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STEPS TOWARD A LONG-TERM STRATEGY OF MATHEMATICAL INSTRUCTION<sup>1)</sup>

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<sup>1)</sup> This paper continues the author's paper presented at the 5th International Conference for the Psychology of Mathematics Education in Grenoble, July 1981

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INTRODUCTION

Use of the appropriate piece of knowledge at the right time is the essence of intelligent behavior... But how is this carried out? Should we only think about concept formation; and arrange knowledge about how concepts are linked as it grows, that is, one "piece" after the other? Or what about "the alphabetical system of arrangement"?

Imagine, for the moment, a not very intelligent owner of "Encyclopaedia Britannica" who has access to the knowledge stored there only by way of the marginals, which may suffice for "Einstein" (where he may look under "E"); but which already fails for "Fibonacci" (where he has to look under "Leonardo of Pisa") --if he does not know about "micropaedia", but unwittingly goes in the "alphabetical mode" through "macropaedia" where the "topical mode" would be appropriate. And how would he access further knowledge upon "a sequence of numbers each of which, after the second, is the sum of the two preceding numbers", if he does not know that "Fibonacci" can be used as an entry? So, to make good use of the knowledge contained in "Encyclopaedia Britannica" he needs particular knowledge about its organizational structure consisting of "Propaedia", "Macropaedia", and "Micropaedia".

And, to come to the point, in order to make good use of any knowledge, and we particularly think of mathematical knowledge, it needs organization at all, as well as an awareness of the organizational principles underlying the mental representations of knowledge.

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Organization yields structure, which itself contains the potential for process the nature of which highly depends on the structure which gives rise to it. But it needs a process again to make structural knowledge effective, to recombine it with other structural knowledge, to combine it with knowledge brought in anew, to operate on the outside world, as well as on internal world models, by the use of the knowledge at one's disposal. And, to return to the opening statement: The degree of being able to make use of appropriate knowledge at the right time depends principally on the organizational principles aggregating structural knowledge. So, what kind of organization makes structural knowledge easily processible?

When querying this I do very much think of a consciously pursued organizing of the knowledge acquired and retained, with the aim of being familiar with it and of knowing the "ports of access", which concerns the phenomenon of understanding; and with the aim of being able to use the knowledge in multiple ways, which concerns the transfiguration into process. All these matters depend to a great extent on the cognitive structures which are established in the individual, i.e. the categories by which one interpretes perceived phenomena, to find them matching certain conceptual "frames" which means to understand (or to find them mismatching which means not to understand them).

But, what is stored in memory is not necessarily the same as what one knows, for a given piece of knowledge, albeit encoded somewhere in the brain, may be inaccessible. So, if we start thinking about how to establish certain structures of mathematical knowledge without thinking about supporting a suitable organizational framework fitting to the structures given in human memory, we are running the risk of putting all the knowledge into a big dump or stack; which is not a good base for retrieving it and putting it into process.

So, as with Encyclopaedia Britannica, we need a topical arrangement, as well as an indexing, cross-references, and a guidance for the translation of knowledge into action.

## DIFFERENT KINDS OF MEMORY STRUCTURES

Now, when speaking all the time of such organizational concerns in human memory and information processing, we must eventually come to what is considered to be the model of an organizing unit: the schema. We could say that schemata set up the structures within the memory system of a given person, and reflect and constitute his knowledge and beliefs, including all his misconceptions and inaccuracies.

The encyclopaedia model might mislead our thinking as to how knowledge becomes processible: If our memory system only possessed a retrieving capability (similar to the "Ready Reference and Index" part of the encyclopaedia), it would not necessarily know what to do about the information accessed. So we can assume that it is linked with knowledge of possible things to do, and of how to put them into process.

As I figured out earlier (Wachsmuth, 1981), often a distinction is found between two different types of "schema", which I called "relational" or type-1, and "operational" or type-2; the first serving as a structural representation unit organizing knowledge declaratively, whereas by the second is meant an instance which directs action (not restricted to the sensorimotor level): for distinction we could therefore speak of an "action schema".

I think everyone will agree with the fact that for instance when I am solving quadratic equations, different sorts of schemas are at work: declarative ones which give me awareness about what I am doing, and more procedural ones which automatically trigger my actions when once activated. So, at least, we have to distinguish skill representations from declaratively stored knowledge.

I myself very much prefer to imagine double-type (conceptual) schemata, which can serve as declarative knowledge stores, allowing conscious inspection of their conceptual relationships by using the names of the symbols involved, as well as they

serve as action guides or, say, director systems in the sense of Skemp (1979), which expand symbols to process when they become activated. (I like best the view that in order to reason about process-like phenomena the processes have to be frozen to symbols, such as limes processes in maths; or think of chess, where ten or even more moves have been assembled to one symbol.)

But: It may happen that one and the same symbol--or we should better say sign--can call upon different schemata, which sometimes, moreover, are employed within the same job. Which schema is then called for first? Can a sign have a demand character, giving preference to calling one schema while being a hindrance to calling the other?

An illustration of what I mean was given earlier (Wachsmuth, 1981) with the task of integrating  $\cos^2 x$ , where the sign "/" once calls process where  $\cos^2 x$  is the operand, as well as declaring an object to be operated on as a whole, and which could be envisioned as being the (unknown) result of the above process. As an organizational principle to pursue in students' schema-building during mathematical instruction I suggested establishing twin representations of a mathematical concept in a schema, using descriptions showing process as well as structure; in order to favor students' actions by their knowing what they can do with it (as a tool), and what they can do about it (as an object; cf. Piaget 1975, p.97).

#### HOLISTIC vs. OPERATIVE STRUCTURES

In comparing psychological research which has influenced mathematical instruction in the last three decades, we find as the two main poles Gestalt psychology on the one hand, postulating a kind of "structured wholes" (organized entities) in perception which become restructured in processes of learning, so that learning was explained by way of changes in perception ("insight", cf. Köhler, 1947); and on the other hand the genetic

psychology of intelligence of Piaget, who stated that the structured wholes of Gestalt psychology do not provide explanation of the general flexibility and reversibility of what he has called "operative structures", which bear upon internalized actions (Piaget, 1948). What we want to point out in the following is that these two dichotomic positions might coexist, each in its own right.

What happens when a student exercises long division by doing quite a number of tasks? By doing this he learns, that is, he acquires particular knowledge about how to do this sort of task, even in those particular instances where he has never done it before. How does it happen?

As we put it, by varying the instances of digits, while performing similar action sequences, he does not precisely store these "experiences" to recall them when needed; but he establishes an action schema which is based upon prototypes of digits rather than particular digits. By this I mean that the core of such an action schema bears upon the skeleton or, say, the invariant pattern of all action sequences, which repeatedly appears when exercising a sample of tasks.

Well; this particular skill knowledge in long division is a priori processible from the fact that it is kind of procedural knowledge which is not too different from sensori-motor kinds of knowledge. But what about that kind of knowledge which is e.g. needed in mathematical problem solving? Could there perhaps be established declarative memory schemas which have a "conceptual skeleton", providing the ability to recognize new problems as belonging to familiar kinds, that is, from the fact that they match the same skeleton? And which could thus call upon suitable prototype action schemata, leading to similar decisions in particular instances, and e.g. apply algorithms that have been successfully used to solve this familiar kind of problem? As a second organizational principle I suggested arranging laterally special context schemata, by evolving links between relational descriptions being of a similar shape. What

do I mean by that? I will try to give a very simple but perhaps illuminating example.

Normally, in the course of mathematical instruction at school, special-context knowledge upon the powers and roots is acquired by the students, and, after quite a period of time, special-context knowledge upon the exponentials and logarithms is acquired which is similar in certain aspects.

At the latest now it is the time to establish an "indexed memory" with lateral entries, for instance to make the new knowledge processible by analogical transfer of the know-how which one has acquired about the "old" knowledge. I would like to illustrate this by figuring out a small portion of the "knowledge web", that is, the semantic network, which could be set up: "Semantic networks presumably are candidates for the role of internal semantic representation--i.e., the notation used to store knowledge inside the head.[...] A semantic network attempts to combine in a single mechanism the ability to store factual knowledge but also to model the associative connections exhibited by humans which make certain items of information accessible from certain others." (cf. Woods, 1975, p.44)

As a particular type of this, a concept network is shown in Figure 1, where "nodes" are joined by "links" which can be linked in turn. The net seems to arrange knowledge only statically, as it shows certain interrelations of the concepts involved; but, by considering a link as a verb ("is"/"are"...) and the nodes it joins as subject and object, some propositions can be drawn from the net:

- $x^2$  and  $\sqrt{x}$  are opposite
- $\exp x$  and  $\log x$  are opposite
- the relations between  $x^2/\sqrt{x}$  and  $\exp x/\log x$  are similar
- $\sqrt{x}$  declares a process ("seek for a number which...")
- $\sqrt{x}$  declares an object ("it is the number which...")
- these two declarations of  $\sqrt{x}$  are opposite
- $\log x$  declares a process
- $\log x$  declares an object

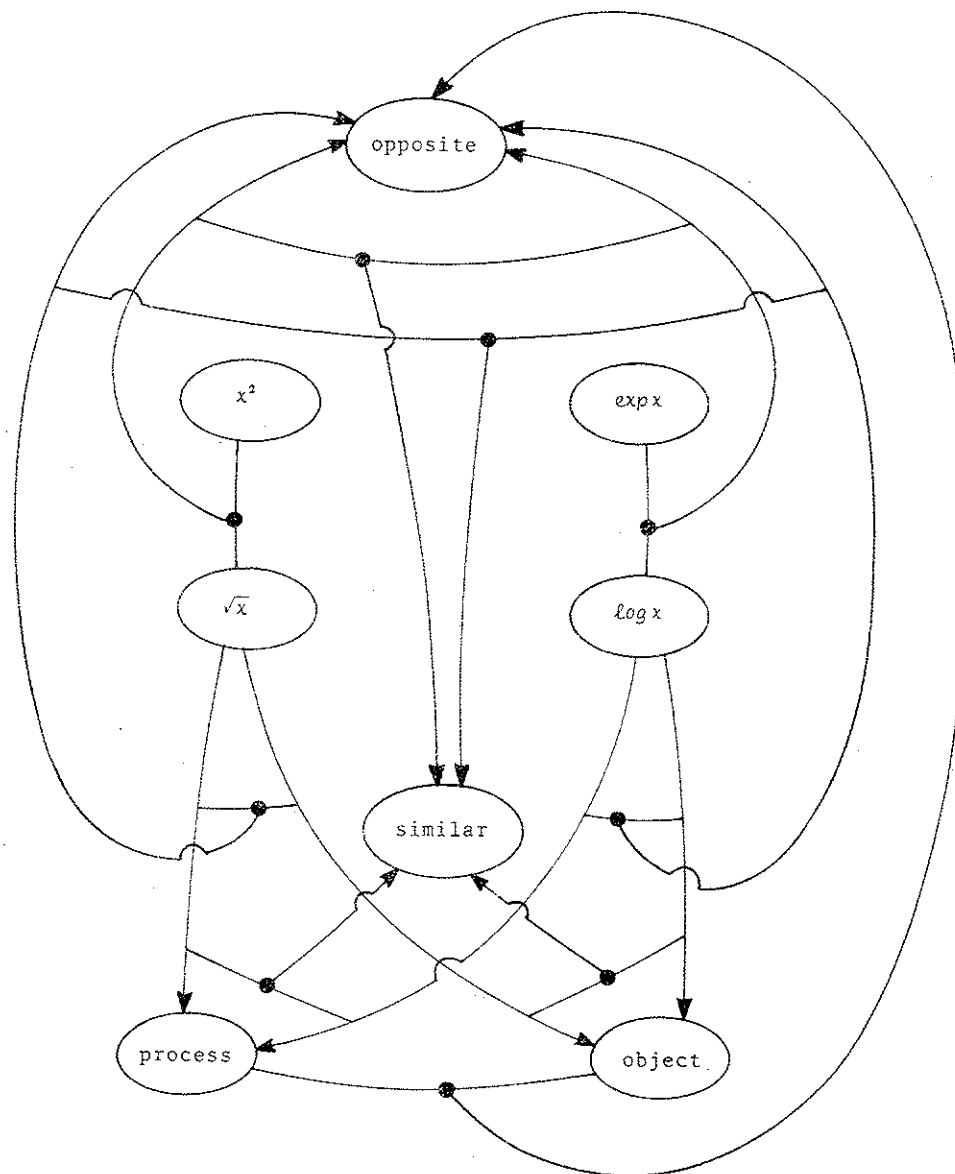


Figure 1

- these two declarations of  $\log x$  are opposite
- the last two opposite statements are similar
- "process" and "object" are opposite
- the declarations of  $\sqrt{x}$  and  $\log x$  of being processes are similar
- the declarations of  $\sqrt{x}$  and  $\log x$  of being objects are similar

By the use of such a kind of description which shows nested relationships and dependencies between the concepts involved, and thus makes aware of "lateral" similarities, an integrative schema of somewhat more generality could be built up; which contains not only the potential of drawing propositions on specific concepts (as "conceptual constants") but which also yields a higher-level view of these matters: Not "constants" (like  $\sqrt{x}$ ) are the instances, but instead, certain sorts of concepts are included (as prototypes of conceptual variables) which in this particular case stand plainly for the concepts of "function" and "inverse function". We could call such a conceptual schema a "prototype schema" (Figure 2).

Note that the prototype schema illustrated in Figure 2 does not contain the word "similar" as a descriptor, as the conceptual net has been simplified, by collapsing it just by the use of the similarity relation. We perhaps could envision this as the application of sort of a "net morphism" which folds the original net into a more abstract, and simpler one, involving  $\sqrt{x}$  as prototype of "some  $f^{-1}$ " rather than as a specific concept.

I think this illustrates one important feature of what Skemp (1979) has called "restructuring of a schema", in this case with the aim of opening a new kind of insight of broader generality; which of course includes the feasibility of taking in new material, for each kind of a conceptual relation which applies to the shape of the prototype schema can now be assimilated into it, by instantiation of the schema variables, whereby the range of what can be specified for the variables is gradually extended.

Now, what I actually did, when once teaching mathematics to a class of students (grade 10) who were very behind, is that to

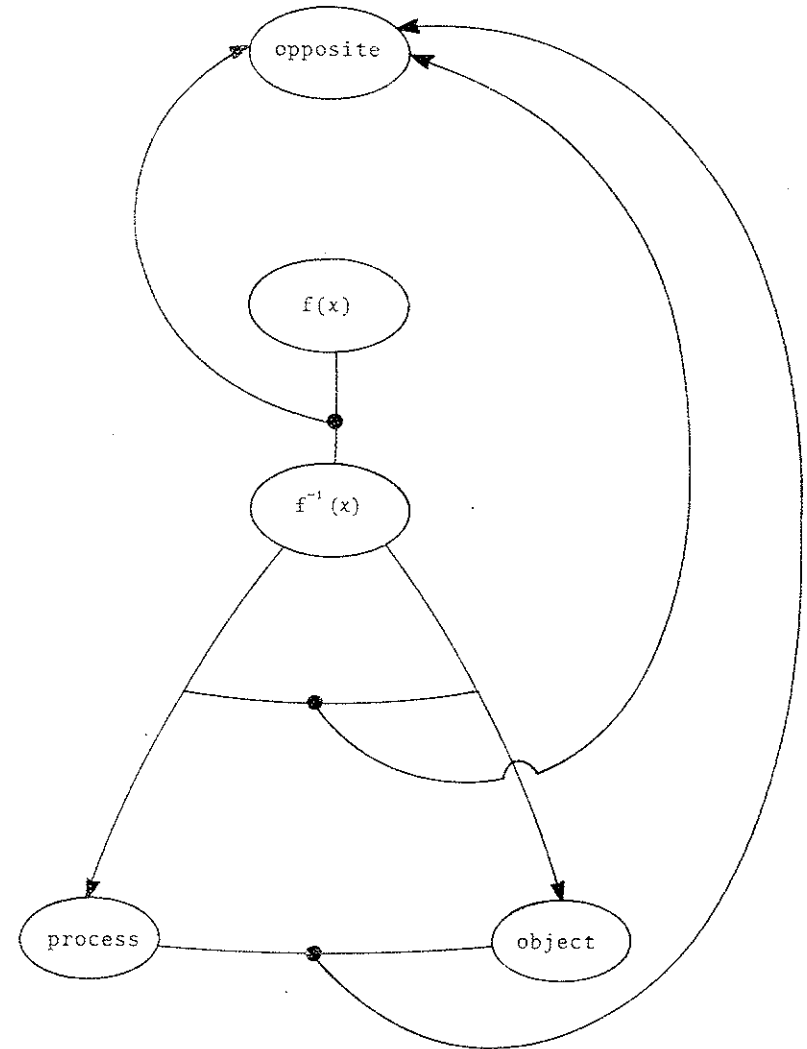


Figure 2

come up with the parallel class we first did a recap of what they had previously learnt about powers and roots, with the emphasis on the general shape of what they actually "knew" also from elementary arithmetic ("+ and - are opposite..."); and then I told them to rediscover such a kind of shape (the "structured whole") in what was coming. (And we managed to come up.)

So, with the last topics, we have come to a very strange and important point: Piaget has led us to believe that "operative structures" are indispensable in any theory of cognition and mathematics learning, whereas Gestalt psychology has spoken of "structured wholes"; both phenomena--dynamic and stationary-- seem to exist in their own right in intelligent thinking, as we have been trying to illustrate. This case might bear upon the fact that all human perception depends on certain grounds of shape: time, as a ground for process, and space, as a ground for structure.

To gain insights of full depth into human intelligent thinking, in my opinion, we have to take account of Piaget's operative structures, as well as of holistic structures as postulated by Gestalt psychology, which for instance can give rise to solving a problem by recognition of its overall structure, that is, to realize how it fits in a certain prototype schema.

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