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ON BALANCED GAMES, CORES, AND PRODUCTION

by

Volker BÖHM

CENTER FOR OPERATIONS RESEARCH & ECONOMETRICS

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CORE, DE CROYLAAN 54, 3030 HEVERLEE, BELGIUM TEL. 016-22 09 21, 016-23 49 31

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1. Since the appearance of the papers by Scarf [10] and Shapley [11] on the core of an N-person game, economists have puzzled over the intuitive meaning of the condition introduced there of balancedness of a family of sets of players and the associated notion of a balanced game. That balancedness provides a sufficient condition for non-emptiness of the core (and, for side payment games even a necessary condition) has made the lack of an intuitive, economic interpretation of this condition all the more bothersome.

In a later paper, Shapley and Shubik [12] together showed that any totally balanced side payment game can be considered as having been generated by an exchange economy with a finite number of agents and commodities and concave, continuous utility functions. Billera and Bixby have, in a recent series of papers [2,3,4], extended this analysis to all convex-valued, totally balanced games. Thus, there is an intimate connection between balancedness notions and exchange economies.

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Another application of balancedness in economics has been in the context of production. The author [5,6] has shown that balancedness of the technology distribution is a sufficient condition, along with other standard assumptions, for the non-emptiness of the core of an economy with production and for the existence of market equilibria with stable profit distributions. The latter equilibrium concept represents a modification of the competitive equilibrium for economies where coalitions of consumers exercise control over firms. In either case the balancedness condition was dictated by necessity of technique rather than by economic assumptions. Shapley and Shubik, in an example in their paper, showed a certain link between the balancedness of a game and a class of production models.

In this paper, we explore further the relationship of balanced distributions of sets and production, and in this process, hope to give a better intuitive understanding of the balancedness assumption. This we pursue in two directions. First we show an intimate relationship of the core of balanced side payment games with a full employment profit maximizing program for a certain class of linear production models. For these models a balanced technology distribution is a sufficient condition to make realized profits in the production model large enough to satisfy payments in the core of an associated game. Furthermore, with convexity of the aggregate production possibility set balancedness is a necessary condition. Second, we show that every totally balanced technology distribution can be generated from a canonical production problem with non-marketed commodities, which in turn implies an alternative representation of all totally balanced games than the one given by Billera [2], since every game is a special case of a technology distribution.

2. Definitions and results on games.

For the purpose of this paper the set $I = \{1,\ldots,n\}$ denotes the set of players, consumers, or productive resources, $2^I = \{S \subseteq I | S \neq \emptyset\}$ the set of coalitions or processes. With each $S \in 2^I$ one can associate in a natural way the members of S with the coordinates of Euclidean space R^n and define $R^S = \{x \in R^n | x_i = 0 \text{ for } i \notin S\}$.

Definition 1. An n-person game is a closed-valued correspondence $V : 2^I \to R^n$ such that for all S

(i) $V(S) \subset R^S$

(ii) if $x \in V(S)$ then $\{x\} - R^S_+ \subset V(S)$.

Definition 2. A game (I,V) is said to be balanced if

$$\sum_{S \in 2^I} d_S V(S) \subset V(I)$$

whenever $d_S \geqq 0$, $S \in 2^I$, and $\sum_{\substack{S \\ i \in S}} d_S = 1$ for all $i \in I$. The set of coefficients (d_S) is a solution to the system of equations

$$\sum_{S \in 2^I} d_S e_S = e_I, \quad d_S \geqq 0$$

where $e_S \in R^S$ with $e^i_S = 1$ for $i \in S$. For simplicity of notation, let E denote the n x (2^n-1)-matrix of vectors (e_S), $S \in 2^I$, and let $D = \{d | Ed = e_I, d \geqq 0\}$, a convex, compact set. The set $\mathcal{S}(d)$ of coalitions associated with the positive components of some $d \in D$ is called a balanced family and its weights are called balanced weights. This notion of a balanced game is stronger than the one introduced originally by Scarf [10]. However, it is the appropriate one for our purposes here. For the case of side payment games the two notions coincide.

Definition 3. A side payment game is a game (I,V) such that there exists a real-valued function $v : 2^I \to R$ and with V defined by

$$V(S) = \{x \in R^S | x.e_S \leqq v(S)\}.$$

Theorem (Shapley). A side payment game has a non-empty core if and only if it is balanced.

This result is an immediate consequence of the standard duality result in linear programming. An intuitive justification for the balancedness assumption will be given in the next section.

The next more general class of games, of which the side payment games are a special case, are the so-called hyperplane games (see Billera [1]). Let $\Pi_S \in R^S_+$ with $\Pi^i_S > 0$ for $i \in S$, and denote by Π the $n \times (2^n-1)$-matrix of vectors (Π_S), $S \in 2^I$.

Definition 4. A Π-hyperplane game is a game such that

$$V(S) = \{x \in R^S | x.\Pi_S \leqq v(S)\}$$

where v is a real-valued function

$$v : 2^I \to R.$$

Definition 5. A Π-balanced family of coalitions $S(\Pi)$ is determined by the positive components $d \in D(\Pi)$ where

$$D(\Pi) = \{d \mid \Pi d = \Pi_I,\ d \geqq 0\}.$$

Again, $D(\Pi)$ is a non-empty convex, compact subset of $R_+^{2^n-1}$.

Definition 6. A Π-hyperplane game is Π-balanced if

$$\sum_{S \in 2^I} d_S V(S) \subset V(I)$$

for every choice of $d \in D(\Pi)$.

As before, this notion of Π-balancedness is stronger than the one originally proposed by Billera [1]. The weaker notion is equivalent to the game having a non-empty core. Hence one has the following result :

Theorem (Billera). A Π-hyperplane game has a non-empty core if it is Π-balanced.

3. Examples.

The first example describes a certain type of productive schedule for which the profit maximizing decision is essentially one of searching for a production plan which guarantees a point in the core of the associated game.

Example 1. Consider a production set-up (e.g. a factory) which for the short run has a certain set $I = \{1,\ldots,n\}$ of fixed resources at its disposal. Typically, these are different types of machines, buildings, but also different types of labor which have been purchased and thus are part of the fixed cost which the factory incurres. Suppose that all of these resources generate services to the factory, which can essentially be measured in units of time, so that, by appropriate normalization, these services deliverable during e.g. a week or a day can be measured on the unit interval. With each subset S of the fixed resources is associated a given production possibility set $Y^S \subset R^k$ where R^k is the space of variable inputs and outputs. Y^S describes all short run net production possibilities if the subset of resources S is used 100% of the time exclusively. The set S with its associated Y^S defines a production process. If for some process S no positive output is possible, then $Y^S = \{0\}$ which essentially implies that the fixed resources associated with this process can be used up without any extra costs. It is assumed that each process is homogeneous in the following sense : if $y \in Y^S$, then, operating process S at an arbitrary level $0 \leqq d_S \leqq 1$, i.e. during d_S percent of the time, implies $d_S y \in Y^S$. This implies that each Y^S is star-shaped relative to the origin.

Now, let $p \in R^k$ denote prices for variable inputs and outputs. Then

$$v_S(p) = \sup pY^S$$

is the maximal profit obtainable from process S exclusively. Then, short run profit maximization to the factory, given the fixed resources $i = 1,\ldots,n$, is a solution of the following problem

$$(1) \qquad \text{Max } \Sigma d_S v_S(p)$$

$$\text{s.t.} \quad Ed = e_I \qquad d \in R_+^{2^n - 1},$$

i.e. $d \in D$, as defined above. The feasible non-zero scales d_S determine a balanced family of processes and, if all $v_S(p)$ are finite, there always exists a solution to the problem since D is compact and $\Sigma d_S v_S(p)$ is continuous. By standard duality arguments, there exists an associated dual problem determining rental values $t_i \geqq 0$ of the fixed resources $i = 1,\ldots,n$ such that for each process the joint rental value of its resources is at least as large as their maximal profit and the total rental value is equal to the total maximal profit, i.e. there exist $t_i^* \geqq 0$, such that

$$(2) \qquad \sum_{i \in S} t_i^* \geqq v_S(p) \qquad S \in 2^I$$

$$\sum_{i \in I} t_i^* = \sum_{S \in 2^I} d_S^* v_S(p)$$

where d^* is a solution to (1).

In game theoretic terms the set of solutions of problem (2) is nothing but the core of a side payment game (I,V) where $V(S) = \{x \in R^S | e_S \cdot x \leq v_S(p)\}$ and if $\sum_{i \in I} t_i^* = v_I(p)$. The latter equality imposes the condition that the game be balanced. For this condition to hold in the game, small coalitions cannot be too strong relative to the all player coalition. For our production model, however, the case where Y^I is relatively inefficient, cannot be excluded a priori since Y^I represents only one particular process and <u>not</u> the set of all feasible production plans. Hence feasibility at the profit maximum requires that there exists a production vector y such that $p.y = \sum_{S \in 2^I} d_S^* v_S(p)$ and such that y can be produced from some processes. Our assumption of homogeneity of the sets Y^S guarantees such that.

The example is actually a special case of the more general problem of determining stable profit distributions in economies where coalitions exercise control over firms (as for example in [6]) or as in the case of coalition production economies (see [5], [9], and [13]). Let $E = \{I, (X_i, e_i, \succsim_i), ((Y^S), Y)\}$ be such an economy where the technology distribution is given by $((Y^S), Y)$, Y being the aggregate production possibility set for the economy as a whole. In [6] Y is assumed to be generated by the collection (Y^S). Sondermann [13] and Oddou [6] assume that $Y = Y^I$, but, in general, this need not be the case*.

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* For some discussion of this problem see [5]. The construction used in [6] which is repeated here gives another convincing example of a situation where Y is different from Y^I.

For a given price system $p \in R^{\ell}$, the commodity space of the economy, a distribution of profit payments (t_i) to the individual consumers is said to be stable if

(1) $\quad t_i \geqq 0 \qquad i \in I$

(2) $\quad \sum_{i \in S} t_i \geqq \sup p.Y^S \qquad S \in 2^I$

(3) $\quad \sum_{i \in I} t_i = p.y \qquad \text{some } y \in Y.$

Different assumptions on $((Y^S), Y)$ have been used to show existence of stable profit distributions. The following lemma demonstrates that the condition of being balanced for the technology distribution comes closest to being necessary, thus indicating that the assumptions by Oddou and Sondermann are special cases.

Definition. A technology distribution $((Y^S), Y)$ is called balanced if

$$\sum_{S \in 2^I} d_S Y^S \subset Y \qquad \text{for all } d \in D.$$

Lemma. Let $0 \in Y$ and Y be convex. Define

$$P = \{p \in R^{\ell} \mid \sup p.Y < \infty\}.$$

If for every $p \in P$ there exists a stable profit distribution, then $((Y^S), Y)$ is balanced.

Proof. Suppose there exist $d \in D$ and $y_S \in Y^S$, $S \in 2^I$, such that $\Sigma d_S y_S \notin Y$. Then, according to the fundamental separation theorem, there exists $\bar{p} \in R^{\ell}$, $\bar{p} \neq 0$, such that

$$\sup \bar{p}.Y < \bar{p}.\Sigma d_S y_S.$$

Clearly $\bar{p} \in P$. Furthermore,

$$\sup \bar{p}Y < \sup\{\Sigma d_S \sup \bar{p}.Y^S | d \in D\}.$$

Hence, for any $y \in Y$ and any distribution (t_i) such that $\bar{p}.y = \sum_{i\in I} t_i$

$$\sum_{i\in I} t_i < \sup\{\Sigma d_S \sup \bar{p}.Y^S | d \in D\}$$

so that (t_i) cannot be stable. Q.E.D.

Examples of balanced technology distributions can be constructed easily.

Let $Y^{\{i\}}$ be a convex, compact set and let each $Y^S = \sum_{i\in S} Y^{\{i\}}$. Then $((Y^S),Y^I)$ is balanced. Let (Y^S) be arbitrary and define Y to be the smallest convex cone at zero containing the union of all Y^S. Then $((Y^S),Y)$ is balanced.

Another class of examples of balanced technology distributions can be described using the procedure to generate Y from (Y^S) introduced in [6]. There each production possibility set Y^S is associated with a firm which operates in a market independently of consumer decisions, but which is controlled by the coalition S. If each firm and each subset of firms can operate independently in the market, then it is natural to assume that

$$Y = \{\bigcup_{J} \sum_{S\in J} Y^S\} \cup \{0\}$$

where $\mathcal{J}$ is any family of non-empty coalitions. This is equivalent to writing

$$Y = \sum_{S \in 2^I} Y^S \cup \{0\}.$$

<u>Lemma</u>. Let (Y^S) be arbitrary and Y be convex and defined as above. Then $((Y^S),Y)$ is balanced.

<u>Proof</u>. Let $\mathcal{S}$ be balanced with weights (d_S) and $y_S \in Y^S$, $S \in \mathcal{S}$. Then

$$\sum_{\mathcal{S}} d_S y_S = \sum_{\mathcal{S}} (d_S y_S + (1-d_S)\{0\}) \in \text{conv} \sum_{\mathcal{S}} Y^S \cup \{0\} \subset Y$$

by convexity of Y. Q.E.D.

Note that for all of these examples neither $0 \in Y^S$ nor convexity on the individual sets is needed, so that a large class of production sets with set-up costs, increasing returns, and indivisibilities is included.

A few more general remarks about balanced technology distributions can be made. Let $\tilde{Y} = \{\sum d_S Y^S \mid d \in D\}$. If each Y^S is convex, then $\tilde{Y}$ is convex. If for some balanced family (e.g. for a partition) $0 \in Y^S$, then $\tilde{Y}$ is starshaped with respect to the origin. If Y is defined as in the last lemma then, in general, $\tilde{Y} \not\subset Y$, even if $0 \in Y^S$ for every S. For existence purposes of equilibria with a stable profit distribution the assumption conv $\tilde{Y} \subset Y$ seems to be the weakest one. For example, combining the techniques in [7] and in [6] one can easily prove the following existence theorem.

<u>Theorem</u>. For the economy $E = \{I,(X_i,e_i,P_i),((Y^S),Y)\}$ assume for each $i \in I$

(1) X_i non-empty, closed, convex and bounded below;

(2) P_i a preference correspondence which is irreflexive, convex-valued with an open graph;

(3) $p.e_i > \inf pX_i$ for all p;

and

(4) $\text{conv}\ \tilde{Y} \subset Y$ which is bounded above.

Then, there exist equilibria with a stable profit distribution.

The second example gives a straightforward generalization of the production model introduced in the first example.

<u>Example 2</u>. Assume that there are n fixed resources available at the amounts $(\alpha_1,\ldots,\alpha_n) = \alpha >> 0$. With each subset S of the resources is associated a production process (Y^S,Π_S) where Y^S is a non-empty production possibility set of the usual type in the space of variable inputs and outputs. Π_S is a vector in R^S_+ indicating the fixed amounts of the resources in S necessary to operate Y^S at unit level. As before, it is assumed that for arbitrary levels $d_S \geqq 0$ the associated production possibilities are $d_S Y^S$ and that excess capacities of any fixed input can be disposed of without costs. This will be the case, for example, if no fixed resource alone can produce non-zero variable outputs, i.e. for each $i = 1,\ldots,n$, $Y^{\{i\}} = \{0\}$. At given prices p for all variable inputs and outputs, the optimal short run allocation of the fixed resources is determined by the following maximization problem

$$\text{Max } \Sigma d_S \sup pY^S$$

subject to $\Sigma d_S \Pi_S = \alpha \quad d_S \geqq 0.$

Since each $\Pi_S \in R^S_+$, the positive weights d_S determine a Π-balanced family relative to α. As in the side payment case one can associate the following dual problem

$$\text{Min } x.\alpha \qquad x \in R^n_+$$

$$\text{s.t. } \quad \Pi_S x \geq \sup pY^S \qquad S \in 2^I.$$

This has a natural interpretation as a solution to the Π-hyperplane game for which the characteristic function is given by

$$V(S) = \{x \in R^S_+ | \Pi_S . x \leqq \sup pY^S\}.$$

Replacing α by Π_I the set of solutions of the dual is nothing but the core of the game which is non-empty, if the game is Π-balanced. On the other hand, in terms of the production model, there has to be an aggregate production vector y such that $p.y = x\Pi_I$. A reasonable assumption, then, on the collection of (Y^S, Π_S) would be that

$$\tilde{Y}(\Pi) = \{\Sigma d_S Y^S | d \in D(\Pi)\}$$

describes aggregate production possibilities. As in the discussion of Example 1 one can demonstrate similar results. The game has a non-empty core for every price vector p if $\tilde{Y}(\Pi) \subset Y^I$. Extending the terminology in a natural way one may call a technology distribution with this property Π-balanced. If Y^I is a convex set, then being Π-balanced is a necessary condition for the core to be non-empty <u>and</u> to be feasible at a production vector $y \in Y^I$ for every price p.

4. Totally balanced technology distributions.

Totally balanced games were introduced first by Shapley and Shubik for the side payment case and later by Billera and Bixby. Their results indicate that all convex valued and totally balanced games can be generated by exchange economies where each consumer has a continuous, concave utility function. A stronger representation result will be given for totally balanced technology distributions.

For the particular case where the aggregate production possibility set coincides with the production possibilities which are enforceable by the all player coalition through collusive action, one may use an analogous definition of a totally balanced technology distribution as for games.

Definition. A technology distribution is totally balanced if for every $T \in 2^I$ and all balanced families $\mathcal{S}(T)$ with respect to T

$$\Sigma d_S Y^S \subset Y^T \qquad d \in D(T)$$

where $D(T) = \{d \in R_+^{2^n - 1} | Ed = e_T\}$.

Within the framework in which production is treated in general equilibrium analysis, there exist only a few attempts of explaining why different groups of consumers have access to different production possibilities and, if so, what is an appropriate rule to determine the production possibilities of a coalition which forms e.g. from a partition of itself. Clearly, given "pure technical knowledge", group efficiency will depend on the managerial ability of its members, their organizational facilities, or other group specific resources or technologies. Taking in principal this approach,it is frequently argued ([9] and [13]) that all coalitions have access to the same technology in the strict sense of a production possibility set, but that there exist group specific resources (non-marketed commodities) which determine certain subsets as the production possibility sets enforceable by coalitions. It will be shown in the following two lemmata that, with an appropriate choice of the aggregate technology in an enlarged commodity space, any distribution of the non-marketed resources will generate a totally balanced technology distribution and, conversely, any such distribution can be generated in the same way from some simple distribution of non-marketed resources.

Let R^{ℓ} denote the space of marketed commodities and let there be m non-marketed commodities. Let $\omega_i \in R^m_+$ denote consumer i's endowment of these resources. Let $\overline{Y} \subset R^{\ell+m}$ denote the aggregate technology and define the technology distribution by

$$Y^S = \{y \in R^{\ell} \mid (y, \sum_{i \in S} \omega_i) \in \overline{Y}\}.$$

Lemma. If $\overline{Y}$ is a convex cone, then $((Y^S), Y^I)$ is totally balanced.

Proof. One only needs to show balancedness for I. Let $\mathcal{S}$ be balanced with weights (d_S). Since $\overline{Y}$ is a convex cone

$$d_S(y_S, \sum_S \omega_i) \in \overline{Y}$$

and $$\sum_{S \in \mathcal{S}} d_S (y_S, \sum_S \omega_i) \in \bar{Y}$$

for all $(y_S, \sum_S \omega_i) \in \bar{Y}$. Furthermore

$$\sum_{\mathcal{S}} d_S \sum_{i \in S} \omega_i = \sum_{i \in I} \sum_{S \ni i} d_S \omega_i = \sum_{i \in I} \omega_i \sum_{S \ni i} d_S = \sum_{i \in I} \omega_i .$$

Hence $\sum_{\mathcal{S}} d_S y_S \in Y^I$. Q.E.D.

Lemma. Let $((Y^S), Y^I)$ be totally balanced. Then there exists an aggregate technology $\bar{Y}$ and endowments of n non-marketed goods given by $e_i \in R^n_+$, $i = 1, \ldots, n$ such that

$$Y^S = \{y \in R^\ell | (y, \sum_{i \in S} e_i) \in \bar{Y}\}.$$

Proof. For each $S \in 2^I$ define the starshaped set in $R^{\ell+m}$

$$\tilde{Y}^S = \{\lambda(y, e_S) | y \in Y^S, 0 \leqq \lambda \leqq 1\}$$

and let

$$\bar{Y} = \{\Sigma d_S \tilde{Y}^S | d \in D\}.$$

It has to be shown that

$$\bar{Y}^S = \{y \in R^\ell | (y, e_S) \in \bar{Y}\}$$

is equal to Y^S. Suppose $y \in Y^S$. Then, for any balanced family with $d_S = 1$ and choosing $0 \in \tilde{Y}^T$, $T \neq S$, $T \in \mathcal{S}$, it is clear that $(y, e_S) \in \bar{Y}$. Conversely, let $(y, e_S) \in \bar{Y}$. Then

$$(y, e_S) = \sum_{\mathcal{J}} d_T \lambda_T (y_T, e_T), \quad y_T \in Y^T$$

where J is a balanced family. Since

$$\sum_{J} d_T \lambda_T e_T = e_S ,$$

the non-zero coefficients $(d_T \lambda_T)$, $T \in J$, determine a balanced family for S. Hence $\sum_{T \in J} d_T \lambda_T y_T \in Y^S$ by assumption. Q.E.D.

It should be noted that convexity of the sets Y^S is not required in the last lemma.

In his most recent contribution [2], Billera conjectures that each convex-valued totally balanced game is a simple market game. One feels that the construction used in the proof of the lemma may be helpful in giving a final answer to this problem.

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