}})}{h_{\overline{\tau}_r}} \geq 1$$

implies $\boldsymbol{e}^{T^r}\boldsymbol{a}^{\oplus} \geq 1$.

Clearly, r can be exchanged with any $\sigma \in \{1, \ldots, r\}$ which results in possible relevant vectors as described.

3^rd^STEP : This vector is seen to satisfy all the equations $\boldsymbol{e}^{T^\rho}\boldsymbol{a} = 1 \quad (\rho \in \mathbf{R})$ as well as $\boldsymbol{c}^0(\boldsymbol{a}) = 1$. Obviously again we have an extremal point of $\boldsymbol{A}$.

q.e.d.

Remark 3.6. For computational purposes and in order to avoid rational expressions of the relevant vectors in terms of the data h_τ, we will sometimes consider another normalization of relevant vectors.

The first possibility described in *item* 1 by (3.17) obviously needs no renormalization. However, the type of relevant vector described within the second *item* via (3.19) can be multiplied by $h_{\overline{\tau}_r}$ obtaining a "renormalized relevant vector" of the shape

(3.25)
$$\boldsymbol{a}^{\oplus} \;=\; (0, \ldots, 0, h_{\overline{\tau}_r}, 0 \ldots, 0, h_{\overline{\tau}_r}, 0, \ldots, 0, 1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}}), 0 \ldots, 0)\ .$$

which yields

$$v(\boldsymbol{a}^{\oplus}) = \boldsymbol{e}^{T^\rho}\boldsymbol{a}^{\oplus} = \boldsymbol{c}^0(\boldsymbol{a}^{\oplus}) = h_{\overline{\tau}_r} \leq \boldsymbol{e}^{T^r}\boldsymbol{a}^{\oplus} \quad (\rho = 1, \ldots, r-1)\ .$$

Similarly, for *item* 3 as specified by (3.22) the alternative version involving the h_τ only in numerators is

(3.26)
$$\begin{aligned} a^{\ominus}{}_{\overline{\tau}_\rho} &= h_{\overline{\overline{\tau}}_r} - h_{\overline{\tau}_r} \quad (\rho \in \mathbf{R} \setminus \{r\}) \\ a^{\ominus}{}_{\overline{\tau}_r} &= 1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_r})\ , \\ a^{\ominus}{}_{\overline{\overline{\tau}}_r} &= (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1\ , \\ a^{\ominus}{}_{\tau} &= 0 \quad \text{otherwise}\ , \end{aligned}$$

(with $a^{\ominus}_{\widehat{\tau}_r} + a^{\ominus}_{\overline{\tau}_r} = h_{\widehat{\tau}_r} - h_{\overline{\tau}_r}$). This version constitutes a "renormalized relevant vector" satisfying

$$v(\boldsymbol{a}^{\ominus}) = \boldsymbol{e}^{T^\rho}\boldsymbol{a}^{\ominus} = \boldsymbol{c}^0(\boldsymbol{a}^{\ominus}) = h_{\widehat{\tau}_r} - h_{\overline{\tau}_r} \quad (\rho \in \mathbf{R})\ .$$

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Example 3.7. Recall the 2×2 Example 2.1. Given the data as specified in (2.8), observe that the vector $\boldsymbol{a}^{14} := (1,0,0,1)$ is always relevant, i.e., extremal in $\boldsymbol{A}$; we have

$$\text{(3.27)}\quad v(\boldsymbol{a}^{14}) := \min\{\boldsymbol{c}^{\rho}\boldsymbol{a}^{14} \,|\, \rho \in \mathbf{R}_0\} = \boldsymbol{e}^{12}\boldsymbol{a}^{14} = \boldsymbol{e}^{34}\boldsymbol{a}^{14} = \boldsymbol{c}^{0}\boldsymbol{a}^{14} = 1 \ .$$

Within this context, we call an $\boldsymbol{\varepsilon}$–$\boldsymbol{a}^{14}$–relevant coalition $T^{14} = T^1 \cup T^4$ to be ***ε*–14–*relevant*** if $\vec{\boldsymbol{\lambda}}(T^{14}) = \varepsilon \boldsymbol{a}^{14}$ holds true. In this case we have

$$\begin{aligned}
\boldsymbol{v}(T^{14}) &= \min\{\boldsymbol{c}^{\rho}\vec{\boldsymbol{\lambda}}(T^{14}) \,\big|\, \rho \in \mathbf{R}_0\} = \varepsilon \min\{\boldsymbol{c}^{\rho}\boldsymbol{a}^{14} \,\big|\, \rho \in \mathbf{R}_0\} \\
&= \boldsymbol{\lambda}^1(T^{14}) = \boldsymbol{\lambda}^2(T^{14}) = \boldsymbol{\lambda}^3(T^{14}) \\
&= \varepsilon \ .
\end{aligned}$$

Consider the case that $h_2 + h_3 \le 1$ and $\lambda_1 + \lambda_3 \le 1$ holds true. We label this case "scarce central commodity" as $\boldsymbol{\lambda}^0$ is small, hence has more influence in the formation of $\boldsymbol{v}$. By contrast, a market with abundant central commodity will be considered later on in Example 4.7 and Theorem 4.9.

Now it turns out that

$$\boldsymbol{a}^{\oplus 23} := (0, \frac{1-h_3}{h_2}, 1, 0) \quad \text{and} \quad \boldsymbol{a}^{\oplus 32} := (0, 1, \frac{1-h_2}{h_3}, 0)$$

are relevant vectors of the second type. Following Remark 3.6 we prefer to normalize these vectors such that the rational expressions are avoided, that is we consider

$$\boldsymbol{a}^{23} := (0, 1-h_3, h_2, 0) \quad \text{and} \quad \boldsymbol{a}^{32} := (0, h_3, 1-h_2, 0) \ .$$

Then

$$\text{(3.28)}\qquad \min\{\boldsymbol{c}^{\rho}\boldsymbol{a}^{23} \,|\, \rho \in \mathbf{R}_0\} = \boldsymbol{e}^{34}\boldsymbol{a}^{23} = \boldsymbol{c}^{3}\boldsymbol{a}^{23} = h_2 < \boldsymbol{e}^{12}\boldsymbol{a}^{23}.$$

Consequently, we speak of an ***ε*–23–*relevant*** coalition $T^{23} \subseteq \boldsymbol{D}^{23}$ if $\vec{\boldsymbol{\lambda}}(T^{23}) = \varepsilon \boldsymbol{a}^{23}$, that is,

$$\begin{aligned}
\boldsymbol{v}(T^{23}) &= \min\{\boldsymbol{c}^{\rho}\vec{\boldsymbol{\lambda}}(T^{23}) \,\big|\, \rho \in \mathbf{R}_0\} = \varepsilon \min\{\boldsymbol{c}^{\rho}\boldsymbol{a}^{23} \,\big|\, \rho \in \mathbf{R}_0\} \\
&= \boldsymbol{\lambda}^2(T^{23}) = \boldsymbol{\lambda}^3(T^{23}) = h_2\varepsilon < \boldsymbol{\lambda}^1(T^{23}) \ .
\end{aligned}$$

Accordingly, if $T^{32} \subseteq \boldsymbol{D}^{23}$ satisfies $\vec{\boldsymbol{\lambda}}(T^{32}) = \varepsilon \boldsymbol{a}^{32}$, then

$$\begin{aligned}
\boldsymbol{v}(T^{32}) &= \min\{\boldsymbol{c}^{\rho}\vec{\boldsymbol{\lambda}}(T^{32}) \,\big|\, \rho \in \mathbf{R}_0\} = \varepsilon \min\{\boldsymbol{c}^{\rho}\boldsymbol{a}^{32} \,\big|\, \rho \in \mathbf{R}_0\} \\
&= \boldsymbol{\lambda}^1(T^{32}) = \boldsymbol{\lambda}^3(T^{32}) = h_3\varepsilon < \boldsymbol{\lambda}^2(T^{32})
\end{aligned}$$

justifies calling T^{32} an ***ε*–32–*relevant*** coalition. A comprehensive discussion of this example will be presented in a separate paper.

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4 The Separating Pre–Coalitions

Within this section we draw some first conclusions based on the Inheritance Theorem. In particular we show that, with some conditions it suffices to focus on the relevant vectors of type $\boldsymbol{a}^{\odot}$ and $\boldsymbol{a}^{\oplus}$ as described in Theorem 3.5.

The following definition is just an extension of our previous conventions.

Definition 4.1. *1. Let $\boldsymbol{x}$ be a pre–imputation and let $\boldsymbol{y} \in \mathbb{R}^{rt}$. We shall say that $\boldsymbol{x}$ **dominates $\boldsymbol{y}$ via $\boldsymbol{a}$** $\in \boldsymbol{A^e}$ if*

$$(4.1) \qquad \boldsymbol{x}\boldsymbol{a} \leq v(\boldsymbol{a}) = 1, \quad \textit{and} \quad x_\tau > y_\tau \ \textit{ for all } \ \tau \ \textit{ with } \ a_\tau > 0 \ .$$

We write $\boldsymbol{x} \operatorname{dom}_{\boldsymbol{a}} \boldsymbol{y}$ to indicate domination.

2. For any nonnegative measurable function $\boldsymbol{\vartheta}$ let

$$(4.2) \qquad m_\tau \ := \ \operatorname{ess\,inf}_{\boldsymbol{D}^\tau} \quad (\tau \in \mathbf{T}) \quad \textit{and} \quad \boldsymbol{m} \ := \ (m_1, \ldots, m_{rt}) \ .$$

*We shall then (somewhat sloppily) refer to $\boldsymbol{m}$ as the vector of essential minima or the **minima vector** of $\boldsymbol{\vartheta}$.*

Lemma 4.2. *Let $\boldsymbol{x}$ be a pre–imputation and let $\boldsymbol{\vartheta}$ be a (nonnegative) measurable function. Let $\boldsymbol{m}$ denote the vector of essential minima of $\boldsymbol{\vartheta}$. If, for some $\boldsymbol{a} \in \boldsymbol{A}^e$ we have $\boldsymbol{x} \operatorname{dom}_{\boldsymbol{a}} \boldsymbol{m}$, then there is $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon < \varepsilon_0$ there is an ε–relevant coalition $T^\varepsilon = T^{\varepsilon \boldsymbol{a}}$ such that*

$$(4.3) \qquad \boldsymbol{\vartheta}^{\boldsymbol{x}} \operatorname{dom}_{T^{\varepsilon \boldsymbol{a}}} \boldsymbol{\vartheta}$$

holds true.

Proof: As $x_\tau > m_\tau$ we can, for all $\tau \in \mathbf{T}$ with $a_\tau > 0$, choose a coalition $S^\tau \in \boldsymbol{D}^\tau$ such that $\dot{\boldsymbol{\vartheta}} < x_\tau$ holds true on S^τ. Therefore, we can choose $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists $T^{\varepsilon\tau} \subseteq S^\tau$ such that $\boldsymbol{\lambda}(T^{\varepsilon\tau}) = \varepsilon a_\tau$ holds true for all τ with $a_\tau > 0$. That is, for $T^\varepsilon \ := \ \cup_{\tau \in \mathbf{T}, a_\tau > 0} T^{\varepsilon\tau}$ we have $\vec{\boldsymbol{\lambda}}(T^\varepsilon) = \varepsilon \boldsymbol{a}$ which implies

$$(4.4) \qquad \boldsymbol{\vartheta}^{\boldsymbol{x}}(T^\varepsilon) \ = \ \varepsilon \boldsymbol{x}\boldsymbol{a} \ = \ \varepsilon \ = \ \varepsilon v(\boldsymbol{a}) \ = \ \boldsymbol{v}(T^\varepsilon) \ .$$

As $T^\tau \subseteq S^\tau$ we conclude

$$(4.5) \qquad \dot{\boldsymbol{\vartheta}} < \dot{\boldsymbol{\vartheta}}^{\boldsymbol{x}} \text{ on } T^\varepsilon \ ,$$

hence $\boldsymbol{\vartheta}^{\boldsymbol{x}} \operatorname{dom}_{T^\varepsilon} \boldsymbol{\vartheta}$,

q.e.d.

Definition 4.3. *A vector (pre–coalition) $\boldsymbol{a} \in \boldsymbol{A}^e$ is said to be **separating** if there is $\bar{\boldsymbol{\tau}} \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$ such that*

$$a_{\bar{\tau}_\rho} > 0 \quad (\rho \in \mathbf{R}) \ \textit{and} \ a_\tau = 0 \ \textit{whenever} \ \tau \neq \tau_\rho \quad .$$

The relevant vectors that are separating are of the shape described in either item 1 *or* item 2 *of Theorem 3.5. Denote*

$$(4.6) \qquad \boldsymbol{A}^s \ := \ \{\boldsymbol{a} \in \boldsymbol{A}^e \,|\, \boldsymbol{a} \textit{ is separating} \}.$$

Thus, for $\boldsymbol{a} \in \boldsymbol{A}^s$, an ε–$\boldsymbol{a}$–coalition $T^{\varepsilon\boldsymbol{a}}$ has an intersection with exactly one $\boldsymbol{D}^\tau$ in every $\boldsymbol{C}^\rho$. The corresponding vector valued measure is either

$$\vec{\boldsymbol{\lambda}}(T^{\varepsilon\boldsymbol{a}}) = \varepsilon\boldsymbol{a} = \varepsilon(0,\ldots,0,1,0\ldots,0,1,0,,\ldots,,\ldots,0,1,0\ldots,0) \ .$$

or

$$\vec{\boldsymbol{\lambda}}(T^{\varepsilon\boldsymbol{a}}) = \varepsilon\boldsymbol{a} = \varepsilon(0,\ldots,0,1,0\ldots,0,,\ldots,,\ldots,0,1,0\ldots,0,\frac{1-\sum_{\rho\in\mathbf{R}\setminus\{r\}} h_{\tau_\rho}}{h_{\overline{\tau}_r}},\ldots,0) \ ,$$

(assuming that r is the index distinguished according to *item 2* of Theorem 3.5). Define

$$\text{(4.7)} \qquad \boldsymbol{H} \ := \ \{\boldsymbol{x} \in \boldsymbol{J} \,|\, \boldsymbol{x}\boldsymbol{a} \geq v(\boldsymbol{a}) = 1, \ \ (\boldsymbol{a} \in \boldsymbol{A}^s)\} \quad .$$

Then $\boldsymbol{H}$ is the set of pre–imputations which cannot be dominated via some separating pre–coalition. Accordingly define

$$\text{(4.8)} \qquad \mathcal{H} \ := \ \{\boldsymbol{\vartheta}^{\boldsymbol{x}} \,|\, \boldsymbol{x} \in \boldsymbol{H}\} \ .$$

Note that $\boldsymbol{C}(v) \subseteq \boldsymbol{H}$ and $\mathcal{C}(\boldsymbol{v}) \subseteq \mathcal{H}$ as $\boldsymbol{e}^{T^\rho} \in \boldsymbol{H}$ holds true for all $\rho \in \mathbf{R}$.

Now it turns out that any imputation with minima vector outside of $\boldsymbol{H}$ *can* be dominated by (itself and) the core.

Theorem 4.4. *Let $\boldsymbol{\vartheta}$ be a imputation and let $\boldsymbol{m} = (m_\tau)_{\tau\in\mathbf{T}}$ denote its vector of minima.*

1. *If, for some $\boldsymbol{a} \in \boldsymbol{A}^s$ we have $\boldsymbol{m}\boldsymbol{a} < 1 = v(\boldsymbol{a})$, then $\boldsymbol{\vartheta}$ is dominated by itself and the core. Hence, $\boldsymbol{\vartheta}$ cannot be an element of a vNM–Stable Set.*

2. *In particular, if $\boldsymbol{F} \subseteq \boldsymbol{J}$ is a set of pre–imputations inducing a vNM–Stable set $\mathcal{F} = \{\boldsymbol{\vartheta}^{\boldsymbol{x}} \,|\, \boldsymbol{x} \in \boldsymbol{F}\}$, then $\mathcal{F} \subseteq \mathcal{H}$.*

Proof: Let $\overline{\boldsymbol{\tau}} \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$ be the sequence defining $\boldsymbol{a}$ via Theorem 3.5. In what follows we write $\overline{h}_{\overline{\tau}} \ := \ \sum_{\rho\in\mathbf{R}} h_{\overline{\tau}_\rho}$.

1$^{\text{st}}$STEP :

As $\boldsymbol{m}\boldsymbol{a} < v(\boldsymbol{a})$ we have

$$\boldsymbol{m}\boldsymbol{a} < \boldsymbol{e}^{T^\rho}\boldsymbol{a} \ \ (\rho \in \mathbf{R}) \ \ .$$

As $\boldsymbol{a}$ is separating this means

$$\sum_{\rho\in\mathbf{R}} m_{\overline{\tau}_\rho} a_{\overline{\tau}_\rho} < a_{\overline{\tau}_\sigma} \quad (\sigma \in \mathbf{R}) \ \ ,$$

and consequently

$$\text{(4.9)} \qquad m_{\overline{\tau}_\rho} < 1 \ \ (\rho \in \mathbf{R}) \ \ .$$

Therefore, we can choose $0 < \varepsilon_1$ such that for all $\varepsilon < \varepsilon_1$ there is an ε–relevant coalition $T^{\varepsilon \boldsymbol{a}}$ (intersecting each $\boldsymbol{C}^\rho$ just in $\boldsymbol{D}^{\overline{\tau}_\rho}$) such that $\boldsymbol{\vartheta}(\bullet) < 1$ on $T^{\varepsilon \boldsymbol{a}}$ and hence, for all $0 < \delta \leq 1$,

$$\boldsymbol{\vartheta}(\bullet) < (1-\delta)\boldsymbol{\vartheta}(\bullet) + \delta \frac{\sum_{\rho \in \mathbf{R}} \boldsymbol{\lambda}^\rho}{r}(\bullet) \quad \text{on } T^{\varepsilon \boldsymbol{a}} \quad . \tag{4.10}$$

Note that $T^{\varepsilon \boldsymbol{a}}$ has measure $\boldsymbol{\lambda}(T^{\varepsilon \boldsymbol{a}}) = \varepsilon \overline{h}_{\overline{\tau}} = \sum_{\rho \in \mathbf{R}} h_{\overline{\tau}_\rho}$.

2nd STEP : Now choose $0 < \varepsilon_2 < \varepsilon_1$ and $0 < \delta_2$ such that, for all $\varepsilon < \varepsilon_2$ and $0 < \delta < \delta_2$, one can find $T^{\varepsilon \boldsymbol{a}}$ satisfying (4.10) and in addition

$$(1-\delta)\boldsymbol{ma} + (1-\delta)\delta + \frac{\delta}{r}\overline{h}_{\overline{\tau}} < v(\boldsymbol{a}) = 1 \tag{4.11}$$

as well as

$$\dot{\boldsymbol{\vartheta}}(\bullet) < m_{\overline{\tau}_\rho} + \frac{\delta}{\overline{h}_{\overline{\tau}}} \quad \text{on} \quad T^{\varepsilon \boldsymbol{a}} \cap \boldsymbol{D}^{\overline{\tau}_\rho} = T^{\varepsilon \boldsymbol{a} \overline{\tau}_\rho} \quad (\rho \in \mathbf{R}) \ . \tag{4.12}$$

That is

$$\dot{\boldsymbol{\vartheta}}(\bullet) < m_{\overline{\tau}_\rho} + \frac{\delta}{\overline{h}_{\overline{\tau}}} \dot{\boldsymbol{\lambda}}^{\overline{\tau}_\rho}(\bullet) \quad \text{on} \quad T^{\varepsilon \boldsymbol{a} \overline{\tau}_\rho} \qquad (\rho \in \mathbf{R}) \ . \tag{4.13}$$

Hence we come up with

$$\begin{aligned} \boldsymbol{\vartheta}(T^{\varepsilon \boldsymbol{a}}) \ &< \ \varepsilon \boldsymbol{ma} + \frac{\delta \boldsymbol{\lambda}(T^{\varepsilon \boldsymbol{a}})}{\overline{h}_{\overline{\tau}}} \\ &= \ \varepsilon \boldsymbol{ma} + \delta \varepsilon \end{aligned} \tag{4.14}$$

3rd STEP : Combining we find, for all $\varepsilon < \varepsilon_2$ and $0 < \delta < \delta_2$:

$$\begin{aligned} &\left[(1-\delta)\boldsymbol{\vartheta} + \delta \frac{\sum_{\rho \in \mathbf{R}} \boldsymbol{\lambda}^\rho}{r}\right](T^{\varepsilon \boldsymbol{a}}) \\ = \ & (1-\delta)\boldsymbol{\vartheta}\,(T^{\varepsilon \boldsymbol{a}}) + \delta \frac{\sum_{\rho \in \mathbf{R}} \boldsymbol{\lambda}^\rho}{r}\,(T^{\varepsilon \boldsymbol{a}}) \\ < \ & (1-\delta)\varepsilon \boldsymbol{ma} + (1-\delta)\delta\varepsilon + \frac{\delta}{r}\varepsilon \overline{h}_{\overline{\tau}} \qquad \text{by (4.14)} \\ = \ & \varepsilon\left((1-\delta)\boldsymbol{ma} + (1-\delta)\delta + \frac{\delta}{r}\overline{h}_{\overline{\tau}}\right) \\ < \ & \varepsilon v(\boldsymbol{a}) \ = \ v(\varepsilon \boldsymbol{a}) \ = \ \boldsymbol{v}(T^{\varepsilon \boldsymbol{a}}) \ . \qquad \text{by (4.11)} \end{aligned} \tag{4.15}$$

Now, the inequalities (4.10) and (4.15) show that the imputation

$$(1-\delta)\boldsymbol{\vartheta} + \delta \frac{\sum_{\rho \in \mathbf{R}} \boldsymbol{\lambda}^\rho}{r}$$

indeed dominates $\boldsymbol{\vartheta}$ via $T^{\varepsilon \boldsymbol{a}}$,

q.e.d.

The previous theorem shows that imputations outside of $\mathcal{H}$ can be dominated by means of the core and themselves. The following exhibits a certain type of candidate which together with the core may dominate imputations via separating vectors.

Theorem 4.5. *Let $\widehat{\boldsymbol{a}}$ be a separating vector (pre–coalition) (with arbitrary coordinate σ for the coordinate $\neq 1$ in the second item if applicable). Let $\widehat{\boldsymbol{\tau}}$ be the sequence specified by Theorem 3.5. Let $\bar{\boldsymbol{x}}$ be an imputation with coordinates $\overline{x}_\tau = h_\tau > 0$ along $\widehat{\boldsymbol{\tau}}$. Let $\boldsymbol{\vartheta}$ be an imputation with minima vector $\boldsymbol{m}$. If*

$$\sum_{\rho \in \mathbf{R}} m_{\widehat{\tau}_\rho} < \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} \ , \tag{4.16}$$

Then, for sufficiently small $\varepsilon > 0$, there exists an ε–relevant coalition $T^\varepsilon = T^{\varepsilon \widehat{\boldsymbol{a}}}$ and a convex combination $\widehat{\boldsymbol{x}}$ of the $\boldsymbol{e}^{T^\rho}$ $(\rho \in \mathbf{R})$ and $\bar{\boldsymbol{x}}$ such that

$$\boldsymbol{\vartheta}^{\widehat{x}} \operatorname{dom}_{T^\varepsilon} \boldsymbol{\vartheta} \tag{4.17}$$

holds true.

Proof:

Assume w.l.g. that r minimizes the quotients $\frac{m_{\widehat{\tau}_\rho}}{h_{\widehat{\tau}_\rho}}$, i.e., $\frac{m_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} \leq \frac{m_{\widehat{\tau}_\rho}}{h_{\widehat{\tau}_\rho}}$, or

$$\frac{m_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} h_{\widehat{\tau}_\rho} \leq m_{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R}) \ . \tag{4.18}$$

Define $\overline{\alpha} := \frac{m_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} < 1$. Now because of

$$1 - \sum_{\rho \in \mathbf{R}} m_{\widehat{\tau}_\rho} > 1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho}$$

it follows that

$$\frac{\left(1 - \sum_{\rho \in \mathbf{R}} m_{\widehat{\tau}_\rho}\right) + m_{\widehat{\tau}_r}}{\left(1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho}\right) + h_{\widehat{\tau}_r}} > \frac{m_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} = \overline{\alpha} \ ,$$

or, equivalently

$$\frac{1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} m_{\widehat{\tau}_\rho}}{1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\widehat{\tau}_\rho}} > \overline{\alpha} \ ,$$

$$1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} m_{\widehat{\tau}_\rho} > \overline{\alpha} \left(1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\widehat{\tau}_\rho} \right)$$

which is

$$1 - \overline{\alpha} > \sum_{\rho \in \mathbf{R} \setminus \{r\}} \left(m_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho} \ . \right) \tag{4.19}$$

Because of (4.18) the terms under sum in (4.19) are all nonnegative. Therefore, (4.19) permits to choose positive reals $\alpha_1, \ldots, \alpha_r$ such that

$$1 - \overline{\alpha} > 1 - \alpha_r > \sum_{\rho \in \mathbf{R} \setminus \{r\}} \left(m_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho} \ . \right) \tag{4.20}$$

$$\alpha_\rho > m_{\widehat{\tau}_\rho} - \alpha_r h_{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R} \setminus \{r\}) \ , \tag{4.21}$$

and

$$1 - \alpha_r \ = \sum_{\rho \in \mathbf{R}\setminus\{r\}} \alpha_\rho \tag{4.22}$$

holds true. In other words, the α_ρ are positive convex coefficients as

$$\sum_{\rho \in \mathbf{R}} \alpha_\rho \ = \ 1 \ . \tag{4.23}$$

Also, we have

$$\alpha_r > \overline{\alpha} = \frac{m_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} \ . \tag{4.24}$$

Now consider the vector

$$\widehat{\boldsymbol{x}} \ := \sum_{\rho \in \mathbf{R}\setminus\{r\}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{T}^\rho} + \alpha_r \bar{\boldsymbol{x}} \ . \tag{4.25}$$

Then clearly for $\rho \in \mathbf{R} \setminus \{r\}$ we have

$$\widehat{x}_{\widehat{\tau}_\rho} \ = \ \alpha_\rho + \alpha_r h_{\widehat{\tau}_\rho} \ > \ m_{\widehat{\tau}_\rho} \tag{4.26}$$

(in view of (4.21)), and for $\rho = r$

$$\widehat{x}_{\widehat{\tau}_r} \ = \ \alpha_r h_{\widehat{\tau}_r} > \overline{\alpha} h_{\widehat{\tau}_r} \ = \ m_{\widehat{\tau}_r} \tag{4.27}$$

(in view of (4.24)). Next

$$\widehat{\boldsymbol{x}}\widehat{\boldsymbol{a}} \ = \sum_{\rho \in \mathbf{R}\setminus\{r\}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{T}^\rho} \widehat{\boldsymbol{a}} + \alpha_r \bar{\boldsymbol{x}} \widehat{\boldsymbol{a}} \ = \ \sum_{\rho \in \mathbf{R}} \alpha_r = 1 = v(\widehat{\boldsymbol{a}}). \tag{4.28}$$

The last 3 equations and inequalities show

$$\widehat{\boldsymbol{x}} \operatorname{dom}_{\widehat{\boldsymbol{a}}} \boldsymbol{m}$$

Now by Lemma 4.2 there exists $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon < \varepsilon_0$ there is an ε–relevant coalition $T^\varepsilon = T^{\varepsilon \boldsymbol{a}}$ such that

$$\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}} \operatorname{dom}_{T^{\varepsilon \boldsymbol{a}} \oplus} \boldsymbol{\vartheta} \tag{4.29}$$

holds true.

q.e.d.

Theorem 4.6. *1. $\mathcal{C}(\boldsymbol{v})$ dominates $\mathcal{J} \setminus \mathcal{H}$.*

2. If $\mathcal{H} \subseteq \mathcal{C}(\boldsymbol{v})$ then $\mathcal{H} = \mathcal{C}(\boldsymbol{v})$ is externally stable, hence the unique vNM–Stable Set.

Proof:

1[st]STEP : Let $\boldsymbol{\vartheta} \notin \mathcal{H}$ be an imputation with minima vector $\boldsymbol{m}$. If $\boldsymbol{m} \in \boldsymbol{H}$, then $\boldsymbol{m}$ is a pre–imputation, hence $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^{\boldsymbol{m}} \in \mathcal{H}$. Hence we can assume $\boldsymbol{m} \notin \boldsymbol{H}$. We show that $\boldsymbol{\vartheta}$ is dominated by the core.

Now, for some $\boldsymbol{a} \in \boldsymbol{A}^s$ we have $\boldsymbol{m}\boldsymbol{a} < v(\boldsymbol{a}) = 1$. Let $\bar{\boldsymbol{\tau}}$ be the sequence specifying $\boldsymbol{a}$ according to 3.5. Then (as the coordinates of $\boldsymbol{a}$ are at least 1 by (3.20))

$$\sum_{\rho \in \mathbf{R}} m_{\overline{\tau}_\rho} \leq \sum_{\rho \in \mathbf{R}} m_{\overline{\tau}_\rho} a_{\overline{\tau}_\rho} = \boldsymbol{m}\boldsymbol{a} < 1 \quad .$$

Now choose $0 < \varepsilon, (\delta_\rho)_{\rho \in \mathbf{R}}$ sufficiently small and an ε–$\boldsymbol{a}$–relevant coalition $T^{\varepsilon \boldsymbol{a}}$ such that

$$\dot{\boldsymbol{\vartheta}}(\bullet) < m_{\overline{\tau}_\rho} + \delta_\rho \;\text{ on }\; T^{\varepsilon \boldsymbol{a}} \cap \boldsymbol{D}^\rho := T^{\varepsilon \boldsymbol{a} \rho} \quad (\rho \in \mathbf{R})$$

as well as

$$\sum_{\rho \in \mathbf{R}} m_{\overline{\tau}_\rho} + \sum_{\rho \in \mathbf{R}} \delta_\rho < 1$$

holds true. Furthermore, choose $0 < \delta$ such that

$$\left(\sum_{\rho \in \mathbf{R}} m_{\overline{\tau}_\rho} + \sum_{\rho \in \mathbf{R}} \delta_\rho \right) + \delta \;=\; 1$$

and put

$$\alpha_\rho \;:=\; m_{\overline{\tau}_\rho} + \delta_\rho + \frac{\delta}{r} \quad (\rho \in \mathbf{R})$$

such $\sum_{\rho \in \mathbf{R}} \alpha_\rho = 1$. Then we have

$$\dot{\boldsymbol{\vartheta}}(\bullet) < m_{\overline{\tau}_\rho} + \delta_\rho < m_{\overline{\tau}_\rho} + \delta_\rho + \frac{\delta}{r} \;=\; \alpha_\rho \;\text{ on }\; T^{\varepsilon \boldsymbol{a}} \cap \boldsymbol{D}^\rho = T^{\varepsilon \boldsymbol{a} \rho} \quad (\rho \in \mathbf{R}) \quad ,$$

that is

$$\dot{\boldsymbol{\vartheta}}(\bullet) < \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{e}^{T^\rho}(\bullet) \;\text{ on }\; T^{\varepsilon \boldsymbol{a}}$$

and

$$\sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho (T^{\varepsilon \boldsymbol{a} \rho}) = \sum_{\rho \in \mathbf{R}} \alpha_\rho \varepsilon (a_{\overline{\tau}_\rho}) = \sum_{\rho \in \mathbf{R}} \alpha_\rho \varepsilon \;=\; \varepsilon = v(\varepsilon \boldsymbol{a}) = \boldsymbol{v}(T^{\varepsilon \boldsymbol{a}}) \;.$$

This way it is seen that $\boldsymbol{\vartheta}$ is dominated by the core.

2[nd]STEP :

The second part follows immediately as $\boldsymbol{C}(v) \subseteq \mathcal{H}$ is generally true. Therefore, we have in this case $\boldsymbol{C}(v) = \mathcal{H}$. that is, the core is externally stable, hence the unique vNM–Stable set. **q.e.d.**

Example 4.7 (The EHMS Example). We consider the case of two intermediate values similarly to SECTION 2, Example 2.1. Thus we have

$$\overset{\bullet}{\boldsymbol{\lambda}}^0 \;=\; h_2\, \mathbb{1}_{\boldsymbol{D}^2} + h_3\, \mathbb{1}_{\boldsymbol{D}^3} + \mathbb{1}_{\boldsymbol{D}^4}\ . \tag{4.30}$$

We focus on $\boldsymbol{h}$ and $\boldsymbol{\lambda}$ satisfying

$$h_1 = 0, \quad h_2, h_3 < 1, \quad h_4 = 1, \quad h_2 + h_3 \geq 1, \quad \lambda_1 + \lambda_3 \leq 1\ . \tag{4.31}$$

Note that the last condition is equivalent to either $\lambda_2 \geq \lambda_3$ or $\lambda_4 \geq \lambda_1$.

EINY, HOLZMAN, MONDERER AND SHITOVITZ ([2], Example 4.3.) consider the special case that $h_1 = 0, h_4 = 1, h_2 = h_3 = \frac{1}{2}$ and $\lambda_1 = \frac{1}{2},\ \lambda_3 = \frac{1}{4}$.

Now, under the condition (4.31) , it turns out that the core is stable. Indeed, we have

$$\boldsymbol{A} \;=\; \left\{\boldsymbol{a} \in \mathbb{R}^{\mathbf{t}} \,\middle|\, \min\{a_1 + a_2,\quad a_3 + a_4,\quad h_2 a_2 + h_3 a_3 + a_4\} \geq 1, \quad \right\}$$

A short computation (using a standard procedure for the generation of extremals of a compact and convex set) yields the relevant vectors:

$$\boldsymbol{A}^e \;=\; \left\{(1,0,0,1),(0,1,0,1),(0,1,1,0),(1,0,\frac{1}{h_3},0)\right\}.$$

Hence, all relevant vectors are separating, we have $\boldsymbol{A}^s \;=\; \boldsymbol{A}^e$. Now,

$$\boldsymbol{H} = \left\{\boldsymbol{x} \in \boldsymbol{J} \,\middle|\, x_1 + x_4 \geq 1, \quad x_2 + x_4 \geq 1, \quad x_2 + x_3 \geq 1, \quad x_1 + \frac{x_3}{h_3} \geq 1\right\}\ . \tag{4.32}$$

By $\lambda_1 + \lambda_3 \leq 1$ the extremals of $\boldsymbol{H}$ turn out to be

$$\{(1,1,0,0),(0,0,1,1)\} \;=\; \left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}\right\} \;\subseteq\; \boldsymbol{C}(v).$$

Indeed, this can either be verified by running though the standard procedure as above or slightly faster as follows.

Suppose $\boldsymbol{x}$ is an extremal in $\boldsymbol{H}$. If

$$x_1 + x_3 < 1 \tag{4.33}$$

Then, adding the first 3 inequalities listed in (4.32) we obtain $(x_1 + x_3) + 2(x_2 + x_4) \geq 3$, hence

$$x_2 + x_4 > 1\ \ . \tag{4.34}$$

Hence at least one of the equations $x_\tau = 0$ has to be involved in the determination of $\boldsymbol{x}$.

Now, $x_4 = 0$ would imply $x_1 \geq 1$ (from the first in (4.32)), hence not compatible with (4.33). Similarly, $x_2 = 0$ would imply $x_1 \geq 1$ and $x_3 = 0$ together with the last one in (4.32) would imply $x_1 \geq 1$ as well.

Finally, if $x_1 = 0$ would be the case, then

$$(4.35)\quad 1 = \sum_{\tau \in \mathbf{T}} x_\tau = \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 \geq \lambda_3(x_2 + x_3) + \lambda_4 x_4 \geq \lambda_3 + \lambda_4 = 1 \ .$$

Now, if $\lambda_3 > \lambda_2$ then there is an immediate contradiction.

If, on the other hand, $\lambda_3 = \lambda_2$ then (4.35) shows that $x_2 + x_3 = 1$ and $x_4 = 1$ (as no strict inequality can occur). Then, together with $x_1 = 0$, there are only 3 independent equations determining $\boldsymbol{x}$. Hence there must be a further equation $x_\tau = 0$ determining $\boldsymbol{x}$ which, in view of the above cannot happen.

Hence, any extremal of $\boldsymbol{H}$ satisfies in addition $x_1 + x_3 \geq 1$. Adding this to the inequalities of $\boldsymbol{H}$, we observe, that all the inequalities of $\boldsymbol{C}(\boldsymbol{v})$ are satisfied, i.e., $\boldsymbol{C}(\boldsymbol{v}) \subseteq \boldsymbol{H}$. Therefore, by Theorem 4.6 the core $\mathcal{C}(\boldsymbol{v})$ is a vNM–Stable Set for $\boldsymbol{v}$.

A slight generalization of the argument provided by EINY ET AL. [2] for the case presented in (4.31). runs as follows.

Let $\boldsymbol{\eta}$ be an imputation and let $\boldsymbol{m}$ denote the vector of minima.

If $m_2 + m_4 < 1$ is true, then, for $\delta := \frac{1}{2}(1 - (m_2 + m_4))$ we can find $\varepsilon > 0$ and coalitions $T^\tau \subseteq \boldsymbol{D}^\tau \quad (\tau = 2, 4)$ such that both are of measure ε and

$$(4.36)\qquad \dot{\boldsymbol{\eta}} < m_2 + \delta \text{ on } T^2, \quad \dot{\boldsymbol{\eta}} < m_4 + \delta \text{ on } T^4 \ .$$

(in other words, $T^{24} := T^2 \cup T^4$ is $\varepsilon - 24$–relevant as $\boldsymbol{\lambda}^3(T) > \varepsilon$). Then clearly the imputation $\boldsymbol{\vartheta} := (m_2 + \delta)\boldsymbol{\lambda}^1 + (m_4 + \delta)\boldsymbol{\lambda}^2 \in \mathcal{C}$ dominates $\boldsymbol{\eta}$ via T^{24}. Hence, if $\boldsymbol{\eta}$ is *not* dominated by the core, then necessarily $m_2 + m_4 \geq 1$ holds true. Analogously we consider $\varepsilon - 14$–relevant and $\varepsilon - 23$–relevant coalitions.

Hence, whenever $\boldsymbol{\eta}$ is *not* dominated by the core, the essential minima derived satisfy the inequalities

$$(4.37)\qquad m_1 + m_4 \geq 1$$

$$(4.38)\qquad m_2 + m_4 \geq 1$$

$$(4.39)\qquad m_2 + m_3 \geq 1 \ .$$

Adding up *suitable "multiples"* of these inequalities we obtain

$$\lambda_1(m_1 + m_4) + \lambda_3(m_2 + m_3) + [1 - (\lambda_1 + \lambda_3)]\,(m_2 + m_4) \geq 1.$$

which after some reshuffling and using $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = 1$ turns out to be

$$(4.40)\qquad \sum_{\tau=1}^{4} \lambda_\tau m_\tau \geq 1 \ .$$

Now, as $\boldsymbol{\eta}$ is an imputation, hence has integral not exceeding 1, we obtain

$$\sum_{\tau=1}^{4} \lambda_\tau m_\tau \leq 1, \tag{4.41}$$

owing to the definition of the m_τ.

Now, none of the inequalities involved can be a strict one, otherwise (4.41) and (4.40) would yield a contradiction. Hence *all* inequalities are equations. Then, from the first three it follows that $m_1 = m_2$ and $m_3 = m_4$ holds true. This shows that $\boldsymbol{\eta} \geq m_1 \mathbb{1}_{\boldsymbol{C}^1} + m_3 \mathbb{1}_{\boldsymbol{C}^2}$ with $m_1 + m_3 = 1$ from (4.40) which is an equation as well. But as $\boldsymbol{\eta}$ is an imputation, we must have $\boldsymbol{\eta} = m_1 \mathbb{1}_{\boldsymbol{C}^1} + m_3 \mathbb{1}_{\boldsymbol{C}^2}$, hence $\boldsymbol{\eta} \in \mathcal{C}$. Therefore, imputations not dominated by the core are elements of the core, that is the core is (the only) vNM–Stable Set. Of course, this procedure is just another approach to showing that $\boldsymbol{H} = \boldsymbol{C}(\boldsymbol{v})$.

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The following generalizes the EHMS Example, however the assumptions are slightly different. As a prerequisite we mention a simple

Lemma 4.8. *Let $\boldsymbol{x} \in \mathbb{R}_+^{rt}$ be such that*

$$x_{\tau_1} + \ldots + x_{\tau_r} \geq 1 \quad (\boldsymbol{\tau} \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r) \tag{4.42}$$

and

$$\sum_{\tau \in \mathbf{T}} \lambda_t x_t \leq 1 \ . \tag{4.43}$$

Then $\boldsymbol{x} = \boldsymbol{e}^{T^\rho}$ for some $\rho \in \mathbf{R}$.

Theorem 4.9. *Let $\boldsymbol{h} > 0$ and assume that there is exactly one sequence $\bar{\boldsymbol{\tau}} \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$ such that*

$$h_{\bar{\tau}_1} + \ldots + h_{\bar{\tau}_r} < 1$$

while

$$h_{\tau_1} + \ldots + h_{\tau_r} \geq 1$$

holds true for all other sequences $\boldsymbol{\tau} \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r$. Then $\mathcal{C}(\boldsymbol{v})$ is the unique vNM–Stable Set.

Proof: 1$^{\text{st}}$STEP : Among the vectors $\boldsymbol{a} \in \boldsymbol{A}^s$ we find all vectors

$$\boldsymbol{a}^{\odot} \; = \; (\ldots, 1, \ldots, 1, \ldots, \ldots, 1, \ldots)$$

with coordinates 1 along any sequence $\boldsymbol{\tau} \neq \bar{\boldsymbol{\tau}}$. There are also vectors of the shape

$$\boldsymbol{a}^{\oplus} \; = \; (\ldots, 1, \ldots, \frac{1 - \sum_{\rho \in \mathbf{R} \setminus \sigma} h_{\bar{\tau}_\rho}}{h_{\bar{\tau}_\sigma}}, \ldots, 1, \ldots) \quad (\sigma \in \mathbf{R})$$

with non vanishing coordinates along the sequence $\bar{\boldsymbol{\tau}}$. Accordingly, for $\boldsymbol{x} \in \boldsymbol{H}$ we have inequalities

$$x_{\tau_1} + \ldots + x_{\tau_r} \quad (\boldsymbol{\tau} \neq \bar{\boldsymbol{\tau}}) \tag{4.44}$$

$$x_{\overline{\tau}_1} + \ldots + x_{\overline{\tau}_\sigma}\left(\frac{1 - \sum_{\rho \in \mathbf{R} \setminus \sigma} h_{\overline{\tau}_\rho}}{h_{\overline{\tau}_\sigma}}\right) + \ldots x_{\overline{\tau}_r} \geq 1 \quad (\sigma \in \mathbf{R}) \ , \tag{4.45}$$

together with the equation

$$\sum_{\tau \in \mathbf{T}} \lambda_t x_t = 1 \tag{4.46}$$

characterizing pre–imputations.

We wish to prove that

$$x_{\overline{\tau}_1} + \ldots + x_{\overline{\tau}_r} \geq 1 \tag{4.47}$$

holds also true, that is, the missing inequality of type (4.44) for $\bar{\boldsymbol{\tau}}$ is present as well. The result will then follow using the lemma.

2$^{\text{nd}}$STEP :

Now, if we add up all the inequalities (4.44), we obtain $r^t - 1$ on the right side as there are r^t sequences $\boldsymbol{\tau}$ and $\bar{\boldsymbol{\tau}}$ is excluded.

Next, on the left side, the variable $x_{\tau_1}(\tau_1 \neq \overline{\tau}_1)$ appears $r^{(t-1)}$ times, as $x_{\tau_2}, \ldots, x_{\tau_r}$ can be chosen freely. On the other hand, the variable $x_{\overline{\tau}_1}$ appears $r^{(t-1)} - 1$ times only, as the sequence $x_{\overline{\tau}_2}, \ldots, x_{\overline{\tau}_r}$ must be avoided. Thus, After adding all inequalities (4.44) and grouping the left side appropriately, we obtain

$$\begin{aligned} (r^{(t-1)} - 1)(x_{\overline{\tau}_1} + \ldots + x_{\overline{\tau}_r}) \\ +(r^{(t-1)})(x_{\tau_2} + \ldots + x_{\tau_r}) \\ + \ldots + \\ +(r^{(t-1)})(x_{\tau'_2} + \ldots + x_{\tau'_r}) \geq r^t - 1 \end{aligned} \tag{4.48}$$

with $r - 1$ terms on the left side apart from the first one. Note that

$$(r^{(t-1)} - 1) + (r - 1)r^{(t-1)} = r^t - 1 \ . \tag{4.49}$$

Thus, if one of the inequalities on the left side yields less than 1 in summation, then another one must exceed 1.

3$^{\text{rd}}$STEP : Consider now an extremal point $\bar{\boldsymbol{x}}$ of $\boldsymbol{H}$. Assume that (4.47) is violated, hence we have

$$x_{\overline{\tau}_1} + \ldots + x_{\overline{\tau}_r} < 1 \ . \tag{4.50}$$

then at least one of the terms in (4.48) has to exceed 1. Hence, in order to generate an extremal point of $\boldsymbol{H}$, there has to be at least one equation

$\overline{x}_t = 0$ involved in the equations selected among the inequalities for $\bar{\boldsymbol{x}}$. Let us assume that $\overline{x}_{\widehat{\tau}_1} = 0$ holds true with $\widehat{\tau}_1 \in \mathbf{T}^1$.

4$^{\text{th}}$STEP : Now, if $\widehat{\tau}_1 \neq \overline{\tau}_1$, then in (4.44) *all* inequalities

$$x_{\tau_2} + \ldots + x_{\tau_r} \geq 1 \quad . \tag{4.51}$$

for any sequence $x_{\tau_2}, \ldots, x_{\tau_r}$ appear as they can be chosen freely. If, on the other hand, $\widehat{\tau}_1 = \overline{\tau}_1$, then consider the first equation (for $\sigma = 1$) in (4.45). As $x_{\overline{\tau}_1} = 0$, this equation reads indeed

$$x_{\overline{\tau}_2} + \ldots + x_{\overline{\tau}_r} \geq 1 \quad . \tag{4.52}$$

Therefore, again *all* inequalities for any sequence $x_{\tau_2}, \ldots, x_{\tau_r}$ appear. Clearly,

$$\sum_{\rho \in \mathbf{R} \setminus 1} \lambda_t \overline{x}_t \leq 1,$$

hence by the lemma $\bar{\boldsymbol{x}}$ restricted to coordinates in $\mathbf{T}^2 \times \ldots \times \mathbf{T}^r$ is one of the $\boldsymbol{e}^{T^\rho}$ for $\rho = 2, \ldots, r$. Then necessarily, $\bar{\boldsymbol{x}}$ has to be zero on all coordinates in $\mathbf{T}^1$.

5$^{\text{th}}$STEP : The above reasoning was done under the assumption that (4.45) is violated. However, if (4.45) holds true, then again by the lemma we have that $\bar{\boldsymbol{x}}$ is one of the $\boldsymbol{e}^{T^\rho}$ for $\rho = 1, \ldots, r$. We conclude, that all the extremals of $\boldsymbol{H}$ are necessarily those of the pre–core, that is $\boldsymbol{H} \subseteq \boldsymbol{C}(v)$ and $\mathcal{H} \subseteq \mathcal{C}(\boldsymbol{v})$. Then the present theorem follows from Theorem 4.6,

q.e.d.

Note that the EHMS–example 4.7 and Theorem 4.9 rely on sightly different assumptions, as in the example we have $h_1 = 0$. The uniqueness of the minimizing sequence is however the same in both cases. There is one extremal point of the shape $\boldsymbol{a}^{\oplus}$ missing and hence we need another device to show that all inequalities $x_{\tau_1} + x_{\tau_2} \geq 1$ appear. This is the requirement towards the density regarding λ_1, λ_3 as specified in (4.31).

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