On Some Classes of Birth And Death Processes in Continuum

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To my Family

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To the memory of
Wacław Derdziuk
1932 – 2010

Contents

Introduction

For over 50 years the interacting particle systems (IPS) have been used to describe various phenomena. The use of IPS was initially motivated by the statistical physics, but soon it became clear that the list of possible applications is long, and includes such fields as, for example, medicine (infection spreading, tumour growth), economy (agent based models), sociology (behavioural systems) and ecology (population models). The latter being in intensive development during last years. Historically, the theory of IPS arose as a part of the probability theory and was initiated by works of F. Spitzer and R. L. Dobrushin in the late 60's with the purpose to study the systems with Gibbs states as equilibrium measures.

A typical IPS consists of a number (finite or infinite) of indistinguishable *particles* located in some *position space*. Sometimes it is more appropriate to use the term *individuals* instead of *particles* to describe the elements of an IPS. Depending on the context, the position space of the particles can be discrete or continuous. In the first case, one considers the so-called *lattice models*, and the standard example of the space is $\mathbb{Z}^d$, although one can also use more general structures such as, e.g., infinite graphs. The lattice systems turned out to be useful and provided the right description for many models and applications (see for example [Lig85]). There are, however, situations when the *continuous* position space is more appropriate or even necessary in order to convey the nature of the considered problem. Thus in this case the position space is assumed to be $\mathbb{R}^d$ or more generally, a Riemannian manifold X (cf. [Kun99]). Many of the lattice models (or their analogues) have been studied in the continuous space case: Glauber and Kawasaki dynamics (cf. [KLR07, FKKZ10, KL05, Ohl07]) and the contact model [KS06]. Also, several new models have been introduced, like for example systems with competition [FKK09] or contact model with jumps [Str09].

We use the *configuration space analysis* as framework for the study of interacting particle systems in continuum throughout this thesis. Given a Riemannian manifold X, the *configuration space* Γ_X over X is defined as the space of all locally finite subsets of X (we call the elements of Γ *config-*

urations). Thus, a configuration $\gamma \in \Gamma$ can be interpreted as an (infinite) *population* of individuals, or a cloud of particles which are indistinguishable and there is at most one element of γ occupying a single site $x \in X$ (therefore, Γ is called the *simple configuration space* as opposed to the *multiple configuration space* $\ddot{\Gamma}$ where this restriction is absent). One can identify a configuration $\gamma \in \Gamma$ with a positive Radon measure via $\gamma = \sum_{x\in\gamma} \delta_x$, where δ_x is the Dirac measure with the mass equal to the unity concentrated on $x \in X$. That allows us to equip the configuration space Γ with the vague topology of the space of all Radon measures on X. The *point processes*, i.e. the measures on Γ, are called the *states* of a given system. And whereas Poisson measures describe the state of a system without interaction, Gibbs measures are used to study models in which the particles interact via, for example, a pair potential. For more detailed description of the configuration spaces including, for example, the geometry of Γ, we refer to [Kun99], [AKR98], [Kut03] and others.

Having in mind two-component systems, we introduce the *two-component configuration space* Γ^2, which is defined as $\Gamma^2 := \{(\gamma^1, \gamma^2) \in \Gamma^+ \times \Gamma^- : \gamma^1 \cap \gamma^2 = \emptyset\} \subset \Gamma \times \Gamma$, where $\Gamma^+ = \Gamma^- = \Gamma$. Nearly all notions and methods used in the single component case can be naturally translated to the two-component framework.

This thesis is devoted to the study of several new IPS models, mainly inspired by the ecological applications. In the first two chapters, modifications of the contact process and the Glauber-type dynamics are considered. These models are examples of spatial birth-and-death processes (see references in the previous paragraph). Their dynamics is described by a heuristic pre-generator, the action of which is defined by:

$$LF(\gamma) := \sum_{x\in\gamma} d(x,\gamma)\,[F(\gamma\setminus x) - F(\gamma)] + \int_X b(x,\gamma)\,[F(\gamma\cup x) - F(\gamma)]\,dx$$

for $\gamma \in \Gamma$ and an appropriate function F. The first part of the operator L describes the "death" of elements of γ according to the *death rate* $d(x, \gamma)$, whereas the second part (*birth rate*) provides the mechanism of offspring production with the function $b(x, \gamma)$ describing the rate at which new elements appear, and their spatial distribution. Thus, the dynamics of a given system is determined by its death and birth rates. However, if we consider for example the evolution of a population of plants, it becomes clear, that the cycle of life of particular individuals depends not only on their age and/or the existing population, but also on a number of external factors such as availability of sun light, resources and diseases. In order to convey the additional influence, we allow the functions b and d to be random, i.e. we consider the above-mentioned models in the *random environment* or, in other words, in the heterogeneous landscape.

The second natural generalization is the introduction of another *type* of population into the model, that is the study of *two-component systems* (see [FKO10]). Having in mind biological applications, we can speak about the *symbiotic* relation between two types of individuals. One of the possible interaction is the *predation*, the illustration of which is the well known Lotka-Volterra model (see [Lot25, Vol26]). In this thesis, however, we study another example of symbiosis, namely the *mutualistic* model (in Chapter 4). The system introduced in Chapter 5 can be considered as the two-component analogue of the Glauber-type dynamics in continuum. The Markov pre-generator of such two (or more) component dynamics should reflect the evolution of each populations in dependence on the other, thus the general form of the informal pre-generator is given by

$$L := L_1 + L_2$$

where each of the operators L_1, L_2 describes the dynamics of one population, taking into account the interaction between them.

This work deals with the following problems. First of all, we study the existence of the evolution of states for a given model. For some particular cases, this can be done using Markov processes corresponding to considered generators. In our case, however, it is convenient to approach the problem in terms of the evolution of corresponding correlation functions. We apply this method to a number of models. In two last chapters we investigate, additionally, the scaling limits for stochastic dynamics, namely the Vlasov-type scaling of the microscopic state evolution to the mesoscopic dynamics, and their convergence for given models.

Overview of the contents

We proceed now to the detailed description of the contents of this thesis.

Configuration spaces

In Chapter 1 we recall some definitions and facts from the configuration spaces theory and the harmonic analysis on configuration spaces. Throughout this thesis we will assume, that the underlying space is the Euclidean space $\mathbb{R}^d$, although it is possible to extend the results to more general cases like, for example, a Riemannian manifold or even more general topological spaces.

The introduction recalls the standard notation which will be used further in this work. After that, we proceed to the definitions of the space of finite and simple configurations over $\mathbb{R}^d$, Γ_0 and Γ, resp.. Next, the topological structures of both spaces are discussed. Whereas the topology of the space Γ_0 is inherited from the topology of the underlying space, the topology on Γ is introduced using the interpretation of configurations as the integer-valued Radon measures over $\mathbb{R}^d$. Thus, we endow Γ with the vague topology of the space of all Radon measures on $\mathbb{R}^d$. Having introduced the topological structure of above-mentioned spaces, we define their corresponding Borel σ-algebras and we proceed to the construction of measures on Γ_0 and Γ. We focus especially on two measures: Lebesgue-Poisson measure $\lambda_{z\sigma}$ on Γ_0 and Poisson measure $\pi_{z\sigma}$ on Γ. We should mention, that the Poisson measure corresponds to the interaction free systems. We also recall useful characterization of the Poisson measure by its Laplace transform and the Minlos lemma (Lemma 1.1), which is one of the main technical tools used in this thesis.

In Section 1.3 we discuss the general framework of the harmonic analysis on the configuration spaces, using mainly [Kun99] and [KK02] as references. First we introduce some classes of functions on Γ_0 and Γ. Then we recollect the definitions of the K-transform and the $\star$-convolution. The K-transform maps the *quasi-observables* (the functions on Γ_0) into functions on Γ (the *observables*). It also has the property that $K(G_1 \star G_2) = KG_1 \cdot KG_2$, hence it can be considered as the Fourier transform on the space of configurations. The *correlation measure* on Γ_0 is defined as the image of a probability measure on Γ (a *state* of the system) under the *dual* K-transform, K^*, with respect to the duality

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)\,(d\eta).$$

Moreover, if the measure μ is locally absolutely continuous w.r.t. Poisson measure, then the correlation measure $K^*\mu$ is absolutely continuous w.r.t. the Lebesgue-Poisson measure on Γ_0 and the corresponding Radon-Nikodym derivative is the *correlation function* of measure μ as known from the statistical physics.

Section 1.4 contains the definition of the *two-component configuration space* Γ^2, which is defined as

$$\Gamma^2 := \left\{(\gamma^1,\gamma^2) \in \Gamma^+ \times \Gamma^- : \quad \gamma^1 \cap \gamma^2 = \emptyset\right\}.$$

The basic definitions and properties are next derived as the straightforward extensions of the proper notions from the single component space. Hence, the two-component K-transform and $\star$-convolution are defined and play similar role as their one-component analogues in our considerations. Also the Minlos lemma for the two-component case is introduced.

Next, we recall the general scheme of investigation for Markov evolutions in configuration spaces (see e.g. [FKO09]). The starting point is the *Kolmogorov equation* for the observables on Γ, and the associated (dual) *Fokker-Planck equation* for the evolution of states of the system. The technical difficulty with these two equations is due to the fact, that both of them are infinite-dimensional and until now, the tools for solving such problems are not sufficiently developed. Therefore, using the K-transform we can "shift" the problems to the finite-dimensional context (of quasi-observables) and try to approach it with classical methods. This gives the equation associated to the *symbol of the generator* L defined by $\hat{L} := K^{-1}LK$, corresponding to the Kolmogorov equation. Also the evolution of correlation functions can be derived and solved explicitly for many models.

We conclude Chapter 1 with the presentation of the Vlasov-type scaling scheme as developed in [FKK10a]. After some historical remarks, we proceed to the introduction of the general algorithm for scaling of a given system. The starting point for the scaling is the Cauchy problem for the evolution of correlation functions (which is associated to the dual of the symbol of generator for the model). The general scheme consists of three steps:

1. scaling of the initial condition,

2. scaling of the generator,

3. renormalizing the scaled generator in a proper way.

After applying three above mentioned steps, we obtain the Cauchy problem for the scaled correlation functions. Using classical theory we can prove that the solutions of the rescaled equation converges to a correlation function of some virtual interacting particle systems, and the Vlasov-type equation is obtained as the first correlation function for this system.

Contact process in random environment

Chapter 2 is devoted to studies of modifications of the contact model in continuum.

The contact model on the lattice was first introduced and studies by T. E. Harris ([Har74]) and its name is due to the interpretation as a model for infection spreading. Namely, given a configuration $\gamma = \{0,1\}^{\mathbb{Z}^d}$, $\gamma(x) = 1$ means that the individual (Harris used the term *creature*) at site $x \in \mathbb{Z}^d$

is infected; the case $\gamma(x) = 0$ means that the individual is healthy. During the time evolution, healthy creatures can get infected with the rate which is proportional to the number of infected neighbours. Note that a creature can be infected only, if there is at least one infected individual on the neighbouring sites. On the other hand, infected individuals will recover after an exponentially distributed time.

In 2006, the continuous version of the contact process was constructed in [KS06] and later some properties of this model were derived in [KKP08]. The heuristic pre-generator of the contact model corresponding to the evolution described in the previous paragraph is given by

$$LF(\gamma) = \sum_{x\in\gamma} [F(\gamma \setminus x) - F(\gamma)] + \lambda \sum_{y\in\gamma} \int_{\mathbb{R}^d} a(x-y)\, [F(\gamma \setminus x) - F(\gamma)]\, dx$$

and the dynamics has been constructed using the branching processes theory for non-negative functions $a \in L^{1+\delta}(\mathbb{R}^d)$, and for a wide class of initial configurations.

After the introduction, we recall the theory of the *extended generator*, as used by [Dav93] and [MT93]. The application of the extended generator is motivated by Lyapunov criteria for the regularity of Markov processes. The standard (strong) generator of a given process usually does not include the unbounded functions in its domain. A Lyapunov-type function, however, is unbounded, and provides a simple and elegant way to prove, that the lifetime of the process is infinite. We also introduce the Lyapunov-type function $\mathbb{V}_\beta$ for the contact process, see (2.10).

In Section 2.2 we recall with details the construction of the contact model in continuum, and apply the scheme presented in the previous section to prove the regularity of the process. First, we construct the process on the space of finite configurations as a pure jump process with generator L (cf. [GS74]). Then, using the branching property of the model, we can extend the construction to the configuration space Γ. However, if we want to prove that the process is regular, we should restrict the class of initial configurations to the space Γ_∞ induced by the Lyapunov-type function $\mathbb{V}_\beta$. This is not a significant restriction, for the space Γ_∞ is big enough to contain supports for a large class of probability measures on Γ (cf. Remark 2.1 in [KS06]). Theorem 2.1 proves, that the contact process constructed previously is non-explosive.

Section 2.3 deals with the theory of Poisson random potentials. We briefly recall some basic estimates of the potential of the form

$$V(x,\omega) := \int_{\mathbb{R}^d} \varphi(x-y)\omega(dy)$$

where ω is a realization of Poisson point process. Following [GKM00], we derive some estimates on the V.

The rest of the chapter is devoted to studies of contact process in random environment.

In Section 2.4 we introduce the *contact model with random establishment*. Heuristically one can think of the heterogeneous landscape with areas, in which the survival rate for the offspring is small compared to other places. Therefore, the pre-generator of this model is given by

$$L_{\omega,b}F(\gamma) = \sum_{x\in\gamma} D_x^- F(\gamma) + \sum_{y\in\gamma} \varkappa \int_{\mathbb{R}^d} a^+(x-y)b(x,\omega)D_x^+ F(\gamma)dx,$$

where

$$b(x,\omega) = e^{-\langle b^+(x-\cdot),\omega\rangle} = \exp\left(-\sum_{y\in\omega} b^+(x-y)\right).$$

After explaining the motivation for such a model, the existence and regularity of this process are proven in a way similar to the non-random case. Next, using the harmonic analysis on the configuration spaces from Chapter 1 we derive the symbol of the generator $L_{\omega,b}$ and its adjoint operator $L^*_{\omega,b}$. The evolution of the correlation functions for the model is governed by the adjoint operator. Using the structure of correlation functions we can apply this operator to each component of the function $k_t := \left(k_t^{(n)}\right)_{n\in\mathbb{N}}$, obtaining for every $n \in \mathbb{N}$ the Cauchy problem of the form:

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) = \hat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) + f_t^{(n)}(x_1,\ldots,x_n)$$
$$k_t^{(n)}(x_1,\ldots,x_n)|_{t=0} := k_0^{(n)}(x_1,\ldots,x_n),$$

in some Banach space X_n. In Proposition 2.3, we give the explicit solution to above mentioned Cauchy problem for each $n \in \mathbb{N}$. Furthermore, assuming that the initial condition satisfies the estimate $k_0^{(n)} \le n!C^n$ (where $k_0^{(n)}$ is the n-th component of the correlation function $k_0 : \Gamma_0 \to \mathbb{R}^d_+$), we can prove an estimate for the solution derived in the Proposition 2.3, see Proposition 2.4. We conclude this section with the Lemma 2.3 which states, that the evolution given by $L^*_{\omega,b}$ preserves the correlation functions, i.e. if the initial condition k_0 is a correlation function for some measure μ_0, then the solution k_t is also a correlation function for some μ_t. The proof is based on the verification of conditions derived by A. Lenard in [Len73]. The equation for the first and the second correlation functions are derived explicitly.

Another example of the contact process in random environment is studied in Section 2.5. This is the *contact process with random fecundity*. Intuitively, we deal with the situation where the rate of offspring production is randomly affected. Thus, the mechanism of evolution for this model is defined as follows:

$$L_{\omega,\varkappa}F(\gamma) = \sum_{x\in\gamma} D_x^- F(\gamma) + \sum_{y\in\gamma} \varkappa(y,\omega) \int_{\mathbb{R}^d} a^+(x-y) D_x^+ F(\gamma) dx,$$

where

$$\varkappa(y,\omega) := \exp\left(- \sum_{x\in\omega} \phi(x-y) \right)$$

for a positive function ϕ. The structure of this section is similar to the structure of the previous one. First, we construct the associated process as a spatial branching process with killing, and using the Lyapunov-type function $\mathbb{V}_\beta$ we prove, that the process is regular. Later, the symbol of the generator $L_{\omega,\varkappa}$ and its adjoint $L^*_{\omega,\varkappa}$ are calculated, and we derive the evolution of correlation functions in terms of a Cauchy problem associated to the adjoint operator $\hat{L}^*_{\omega,\varkappa}$. Using the theory of evolution equations, the solution of this equation is presented in Proposition 2.5. The *a priori* estimates for the solution are proven in Proposition 2.6, assuming that the initial condition $k_0^{(n)}$ satisfies the bound $k_0^{(n)} \le n!C^n$ for some $C > 0$ and each $n \in \mathbb{N}$. The rest of this section is devoted to the proof of the fact, that the evolution given by $L^*_{\omega,\varkappa}$ preserves the correlation functions (cf. Section 2.4).

We conclude this chapter with the description of the *contact process with random mortality*. Here, the death rate of a particle is dependent on the random influence. Thus, the Markov pre-generator of the model is given by:

$$L_{\omega,m}F(\gamma) = \sum_{x\in\gamma} m(x,\omega) D_x^- F(\gamma) + \varkappa \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y) D_x^+ F(\gamma) dx,$$

where $m(x,\omega) = \sum_{x'\in\omega} \varphi(x-x')$. Note that the methods used in two previous sections cannot be applied in this case, and the question of the existence of this process in Γ remains open. However, using the harmonic analysis on Γ we are able to calculate the symbol of the generator $L_{\omega,m}$, and its dual operator $L^*_{\omega,m}$. Again, the system of evolution equations for correlation function is derived. Moreover, using the perturbation theory for linear operators, we can solve the Cauchy problem for each of the components obtaining the explicit form of the solution:

$$k_t^{(n)}(x_1,\dots,x_n) = e^{t\hat{L}^*_n} k_0^{(n)}(x_1,\dots,x_n) + \int_0^t e^{(t-s)L^*_n} f_s^{(n)}(x_1,\dots,x_n) ds.$$

There are number of open questions arising from the analysis of three models presented above. For example, the first correlation function for the contact process with random mortality satisfies the equation

$$\frac{\partial k_t^{(1)}}{\partial t}(x) = L^1 k_t^{(1)}(x) - V(x,\omega) k_t^{(1)}(x)$$

which is nothing else but the evolution of a jumping particle among Poisonnian obstacles. In the case of Brownian motion instead of jumps, this equation is called the *parabolic Anderson problem* and has been widely studied for example by A. S. Sznitman, S. Molchanov, J. Gärtner, W. König et al. (see e.g. [Szn06, Szn98, GKM00, ABMY00]).

Glauber-type dynamics in random environment

In the present chapter we apply the perturbation theory to construct a semigroup associated to the symbol of the Glauber dynamics in the random environment.

In the introduction, we recall some known facts about the Glauber dynamics on the lattice and in the continuous space case. If we consider the classical Ising model with the spin space $S = \{-1, 1\}$, then the Glauber dynamics of the systems means, that the particles placed on the sites $x \in \mathbb{Z}^d$ randomly change their spin value (it's called the spin-flip dynamics). We refer to [Lig85] for the detailed discussion of the Glauber dynamics on the lattice. Also the continuous space analogue of the Glauber dynamics was constructed in both equilibrium and non-equilibrium case (see e.g. [KL05, KKZ06]). The Glauber-type dynamics in continuum is a process where the particles randomly appear and disappear in the space.

In Section 3.1 we recollect some basic facts from the theory of Gibbs measures associated to the pair potential ϕ. We remind the definitions of the *Hamiltonian* E^ϕ, and the *relative energy of interaction* $E^\phi(x,\gamma)$ between a particle located at site $x \in \mathbb{R}^d$ and the configuration $\gamma \in \Gamma$. After some preparations, we recall the *Dobrushin-Lanford-Ruelle* (DLR) equation to define the Gibbs measure μ associated to the pair potential ϕ, inverse temperature β and the parameter $z > 0$ (see Definition 3.1).

Some classical facts from the perturbation theory are stated in Section 3.2. We focus here on the perturbation theory for holomorphic semigroups, generated by operators belonging to the set $\mathcal{H}(\omega, 0)$ (for $\omega > 0$) of all closed and densely defined operators T, the resolvent of which contains the sector

$$Sect(\frac{\pi}{2} + \omega) = \{\zeta \in \mathbb{C} : |arg\zeta| < \frac{\pi}{2} + \omega\} \setminus \{0\},$$

and such that for any $\varepsilon > 0$

$$\|(T - \zeta \mathbb{1})^{-1}\| \leq \frac{M_\varepsilon}{|\zeta|},$$

and M_ε does not depend on ζ. It is known, that every operator $T \in \mathcal{H}(\omega, \theta)$ is the generator of a holomorphic semigroup (cf. Remark 3.1). We apply the perturbation theory presented in this section to construct the semigroups corresponding to the symbols of two pre-generators introduced in Section 3.3. The action of the first one is given by:

$$L_\omega^{ext} F(\gamma) := \sum_{x \in \gamma} D_x^- F(\gamma) + \varkappa \int_{\mathbb{R}^d} e^{-\beta E^\phi(x,\gamma)} D_x^+ F(\gamma) e^{-E^h(x,\omega)} dx,$$

with the external field interaction, and

$$L_\omega F(\gamma) := \sum_{x \in \gamma} e^{-E^h(x,\omega)} D_x^- F(\gamma) + \varkappa \int_{\mathbb{R}^d} e^{-\beta E^\phi(x,\gamma)} e^{-E^h(x,\omega)} D_x^+ F(\gamma) dx$$

where we have random perturbation of the rates. Both of these operators satisfy the *detailed balance condition* (cf. 3.14 and [Glo81]), hence they have Gibbs states as symmetrizing measures. In the case of L_ω^{ext}, it is Gibbs measure $\mu \in \mathcal{G}(\phi, \varkappa, \beta)$ which is associated to the Lebesgue-Poisson measure with the random intensity measure: $\sigma_\omega(dx) := e^{-E^h(x,\omega)} dx$, heuristically given by

$$\lambda_{\varkappa,\omega} = \sum_{n=0}^{\infty} \frac{\varkappa^n}{n!} \sigma_\omega^{(n)}.$$

Whereas the symmetrizing measure for the operator L_ω is just the Gibbs state with the Lebesgue-Poisson measure as the reference measure (cf. Section 3.1). Next, the symbols of the two generators above are calculated using the K-transform.

In Section 3.4 we construct semigroups associated to the symbols $\hat{L}_\omega^{ext}$ and $\hat{L}_\omega$ in the space

$$\mathcal{L}_{C,\beta} := L^1\left(\Gamma_0, C^{|\eta|} e^{-\beta E(\eta)} \lambda(d\eta)\right).$$

This is carried out using the perturbation methods introduced in the Section 3.2, and the constructed semigroups turn out to be holomorphic in the sector $|\arg t| < \omega$ for some $\omega > 0$.

Two-component ecological model

Chapter 4 deals with the two-component ecological model which is an example of the process with mutualistic interaction between two populations. After short introduction we proceed to the construction of the semigroup associated to the symbol of the generator as in Chapter 3.

The mutual interaction between two populations of individuals means that both of them contribute to the creation of new members of each populations but also have the influence on the death rate of existing individuals. As mentioned before, the generator of such a process has the form

$$L = L_1 + L_2$$

and operators L_1 and L_2 are given as follows:

$$\begin{aligned}(L^1F)(\gamma^1,\gamma^2) :=& \sum_{x\in\gamma^1} d^1(x,\gamma^1\setminus x,\gamma^2)\left[F(\gamma^1\setminus x,\gamma^2)-F(\gamma^1,\gamma^2)\right]\\ &+\int_{\mathbb{R}^d} b^1(x,\gamma^1,\gamma^2)\left[F(\gamma^1\cup x,\gamma^2)-F(\gamma^1,\gamma^2)\right]dx,\end{aligned}$$

describes the evolution of the first population (type 1), and

$$\begin{aligned}(L^2F)(\gamma^1,\gamma^2) :=& \sum_{y\in\gamma^2} d^2(y,\gamma^1,\gamma^2\setminus y)\left[F(\gamma^1,\gamma^2\setminus y)-F(\gamma^1,\gamma^2)\right]\\ &+\int_{\mathbb{R}^d} b^2(y,\gamma^1,\gamma^2)\left[F(\gamma^1,\gamma^2\cup y)-F(\gamma^1,\gamma^2)\right]dy.\end{aligned}$$

characterizes the second population (type 2). The birth and death coefficients reflect the mutualistic nature of the model thus they are given by:

$$\begin{aligned}d^1(x,\gamma^1,\gamma^2) =& \quad m^+ + A_1^-\sum_{x'\in\gamma^1} a_1^-(x-x') + B_1^-\sum_{y\in\gamma^2} b_1^-(x-y),\\ b^1(x,\gamma^1,\gamma^2) =& \quad A_1^+\sum_{x'\in\gamma^1} a_1^+(x-x') + B_1^+\sum_{y\in\gamma^2} b_1^+(x-y),\\ d^2(y,\gamma^1,\gamma^2) =& \quad m^- + A_2^-\sum_{y'\in\gamma^2} a_2^-(y-y') + B_2^-\sum_{x\in\gamma^1} b_2^-(y-x),\\ b^2(y,\gamma^1,\gamma^2) =& \quad A_2^-\sum_{y'\in\gamma^2} a_2^+(y-y') + B_2^+\sum_{x\in\gamma^1} b_2^+(y-x).\end{aligned}$$

Next, we calculate the symbol of the generator L, and in the series of propositions we show that one part of the symbol plays role of the leading operator, and the rest is relatively bounded with respect to it. That allows us to apply the classical result (cf. Theorem 3.1) and to establish the existence of the semigroup associated to the symbol $\hat{L}$ in the space

$$\mathcal{L}_C := L^1\left(\Gamma_0^+ \times \Gamma_0^-, C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)\right),$$

see Theorem 4.1.

In Section 4.3 we introduce the space of the *so-called correlation functions*:

$$\mathcal{Q}_C := \left\{k : \Gamma_0 \times \Gamma_0 \to \mathbb{R} |\ k \cdot C^{-(|\eta^1|+|\eta^2|)} \in L^\infty(\Gamma_0 \times \Gamma_0, \lambda^2)\right\}.$$

and derive the dual of the operator $\hat{L}$ in the space $\mathcal{Q}_C$. Then the evolution of the correlation functions is given in terms of the dual semigroup in the weak sense, i.e. in the sense of the duality (4.19).

Finally, in the last section of this chapter we apply the Vlasov-type scaling scheme introduced in Section 1.6 to the operator L. For $\varepsilon > 0$ we consider the scaled operator L_ε. The scaling is as follows: the birth coefficients remain unchanged and the death coefficients of the operator L are scaled in the following manner:

$$d_\varepsilon^1(x, \gamma^1, \gamma^2) = m^+ + \varepsilon A_1^- \sum_{x' \in \gamma^1} a_1^-(x - x') + \varepsilon B_1^- \sum_{y \in \gamma^2} b_1^-(x - y),$$
$$d_\varepsilon^2(y, \gamma^1, \gamma^2) = m^- + \varepsilon A_2^- \sum_{y' \in \gamma^2} a_2^-(y - y') + \varepsilon B_2^- \sum_{x \in \gamma^1} b_2^-(y - x).$$

Then, the form of the symbol $\hat{L}_\varepsilon$ is obtained using the harmonic analysis, and Theorem 4.2 shows that the scaled and renormalized symbol $\hat{L}_{\varepsilon,ren}$ is the generator of a holomorphic semigroup $\hat{U}_\varepsilon(t)$ in $\mathcal{L}_C$. In Theorem 4.4 we prove the strong convergence of $\hat{U}_\varepsilon(t)$ to the semigroup $\hat{U}^V(t)$ generated by the pointwise limit of the operators $\hat{L}_{\varepsilon,ren}$ (denoted by $\hat{L}^V$). We conclude this chapter with the derivation of the *Vlasov-type equations* for the model, that is the system of two equations for the densities of both populations:

$$\begin{cases} \frac{\partial}{\partial t}\rho_t^1(x) &= -m^+\rho_t^1(x) \\ & \quad - A_1^-\rho_t^1(x)\left(a_1^- * \rho_t^1\right)(x) - B_1^-\rho_t^1(x)\left(b_1^- * \rho_t^2\right)(x) \\ & \quad + A_1^+\left(a_1^+ * \rho_t^1\right)(x) + B_1^+\left(b_1^+ * \rho_t^2\right)(x) \\ \rho_t^1(x)|_{t=0} &= \rho_0^1(x), \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial t}\rho_t^2(y) &= -m^-\rho_t^2(y) \\ & \quad - B_2^-\rho_t^2(y)\left(b_2^- * \rho_t^1\right)(y) - A_2^-\rho_t^2(y)\left(a_2^- * \rho_t^2\right)(y) \\ & \quad + A_2^+\left(a_2^+ * \rho_t^2\right)(y) + B_2^+\left(b_2^+ * \rho_t^1\right)(y) \\ \rho_t^2(y)|_{t=0} &= \rho_0^2(y), \end{cases}$$

Note, that both densities depend on each other and that they cannot be separated.

Potts-type model

The last chapter is devoted to the study of Potts-type model. Heuristically, the system consists of two interacting clouds of particles. Note, that there is no interaction between the particles of the same type, and the dynamics of each cloud is of *Glauber type*, hence the form of the pre-generator:

$$\begin{aligned} LF(\gamma^1,\gamma^2) :=& \sum_{x\in\gamma^1} D_x^{1-}F(\gamma^1,\gamma^2) + \varkappa\int_{\mathbb{R}^d} e^{-\beta E^\phi(x,\gamma^2)} D_x^{1+}F(\gamma^1,\gamma^2)dx \\ &+ \sum_{y\in\gamma^2} D_y^{2-}F(\gamma^1,\gamma^2) + \varkappa\int_{\mathbb{R}^d} e^{-\beta E^\phi(y,\gamma^1)} D_y^{2+}F(\gamma^1,\gamma^2)dy, \end{aligned}$$

where D_x^{1-}, D_x^{1+}, D_x^{2-}, D_x^{2+} denote the corresponding gradients. In Section 5.2 we focus on the symbol of the generator L, and construct the associated semigroup in the space $\mathcal{L}_C$ introduced in the previous chapter. The form of the symbol is derived in Proposition 5.1 and the next proposition shows, that $\hat{L}$ with its domain is a linear operator in $\mathcal{L}_C$. Next we use the approach developed in [FKKZ10] to construct the semigroup associated with the symbol. In order to do that, for $\delta > 0$ we introduce the approximation operator:

$$\begin{aligned} \hat{P}_\delta G(\eta^1,\eta^2) = & \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa_1\delta)^{|\sigma^1|}(\varkappa_2\delta)^{|\sigma^2|} \\ \times & \prod_{x\in\zeta^1} e^{-\beta E^\phi(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left(e^{-\beta E^\phi(x',\sigma^2)}-1\right) \\ \times & \prod_{y\in\zeta^2} e^{-\beta E^\phi(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left(e^{-\beta E^\phi(y',\sigma^1)}-1\right) \\ \times & \; G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2), \end{aligned}$$

and show, that it is a contraction in $\mathcal{L}_C$ (see Lemma 5.2). Then, after establishing additional properties of $\hat{P}_\delta$ we can use Corollary 3.8 from [EK05] (cf. Lemma 5.1) to show, that the closure of $(\hat{L}, \mathcal{L}_{2C})$ generates a strongly continuous contraction semigroup on $\mathcal{L}_C$ (Theorem 5.1).

Finally, Section 5.2 is devoted to the Vlasov-type scaling of the considered model. The proper scaling for this model yields the following form of the scaled generator:

$$\begin{aligned} L_\varepsilon F(\gamma^1, \gamma^2) := \quad & \sum_{x\in\gamma^1} D_x^{1-} F(\gamma^1, \gamma^2) + \frac{\varkappa}{\varepsilon} \int_{\mathbb{R}^d} e^{-\varepsilon\beta E^\phi(x,\gamma^2)} D_x^{1+} F(\gamma^1, \gamma^2) dx \\ & \sum_{y\in\gamma^2} D_y^{2-} F(\gamma^1, \gamma^2) + \frac{\varkappa}{\varepsilon} \int_{\mathbb{R}^d} e^{-\varepsilon\beta E^\phi(y,\gamma^1)} D_y^{2+} F(\gamma^1, \gamma^2) dy. \end{aligned}$$

After calculating the symbol of L_ε, we consider the scaled and renormalized generator $\hat{L}_{\varepsilon,ren}$ and its weak limit as $\varepsilon \downarrow 0$ denoted by $\hat{L}^V$. Using the approximations introduced in Section 5.2 we are able to show, that the closures of both operators generate contraction semigroups which we denote, respectively, with $\hat{U}_{\varepsilon,ren}(t)$ and $\hat{U}^V$ (see Theorem 5.2). As contrasted with the previous chapter, in the case of the Potts-type model we focus on the convergence of the dual semigroups (thus on the convergence of the solutions to the corresponding Cauchy problems associated with the dual operators $\hat{L}^*_{\varepsilon,ren}$ and $\hat{L}^*_V$). Unfortunately, even if $\hat{U}_{\varepsilon,ren}(t)$ and $\hat{U}^V$ are strongly continuous, the dual semigroups are not strongly continuous in the dual space $\mathcal{Q}_C$. To circumvent this problem, we consider these dual semigroups on a proper subspace of $\mathcal{Q}_C$ (see Theorem 5.3), in which they are strongly continuous and their generators can be described in terms of the adjoint generators $\hat{L}^*_{\varepsilon,ren}$ and $\hat{L}^*_V$ (cf. equations (5.38) to (5.41)). Theorem 5.4 states, that the dual of the scaled semigroup (defined above) converges in $\mathcal{Q}_C$ to the corresponding Vlasov semigroup. We conclude this chapter with the derivation of the Vlasov-type equations for densities corresponding to both types of particles. As result, we get the following Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t}\rho_t^1(x) &= -\rho_t^1(x) + \varkappa e^{-\beta\left(\rho_t^2 * \phi\right)(x)} \\ \rho_t^1(x)|_{t=0} &= \rho_0^1(x), \\ \frac{\partial}{\partial t}\rho_t^2(y) &= -\rho_t^2(y) + \varkappa e^{-\beta\left(\rho_t^1 * \phi\right)(y)} \\ \rho_t^2(y)|_{t=0} &= \rho_0^2(y). \end{cases}$$

Note, that the corresponding densities evolve in dependence on each other and cannot be separated. As in the previous chapter, we assume that the initial state of the system is a Poisson measure (not necessarily homogeneous). Then we have the chaos preservation property, i.e., the Poissonian structure is preserved during the evolution and corresponding densities solve the system of two equations above.

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Chapter 1

Configuration spaces

We devote this chapter to recall some known facts from the theory of configuration spaces and to the introduction of the two-component configuration spaces.

One can define the configuration space over a general connected, oriented Riemannian C^∞-manifold X which we call the *position space* of the particles (or individuals). The following notation will be used throughout this thesis:

$\mathcal{O}(X)$: the family of all open subsets of X,

$\mathcal{B}(X)$: the Borel σ-algebra on X,

$\mathcal{O}_c(X), \mathcal{B}_c(X)$: the family of open (Borel-measurable, resp.) sets in X with compact closure,

$B(X)$: the family of all measurable bounded functions on X,

$C_0(X)$: the set of all continuous functions with compact support.

In the present work we consider only the case where $X = \mathbb{R}^d$, which is the natural choice if one considers the ecological applications of the investigated models. For more general theory of configuration spaces we refer to [Kun99] and [AKR98].

1.1 One-component configuration spaces

Let $n \in \mathbb{N} \cup \{0\}$ and $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, define the *space of n-point configurations over* Λ as follows:

$$\Gamma_{0,\Lambda}^{(n)} := \{\eta \subset \Lambda : \quad |\eta| = n\}, \quad \Gamma_{0,\Lambda}^{(0)} := \{\emptyset\}, \tag{1.1}$$

where $|A|$ denotes the cardinality of the set A. We call the elements $\eta \in \Gamma_{0,\Lambda}^{(n)}$ *configurations*. Now let $Y \in \mathcal{B}_c(\Lambda)$ and denote by $\eta_Y := \eta \cap Y$. Introduce also the mapping $N_Y : \Gamma_{0,\Lambda}^{(n)} \to \mathbb{N} \cup \{0\}$, $N_Y(\eta) = |\eta_Y|$, the number of the points in the configuration η in Y. The topological structure on $\Gamma_{0,\Lambda}^{(n)}$ may be defined using the *symmetrizing mapping* from

$$\widetilde{\Lambda^n} := \{(x_1, \ldots, x_n) \in \Lambda^n : \quad x_k \neq x_j \text{ for } k \neq j\} \tag{1.2}$$

onto $\Gamma_{0,\Lambda}^{(n)}$, defined as

$$\begin{aligned} &\mathrm{sym}_\Lambda^n : \widetilde{\Lambda^n} \to \Gamma_{0,\Lambda}^{(n)}, \\ &\mathrm{sym}_\Lambda^n(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}. \end{aligned} \tag{1.3}$$

Denote with $\mathcal{O}(\Gamma_{0,\mathbb{R}^d}^{(n)})$ the topology on $\Gamma_{0,\mathbb{R}^d}^{(n)}$ generated by the map sym_Λ^n and the corresponding Borel σ-algebra by $\mathcal{B}(\Gamma_{0,\mathbb{R}^d}^{(n)})$. It can be shown (see e.g. [Len75]), that $\mathcal{B}(\Gamma_{0,\mathbb{R}^d}^{(n)})$ coincides with the σ-algebra generated by the mappings N_Λ, i.e.

$$\mathcal{B}(\Gamma_{0,\mathbb{R}^d}^{(n)}) = \sigma\left(N_\Lambda | \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\right). \tag{1.4}$$

Finally, define the *space of finite configurations* :

$$\Gamma_{0,\mathbb{R}^d} := \bigsqcup_{n \in \mathbb{N} \cup \{0\}} \Gamma_{0,\mathbb{R}^d}^{(n)}. \tag{1.5}$$

It is equipped with the topology of disjoint union. In the sequel, we will simply write Γ_0 instead of $\Gamma_{0,\mathbb{R}^d}$.

The *configuration space* $\Gamma(:= \Gamma_{\mathbb{R}^d})$ is defined as the space of all locally finite subsets of $\mathbb{R}^d$, i.e.:

$$\Gamma := \left\{\gamma \subset \mathbb{R}^d : \quad |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\right\}. \tag{1.6}$$

Using the representation

$$\gamma = \sum_{x \in \gamma} \delta_x$$

where δ_x is the Dirac measure with unit mass at x, we can consider the configuration space as the subset of the space of all positive Radon measures on $\mathbb{R}^d$ - note that we do not allow more than one particle at the same site $x \in \mathbb{R}^d$, therefore we call the elements $\gamma \in \Gamma$ *simple configurations.* We equip Γ with the vague topology of the space $\mathcal{M}(\mathbb{R}^d)$ of all Radon measures, i.e. the weakest topology in which mappings

$$\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \int f(x) d\gamma(x) = \sum_{x \in \gamma} f(x), \quad f \in C_0(\mathbb{R}^d) \tag{1.7}$$

are continuous. In the following we will use the notation $\langle f, \gamma \rangle$ for all functions f for which it makes sense.

One can show (see e.g. [Kut03]), that Γ equipped with the vague topology can be metrized so that it becomes Polish space.

1.2 Measures on Γ_0 and Γ

We will now recall the definitions of the Lebesgue-Poisson and the Poisson measure on Γ in the free case (without interaction between particles). Fix a non-atomic and locally finite measure σ on $\mathbb{R}^d$, we will call it the *intensity measure.*

1.2.1 Lebesgue-Poisson and Poisson measure

Let $n \in \mathbb{N}$. Recall the definition

$$\widetilde{(\mathbb{R}^d)^n} := \left\{(x_1, \dots, x_n \in \left(\mathbb{R}^d\right)^n : \quad x_k \neq x_j \text{ for } k \neq j\right\}.$$

Consider the restriction of $\sigma^{\otimes n}$ to the space $\left(\widetilde{(\mathbb{R}^d)^n}, \mathcal{B}(\widetilde{(\mathbb{R}^d)^n})\right)$ (note that $\sigma^{\otimes n}\left((\mathbb{R}^d)^n \setminus \widetilde{(\mathbb{R}^d)^n}\right) = 0$) and denote by $\sigma^{(n)} := \sigma^{\otimes n} \circ (\mathrm{sym}^n)^{-1}$ the corresponding measure on $\Gamma_0^{(n)}$ (with $\sigma^{(0)}(\{\emptyset\}) := 1$).

Define the *Lebesgue-Poisson measure* on Γ_0 as

$$\lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)} \tag{1.8}$$

where $z > 0$ is called the *activity parameter.*

For $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we have $\lambda_{z\sigma}(\Gamma_{0,\Lambda}) = e^{z\sigma(\Lambda)}$, and if we will consider the restriction of $\lambda_{z\sigma}$ to a set $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ (which we also denote by $\lambda_{z\sigma}$), then we can define a probability measure on Γ_Λ by

$$\pi_{z\sigma}^{\Lambda} := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}. \tag{1.9}$$

One can check, that the family $\left(\pi_{z\sigma}^{\Lambda}\right)_{\Lambda \in \mathcal{B}(\mathbb{R}^d)}$ is consistent (cf. [Kun99]). Thus, by (a version of) Kolmogorov theorem there exists a measure $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ such that $\pi_{z\sigma}^{\Lambda} = \pi_{z\sigma} \circ p_\Lambda^{-1}$, where p_Λ is a projection $p_\Lambda : \Gamma \mapsto \Gamma_\Lambda$, $p_\Lambda(\gamma) = \gamma_\Lambda$. The measure $\pi_{z\sigma}$ is called the *Poisson measure* on $(\Gamma, \mathcal{B}(\Gamma))$ with the intensity measure $z\sigma$.

Remark 1.1 ([Oli02]). *One can also define the Poisson measure on Γ by its Laplace transform in the following way:*

$$\int_{\Gamma} e^{\langle f,\gamma\rangle}\pi_{z\sigma}(d\gamma) = e^{z\int_{\mathbb{R}^d}\left(e^{f(x)}-1\right)\sigma(dx)} \tag{1.10}$$

for any infinitely differentiable real-valued function f with compact support.

We say, that a given measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ has the *correlation functions* $k_\mu := \left(k_\mu^{(n)}\right)_{n\in\mathbb{N}}$, if for any $n \in \mathbb{N}$ there exists a non-negative, symmetric and measurable function $k_\mu^{(n)}$ on $(\mathbb{R}^d)^n$ such that

$$\int_{\Gamma} \sum_{\{x_1,\ldots,x_n\}\subset\gamma} f^{(n)}(x_1,\ldots,x_n)\mu(d\gamma) = \tag{1.11}$$
$$\frac{1}{n!}\int_{\left(\mathbb{R}^d\right)^n} f^{(n)}(x_1,\ldots,x_n)k_\mu^{(n)}(x_1,\ldots,x_n)\sigma(dx_1)\cdots\sigma(dx_n),$$

for any measurable, symmetric function $f^{(n)}$.

The next lemma is one of the main technical tools in our considerations, its proof can be found in [Oli02].

Lemma 1.1 (Minlos lemma). *Let $G : \Gamma_0 \mapsto \mathbb{R}$, $H : \Gamma_0 \times \cdots \times \Gamma_0 \mapsto \mathbb{R}$ be positive and measurable, then for $n \in \mathbb{N}$, $n \geq 2$:*

$$\int_{\Gamma_0}\cdots\int_{\Gamma_0} G(\eta_1\cup\ldots\cup\eta_n)H(\eta_1,\ldots,\eta_n)\lambda_{z\sigma}(d\eta_1)\cdots\lambda_{z\sigma}(d\eta_n) \tag{1.12}$$
$$= \int_{\Gamma_0} G(\eta)\sum_{(\eta_1,\ldots,\eta_n)\in\mathcal{P}_n^{\emptyset}(\eta)} H(\eta_1,\ldots,\eta_n)\lambda_{z\sigma}(d\eta),$$

where $\mathcal{P}_n^{\emptyset}(\eta)$ denotes the family of all ordered partitions of η in n parts, which may be empty.

From now on, fix the parameter $z = 1$ and let the measure σ be the Lebesgue measure on $\mathbb{R}^d$. In this case we will write λ instead of $\lambda_{z\sigma}$ to denote the Lebesgue-Poisson measure.

1.3 Harmonic analysis on configuration spaces

We will recall some facts which will be used often in this thesis, for proofs and more detailed description we refer e.g. to [FKO09, Oli02, Kun99, KK02].

1.3.1 Functions on Γ_0 and Γ

Let $L^0(\Gamma_0)$ denote the set of all $\mathcal{B}(\Gamma_0)$-measurable real-valued functions on Γ_0, and let $B(\Gamma_0) \subset L^0(\Gamma_0)$ denote the set of those measurable functions which are bounded.

Definition 1.1. *Denote with $L^0_{ls}(\Gamma_0)$ the set of all measurable functions with local support, i.e.: $G \in L^0_{ls}(\Gamma_0)$ iff $G \in L^0(\Gamma_0)$ and there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ such that $G \upharpoonright_{\Gamma_0 \setminus \Gamma_\Lambda} = 0$.*

Denote with $L^0_{bs}(\Gamma_0)$ the set of all measurable functions with bounded support, i.e. $G \in L^0_{bs}(\Gamma_0)$ iff $G \in L^0(\Gamma_0)$ and there exists $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$, such that $G \upharpoonright_{\Gamma_0 \setminus \left(\sqcup_{n=0}^N \Gamma_\Lambda^{(n)}\right)} = 0$.

We define the family of bounded functions with local support $B_{ls}(\Gamma_0)$ and the family of bounded functions with bounded support $B_{bs}(\Gamma_0)$ in the similar way.

Let for $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$,

$$\mathcal{B}_\Lambda(\Gamma) := \sigma\left(N_{\Lambda'} : \Lambda' \in \mathcal{B}_c(\mathbb{R}^d) \text{ with } \Lambda' \subset \Lambda\right).$$

Denote by $L^0(\Gamma)$ the set of all $\mathcal{B}(\Gamma)$-measurable functions, and define the σ-algebra of cylinder sets

$$\mathcal{B}_{cyl}(\Gamma) := \bigcup_{\Lambda \in \mathcal{B}_c(\mathbb{R}^d)} \mathcal{B}_\Lambda(\Gamma). \tag{1.13}$$

A cylinder function $F \in L^0(\Gamma)$ is a function which is measurable w.r.t. $\mathcal{B}_{cyl}(\Gamma)$. We will denote the set of all cylinder functions by $\mathcal{F}L^0(\Gamma, \mathcal{B}(\Gamma))$. In particular, $F \in \mathcal{F}L^0(\Gamma, \mathcal{B}(\Gamma))$ means, that F is $\mathcal{B}_\Lambda(\Gamma)$-measurable for some $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and

$$F(\gamma) = F \upharpoonright_{\Gamma_\Lambda} (\gamma_\Lambda). \tag{1.14}$$

Let $\mathcal{F}C_b = \mathcal{F}C_b(C_0(\mathbb{R}^d), \Gamma)$ denote the set of all bounded continuous cylinder functions, i.e. those functions F on Γ which have the representation:

$$F(\gamma) = g_F\left(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle\right)$$

for some $N \in \mathbb{N}$, $g_F \in C_b(\mathbb{R}^N)$ and $\varphi \in C_0(\mathbb{R}^d)$. Note, that this representation is not unique.

We will call the functions on Γ_0 *quasi-observables*, and those on Γ *observables.*

1.3.2 K-transform

The following mapping between quasi-observables and observables plays crucial role in our further considerations. Its introduction was motivated by the concepts of additive type observables from statistical mechanics, it was also used by Lenard to define the correlation functions, see e.g. [Len75, Bog46]. Let $G \in L^0_{ls}(\Gamma_0)$, $\gamma \in \Gamma$ and define

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta). \tag{1.15}$$

Here and throughout this thesis, $\eta \Subset \gamma$ means, that η is a finite subset of γ. Note that this sum is well defined, because only finite number of summands is unequal to zero. Below we present some properties of the K-transform. Their proofs can be found in [KK02].

Remark 1.2. *The K-transform is linear and preserves positivity, it maps $L^0_{ls}(\Gamma_0)$ into $\mathcal{F}L^0(\Gamma)$.*

The K-transform is invertible, with the inverse defined by

$$K^{-1}F(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0 \tag{1.16}$$

for a cylindrical function $F : \Gamma \mapsto \mathbb{R}$.

Below we give an example of K-transform of the so called *coherent state* $e_\lambda(f, \cdot)$ corresponding to a measurable function $f : \mathbb{R}^d \mapsto \mathbb{R}$, i.e.

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0 \tag{1.17}$$

and $e_\lambda(f, \emptyset) := 1$. Assume now, that f has a compact support, then

$$\left(Ke_\lambda(f, \cdot)\right)(\gamma) = \prod_{x \in \gamma} \left(1 + f(x)\right), \quad \gamma \in \Gamma. \tag{1.18}$$

We define now, for $G_1, G_2 \in L^0_{ls}(\Gamma_0)$, the $\star$-*convolution*:

$$G_1 \star G_2(\eta) := \sum_{(\eta_1, \eta_2, \eta_3) \in \mathcal{P}^\emptyset_3(\eta)} G_1(\eta_1 \cup \eta_2) \cdot G_2(\eta_2 \cup \eta_3). \tag{1.19}$$

One of the most important properties of the $\star$-convolution is stated in the following remark, the proof of which can be found in [KK02]:

Remark 1.3. *For $G_1, G_2 \in L^0_{ls}(\Gamma_0)$ we have*

$$K\left(G_1 \star G_2\right)(\eta) = KG_1(\eta) \cdot KG_2(\eta). \tag{1.20}$$

Due to this property the K-transform is analogous to the (classical) Fourier transform in the case of configuration space analysis.

1.3.3 Correlation measures

Using the K-transform, we can define a measure on Γ_0. Denote with $\mathcal{M}^1_{fm}(\Gamma)$ the set of all probability measures on Γ which have *finite local moments* (of all orders), that is

$$\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < \infty \tag{1.21}$$

for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and all $n \in \mathbb{N}$. Next, with $\mathcal{M}_{lf}(\Gamma_0)$ denote the set of all *locally finite* measures on Γ_0, i.e. $\rho(\Lambda) < \infty$ for all $\rho \in \mathcal{M}_{lf}(\Gamma_0)$ and all bounded sets Λ from $\mathcal{B}(\Gamma_0)$.

Let now $\mu \in \mathcal{M}^1_{fm}(\Gamma)$ and define the dual of K-transform (denoted by K^*) as follows:

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)\,(d\eta). \tag{1.22}$$

We call $\rho_\mu := K^*\mu$, the *correlation measure* of the measure μ.

Remark 1.4. *A useful example of such a dualism is given in [FKO09]. The correlation measure corresponding to the Poisson measure $\pi_{z\sigma}$ is the Lebesque-Poisson measure $\lambda_{z\sigma}$.*

Having defined the correlation measure, we can recall the important fact about the extension of the K-transform defined in (1.15):

Theorem 1.1 ([KK02], Thm. 4.1)**.** *Let $\mu \in \mathcal{M}^1_{fm}(\Gamma)$ be given. For any $G \in L^1(\Gamma_0, \rho_\mu)$ we define*

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \tag{1.23}$$

where the latter series is μ-a.s. absolutely convergent. Furthermore one can show that $KG \in L^1(\Gamma, \mu)$,

$$\|KG\|_{L^1(\mu)} \leq \|K|G|\|_{L^1(\mu)} = \|G\|_{L^1(\rho_\mu)} \tag{1.24}$$

and

$$\int_{\Gamma_0} G(\eta)\rho_\mu(d\eta) = \int_\Gamma KG(\gamma)\mu(d\gamma). \tag{1.25}$$

We have already defined the correlation functions in (1.11). One can also introduce them using K^*, if the measure μ and its correlation measure have densities.

We say, that a measure $\mu \in \mathcal{M}^1_{fm}(\Gamma)$ is *locally absolutely continuous* w.r.t. measure π_σ, iff for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, the measure $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous w.r.t. $\pi_\sigma^\Lambda := \pi_\sigma \circ p_\Lambda^{-1}$. In this case, $\rho_\mu = K^*\mu$ is absolutely continuous w.r.t. λ_σ and we have

$$k_\mu(\eta) = \frac{d\rho_\mu}{d\lambda_\sigma}(\eta).. \tag{1.26}$$

As we will see later, the correlation functions are very useful to describe the evolution of certain (Markov) processes on configuration space.

1.4 Two-component configuration space

Having in mind the motivation source of the considered models (e.g. ecological applications) we introduce now the *two-component configuration space* as the Cartesian product of two identical copies of the space Γ (cf. [FKO10]). Again, for the above mentioned reason, we distinguish the elements of each of the two spaces as different population types, i.e. Γ^+ and Γ^-, thus

$$\Gamma^2 := \left\{(\gamma^1, \gamma^2) \in \Gamma^+ \times \Gamma^- : \quad \gamma^1 \cap \gamma^2 = \emptyset\right\}. \tag{1.27}$$

Similarly we can define the *two-component space of finite configurations*:

$$\Gamma_0^2 := \left\{(\eta^1, \eta^2) \in \Gamma_0^+ \times \Gamma_0^- : \quad \eta^1 \cap \eta^2 = \emptyset\right\}. \tag{1.28}$$

The two-component spaces are equipped with the product topologies of the (finite) configuration spaces, their structure is inherited from the underlying one-component spaces. This applies also to the Lebesgue-Poisson and Poisson measure, thus we can consider spaces $(\Gamma^2, \pi_{z\sigma} \otimes \pi_{z\sigma})$ and $(\Gamma_0^2, \lambda_{z\sigma} \otimes \lambda_{z\sigma})$.

In the following, we define the two-component analogues of K-transform, the $\star$-convolution and the Minlos lemma. Let $G \in L^0_{ls}(\Gamma_0^2)$ and define

$$\mathcal{K}G(\gamma^1, \gamma^2) := \sum_{\eta^1 \Subset \gamma^1} \sum_{\eta^2 \Subset \gamma^2} G(\eta^1, \eta^2), \quad (\gamma^1, \gamma^2) \in \Gamma^2. \tag{1.29}$$

As in the one-component case it is invertible with the inverse given by

$$\mathcal{K}^{-1}F(\eta^1, \eta^2) = \sum_{\xi^1 \subset \eta^1} \sum_{\xi^2 \subset \eta^2} (-1)^{|\eta^1 \setminus \xi^1| + |\eta^2 \setminus \xi^2|} F(\xi^1, \xi^2) \tag{1.30}$$

for F cylindrical and $(\eta^1,\eta^2)\in\Gamma_0^2$.

Let $\eta := (\eta^1,\eta^2)\in\Gamma_0^2$, then the $\star$-convolution is defined as

$$G_1\star G_2(\eta) := \sum_{\substack{(\eta_1^1,\eta_2^1,\eta_3^1)\in\mathcal{P}_3^\emptyset(\eta^1)\\(\eta_1^2,\eta_2^2,\eta_3^2)\in\mathcal{P}_3^\emptyset(\eta^2)}} G_1(\eta_1^1\cup\eta_2^1,\eta_1^2\cup\eta_2^2)G_2(\eta_2^1\cup\eta_3^1,\eta_2^2\cup\eta_3^2), \quad (1.31)$$

for $G_1,G_2\in L_{ls}^0(\Gamma_0^2)$. As in the one-component case, the following property holds:

Lemma 1.2. *Let $G_1,G_2\in L_{ls}^0(\Gamma_0^2)$, then*

$$\mathcal{K}(G_1\star G_2)(\eta) = \mathcal{K}G_1(\eta)\cdot\mathcal{K}G_2(\eta). \quad (1.32)$$

Proof. Let $G_1,G_2\in L_{ls}^0(\Gamma_0^2)$. Then we have

$$\mathcal{K}G_1(\gamma^1,\gamma^2)\cdot\mathcal{K}G_2(\gamma^1,\gamma^2) = \sum_{\eta_1^1\Subset\gamma^1}\sum_{\eta_1^2\Subset\gamma^2} G_1(\eta_1^1,\eta_1^2)\cdot\sum_{\eta_2^1\Subset\gamma^1}\sum_{\eta_2^2\Subset\gamma^2} G_2(\eta_2^1,\eta_2^2)$$

and because of the assumptions about G_1,G_2 those sums are finite, hence the latter is equal to

$$\sum_{\eta_1^1\Subset\gamma^1}\sum_{\eta_2^1\Subset\gamma^1}\sum_{\eta_1^2\Subset\gamma^2}\sum_{\eta_2^2\Subset\gamma^2} G_1(\eta_1^1,\eta_1^2)\cdot G_2(\eta_2^1,\eta_2^2).$$

For $i=1,2$, we can decompose γ^i into four sets: $\xi_1^i := \eta_1^i\setminus\eta_2^i$, $\xi_2^i := \eta_2^i\setminus\eta_1^i$, $\xi_3^i := \eta_1^i\cap\eta_2^i$ and $\xi_4^i := \gamma^i\setminus(\eta_1^i\cup\eta_2^i)$. Then we obtain

$$\sum_{\substack{(\xi_1^1,\xi_2^1,\xi_3^1,\xi_4^1)\in\mathcal{P}_4^\emptyset(\gamma^1)\\(\xi_1^2,\xi_2^2,\xi_3^2,\xi_4^2)\in\mathcal{P}_4^\emptyset(\gamma^2)}} G_1(\xi_1^1\cup\xi_3^1,\xi_1^2\cup\xi_3^2)\cdot G_2(\xi_2^1\cup\xi_3^1,\xi_2^2\cup\xi_3^2)$$

but this is the same as

$$\sum_{\xi_4^1\Subset\gamma^1}\sum_{\xi_4^2\Subset\gamma^2}\sum_{\substack{(\xi_1^1,\xi_2^1,\xi_3^1)\in\mathcal{P}_3^\emptyset(\gamma^1\setminus\xi_4^1)\\(\xi_1^2,\xi_2^2,\xi_3^2)\in\mathcal{P}_3^\emptyset(\gamma^2\setminus\xi_4^2)}} G_1(\xi_1^1\cup\xi_3^1,\xi_1^2\cup\xi_3^2)\cdot G_2(\xi_2^1\cup\xi_3^1,\xi_2^2\cup\xi_3^2)$$

which is equal to

$$\sum_{\xi_4^1\Subset\gamma^1}\sum_{\xi_4^2\Subset\gamma^2}(G_1\star G_2)(\gamma^1\setminus\xi_4^1,\gamma^2\setminus\xi_4^2) = \mathcal{K}(G_1\star G_2)(\gamma^1,\gamma^2).$$

□

Denote now

$$e_\lambda(f,g,\eta^1,\eta^2) := e_\lambda(f,\eta^1)e_\lambda(g,\eta^2), \tag{1.33}$$

then for $f,g \in L^1(\Gamma_0^2,\rho_\mu)$ we have

$$\left(\mathcal{K}e_\lambda(f,g,\cdot^1,\cdot^2)\right)(\gamma^1,\gamma^2) = \prod_{x\in\gamma^1}(1+f(x))\prod_{y\in\gamma^2}(1+g(y)) \tag{1.34}$$

We will also need the Minlos lemma in the two-dimensional case:

Lemma 1.3. *Let $n \geq 1$, and for each $i = 1,\dots,n$, $\eta_i = (\eta_i^1,\eta_i^2) \in \Gamma_0^2$. Let $\lambda^2 := \lambda\otimes\lambda$ be the product measure on $(\Gamma_0^2,\mathcal{B}(\Gamma_0^2))$. Then*

$$\int_{\Gamma_0^2} d\lambda^2(\eta_1)\dots\int_{\Gamma_0^2} d\lambda^2(\eta_n)G(\eta_1^1\cup\dots\cup\eta_n^1,\eta_1^2\cup\dots\cup\eta_n^2)H(\eta_1,\dots,\eta_n)$$
$$= \int_{\Gamma_0^2} d\lambda^2(\eta)G(\eta)\sum_{\substack{(\eta_1^1,\dots,\eta_n^1)\in\mathcal{P}_n^\emptyset(\eta^1)\\(\eta_1^2,\dots,\eta_n^2)\in\mathcal{P}_n^\emptyset(\eta^2)}} H((\eta_1^1,\eta_1^2),\dots,(\eta_n^1,\eta_n^2)),$$

for all functions G,H for which both sides of the equality make sense.

The definitions introduced in this section can be further generalized to systems which consist of more than two populations, i.e. we can define the *multicomponent* configuration spaces in a similar way.

1.5 Markov evolutions in CS

In this section we present the general investigation scheme for the infinite particle systems, using the framework of the configuration spaces analysis (see e.g. [KK02, FKK09, Str09, FKO09, FKO10]). As we mentioned before, the functions F on Γ are called observables. The measure μ on Γ will be then the *state* of a system. Note, that the number of particles (individuals) of the system is infinite. This fact is the source of many technical difficulties, as well as of some interesting questions. We will denote by $\langle\cdot,\cdot\rangle$ the expected value of an observable F w.r.t. to the state μ:

$$\langle F,\mu\rangle = \int_\Gamma F(\gamma)\mu(d\gamma). \tag{1.35}$$

Let L denote the heuristic (Markov) pre-generator which describes the infinitesimal behaviour of a given model. The mechanism of evolution of the system is determined by the action of L. Having in mind applications, the possible events include:

- birth,
- death,
- jump,
- diffusion (motion)

of a particle (or site $x \in \mathbb{R}^d$) during the infinitesimal time interval $[t, t+dt]$. Because of the very complex structure of the space Γ, it is often difficult to give precise description of the operator L, i.e. to specify its domain and thus to consider L as a generator of a strongly continuous contraction semigroup associated with a Markov process using the standard methods (as in, e.g. [MR92]). We will use different approach to the problem. If LF is (at least) point-wisely well defined for a function $F \in \mathcal{F}C_b$, then we can write the so-called *Kolmogorov equation* for observables

$$\frac{\partial}{\partial t} F_t = LF_t. \tag{1.36}$$

The equation for the associated state μ_t would be the dual Kolmogorov equation (or *Fokker-Planck equation*)

$$\frac{\partial}{\partial t} \mu_t = L^* \mu_t, \tag{1.37}$$

where L^* is the adjoint of the operator L with respect to the duality (1.35). In this situation, we are still in the infinite-dimensional context, which makes it complicated to even formulate the problem rigorously. However, using the tools presented earlier in this chapter, we can rewrite infinite dimensional evolutional equation as an infinite system of finite dimensional equations, namely as an evolutional equations of quasi-observables. Define the *symbol* $\hat{L}$ of the operator L, $\hat{L} := K^{-1}LK$. Using the symbol we can obtain the equation for quasi-observables corresponding to the Kolmogorov equation:

$$\frac{\partial}{\partial t} G_t = \hat{L} G_t. \tag{1.38}$$

We can, again, deduce the dual equation on correlation functions

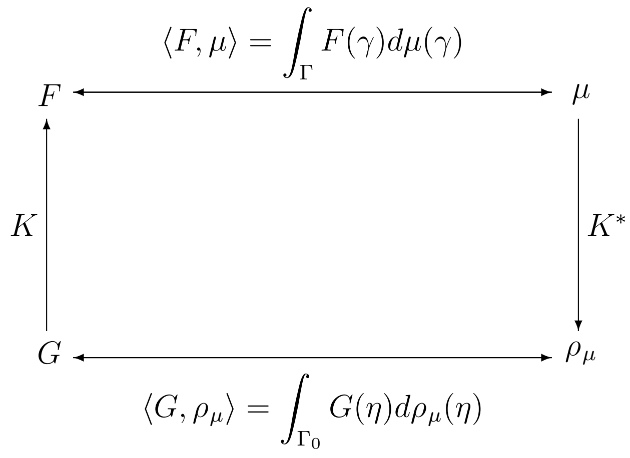
$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t, \tag{1.39}$$

where the operator $\hat{L}^*$ is defined via the duality

$$\langle \hat{L}G, k \rangle = \int_{\Gamma_0} \hat{L}G(\eta) k(\eta) \lambda_{z\sigma}(d\eta) = \int_{\Gamma_0} G(\eta) \hat{L}^* k(\eta) \lambda_{z\sigma}(d\eta) = \langle G, \hat{L}^* k \rangle. \tag{1.40}$$

In order to do this, we should assure that the corresponding correlation measure ρ_μ is absolutely continuous w.r.t. the Lebesgue-Poisson measure $\lambda_{z\sigma}$ for every time $t \geq 0$. We should also mention that the solution to the equation (1.40) is not necessary a correlation function associated to some measure. There exist, however, conditions which assure the existence of such a measure (see, e.g. [BKKL99, Len73]).

To summarize this section, let us present the latter considerations on the diagram ([FKO09]):



1.6 Vlasov type scaling

The Vlasov equation was introduced by A. Vlasov in 1938 in the context of plasma physics, to describe the evolution of density of plasma particles with long-range interaction (see [Vla68] for English translation). It also plays an important role in the stellar dynamics (see e.g. [Spo80]). Later development and applications of Vlasov scaling are due to the works of Braun and Hepp ([BH77]), Dobrushin ([Dob79]) and Kozlov ([Koz08]).

The Vlasov equation can be obtained by a proper scaling of a system. In this work we study a type of Vlasov scaling for two-component interacting particle systems. However, as it was mentioned in [FKK10a], the methods used by the authors above cannot be simply used in our case (one of the reasons for that is for example lack of the description of a given model in terms of a stochastic differential equation describing the evolution), therefore we will recall here the Vlasov-type scaling scheme developed in [FKK10a]. The description presented here is general and does not contain all technical details needed to properly formulate the statements. Those details are model-dependent thus we will give them later in the proper chapters.

In the previous section we have presented the general scheme to obtain the evolution of correlation functions for a given system. The starting point was a Markov pre-generator L and the equation (1.36). Then, using the K-transform we could derive the evolution equation for the system of correlation functions (1.39) corresponding to the states $(\mu_t)_{t\geq 0}$. Note, that depending on the considered model we will later specify the formal conditions for such an evolution to exist. Usually, the Vlasov scaling is realized in terms of correlation functions, thus our starting point is the following Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t} k_t = \hat{L}^* k_t \\ \\ k_t|_{t=0} = k_0. \end{cases} \tag{1.41}$$

Recall that if $\hat{L}^*$ generates a semigroup $\hat{U}^*(t)$ in some space then the solution to (1.41) is given by $k_t = \hat{U}^*(t)k_0$, $t \geq 0$.

The general scheme of Vlasov type scaling introduced in [FKK10a] is as follows:

Step 1. We scale the initial function k_0 with $\varepsilon > 0$ in such a way, that $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\varepsilon \to 0$, $\eta \in \Gamma_0$ and the function r_0 is independent of ε. The choice of this initial density r_0 is usually motivated by the considered model. As it will become clear later, the function $r_0(\eta) := e_\lambda(\rho_0, \eta)$ plays essential role in our considerations, moreover we expect that the scaled dynamics preserves the factorized form of such initial density r_0, i.e. $r_t(\eta) = e_\lambda(\rho_t, \eta)$ for some ρ_t, and

$$\frac{\partial}{\partial t} \rho_t(x) = v(\rho_t)(x), \tag{1.42}$$

which is the *Vlasov-type equation* in our case. Although the equation (1.41) is linear, the equation (1.42) can be much more complicated (it is usually not linear any more).

Step 2. Now we should scale the generator $\hat{L}^*$ in a proper way. Again, the exact form of this scaling depends deeply on particular models. After scaling, we obtain a generator $\hat{L}^*_\varepsilon$ and the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} k_t^{(\varepsilon)} = \hat{L}^*_\varepsilon k_t^{(\varepsilon)} \\ \\ k_t^{(\varepsilon)}|_{t=0} = k_0^{(\varepsilon)}. \end{cases} \tag{1.43}$$

The idea of the scaling of the generator is very much related to the next Step.

Step 3. We impose that the scaled evolution preserves the order of singularity in ε, hence we need to *renormalize* $k_t^{(\varepsilon)}$ setting $k_{t,ren}^{(\varepsilon)}(\eta) := \varepsilon^{|\eta|} k_t^{(\varepsilon)}(\eta)$, $\eta \in \Gamma_0$ so that

$$k_{t,ren}^{(\varepsilon)}(\eta) \to r_t(\eta), \quad \varepsilon \to 0. \tag{1.44}$$

As result, we consider the renormalized version of the operator $\hat{L}_\varepsilon^*$,

$$\hat{L}_{\varepsilon,ren}^* := e^{|\eta|} \hat{L}_\varepsilon^* e^{-|\eta|}$$

and thus the equation (1.43) becomes

$$\begin{cases} \frac{\partial}{\partial t} k_{t,ren}^{(\varepsilon)} = \hat{L}_{\varepsilon,ren}^* k_{t,ren}^{(\varepsilon)} \\ \\ k_{t,ren}^{(\varepsilon)}\big|_{t=0} = k_{0,ren}^{(\varepsilon)}. \end{cases} \tag{1.45}$$

Hence, informally, letting ε tend to 0 we are looking for the solution of the following equation

$$\begin{cases} \frac{\partial}{\partial t} r_t = L^V r_t \\ \\ r_t\big|_{t=0} = r_0. \end{cases} \tag{1.46}$$

The natural candidate for the operator L^V is the pointwise limit of operators $\hat{L}_{\varepsilon,ren}^*$. The Vlasov equation (1.42) can be deduced heuristically from the equation (1.43), which is the analogue of the BBGKY hierarchy in the case of Hamiltonian systems (see e.g. [Spo80]).

This type of scaling has been studied e.g. in the case of individual based models with competition ([FKK10c]) and Glauber-type dynamics in continuum ([FKK10d]). In what follows, we will present the results of the Vlasov-type scaling for the following two-component systems: ecological model and the Glauber-Potts model.

Chapter 2

Continuous contact model in random environment

2.1 Introduction

In this chapter we study the modified version of the contact model in continuum introduced in [KS06] and later on investigated for example in [KKP08] and [Str09]. We consider three versions of the contact model in random environment which can be described as *random fecundity*, *random establishment* and *random mortality*. Before proceeding to the construction and investigation of the above-mentioned models, we recall some useful facts known from the theory of stability for Markov processes.

2.1.1 Extended generator

In the standard theory of Markov processes, the latter are characterized in terms of the associated semigroup or the *strong generator* together with its domain (see e.g. [MR92]). In the classical case, the domain typically consists (for example) of bounded functions with some additional properties. The use of unbounded functions is in general problematic. In this case, in order to show regularity of the considered models, we are compelled to work with functions which are not necessarily bounded (i.e. we should 'include' those functions into the domain of the generator of the process). To do so, we will use the so-called *extended generator* of the process. More detailed description of the theory presented here can be found e.g. in [MT93] and [Dav93].

Let X_t be a time-homogeneous Markov process with state space $(S, \mathcal{B}(S))$. We assume that S is a Borel space with the Borel σ-algebra $\mathcal{B}(S)$. Denote by $D(\mathcal{A})$ the set of all functions $V : S \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ for which there exists a measurable function $U : S \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ such that

$$E_x \left[V(X_t, t) \right] = V(x, 0) + E_x \left[\int_0^t U(X_s, s) ds \right] \tag{2.1}$$

and

$$\int_0^t E_x \left[|U(X_s, s)| \right] ds < \infty \tag{2.2}$$

for all $x \in S$, $t > 0$. We call $\mathcal{A}$ defined by $\mathcal{A}V := U$, the extended generator of the process X_t. In the next subsection we describe one of the possible ways of determining whether a given function is in the domain of the extended generator.

2.1.2 Truncation of the process X_t

Let $(O^m)_{m \in \mathbb{N}}$ be a family of open pre-compact sets in S, such that $S = \bigcup_m O^m$ and $O^m \subset O^{m+1}$ for any $m \in \mathbb{N}$. Let T^m be the first-entrance time of the process X_t to the set $(O^m)^c = S \setminus O^m$.
Denote by ζ the lifetime of the process, i.e.,

$$\zeta := \lim_{m \to \infty} T^m.$$

We introduce the truncations of X_t in the following way:

$$X_t^m := \begin{cases} X_t, & t < T^m \\ \triangle_m, & t \geq T^m \end{cases}$$

where $\triangle_m \in (O^m)^c$ is called the *cemetery* or the *graveyard state*.
Now let $\mathcal{A}_m$ denote the extended generator of the truncated process X_t^m, and define the domain of its *weak infinitesimal generator* (denoted by $D(\tilde{\mathcal{A}}_m)$) as the set of all measurable functions $W : S \to \mathbb{R}$ such that the pointwise limit

$$\tilde{\mathcal{A}}_m W(x) := \lim_{h \to 0} \frac{\mathbb{E}_x \left[W(X_h^m) \right] - W(x)}{h} \tag{2.3}$$

exists for $x \in S$ and satisfies

$$\lim_{h \to 0} \mathbb{E}_x \left[\tilde{\mathcal{A}}_m W(X_h^m) \right] = \tilde{\mathcal{A}}_m W(x). \tag{2.4}$$

In addition, if the following holds

$$\sup_{x \in C} \left| \tilde{\mathcal{A}}_m W(x) \right| < \infty \tag{2.5}$$

for any compact set $C \subset S$, then $D(\tilde{\mathcal{A}}_m) \subset D(\mathcal{A}_m)$ (see [Kus67]).

2.1.3 Lyapunov-type function for the process

Let $\beta > 0$ and $x, y \in \mathbb{R}^d$. Define

$$e_\beta(x) := e^{-\beta|x|} \tag{2.6}$$

and

$$\Psi_\beta(x,y) := e_\beta(x)e_\beta(y)\frac{|x-y|+1}{|x-y|}\mathbb{1}_{\{x\neq y\}}(x,y). \tag{2.7}$$

Now for $\gamma \in \Gamma$ define the following functions:

$$\mathbb{L}_\beta(\gamma) := \sum_{x\in\gamma} e_\beta(x) = \langle e_\beta, \gamma\rangle\,, \tag{2.8}$$

and

$$\mathbb{E}_\beta(\gamma) := \sum_{\{x,y\}\subset\gamma} \Psi_\beta(x,y) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi_\beta(x,y)\gamma(dx)\gamma(dy). \tag{2.9}$$

Finally, let

$$\mathbb{V}_\beta(\gamma) := \mathbb{E}_\beta(\gamma) + \mathbb{L}_\beta(\gamma). \tag{2.10}$$

In the sequel, function $\mathbb{V}_\beta$ will play role of the Lyapunov function for the contact process. It can be shown, that the sets

$$\{\gamma \in \Gamma | \mathbb{V}_\beta(\gamma) \leq C\}$$

are precompact in Γ for every $C > 0$ (see e.g. [KKP08]).
Introduce the spaces induced by the function $\mathbb{V}_\beta$:

$$\Gamma_\beta := \{\gamma \in \Gamma : \quad \mathbb{V}_\beta(\gamma) < \infty\} \tag{2.11}$$

and

$$\Gamma_\infty := \bigcup_{\beta>0} \Gamma_\beta. \tag{2.12}$$

Remark 2.1 (cf. Remark 2.1 in [KS06]). *Note, that $\mu(\Gamma_\infty) = 1$ for all probability measures μ on $\mathcal{B}(\Gamma)$ which have second local moment finite, i.e.*

$$\int_\Lambda |\gamma_\Lambda|^2 \mu(d\gamma) < \infty$$

for all compact sets $\Lambda \subset \mathbb{R}^d$.

2.2 Contact process in continuum

In this section we recall the construction of the contact process in continuum as in [KS06]. Using the framework introduced in the previous sections we show, that the lifetime of the process is equal to infinity, i.e., the explosion does not occur.

The heuristic pre-generator of the contact process has the following form:

$$LF(\gamma) = \sum_{x\in\gamma} D_x^- F(\gamma) + \varkappa \sum_{y\in\gamma} \int_{\mathbb{R}^d} a(x-y) D_x^+ F(\gamma) dx \tag{2.13}$$

where

$$D_x^- F(\gamma) := F(\gamma \setminus x) - F(\gamma),\ D_x^+ F(\gamma) := F(\gamma \cup x) - F(\gamma).$$

The operator (2.13) is well defined e.g. for cylinder functions F almost surely w.r.t. the appropriate measure on Γ, see [FKO09]. Throughout this chapter we assume that $a \in L^\infty(\mathbb{R}^d)$ and that a has bounded support, i.e. there exists a $R > 0$ such that supp $a \subset B_R(0)$.

2.2.1 Construction of the process

We construct the contact process as a spatial branching process with killing in the space $\mathbb{R}^d$. In order to do so note, that for any $\eta \in \Gamma_0$ we can rewrite pre-generator L as follows:

$$LF(\eta) = \lambda(\eta) \int_{\Gamma_0} \left(F(\eta') - F(\eta)\right) Q(\eta, d\eta') \tag{2.14}$$

with $\lambda(\eta) = |\eta|(1+\varkappa)$, and

$$Q(\eta, d\eta') = \frac{1}{\lambda(\eta)} \left[\sum_{x\in\eta} \delta_{\eta\setminus x}(d\eta') + \varkappa \sum_{y\in\eta} \int_{\mathbb{R}^d} a(x-y) \delta_{\eta\cup x}(d\eta') dx \right]. \tag{2.15}$$

From the theory of pure jump processes follows, that there exists a jump process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t^\eta)_{t\geq 0}, \mathbb{P}_\eta)$ starting from $\eta \in \Gamma_0$ with lifetime $\zeta(\omega)$. Such a process can be constructed by means of the associated Markov chain and the sequence of the stopping times, see e.g. [GS74, EK05]. As it was shown in [FM04], the lifetime of this process (starting from finite configuration) is infinite.

Having constructed the contact process for a given finite configuration η, let us proceed to the construction of the process starting with any initial configuration $\gamma \in \Gamma_\beta$. Denote by $\gamma_n := \gamma \cap B(0,n) \in \Gamma_0$, $n \in \mathbb{N}$ and consider the non-decreasing sequence of Markov processes $((X_t^{\gamma_n})_{t\geq 0})$ defined on joint probability space, with the property

$$\forall_{n\in\mathbb{N},t\geq 0} \quad X_t^{\gamma_n} \subset X_t^{\gamma_{n+1}} \text{ a.s.} \tag{2.16}$$

Such birth-and-death processes with the finite initial configuration were constructed for example in [FM04] using Poisson stochastic equations. Because of the additive structure of the operator L, $X_t^{\gamma_n}$ and $X_t^{\gamma_{n+1}\setminus\gamma_n}$ are independent Markov processes with $X_t^{\gamma_n} \cap X_t^{\gamma_{n+1}\setminus\gamma_n} = \emptyset$ a.s., hence $X_t^{\gamma_{n+1}} = X_t^{\gamma_n} \cup X_t^{\gamma_{n+1}\setminus\gamma_n}$ a.s. and we can introduce the limiting process

$$X_t^\gamma(\omega) = \left(\bigcup_{n\in\mathbb{N}} X_t^{\gamma_n}\right)(\omega) \tag{2.17}$$

with the lifetime $\zeta(\omega)$. For more detailed discussion see e.g. [AN72] and [Isc86].

2.2.2 Regularity of the process

In what follows we will show that the lifetime of the process X_t constructed in the previous section is almost surely infinite.

Take $\gamma \in \Gamma_\infty$. That means that there exists $\beta > 0$ such that $\gamma \in \Gamma_\beta$. From now on we will fix β and consider X_t with the initial configuration γ, i.e. with probability one we have that $X_0 = \gamma$.

Set $O^m := \{\gamma \in \Gamma_\beta : \mathbb{V}_\beta(\gamma) \leq m\}$. Then obviously $O^m \uparrow \Gamma_\beta$ and each set O^m is relatively compact. Note, that the sets O^m, $m \in \mathbb{N}$ depend on β, but this is fixed here so we omit this dependence in the notation.

Recall the truncated process defined above, X_t^m with the initial configuration $\gamma^m := \gamma \cap O^m$ and its extended generator L_m. For functions F in the domain of the extended generator $D(L)$ we clearly have $L_m F(\gamma) = LF(\gamma)$ for all $\gamma \in O^m$.

Proposition 2.1. *For every $m \in \mathbb{N}$, there exists $C_m > 0$ such that the following holds:*

$$\sup_{\eta\in O^m} |L_m \mathbb{V}_\beta(\eta)| \leq C_m < \infty \tag{2.18}$$

for all $\beta > 0$.

Proof. Let $\eta \in O^m$ and recall that $L_m F(\gamma) = LF(\gamma)$ on O^m. First let us estimate

$$\begin{aligned}|L_m \mathbb{L}_\beta(\eta)| &= \Bigg| \sum_{x\in\eta} \Bigg(\sum_{y\in\eta\setminus x} e_\beta(y) - \sum_{y\in\eta} e_\beta(y) \Bigg) \\ &\quad + \varkappa \sum_{y\in\eta} \int_{\mathbb{R}^d} a(x-y) \Bigg(\sum_{y\in\eta\cup x} e_\beta(y) - \sum_{y\in\eta} e_\beta(y) \Bigg) dx \Bigg| \\ &= \Bigg| -\sum_{x\in\eta} e_\beta(x) + \varkappa \sum_{y\in\eta} \int_{\mathbb{R}^d} a(x-y) e_\beta(x) dx \Bigg| \\ &\le |\mathbb{L}_\beta(\eta)| + \Bigg| C_1 \varkappa \sum_{y\in\eta} e_\beta(y) \Bigg| = (\varkappa C_1 + 1)\, \mathbb{L}_\beta(\eta)\end{aligned}$$

where we have used the fact, that function a has bounded support. Hence, we obtain

$$\sup_{\eta\in O^m} |L_m \mathbb{L}_\beta(\eta)| \le (\varkappa C_1 + 1) \sup_{\eta\in O^m} \mathbb{L}_\beta(\eta) \le (\varkappa C_1 + 1)\, m < \infty$$

because of the definition of O^m. Next

$$\begin{aligned}\big|L_m \mathbb{E}_\beta(\eta)\big| = \Bigg| &\sum_{x\in\eta} \Bigg(\sum_{\{z_1,z_2\}\subset\eta\setminus x} \Psi_\beta(z_1,z_2) - \sum_{\{z_1,z_2\}\subset\eta} \Psi_\beta(z_1,z_2) \Bigg) \\ &+ \varkappa \sum_{y\in\eta} \int_{\mathbb{R}^d} a(x-y) \Bigg(\sum_{\{z_1,z_2\}\subset\eta\cup x} \Psi_\beta(z_1,z_2) - \sum_{\{z_1,z_2\}\subset\eta} \Psi_\beta(z_1,z_2) \Bigg) dx \Bigg|\end{aligned}$$

and this is equal to

$$\Bigg| -2\mathbb{E}_\beta(\eta) + \varkappa \sum_{y\in\eta} \sum_{z\in\eta} \int_{\mathbb{R}^d} a(x-y) \Psi_\beta(x,z) dx \Bigg| .$$

Using the definition of Ψ_β and the properties of a there exists a constant C_2 such that the latter can be estimated by

$$|2\mathbb{E}_\beta(\eta)| + C_2 \varkappa \Bigg| \sum_{y\in\eta} \sum_{z\in\eta} e_\beta(y) e_\beta(z) \int_{B_R(0)} \frac{dx}{|x-(z+y)|} \Bigg|$$

but the integral above is uniformly bounded when $d \ge 2$, thus

$$\big|L_m \mathbb{E}_\beta(\eta)\big| \le |2\mathbb{E}_\beta(\eta)| + C_3 \varkappa \Bigg| \sum_{y\in\eta} \sum_{z\in\eta} e_\beta(y) e_\beta(z) \Bigg|$$

for some $C_3 > 0$. Note that

$$\sum_{y\in\eta}\sum_{z\in\eta} e_\beta(y)e_\beta(z) = 2\sum_{\{y,z\}\subset\eta} e_\beta(y)e_\beta(z) + \sum_{x\in\eta} e^{-2\beta|x|} \leq 2\mathbb{E}_\beta(\eta) + \mathbb{L}_\beta(\eta)$$

which gives

$$\begin{aligned}\sup_{\eta\in O^m} \left|L_m\mathbb{E}_\beta(\eta)\right| &\leq \sup_{\eta\in O^m} [\varkappa C_3\mathbb{L}_\beta(\eta) + 4\varkappa C_3\mathbb{E}_\beta(\eta)] \\ &\leq \max\{\varkappa C_3, 4\varkappa C_3\} \sup_{\eta\in O^m} \mathbb{V}_\beta(\eta) \\ &\leq \max\{\varkappa C_3, 4\varkappa C_3\} m < \infty.\end{aligned}$$

Thus, there exists a constant $C_m > 0$ such that

$$\sup_{\eta\in O^m} |L_m\mathbb{V}_\beta(\eta)| \leq C_m < \infty$$

for every $\beta > 0$ and $m \in \mathbb{N}$. That concludes the proof. □

Hence the condition (2.5) is fulfilled and we have the following

Corollary 2.1. *For every $\beta > 0$ and $m \in \mathbb{N}$ function $\mathbb{V}_\beta$ is in the domain of the extended generator L_m.*

Notice that from the proof of Proposition 2.1 we can conclude:

Corollary 2.2. *There exists a constant $C > 0$ such that for every $m \in \mathbb{N}$ and every $\gamma \in O^m$ the following inequality holds:*

$$L_m\mathbb{V}_\beta(\gamma) \leq C\mathbb{V}_\beta(\gamma). \tag{2.19}$$

Finally we are able to show the following

Theorem 2.1. *Contact process (X_t^γ) is non-explosive for each $\gamma \in \Gamma_\infty$, i.e. $\zeta = \infty$ with probability one.*

This result is a direct consequence of Theorem 2.1. in [MT93]:

Theorem 2.2 ([MT93])**.** *If $(X_t^\gamma)_{t\geq 0}$ is a right process and* (2.19) *is satisfied, then*

1. *$\zeta = \infty$, so that $(X_t^\gamma)_{t\geq 0}$ is non-explosive.*

2. *There exists an a.s. finite random variable $\tilde{D}$ such that*

$$\mathbb{V}_\beta(X_t^\gamma) \le \tilde{D}e^{ct}, \quad 0 \le t < \infty \tag{2.20}$$

The random variable $\tilde{D}$ satisfies the bound

$$P_\gamma(\tilde{D} \ge a) \le \frac{\mathbb{V}_\beta(\gamma)}{a}, \quad a > 0, \gamma \in \Gamma_\infty$$

3. *The expectation $E(\mathbb{V}_\beta(X_t^\gamma))$ is finite for each γ and t, and the following bound holds*

$$E(\mathbb{V}_\beta(X_t^\gamma)) \le e^{ct}\mathbb{V}_\beta(\gamma).$$

2.3 Properties of the random potential

Before considering the contact process in random environment we will introduce the random potential and show some of its properties.

We investigate the random potential corresponding to Poisson random field ω, and the potential function φ. There exist also other possibilities for realization of the random influence in our model, for example the Gaussian potential (see e.g. [GKM00]), but we will be focused only on the Poissonian case.

In the following, we denote by P and $\langle\cdot\rangle$ the probability and the expectation value with respect to the law of Poisson point process with intensity parameter λ. That is, the probability of the number of points of ω in a set $A \in \mathcal{B}(\mathbb{R}^d)$, $N_\omega(A)$ is given by Poisson distribution

$$P\left[N_\omega(A) = k\right] = \frac{(\lambda|A|)^k}{k!}e^{-\lambda|A|}.$$

We consider the random potential which, for a fixed ω, has the following form:

$$V(x) := V(x,\omega) = \int_{\mathbb{R}^d} \varphi(x-y)\omega(dy), \tag{2.21}$$

for $x \in \mathbb{R}^d$. Let $r > 0$ and define $Q_r := [-r,r]^d$. We impose the following assumptions on the function φ:

- $\varphi \ge 0$ and φ is a continuous even function with compact support, i.e. there exists $R > 0$ such that $\varphi(x-y) = 0$ if $|x-y| > R$,
- $\varphi(0) > \varphi(x)$ for all $x \in \mathbb{R}^d$, and $\varphi(0) > 0$.

Denote by $\hat{\varphi}(x) = \max_{y\in Q_1} |\varphi(x-y)|$. It is clear that $\hat{\varphi}(y) = 0$ for all $y \in Q^c_{R+1}$, and $\hat{\varphi}(x) < \varphi(0)$ for all $x \in Q_{R+1}$. The following lemma describes the behaviour of $V(x)$ in the unit cube:

Lemma 2.1. *For all* $\alpha > 1$

$$P\left(\max_{x\in Q_1} |V(x)| > \alpha\right) \le e^{-C\alpha\log\alpha} \tag{2.22}$$

where $C > 0$ *is independent of* α.

Proof. Using Chernoff's inequality and the explicit form of the moment generating function for Poisson measure, we obtain for all $\beta > 0$

$$\begin{aligned} P\left(\max_{x\in Q_1} |V(x)| > \alpha\right) &\le P\left(\int_{\mathbb{R}^d} \hat{\varphi}(x)\omega(dx) > \alpha\right) \\ &\le e^{-\alpha\beta}\left\langle e^{\beta\int_{\mathbb{R}^d}\hat{\varphi}(x)\omega(dx)}\right\rangle \end{aligned}$$

but this is equal to

$$\begin{aligned} e^{-\alpha\beta}\exp\left[\lambda\int_{\mathbb{R}^d}\left(e^{\beta\hat{\varphi}(x)}-1\right)dx\right] &= e^{-\alpha\beta}\exp\left[\lambda\int_{\mathbb{R}^d}\int_0^\beta e^{r\hat{\varphi}(x)}\hat{\varphi}(x)dr dx\right] \\ &\le e^{-\alpha\beta}\exp\left[\lambda\beta e^{\beta\varphi(0)}\int_{\mathbb{R}^d}\hat{\varphi}(x)dx\right] \\ &\le \exp\left[-\alpha\beta+\lambda\beta e^{\beta\varphi(0)}\varphi(0)|Q_{R+1}|\right]. \end{aligned}$$

now let $\beta = \frac{\log\alpha}{\varphi(0)}$, then the last line is equal to

$$\exp\left[-\frac{\alpha\log\alpha}{\varphi(0)}\left(1+\varphi(0)\lambda|Q_{R+1}|\right)\right], \tag{2.23}$$

thus we have obtained (2.22). □

Using this lemma we are able to show that the potential V is bounded almost surely in the cube of size L.

Lemma 2.2. *With probability one, we have*

$$\max_{x\in Q_L} |V(x)| \le C\frac{\log L}{\log\log L} \tag{2.24}$$

for L *large enough.*

Proof. Let $\alpha > 1$ and $L_n = 2^n$ for some $n \in \mathbb{N}$. Using the translation invariance of the Poisson measure and Lemma 2.1 we obtain

$$\begin{aligned} P\left(\max_{x\in Q_{L_n}} |V(x)| > \alpha\right) &\leq 2^{nd} P\left(\max_{x\in Q_1} |V(x)| > \alpha\right) \\ &\leq 2^{nd} e^{-C\alpha\log\alpha}. \end{aligned}$$

Let now $\alpha = \frac{n}{\log n}$. The application of the first Borel-Cantelli lemma for the sequence L_n will give us the required result. □

We proceed now to the contact process in random environment.

2.4 Contact model with random establishment

2.4.1 Introduction

In this section the contact model with random spatial offspring distribution is studied. The birth rate in this case will be random, and has the following form: $b(x,\gamma) = \sum_{y\in\gamma} a(x-y)b(x,\omega)$. We assume that ω is a (fixed) realisation of the Poisson point process. This additional factor has an influence on the location for the newly created individuals, i.e. the presence of many points of ω in the area makes it unattractive and the probability that a "parent" will send its offspring to that area is relatively small. Thus the heuristic pre-generator of the contact process with random establishment has the following form:

$$L_{\omega,b}F(\gamma) = \sum_{x\in\gamma} D_x^- F(\gamma) + \sum_{y\in\gamma} \varkappa \int_{\mathbb{R}^d} a^+(x-y)b(x,\omega)D_x^+ F(\gamma)dx, \tag{2.25}$$

and the random function b has the following form

$$b(x,\omega) = e^{-\langle b^+(x-\cdot),\omega\rangle} = \exp\left(-\sum_{y\in\omega} b^+(x-y)\right)$$

for a non-negative function b^+ with compact support.

2.4.2 Existence and regularity

The process can be constructed similarly to the classical case, as a branching process with killing. Notice also, that for each $\beta > 0$ and $m \in \mathbb{N}$, there exists a constant C_m such that

$$\left|\sup_{\gamma\in O^m} L_{\omega,b}\mathbb{V}_\beta(\gamma)\right| \leq C_m < \infty. \tag{2.26}$$

To see this, take $\gamma \in O^m$ and calculate:

$$\begin{aligned}
|L_{\omega,b}\mathbb{L}_\beta(\gamma)| = & \Bigg| \sum_{x\in\gamma} \left(\sum_{y\in\gamma\setminus x} e_\beta(y) - \sum_{y\in\gamma} e_\beta(y) \right) \\
& + \varkappa \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y) b(x,\omega) \left(\sum_{y\in\gamma\cup x} e_\beta(y) - \sum_{y\in\gamma} e_\beta(y) \right) dx \Bigg| \\
= & \left| - \sum_{x\in\gamma} e_\beta(x) + \varkappa \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y) e_\beta(x) b(x,\omega) dx \right| \\
\leq & \mathbb{L}_\beta(\gamma) + C_1 \sum_{y\in\gamma} e_\beta(y) \int_{B_R(0)} b(x+y,\omega) dx \\
\leq & (1 + C_1|B_R(0)|)\, \mathbb{L}_\beta(\gamma) < (1 + C_1|B_R(0)|)\, m
\end{aligned}$$

where $|B_R(0)|$ denotes the volume of the ball $B_R(0)$ in $\mathbb{R}^d$. Similarly

$$\begin{aligned}
&|L\mathbb{E}_\beta(\gamma)| = \left| - \sum_{x\in\gamma} \sum_{y\in\gamma\setminus x} \Psi_\beta(x,y) + \varkappa \sum_{y\in\gamma} \sum_{z\in\gamma} \int_{\mathbb{R}^d} a^+(x-y) b(x,\omega) \Psi_\beta(x,z) dx \right| \\
&\leq \mathbb{E}_\beta(\gamma) + \varkappa \sum_{y\in\gamma} \sum_{z\in\gamma} e_\beta(z) \int_{\mathbb{R}^d} a^+(x-y) e_\beta(x) b(x,\omega) \frac{1+|x-z|}{|x-z|} \mathbb{1}_{\{x\neq z\}} dx \\
&= \mathbb{E}_\beta(\gamma) + \varkappa \sum_{y\in\gamma} \sum_{z\in\gamma} e_\beta(y) e_\beta(z) \int_{\mathbb{R}^d} a^+(x) e_\beta(x) b(x-y,\omega) \frac{1+|x-y-z|}{|x-y-z|} dx \\
&\leq \mathbb{E}_\beta(\gamma) + A\varkappa \sum_{y\in\gamma} \sum_{z\in\gamma} e_\beta(y) e_\beta(z) \left(\int_{B_R(0)} b(x,\omega) \frac{1}{|x-y-z|} dx + |B_R(0)| \right) \\
&\leq \mathbb{E}_\beta(\gamma) + C_1 \left(2\mathbb{E}_\beta(\gamma) + \mathbb{L}_\beta(\gamma) \right) \leq C_2 \mathbb{V}_\beta(\gamma) \leq C_2 m.
\end{aligned}$$

Thus, the function $\mathbb{V}_\beta$ is in the domain of the extended generator $L^m_{\omega,b}$ for every m (see Corollary 2.1). Moreover, it follows from the latter calculation, that for each $\gamma \in O^m$ we have

$$L_{\omega,b}\mathbb{V}_\beta(\gamma) \leq C\mathbb{V}_\beta(\gamma)$$

for some $C > 0$. This together with Theorem 2.2 gives the regularity of the process associated with the operator $L_{\omega,b}$.

2.4.3 The symbol of the generator

We now apply the scheme introduced in Chapter 1 to derive the corresponding evolution of correlation functions for the considered system. In the first step, we calculate the symbol $\hat{L}_{\omega,b}$ of the operator (2.25). Recall that $\hat{L}_{\omega,b} := K^{-1}L_{\omega,b}K$, then:

$$\hat{L}_{\omega,b}G(\eta) = -|\eta|G(\eta) + \varkappa\int_{\mathbb{R}^d} b(x,\omega)\sum_{y\in\eta} a^+(x-y)G(\eta\setminus y\cup x)dx + \varkappa\int_{\mathbb{R}^d} b(x,\omega)\sum_{y\in\eta} a^+(x-y)G(\eta\cup x)dx. \tag{2.27}$$

To show the latter fact, we will calculate the symbol directly. The definition of $\hat{L}_{\omega,b}$ yields: $\hat{L}G(\eta) = I_1(\eta) + I_2(\eta)$ where

$$\begin{aligned}
I_1(\eta) &:= K^{-1}\left(\sum_{x\in\cdot}[KG(\cdot\setminus x) - KG(\cdot)]\right)(\eta)\\
&= K^{-1}\left(-\sum_{x\in\cdot}\sum_{\xi\subset\cdot\setminus x} G(\xi\cup x)\right)(\eta)\\
&= -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta}\sum_{\xi\in\zeta\setminus x} G(\xi\cup x)(\eta)\\
&= -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta} KG(\cdot\cup x)(\zeta\setminus x)\\
&= -\sum_{x\in\eta}\sum_{\zeta\subset\eta\setminus x}(-1)^{|\eta\setminus(\zeta\cup x)|}KG(\cdot\cup x)(\zeta)\\
&= -\sum_{x\in\eta} K^{-1}\left(KG(\cdot\cup x)(\eta\setminus x)\right) = -|\eta|G(\eta),
\end{aligned}$$

and

$$\begin{aligned}
I_2(\eta) :=& K^{-1}\left(\varkappa\int_{\mathbb{R}^d}\sum_{y\in\cdot} a^+(x-y)b(x,\omega)\left[KG(\cdot\cup x) - KG(\cdot)\right]dx\right)(\eta)\\
=&\varkappa\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\int_{\mathbb{R}^d} b(x,\omega)\sum_{y\in\zeta} a^+(x-y)\sum_{\xi\subset\zeta} G(\xi\cup x)dx\\
=&\varkappa\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\int_{\mathbb{R}^d} b(x,\omega)K(a^+(x-\cdot)\mathbb{1}_{|\cdot|=1})(\zeta)\cdot KG(\cdot\cup x)(\zeta)dx\\
=&\varkappa\int_{\mathbb{R}^d} b(x,\omega)[a^+(x-\cdot)\mathbb{1}_{|\cdot|=1}\star G(\cdot\cup x)](\zeta)dx\\
=&\varkappa\int_{\mathbb{R}^d} b(x,\omega)\sum_{(\eta_1,\eta_2,\eta_3)\in\mathcal{P}^3_\emptyset(\eta)} a^+(x-\cdot)\mathbb{1}_{|\cdot|=1}(\eta_1\cup\eta_2)G(\cdot\cup x)(\eta_2\cup\eta_3)dx.
\end{aligned}$$

The latter sum has only two non-zero terms, that is: $\eta_1 = \emptyset$ and $|\eta_2| = 1$ or $\eta_2 = \emptyset$ and $|\eta_1| = 1$ thus we obtain

$$I_2(\eta) = \varkappa \int_{\mathbb{R}^d} b(x,\omega) \sum_{y\in\eta} a^+(x-y) G(\eta \setminus y \cup x) dx \\ + \varkappa \int_{\mathbb{R}^d} b(x,\omega) \sum_{y\in\eta} a^+(x-y) G(\eta \cup x) dx,$$

and hence the symbol $\hat{L}_{\omega,b}$ is given as above.

2.4.4 The adjoint operator

We will show now, that the adjoint operator $\hat{L}^*_{\omega,b}$ w.r.t. the relation (1.40) has the following form:

$$\hat{L}^* k(\eta) = -|\eta| k(\eta) + \varkappa \sum_{x\in\eta} k(\eta\setminus x) \sum_{y\in\eta\setminus x} a^+(x-y) b(x,\omega) \\ + \varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a^+(x-y) b(x,\omega) k((\eta\setminus x)\cup y) dy. \tag{2.28}$$

It is easy to see that,

$$\int_{\Gamma_0} I_1(\eta) k(\eta) \lambda(d\eta) = -\int_{\Gamma_0} |\eta| G(\eta) k(\eta) \lambda(d\eta) \\ = \int_{\Gamma_0} G(\eta) \left(-|\eta| k(\eta)\right) \lambda(d\eta).$$

This identity gives us the first part of the formula. Now

$$\int_{\Gamma_0} I_2(\eta) k(\eta) \lambda(d\eta) = J_1 + J_2$$

where

$$J_1 := \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y) b(x,\omega) G((\eta\setminus y)\cup x) dx k(\eta) \lambda(d\eta),$$
$$J_2 := \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta\cup x) \sum_{y\in\eta} a^+(x-y) b(x,\omega) dx k(\eta) \lambda(d\eta).$$

We will rewrite two expressions above using Lemma 1.1. We start with J_1:

$$\begin{aligned}
J_1 &= \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} k(\eta \cup y) \left(\int_{\mathbb{R}^d} a^+(x-y) b(x,\omega) G(\eta \cup x) dx \right) dy \lambda(d\eta) \\
&= \varkappa \int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta \cup x) \left(\int_{\mathbb{R}^d} a^+(x-y) b(x,\omega) k(\eta \cup y) dy \right) dx \lambda(d\eta) \\
&= \int_{\Gamma_0} G(\eta) \left(\varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a^+(x-y) b(x,\omega) k((\eta \setminus x) \cup y) dy \right) \lambda(d\eta)
\end{aligned}$$

similarly in the case of J_2 after applying Lemma 1.1 we get

$$J_2 = \int_{\Gamma_0} G(\eta) \left(\varkappa \sum_{x\in\eta} k(\eta \setminus x) \sum_{y\in\eta\setminus x} a^+(x-y) b(x,\omega) \right) \lambda(d\eta)$$

which gives us precisely (2.28).

2.4.5 Time evolution of the correlation functions

Having calculated the symbol $\hat{L}_{\omega,b}$, we proceed to the evolution equations associated with this operator. The starting point for our consideration will be the following equation:

$$\begin{aligned}
\frac{\partial k_t}{\partial t}(\eta) = \hat{L}^*_{\omega,b} k_t(\eta) = &- |\eta| k_t(\eta) + \varkappa \sum_{x\in\eta} k_t(\eta \setminus x) \sum_{y\in\eta\setminus x} a^+(x-y) b(x,\omega) \\
&+ \varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a^+(x-y) b(x,\omega) k_t((\eta \setminus x) \cup y) dy.
\end{aligned}$$

Below we will give formal meaning to this equation. First note, that one can rewrite the latter equation component-wise taking into account the structure of correlation functions:

$$\begin{aligned}
\frac{\partial k_t^{(n)}}{\partial t}(x_1,\dots,x_n) &= -n k_t^{(n)}(x_1,\dots,x_n) \\
&\quad + \varkappa \sum_{i=1}^n k_t^{(n-1)}(x_1,\dots,\check{x}_i,\dots,x_n) \sum_{j:j\neq i} a^+(x_i - x_j) b(x_i,\omega) \\
&\quad + \sum_{i=1}^n \varkappa \int_{\mathbb{R}^d} a^+(x_i - y) b(x_i,\omega) k_t^{(n)}(x_1,\dots,x_{i-1},y,\dots,x_n) dy \\
&= \hat{L}^*_n k_t^{(n)}(x_1,\dots,x_n) + f_t^{(n)}(x_1,\dots,x_n)
\end{aligned}$$

where, for $n \geq 1$

$$\hat{L}_n^* k_t^{(n)}(x_1, \ldots, x_n) = -n k_t^{(n)}(x_1, \ldots, x_n) + \sum_{i=1}^{n} \varkappa \int_{\mathbb{R}^d} a^+(x_i - y) b(x_i, \omega) k_t^{(n)}(x_1, \ldots, x_{i-1}, y, \ldots, x_n) dy$$

and

$$f_t^{(n)}(x_1, \ldots, x_n) = \varkappa \sum_{i=1}^{n} k_t^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:j\neq i} a^+(x_i - x_j) b(x_i, \omega),$$

with $f_t^{(1)} \equiv 0$. Recall, that every function $k^{(n)}$ is symmetric and defined on $\widetilde{(\mathbb{R}^d)^n}$ or, by construction on $(\mathbb{R}^d)^n$ putting $k^{(n)} \equiv 0$ on $\widetilde{(\mathbb{R}^d)^n}^c$. Hence, for each $n \in \mathbb{N}$, we consider a linear Cauchy problem in some Banach space X_n given by

$$\begin{cases} \frac{\partial k_t^{(n)}}{\partial t}(x_1, \ldots, x_n) = \hat{L}_n^* k_t^{(n)}(x_1, \ldots, x_n) + f_t^{(n)}(x_1, \ldots, x_n), \\ k_t^{(n)}(x_1, \ldots, x_n)|_{t=0} = k_0^{(n)}(x_1, \ldots, x_n). \end{cases} \tag{2.29}$$

Notice also, that we can rewrite operator $\hat{L}_n^*$ in the following way

$$\hat{L}_n^* k_t^{(n)}(x_1, \ldots, x_n) = \left(\varkappa \langle a^+ \rangle \sum_{i=1}^{n} b(x_i, \omega) - n \right) k_t^{(n)}(x_1, \ldots, x_n) + \sum_{i=1}^{n} L_\omega^i k_t^{(n)}(x_1, \ldots, x_n),$$

where

$$L_\omega^i k^{(n)}(x_1, \ldots, x_n) = \varkappa b(x_i, \omega) \times \int_{\mathbb{R}^d} a^+(x_i - y) \left[k^{(n)}(x_1, \ldots, x_{i-1}, y, \ldots, x_n) - k^{(n)}(x_1, \ldots, x_n) \right] dy$$

is a generator of Markov jump process with random jump intensity.
Now set $X_n := B_b((\mathbb{R}^d)^n)$, the space of real valued bounded functions on $\mathbb{R}^d$ with the supremum norm. Notice that for $k \in X_n$:

$$\left| \hat{L}_n^* k(x_1, \ldots, x_n) \right| = \left| - n k(x_1, \ldots, x_n) + \varkappa \sum_{i=1}^{n} b(x_i, \omega) \int_{\mathbb{R}^d} a^+(x_i - y) k(x_1, \ldots, x_{i-1}, y, \ldots, x_n) dy \right| \leq n \left(1 + \varkappa \langle a \rangle\right) \|k\|_{X^n} < \infty.$$

From this we can conclude:

Proposition 2.2. *The operator $\hat{L}_n^*$ is bounded in X_n. Moreover, L_ω^i is a generator of a contraction semigroup in X_n.*

Thus, we also have the following

Proposition 2.3. *For each $n \in \mathbb{N}$ the solution to the Cauchy problem* (2.29) *in the space X_n is given by:*

$$k_t^{(n)}(x_1, \dots, x_n) = e^{t\hat{L}_n^*} k_0(x_1, \dots, x_n) + \int_0^t e^{(t-s)\hat{L}_n^*} f_s^{(n)}(x_1, \dots, x_n) ds. \quad (2.30)$$

Proof. The statement follows from the classical theory, see e.g. [IK02]. □

We can *a priori* estimate the solution (2.30). Let

$$\varkappa(t) := \max \left\{ 1, \varkappa, \varkappa e^{-t\left(\varkappa \langle a^+ \rangle - 1\right)} \right\}.$$

Proposition 2.4. *Let $a^+ \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be even, positive function and recall that $A := \|a^+\|_\infty$. Let $C > 0$ be a constant independent of n such that*

$$k_0^{(n)}(x_1, \dots, x_n) \le n! C^n \quad (2.31)$$

for all $(x_1, \dots, x_n) \in \left(\mathbb{R}^d\right)^n$. Then for all $n \in \mathbb{N}$ and $t \ge 0$ the following inequality holds:

$$k_t^{(n)}(x_1, \dots, x_n) \le \varkappa(t)^n (1 + A)^n e^{n\left(\langle a^+ \rangle \varkappa - 1\right)t} (C + t)^n n! \quad (2.32)$$

for all $(x_1, \dots, x_n) \in \left(\mathbb{R}^d\right)^n$.

Proof. We will argue by induction. For $n = 1$ we have:

$$k_t^{(1)}(x) = e^{t(\varkappa\langle a^+ \rangle - 1)} e^{tL_\omega^1} k_0^{(1)}(x) \le e^{t(\varkappa\langle a^+ \rangle - 1)} C$$

and (2.32) is satisfied. Now assume that (2.32) holds for $n - 1$, then using (2.30) we obtain:

$$\begin{aligned} k_t^{(n)}(x_1, \dots, x_n) = {} & e^{t\left(\varkappa\langle a^+ \rangle \sum_{i=1}^n b(x_i, \omega) - n\right)} \left(\bigotimes_{i=1}^n e^{tL_\omega^i} \right) k_0^{(n)}(x_1, \dots, x_n) \\ & + \varkappa e^{t\left(\varkappa\langle a^+ \rangle \sum_{i=1}^n b(x_i, \omega) - n\right)} \int_0^t e^{-s\left(\varkappa\langle a^+ \rangle \sum_{i=1}^n b(x_i, \omega) - n\right)} \\ & \times \left(\bigotimes_{i=1}^n e^{(t-s)L_\omega^i} \right) \sum_{i=1}^n b(x_i, \omega) k_s^{(n-1)}(x_1, \dots, \check{x}_i, \dots, x_n) \\ & \times \sum_{j: j \ne i} a^+(x_i - x_j) ds. \end{aligned}$$

Using (2.31) and (2.32) for $n-1$ we obtain

$$\begin{aligned}
k_t^{(n)}(x_1,\dots,x_n) \le{}& e^{t\left(\varkappa\langle a^+\rangle-1\right)n}C^n n! \\
&+ \varkappa e^{t\left(\varkappa\langle a^+\rangle-1\right)n}\int_0^t e^{-s\left(\varkappa\langle a^+\rangle-1\right)n} n(n-1) \\
&\times \left[\varkappa(s)^{n-1}(1+A)^n e^{(n-1)\left(\varkappa\langle a^+\rangle-1\right)s}(n-1)!(C+s)^{n-1}\right] ds \\
\le{}& e^{t\left(\varkappa\langle a^+\rangle-1\right)n}C^n n! \\
&+ \varkappa e^{t\left(\varkappa\langle a^+\rangle-1\right)n} n(1+A)^n\varkappa(t)^{n-1}n! \\
&\times \int_0^t e^{-s\left(\varkappa\langle a^+\rangle-1\right)}(C+s)^{n-1}ds.
\end{aligned}$$

Note that for $0\le s\le t$ we have $1\le \varkappa e^{-s\left(\varkappa\langle a^+\rangle-1\right)}\le\varkappa(s)\le\varkappa(t)$ thus

$$k_t^{(n)}(x_1,\dots,x_n) \le e^{t\left(\varkappa\langle a^+\rangle-1\right)n}(1+A)^n\varkappa(t)^n n!\left(C^n + n\int_0^t (C+s)^{n-1}ds\right)$$

and the assertion is proved. □

Let the initial condition $\left(k_0^{(n)}\right)_{n\in\mathbb{N}}$ in (2.29) be a system of correlation functions, i.e., there exists a measure $\mu_0\in\mathcal{M}^1_{fm}(\Gamma)$ (locally absolutely continuous w.r.t. the Poisson measure on Γ) the correlation functions of which are exactly $\left(k_0^{(n)}\right)_{n\in\mathbb{N}}$ (see e.g. [KK02] and Section 1.2). Natural question arises: does the time evolution of $\left(k_0^{(n)}\right)_{n\in\mathbb{N}}$ preserves this property? In other words, is $\left(k_t^{(n)}\right)_{n\in\mathbb{N}}$ a system of correlation functions of some measure $\mu_t\in\mathcal{M}^1_{fm}(\Gamma)$ for each $t>0$? One of the possible ways to assure that the solution of (2.29) is a correlation function is the result of A. Lenard ([Len73]). Namely, let $\rho\in\mathcal{M}(\Gamma_0)$ be a locally finite and normalized (i.e. $\rho(\Gamma_0^{(0)})=1$) measure with corresponding system of correlation functions $\left(k_t^{(n)}\right)_{n\in\mathbb{N}}$. Then ρ is a correlation function of some measure $\mu\in\mathcal{M}^1_{fm}(\Gamma)$ if the following conditions are satisfied:

(P) For any $G\in B_{bs}(\Gamma_0)$ such that $KG\ge 0$:

$$\int_{\Gamma_0} G(\eta)\rho(d\eta)\ge 0. \tag{2.33}$$

(M) For any bounded set $\Lambda \subset \mathbb{R}^d$ and $j \geq 0$:

$$\sum_{n=0}^{\infty} \left(m_{n+j}^{\Lambda}\right)^{-\frac{1}{n}} = +\infty, \tag{2.34}$$

where

$$m_n^{\Lambda} := \frac{1}{n!} \int_{\Lambda} \cdots \int_{\Lambda} k^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

Remark 2.2. *Condition **(P)** is called **Lenard positivity** and provides the existence of the measure μ above, whereas **(M)** is called the **moment growth condition** and ensures the uniqueness of such measure (see e.g. [Len73, KK02]).*

We have also the following remark (cf. Proposition 2.4):

Remark 2.3. *Condition **(M)** is satisfied in particular for a system of functions $\left(k^{(n)}\right)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ the following inequality*

$$k^{(n)}(x_1, \ldots, x_n) \leq C^n n!$$

holds for some constant $C > 0$ independent of n, and all $(x_1, \ldots, x_n) \in \left(\mathbb{R}^d\right)^n$.

We will now show that the latter conditions are fulfilled in the case of considered model.

Lemma 2.3. *Let a^+ be as in Proposition 2.4. Then the solution of (2.29) satisfies condition **(P)**.*

Proof. By the definition of correlation measure we have to show the following for all $G \in B_{bs}(\Gamma_0)$ with $KG \geq 0$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} G^{(n)}(x_1, \ldots, x_n) k_t^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n \geq 0 \tag{2.35}$$

Let $\mu_0 \in \mathcal{M}_{fm}^1(\Gamma)$ have the correlation measure which is absolutely continuous w.r.t. the Lebesgue-Poisson measure (that is the case if for example μ_0 is locally absolutely continuous w.r.t. the Poisson measure on Γ) and such that its correlation functions $\left(k^{(n)}\right)_{n\in\mathbb{N}}$ are bounded.
Define for $n \in \mathbb{N}$, $\beta > 0$:

$$F^{(n)}(\gamma) := \sum_{\{x_1, \ldots, x_n\} \subset \gamma} e^{-\beta|x_1|} \cdots e^{-\beta|x_n|}$$

for $\gamma \in \Gamma$ with $|\gamma| \geq n$, and $F^{(n)}(\gamma) = 0$ otherwise. Note that for the measure μ_0 as above we have

$$\int_\Gamma F^{(n)}(\gamma)\mu_0(d\gamma) = \frac{1}{n!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} e^{-\beta|x_1|}\cdots e^{-\beta|x_n|}k^{(n)}(x_1,\ldots,x_n)dx_1\cdots dx_n < \infty. \tag{2.36}$$

As it was shown before in this section, there exists a Markov process X_t^γ associated to the generator $L_{\omega,b}$ and such that $X_t^\gamma \in \Gamma_\beta$ almost surely for all $t \geq 0$. For $n \geq 2$ we have

$$\begin{aligned}
L_{\omega,b}F^n(\gamma) =& \sum_{x\in\gamma}\left(F^{(n)}(\gamma\setminus x) - F^{(n)}(\gamma)\right)\\
&+ \varkappa\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^+(y-x)b(x,\omega)\left(F^{(n)}(\gamma\cup x) - F^{(n)}(\gamma)\right)dx\\
=& - F^{(n)}(\gamma)\\
&+ \varkappa\sum_{y\in\gamma}\sum_{\{x_1,\ldots,x_n\}\subset\gamma} e^{-\beta|x_1|}\cdots e^{-\beta|x_1|}\int_{\mathbb{R}^d} a^+(y-x)b(x,\omega)e^{-\beta|x|}dx\\
\leq& - F^{(n)}(\gamma) + \varkappa\langle a^+\rangle\sum_{y\in\gamma}\sum_{\{x_1,\ldots,x_n\}\subset\gamma} e^{-\beta|y|}e^{-\beta|x_1|}\cdots e^{-\beta|x_1|}\\
\leq& \left(\varkappa\langle a^+\rangle - 1\right)F^{(n)}(\gamma) + \varkappa\langle a^+\rangle F^{(n-1)}(\gamma).
\end{aligned}$$

Now let

$$\mathbb{F}^{(N)}(\gamma) := \sum_{n=1}^N F^{(n)}(\gamma),$$

and from the previous calculation follows that there exists $C > 0$ such that

$$L_{\omega,b}\mathbb{F}^{(N)}(\gamma) \leq C\mathbb{F}^{(N)}(\gamma). \tag{2.37}$$

As in the construction of the classical contact model ([KS06]), the Markov property together with the Gronwall inequality give us then:

$$\mathbb{E}\left[\mathbb{F}^{(N)}(X_t^\gamma)\right] \leq \mathbb{F}^{(N)}(\gamma)e^{Ct}. \tag{2.38}$$

Recall from Section 1.5 that the evolution of the initial measure (state) μ_0 associated to the process X_t^γ is given by the dual operator $L_{\omega,b}^*$ (with respect to duality (1.35)), i.e.:

$$\begin{aligned}
\frac{\partial}{\partial t}\mu_t &= L_{\omega,b}^*\mu_t,\\
\mu_t|_{t=0} &= \mu_0.
\end{aligned}$$

Let $(\mu_t)_{t\geq 0}$ denote the evolution of μ_0 given by the equation above. If for every $t \geq 0$, $\mu_t \in \mathcal{M}^1_{fm}(\Gamma)$, then the Markov evolution of correlation measures ρ_t corresponding to μ_t exists in $\mathcal{M}_{lf}(\Gamma_0)$ and (2.35) will follow trivially because of the Markov property of the corresponding semigroup.
For the function $\mathbb{F}^{(N)}$ and a bounded set $\Lambda \subset \mathbb{R}^d$ we have (see [KKP08, Str09]):

$$\begin{aligned}\mathbb{F}^{(N)}(\gamma) \geq \mathbb{F}^{(N)}(\gamma_\Lambda) &= \sum_{n=1}^{N} \sum_{\{x_1,\ldots,x_n\}\subset\gamma_\Lambda} e^{-\beta|x_1|}\cdots e^{-\beta|x_n|} \\ &\geq \sum_{n=1}^{N} \left(\min_{x\in\Lambda} e^{-\beta|x|}\right)^n \binom{|\gamma_\Lambda|}{n}.\end{aligned}$$

In the case $|\gamma_\Lambda| \leq N$ we obtain

$$\mathbb{F}^{(N)}(\gamma) \geq \left(1 + \min_{x\in\Lambda} e^{-\beta|x|}\right)^{|\gamma_\Lambda|} - 1,$$

and for $|\gamma_\Lambda| > N$, using Sterling's formula:

$$\mathbb{F}^{(N)}(\gamma) \geq \left(\min_{x\in\Lambda} e^{-\beta|x|}\right)^N C_N|\gamma_\Lambda|^N$$

with $0 < C_N < \frac{1}{N^N}$. Thus

$$\begin{aligned}\int_\Gamma |\gamma_\Lambda|^N \mu_t(d\gamma) &= \int_{|\gamma_\Lambda|\leq N} |\gamma_\Lambda|^N \mu_t(d\gamma) + \int_{|\gamma_\Lambda|>N} |\gamma_\Lambda|^N \mu_t(d\gamma) \\ &\leq N^N + \left(\min_{x\in\Lambda} e^{-\beta|x|}\right)^{-N} C_N^{-1} \int_\Gamma \mathbb{E}\left[\mathbb{F}^{(N)}(X_t^\gamma)\right]\mu_0(d\gamma) \\ &\leq N^N + \left(\min_{x\in\Lambda} e^{-\beta|x|}\right)^{-N} C_N^{-1} e^{Ct} \int_\Gamma \mathbb{F}^{(N)}(\gamma)\mu_0(d\gamma) \\ &< +\infty\end{aligned}$$

because (2.36). Hence for all $t \geq 0$, $\mu_t \in \mathcal{M}^1_{fm}(\Gamma)$ and taking into account previous considerations, (2.35) holds. □

2.4.6 1st and 2nd correlation functions

We will now derive the equations for the first and the second correlation functions. The first correlation function is the solution to the following equation:

$$\frac{\partial k_t^{(1)}}{\partial t}(x) = L^1_\omega k_t^{(1)}(x) - V(x,\omega)k_t^{(1)}(x)$$

where

$$V(x,\omega)f(x) = (1 - \varkappa b(x,\omega))\, f(x).$$

For the second correlation function, equation is as follows

$$\begin{aligned}\frac{\partial k_t^{(2)}}{\partial t}(x_1,x_2) = {} & (2 - \varkappa b(x_1,\omega) - \varkappa b(x_2,\omega))\, k_t^{(2)}(x_1,x_2)\\ & + L_\omega^1 k_t^{(2)}(x_1,x_2) + L_\omega^2 k_t^{(2)}(x_1,x_2)\\ & + \varkappa k_t^{(1)}(x_2)a^+(x_1-x_2)b(x_1,\omega)\\ & + \varkappa k_t^{(1)}(x_1)a^+(x_2-x_1)b(x_2,\omega).\end{aligned}$$

2.5 Contact process with random fecundity

2.5.1 Introduction

Let us now consider another modification of the classical contact process in continuum. Namely, we allow the rate of offspring production to be random, i.e. we replace the constant parameter $\varkappa$ in the second part of the operator (2.13) with a random function $x \mapsto \varkappa(x,\omega)$. This can be considered as the random influence on the fecundity of members of population – the offspring production rate can change depending on the presence of random factors.

Taking into account the discussion above, the mechanism of evolution for the contact process with random fecundity is described by the following heuristic formula:

$$L_{\omega,\varkappa}F(\gamma) = \sum_{x\in\gamma} D_x^- F(\gamma) + \sum_{y\in\gamma} \varkappa(y,\omega) \int_{\mathbb{R}^d} a^+(x-y)D_x^+F(\gamma)dx, \qquad (2.39)$$

where

$$\varkappa(y,\omega) := \exp\left(-\sum_{x\in\omega}\phi(x-y)\right) \qquad (2.40)$$

for a positive function ϕ. Recall, that ω is a realization of Poisson point process. Furthermore we assume, that the function ϕ has bounded support, so that $\varkappa(y,\omega) > 0$ for all $y \in \mathbb{R}^d$ and almost all ω.

2.5.2 Construction, regularity

The construction of the process can be carried out as in the classical case, just notice that for all $x \in \mathbb{R}^d$ we have $\varkappa(x,\omega) \le 1$ and thus the birth part of $L_{\omega,\varkappa}$ is bounded by the birth part of the operator L defined in (2.13). In order to show the regularity of it, let us show that condition (2.5) is fulfilled and thus the function $\mathbb{V}_\beta$ is in the domain of the extended generator $L_{\omega,\varkappa}$. We start with the estimation:

$$\begin{aligned} |L_{\omega,\varkappa}\mathbb{L}_\beta(\gamma)| = & \Bigg| \sum_{x\in\gamma} \left(\sum_{y\in\gamma\setminus x} e_\beta(y) - \sum_{y\in\gamma} e_\beta(y) \right) \\ & + \sum_{y\in\gamma} \varkappa(y,\omega) \int_{\mathbb{R}^d} a^+(x-y) \left(\sum_{y\in\gamma\cup x} e_\beta(y) - \sum_{y\in\gamma} e_\beta(y) \right) dx \Bigg| \\ = & \left| -\sum_{x\in\gamma} e_\beta(x) + \sum_{y\in\gamma} \varkappa(y,\omega) \int_{\mathbb{R}^d} a^+(x-y) e_\beta(x) dx \right| \\ \leq & \ \mathbb{L}_\beta(\gamma) + C_1 \sum_{y\in\gamma} e_\beta(y) \\ \leq & \ (1+C_1)\mathbb{L}_\beta(\gamma) \end{aligned}$$

and

$$\begin{aligned} |L_{\omega,\varkappa}\mathbb{E}_\beta(\gamma)| = & \Bigg| - \sum_{x\in\gamma} \sum_{y\in\gamma\setminus x} \Psi_\beta(x,y) \\ & + \sum_{y\in\gamma} \varkappa(y,\omega) \sum_{z\in\gamma} \int_{\mathbb{R}^d} a^+(x-y) \Psi_\beta(x,z) dx \Bigg| \\ & \leq 2\mathbb{E}_\beta(\gamma) + C_1 \sum_{y\in\gamma} \sum_{z\in\gamma} e_\beta(y) e_\beta(z) \leq C_2 \mathbb{V}_\beta(\gamma). \end{aligned}$$

Hence we have obtained, that for every $m \in \mathbb{N}$ there exists a constant $C_m > 0$ such that:

$$\sup_{\gamma\in O^m} |L_{\omega,\varkappa}\mathbb{V}_\beta(\gamma)| \leq C_m < \infty. \tag{2.41}$$

Moreover, we have the following inequality

$$L_{\omega,\varkappa}\mathbb{V}_\beta(\gamma) \leq C\mathbb{V}_\beta(\gamma) \tag{2.42}$$

for some $C > 0$ and all $\gamma \in O^m$ for $m \in \mathbb{N}$. Thus, using Theorem 2.2 we obtain the following

Corollary 2.3. *The contact process associated with $L_{\omega,\varkappa}$ is regular.*

2.5.3 The symbol of the generator

The direct calculation yields the following form of the symbol:

$$\hat{L}_{\omega,\varkappa}G(\eta) = -|\eta|G(\eta) + \int_{\mathbb{R}^d}\sum_{y\in\eta}\varkappa(y,\omega)a^+(x-y)G(\eta\setminus y\cup x)dx + \int_{\mathbb{R}^d}\sum_{y\in\eta}\varkappa(y,\omega)a^+(x-y)G(\eta\cup x)dx$$

As in the previous case we have

$$\hat{L}_{\omega,\varkappa}G(\eta) = I_1(\eta) + I_2(\eta), \tag{2.43}$$

and because the first part has been calculated in the previous section, we will only show the second part. Notice that

$$\sum_{y\in\eta}\varkappa(y,\omega)a^+(x-y) = K(\varkappa(\cdot,\omega)a^+(x-\cdot)\mathbb{1}_{|\cdot|=1})(\eta),$$

and thus we obtain

$$\begin{aligned} I_2(\eta) &= K^{-1}\left(\int_{\mathbb{R}^d}\sum_{y\in\cdot}\varkappa(y,\omega)a^+(x-y)\sum_{\xi\subset\cdot}G(\xi\cup x)dx\right)(\eta) \\ &= \sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\int_{\mathbb{R}^d}\sum_{y\in\zeta}\varkappa(y,\omega)a^+(x-y)\sum_{\xi\subset\zeta}G(\xi\cup x)dx. \end{aligned}$$

Next, using the definition of the K-transform the latter is equal to

$$\begin{aligned} &\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\int_{\mathbb{R}^d}K(\varkappa(\cdot,\omega)a^+(x-\cdot)\mathbb{1}_{|\cdot|=1})(\zeta)\cdot KG(\cdot\cup x)(\zeta)dx \\ &\quad= \int_{\mathbb{R}^d}[\varkappa(\cdot,\omega)a^+(x-\cdot)\mathbb{1}_{|\cdot|=1}\star G(\cdot\cup x)](\zeta)dx \\ &\quad= \int_{\mathbb{R}^d}\sum_{(\eta_1,\eta_2,\eta_3)\in\mathcal{P}^3_\emptyset(\eta)}\varkappa(\cdot,\omega)a^+(x-\cdot)\mathbb{1}_{|\cdot|=1}(\eta_1\cup\eta_2)G(\cdot\cup x)(\eta_2\cup\eta_3)dx \end{aligned}$$

and because this sum has only two non-zero cases, i.e. $\eta_1=\emptyset$ and $|\eta_2|=1$ or $\eta_2=\emptyset$ and $|\eta_1|=1$ we obtain

$$I_2(\eta) = \int_{\mathbb{R}^d}\sum_{y\in\eta}\varkappa(y,\omega)a^+(x-y)G(\eta\setminus y\cup x)dx + \int_{\mathbb{R}^d}\sum_{y\in\eta}\varkappa(y,\omega)a^+(x-y)G(\eta\cup x)dx,$$

and hence the symbol $\hat{L}_{\omega,\varkappa}$ is given as above.

2.5.4 The adjoint operator

The calculation of the adjoint operator $\hat{L}^*_{\omega,\varkappa}$ gives the following form of the adjoint operator:

$$\hat{L}^*_{\omega,\varkappa}k(\eta) = -|\eta|k(\eta) + \sum_{x\in\eta} k(\eta\setminus x) \sum_{y\in\eta\setminus x} \varkappa(y,\omega)a^+(x-y) + \sum_{x\in\eta}\int_{\mathbb{R}^d} \varkappa(y,\omega)a^+(x-y)k((\eta\setminus x)\cup y)dy \tag{2.44}$$

Because the death part of the generator $L_{\omega,\varkappa}$ is identical to the death part of $L_{\omega,b}$ we will focus on the birth part of $L_{\omega,\varkappa}$. Recall $I_2(\eta)$ from (2.43). The definition of the adjoint generator yields:

$$\int_{\Gamma_0} I_2(\eta)k(\eta)\lambda(d\eta) = J_1 + J_2$$

where

$$J_1 := \int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{y\in\eta} \varkappa(y,\omega)a^+(x-y)G((\eta\setminus y)\cup x)dxk(\eta)\lambda(d\eta),$$

and

$$J_2 := \int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\sum_{y\in\eta} \varkappa(y,\omega)a^+(x-y)dxk(\eta)\lambda(d\eta).$$

Using Minlos Lemma we obtain

$$\begin{aligned} J_1 &= \int_{\Gamma_0}\int_{\mathbb{R}^d} k(\eta\cup y)\left(\int_{\mathbb{R}^d} \varkappa(y,\omega)a^+(x-y)G(\eta\cup x)dx\right)dy\lambda(d\eta) \\ &= \int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\left(\int_{\mathbb{R}^d} \varkappa(y,\omega)a^+(x-y)k(\eta\cup y)dy\right)dx\lambda(d\eta) \\ &= \int_{\Gamma_0} G(\eta)\left(\sum_{x\in\eta}\int_{\mathbb{R}^d} \varkappa(y,\omega)a^+(x-y)k((\eta\setminus x)\cup y)dx\right)\lambda(d\eta). \end{aligned}$$

And similarly in the case of J_2 we get

$$J_2 = \int_{\Gamma_0} G(\eta)\left(\sum_{x\in\eta} k(\eta\setminus x)\sum_{y\in\eta\setminus x} \varkappa(y,\omega)a^+(x-y)\right)\lambda(d\eta)$$

which gives us the formula (2.44).

2.5.5 Time evolution of the correlation function

We proceed to the evolution equations associated with the operator $\hat{L}^*_{\omega,\varkappa}$. For a function $k : \Gamma_0 \times \mathbb{R}^d_+ \to 0$, $k_t(\eta) := k(t,\eta)$ it has the following form:

$$\frac{\partial k_t}{\partial t}(\eta) = \hat{L}^*_{\omega,\varkappa} k_t(\eta) = -\,|\eta| k_t(\eta) + \sum_{x\in\eta} k_t(\eta\setminus x) \sum_{y\in\eta\setminus x} \varkappa(y,\omega) a^+(x-y)$$
$$+ \sum_{x\in\eta} \int_{\mathbb{R}^d} \varkappa(y,\omega) a^+(x-y) k_t((\eta\setminus x)\cup y) dy. \tag{2.45}$$

Using the hierarchical structure of the functions on Γ_0, we can rewrite the equation above component-wisely as a system of equations. For $n \in \mathbb{N}$, the n-th component of the correlation function $k_t(\eta)$ satisfies the following equation:

$$\begin{aligned}\frac{\partial k_t^{(n)}}{\partial t}(x_1,\dots,x_n) &= -n k_t^{(n)}(x_1,\dots,x_n) \\ &\quad + \sum_{i=1}^n k_t^{(n-1)}(x_1,\dots,\check{x}_i,\dots,x_n) \sum_{j:j\neq i} \varkappa(x_j,\omega) a^+(x_i-x_j) \\ &\quad + \sum_{i=1}^n \int_{\mathbb{R}^d} \varkappa(y,\omega) a^+(x_i-y) k_t^{(n)}(x_1,\dots,x_{i-1},y,\dots,x_n) dy \\ &= \hat{L}^*_n k_t^{(n)}(x_1,\dots,x_n) + f_t^{(n)}(x_1,\dots,x_n)\end{aligned}$$

where, for $n \geq 1$

$$\begin{aligned}\hat{L}^*_n k_t^{(n)}(x_1,\dots,x_n) &= -n k_t^{(n)}(x_1,\dots,x_n) \\ &\quad + \sum_{i=1}^n \int_{\mathbb{R}^d} \varkappa(y,\omega) a^+(x_i-y) k_t^{(n)}(x_1,\dots,x_{i-1},y,\dots,x_n) dy\end{aligned}$$

with $f_t^{(1)} \equiv 0$, and

$$f_t^{(n)}(x_1,\dots,x_n) = \sum_{i=1}^n k_t^{(n-1)}(x_1,\dots,\check{x}_i,\dots,x_n) \sum_{j:j\neq i} \varkappa(x_j,\omega) a^+(x_i-x_j),$$

for $n \geq 2$. To give meaning to (2.45) for each $n \in \mathbb{N}$ we consider a linear Cauchy problem given by

$$\begin{cases} \frac{\partial}{\partial t} k_t^{(n)}(x_1,\dots,x_n) = \hat{L}^*_n k_t^{(n)}(x_1,\dots,x_n) + f_t^{(n)}(x_1,\dots,x_n) \\ k_t^{(n)}(x_1,\dots,x_n)|_{t=0} = k_0^{(n)}(x_1,\dots,x_n), \end{cases} \tag{2.46}$$

in some Banach space, which will be defined later.

Notice also, that we can rewrite operator $\hat{L}_n^*$ in the following way

$$\begin{aligned}\hat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) &= \left(\sum_{i=1}^n \int_{\mathbb{R}^d} \varkappa(y,\omega) a^+(x_i-y)dy - n\right) k_t^{(n)}(x_1,\ldots,x_n)\\ &\quad + \sum_{i=1}^n L_\omega^i k_t^{(n)}(x_1,\ldots,x_n),\end{aligned}$$

where

$$\begin{aligned}&L_\omega^i k^{(n)}(x_1,\ldots,x_n)\\ &= \int_{\mathbb{R}^d} a^+(x_i-y)\left[k^{(n)}(x_1,\ldots,x_{i-1},y,\ldots,x_n) - k^{(n)}(x_1,\ldots,x_n)\right]\varkappa(y,\omega)dy\end{aligned}$$

is a generator of Markov jump process. To see that, let

$$\lambda_{a^+,\varkappa}(x,\omega) := \int_{\mathbb{R}^d} a^+(x-y)\varkappa(y,\omega)dy.$$

Then $0 < \lambda_{a^+,\varkappa}(x,\omega) \le 1$ for all $x \in \mathbb{R}^d$ and a.a. ω. Moreover

$$\begin{aligned}&L_\omega^i k^{(n)}(x_1,\ldots,x_n) = \lambda_{a^+,\varkappa}(x_i,\omega)\\ &\times \int_{\mathbb{R}^d} \frac{a^+(x_i-y)\varkappa(y,\omega)}{\lambda_{a^+,\varkappa}(x_i,\omega)}\left[k^{(n)}(\ldots,x_{i-1},y,\ldots) - k^{(n)}(x_1,\ldots,x_n)\right]dy.\end{aligned}$$

Now set $X_n := B_b\left((\mathbb{R}^d)^n\right)$ and notice that for $k \in X_n$:

$$\begin{aligned}\left|\hat{L}_n^* k(x_1,\ldots,x_n)\right| &= \Bigg| - nk(x_1,\ldots,x_n)\\ &\quad + \sum_{i=1}^n \int_{\mathbb{R}^d} \varkappa(y,\omega)a^+(x_i-y)k(x_1,\ldots,x_{i-1},y,\ldots,x_n)dy\Bigg|\\ &\le n\left(1+\langle a\rangle\right)\|k\|_{X^n} < \infty.\end{aligned}$$

Then from the classical result we can derive:

Proposition 2.5. *For each $n \in \mathbb{N}$ the solution to the Cauchy problem* (2.46) *in the space X_n is given by:*

$$k_t^{(n)}(x_1,\ldots,x_n) = e^{t\hat{L}_n^*}k_0(x_1,\ldots,x_n) + \int_0^t e^{(t-s)\hat{L}_n^*} f_s^{(n)}(x_1,\ldots,x_n)ds. \quad (2.47)$$

Proof. See the proof of Proposition 2.3. □

Similarly to the continuous contact model in continuum considered by Kondratiev and Skorokhod in [KS06] we can introduce some *a priori* estimates for the solution (2.47) as in [KKP08]. Recall that $A := \sup_{x\in\mathbb{R}^d} |a^+(x)|$.

Proposition 2.6. *Assume that there exists a constant $C > 0$ such that for all $(x_1, \ldots, x_n) \in \mathbb{R}^{dn}$ and all $n \in \mathbb{N}$ we have:*

$$k_0^{(n)}(x_1, \ldots, x_n) \le n! C^n. \tag{2.48}$$

Then for every $t \ge 0$,

$$k_t^{(n)}(x_1, \ldots, x_n) \le \left(1 \vee e^{-\left(\langle a^+\rangle -1\right)t}\right)^n (1+A)^n e^{n\left(\langle a^+\rangle -1\right)t} (C+t)^n\, n! \tag{2.49}$$

for $(x_1, \ldots, x_n) \in \left(\mathbb{R}^d\right)^n$ and all $n \in \mathbb{N}$.

Proof. The proof uses induction over the number of particles n. Let us first calculate the recurrent bound on the function $k_t^{(n)}$ assuming (2.48). Note that for all $(x_1, \ldots, x_n) \in \left(\mathbb{R}^d\right)^n$:

$$\sum_{i=1}^n \int_{\mathbb{R}^d} a^+(x_i - y)\varkappa(y,\omega)dy - n \le \sum_{i=1}^n \int_{\mathbb{R}^d} a^+(x_i - y)dy - n \le n\left(\langle a^+\rangle - 1\right).$$

Thus, using (2.47), we obtain:

$$\begin{aligned} k_t^{(n)}(x_1, \ldots, x_n) &\le e^{t\left(\langle a^+\rangle -1\right)n} n! C^n \\ &\quad + e^{t\left(\langle a^+\rangle -1\right)n} \int_0^t e^{-s\left(\langle a^+\rangle -1\right)n} \\ &\quad \times \left[\sum_{i=1}^n k_s^{(n-1)}(x_1, \ldots, x_n) \sum_{j: i\ne j} \varkappa(x_j,\omega) a^+(x_i - x_j)\right] ds \\ &\le e^{t\left(\langle a^+\rangle -1\right)n} n! C^n \\ &\quad + e^{t\left(\langle a^+\rangle -1\right)n} (n-1)(1+A) \\ &\quad \times \int_0^t e^{-s\left(\langle a^+\rangle -1\right)n} \sum_{i=1}^n k_s^{(n-1)}(x_1, \ldots, x_n) ds. \end{aligned}$$

Let now $n = 1$. Then from the latter calculation we get

$$k_t^{(1)}(x_1, \ldots, x_n) \le e^{t\left(\langle a^+\rangle -1\right)} C$$

hence the inequality (2.49) is trivially satisfied.

Assume now, that (2.49) holds for $n-1$. Then

$$\begin{aligned}
k_t^{(n)}(x_1,\dots,x_n) \le\ & e^{t\left(\langle a^+\rangle -1\right)n} n! C^n \\
& +\ e^{t\left(\langle a^+\rangle -1\right)n}(n-1)(1+A) \\
& \times \int_0^t e^{-s\left(\langle a^+\rangle -1\right)n} n \left(1 \vee e^{-\left(\langle a^+\rangle -1\right)t}\right)^{n-1} \\
& \times (1+A)^{n-1} e^{(n-1)\left(\langle a^+\rangle -1\right)s} \left(C+s\right)^{n-1}(n-1)! ds \\
\le\ & e^{t\left(\langle a^+\rangle -1\right)n} n! C^n \\
& +\ e^{t\left(\langle a^+\rangle -1\right)n} n(1+A)^n n! \left(1 \vee e^{-\left(\langle a^+\rangle -1\right)s}\right)^{n-1} \\
& \times \int_0^t e^{-s\left(\langle a^+\rangle -1\right)} \left(C+s\right)^{n-1} ds
\end{aligned}$$

but for $0 \le s \le t$ we have

$$e^{-s\left(\langle a^+\rangle -1\right)} \le \left(1 \vee e^{-\left(\langle a^+\rangle -1\right)t}\right)$$

thus

$$\begin{aligned}
k_t^{(n)}(x_1,\dots,x_n) \le\ & e^{t\left(\langle a^+\rangle -1\right)n} n! \left(1 \vee e^{-\left(\langle a^+\rangle -1\right)s}\right)^{n} (1+A)^n \\
& \times \left[C^n + n\int_0^t \left(C+s\right)^{n-1} ds\right]
\end{aligned}$$

and the expression in last bracket is equal to $\left(C+t\right)^n$.
This concludes the proof. □

2.5.6 Preservation of correlation functions

Recall from the previous Section that even if the initial condition in (2.46) is a correlation function for some measure $\mu_0 \in \mathcal{M}^1_{fm}(\Gamma)$ then the preservation of this property in time is a non-trivial question. Fortunately also in the case of the contact process with random fecundity something akin to the Lemma 2.3 holds and we can use Lenard's criterion to show that the evolution given by $\hat{L}^*_{\omega,\varkappa}$ preserves the correlation functions. Because the proof is almost identical to the proof of Lemma 2.3, we will omit most of it here, leaving just the calculations which differ from those mentioned above. Recall conditions **(M)** and **(P)** defined in (2.33) and (2.34) resp. Then condition **(M)** is satisfied if (2.48) holds (c.f. Remark 2.3). It remains to show that the condition **(P)** is satisfied in the case of considered model. As we mentioned before, we have the following

Lemma 2.4. *For $a^+ \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ the solution of* (2.46) *satisfies condition **(P)**.*

Proof. The proof of this Lemma is analogous to the one of Lemma 2.7 hence we present here only the parts which are directly connected to this specific model.
Recall the function defined for $n \in \mathbb{N}$, $\beta > 0$ as:

$$F^{(n)}(\gamma) := \sum_{\{x_1,\dots,x_n\}\subset\gamma} e^{-\beta|x_1|}\cdots e^{-\beta|x_n|}$$

for $\gamma \in \Gamma$ with $|\gamma| \geq n$, and $F^{(n)}(\gamma) = 0$ otherwise. And note that for $n \geq 2$ we have

$$\begin{aligned}
L_{\omega,\varkappa}F^n(\gamma) =& \sum_{x\in\gamma}\left(F^{(n)}(\gamma\setminus x) - F^{(n)}(\gamma)\right)\\
&+ \sum_{y\in\gamma}\varkappa(y,\omega)\int_{\mathbb{R}^d} a^+(y-x)\left(F^{(n)}(\gamma\cup x) - F^{(n)}(\gamma)\right)dx\\
=& - F^{(n)}(\gamma)\\
&+ \sum_{y\in\gamma}\varkappa(y,\omega)\sum_{\{x_1,\dots,x_n\}\subset\gamma} e^{-\beta|x_1|}\cdots e^{-\beta|x_1|}\int_{\mathbb{R}^d} a^+(y-x)e^{-\beta|x|}dx\\
\leq& - F^{(n)}(\gamma) + \langle a^+\rangle\sum_{y\in\gamma}\sum_{\{x_1,\dots,x_n\}\subset\gamma} e^{-\beta|y|}e^{-\beta|x_1|}\cdots e^{-\beta|x_1|}\\
\leq& \left(\langle a^+\rangle - 1\right)F^{(n)}(\gamma) + \langle a^+\rangle F^{(n-1)}(\gamma).
\end{aligned}$$

This gives an estimate for the function

$$\mathbb{F}^{(N)}(\gamma) := \sum_{n=1}^{N} F^{(n)}(\gamma),$$

that is there exists $C > 0$ such that

$$L_{\omega,\varkappa}\mathbb{F}^{(N)}(\gamma) \leq C\mathbb{F}^{(N)}(\gamma).$$

The Markov property together with the Gronwall inequality give us then

$$\mathbb{E}\left[\mathbb{F}^{(N)}(X_t^\gamma)\right] \leq \mathbb{F}^{(N)}(\gamma)e^{Ct}. \tag{2.50}$$

The rest of the proof is the same as in Lemma 2.3 hence we omit it here. □

2.5.7 Equations for the first and second correlation functions

The evolution of the first and second correlation functions has the following form:

$$\frac{\partial k_t^{(1)}}{\partial t}(x) = L_\omega^1 k^{(1)}(x) - V(x,\omega)k_t^{(1)}(x) \tag{2.51}$$

where

$$V(x,\omega)f(x) = \left(1 - \left\langle \varkappa(\cdot,\omega)a^+(x-\cdot)\right\rangle\right) f(x), \tag{2.52}$$

and

$$\begin{aligned}\frac{\partial k_t^{(2)}}{\partial t}(x_1,x_2) = {} & \left(2 - \left(\left\langle \varkappa(\cdot,\omega)a^+(x_1-\cdot)\right\rangle + \left\langle \varkappa(\cdot,\omega)a^+(x_2-\cdot)\right\rangle\right)\right) k_t^{(2)}(x_1,x_2) \\ & + L_\omega^1 k_t^{(2)}(x_1,x_2) + L_\omega^2 k_t^{(2)}(x_1,x_2) \\ & + k_t^{(1)}(x_2)\varkappa(x_2,\omega)a^+(x_1-x_2) \\ & + k_t^{(1)}(x_1)\varkappa(x_1,\omega)a^+(x_2-x_1).\end{aligned}$$

The analysis of those equations and their long time asymptotic are missing and pose non-trivial open problems.

2.6 Contact process with random mortality

2.6.1 Introduction

The third model studied in this chapter is the contact process with random mortality rate. The random influence of the environment contributes to the constant rate of death (equal to 1) in terms of a random positive function $m(x,\omega)$ of the form $m(x,\omega) := \sum_{x'\in\omega}\varphi(x-x')$. Unfortunately the methods used before cannot be applied to construct the process and until now we are not able to show rigorously that the process exists in Γ (or in the subset of Γ). The technical reason for that is the unboundedness of the death rate. We proceed now to the description of the model.

Let ω be a realization of the homogeneous Poisson point process on $\mathbb{R}^d$ with the intensity measure being Lebesgue measure on $\mathbb{R}^d$ and let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be non-negative, even and continuous function such that

$$\int_{\mathbb{R}^d} \varphi(x)dx < \infty. \tag{2.53}$$

Define now for $x \in \mathbb{R}^d$

$$m(x,\omega) = \int_{\mathbb{R}^d} \varphi(x-y)\omega(dy). \tag{2.54}$$

Condition (2.53) assures, that $m(x,\omega)$ is well defined for each $x \in \mathbb{R}^d$ and almost all ω (see e.g. [Kal10]). The mechanism of evolution for the contact process with random mortality is then given by

$$L_{\omega,m}F(\gamma) = \sum_{x\in\gamma} m(x,\omega)D_x^- F(\gamma) + \varkappa \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y)D_x^+ F(\gamma)dx. \tag{2.55}$$

We assume that the function a^+ is continuous and has bounded support, denote with A the maximum of the function a^+.

As we mentioned before, the construction and the regularity of this model are still open problems. On the other hand, we are able to derive the equation for the evolution of correlation functions in some Banach space. The remaining part of this chapter is devoted to the latter problem. We will start with the calculation of the symbol of generator L_ω. Recall that $\hat{L}_\omega := K^{-1}L_\omega K$, then we have the following

Lemma 2.5. *The symbol of the generator L_ω is given by*

$$\begin{aligned}\hat{L}_{\omega,m}G(\eta) = &-\sum_{x\in\eta} m(x,\omega)G(\eta) + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y)G(\eta\setminus y\cup x)dx \\ &+ \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y)G(\eta\cup x)dx.\end{aligned} \tag{2.56}$$

Proof. From direct calculation follows that $\hat{L}_\omega G(\eta) = I_1(\eta) + I_2(\eta)$, where

$$\begin{aligned}
I_1(\eta) &:= K^{-1}\left(\sum_{x\in\cdot} m(x,\omega)\left[KG(\cdot\setminus x) - KG(\cdot)\right]\right)(\eta) \\
&= K^{-1}\left(-\sum_{x\in\cdot} m(x,\omega)\sum_{\xi\subset\cdot\setminus x} G(\xi\cup x)\right)(\eta) \\
&= -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta}\sum_{\xi\in\zeta\setminus x} m(x,\omega)G(\xi\cup x)(\eta) \\
&= -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta} m(x,\omega)KG(\cdot\cup x)(\zeta\setminus x) \\
&= -\sum_{x\in\eta} m(x,\omega)\sum_{\zeta\subset\eta\setminus x}(-1)^{|\eta\setminus(\zeta\cup x)|}KG(\cdot\cup x)(\zeta) = -\sum_{x\in\eta} m(x,\omega)G(\eta),
\end{aligned}$$

$$\begin{aligned}
I_2(\eta) &:= K^{-1}\left(\varkappa\int_{\mathbb{R}^d}\sum_{y\in\cdot}a^+(x-y)\left[KG(\cdot\cup x)-KG(\cdot)\right]dx\right)(\eta) \\
&= \varkappa\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\int_{\mathbb{R}^d}\sum_{y\in\zeta}a^+(x-y)\sum_{\xi\subset\zeta}G(\xi\cup x)dx \\
&= \varkappa\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\int_{\mathbb{R}^d}K(a^+(x-\cdot)\mathbb{1}_{|\cdot|=1})(\zeta)\cdot KG(\cdot\cup x)(\zeta)dx \\
&= \varkappa\int_{\mathbb{R}^d}\sum_{(\eta_1,\eta_2,\eta_3)\in\mathcal{P}^3_\emptyset(\eta)}a^+(x-\cdot)\mathbb{1}_{|\cdot|=1}(\eta_1\cup\eta_2)G(\cdot\cup x)(\eta_2\cup\eta_3)dx.
\end{aligned}$$

Notice, that the latter sum has only two non-zero cases, i.e. $\eta_1=\emptyset$ and $|\eta_2|=1$ or $\eta_2=\emptyset$ and $|\eta_1|=1$ thus we obtain

$$I_2(\eta)=\varkappa\int_{\mathbb{R}^d}\sum_{y\in\eta}a^+(x-y)G(\eta\setminus y\cup x)dx+\varkappa\int_{\mathbb{R}^d}\sum_{y\in\eta}a^+(x-y)G(\eta\cup x)dx,$$

and hence the symbol $\hat{L}_{\omega,m}$ is given as above. □

2.6.2 The adjoint operator

Recall from Section 1.5 the duality relation:

$$\int_{\Gamma_0}\hat{L}G(\eta)k(\eta)\lambda(d\eta)=\int_{\Gamma_0}G(\eta)\hat{L}^*k(\eta)\lambda(d\eta)$$

with respect to which we define the adjoint of the operator $L_{\omega,m}$. In our case, the adjoint operator $\hat{L}^*_{\omega,m}$ has the form:

$$\begin{aligned}
\hat{L}^*_{\omega,m}k(\eta) &= -\sum_{x\in\eta}m(x,\omega)k(\eta)+\varkappa\sum_{x\in\eta}k(\eta\setminus x)\sum_{y\in\eta\setminus x}a^+(x-y) \\
&\quad+\varkappa\sum_{x\in\eta}\int_{\mathbb{R}^d}a^+(x-y)k((\eta\setminus x)\cup y)dy.
\end{aligned}\tag{2.57}$$

Consider first

$$\begin{aligned}
\int_{\Gamma_0}I_1(\eta)k(\eta)\lambda(d\eta) &= -\int_{\Gamma_0}\sum_{x\in\eta}m(x,\omega)G(\eta)k(\eta)\lambda(d\eta) \\
&= \int_{\Gamma_0}G(\eta)\left(-\sum_{x\in\eta}m(x,\omega)k(\eta)\right)\lambda(d\eta)
\end{aligned}$$

which gives us the first part of the formula. Now

$$\int_{\Gamma_0} I_2(\eta)k(\eta)\lambda(d\eta) = J_1 + J_2$$

where

$$\begin{aligned} J_1 &:= \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{y\in\eta} a^+(x-y)G((\eta\setminus y)\cup x)dxk(\eta)\lambda(d\eta), \\ J_2 &:= \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\sum_{y\in\eta} a^+(x-y)dxk(\eta)\lambda(d\eta). \end{aligned}$$

We will calculate two expressions above using Lemma 1.1. We start with J_1:

$$\begin{aligned} J_1 &= \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} k(\eta\cup y)\left(\int_{\mathbb{R}^d} a^+(x-y)G(\eta\cup x)dx\right)dy\lambda(d\eta) \\ &= \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\left(\int_{\mathbb{R}^d} a^+(x-y)k(\eta\cup y)dy\right)dx\lambda(d\eta) \\ &= \varkappa \int_{\Gamma_0} G(\eta)\left(\sum_{x\in\eta}\int_{\mathbb{R}^d} a^+(x-y)k((\eta\setminus x)\cup y)dx\right)\lambda(d\eta) \end{aligned}$$

and similarly in the case of J_2:

$$J_2 = \varkappa \int_{\Gamma_0} G(\eta)\left(\sum_{x\in\eta} k(\eta\setminus x)\sum_{y\in\eta\setminus x} a^+(x-y)\right)\lambda(d\eta)$$

which gives us the formula (2.57).

2.6.3 Time evolution of the correlation function

We proceed to the evolution equations associated with the operator $\hat{L}^*_\omega$. It has the following form

$$\begin{aligned} \frac{\partial k_t}{\partial t}(\eta) = \hat{L}^*_\omega k_t(\eta) = -\sum_{x\in\eta} m(x,\omega)k_t(\eta) + \varkappa\sum_{x\in\eta} k_t(\eta\setminus x)\sum_{y\in\eta\setminus x} a^+(x-y) \\ + \varkappa\sum_{x\in\eta}\int_{\mathbb{R}^d} a^+(x-y)k_t((\eta\setminus x)\cup y)dy \end{aligned} \tag{2.58}$$

Knowing the hierarchical structure of the functions on Γ_0 we can rewrite the equation above component-wisely as a system of equations, namely for $n \in \mathbb{N}$:

$$\begin{aligned}\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) = & -\sum_{i=1}^n m(x_i,\omega)k_t^{(n)}(x_1,\ldots,x_n)\\ & + \varkappa\sum_{i=1}^n k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n)\sum_{j:j\neq i} a^+(x_i-x_j)\\ & + \varkappa\sum_{i=1}^n \int_{\mathbb{R}^d} a^+(x_i-y)k_t^{(n)}(x_1,\ldots,x_{i-1},y,\ldots,x_n)dy\\ = & \hat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) + f_t^{(n)}(x_1,\ldots,x_n)\end{aligned}$$

where, for $n \geq 1$

$$\begin{aligned}\hat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) = & -\sum_{i=1}^n m(x_i,\omega)k_t^{(n)}(x_1,\ldots,x_n)\\ & + \varkappa\sum_{i=1}^n \int_{\mathbb{R}^d} a^+(x_i-y)k_t^{(n)}(x_1,\ldots,x_{i-1},y,\ldots,x_n)dy\end{aligned}$$

and

$$f_t^{(n)}(x_1,\ldots,x_n) = \varkappa\sum_{i=1}^n k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n)\sum_{j:j\neq i} a^+(x_i-x_j),\ n\geq 2$$

with $f_t^{(1)} \equiv 0$. So, for each $n \in \mathbb{N}$, we consider a linear Cauchy problem in some Banach space X_n:

$$\begin{aligned}\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) &= \hat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) + f_t^{(n)}(x_1,\ldots,x_n) \qquad (2.59)\\ k_t^{(n)}(x_1,\ldots,x_n)|_{t=0} &:= k_0^{(n)}(x_1,\ldots,x_n).\end{aligned}$$

Notice also, that we can rewrite operator $\hat{L}_n^*$ in the following way

$$\begin{aligned}\hat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) = & (n\varkappa - \sum_{i=1}^n m(x_i,\omega))k_t^{(n)}(x_1,\ldots,x_n)\\ & + \sum_{i=1}^n L^i k_t^{(n)}(x_1,\ldots,x_n),\end{aligned}$$

where

$$L^i k^{(n)}(x_1,\dots,x_n) = \varkappa \int_{\mathbb{R}^d} a^+(x_i - y)\left[k^{(n)}(x_1,\dots,x_{i-1},y,\dots,x_n) - k^{(n)}(x_1,\dots,x_n)\right] dy$$

is the pure jump Markov process generator with jumps distribution a^+ and intensity $\varkappa$, see e.g. [EK05, Kal10, GS74].

Define now for $M,\zeta \in \mathbb{R}$ the space $\mathcal{G}(M,\zeta)$ as the space of all operators T on some Banach space X such that:

1. domain $D(L)$ is dense in X,
2. semi-infinite interval $\xi > \zeta$ belongs to the resolvent set of $-T$ and let

$$\left\|(T+\xi)^{-k}\right\| \le M(\xi-\zeta)^{-k}, \quad k = 1, 2\dots. \tag{2.60}$$

Then we have the following (see [Kat95]):

Theorem 2.3. *Let $T \in \mathcal{G}(M,\beta)$ and let $f(t)$ be continuously differentiable for $t \ge 0$ and let $U(t) = e^{tT}$. For any $u_0 \in D(T)$, the $u(t)$ given by*

$$u(t) = U(t)u_0 + \int_0^t U(t-s)f(s)ds, \quad u_0 = u(0)$$

is continuously differentiable for $t \ge 0$ and is a solution of

$$du/dt = Tu + f(t), \quad t > 0$$

with the initial value $u(0) = u_0$.

Thus to obtain the solution to the Cauchy problem (2.59) we should show that the operator $\hat{L}_n^*$ is in $\mathcal{G}(M,\beta)$ for all $n \in \mathbb{N}$ and some $M,\beta \in \mathbb{R}$. From now on, set $X_n := L^1\left((\mathbb{R}^d)^n, dx^{\otimes n}\right)$ and define

$$M_\omega^n k(x_1,\dots,x_n) := -\sum_{i=1}^n m(x_i,\omega)k(x_1,\dots,x_n)$$

together with the domain $D(M_\omega^n) = C_0((\mathbb{R}^d)^n)$. Remark, that for $\psi \in D(M_\omega^n)$ with supp $\psi =: \Lambda$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{(\mathbb{R}^d)^n} |M_\cdot^n \psi(x_1,\dots,x_n)|\, dx_1\cdots dx_n\right] \\
&\qquad \le \mathbb{E}\left[\sum_{i=1}^n \int_{(\mathbb{R}^d)^n} m(x_i,\cdot)|\psi(x_1,\dots,x_n)|dx_1\cdots dx_n\right] \\
&\qquad = \sum_{i=1}^n \int_\Lambda \int_{\mathbb{R}^d} \varphi(x_i-y)dy|\psi(x_1,\dots,x_n)|dx_i\cdots dx_n < \infty.
\end{aligned}$$

This implies, that for $\psi \in D(M^n_\omega)$, $\|M^n_\omega \psi\|_{X_n} < \infty$, for almost all ω. Because $C_0((\mathbb{R}^d)^n)$ is dense in X_n, it is easy to see that $M^n_\omega \in \mathcal{G}(1,0)$. Furthermore, we have for $k \in C_c((\mathbb{R}^d)^n)$:

$$\left\| \varkappa \sum_{i=1}^n \int_{\mathbb{R}^d} a^+(x_i - y) k(x_1, \dots, x_{i-1}, y, \dots, x_n) dy \right\|_{X_n} \leq n\varkappa A |\text{suppa}^+| \|k\|_{X_n}$$

hence the operator $L^*_n - M^n_\omega$ is bounded in X_n. Thus, from the Theorem 2.1 in [Kat95] follows, that $L^*_n \in \mathcal{G}(1, n\varkappa\langle a^+ \rangle)$ and we can apply Theorem 2.3. As result we obtain:

Corollary 2.4. *For each $n \in \mathbb{N}$ and almost all ω, the solution of the Cauchy problem* (2.59) *is given by:*

$$k^{(n)}_t(x_1, \dots, x_n) = e^{t\hat{L}^*_n} k^{(n)}_0(x_1, \dots, x_n) + \int_0^t e^{(t-s)\hat{L}^*_n} f^{(n)}_s(x_1, \dots, x_n) ds. \tag{2.61}$$

Let us write explicitly the equations for the first and second correlation functions. The evolution of the first correlation function describes the evolution of the density for the process. It has the following form:

$$\frac{\partial k^{(1)}_t}{\partial t}(x) = L^1 k^{(1)}_t(x) - V(x,\omega) k^{(1)}_t(x) \tag{2.62}$$

where

$$V(x,\omega) f(x) = (m(x,\omega) - \varkappa) f(x). \tag{2.63}$$

For the second correlation function, equation is as follows:

$$\begin{aligned} \frac{\partial k^{(2)}_t}{\partial t}(x_1, x_2) = {} & (2\varkappa - m(x_1,\omega) - m(x_2,\omega)) k^{(2)}_t(x_1, x_2) \\ & + L^1 k^{(2)}_t(x_1, x_2) + L^2 k^{(2)}_t(x_1, x_2) \\ & + \varkappa \left(k^{(1)}_t(x_2) a^+(x_1 - x_2) + k^{(1)}_t(x_1) a^+(x_2 - x_1) \right). \end{aligned}$$

Chapter 3

Glauber-type dynamics in random environment

This chapter is devoted to the study of the Glauber dynamics in random environment. Let us recall, that Glauber dynamics has been extensively studied in the case of the lattice spin systems (see e.g. [BMP04] and [Lig85]). The Glauber dynamics for such systems can be interpreted as, for example, the spin-flip of particles or, in the case of lattice gas models, the dynamics in which particles randomly appear and disappear from the sites of the lattice. In this work we are concerned with the continuous space models. The Glauber dynamics in our case is a special case of spatial birth and death processes which have Gibbs measure as stationary one. The general form of the Markov pre-generator for such processes is given by:

$$LF(\gamma) = \sum_{x\in\gamma} d(x,\gamma)\,[F(\gamma\setminus x) - F(\gamma)] + \int_{\mathbb{R}^d} b(x,\gamma)\,[F(\gamma\cup x) - F(\gamma)]\,dx. \tag{3.1}$$

The equilibrium Glauber dynamics for the continuous systems was constructed in [KL05]. The existence of the corresponding non-equilibrium dynamics was shown in [KKZ06]. In the present thesis, we consider the modification of the classical Glauber dynamics in which a random field influences the birth and/or the death mechanisms. Depending on the type of influence, we consider two different modifications of the GD and using perturbation theory we show the existence of the corresponding evolutions on the level of quasi-observables.

3.1 Gibbs measures

For the completeness of this thesis we will shortly recall the basic definitions from the theory of Gibbs measures, although that is not the main subject of this chapter. Gibbs measures play a significant role in the studies of Glauber dynamics: in the equilibrium case it is the stationary measure for the process (see for example [KLR07]). For more detailed and general discussion of Gibbs measures we refer to [Geo88, Tek10, Kun99].

Introduce the *pair potential*, that is a Borel measurable, even function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. We assume, that it satisfies the following conditions (cf. [KKZ06]):

(I) (Integrability) For any $\beta > 0$,

$$C(\beta) := \int_{\mathbb{R}^d} |1 - \exp(-\beta\phi(x))| dx < +\infty. \tag{3.2}$$

(P) (Positivity) $\phi(x) > 0$ for all $x \in \mathbb{R}^d$.

Define the Hamiltonian (or the *energy of configuration* $\eta \in \Gamma_0$, $|\eta| \geq 2$) corresponding to the potential ϕ as

$$E^\phi(\eta) := \sum_{\{x,y\}\subset\eta} \phi(x-y). \tag{3.3}$$

Next, the *relative energy of interaction* is defined for $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$ in the following way:

$$E^\phi(x,\gamma) := \begin{cases} \sum_{y\in\gamma} \phi(x-y), & \text{if } \sum_{y\in\gamma} |\phi(x-y)| < \infty, \\ \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.4}$$

Let now ω be fixed realization of a spatial point process (for example Poisson PP in $\mathbb{R}^d$) and assume that function $h : \mathbb{R}^d \to \mathbb{R}$ is non-negative. The influence of the random environment on the dynamics is realized in terms of the interaction energy corresponding to a potential function h, namely:

$$E^h(x,\omega) := \sum_{x'\in\omega} h(x-x') \leq \infty, \quad x \in \mathbb{R}^d. \tag{3.5}$$

For a given set $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ define the Hamiltonian $E^\phi_\Lambda : \Gamma_\Lambda \to \mathbb{R}$ by

$$E^\phi_\Lambda(\eta) = \sum_{\{x,y\}\subset\eta} \phi(x-y),\ \eta \in \Gamma_\Lambda,\ |\eta| \geq 2. \tag{3.6}$$

The *interaction energy* between $\eta \in \Gamma_\Lambda$ and $\bar{\gamma}_{\Lambda^c} := \bar{\gamma} \cap \Lambda^c, \bar{\gamma} \in \Gamma$ is defined as follows:

$$W_\Lambda(\eta|\bar{\gamma}) = \sum_{x \in \eta, y \in \bar{\gamma}_{\Lambda^c}} \phi(x-y). \tag{3.7}$$

Finally, let $\beta > 0$ and

$$E_\Lambda(\eta|\bar{\gamma}) := E_\Lambda(\eta) + W_\Lambda(\eta|\bar{\gamma}). \tag{3.8}$$

The *partition function* is defined by:

$$Z_\Lambda(\bar{\gamma}) := \int_{\Gamma_\Lambda} e^{-\beta E_\Lambda(\eta|\bar{\gamma})} \lambda_z(d\eta). \tag{3.9}$$

Let now $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $\beta > 0$ and $\bar{\gamma} \in \Gamma$. The finite volume Gibbs measure on the space Γ_Λ with the boundary condition $\bar{\gamma}$ is given as:

$$P_{\Lambda,\bar{\gamma}}(d\eta) = \frac{1}{Z_\Lambda(\bar{\gamma})} e^{-\beta E_\Lambda(\eta|\bar{\gamma})} \lambda_z(d\eta),$$

and for $\bar{\gamma} = \emptyset$ we set $P_{\Lambda,\emptyset} =: P_\Lambda$.

Let $\{\pi_\Lambda\}$ denote the specification associated with z and the Hamiltonian E^ϕ, that is:

$$\pi_{\Lambda,\bar{\gamma}}(A) = \int_{A'} P_{\Lambda,\bar{\gamma}}(d\eta) \tag{3.10}$$

where $A' = \{\eta \in \Gamma_\Lambda : \eta \cup \bar{\gamma}_{\Lambda^c} \in A\}$, $A \in \mathcal{B}(\Gamma)$, $\bar{\gamma} \in \Gamma$.

Definition 3.1. *A Gibbs measure for E^ϕ and z is any probability measure μ on Γ, for which the following holds:*

$$\mu(\pi_{\Lambda,\bar{\gamma}}(A)) = \mu(A)$$

for any $A \in \mathcal{B}(\Gamma)$ and every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. This identity is called the Dobrushin-Lanford-Ruelle (DLR) equation (cf. [Geo88]).

We will denote the set of all Gibbs measures corresponding to the potential ϕ, parameter $z > 0$ and inverse temperature $\beta > 0$ by $\mathcal{G}(\phi, z, \beta)$. This set is not empty for any potential ϕ satisfying the conditions (I) and (P) (see [Kun99]).

3.2 Perturbation theory

In this section we will recall some classical results which will be used in the sequel.

For $\omega > 0$, introduce the set $\mathcal{H}(\omega, 0)$ of all closed and densely defined operators T, the resolvent of which contains the sector

$$Sect(\frac{\pi}{2} + \omega) = \left\{ \zeta \in \mathbb{C} : |arg\zeta| < \frac{\pi}{2} + \omega \right\} \setminus \{0\},$$

and such that for any $\varepsilon > 0$

$$\|(T - \zeta \mathbb{1})^{-1}\| \leq \frac{M_\varepsilon}{|\zeta|}, \tag{3.11}$$

and M_ε doesn't depend on ζ.

Remark 3.1. *Any operator $T \in \mathcal{H}(\omega, \theta)$ is a generator of a semigroup $U(t)$ homolorphic in the sector $|\arg t| < \omega$ and (at least) quasi-bounded, i.e.*

$$\|U(t)\| \leq Ae^{\theta t}$$

for some $A > 0$ and $|\arg t| \leq \omega - \varepsilon$, see e.g. [Kat95, KKZ06].

The next theorem turns out to be a useful tool in our considerations:

Theorem 3.1 ([Kat95], Thm. 2.4)**.** *For any $T \in \mathcal{H}(\omega, \theta)$ and $\varepsilon > 0$, there exist positive constants γ, δ with the following properties. If A is relatively bounded with respect to T so that*

$$\|Au\| \leq a\|u\| + b\|Tu\|, \quad u \in D(T) \subset D(A), \tag{3.12}$$

with $a < \delta$, $b < \delta$, then $T + A \in \mathcal{H}(\omega - \varepsilon, \gamma)$. If, in particular, $\theta = 0$ and $a = 0$, then $T + A \in \mathcal{H}(\omega - \varepsilon, 0)$.

3.3 Pre-generators of Glauber type dynamics in RE

We study two types of random modifications of the Glauber dynamics constructed in [KKZ06] in which random birth and/or death rates are allowed.

The Markov pre-generator for the **first model** is as follows:

$$L_\omega^{ext} F(\gamma) := \sum_{x \in \gamma} D_x^- F(\gamma) + \varkappa \int_{\mathbb{R}^d} e^{-\beta E^\phi(x,\gamma)} D_x^+ F(\gamma) e^{-E^h(x,\omega)} dx, \tag{3.13}$$

This model can be interpreted as birth and death dynamics in Γ with the death rate $d(x,\gamma) = 1$, the birth rate $b(x,\gamma) = e^{-E^{\phi}(x,\gamma)}$ and the stationary Gibbs measure with the random intensity $z := z(x,\varkappa,\omega) = \varkappa e^{-E^{h}(x,\omega)}$. Indeed, the coefficients b and d in this case satisfy the *detailed balance condition*, i.e.

$$b(x,\gamma) = d(x,\gamma)e^{-E^{\phi}(x,\gamma)}. \tag{3.14}$$

As result, the symmetrizing (and thus invariant) measure for this process will be a Gibbs state $\mu \in \mathcal{G}(\phi, z, \beta)$ associated with the Lebesgue-Poisson measure with the random intensity measure: $\sigma_{\omega,\varkappa}(dx) := z(x,\varkappa,\omega)dx$, that is

$$\lambda_{\varkappa,\omega} = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_{\omega,\varkappa}^{(n)}.$$

See [Glo81] for the proof of the latter statement.

The second model includes the same kind of random interaction in the death and the birth mechanisms of the system, i.e. the corresponding pre-generator has form:

$$L_\omega F(\gamma) := \sum_{x\in\gamma} e^{-E^{h}(x,\omega)} D_x^- F(\gamma) + \varkappa \int_{\mathbb{R}^d} e^{-\beta E^{\phi}(x,\gamma)} e^{-E^{h}(x,\omega)} D_x^+ F(\gamma) dx, \tag{3.15}$$

We assume, that the function $h : \mathbb{R}^d \to \mathbb{R}$ is measurable and satisfies (2.53) so that $E^h(x,\omega)$ is almost surely finite. Also in this case, condition (3.14) holds and the Gibbs measure (with the Lebesgue-Poisson measure λ_z as reference measure) is again the symmetrizing measure for this model.

We proceed now to the construction of semigroups associated with the symbols of the operators defined above.

3.3.1 Symbols of the generators

Let us start with calculation of the symbols corresponding to the operators L_ω^{ext} and L_ω respectively.

Proposition 3.1. *The symbol of the operator L_ω^{ext} is given by:*

$$\hat{L}_\omega^{ext} G(\eta) = L_0^{ext} G(\eta) + L_1^{ext} G(\eta), \quad G \in B_{bs}(\Gamma_0), \tag{3.16}$$

where

$$L_0^{ext}G(\eta) := -|\eta|G(\eta),$$
$$L_1^{ext}G(\eta) := \varkappa\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x)\prod_{y\in\eta\setminus\xi}\left(e^{-\beta\phi(x-y)}-1\right)e^{-\beta E^{\phi}(x,\xi)}e^{-E^h(x,\omega)}dx.$$

Proof. We will start with the first part:

$$\begin{aligned}
L_0^{ext}G(\eta) &:= K^{-1}\left(\sum_{x\in\cdot}\left[KG(\cdot\setminus x)-KG(\cdot)\right]\right)(\eta)\\
&= K^{-1}\left(-\sum_{x\in\cdot}\sum_{\xi\subset\cdot\setminus x}G(\xi\cup x)\right)(\eta)\\
&= -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta}\sum_{\xi\in\zeta\setminus x}G(\xi\cup x)\\
&= -\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\zeta|}\sum_{x\in\zeta}KG(\cdot\cup x)(\zeta\setminus x)\\
&= -\sum_{x\in\eta}\sum_{\zeta\subset\eta\setminus x}(-1)^{|\eta\setminus(\zeta\cup x)|}KG(\cdot\cup x)(\zeta) = -|\eta|G(\eta).
\end{aligned}$$

As for the second part, note that

$$e^{-\beta E^{\phi}(x,\gamma)} = \prod_{y\in\gamma}e^{-\beta\phi(x-y)} = Ke_{\lambda}\left(e^{-\beta\phi(x-\cdot)}-1\right)(\gamma),$$

then

$$\begin{aligned}
L_1^{ext}G(\eta) &= K^{-1}\left(\varkappa\int_{\mathbb{R}^d}e^{-\beta E^{\phi}(x,\cdot)}\left[KG(\cdot\cup x)-KG(\cdot)\right]e^{-E^h(x,\omega)}dx\right)(\eta)\\
&= \varkappa K^{-1}\left(\int_{\mathbb{R}^d}e^{-\beta E^{\phi}(x,\cdot)}\left[\sum_{\xi\subset\cdot}G(\xi\cup x)\right]e^{-E^h(x,\omega)}dx\right)(\eta)\\
&= \varkappa K^{-1}\left(\int_{\mathbb{R}^d}Ke_{\lambda}(e^{-\beta\phi(x-\cdot)}-1)\cdot KG(\cdot\cup x)e^{-E^h(x,\omega)}dx\right)(\eta)\\
&= \varkappa\int_{\mathbb{R}^d}\left[e_{\lambda}(e^{-\beta\phi(x-\cdot)}-1)\star G(\cdot\cup x)\right](\eta)e^{-E^h(x,\omega)}dx\\
&= \varkappa\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}G(\xi\cup x)\prod_{y\in\eta\setminus\xi}\left(e^{-\beta\phi(x-y)}-1\right)e^{-\beta E^{\phi}(x,\xi)}e^{-E^h(x,\omega)}dx.
\end{aligned}$$

$\square$

With a slight modification of the calculations above we get similar result in the case of L_ω.

Proposition 3.2. *The symbol of the operator L_ω is given as follows:*

$$\hat{L}_\omega G(\eta) = L_0 G(\eta) + L_1 G(\eta), \quad G \in B_{bs}(\Gamma_0),$$

where

$$L_0 G(\eta) := -\left(\sum_{x\in\eta} e^{-E^h(x,\omega)}\right) G(\eta),$$
$$L_1 G(\eta) := \varkappa \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} G(\xi \cup x) \prod_{y\in\eta\setminus\xi} \left(e^{-\beta\phi(x-y)} - 1\right) e^{-\beta E^\phi(x,\xi)} e^{-E^h(x,\omega)} dx.$$

3.4 Construction of the associated semigroups

Using results from Section 3.2 we construct two semigroups corresponding to the operators $\hat{L}^{ext}_\omega$ and $\hat{L}_\omega$ derived above.
First, for $C > 0$, $\beta > 0$ introduce the space

$$\mathcal{L}_{C,\beta} := L^1\left(\Gamma_0, C^{|\eta|} e^{-\beta E(\eta)} \lambda(d\eta)\right) \tag{3.17}$$

together with the norm defined for $G \in \mathcal{L}_{C,\beta}$ by

$$\|G\|_C := \int_{\Gamma_0} |G(\eta)| e^{-\beta E(\eta)} C^{|\eta|} \lambda(d\eta). \tag{3.18}$$

Proposition 3.3. *For any $C > 0$, $\beta > 0$, the operator*

$$L^{ext}_0 G(\eta) = -|\eta| G(\eta)$$

with $D(L^{ext}_0) = \{G \in \mathcal{L}_{C,\beta} : \ |\eta| G(\eta) \in \mathcal{L}_{C,\beta}\}$, is the generator of a contraction semigroup on $\mathcal{L}_{C,\beta}$. Moreover, $L^{ext}_0 \in \mathcal{H}(\theta, 0)$.

Proof. Fix $\omega \in \left(0, \frac{\pi}{2}\right)$ and take $\zeta \in Sect(\frac{\pi}{2} + \omega)$. The operator L^{ext}_0 is densely defined in $\mathcal{L}_{C,\beta}$. On the other hand for $\zeta \in Sect\left(\frac{\pi}{2} + \omega\right)$ we have $||\eta| + \zeta| > 0$ for all $\eta \in \Gamma_0$ and thus the operator

$$\left(L^{ext}_0 - \zeta \mathbb{1}\right)^{-1} G(\eta) = -\frac{1}{|\eta| + \zeta} G(\eta)$$

is well defined for every $G \in \mathcal{L}_C$. It remains to prove the inequality (3.11). Let $Re\zeta \geq 0$ then obviously

$$\left\| \left(L_0^{ext} - \zeta 1\!\!1\right)^{-1} G \right\|_{\mathcal{L}_C} = \left\| \frac{1}{|\eta| + \zeta} |G| \right\|_{\mathcal{L}_C} \leq \frac{1}{|\zeta|} \|G\|_{\mathcal{L}_C}.$$

In the case of $Re\zeta < 0$ notice, that

$$||\eta| + \zeta| \geq |Im\zeta| \geq |\zeta| \cos\omega,$$

thus

$$\left\| \left(L_0^{ext} - \zeta 1\!\!1\right)^{-1} G \right\|_{\mathcal{L}_C} \leq \frac{1}{|\zeta| \cos\omega} \|G\|_{\mathcal{L}_C}.$$

Summarizing, we get

$$\left\| \left(L_0^{ext} - \zeta 1\!\!1\right)^{-1} G \right\|_{\mathcal{L}_C} \leq \max\left\{ \frac{1}{|\zeta|}, \frac{1}{|\zeta| \cos\omega} \right\}.$$

The statement follows now from the Hille-Yosida theorem (see e.g. [Paz83, Kat95]). □

We can prove similar result also for the operator L_0.

Proposition 3.4. *Let any $C > 0$, $\beta > 0$. Then the operator*

$$L_0 G(\eta) = -\left(\sum_{x \in \eta} e^{-E^h(x,\omega)} \right) G(\eta)$$

together with its domain defined as $D(L_0) = \{G \in \mathcal{L}_{C,\beta} : \ L_0 G(\eta) \in \mathcal{L}_{C,\beta}\}$, generates a contraction semigroup on $\mathcal{L}_{C,\beta}$. Furthermore $L_0 \in \mathcal{H}(\theta, 0)$.

Proof. See the proof of Proposition 3.3. □

In order to apply Theorem 3.1 we need to show that operators L_1^{ext} and L_1 are relatively bounded with respect to L_0^{ext} and L_0, respectively. Indeed, we have the following results:

Proposition 3.5. *Consider the operator*

$$L_1^{ext} G(\eta) = \varkappa \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) \prod_{y \in \eta \setminus \xi} \left(e^{-\beta\phi(x-y)} - 1\right) e^{-\beta E^\phi(x,\xi)} e^{-E^h(x,\omega)} dx$$

with the domain $D(L_1^{ext}) := D(L_0^{ext})$. Then, for all $\varkappa, C, \beta > 0$ and $G \in D(L_0)$ the following inequality holds:

$$\|L_1^{ext} G\|_C \leq \varkappa C^{-1} e^{CC(\beta)} \|L_0^{ext} G\|_C. \tag{3.19}$$

Proof. Define
$$\mathcal{K}(x,\eta) := \prod_{y\in\eta}\left(e^{-\beta\phi(x-y)}-1\right)$$
for $x\in\mathbb{R}^d$, $\eta\in\Gamma_0$. Then for $G\in D(L_1^{ext})$ the norm $\|L_1^{ext}G\|_C$ is equal to

$$\varkappa\int_{\Gamma_0}\left|\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}G(\xi\cup x)\mathcal{K}(x,\eta\setminus\xi)e^{-\beta E^{\phi}(x,\xi)}e^{-E^h(x,\omega)}dx\right|C^{|\eta|}e^{-\beta E^{\phi}(\eta)}\lambda(d\eta).$$

This can be estimated from above by

$$\varkappa\int_{\Gamma_0}\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}|G(\xi\cup x)|\mathcal{K}(x,\eta\setminus\xi)e^{-\beta E^{\phi}(x,\xi)}e^{-E^h(x,\omega)}dxC^{|\eta|}e^{-\beta E^{\phi}(\eta)}\lambda(d\eta)$$

and using Minlos lemma, the latter is equal to

$$\varkappa\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}|G(\xi\cup x)|\mathcal{K}(x,\eta)e^{-\beta E^{\phi}(x,\xi)}e^{-E^h(x,\omega)} \times\ C^{|\eta\cup\xi|}e^{-\beta E^{\phi}(\eta\cup\xi)}dx\lambda(d\xi)\lambda(d\eta).$$

Using Minlos lemma again we obtain

$$\varkappa\int_{\Gamma_0}\int_{\Gamma_0}\sum_{x\in\xi}|G(\xi)|\mathcal{K}(x,\eta)e^{-\beta E^{\phi}(x,\xi\setminus x)}e^{-E^h(x,\omega)} \times\ C^{|\eta\cup\xi\setminus x|}e^{-\beta E^{\phi}(\eta\cup\xi\setminus x)}\lambda(d\xi)\lambda(d\eta). \tag{3.20}$$

Now notice that

$$E^{\phi}(x,\xi\setminus x) = E^{\phi}(\xi) - E^{\phi}(\xi\setminus x),$$

and because $\phi>0$ we can bound (3.20) by

$$\varkappa C^{-1}\int_{\Gamma_0}|G(\xi)|e^{-\beta E^{\phi}(\xi)}C^{|\xi|}\sum_{x\in\xi}e^{-E^h(x,\omega)}\int_{\Gamma_0}\mathcal{K}(x,\eta)C^{|\eta|}\lambda(d\eta)\lambda(d\xi).$$

Finally, using the fact that $h>0$ we can bound the latter by

$$\varkappa C^{-1}e^{CC(\beta)}\int_{\Gamma_0}||\xi|G(\xi)|\,e^{-\beta E^{\phi}(\xi)}C^{|\xi|}\lambda(d\xi),$$

hence we obtain
$$\|L_1^{ext}G\|_C \le \varkappa C^{-1}e^{CC(\beta)}\|L_0^{ext}G\|_C.$$

□

Similar calculation of the norm $\|L_1 G\|_C$ yields that

$$\varkappa \int_{\Gamma_0} \left| \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) \mathcal{K}(x, \eta \setminus \xi) e^{-\beta E^{\phi}(x,\xi)} e^{-E^h(x,\omega)} dx \right| C^{|\eta|} e^{-\beta E^{\phi}(\eta)} \lambda(d\eta)$$

is bounded by

$$\varkappa C^{-1} \int_{\Gamma_0} |G(\xi)| e^{-\beta E^{\phi}(\xi)} C^{|\xi|} \sum_{x \in \xi} e^{-E^h(x,\omega)} \int_{\Gamma_0} \mathcal{K}(x, \eta) C^{|\eta|} \lambda(d\eta) \lambda(d\xi).$$

This is equal to

$$\varkappa C^{-1} e^{CC(\beta)} \int_{\Gamma_0} \left| \sum_{x \in \xi} e^{-E^h(x,\omega)} G(\xi) \right| e^{-\beta E^{\phi}(\xi)} C^{|\xi|} \lambda(d\xi),$$

and we obtain the following

Corollary 3.1. *Define the operator*

$$L_1 G(\eta) = \varkappa \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) \prod_{y \in \eta \setminus \xi} \left(e^{-\beta \phi(x-y)} - 1 \right) e^{-\beta E^{\phi}(x,\xi)} e^{-E^h(x,\omega)} dx$$

together with its domain $D(L_1) := D(L_0)$. Then, for all $\varkappa, C, \beta > 0$ and $G \in D(L_0)$ the following inequality holds:

$$\|L_1 G\|_C \leq \varkappa C^{-1} e^{CC(\beta)} \|L_0 G\|_C. \tag{3.21}$$

Proposition 3.3 together with Proposition 3.5 and Corollary 3.1 give us the following result:

Theorem 3.2. *Let $C > 0$, then for any $\varkappa, \beta > 0$ which satisfy the following inequality:*

$$2\varkappa C^{-1} e^{CC(\beta)} < 1,$$

and for almost all ω we have:

- *the operator $\left(\hat{L}^{ext}_{\omega}, D(L^{ext}_0) \right)$ is the generator of a holomorphic semigroup $\hat{U}^{ext}_{\omega}(t)$ in $\mathcal{L}_{C,\beta}$, and also*
- *$\left(\hat{L}_{\omega}, D(L_0) \right)$ generates a holomorphic semigroup which we denote by $\hat{U}_{\omega}(t)$ in $\mathcal{L}_{C,\beta}$.*

3.5 Evolution of the correlation functions

Fix now $\varkappa$, C and β in such way, that the assumptions of Theorem 3.2 are satisfied. Define the Banach space

$$\mathcal{Q}_{C,\beta} := \left\{ k : \Gamma_0 \to \mathbb{R} : \quad k(\cdot)C^{-|\cdot|}e^{\beta E^{\phi}(\cdot)} \in L^{\infty}(\Gamma_0, \lambda) \right\}$$

dual to the space $\mathcal{L}_{C,\beta}$ with respect to the duality defined by:

$$\left\langle\left\langle G, k \right\rangle\right\rangle := \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta) \tag{3.22}$$

for $G \in \mathcal{L}_{C,\beta}$ and $k \in \mathcal{Q}_{C,\beta}$. This duality is well defined since

$$\begin{aligned}\int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta) = &\int_{\Gamma_0} G(\eta)C^{|\eta|}e^{-E^{\phi}(\eta)}k(\eta)C^{-|\eta|}e^{E^{\phi}(\eta)}\lambda(d\eta)\\ \leq &\|k\|_{\mathcal{Q}_{C,\beta}}\|G\|_{\mathcal{L}_{C,\beta}} < \infty.\end{aligned}$$

Having constructed the semigroups on the space $\mathcal{L}_{C,\beta}$ and using the duality defined above, we can determine the corresponding dual semigroups on $\mathcal{Q}_{C,\beta}$. Namely, recall the semigroups generated by $\left(\hat{L}^{ext}_{\omega}, D(L^{ext}_0)\right)$ and $\left(\hat{L}_{\omega}, D(L_0)\right)$, that is $\hat{U}^{ext}_{\omega}(t)$ and $\hat{U}_{\omega}(t)$, respectively. One can easily show that these two semigroups determine their duals via (3.22), i.e.:

$$\left\langle\left\langle \hat{U}^{ext}_{\omega}(t)G, k \right\rangle\right\rangle = \left\langle\left\langle G, (\hat{U}^{ext}_{\omega})^*(t)k \right\rangle\right\rangle$$

and

$$\left\langle\left\langle \hat{U}_{\omega}(t)G, k \right\rangle\right\rangle = \left\langle\left\langle G, (\hat{U}_{\omega})^*(t)k \right\rangle\right\rangle.$$

Now assume that function $k_0 \in \mathcal{Q}_{C,\beta}$ is a correlation function of some probability measure μ_0 (with finite local moments) on Γ. Then we can check (see [FKKZ10] and [KKZ06]), that the evolution given by $(\hat{U}^{ext}_{\omega})^*(t)$ preserves this property, i.e. there exists a measure $\mu_t \in \mathcal{M}^1_{fm}(\Gamma)$ having $k_t := (\hat{U}^{ext}_{\omega})^*(t)k_0$ as correlation function.

Chapter 4

Two-component ecological model

4.1 Introduction

This chapter is devoted to the study of two-component ecological process in continuum. The introduction of such model is motivated by various biological applications in which different types of particles represent distinct populations of individuals. The structure of the chapter is as follows: first we construct a semigroup associated to the symbol of the pre-generator of the process in some functional space over Γ_0 (i.e. evolution of quasi-observables), together with the evolution (in weak sense) of the correlation functions. Next, we scale the model in the way introduced in Section 1.6 and show the strong convergence of the rescaled semigroup.

4.2 Construction of the semigroup in Γ_0

In the considered model the interaction between two types of individuals is of mutual type. That is, the birth and death rates of each type depend on both populations. Thus, the evolution of one population can influence the expansion and the reduction of the other. This model can be considered as the extension of the so-called Bolker, Pacala, Dieckmann and Law model of plant competition (see [BP, DL02]).

4.2.1 The mechanism of the evolution

The Markov pre-generator of a two-component birth and death process consist of two parts. Recall from Section 1.4, that the elements of two-component configuration space Γ^2 are denoted by (γ^1, γ^2) and although each of the components belongs to Γ, we make the distinction between them in order to emphasize different types of the populations. Thus, the Markov pre-generator has the following form:

$$L = L^1 + L^2, \tag{4.1}$$

in which

$$\begin{aligned}(L^1F)(\gamma^1,\gamma^2) := &\sum_{x\in\gamma^1} d^1(x,\gamma^1\setminus x,\gamma^2)\left[F(\gamma^1\setminus x,\gamma^2) - F(\gamma^1,\gamma^2)\right] \\ &+ \int_{\mathbb{R}^d} b^1(x,\gamma^1,\gamma^2)\left[F(\gamma^1\cup x,\gamma^2) - F(\gamma^1,\gamma^2)\right]dx,\end{aligned} \tag{4.2}$$

describes the evolution of the first population (type 1), and

$$\begin{aligned}(L^2F)(\gamma^1,\gamma^2) := &\sum_{y\in\gamma^2} d^2(y,\gamma^1,\gamma^2\setminus y)\left[F(\gamma^1,\gamma^2\setminus y) - F(\gamma^1,\gamma^2)\right] \\ &+ \int_{\mathbb{R}^d} b^2(y,\gamma^1,\gamma^2)\left[F(\gamma^1,\gamma^2\cup y) - F(\gamma^1,\gamma^2)\right]dy.\end{aligned} \tag{4.3}$$

characterizes the second population (type 2). In this particular case, taking into account the mutual relation between two types of individuals, birth and death rates are defined as follows:

$$\begin{aligned}
d^1(x,\gamma^1,\gamma^2) = &\quad m^+ + A_1^-\sum_{x'\in\gamma^1} a_1^-(x-x') + B_1^-\sum_{y\in\gamma^2} b_1^-(x-y),\\
b^1(x,\gamma^1,\gamma^2) = &\quad A_1^+\sum_{x'\in\gamma^1} a_1^+(x-x') + B_1^+\sum_{y\in\gamma^2} b_1^+(x-y),\\
d^2(y,\gamma^1,\gamma^2) = &\quad m^- + A_2^-\sum_{y'\in\gamma^2} a_2^-(y-y') + B_2^-\sum_{x\in\gamma^1} b_2^-(y-x),\\
b^2(y,\gamma^1,\gamma^2) = &\quad A_2^+\sum_{y'\in\gamma^2} a_2^+(y-y') + B_2^+\sum_{x\in\gamma^1} b_2^+(y-x).
\end{aligned}$$

We assume that all functions $a_i^\#, b_i^\#$ are probability densities, and $A_i^\#, B_i^\# > 0$ for $\# \in \{+,-\}$, $i = 1, 2$.

4.2.2 The symbol of L

Recall that the $\mathcal{K}$-transform plays the role of Fourier transform in the configuration space analysis. The symbol of the Markov pre-generator L is defined as its $\mathcal{K}$-image. Namely

$$\hat{L}G := \mathcal{K}^{-1}L\mathcal{K}G$$

for measurable functions G on the space of finite configurations Γ_0^2. Below we show the explicit form of $\hat{L}$.

Proposition 4.1. *The symbol of the generator (4.1) is given by:*

$$\begin{aligned}
\hat{L}G(\eta^1,\eta^2) = &-\left(m^+|\eta^1|+m^-|\eta^2|\right)G(\eta^1,\eta^2)\\
&-\sum_{x\in\eta^1}\left[A_1^-\sum_{x'\in\eta^1\setminus x}a_1^-(x-x')+B_1^-\sum_{y\in\eta^2}b_1^-(x-y)\right]G(\eta^1,\eta^2)\\
&-\sum_{y\in\eta^2}\left[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y')+B_2^-\sum_{x\in\eta^1}b_2^-(y-x)\right]G(\eta^1,\eta^2)\\
&-\sum_{x\in\eta^1}\left[A_1^-\sum_{x'\in\eta^1\setminus x}a_1^-(x-x')+B_2^-\sum_{y\in\eta^2}b_2^-(y-x)\right]G(\eta^1\setminus x,\eta^2)\\
&-\sum_{y\in\eta^2}\left[B_1^-\sum_{x\in\eta^1}b_1^-(x-y)+A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y')\right]G(\eta^1,\eta^2\setminus y)\\
&+\int_{\mathbb{R}^d}\left[A_1^+\sum_{x'\in\eta^1}a_1^+(x-x')+B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]G(\eta^1\cup x,\eta^2)dx\\
&+\int_{\mathbb{R}^d}\left[A_2^+\sum_{y'\in\eta^2}a_2^+(y-y')+B_2^+\sum_{x\in\eta^1}b_2^+(y-x)\right]G(\eta^1,\eta^2\cup y)dy\\
&+A_1^+\int_{\mathbb{R}^d}\sum_{x'\in\eta^1}a_1^+(x-x')G(\eta^1\setminus x'\cup x,\eta^2)dx\\
&+B_1^+\int_{\mathbb{R}^d}\sum_{y\in\eta^2}b_1^+(x-y)G(\eta^1\cup x,\eta^2\setminus y)dx\\
&+A_2^+\int_{\mathbb{R}^d}\sum_{y'\in\eta^2}a_2^+(y-y')G(\eta^1,\eta^2\setminus y'\cup y)dy\\
&+B_2^+\int_{\mathbb{R}^d}\sum_{x\in\eta^1}b_2^+(y-x)G(\eta^1\setminus x,\eta^2\cup y)dy,
\end{aligned}$$

for all functions $G\in B_{bs}(\Gamma_0^2)$ and $(\eta^1,\eta^2)\in\Gamma_0^2$.

Proof. First of all we simplify the form of the operator L. Note that we can rewrite the birth and death coefficients using $\mathcal{K}$-transform:

$$\begin{aligned}
b^i(x,\gamma^1,\gamma^2) &= \mathcal{K}B_x^i(\gamma^1,\gamma^2),\\
d^i(x,\gamma^1,\gamma^2) &= \mathcal{K}D_x^i(\gamma^1,\gamma^2)
\end{aligned}\qquad(4.4)$$

for $i = 1, 2$, where

$$
\begin{aligned}
D^1_x(\gamma^1,\gamma^2) &= e_\lambda(0,0,\gamma^1,\gamma^2)m^+ + A_1^- e_\lambda(0,\gamma^2)\mathbb{1}_{\{|\gamma^1|=1\}}(\gamma^1)\langle a_1^-(x-\cdot),\gamma^1\rangle \\
&\quad + B_1^- e_\lambda(0,\gamma^1)\mathbb{1}_{\{|\gamma^2|=1\}}(\gamma^2)\langle b_1^-(x-\cdot),\gamma^2\rangle, \\
B^1_x(\gamma^1,\gamma^2) &= A_1^+ e_\lambda(0,\gamma^2)\mathbb{1}_{\{|\gamma^1|=1\}}(\gamma^2)\langle a_1^+(x-\cdot),\gamma^1\rangle \\
&\quad + B_1^+ \mathbb{1}_{\{|\gamma^2|=1\}}(\gamma^2) e_\lambda(0,\gamma^1)\langle b_1^+(x-\cdot),\gamma^2\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
D^2_y(\gamma^1,\gamma^2) &= e_\lambda(0,0,\gamma^1,\gamma^2)m^- + A_2^- e_\lambda(0,\gamma^1)\mathbb{1}_{\{|\gamma^2|=1\}}(\gamma^2)\langle a_2^-(y-\cdot),\gamma^2\rangle \\
&\quad + B_2^- e_\lambda(0,\gamma^2)\mathbb{1}_{\{|\gamma^1|=1\}}(\gamma^1)\langle b_2^-(y-\cdot),\gamma^1\rangle, \\
B^2_y(\gamma^1,\gamma^2) &= A_2^+ e_\lambda(0,\gamma^1)\mathbb{1}_{\{|\gamma^2|=1\}}(\gamma^2)\langle a_2^+(y-\cdot),\gamma^2\rangle \\
&\quad + B_2^+ e_\lambda(0,\gamma^1)\mathbb{1}_{\{|\gamma^1|=1\}}(\gamma^1)\langle b_2^+(y-\cdot),\gamma^1\rangle.
\end{aligned}
$$

Now denote the gradients with

$$
\begin{aligned}
D_x^{1-}F(\gamma^1,\gamma^2) &= F(\gamma^1\setminus x,\gamma^2) - F(\gamma^1,\gamma^2), \\
D_x^{1+}F(\gamma^1,\gamma^2) &= F(\gamma^1\cup x,\gamma^2) - F(\gamma^1,\gamma^2), \\
D_x^{2-}F(\gamma^1,\gamma^2) &= F(\gamma^1,\gamma^2\setminus x) - F(\gamma^1,\gamma^2), \\
D_x^{2+}F(\gamma^1,\gamma^2) &= F(\gamma^1,\gamma^2\cup x) - F(\gamma^1,\gamma^2).
\end{aligned}
$$

Using this notation, we can rewrite the pre-generator L in more compact form:

$$
\begin{aligned}
LF(\gamma^1,\gamma^2) = &\sum_{x\in\gamma^1} \mathcal{K}D_x^+(\gamma^1,\gamma^2)D_x^{1-}F(\gamma^1,\gamma^2) \\
&+ \int_{\mathbb{R}^d} \mathcal{K}B_x^+(\gamma^1,\gamma^2)D_x^{1+}F(\gamma^1,\gamma^2)dx \\
&+ \sum_{y\in\gamma^2} \mathcal{K}D_y^- F(\gamma^1,\gamma^2)D_x^{2-}F(\gamma^1,\gamma^2) \\
&+ \int_{\mathbb{R}^d} \mathcal{K}B_y^-(\gamma^1,\gamma^2)D_y^{2-}F(\gamma^1,\gamma^2)dy.
\end{aligned}
$$

Below we show how to calculate the symbol for the first part of the pre-generator L_1, which describes the birth and death of particles in the population γ^1. Let us start with

$$\begin{aligned}
D_x^{1-}\mathcal{K}G(\gamma^1,\gamma^2) &= \mathcal{K}G(\gamma^1\setminus x,\gamma^2)-\mathcal{K}G(\gamma^1,\gamma^2)\\
&= \sum_{\eta^1\Subset\gamma^1\setminus x}\sum_{\eta^2\Subset\gamma^2} G(\eta^1,\eta^2)-\sum_{\eta^1\Subset\gamma^1}\sum_{\eta^2\Subset\gamma^2} G(\eta^1,\eta^2)\\
&= \sum_{\eta^2\Subset\gamma^2}\left[\sum_{\eta^1\Subset\gamma^1\setminus x} G(\eta^1,\eta^2)-\sum_{\eta^1\Subset\gamma^1} G(\eta^1,\eta^2)\right]\\
&= -\sum_{\eta^2\Subset\gamma^2}\sum_{\eta^1\Subset\gamma^1\setminus x} G(\eta^1\cup x,\eta^2)\\
&= -\ \mathcal{K}G(\cdot\cup x,\cdot)(\gamma^1\setminus x,\gamma^2),
\end{aligned}$$

and

$$\begin{aligned}
D_x^{1+}\mathcal{K}G(\gamma^1,\gamma^2) &= \mathcal{K}G(\gamma^1\cup x,\gamma^2)-\mathcal{K}G(\gamma^1,\gamma^2)\\
&= \sum_{\eta^1\Subset\gamma^1\cup x}\sum_{\eta^2\Subset\gamma^2} G(\eta^1,\eta^2)-\sum_{\eta^1\Subset\gamma^1}\sum_{\eta^2\Subset\gamma^2} G(\eta^1,\eta^2)\\
&= \sum_{\eta^2\Subset\gamma^2}\sum_{\eta^1\Subset\gamma^1} G(\eta^1\cup x,\eta^2)\\
&= \mathcal{K}G(\cdot\cup x,\cdot)(\gamma^1,\gamma^2).
\end{aligned}$$

Hence for the pre-generator L_1 we have

$$\begin{aligned}
L^1\mathcal{K}G(\gamma^1,\gamma^2) =&\ -\sum_{x\in\gamma^1}\mathcal{K}D_x^1(\gamma^1\setminus x,\gamma^2)\left(\mathcal{K}G(\cdot\cup x,\cdot)\right)(\gamma^1\setminus x,\gamma^2)\\
&+\int_{\mathbb{R}^d}\mathcal{K}B_x^1(\gamma^1,\gamma^2)\left(\mathcal{K}G(\cdot\cup x,\cdot)\right)(\gamma^1,\gamma^2)dx\\
=&\ -\sum_{x\in\gamma^1}\mathcal{K}\left[D_x^1\star G(\cdot\cup x,\cdot)\right](\gamma^1\setminus x,\gamma^2)\\
&+\int_{\mathbb{R}^d}\mathcal{K}\left[B_x^1\star G(\cdot\cup x,\cdot)\right](\gamma^1,\gamma^2)dx.
\end{aligned}$$

Applying $\mathcal{K}^{-1}$ to the object above we obtain the symbol of L^1:

$$\begin{aligned}
\hat{L}^1G(\eta^1,\eta^2) &= \mathcal{K}^{-1}L^+\mathcal{K}G(\eta^1,\eta^2)\\
&= -\sum_{x\in\gamma^1}\left[D_x^1\star G(\cdot\cup x,\cdot)\right](\eta^1\setminus x,\eta^2)\\
&\quad+\int_{\mathbb{R}^d}\left[B_x^1\star G(\cdot\cup x,\cdot)\right](\eta^1,\eta^2)dx.
\end{aligned}$$

It remains to determine the values of two convolutions above. In order to make the calculation more readable, below we use the notation:

$$\sum := \sum_{\substack{(\eta_1^1,\eta_2^1,\eta_3^1)\in\mathcal{P}_3^\emptyset(\eta^1)\\(\eta_1^2,\eta_2^2,\eta_3^2)\in\mathcal{P}_3^\emptyset(\eta^2)}} .$$

Then we have

$$\begin{aligned}
&\Big[D_x^1\star G(\cdot\cup x,\cdot)\Big](\eta^1,\eta^2)\\
&\quad= m^+\sum e_\lambda(0,\eta_1^1\cup\eta_2^1)e_\lambda(0,\eta_1^2\cup\eta_2^2)G(\eta_2^1\cup\eta_3^1\cup x,\eta_2^2\cup\eta_3^2)\\
&\qquad+\; A_1^+\sum e_\lambda(0,\eta_1^2\cup\eta_2^2)1\!\!1_{\{|\eta_1^1\cup\eta_2^1|=1\}}(\eta_1^1\cup\eta_2^1)\langle a_1^+(x-\cdot),\eta_1^1\cup\eta_2^1\rangle\\
&\qquad\qquad\times G(\eta_2^1\cup\eta_3^1\cup x,\eta_2^2\cup\eta_3^2)\\
&\qquad+\; B_1^-\sum e_\lambda(0,\eta_1^1\cup\eta_2^1)1\!\!1_{\{|\eta_1^2\cup\eta_2^2|=1\}}(\eta_1^2\cup\eta_2^2)\langle b_1^-(x-\cdot),\eta_1^2\cup\eta_2^2\rangle\\
&\qquad\qquad\times G(\eta_2^1\cup\eta_3^1\cup x,\eta_2^2\cup\eta_3^2),
\end{aligned}$$

using the properties of the coherent states and those of the indicator function we notice, that in fact many of the terms in the three sums above vanish, and the rest is equal to

$$\begin{aligned}
\Big[D_x^1\star G(\cdot\cup x,\cdot)\Big](\eta^1,\eta^2) &= m^+G(\eta^1\cup x,\eta^2)\\
&\quad+A_1^+\sum_{x'\in\eta^1}a_1^+(x-x')G(\eta^1\cup x,\eta^2)\\
&\quad+A_1^+\sum_{x'\in\eta^1}a_1^+(x-x')G(\eta^1\setminus x'\cup x,\eta^2)\\
&\quad+B_1^-\sum_{y\in\eta^2}b_1^-(x-y)G(\eta^1\cup x,\eta^2)\\
&\quad+B_1^-\sum_{y\in\eta^2}b_1^-(x-y)G(\eta^1\cup x,\eta^2\setminus y).
\end{aligned}$$

Similarly we can calculate

$$\begin{aligned}
&\Big[B_x^1\star G(\cdot\cup x,\cdot)\Big](\eta^1,\eta^2)=\\
&\quad= A_1^+\sum e_\lambda(0,\eta_1^2\cup\eta_2^2)1\!\!1_{\{|\eta_1^1\cup\eta_2^1|=1\}}(\eta_1^1\cup\eta_2^1)\langle a_1^+(x-\cdot),\eta_1^1\cup\eta_2^1\rangle\\
&\qquad\qquad\times G(\eta_2^1\cup\eta_3^1\cup x,\eta_2^2\cup\eta_3^2)\\
&\qquad+\; B_1^+\sum 1\!\!1_{\{|\eta_1^2\cup\eta_2^2|=1\}}(\eta_1^2\cup\eta_2^2)(\eta_1^2\cup\eta_2^2)\langle b_1^+(x-\cdot),\eta_1^2\cup\eta_2^2\rangle\\
&\qquad\qquad\times e_\lambda(0,\eta_1^1\cup\eta_2^1)G(\eta_2^1\cup\eta_3^1\cup x,\eta_2^2\cup\eta_3^2)
\end{aligned}$$

which gives

$$A_1^+ \sum_{x'\in\eta^1} a_1^+(x-x')G(\eta^1\cup x,\eta^2) + A_1^+ \sum_{x'\in\eta^1} a_1^+(x-x')G(\eta^1\setminus x'\cup x,\eta^2)$$
$$+ \; B_1^+ \sum_{y\in\eta^2} b_1^+(x-y)G(\eta^1\cup x,\eta^2) + B_1^+ \sum_{y\in\eta^2} b_1^+(x-y)G(\eta^1\cup x,\eta^2\setminus y).$$

The symbol of L_2 can be calculated analogously. Summing this up, we obtain the form of the symbol $\hat{L}$. □

Remark 4.1. *Note, that we are given the family of generators depending on parameters* $m^+, m^-, A_1^-, A_2^-, A_1^+, A_2^+, B_1^-, B_2^-, B_1^+, B_2^+ > 0$*, so formally*

$$\hat{L} := \hat{L}(m^+, m^-, A_1^-, A_2^-, A_1^+, A_2^+, B_1^-, B_2^-, B_1^+, B_2^+).$$

Throughout the rest of this chapter we write simply $\hat{L}$ *when it doesn't lead to confusion.*

4.2.3 Semigroup associated to $\hat{L}$

We proceed now to the construction of the semigroup associated to $\hat{L}$ using the method which was applied in Chapter 3. Let $C > 0$ and recall the definition of the space

$$\mathcal{L}_C := L^1\left(\Gamma_0^+ \times \Gamma_0^-, C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)\right), \tag{4.5}$$

$\|\cdot\|_C = \int_{\Gamma_0^2} |\cdot| C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)$. Then we have the following:

Proposition 4.2. *For every* $C > 0$, $m^+, m^- > 0$*, the operator*

$$\begin{aligned} L_0 G(\eta^1,\eta^2) := &- \left(m^+|\eta^1| + m^-|\eta^2|\right) G(\eta^1,\eta^2) \\ &- \sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^- \sum_{y\in\eta^2} b_1^-(x-y)\right] G(\eta^1,\eta^2) \\ &- \sum_{y\in\eta^2}\left[A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^- \sum_{x\in\eta^1} b_2^-(y-x)\right] G(\eta^1,\eta^2) \end{aligned}$$

with

$$D(L_0) := \{G \in \mathcal{L}_C : L_0 G \in \mathcal{L}_C\}$$

is the generator of a contraction semigroup on $\mathcal{L}_C$*. Moreover,* $L_0 \in \mathcal{H}(\omega, 0)$ *for all* $\omega \in \left(0, \frac{\pi}{2}\right)$*. See Section 3.2 for the corresponding definitions.*

Proof. Fix $\omega \in \left(0, \frac{\pi}{2}\right)$ and take $\zeta \in Sect(\frac{\pi}{2} + \omega)$. Denote with

$$\begin{aligned}\Xi(\eta^1, \eta^2) \quad &:= \quad m^+|\eta^1| + m^-|\eta^2| \\ &+ \sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^- \sum_{y\in\eta^2} b_1^-(x-y)\right] \\ &+ \sum_{y\in\eta^2}\left[A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^- \sum_{x\in\eta^1} b_2^-(y-x)\right].\end{aligned}$$

Then $0 \le \Xi(\eta^1, \eta^2) < +\infty$ for all $(\eta^1, \eta^2) \in \Gamma_0^2$ and

$$L_0 G(\eta^1, \eta^2) = -\Xi(\eta^1, \eta^2) G(\eta^1, \eta^2).$$

It is easy to see that the operator $(L_0, D(L_0))$ is densely defined in $\mathcal{L}_C$.

On the other hand for $\zeta \in Sect\left(\frac{\pi}{2} + \omega\right)$ we have $|\Xi(\eta^1, \eta^2) + \zeta| > 0$ for all $(\eta^1, \eta^2) \in \Gamma_0^2$ and thus the operator

$$\left(L_0 - \zeta 1\!\!1\right)^{-1} G = -\frac{1}{\Xi(\eta^1, \eta^2) + \zeta} G(\eta^1, \eta^2)$$

is well defined for every $G \in \mathcal{L}_C$. It remains us to prove the inequality (3.11). Let $Re\zeta \ge 0$ then obviously

$$\begin{aligned}\left\|\left(L_0 - \zeta 1\!\!1\right)^{-1} G\right\|_{\mathcal{L}_C} &= \left\| \frac{1}{|\Xi(\eta^1, \eta^2) + \zeta|} |G| \right\|_{\mathcal{L}_C} \\ &\le \frac{1}{|\zeta|} \|G\|_{\mathcal{L}_C}.\end{aligned}$$

In the case of $Re\zeta < 0$ notice, that

$$\left|\Xi(\eta^2, \eta^2) + \zeta\right| > |Im\zeta| > |\zeta| \cos\omega,$$

thus

$$\left\|\left(L_0 - \zeta 1\!\!1\right)^{-1} G\right\|_{\mathcal{L}_C} \le \frac{1}{|\zeta| \cos\omega} \|G\|_{\mathcal{L}_C}.$$

Summarizing, we get

$$\left\|\left(L_0 - \zeta 1\!\!1\right)^{-1} G\right\|_{\mathcal{L}_C} \le \max\left\{\frac{1}{|\zeta|}, \frac{1}{|\zeta| \cos\omega}\right\}.$$

The statement follows now from the Hille-Yosida theorem (see e.g. [Paz83, Kat95]). □

Below we prove a series of technical propositions showing that the remaining part of the operator $\hat{L}$ is relatively bounded with respect to L_0. We will sometimes write $\eta := (\eta^1, \eta^2)$ to make the text more readable.

Proposition 4.3. *Define*

$$L_1 G(\eta) := -\sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right] G(\eta^1\setminus x, \eta^2)$$

with $D(L_1) = D(L_0)$*, then, for any* $A_1^-, B_2^-, C > 0$ *such that the following hold:*

$$\begin{aligned} C\left[A_1^-|\eta^1| + B_2^-|\eta^2|\right] \;\le\; & \delta_1\Big(\left(m^+|\eta^1| + m^-|\eta^2|\right) \\ & + \sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^- \sum_{y\in\eta^2} b_1^-(x-y)\right] \\ & + \sum_{y\in\eta^2}\left[A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^- \sum_{x\in\eta^1} b_2^-(y-x)\right]\Big) \end{aligned}$$

for some $\delta_1 > 0$ *and all* $(\eta^1, \eta^2) \in \Gamma_0^2$*, the following inequality is fulfilled:*

$$\|L_1 G\|_C \le \delta_1 \|L_0 G\|_C. \tag{4.6}$$

Proof. Let $G \in D(L_1)$ and $C > 0$, then

$$\begin{aligned} \|L_1 G\|_C \;=\; & \int_{\Gamma_0^2}\Bigg|\sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right] \\ & \times G(\eta^1\setminus x, \eta^2)\Bigg|\, C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2). \end{aligned}$$

Using Lemma 1.3 we can estimate the latter by

$$\begin{aligned} & \int_{\Gamma_0^-}\int_{\Gamma_0^+}\int_{\mathbb{R}^d} dx\left[A_1^- \sum_{x'\in\eta^1} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right] \\ & \qquad \times\; \left|G(\eta^1, \eta^2)\right| C^{|\eta^1|+|\eta^2|+1}\lambda(d\eta^1)\lambda(d\eta^2). \end{aligned}$$

Hence

$$\begin{aligned}\|L_0G\|_C &\le C\int_{\Gamma_0^-}\int_{\Gamma_0^+}\left[A_1^-|\eta^1|+B_2^-|\eta^2|\right]|G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda d\eta^2\\ &\le \delta_1\|L_0G\|_C.\end{aligned}$$

□

Remark 4.2. *In particular, the following estimate is also true:*

$$\|L_1G\|_C \le C\left[\frac{A_1^-}{m^+}+\frac{B_2^-}{m^-}\right]\|L_0G\|_C.$$

Due to a similar structure of the operator L_2 defined below, we will omit the proof of the next proposition for it's analogous to the proof of the Proposition 4.3.

Proposition 4.4. *Let*

$$L_2G(\eta) := -\sum_{y\in\eta^2}\left[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y')+B_1^-\sum_{x\in\eta^1}b_1^-(y-x)\right]G(\eta^1,\eta^2\setminus y)$$

with $D(L_2):=D(L_0)$. Then, if for $A_2^-,B_1^->0$ and for $C>0$ there exists $\delta_2>0$ such that,

$$\begin{aligned}C\left[A_2^-|\eta^2|+B_1^-|\eta^1|\right] \;\le\; & \delta_2\Big(\left(m^+|\eta^1|+m^-|\eta^2|\right)\\ &+\sum_{x\in\eta^1}\Big[A_1^-\sum_{x'\in\eta^1\setminus x}a_1^-(x-x')+B_1^-\sum_{y\in\eta^2}b_1^-(x-y)\Big]\\ &+\sum_{y\in\eta^2}\Big[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y')+B_2^-\sum_{x\in\eta^1}b_2^-(y-x)\Big]\Big)\end{aligned}$$

for all $(\eta^1,\eta^2)\in\Gamma_0^2$, then

$$\|L_2G\|_C \le \delta_2\|L_0G\|_C. \tag{4.7}$$

Remark 4.3. *As in the case of L_1, the following inequality holds:*

$$\|L_2G\|_C \le C\left[\frac{A_2^-}{m^-}+\frac{B_1^-}{m^+}\right]\|L_0G\|_C.$$

Two next proposition give the bounds of two other parts of the operator $\hat{L}$ relatively with respect to the operator L_0. Due to the same reasons as in the previous case, we will give proof for only first of the following propositions and we will omit the proof of the second one.

Proposition 4.5. *Define*

$$L_3G(\eta^1,\eta^2) := \int_{\mathbb{R}^d}\left[A_1^+\sum_{x'\in\eta^1}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]G(\eta^1\cup x,\eta^2)dx$$

and $D(L_3) := D(L_0)$. *Then, for all functions* a_1^+ *and* b_1^+, *and all* $C>0$ *for which the following estimate*

$$\begin{aligned}\frac{1}{C}\sum_{x\in\eta^1}&\left[A_1^+\sum_{x'\in\eta^1\setminus x}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]\\
&\le \delta_3\Big(\big(m^+|\eta^1| + m^-|\eta^2|\big)\\
&\quad + \sum_{x\in\eta^1}\Big[A_1^-\sum_{x'\in\eta^1\setminus x}a_1^-(x-x') + B_1^-\sum_{y\in\eta^2}b_1^-(x-y)\Big]\\
&\quad + \sum_{y\in\eta^2}\Big[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y') + B_2^-\sum_{x\in\eta^1}b_2^-(y-x)\Big]\Big)\end{aligned}$$

holds with some $\delta_3>0$, *we have*

$$\|L_3G\|_C \le \delta_3\|L_0G\|_C. \tag{4.8}$$

Proof. Using properties of the modulus and Minlos Lemma, we obtain

$$\begin{aligned}\|L_3G\|_C &= \int_{\Gamma_0^2}\int_{\mathbb{R}^d}\left[A_1^+\sum_{x'\in\eta^1}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]\\
&\quad\times |G(\eta^1\cup x,\eta^2)|C^{|\eta^1|+|\eta^2|}dx\lambda(d\eta^1)\lambda(d\eta^2)\\
&= \int_{\Gamma_0^-}\lambda(d\eta^2)\int_{\Gamma_0^+}\lambda(d\eta^1)C^{|\eta^1|+|\eta^2|-1}|G(\eta^1,\eta^2)|\\
&\quad\times\sum_{x\in\eta^1}\left[A_1^+\sum_{x'\in\eta^1\setminus x}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]\end{aligned}$$

and using the assumptions, we obtain

$$\|L_3G\|_C \le \delta_3\|L_0G\|_C.$$

□

Proposition 4.6. *Define*

$$L_4G(\eta^1,\eta^2) := \int_{\mathbb{R}^d}\left[A_2^+\sum_{y'\in\eta^2}a_2^+(y-y')+B_2^+\sum_{x\in\eta^1}b_2^+(y-x)\right]G(\eta^1,\eta^2\cup y)dy$$

with $D(L_4)=D(L_0)$. *Then, for all functions* a_2^+, b_2^+ *and all* $C>0$ *fulfilling*

$$\begin{aligned}\frac{1}{C}\sum_{y\in\eta^2}&\left[A_2^+\sum_{y'\in\eta^2\setminus y}a_2^+(y-y')+B_2^+\sum_{x\in\eta^1}b_2^+(y-x)\right]\\
&\le \delta_4\Big(\left(m^+|\eta^1|+m^-|\eta^2|\right)\\
&\quad+\sum_{x\in\eta^1}\Big[A_1^-\sum_{x'\in\eta^1\setminus x}a_1^-(x-x')+B_1^-\sum_{y\in\eta^2}b_1^-(x-y)\Big]\\
&\quad+\sum_{y\in\eta^2}\Big[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y')+B_2^-\sum_{x\in\eta^1}b_2^-(y-x)\Big]\Big)\end{aligned}$$

for some $\delta_4>0$ *the following inequality holds:*

$$\|L_4G\|_C \le \delta_4\|L_0G\|_C. \tag{4.9}$$

Last four operators are of slightly different nature. They are however also relatively bounded with respect to L_0.

Proposition 4.7. *Let* $C>0$ *and define*

$$L_5G(\eta^1,\eta^2) := A_1^+\sum_{x'\in\eta^1}\int_{\mathbb{R}^d}a_1^+(x-x')G(\eta^1\setminus x'\cup x,\eta^2)dx$$

with $D(L_5):=D(L_0)$. *Then for any* $A_1^+>0$, *such that the following estimate*

$$\begin{aligned}A_1^+|\eta^1|\le\quad&\delta_5\Big(\left(m^+|\eta^1|+m^-|\eta^2|\right)\\
&+\sum_{x\in\eta^1}\Big[A_1^-\sum_{x'\in\eta^1\setminus x}a_1^-(x-x')+B_1^-\sum_{y\in\eta^2}b_1^-(x-y)\Big]\\
&+\sum_{y\in\eta^2}\Big[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y')+B_2^-\sum_{x\in\eta^1}b_2^-(y-x)\Big]\Big)\end{aligned}$$

holds with some $\delta_5>0$ *for all* $(\eta^1,\eta^2)\in\Gamma_0^2$, *the following inequality holds*

$$\|L_5G\|_C \le \delta_5\|L_0G\|_C. \tag{4.10}$$

Proof. The norm $\|L_5G\|_C$ is equal to

$$
\begin{aligned}
&A_1^+\int_{\Gamma_0^2}\sum_{x'\in\eta^1}\int_{\mathbb{R}^d}a_1^+(x-x')|G(\eta^1\setminus x'\cup x,\eta^2)|C^{|\eta^1|+|\eta^2|}dx\lambda(d\eta^1)\lambda(d\eta^2)\\
&\quad= A_1^+\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}a_1^+(x-x')|G(\eta^1\cup x,\eta^2)dxdx'|C^{|\eta^1|+|\eta^2|+1}\lambda(d\eta^1)\lambda(d\eta^2)\\
&\quad= A_1^+\int_{\Gamma_0^2}\sum_{x\in\eta^1}\int_{\mathbb{R}^d}a_1^+(x-x')dx'|G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)\\
&\quad= A_1^+\int_{\Gamma_0^2}|\eta^1||G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)\le\delta_5\|L_0G\|_C.
\end{aligned}
$$

□

Proposition 4.8. *Define*

$$
L_6G(\eta^1,\eta^2):=B_1^+\sum_{y\in\eta^2}\int_{\mathbb{R}^d}b_1^+(x-y)G(\eta^1\cup x,\eta^2\setminus y)dx
$$

with $D(L_6):=D(L_0)$. Then, for all $B_1^+>0$ for which there exists $\delta_6>0$ such that for every pair $(\eta^1,\eta^2)\in\Gamma_0^2$:

$$
\begin{aligned}
B_1^+|\eta^1|\le\quad&\delta_6\Big(\big(m^+|\eta^1|+m^-|\eta^2|\big)\\
&+\sum_{x\in\eta^1}\Big[A_1^-\sum_{x'\in\eta^1\setminus x}a_1^-(x-x')+B_1^-\sum_{y\in\eta^2}b_1^-(x-y)\Big]\\
&+\sum_{y\in\eta^2}\Big[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y')+B_2^-\sum_{x\in\eta^1}b_2^-(y-x)\Big]\Big)
\end{aligned}
$$

we have the following estimate

$$
\|L_6G\|_C\le\delta_6\|L_0G\|_C. \tag{4.11}
$$

Proof. We can calculate the norm $\|L_6G\|_C$:

$$
\begin{aligned}
&B_1^+\int_{\Gamma_0}\int_{\Gamma_0}\sum_{y\in\eta^2}\int_{\mathbb{R}^d}b_1^+(x-y)|G(\eta^1\cup x,\eta^2\setminus y)|C^{|\eta^1|+|\eta^2|}dx\lambda d\eta^1\lambda d\eta^2\\
&\quad= B_1^+\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}b_1^+(x-y)|G(\eta^1\cup x,\eta^2)|C^{|\eta^1|+|\eta^2|+1}dxdy\lambda(d\eta^1)\lambda(d\eta^2)\\
&\quad= B_1^+\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}\sum_{x\in\eta^1}b_1^+(x-y)|G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}dy\lambda(d\eta^1)\lambda(d\eta^2)\\
&\quad= B_1^+\int_{\Gamma_0}\int_{\Gamma_0}|\eta^1||G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)\le\delta_6\|L_0G\|_C.
\end{aligned}
$$

□

Proposition 4.9. *Define*

$$L_7G(\eta^1,\eta^2) := A_2^+ \sum_{y'\in\eta^2} \int_{\mathbb{R}^d} a_2^+(y-y')G(\eta^1,\eta^2\setminus y'\cup y)dy$$

with $D(L_7) := D(L_0)$. *Then for any* $A_2^+>0$, *such that the following estimate*

$$\begin{aligned} A_2^+|\eta^2| \leq\ & \delta_7\Big(\big(m^+|\eta^1| + m^-|\eta^2|\big) \\ & + \sum_{x\in\eta^1}\Big[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^- \sum_{y\in\eta^2} b_1^-(x-y)\Big] \\ & + \sum_{y\in\eta^2}\Big[A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^- \sum_{x\in\eta^1} b_2^-(y-x)\Big]\Big) \end{aligned}$$

holds with some $\delta_7>0$ *for all* $(\eta^1,\eta^2)\in\Gamma_0^2$, *the following inequality holds*

$$\|L_7G\|_C \leq \delta_7\|L_0G\|_C \tag{4.12}$$

for all $C>0$.

The proof of this proposition is analogous to the case of L_5 in Proposition 4.7. Hence we omit it here.

Proposition 4.10. *Define*

$$L_8G(\eta^1,\eta^2) := B_2^+ \sum_{x\in\eta^1} \int_{\mathbb{R}^d} b_2^+(x-y)G(\eta^1\setminus x,\eta^2\cup y)dy$$

with $D(L_8) := D(L_0)$. *Then for any* $B_2^+>0$, *such that the following estimate*

$$\begin{aligned} B_2^+|\eta^2| \leq\ & \delta_8\Big(\big(m^+|\eta^1| + m^-|\eta^2|\big) \\ & + \sum_{x\in\eta^1}\Big[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^- \sum_{y\in\eta^2} b_1^-(x-y)\Big] \\ & + \sum_{y\in\eta^2}\Big[A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^- \sum_{x\in\eta^1} b_2^-(y-x)\Big]\Big) \end{aligned}$$

holds with some $\delta_8>0$ *for all* $(\eta^1,\eta^2)\in\Gamma_0^2$, *the following inequality holds*

$$\|L_8G\|_C \leq \delta_8\|L_0G\|_C \tag{4.13}$$

for all $C>0$.

Proof. As in previous cases, we calculate the norm $\|L_8 G\|_C$:

$$\begin{aligned} B_2^+ &\int_{\Gamma_0}\int_{\Gamma_0}\sum_{x\in\eta^1}\int_{\mathbb{R}^d} b_2^+(x-y)|G(\eta^1\setminus x,\eta^2\cup y)|dy C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2) \\ &= B_2^+\int_{\Gamma_0}\int_{\Gamma_0}\sum_{x\in\eta^1}\sum_{y\in\eta^2} b_2^+(x-y)|G(\eta^1\setminus x,\eta^2)|C^{|\eta^1|+|\eta^2|-1}\lambda(d\eta^1)\lambda(d\eta^2) \\ &= B_2^+\int_{\Gamma_0}\int_{\Gamma_0}\sum_{y\in\eta^2}\int_{\mathbb{R}^d} b_2^+(x-y)|G(\eta^1,\eta^2)|dx C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2) \\ &= B_2^+\int_{\Gamma_0}\int_{\Gamma_0}|\eta^2||G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2) \\ &\le \delta_8\|L_0 G\|_C. \end{aligned}$$

□

Denote now for $\eta := (\eta^1,\eta^2)\in\Gamma_0\times\Gamma_0$:

$$\begin{aligned} \Upsilon(\eta) :=& \big(C\left[A_1^- + B_1^-\right] + A_1^+ + B_1^+\big)|\eta^1| + \big(C\left[A_2^- + B_2^-\right] + A_2^+ + B_2^+\big)|\eta^2| \\ &+ \frac{1}{C}\sum_{x\in\eta^1}\left[A_1^+\sum_{x'\in\eta^1\setminus x} a_1^+(x-x') + B_1^+\sum_{y\in\eta^2} b_1^+(x-y)\right] \\ &+ \frac{1}{C}\sum_{y\in\eta^2}\left[A_2^+\sum_{y'\in\eta^2\setminus y} a_2^+(y-y') + B_2^+\sum_{x\in\eta^1} b_2^+(y-x)\right] \end{aligned}$$

Putting together previous results we get the relative bound of the operator $\hat{L} - L_0$ in terms of the L_0:

Corollary 4.1. *Let*

$$\begin{aligned} \Upsilon(\eta^1,\eta^2) \le\, & \delta\big(m^+|\eta^1| + m^-|\eta^2|\big) \\ &+ \sum_{x\in\eta^1}\Big[A_1^-\sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^-\sum_{y\in\eta^2} b_1^-(x-y)\Big] \\ &+ \sum_{y\in\eta^2}\Big[A_2^-\sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^-\sum_{x\in\eta^1} b_2^-(y-x)\Big] \end{aligned} \tag{4.14}$$

For some $\delta > 0$ and all $(\eta^1,\eta^2)\in\Gamma_0\times\Gamma_0$. Then the following inequality holds:

$$\left\|\sum_{i=1}^{8} L_i G\right\|_C \le \delta\|L_0 G\|_C,$$

for all $G\in D(L_0)$.

Now we are ready to prove one of the main results of this chapter, that is the existence of the semigroup associated with the generator $\hat{L}$.

Theorem 4.1. *If* (4.14) *holds, then the operator* $\hat{L}$ *is the generator of a holomorphic semigroup on* $\mathcal{L}_C$ *which we will denote* $\hat{U}(t), t \geq 0$.

Proof. Proposition 4.2 together with Corollary 4.1 show that the operator $\hat{L}$ verifies the assumptions of the Theorem 3.1 and thus the statement of the Theorem follows now trivially. □

Remark 4.4. *From the proof of the Theorem 2.4 in [Kat95] one can conclude that in fact the* δ *can be chosen to be equal to* $\frac{1}{2}$. *See also [KKZ06].*

Having constructed the semigroup $\hat{U}(t)$ acting on quasi-observables we proceed to the description of the evolution of the system of correlation functions associated with our model.

4.3 The evolution of correlation functions

From now on, we fix the parameters of our system: $m^+, m^-, A_1^-, A_2^-, A_1^+, A_2^+, B_1^-, B_2^-, B_1^+, B_2^+ > 0$, as well as $C > 0$ such that the operator $\hat{L}$ fulfils the conditions of Theorem 4.1 and thus generates a holomorphic semigroup $\hat{U}(t)$ on $\mathcal{L}_C$.

4.3.1 Space $\mathcal{Q}_C$

Consider the space $\mathcal{Q}_C$ of the *so-called correlation functions* defined as

$$\mathcal{Q}_C := \left\{ k : \Gamma_0 \times \Gamma_0 \to \mathbb{R} |\ k(\cdot^1, \cdot^2) \cdot C^{-(|\cdot^1| + |\cdot^2|)} \in L^\infty(\Gamma_0 \times \Gamma_0, \lambda \otimes \lambda) \right\}. \tag{4.15}$$

It is a Banach space with the norm

$$\|k\| := \text{ess sup} \left| k(\eta^1, \eta^2) C^{-(|\eta^1| + |\eta^2|)} \right|, \tag{4.16}$$

where the ess sup is calculated with respect to the measure $\lambda^{\otimes 2}$. Note, that any function $k \in \mathcal{Q}_C$ satisfies the bound

$$|k(\eta^1, \eta^2)| \leq C_1 C^{(|\eta^1| + |\eta^2|)} \tag{4.17}$$

for $\lambda^{\otimes 2}$-a.a. $(\eta^1, \eta^2) \in \Gamma_0^2$ and for some $C_1 > 0$.

Define now following duality between $\mathcal{L}_C$ and $\mathcal{Q}_C$:

$$\langle\langle G, k\rangle\rangle := \int_{\Gamma_0^2} G(\eta^1,\eta^2)\cdot k(\eta^1,\eta^2)\lambda(d\eta^1)\lambda(d\eta^2) \tag{4.18}$$

for $G \in \mathcal{L}_C$ and $k \in \mathcal{Q}_C$. The duality (4.18) is well defined since $G \in \mathcal{L}_C$, $G(\cdot^1,\cdot^2)C^{|\cdot^1|+|\cdot^2|} \in L^1(\Gamma_0^2, \lambda^{\otimes 2})$ and additionally (4.17) holds.

4.3.2 The dual of the operator $\hat{L}$

With help of the duality (4.18) one can define the dual operator $\hat{L}^*$ to the generator $\hat{L}$ (c.f. Chapter 1, Section 1.5), namely

$$\int_{\Gamma_0^2} \hat{L}G\cdot k d\lambda^{\otimes 2} = \int_{\Gamma_0^2} G\cdot \hat{L}^* k d\lambda^{\otimes 2}. \tag{4.19}$$

Using the fact that

$$\hat{L} = L_0 + \ldots + L_8$$

where the operators L_i, $i = 0...8$ were defined in the previous section, we calculate the dual operator $\hat{L}^*$ as the sum of the respective duals of operators $L_1, \ldots, L_8$ in the series of lemmas.

Lemma 4.1. *The operator L_0 is "self-dual", hence $L_0^* = L_0$.*

We skip the proof for L_0 is just the multiplication operator.

Lemma 4.2. *The operator adjoint to the operator defined by*

$$L_1 G(\eta) = -\sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right] G(\eta^1\setminus x, \eta^2)$$

with respect to the duality (4.18) *has the following form*

$$L_1^* k(\eta) = -\int_{\mathbb{R}^d}\left(A_1^- \sum_{x'\in\eta^1} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right) k(\eta^1\cup x, \eta^2)dx.$$

Proof. By the definition and using the Minlos lemma, we have

$$\begin{aligned}
&-\int_{\Gamma_0^2}\sum_{x\in\eta^1}\left(A_1^- \sum_{x'\in\eta^1} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right) G(\eta^1\setminus x, \eta^2)\\
&\qquad\qquad \times k(\eta^1,\eta^2)\lambda^2(d\eta^1, d\eta^2)\\
&= -\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\left(A_1^- \sum_{x'\in\eta^1} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right) k(\eta^1\cup x, \eta^2)dx\\
&\qquad\qquad \times G(\eta^1,\eta^2)\lambda^2(d\eta^1, d\eta^2).
\end{aligned}$$

□

In the similar way we can obtain the following

Lemma 4.3. *The operator adjoint to the operator defined by*

$$L_2G(\eta) = -\sum_{y\in\eta^2}\left[A_2^-\sum_{y'\in\eta^2\setminus y}a_2^-(y-y') + B_1^-\sum_{x\in\eta^1}b_1^-(y-x)\right]G(\eta^1,\eta^2\setminus y)$$

with respect to the duality (4.18) *has the following form*

$$L_2^*k(\eta) = -\int_{\mathbb{R}^d}\left(A_2^-\sum_{y'\in\eta^2}a_2^-(y-y') + B_1^-\sum_{x\in\eta^1}b_1^-(y-x)\right)k(\eta^1,\eta^2\cup y)dy.$$

Lemma 4.4. *The adjoint of the operator*

$$L_3G(\eta^1,\eta^2) = \int_{\mathbb{R}^d}\left[A_1^+\sum_{x'\in\eta^1}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]G(\eta^1\cup x,\eta^2)dx$$

with respect to the duality (4.18) *has the following form*

$$L_3^*k(\eta^1,\eta^2) = \sum_{x\in\eta^1}\left[A_1^+\sum_{x'\in\eta^1\setminus x}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]k(\eta^1\setminus x,\eta^2).$$

Proof. The direct calculation yields

$$\begin{aligned}\int_{\Gamma_0^2}\int_{\mathbb{R}^d}&\left[A_1^+\sum_{x'\in\eta^1}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]G(\eta^1\cup x,\eta^2)dx\\ &\qquad\times\ k(\eta^1,\eta^2)\lambda^2(d\eta^1,d\eta^2)\\ =\int_{\Gamma_0^2}\sum_{x\in\eta^1}&\left[A_1^+\sum_{x'\in\eta^1}a_1^+(x-x') + B_1^+\sum_{y\in\eta^2}b_1^+(x-y)\right]k(\eta^1\setminus x,\eta^2)dx\\ &\qquad\times\ G(\eta^1,\eta^2)\lambda^2(d\eta^1,d\eta^2).\end{aligned}$$

□

Lemma 4.5. *The operator adjoint to the operator defined by*

$$L_4G(\eta^1,\eta^2) = \int_{\mathbb{R}^d} \left[A_2^+ \sum_{y'\in\eta^2} a_2^+(y-y') + B_2^+ \sum_{x\in\eta^1} b_2^+(y-x) \right] G(\eta^1,\eta^2\cup y)dy$$

with respect to the duality (4.18) *is as follows*

$$L_4^*k(\eta^1,\eta^2) = \sum_{y\in\eta^2} \left[A_2^+ \sum_{y'\in\eta^2\setminus y} a_2^+(y-y') + B_2^+ \sum_{x\in\eta^1} b_1^+(y-x) \right] k(\eta^1,\eta^2\setminus y).$$

Again, the proof is similar to the previous case and we omit it here.

Lemma 4.6. *The operator adjoint to the operator*

$$L_5G(\eta^1,\eta^2) = A_1^+ \sum_{x'\in\eta^1} \int_{\mathbb{R}^d} a_1^+(x-x')G(\eta^1\setminus x'\cup x,\eta^2)dx$$

with respect to the duality (4.18) *has the following form*

$$L_5^*k(\eta^1,\eta^2) = A_1^+ \sum_{x\in\eta^1} \int_{\mathbb{R}^d} a_1^+(x-x')k(\eta^1\cup x'\setminus x,\eta^2)dx'.$$

Proof. Using the definition we obtain

$$\begin{aligned}
&A_1^+ \int_{\Gamma_0^2}\int_{\mathbb{R}^d} \sum_{x'\in\eta^1} a_1^+(x-x')G(\eta^1\setminus x'\cup x,\eta^2)dxk(\eta^1,\eta^2)\lambda^2(d\eta^1,d\eta^2) \\
&\quad = A_1^+ \int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a_1^+(x-x')G(\eta^1\cup x,\eta^2)k(\eta^1\cup x',\eta^2)dx'dx\lambda^2(d\eta^1,d\eta^2) \\
&\quad = A_1^+ \int_{\Gamma_0^2} G(\eta^1,\eta^2) \sum_{x\in\eta^1}\int_{\mathbb{R}^d} a_1^+(x-x')k(\eta^1\cup x'\setminus x,\eta^2)dx'\lambda^2(d\eta^1,d\eta^2).
\end{aligned}$$

□

Lemma 4.7. *The adjoint with respect to the duality* (4.18) *of the operator defined by*

$$L_6G(\eta^1,\eta^2) = B_1^+ \sum_{y\in\eta^2} \int_{\mathbb{R}^d} b_1^+(x-y)G(\eta^1\cup x,\eta^2\setminus y)dx$$

is given by

$$L_6^*k(\eta^1,\eta^2) = B_1^+ \sum_{x\in\eta^1} \int_{\mathbb{R}^d} b_1^+(x-y)k(\eta^1\setminus x,\eta^2\cup y)dy.$$

Proof. Applying the definition of duality and using Minlos lemma we obtain

$$\begin{aligned}
&B_1^+\int_{\Gamma_0^2}\sum_{y\in\eta^2}\int_{\mathbb{R}^d} b_1^+(x-y)G(\eta^1\cup x,\eta^2\setminus y)dxk(\eta^1,\eta^2)\lambda^2(d\eta^1,d\eta^2)\\
&=B_1^+\int_{\Gamma_0^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b_1^+(x-y)G(\eta^1\cup x,\eta^2)k(\eta^1,\eta^2\cup y)dxdy\lambda^2(d\eta^1,d\eta^2)\\
&=B_1^+\int_{\Gamma_0^2} G(\eta^1,\eta^2)\sum_{x\in\eta^2}\int_{\mathbb{R}^d} b_1^+(x-y)k(\eta^1\setminus x,\eta^2\cup y)dy\lambda^2(d\eta^1,d\eta^2).
\end{aligned}$$

□

Lemma 4.8. *Define*

$$L_7G(\eta^1,\eta^2)=A_2^+\sum_{y'\in\eta^2}\int_{\mathbb{R}^d} a_2^+(y-y')G(\eta^1,\eta^2\setminus y'\cup y)dy.$$

Then the adjoint of the operator L_7 with respect to the duality (4.18) *has the following form*

$$L_7^*k(\eta^1,\eta^2)=A_2^+\sum_{y\in\eta^2}\int_{\mathbb{R}^d} a_2^+(y-y')k(\eta^1,\eta^2\cup y'\setminus y)dy'.$$

Proof. The proof is similar to the case of L_5^*. □

The calculation of the adjoint of operator L_8 is similar to the proof of Lemma 4.7. Thus the next statement will be given without proof.

Lemma 4.9. *Let the operator L_8 be defined by*

$$L_8G(\eta^1,\eta^2):=B_2^+\sum_{x\in\eta^1}\int_{\mathbb{R}^d} b_2^+(x-y)G(\eta^1\setminus x,\eta^2\cup y)dy.$$

Then the adjoint of L_8 in the space $\mathcal{Q}_C$ with respect to the duality (4.18) *is as follows:*

$$L_8^*k(\eta^1,\eta^2)=B_2^+\sum_{y\in\eta^2}\int_{\mathbb{R}^d} b_2^+(y-x)k(\eta^1\cup x,\eta^2\setminus y)dx.$$

Taking into account all previous calculations we can derive the form of the adjoint operator $\hat{L}^*$ with respect to the duality (4.18):

$$\begin{aligned}
\hat{L}^* k(\eta^1,\eta^2) = & -\left(m^+|\eta^1| + m^-|\eta^2|\right) k(\eta^1,\eta^2) \qquad (4.20)\\
& - \sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^- \sum_{y\in\eta^2} b_1^-(x-y)\right] k(\eta^1,\eta^2)\\
& - \sum_{y\in\eta^2}\left[A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^- \sum_{x\in\eta^1} b_2^-(y-x)\right] k(\eta^1,\eta^2)\\
& - \int_{\mathbb{R}^d}\left(A_1^- \sum_{x'\in\eta^1} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(x-y)\right) k(\eta^1\cup x,\eta^2)dx\\
& - \int_{\mathbb{R}^d}\left(A_2^- \sum_{y'\in\eta^2} a_2^-(y-y') + B_1^- \sum_{x\in\eta^1} b_1^-(y-x)\right) k(\eta^1,\eta^2\cup y)dy\\
& + \sum_{x\in\eta^1}\left[A_1^+ \sum_{x'\in\eta^1\setminus x} a_1^+(x-x') + B_1^+ \sum_{y\in\eta^2} b_1^+(x-y)\right] k(\eta^1\setminus x,\eta^2)\\
& + \sum_{y\in\eta^2}\left[A_2^+ \sum_{y'\in\eta^2\setminus y} a_2^+(y-y') + B_2^+ \sum_{x\in\eta^1} b_1^+(y-x)\right] k(\eta^1,\eta^2\setminus y)\\
& + A_1^+ \sum_{x\in\eta^1}\int_{\mathbb{R}^d} a_1^+(x-x') k(\eta^1\cup x'\setminus x,\eta^2)dx'\\
& + B_1^+ \sum_{x\in\eta^1}\int_{\mathbb{R}^d} b_1^+(x-y) k(\eta^1\setminus x,\eta^2\cup y)dy\\
& + A_2^+ \sum_{y\in\eta^2}\int_{\mathbb{R}^d} a_2^+(y-y') k(\eta^1,\eta^2\cup y'\setminus y)dy'\\
& + B_2^+ \sum_{y\in\eta^2}\int_{\mathbb{R}^d} b_2^+(y-x) k(\eta^1\cup x,\eta^2\setminus y)dx.
\end{aligned}$$

for $k \in \mathcal{Q}_C$.

The duality (4.18) determines a semigroup, which we will denote $\hat{U}(t)^*$, i.e.

$$\left\langle\left\langle \hat{U}(t)G, k\right\rangle\right\rangle = \left\langle\left\langle G, \hat{U}^*(t)k\right\rangle\right\rangle.$$

The question arises: assume we start from a proper state of the system μ_0 which has the correlation function k_{μ_0}, is then $k_t := \hat{U}^*(t)k_{\mu_0}$ also a correlation function for some measure μ_t? Existence of such a measure is sometimes very difficult to prove and in this chapter we don't investigate this problem in details. For more information and some technical tools we refer e.g. to [KK02], and for the application of such a framework, see e.g. [KKZ06] or [KL05] in the case of Glauber dynamics.

4.4 Vlasov-type scaling for the model

The general scheme of Vlasov scaling was introduced in Chapter 1. In this section we apply this scheme to our model giving the formal meaning to the considerations from Section 1.6. More precisely, we scale the pre-generator L introduced in (4.1) obtaining an operator L_ε and then we calculate its symbol $\hat{L}_\varepsilon$. Next, we show the strong convergence of the associated semigroup to the Vlasov semigroup $\hat{U}^V(t)$. Finally we derive a Vlasov-type equation for the two-component ecological model and give its mild solution.

4.4.1 Scaling of the operator L and its symbol

The Markov birth-and-death pre-generator L can be represented as a sum of two operators. The first one, L^+, corresponds to the birth of the individuals in the system whereas the second one, L^- describes the death of individuals. As it was stated in [FKK10a], the right Vlasov scaling for L has the following form:

$$L_\varepsilon := L^-(d^1_\varepsilon, d^2_\varepsilon) + \varepsilon^{-1} L^+(b^1_\varepsilon, b^2_\varepsilon) \tag{4.21}$$

where d^1_ε, d^2_ε, b^1_ε, b^2_ε are some scalings of the rates d^1, d^2, b^1, b^2 which will be described later. An additional increasing of the intensity of birth is used to preserve the influence of the birth part in the limiting Vlasov hierarchy. Moreover, the real necessity of the factor ε^{-1} in (4.21) will become clear *a posteriori*.

Recall, that the Markov pre-generator in our case has the form $L = L^1 + L^2$. Let $\varepsilon > 0$. We define the scaled operator L_ε as $L^1_\varepsilon + L^2_\varepsilon$, where

$$\begin{aligned}(L^1_\varepsilon F)(\gamma^1,\gamma^2) :=& \sum_{x\in\gamma^1} d^1_\varepsilon(x,\gamma^1\setminus x,\gamma^2)\left[F(\gamma^1\setminus x,\gamma^2) - F(\gamma^1,\gamma^2)\right] \\ &+ \varepsilon^{-1}\int_{\mathbb{R}^d} b^1_\varepsilon(x,\gamma^1,\gamma^2)\left[F(\gamma^1\cup x,\gamma^2) - F(\gamma^1,\gamma^2)\right] dx,\end{aligned} \tag{4.22}$$

and

$$(L_\varepsilon^2 F)(\gamma^1,\gamma^2) := \sum_{y\in\gamma^2} d_\varepsilon^2(y,\gamma^1,\gamma^2\setminus y)\left[F(\gamma^1,\gamma^2\setminus y) - F(\gamma^1,\gamma^2)\right] \\ + \varepsilon^{-1}\int_{\mathbb{R}^d} b_\varepsilon^2(y,\gamma^1,\gamma^2)\left[F(\gamma^1,\gamma^2\cup y) - F(\gamma^1,\gamma^2)\right] dy. \tag{4.23}$$

The proper scaling of coefficients is making all interactions in the system weaker and has the following form:

$$d_\varepsilon^1(x,\gamma^1,\gamma^2) = m^+ + \varepsilon A_1^- \sum_{x'\in\gamma^1} a_1^-(x-x') + \varepsilon B_1^- \sum_{y\in\gamma^2} b_1^-(x-y),$$
$$d_\varepsilon^2(y,\gamma^1,\gamma^2) = m^- + \varepsilon A_2^- \sum_{y'\in\gamma^2} a_2^-(y-y') + \varepsilon B_2^- \sum_{x\in\gamma^1} b_2^-(y-x),$$
$$b_\varepsilon^1(x,\gamma^1,\gamma^2) = \varepsilon A_1^+ \sum_{x'\in\gamma^1} a_1^+(x-x') + \varepsilon B_1^+ \sum_{y\in\gamma^2} b_1^+(x-y),$$
$$b_\varepsilon^2(y,\gamma^1,\gamma^2) = \varepsilon A_2^- \sum_{y'\in\gamma^2} a_2^+(y-y') + \varepsilon B_2^+ \sum_{x\in\gamma^1} b_2^+(y-x).$$

Effectively, the part of L which corresponds to the birth is not changing in the considered system. As in the previous section, we prove that for all $\varepsilon > 0$, the symbol of the operator L_ε generates a holomorphic semigroup in $\mathcal{L}_C$. For $\varepsilon > 0$ recall the following renormalization:

$$R_\varepsilon G(\eta)(\eta^1,\eta^2) = \varepsilon^{|\eta^1|+|\eta^2|} G(\eta^1,\eta^2) \tag{4.24}$$

for $(\eta^1,\eta^2) \in \Gamma_0^2$. Below we give the form of the symbol $\hat{L}_\varepsilon$ and the renormalized operator $\hat{L}_{\varepsilon,ren} := R_\varepsilon^{-1}\hat{L}_\varepsilon R_\varepsilon = R_{\varepsilon^{-1}}\hat{L}_\varepsilon R_\varepsilon$.
For any $\varepsilon > 0$, the symbol of the operator L_ε is given by

$$\begin{aligned}\hat{L}_\varepsilon G(\eta^1,\eta^2) = &-\left(m^+|\eta^1| + m^-|\eta^2|\right) G(\eta^1,\eta^2) \\ &- \sum_{x\in\eta^1}\left[\varepsilon A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + \varepsilon B_1^- \sum_{y\in\eta^2} b_1^-(x-y)\right] G(\eta^1,\eta^2) \\ &- \sum_{y\in\eta^2}\left[\varepsilon A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + \varepsilon B_2^- \sum_{x\in\eta^1} b_2^-(y-x)\right] G(\eta^1,\eta^2) \\ &- \sum_{x\in\eta^1}\left[\varepsilon A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + \varepsilon B_2^- \sum_{y\in\eta^2} b_2^-(y-x)\right] G(\eta^1\setminus x,\eta^2) \\ &+ \ldots\end{aligned}$$

$$\ldots - \sum_{y\in\eta^2}\left[\varepsilon B_1^- \sum_{x\in\eta^1} b_1^-(x-y) + \varepsilon A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y')\right] G(\eta^1,\eta^2\setminus y)$$
$$+ \int_{\mathbb{R}^d}\left[A_1^+ \sum_{x'\in\eta^1} a_1^+(x-x') + A_1^+ \sum_{y\in\eta^2} b_1^+(x-y)\right] G(\eta^1\cup x,\eta^2)dx$$
$$+ \int_{\mathbb{R}^d}\left[A_2^+ \sum_{y'\in\eta^2} a_2^+(y-y') + B_2^+ \sum_{x\in\eta^1} b_2^+(y-x)\right] G(\eta^1,\eta^2\cup y)dy$$
$$+ A_1^+ \int_{\mathbb{R}^d} \sum_{x'\in\eta^1} a_1^+(x-x') G(\eta^1\setminus x'\cup x,\eta^2)dx$$
$$+ B_1^+ \int_{\mathbb{R}^d} \sum_{y\in\eta^2} b_1^+(x-y) G(\eta^1\cup x,\eta^2\setminus y)dx$$
$$+ A_2^+ \int_{\mathbb{R}^d} \sum_{y'\in\eta^2} a_2^+(y-y') G(\eta^1,\eta^2\setminus y'\cup y)dy$$
$$+ B_2^+ \int_{\mathbb{R}^d} \sum_{x\in\eta^1} b_2^+(y-x) G(\eta^1\setminus x,\eta^2\cup y)dy$$

for $G \in L^0_{ls}(\Gamma^2_0)$ and $(\eta^1,\eta^2)\in\Gamma^2_0$. Next, we renormalise "line by line" the operator above obtaining for any $\varepsilon > 0$ (recall $\eta = (\eta^1,\eta^2)$):

$$\hat{L}_{\varepsilon,ren} = \sum_{i=1}^{7} E_i + \varepsilon \sum_{j=1}^{4} F_j \tag{4.25}$$

where

$$E_1G(\eta) := -\left(m^+|\eta^1| + m^-|\eta^2|\right) G(\eta^1,\eta^2)$$
$$E_2G(\eta) := -\sum_{x\in\eta^1}\left[A_1^- \sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_2^- \sum_{y\in\eta^2} b_2^-(y-x)\right] G(\eta^1\setminus x,\eta^2)$$
$$E_3G(\eta) := -\sum_{y\in\eta^2}\left[A_2^- \sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_1^- \sum_{x\in\eta^1} b_1^-(x-y)\right] G(\eta^1,\eta^2\setminus y)$$
$$E_4G(\eta) := A_1^+ \int_{\mathbb{R}^d} \sum_{x'\in\eta^1} a_1^+(x-x') G(\eta^1\setminus x'\cup x,\eta^2)dx$$

and

$$E_5G(\eta) := B_1^+ \int_{\mathbb{R}^d} \sum_{y\in\eta^2} b_1^+(x-y)G(\eta^1\cup x,\eta^2\setminus y)dx$$

$$E_6G(\eta) := A_2^+ \int_{\mathbb{R}^d} \sum_{y'\in\eta^2} a_2^+(y-y')G(\eta^1,\eta^2\setminus y'\cup y)dy$$

$$E_7G(\eta) := B_2^+ \int_{\mathbb{R}^d} \sum_{x\in\eta^1} b_2^+(y-x)G(\eta^1\setminus x,\eta^2\cup y)dy$$

$$F_1G(\eta) := -\sum_{x\in\eta^1}\left[A_1^-\sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^-\sum_{y\in\eta^2} b_1^-(x-y)\right]G(\eta^1,\eta^2)$$

$$F_2G(\eta) := -\sum_{y\in\eta^2}\left[A_2^-\sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^-\sum_{x\in\eta^1} b_2^-(y-x)\right]G(\eta^1,\eta^2)$$

$$F_3G(\eta) := \int_{\mathbb{R}^d}\left[A_1^+\sum_{x'\in\eta^1} a_1^+(x-x') + B_1^+\sum_{y\in\eta^2} b_1^+(x-y)\right]G(\eta^1\cup x,\eta^2)dx$$

$$F_4G(\eta) := \int_{\mathbb{R}^d}\left[A_2^+\sum_{y'\in\eta^2} a_2^+(y-y') + B_2^+\sum_{x\in\eta^1} b_2^+(y-x)\right]G(\eta^1,\eta^2\cup y)dy.$$

We consider maximal domain for $\hat{L}_{\varepsilon,ren}$ defined as

$$D(\hat{L}_{\varepsilon,ren}) := \{G\in\mathcal{L}_C|\ \hat{L}_{\varepsilon,ren}G\in\mathcal{L}_C\} \tag{4.26}$$

for $\varepsilon > 0$, and

$$D(\hat{L}_{ren}) := \{G\in\mathcal{L}_C|\ \left(m^+|\cdot^1| + m^-|\cdot^2|\right)G(\cdot^1,\cdot^2)\in\mathcal{L}_C\} \tag{4.27}$$

in the case when $\varepsilon = 0$.

Remark 4.5. *It is easy to see that $D(\hat{L}_{\varepsilon,ren})\subset D(\hat{L}_{ren})$.*

Similarly to the unscaled case (cf. Theorem 4.1), operator $\hat{L}_{\varepsilon,ren}$ with the domain defined above generates a semigroup in $\mathcal{L}_C$:

Theorem 4.2. *For all $m^\#$, $A_i^\#$, $B_i^\#$, $\#\in\{+,-\}$, $i=1,2$ such that the following assumptions hold for all $(\eta^1,\eta^2)\in\Gamma_0\times\Gamma_0$:*

1. *there exists $\vartheta_1 > 0$ such that*

$$\left[C\left(A_1^- + B_1^-\right) + A_1^+ + B_1^+\right]|\eta^1| + \left[C\left(B_2^- + A_2^-\right) + A_2^+ + B_2^+\right]|\eta^2| \le \vartheta_1\left(m^+|\eta^1| + m^-|\eta^2|\right),$$

2. *there exists* $\vartheta_2 > 0$ *such that*

$$\begin{aligned}
&\frac{\varepsilon}{C}\sum_{x\in\eta^1}\Big[A_1^+\sum_{x'\in\eta^1\setminus x} a_1^+(x-x') + B_1^+\sum_{y\in\eta^2} b_1^+(x-y)\Big] \\
&\quad + \frac{\varepsilon}{C}\sum_{y\in\eta^2}\Big[A_2^+\sum_{y'\in\eta^2\setminus y} a_2^+(y-y') + B_2^+\sum_{x\in\eta^1} b_2^+(y-x)\Big] \\
&\qquad \le \vartheta_2\Big(\big(m^+|\eta^1| + m^-|\eta^2|\big) \\
&\qquad\quad + \varepsilon\sum_{x\in\eta^1}\Big[A_1^-\sum_{x'\in\eta^1\setminus x} a_1^-(x-x') + B_1^-\sum_{y\in\eta^2} b_1^-(x-y)\Big] \\
&\qquad\quad + \varepsilon\sum_{y\in\eta^2}\Big[A_2^-\sum_{y'\in\eta^2\setminus y} a_2^-(y-y') + B_2^-\sum_{x\in\eta^1} b_2^-(y-x)\Big]\Big),
\end{aligned}$$

3. *and additionally the following holds for some* $\delta > 0$*:*

$$\vartheta_1 + \vartheta_2 \le 1 - \delta.$$

Then, for every $\varepsilon > 0$*,* $\left(\hat{L}_{\varepsilon,ren}, D(\hat{L}_{\varepsilon,ren}\right)$ *generates a holomorphic semigroup* $\hat{U}_\varepsilon(t), t \ge 0$ *in* $\mathcal{L}_C$ *and moreover* $\hat{L}_{\varepsilon,ren} \in \mathcal{H}(\omega)$ *for all* $\omega \in (0, \frac{\pi}{2})$*.*

Proof. Proof is very similar to the unscaled case (Theorem 4.1). Note only the following facts:

1. operator $E_1 + \varepsilon(F_1 + F_2)$ with domain $D(\hat{L}_{\varepsilon,ren})$ is the generator of a contraction semigroup on $\mathcal{L}_C$ and belongs to $\mathcal{H}(\omega)$ for all $\omega \in (0, \frac{\pi}{2})$,
2. operator $\hat{L}_{ren,\varepsilon} - (E_1 + \varepsilon(F_1 + F_2))$ with the same domain is relatively bounded with respect to the operator $E_1 + \varepsilon(F_1 + F_2)$.

Hence the operator $\hat{L}_{\varepsilon,ren}$ fulfils the assumptions of the Theorem 3.1 and the statement follows trivially. □

Summarizing, we have constructed holomorphic semigroup $\hat{U}_\varepsilon(t)$ on $\mathcal{L}_C$ which is generated by the operator $\hat{L}_{\varepsilon,ren}$. The next problem is to examine the behaviour of aforementioned semigroup as $\varepsilon \to 0$.

4.4.2 Convergence of the rescaled semigroup

The natural candidate for the limiting operator (as ε tends to 0) is the pointwise limit of the operator $\hat{L}_{\varepsilon,ren}$ which we denote by $\hat{L}^V$, i.e.

$$\lim_{\varepsilon\to 0}\hat{L}_{\varepsilon,ren}G(\eta^1,\eta^2) =: \hat{L}^V G(\eta^1,\eta^2) = E_1 G(\eta^1,\eta^2) + \sum_{i=2}^{7} E_i G(\eta^1,\eta^2) \quad (4.28)$$

for all $(\eta^1, \eta^2) \in \Gamma_0^2$. Define the domain $D(\hat{L}^V)$ as

$$D(\hat{L}^V) := \{G \in \mathcal{L}_C | \ \left(m^+ | \cdot^1 | + m^- | \cdot^2 |\right) G(\cdot^1, \cdot^2) \in \mathcal{L}_C\}.$$

Proceeding in the same manner as previously we immediately get the following

Corollary 4.2. *Under the assumption (1) of Theorem 4.2, $\hat{L}^V$ generates a holomorphic semigroup $\hat{U}^V(t), t \geq 0$ on $\mathcal{L}_C$.*

The question remains, whether the semigroup $\hat{U}_{t,\varepsilon}$ converges strongly to $\hat{U}_t^V$ as $\varepsilon \to 0$. One of the possible ways to answer this question, is to show the convergence of the corresponding resolvents as in the following theorem:

Theorem 4.3 (see e.g. [Kat95], Chap. IX, Thm. 2.16)**.** *Let T and T_n, $n = 1, 2, \ldots$ generate quasi-bounded semigroups $U^T(t)$, $U^{T_n}(t)$ respectively. If there exists $\beta > 0$ and $\lambda :$ $Re\lambda > \beta$ such that*

$$(T_n - \lambda 1\!\!1)^{-1} \xrightarrow{s} (T - \lambda 1\!\!1)^{-1}, \tag{4.29}$$

then

$$U^{T_n}(t) \xrightarrow{s} U^T(t) \tag{4.30}$$

uniformly in any finite interval of $t \geq 0$. Conversely, if (4.30) holds for all t such that $0 \leq t \leq b, b > 0$, then (4.29) holds for every λ with $Re\lambda > \beta$.

It is enough to assure the condition (4.29). Using the results from [FKK10c], one can show that the following conditions are sufficient for equation (4.29) to be satisfied:

A1 For any $\varepsilon \geq 0$ the operator $(\hat{L}_{\varepsilon,ren}, D(\hat{L}_{\varepsilon,ren}))$ can be represented as

$$\hat{L}_{\varepsilon,ren} := A_1(\varepsilon) + A_2(\varepsilon),$$

where $D(A_1(\varepsilon)) = D(A_2(\varepsilon)) := D(\hat{L}_{\varepsilon,ren})$.

A2 There exists $\beta > 0$ and $\lambda > \beta$ such that

1. λ belongs to the resolvent set of $A_1(\varepsilon)$ for any $\varepsilon \geq 0$ and

$$(A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \xrightarrow{s} (A_1(0) - \lambda 1\!\!1)^{-1}, \varepsilon \to 0,$$

2.

$$\sup_{\varepsilon > 0} \left\| (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \right\|_C \leq \left\| (A_1(0) - \lambda 1\!\!1)^{-1} \right\|_C,$$

3. for any $\varepsilon \geq 0$
$$\left\| A_2(\varepsilon)(A_1(\varepsilon) - \lambda \mathbb{1})^{-1} \right\|_C < 1,$$

4. $\left(A_2(\varepsilon)(A_1(\varepsilon) - \lambda \mathbb{1})^{-1} + \mathbb{1}\right)^{-1}$ converges strongly to the operator $\left(A_2(0)(A_1(0) - \lambda \mathbb{1})^{-1} + \mathbb{1}\right)^{-1}$ as $\varepsilon \to 0$.

We are now ready to state the main result of this section:

Theorem 4.4. *$\hat{U}_\varepsilon(t)$ converges strongly to $\hat{U}^V(t)$ as $\varepsilon \to 0$ on any finite interval of time, provided that the conditions (1)-(3) of Theorem 4.2 are satisfied.*

Proof. We will prove, that $\hat{L}_{\varepsilon,ren}$ fulfils the conditions A1 and A2 stated above.
A1. Denote

$$\Psi_1(x, \eta^1, \eta^2) := A_1^- \sum_{x' \in \eta^1 \setminus x} a_1^-(x - x') + B_1^- \sum_{y \in \eta^2} b_1^-(x - y) \tag{4.31}$$

$$\Psi_2(y, \eta^1, \eta^2) := A_2^- \sum_{y' \in \eta^2 \setminus y} a_2^-(y - y') + B_2^- \sum_{x \in \eta^1} b_2^-(y - x) \tag{4.32}$$

and for $\varepsilon \geq 0$, define

$$\begin{aligned} A_1(\varepsilon) G(\eta^1, \eta^2) := & - \left(m^+ |\eta^1| + m^- |\eta^2|\right) G(\eta^1, \eta^2) \\ & - \varepsilon \left(\sum_{x \in \eta^1} \Psi_1(x, \eta^1, \eta^2) + \sum_{y \in \eta^2} \Psi_2(y, \eta^1, \eta^2) \right) G(\eta^1, \eta^2) \end{aligned}$$

with $D(A_1(\varepsilon)) := D(\hat{L}_{ren,\varepsilon})$ and

$$A_2(\varepsilon) = \hat{L}_{\varepsilon,ren} - A_1(\varepsilon) \tag{4.33}$$

with $D(A_2(\varepsilon)) := D(A_1(\varepsilon))$. Note $A_2(0) = \hat{L}^V - A_1(0)$. It's obvious, that the assumption **A1** is satisfied.
A2-(1). Let $G \in \mathcal{L}_C$ and $\lambda > 0$, then the calculation of the norm

$$\left\| (A_1(\varepsilon) - \lambda \mathbb{1})^{-1} G - (A_1(0) - \lambda \mathbb{1})^{-1} G \right\|_C \tag{4.34}$$

yields

$$\int_{\Gamma_0^2} \left| \frac{1}{m^+|\eta^1| + m^-|\eta^2| + \varepsilon \left(\sum_{x \in \eta^1} \Psi_1(x, \eta^1, \eta^2) + \sum_{y \in \eta^2} \Psi_2(y, \eta^1, \eta^2) \right) + \lambda} + \frac{1}{-(m^+|\eta^1| + m^-|\eta^2| + \lambda)} \right| |G(\eta^1, \eta^2)| C^{|\eta^1| + |\eta^2|} \lambda^2(d\eta^1, d\eta^2)$$

which we shortly write as

$$\int_{\Gamma_0^2} |F(\varepsilon,\eta^1,\eta^2)G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}\lambda^2(d\eta^1,d\eta^2)$$

where

$$F(\varepsilon,\eta^1,\eta^2) := \frac{\varepsilon\left(\sum_{x\in\eta^1}\Psi_1(x,\eta^1,\eta^2)+\sum_{y\in\eta^2}\Psi_2(y,\eta^1,\eta^2)\right)}{(m^+|\eta^1|+m^-|\eta^2|+\lambda)}$$

$$\times \frac{1}{m^+|\eta^1|+m^-|\eta^2|+\varepsilon\left(\sum\limits_{x\in\eta^1}\Psi_1(x,\eta^1,\eta^2)+\sum\limits_{y\in\eta^2}\Psi_2(y,\eta^1,\eta^2)\right)+\lambda}$$

and since $0 < F \leq \frac{1}{\lambda}$ and $\lim_{\varepsilon\to 0} F(\varepsilon,\eta^1,\eta^2) = 0$ for all $(\eta^1,\eta^2) \in \Gamma_0^2$ the condition A2-(1) holds.
A2-(2). Let $G \in \mathcal{L}_C$, $\lambda > 0$. Then the norm

$$\left\|A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\right\|_C \tag{4.35}$$

is equal to

$$\int\limits_{\Gamma_0\times\Gamma_0}\left|\frac{1}{m^+|\eta^1|+m^-|\eta^2|+\varepsilon\left(\sum_{x\in\eta^1}\Psi_1(x,\eta^1,\eta^2)+\sum_{y\in\eta^2}\Psi_2(y,\eta^1,\eta^2)\right)+\lambda}\right|$$

$$\times|G(\eta^1,\eta^2)|C^{|\eta^1|+|\eta^2|}\lambda^2(d\eta^1,d\eta^2)$$

and we obviously have

$$\begin{aligned}\left\|A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\right\|_C &\leq \int\limits_{\Gamma_0^2}\frac{|G(\eta^1,\eta^2)|}{|m^+|\eta^1|+m^-|\eta^2|+\lambda|}C^{|\eta^1|+|\eta^2|}\lambda^2(d\eta^1,d\eta^2)\\ &= \left\|A_1(0)-\lambda 1\!\!1)^{-1}\right\|_C,\end{aligned}$$

hence the condition A2-(2) holds.
A3-(3). Let $\lambda > 0$. To show that

$$\left\|A_2(\varepsilon)(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\right\|_C < 1, \tag{4.36}$$

notice first, that from the assumptions of the Theorem 4.2 we obtain that for every $\varepsilon > 0$

$$\|A_2(\varepsilon)\|_C \leq (\vartheta_1+\vartheta_2)\,\|A_1(\varepsilon)\|_C, \tag{4.37}$$

and for every $\lambda > 0$ the following inequality holds:

$$\|A_1(\varepsilon)\|_C \leq \|A_1(\varepsilon) - \lambda \mathbb{1}\|_C. \tag{4.38}$$

Using the latter facts, we obtain

$$\begin{aligned} \left\|A_2(\varepsilon)(A_1(\varepsilon) - \lambda\mathbb{1})^{-1}\right\|_C &\leq (\vartheta_1 + \vartheta_2)\left\|A_1(\varepsilon)(A_1(\varepsilon) - \lambda\mathbb{1})^{-1}\right\|_C \\ &\leq (\vartheta_1 + \vartheta_2) < 1. \end{aligned}$$

A2-(4). We will show that $(A_2(\varepsilon)(A_1(\varepsilon) - \lambda\mathbb{1})^{-1} + \mathbb{1})^{-1}$ converges strongly to the operator $(A_2(0)(A_1(0) - \lambda\mathbb{1})^{-1} + \mathbb{1})^{-1}$ as $\varepsilon \to 0$. First denote with

$$C_\varepsilon := A_2(\varepsilon)(A_1(\varepsilon) - \lambda\mathbb{1})^{-1},$$

and

$$Q := \left(A_2(0)(A_1(0) - \lambda\mathbb{1})^{-1} + \mathbb{1}\right)^{-1}.$$

Note that the condition A2-(4) is equivalent to $(C_\varepsilon + 1)^{-1} \xrightarrow{s} Q$, as $\varepsilon \to 0$, but

$$\begin{aligned} (C_\varepsilon + 1)^{-1} - Q &= (C_\varepsilon + 1)^{-1}(Q^{-1} - C_\varepsilon - 1)Q \\ &= (C_\varepsilon + 1)^{-1}\left(A_2(0)(A_1(0) - \lambda)^{-1} - C_\varepsilon\right)\left(A_2(0)(A_1(0) - \lambda)^{-1} + 1\right)^{-1} \end{aligned}$$

and $\|C_\varepsilon + 1\|_C < \frac{1}{\delta}$ (assumption A2-(3)). Hence it is sufficient to show that

$$A_2(\varepsilon)(A_1(\varepsilon) - \lambda\mathbb{1})^{-1} \xrightarrow{s} (A_2(0)(A_1(0) - \lambda\mathbb{1})^{-1} \tag{4.39}$$

as $\varepsilon \to 0$. Now set $A_1 := A_1(0)$ and $A_2 := A_2(0)$. It is clear, that

$$A_2(\varepsilon) = A_2 + \varepsilon B_2$$

for every $\varepsilon > 0$. The convergence of (4.39) is then equivalent to

$$\begin{aligned} &A_2(A_1(\varepsilon) - \lambda\mathbb{1})^{-1} \xrightarrow{s} A_2(A_1 - \lambda\mathbb{1})^{-1}, \text{ and} \\ &\varepsilon B_2(A_1(\varepsilon) - \lambda\mathbb{1})^{-1} \xrightarrow{s} 0, \text{ as } \varepsilon \to 0. \end{aligned}$$

First, because $D(A_1(\varepsilon)) \subset D(A_1) = D(A_2)$, the following identity is true:

$$A_2(A_1(\varepsilon) - \lambda\mathbb{1})^{-1} = A_2(A_1 - \lambda\mathbb{1})^{-1}(A_1 - \lambda\mathbb{1})(A_1(\varepsilon) - \lambda\mathbb{1})^{-1},$$

and we need to show that

$$(A_1 - \lambda\mathbb{1})(A_1(\varepsilon) - \lambda\mathbb{1})^{-1} \xrightarrow{s} \mathbb{1}.$$

Take $G \in \mathcal{L}_C$, then $\| \left((A_1 - \lambda 1\!\!1)(A_1(\varepsilon) - \lambda 1\!\!1)^{-1} - 1 \right) G\|_C$ is equal to

$$\int_{\Gamma_0^2} \left| \frac{m^+|\eta^1| + m^-|\eta^2| + \lambda}{m^+|\eta^1| + m^-|\eta^2| + \varepsilon \left(\sum_{x\in\eta^1} \Psi_1(x,\eta^1,\eta^2) + \sum_{y\in\eta^2} \Psi_2(y,\eta^1,\eta^2) \right) + \lambda} - 1 \right| |G(\eta^1,\eta^2)| C^{|\eta^1|+|\eta^2|} \lambda^2(d\eta^1, d\eta^2)$$

and hence to

$$\int_{\Gamma_0^2} \left| \frac{\varepsilon \left(\sum_{x\in\eta^1} \Psi_1(x,\eta^1,\eta^2) + \sum_{y\in\eta^2} \Psi_2(y,\eta^1,\eta^2) \right)}{m^+|\eta^1| + m^-|\eta^2| + \varepsilon \left(\sum_{x\in\eta^1} \Psi_1(x,\eta^1,\eta^2) + \sum_{y\in\eta^2} \Psi_2(y,\eta^1,\eta^2) \right) + \lambda} \right| \times |G(\eta^1,\eta^2)| C^{|\eta^1|+|\eta^2|} \lambda^2(d\eta^1, d\eta^2)$$

which converges to 0 when $\varepsilon \to 0$. On the other hand, we have

$$\|\varepsilon B_2 G\|_C \leq \vartheta_2 \|A_1(\varepsilon)G\|_C$$

for all $G \in \mathcal{L}_C$, hence

$$\begin{aligned} \left\|\varepsilon B_2 (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} G\right\|_C \leq & \quad \varepsilon\vartheta_2 \left\|A_1(\varepsilon)(A_1(\varepsilon) - \lambda 1\!\!1)^{-1} G\right\|_C \\ \leq & \quad \varepsilon\vartheta_2 \left\|A_1(\varepsilon)(A_1(\varepsilon) - \lambda 1\!\!1)^{-1}\right\| \|G\|_C \\ \leq & \quad \varepsilon\vartheta_2 \|G\|_C \to 0, \, \varepsilon \to 0. \end{aligned}$$

Thus we have proved that all the assumptions A1 and A2 are fulfilled. The assertion of the Theorem 4.4 follows now from the Theorem 3.1. in [FKK10c]. □

4.4.3 Vlasov-type equation for the model

We will now derive a type of the Vlasov equation for the scaling realized above. The dual Vlasov operator in the sense of duality given in the previous section (see (4.19)) can be easily calculated:

$$\begin{aligned}
\hat{V}^* k(\eta) = & - \left(m^+|\eta^1| + m^-|\eta^2|\right) k(\eta^1, \eta^2) \\
& - \int_{\mathbb{R}^d} \left(A_1^- \sum_{x' \in \eta^1} a_1^-(x - x') + B_2^- \sum_{y \in \eta^2} b_2^-(x - y) \right) k(\eta^1 \cup x, \eta^2) dx \\
& - \int_{\mathbb{R}^d} \left(A_2^- \sum_{y' \in \eta^2} a_2^-(y - y') + B_1^- \sum_{x \in \eta^1} b_1^-(y - x) \right) k(\eta^1, \eta^2 \cup y) dy \\
& + A_1^+ \sum_{x \in \eta^1} \int_{\mathbb{R}^d} a_1^+(x - x') k(\eta^1 \cup x' \setminus x, \eta^2) dx' \\
& + B_1^+ \sum_{x \in \eta^1} \int_{\mathbb{R}^d} b_1^+(x - y) k(\eta^1 \setminus x, \eta^2 \cup y) dy \\
& + A_2^+ \sum_{y \in \eta^2} \int_{\mathbb{R}^d} a_2^+(y - y') k(\eta^1, \eta^2 \cup y' \setminus y) dy' \\
& + B_2^+ \sum_{y \in \eta^2} \int_{\mathbb{R}^d} b_2^+(y - x) k(\eta^1 \cup x, \eta^2 \setminus y) dx.
\end{aligned}$$

We consider now the Cauchy problem associated with the generator defined above, i.e.:

$$\begin{cases} \frac{\partial}{\partial t} k_t = \hat{V}^* k_t, \\ k_t|_{t=0} = k_0. \end{cases} \tag{4.40}$$

Assume that

$$k_0(\eta^1, \eta^2) = e_\lambda(\rho_0^1, \eta^1) \cdot e_\lambda(\rho_0^2, \eta^2)$$

and ρ_0^1, ρ_0^2 are measurable functions on $\mathbb{R}^d$ such that the following two conditions hold:

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |\rho_0^1(x)| \le C, \quad \text{and} \quad \operatorname*{ess\,sup}_{y \in \mathbb{R}^d} |\rho_0^2(y)| \le C.$$

Then the Cauchy problem (4.40) has a mild solution

$$k_t(\eta^1, \eta^2) = e_\lambda(\rho_t^1, \eta^1) \cdot e_\lambda(\rho_t^2, \eta^2) \in \mathcal{Q}_C$$

where ρ_t^1 and ρ_t^2 are solutions to the following equations (provided they exist):

$$\begin{cases} \frac{\partial}{\partial t}\rho_t^1(x) &= -m^+\rho_t^1(x) \\ &\quad - A_1^-\rho_t^1(x)\left(a_1^- * \rho_t^1\right)(x) - B_1^-\rho_t^1(x)\left(b_1^- * \rho_t^2\right)(x) \\ &\quad + A_1^+\left(a_1^+ * \rho_t^1\right)(x) + B_1^+\left(b_1^+ * \rho_t^2\right)(x) \\ \rho_t^1(x)|_{t=0} &= \rho_0^1(x), \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial t}\rho_t^2(y) &= -m^-\rho_t^2(y) \\ &\quad - B_2^-\rho_t^2(y)\left(b_2^- * \rho_t^1\right)(y) - A_2^-\rho_t^2(y)\left(a_2^- * \rho_t^2\right)(y) \\ &\quad + A_2^+\left(a_2^+ * \rho_t^2\right)(y) + B_2^+\left(b_2^+ * \rho_t^1\right)(y) \\ \rho_t^2(y)|_{t=0} &= \rho_0^2(y). \end{cases}$$

To see that, let $k_t(\eta^1, \eta^2) := e_\lambda(\rho_t^1, \eta^1)e_\lambda(\rho^2, \eta^2)$. Then $\frac{\partial}{\partial t}k_t(\eta^1, \eta^2)$ becomes:

$$\sum_{x\in\eta^1} \frac{\partial}{\partial t}\rho_t^1(x)e_\lambda(\rho_t^1, \eta^1 \setminus x)e_\lambda(\rho_t^2, \eta^2) + \sum_{y\in\eta^2} \frac{\partial}{\partial t}\rho_t^2(y)e_\lambda(\rho_t^2, \eta^2 \setminus y)e_\lambda(\rho_t^1, \eta^1).$$

On the other hand, equation (4.40) yields:

$$\begin{aligned} \frac{\partial}{\partial t}k_t(\eta^1,\eta^2) = &-\left(m^+|\eta^1| + m^-|\eta^2|\right) e_\lambda(\rho_t^1, \eta^1)e_\lambda(\rho^2, \eta^2) \\ &- e_\lambda(\rho_t^1, \eta^1)e_\lambda(\rho^2, \eta^2)\left(A_1^- \sum_{x'\in\eta^1}(a_1^- * \rho_t^1)(x') + B_2^- \sum_{y\in\eta^2}(b_2^- * \rho_t^1)(y)\right) \\ &- e_\lambda(\rho_t^1, \eta^1)e_\lambda(\rho^2, \eta^2)\left(A_2^- \sum_{y'\in\eta^2}(a_2^- * \rho_t^2)(y') + B_1^- \sum_{x\in\eta^1}(b_1^- * \rho_t^2)(x)\right) \\ &+ A_1^+ e_\lambda(\rho^2, \eta^2)\sum_{x\in\eta^1} e_\lambda(\rho_t^1, \eta^1 \setminus x)(a_1^+ * \rho_t^1)(x) \\ &+ B_1^+ e_\lambda(\rho_t^2, \eta^2)\sum_{x\in\eta^1} e_\lambda(\rho_t^1, \eta^1 \setminus x)(b_1^+ * \rho_t^2)(x) \\ &+ A_2^+ e_\lambda(\rho_t^1, \eta^1)\sum_{y\in\eta^2} e_\lambda(\rho_t^2, \eta^2 \setminus y)(a_2^+ * \rho_t^2)(y) \\ &+ B_2^+ e_\lambda(\rho_t^1, \eta^1)\sum_{y\in\eta^2} e_\lambda(\rho_t^2, \eta^2 \setminus y)(b_2^+ * \rho_t^1)(y). \end{aligned}$$

Easy calculation shows that (4.40) is satisfied if only ρ_t^1 and ρ_t^2 satisfy two equations above.

Thus we have obtained the Vlasov type equation for the associated density (and first correlation function). Note, that the densities of two population are dependent and one can not separate them in general. We also have the *chaos preservation property* of the Vlasov operator, i.e. it preserves the product form of the initial condition k_0.

Chapter 5

Potts-type model

5.1 Introduction

In this chapter, we prove the existence of a strongly continuous contraction semigroup associated with the symbol of the pre-generator of the two-component analogue of the Glauber dynamics, which we will call the Potts-type model with two types of particles (see e.g. [FKO10]). We also derive a Vlasov-type equation for the process.

Define the interaction energy as

$$E^{\phi}(x,\gamma) := \sum_{y\in\gamma} \phi(x-y)\,(\leq \infty) \tag{5.1}$$

for a positive function $\phi : \mathbb{R}^d \to \mathbb{R}$. The pre-generator of Potts-type model is given as follows:

$$\begin{aligned} LF(\gamma^1,\gamma^2) := &\sum_{x\in\gamma^1} D_x^{1-}F(\gamma^1,\gamma^2) + \varkappa \int_{\mathbb{R}^d} e^{-\beta E^{\phi}(x,\gamma^2)} D_x^{1+}F(\gamma^1,\gamma^2)dx \\ &+ \sum_{y\in\gamma^2} D_y^{2-}F(\gamma^1,\gamma^2) + \varkappa \int_{\mathbb{R}^d} e^{-\beta E^{\phi}(y,\gamma^1)} D_y^{2+}F(\gamma^1,\gamma^2)dy, \end{aligned} \tag{5.2}$$

for $\varkappa > 0$. Recall the notation:

$$\begin{aligned} D_x^{1-}F(\gamma^1,\gamma^2) &= F(\gamma^1\setminus x,\gamma^2) - F(\gamma^1,\gamma^2),\\ D_x^{1+}F(\gamma^1,\gamma^2) &= F(\gamma^1\cup x,\gamma^2) - F(\gamma^1,\gamma^2),\\ D_x^{2-}F(\gamma^1,\gamma^2) &= F(\gamma^1,\gamma^2\setminus x) - F(\gamma^1,\gamma^2),\\ D_x^{2+}F(\gamma^1,\gamma^2) &= F(\gamma^1,\gamma^2\cup x) - F(\gamma^1,\gamma^2). \end{aligned}$$

Heuristically it means that while the particles in each population die independently of each other, the appearance of a new particle depends not only on its population but also on the particles of the other type.

Throughout this chapter we will always assume that the potential ϕ satisfies

$$\phi(x) \geq 0, \quad x \in \mathbb{R}^d. \tag{5.3}$$

Moreover, the following condition holds:

$$\Phi := \int_{\mathbb{R}^d} \phi(x) dx < +\infty. \tag{5.4}$$

Note, that (5.4) implies

$$C(\beta) := \int\limits_{\mathbb{R}^d} \left(e^{-\beta\phi(x)} - 1\right) dx < \infty. \tag{5.5}$$

5.2 Construction of the process on Γ_0^2

Using the approximation methods (see e.g. [FKKZ10]), we will construct the semigroup associated to the symbol of the operator L on the linear space introduced in the previous chapter, namely the space $\mathcal{L}_C$ which is defined as

$$\mathcal{L}_C = L^1 \left(\Gamma_0 \times \Gamma_0, C^{(|\eta^1|+\eta^2|)} \lambda(d\eta^1)\lambda(d\eta^2)\right)$$

for $C > 0$ and equipped with the norm

$$\|G\|_C := \int_{\Gamma_0^2} \left|G(\eta^1, \eta^2)\right| C^{(|\eta^1|+|\eta^2|)} \lambda(d\eta^1)\lambda(d\eta^2).$$

5.2.1 Symbol of the generator L

Notice, that to calculate the symbol for Glauber-Potts model we use the one-dimensional K-transform instead of its multicomponent analogue $\mathcal{K}$. The reason for that is that the birth coefficients are in fact images of certain functions under K-transform. The details will be given below. Recall also, that we write $\eta = (\eta^1, \eta^2)$ if it does not lead to confusion.

Proposition 5.1. *The symbol of the operator L, i.e. $\hat{L} := K^{-1}LK$, has the following form*

$$\hat{L}G(\eta) = -(|\eta^1| + |\eta^2|)G(\eta^1,\eta^2)$$
$$+\varkappa \sum_{\xi^2\subset\eta^2} \int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2)e_\lambda(e^{-\beta\phi(x-\cdot)}-1,\eta^2\setminus\xi^2)e^{-\beta E^\phi(x,\xi^2)}dx$$
$$+\varkappa \sum_{\xi^1\subset\eta^1} \int_{\mathbb{R}^d} G(\xi^1,\eta^2\cup y)e_\lambda(e^{-\beta\phi(y-\cdot)}-1,\eta^1\setminus\xi^1)e^{-\beta E^\phi(y,\xi^1)}dy.$$

Proof. We will calculate the components of the symbol. Denote

$$L_0^1G(\eta^1,\eta^2) := K^{-1}\left(\sum_{x\in\cdot^1} D_x^{1-}KG(\cdot^1,\cdot^2)\right)(\eta^1,\eta^2).$$

Then, we have

$$\begin{aligned}
L_0^1G(\eta^1,\eta^2) &= K^{-1}\left(\sum_{x\in\cdot^1}\left[KG(\cdot^1\setminus x,\cdot^2)-KG(\cdot^1,\cdot^2)\right]\right)(\eta^1,\eta^2)\\
&= K^{-1}\left(\sum_{x\in\cdot^1}\left[\sum_{\xi^1\subset\cdot^1\setminus x}G(\xi^1,\cdot^2)-\sum_{\xi^1\subset\cdot^1}G(\xi^1,\cdot^2)\right]\right)(\eta^1,\eta^2)\\
&= K^{-1}\left(-\sum_{x\in\cdot^1}\sum_{\xi^1\subset\cdot^1\setminus x}G(\xi^1\cup x,\cdot^2)\right)(\eta^1,\eta^2)\\
&= -\sum_{\zeta^1\subset\eta^1}(-1)^{|\eta^1\setminus\zeta^1|}\sum_{x\in\zeta^1}\sum_{\xi^1\in\zeta^1\setminus x}G(\xi^1\cup x,\eta^2)\\
&= -\sum_{\zeta^1\subset\eta^1}(-1)^{|\eta^1\setminus\zeta^1|}\sum_{x\in\zeta^1}KG(\cdot\cup x,\eta^2)(\zeta^1\setminus x)\\
&= -\sum_{x\in\eta^1}\sum_{\zeta^1\subset\eta^1\setminus x}(-1)^{|\eta^1\setminus(\zeta^1\cup x)|}KG(\cdot\cup x)(\zeta^1)\\
&= -|\eta^1|G(\eta^1,\eta^2).
\end{aligned}$$

In the case of $L_0^2G(\eta^1,\eta^2) := K^{-1}\left(\sum_{y\in\cdot^2} D_y^{2-}KG(\cdot^1,\cdot^2)\right)(\eta^1,\eta^2)$ the similar calculation yields $L_0^2G(\eta^1,\eta^2) := -|\eta^2|G(\eta^1,\eta^2)$.

Before we will calculate the second part,define

$$L_1^1G(\eta^1,\eta^2) := K^{-1}\left[\varkappa\int_{\mathbb{R}^d} e^{-\beta E^\phi(x,\cdot^2)}D_x^{1+}KG(\cdot^1,\cdot^2)dx\right](\eta^1,\eta^2)$$

and notice that for $\eta \in \Gamma_0$, $x \in \mathbb{R}^d$:

$$e^{-\beta E^{\phi}(x,\eta)} = \prod_{y\in\eta} e^{-\beta\phi(x-y)} = Ke_{\lambda}\left(e^{-\beta\phi(x-\cdot)} - 1\right)(\eta), \tag{5.6}$$

then, using (5.6), we can calculate

$$\begin{aligned}
L_1^1 G(\eta) &= \varkappa K^{-1}\left(\int_{\mathbb{R}^d} e^{-\beta E^{\phi}(x,\cdot^2)}\left[KG(\cdot^1\cup x,\cdot^2) - KG(\cdot^1,\cdot^2)\right]dx\right)(\eta^1,\eta^2)\\
&= \varkappa K^{-1}\left(\int_{\mathbb{R}^d} K\prod_{y\in\cdot^2}(e^{-\beta\phi(x-y)}-1)\cdot KG(\cdot^1\cup x,\cdot^2)dx\right)(\eta^1,\eta^2)\\
&= \varkappa K^{-1}\left(\int_{\mathbb{R}^d} K\left[\prod_{y\in\cdot^2}(e^{-\beta\phi(x-y)}-1)\star G(\cdot^1\cup x,\cdot^2)\right]dx\right)(\eta^1,\eta^2)\\
&= \varkappa\int_{\mathbb{R}^d}\left[\prod_{y\in\cdot^2}(e^{-\beta\phi(x-y)}-1)\star G(\cdot^1\cup x,\cdot^2)\right](\eta^1,\eta^2)dx\\
&= \varkappa\sum_{\xi^2\subset\eta^2}\int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2)\prod_{y\in\eta^2\setminus\xi^2}\left(e^{-\beta\phi(x-y)}-1\right)e^{-\beta E^{\phi}(x,\xi^2)}dx.
\end{aligned}$$

The analogous calculation can be done for the operator defined as

$$L_1^2 G(\eta^1,\eta^2) := K^{-1}\left[\varkappa\int_{\mathbb{R}^d} e^{-\beta E^{\phi}(x,\cdot^1)}D_x^{2+}KG(\cdot^1,\cdot^2)dy\right](\eta^1,\eta^2),$$

giving

$$L_1^2 G(\eta^1,\eta^2) = \varkappa\sum_{\xi^1\subset\eta^1}\int_{\mathbb{R}^d} G(\xi^1,\eta^2\cup y)e_{\lambda}(e^{-\beta\phi(y-\cdot)}-1,\eta^1\setminus\xi^1)e^{-\beta E^{\phi}(y,\xi^1)}dy$$

Clearly $\hat{L} = L_0 + L_1$, where $L_{\#} = L_{\#}^1 + L_{\#}^2$, $\# \in \{0,1\}$ thus the Proposition is proven. □

To give the proper meaning to the operator $\hat{L}$, we define the domain of $\hat{L}$ by

$$D(\hat{L}) := \mathcal{L}_{2C} \subset \mathcal{L}_C$$

for $C > 0$. Notice that the embedding $\mathcal{L}_{2C} \subset \mathcal{L}_C$ is dense. In the sequel the following property of $\hat{L}$ will be useful:

Proposition 5.2. *Operator $(\hat{L}, D(\hat{L}))$ defines a linear operator in $\mathcal{L}_C$.*

Proof. It is obvious that $\hat{L}$ is linear, hence we should prove that for $G \in \mathcal{L}_{2C}$ we have $\|\hat{L}G\|_{2C} < \infty$. Let $G \in \mathcal{L}_{2C}$ and denote

$$\begin{aligned}
L_0G(\eta^1,\eta^2) &:= -(|\eta^1|+|\eta^2|)G(\eta^1,\eta^2),\\
L_1G(\eta^1,\eta^2) &:= \varkappa \sum_{\xi^2\subset\eta^2}\int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2)e_\lambda(e^{-\beta\phi(x-\cdot)},\eta^2\setminus\xi^2)e^{-\beta E^\phi(x,\xi^2)}dx,\\
L_2G(\eta^1,\eta^2) &:= \varkappa \sum_{\xi^1\subset\eta^1}\int_{\mathbb{R}^d} G(\xi^1,\eta^2\cup y)e_\lambda(e^{-\beta\phi(y-\cdot)},\eta^1\setminus\xi^1)e^{-\beta E^\phi(y,\xi^1)}dy.
\end{aligned}$$

Clearly $\hat{L} = L_0 + L_1 + L_2$. And we have for L_0:

$$\begin{aligned}
L_0G(\eta^1,\eta^2) &= \int_{\Gamma_0\times\Gamma_0}\big(|\eta^1|+|\eta^2|\big)\left|G(\eta^1,\eta^2)\right|C^{(|\eta^1|+|\eta^2|)}\lambda(d\eta^1)\lambda(d\eta^2)\\
&\le \int_{\Gamma_0\times\Gamma_0} 2^{(|\eta^1|+|\eta^2|)}\left|G(\eta^1,\eta^2)\right|C^{(|\eta^1|+|\eta^2|)}\lambda(d\eta^1)\lambda(d\eta^2)\\
&= \|G\|_{2C} < \infty.
\end{aligned}$$

It remains to prove, that $\|L_1G\|_C < \infty$ and $\|L_2G\|_C < \infty$. But due to the similarity of two operators, we only prove the first inequality. Using Minlos lemma we can calculate $\|L_1G\|_C$ as follows:

$$\begin{aligned}
\|L_1G\|_C &= \varkappa\int_{\Gamma_0}\int_{\Gamma_0}\sum_{\xi^2\subset\eta^2}\int_{\mathbb{R}^d}\left|G(\eta^1\cup x,\xi^2)\right|e_\lambda(e^{-\beta\phi(x-\cdot)}-1,\eta^2\setminus\ \xi^2)\\
&\qquad\qquad\times e^{-\beta E^\phi(x,\xi^2)}C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)\\
&= \varkappa\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}\left|G(\eta^1\cup x,\xi^2)\right|e_\lambda(e^{-\beta\phi(x-\cdot)}-1,\eta^2)\\
&\qquad\qquad\times e^{-\beta E^\phi(x,\xi^2)}C^{|\eta^1|+|\eta^2|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1)\lambda(d\eta^2)\\
&= \varkappa\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}\sum_{x\in\eta^1}\left|G(\eta^1,\xi^2)\right|e_\lambda(e^{-\beta\phi(x-\cdot)}-1,\eta^2)\\
&\qquad\qquad\times e^{-\beta E^\phi(x,\xi^2)}C^{|\eta^1|+|\eta^2|+|\xi^2|-1}\lambda(d\eta^1)\lambda(d\xi^2)\lambda(d\eta^2)\\
&= \frac{\varkappa}{C}e^{CC(\beta)}\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta^1,\xi^2)|\sum_{x\in\eta^1}e^{-\beta E^\phi(x,\xi^2)}C^{|\eta^1|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1)\\
&\le \frac{\varkappa}{C}e^{CC(\beta)}\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta^1,\xi^2)||\eta^1|C^{|\eta^1|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1)\\
&\le \frac{\varkappa}{C}e^{CC(\beta)}\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta^1,\xi^2)|2^{|\eta^1|}C^{|\eta^1|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1)\\
&\le \frac{\varkappa}{C}e^{CC(\beta)}\|G\|_{2C} < \infty.
\end{aligned}$$

Putting this up together we obtain

$$\begin{aligned}\|\hat{L}G\|_C \leq & \quad \|L_0G\|_C + \|L_1G\|_C + \|L_2G\|_C \\ \leq & \quad \Big(1 + \frac{\varkappa}{C}e^{CC(\beta)} + \frac{\varkappa}{C}e^{CC(\beta)}\Big)\, \|G\|_{2C} < \infty\end{aligned}$$

for all $G \in \mathcal{L}_{2C}$. □

5.2.2 Approximation operator and its symbol

Let $\delta \in (0,1)$ be fixed, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and define the following linear operator on $F \in \mathcal{F}_{cyl}(\Gamma_0 \times \Gamma_0)$:

$$\begin{aligned}(P_\delta^\Lambda F)(\gamma^1,\gamma^2) := & \sum_{\eta^1\subset\gamma^1}\sum_{\eta^2\subset\gamma^2} \delta^{|\eta^1|+|\eta^2|}(1-\delta)^{|\gamma^1\setminus\eta^1|+|\gamma^2\setminus\eta^2|}\left(\Xi_\delta^\Lambda(\gamma^1,\gamma^2)\right)^{-1} \\ \times & \int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\omega^1|}(\varkappa\delta)^{|\omega^2|}\prod_{x\in\omega^1}e^{-\beta E^\phi(x,\gamma^2)}\prod_{y\in\omega^2}e^{-\beta E^\phi(y,\gamma^1)} \\ \times & \quad F((\gamma^1\setminus\eta^1)\cup\omega^1,(\gamma^2\setminus\eta^2)\cup\omega^2)\lambda(d\omega^1)\lambda(d\omega^2),\end{aligned}$$

where

$$\begin{aligned}\Xi_\delta^\Lambda(\gamma^1,\gamma^2) := & \int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\omega^1|}(\varkappa\delta)^{|\omega^2|} \\ & \times \prod_{x\in\omega^1}e^{-\beta E^\phi(x,\gamma^2)}\prod_{y\in\omega^2}e^{-\beta E^\phi(y,\gamma^1)}\lambda(d\omega^1)\lambda(d\omega^2).\end{aligned}$$

The operator P_δ^Λ can be considered as the transition operator of a discrete-time Markov chain, the continuous version of which is the process with the evolution defined by (5.2). In other words, the probability of transition from the state (γ^1,γ^2) to $((\gamma^1\setminus\eta^1)\cup\omega^1,(\gamma^2\setminus\eta^2)\cup\omega^2)$ after time δ is equal to:

$$\begin{aligned}&\left(\Xi_\delta^\Lambda(\gamma^1,\gamma^2)\right)^{-1}\delta^{|\eta^1|+|\eta^2|}(1-\delta)^{|\gamma^1\setminus\eta^1|+|\gamma^2\setminus\eta^2|}(\varkappa\delta)^{|\omega^1|} \\ &\qquad \times (\varkappa\delta)^{|\omega^2|}\prod_{x\in\omega^1}e^{-\beta E^\phi(x,\gamma^2)}\prod_{y\in\omega^2}e^{-\beta E^\phi(y,\gamma^1)}.\end{aligned}$$

Before we proceed to the construction of the process, let us remind that the 2-dimensional analog of the K-transform is defined as:

$$\mathcal{K}G(\gamma^1,\gamma^2) := \sum_{\eta^1\Subset\gamma^1}\sum_{\eta^2\Subset\gamma^2}G(\eta^1,\eta^2),$$

together with its inverse:

$$\mathcal{K}^{-1}F(\eta^1,\eta^2) = \sum_{\xi^1\subset\eta^1}\sum_{\xi^2\subset\eta^2}(-1)^{|\eta^1\setminus\xi^1|+|\eta^2\setminus\xi^2|}F(\xi^1,\xi^2).$$

Next proposition allows us to rewrite the operator P_δ^Λ in a more friendly form.

Proposition 5.3. *Operator P_δ^Λ defined above has the following representation:*

$$\begin{aligned} P_\delta^\Lambda F(\gamma^1,\gamma^2) = & \sum_{\zeta^1\subset\gamma^1}\sum_{\zeta^2\subset\gamma^2}(1-\delta)^{|\zeta^1|+|\zeta^2|} \\ \times & \int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\prod_{x\in\sigma^1}e^{-\beta E^\phi(x,\gamma^2)}\prod_{y\in\sigma^2}e^{-\beta E^\phi(y,\gamma^1)} \\ \times & \mathcal{K}^{-1}F(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2). \end{aligned}$$

Proof. Let $G=\mathcal{K}^{-1}F$. We can rewrite the operator P_δ^Λ in the following way:

$$\begin{aligned} P_\delta^\Lambda F(\gamma^1,\gamma^2) = & \left(\Xi_\delta^\Lambda(\gamma^1,\gamma^2)\right)^{-1}\int\limits_{\Gamma_\Lambda}\int\limits_{\Gamma_\Lambda}(\varkappa\delta)^{|\omega^1|}(\varkappa\delta)^{|\omega^2|} \\ & \times \prod_{x\in\omega^1}e^{-\beta E^\phi(x,\gamma^2)}\prod_{y\in\omega^2}e^{-\beta E^\phi(y,\gamma^1)} \\ & \times \sum_{\eta^1\subset\gamma^1}\delta^{|\gamma^1\setminus\eta^1|}(1-\delta)^{|\eta^1|}\sum_{\eta^2\subset\gamma^2}\delta^{|\gamma^2\setminus\eta^2|}(1-\delta)^{|\eta^2|} \\ & \times F(\eta^1\cup\omega^1,\eta^2\cup\omega^2)\lambda(d\omega^1)\lambda(d\omega^2). \end{aligned}$$

Now, using the fact that $F=\mathcal{K}G$, the expression

$$\sum_{\eta^1\subset\gamma^1}\delta^{|\gamma^1\setminus\eta^1|}(1-\delta)^{|\eta^1|}\sum_{\eta^2\subset\gamma^2}\delta^{|\gamma^2\setminus\eta^2|}(1-\delta)^{|\eta^2|}F(\eta^1\cup\omega^1,\eta^2\cup\omega^2)$$

is equal to

$$\sum_{\eta^1\subset\gamma^1}\delta^{|\gamma^1\setminus\eta^1|}(1-\delta)^{|\eta^1|}\sum_{\eta^2\subset\gamma^2}\delta^{|\gamma^2\setminus\eta^2|}(1-\delta)^{|\eta^2|}\sum_{\zeta^1\subset\eta^1\cup\omega^1}\sum_{\zeta^2\subset\eta^2\cup\omega^2}G(\zeta^1,\zeta^2).$$

Using basic set-theoretical facts, we can rewrite the latter as follows:

$$\begin{aligned} & \sum_{\eta^1\subset\gamma^1}\delta^{|\gamma^1\setminus\eta^1|}(1-\delta)^{|\eta^1|}\sum_{\eta^2\subset\gamma^2}\delta^{|\gamma^2\setminus\eta^2|}(1-\delta)^{|\eta^2|} \\ & \qquad\times\sum_{\zeta^1\subset\eta^1}\sum_{\sigma^1\subset\omega^1}\sum_{\zeta^2\subset\eta^2}\sum_{\sigma^2\subset\omega^2}G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2) \end{aligned}$$

and then

$$\sum_{\zeta^1\subset\eta^1}\sum_{\sigma^1\subset\omega^1}\sum_{\zeta^2\subset\eta^2}\sum_{\sigma^2\subset\omega^2} G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)$$
$$\times \sum_{\alpha^1\subset\gamma^1\setminus\zeta^1}\delta^{|\gamma^1\setminus(\alpha^1\cup\zeta^1)|}(1-\delta)^{|\alpha^1\cup\zeta^1|}\sum_{\alpha^2\subset\gamma^2\setminus\zeta^2}\delta^{|\gamma^2\setminus(\alpha^2\cup\zeta^2)|}(1-\delta)^{|\alpha^2\cup\zeta^2|}.$$

The latter is equal to

$$\sum_{\zeta^1\subset\eta^1}\sum_{\sigma^1\subset\omega^1}\sum_{\zeta^2\subset\eta^2}\sum_{\sigma^2\subset\omega^2} G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)(1-\delta)^{|\zeta^1|}(1-\delta)^{|\zeta^2|}$$
$$\times \sum_{\alpha^1\subset\gamma^1\setminus\zeta^1}\delta^{|(\gamma^1\setminus\zeta^1)\setminus\alpha^1|}(1-\delta)^{|\alpha^1|}\sum_{\alpha^2\subset\gamma^2\setminus\zeta^2}\delta^{|(\gamma^2\setminus\zeta^2)\setminus\alpha^2|}(1-\delta)^{|\alpha^2|}.$$

Notice, that by the binomial formula, the two sums in the second line of the latter expression are equal to 1 each, hence

$$\begin{aligned} P_\delta^\Lambda F(\gamma^1,\gamma^2) = {} & \left(\Xi_\delta^\Lambda(\gamma^1,\gamma^2)\right)^{-1}\sum_{\zeta^1\subset\gamma^1}\sum_{\zeta^2\subset\gamma^2}(1-\delta)^{|\zeta^1|+|\zeta^2|} \\ & \times \int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\omega^1|}(\varkappa\delta)^{|\omega^2|}\prod_{x\in\omega^1}e^{-\beta E^\phi(x,\gamma^2)}\prod_{y\in\omega^2}e^{-\beta E^\phi(y,\gamma^1)} \\ & \times \sum_{\sigma^1\subset\omega^1}\sum_{\sigma^2\subset\omega^2}G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\omega^1)\lambda(d\omega^2). \end{aligned}$$

Using the Minlos lemma, we obtain

$$\begin{aligned} P_\delta^\Lambda F(\gamma^1,\gamma^2) &= \left(\Xi_\delta^\Lambda(\gamma^1,\gamma^2)\right)^{-1}\sum_{\zeta^1\subset\gamma^1}\sum_{\zeta^2\subset\gamma^2}(1-\delta)^{|\zeta^1|+|\zeta^2|} \\ & \times \int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\omega^1|+|\sigma^1|}(\varkappa\delta)^{|\omega^2|+|\sigma^2|} \\ & \times \prod_{x\in\omega^1\cup\sigma^1}e^{-\beta E^\phi(x,\gamma^2)}\prod_{y\in\omega^2\cup\sigma^2}e^{-\beta E^\phi(y,\gamma^1)} \\ & \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\omega^1)\lambda(d\sigma^1)\lambda(d\omega^2)\lambda(d\sigma^2). \end{aligned}$$

Using the definition of $\Xi_\delta^\Lambda(\gamma^1,\gamma^2)$ and the fact that $G=\mathcal{K}^{-1}F$, the statement of the proposition is proven. □

Denote the symbol of the operator P_δ^Λ with $\hat{P}_\delta^\Lambda(:=\mathcal{K}^{-1}P_\delta^\Lambda\mathcal{K})$. Using Proposition 5.3 we can easily calculate the form of the symbol.

Proposition 5.4. *The symbol of the operator* $\hat{P}_\delta^\Lambda$ *is given as*

$$\begin{aligned}\hat{P}_\delta^\Lambda G(\eta^1,\eta^2) = \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}&(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\\ \times&\prod_{x\in\zeta^1}e^{-\beta E^\phi(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left(e^{-\beta E^\phi(x',\sigma^2)}-1\right)\\ \times&\prod_{y\in\zeta^2}e^{-\beta E^\phi(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left(e^{-\beta E^\phi(y',\sigma^1)}-1\right)\\ \times&\,G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2)\end{aligned}$$

for all functions $G\in B_{ls}(\Gamma_0^2)$.

Proof. Using Proposition 5.3 and the definition of $\mathcal{K}^{-1}$, we obtain:

$$\begin{aligned}\hat{P}_\delta^\Lambda G(\eta^1,\eta^2) =& \sum_{\xi^1\subset\eta^1}\sum_{\xi^2\subset\eta^2}(-1)^{|\eta^1\setminus\xi^1|+|\eta^2\setminus\xi^2|}\sum_{\zeta^1\subset\xi^1}\sum_{\zeta^2\subset\xi^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\\ &\times\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\prod_{x\in\sigma^1}e^{-\beta E^\phi(x,\xi^2)}\prod_{y\in\sigma^2}e^{-\beta E^\phi(y,\xi^1)}\\ &\times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2)\\ =& \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\\ &\times\sum_{\xi^1\subset\eta^1\setminus\zeta^1}\sum_{\xi^2\subset\eta^2\setminus\zeta^2}(-1)^{|\eta^1\setminus(\zeta^1\cup\xi^1)|+|\eta^2\setminus(\zeta^2\cup\xi^2)|}\\ &\times\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\prod_{x\in\sigma^1}e^{-\beta E^\phi(x,\zeta^2\cup\xi^2)}\\ &\times\prod_{y\in\sigma^2}e^{-\beta E^\phi(y,\zeta^1\cup\xi^1)}G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2).\end{aligned}$$

Now, having in mind the definition of $E^\phi(x,\gamma)$, we can calculate:

$$\begin{aligned}\prod_{x\in\sigma^1}e^{-\beta E^\phi(x,\zeta^2\cup\xi^2)} &= \prod_{y\in\zeta^2}e^{-\beta E^\phi(y,\sigma^1)}\prod_{y'\in\xi^2}e^{-\beta E^\phi(y',\sigma^1)},\text{ and}\\ \prod_{y\in\sigma^2}e^{-\beta E^\phi(y,\zeta^1\cup\xi^1)} &= \prod_{x\in\zeta^1}e^{-\beta E^\phi(x,\sigma^2)}\prod_{x'\in\xi^1}e^{-\beta E^\phi(x',\sigma^2)},\end{aligned}$$

which gives us

$$
\begin{aligned}
\hat{P}_\delta^\Lambda G(\eta^1,\eta^2) = & \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|} \\
& \times \int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\sum_{\substack{\xi^1\subset\eta^1\setminus\zeta^1\\ \xi^2\subset\eta^2\setminus\zeta^2}}(-1)^{|(\eta^1\setminus\zeta^1)\setminus\xi^1|+|(\eta^2\setminus\zeta^2)\setminus\xi^2|} \\
& \times \prod_{x'\in\xi^1}e^{-\beta E^\phi(x',\sigma^2)}\prod_{y'\in\xi^2}e^{-\beta E^\phi(y',\sigma^1)}\prod_{y\in\zeta^2}e^{-\beta E^\phi(y,\sigma^1)} \\
& \times \prod_{x\in\zeta^1}e^{-\beta E^\phi(x,\sigma^2)}G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2).
\end{aligned}
$$

The third line of the latter expression is by definition equal to

$$
\begin{aligned}
&\left(\mathcal{K}^{-1}\prod_{x'\in\cdot^1}e^{-\beta E^\phi(x',\sigma^2)}\prod_{y'\in\cdot^2}e^{-\beta E^\phi(y',\sigma^1)}\right)(\eta^1\setminus\zeta^1,\eta^2\setminus\zeta^2) \\
&\qquad = \prod_{x'\in\eta^1\setminus\zeta^1}\left(e^{-\beta E^\phi(x',\sigma^2)}-1\right)\prod_{y'\in\eta^2\setminus\zeta^2}\left(e^{-\beta E^\phi(y',\sigma^1)}-1\right),
\end{aligned}
$$

thus the symbol of the operator P_δ^Λ has the form:

$$
\begin{aligned}
\hat{P}_\delta^\Lambda G(\eta^1,\eta^2) = & \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|} \\
& \times \prod_{x\in\zeta^1}e^{-\beta E^\phi(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left(e^{-\beta E^\phi(x',\sigma^2)}-1\right) \\
& \times \prod_{y\in\zeta^2}e^{-\beta E^\phi(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left(e^{-\beta E^\phi(y',\sigma^1)}-1\right) \\
& \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2).
\end{aligned}
$$

□

5.2.3 Construction of the semigroup

We now proceed to the construction of the semigroup associated with the operator $\hat{L}$. In what follows we introduce the approximation operator P_δ^Λ in the similar way to the one presented in [FKKZ10]. Later we show that the under certain assumptions on the coefficients $\varkappa$ and C the approximation operator converges to $\hat{L}$ and from this we can conclude some facts about the generated semigroup.

The main result of this part is the following

Theorem 5.1. *Let*

$$\varkappa \leq \min\left\{2Ce^{-2CC(\beta)}, Ce^{-CC(\beta)}\right\}, \tag{5.7}$$

then $(\hat{L}, \mathcal{L}_{2C})$ is a closable linear operator in $\mathcal{L}_C$ and its closure generates a strongly continuous contraction semigroup $\hat{U}_t$ on $\mathcal{L}_C$.

The proof of the Theorem 5.1 is based on the following result:

Lemma 5.1 ([EK05], Corollary 3.8)**.** *Let A be linear operator on Banach space L with $D(A)$ dense in L, and let $\|\cdot\|_{D(A)}$ be a norm on $D(A)$ with respect to which $D(A)$ is a Banach space. For $n \in \mathbb{N}$ let T_n be a linear $\|\cdot\|_L$-contraction on L such that $T_n : D(A) \to D(A)$, and define $A_n = n(T_n - 1)$. Suppose there exist $\omega \geq 0$ and a sequence $\{\varepsilon_n\} \subset (0, +\infty)$ tending to zero such that for $n \in \mathbb{N}$*

$$\|(A_n - A)f\|_L \leq \varepsilon_n \|f\|_{D(A)}, \quad f \in D(A), \tag{5.8}$$

and

$$\|T_n|_{D(A)}\| \leq 1 + \frac{\omega}{n}. \tag{5.9}$$

Then A is closable and the closure of A generates a strongly continuous contraction semigroup on L.

By analogy to the operator $\hat{P}_\delta^\Lambda$, we can define a linear operator on $\mathcal{L}_C$ by:

$$\begin{aligned}
\hat{P}_\delta G(\eta^1, \eta^2) = & \sum_{\zeta^1 \subset \eta^1} \sum_{\zeta^2 \subset \eta^2} (1-\delta)^{|\zeta^1| + |\zeta^2|} \int_{\Gamma_0} \int_{\Gamma_0} (\varkappa\delta)^{|\sigma^1|} (\varkappa\delta)^{|\sigma^2|} \\
\times & \prod_{x \in \zeta^1} e^{-\beta E^\phi(x, \sigma^2)} \prod_{x' \in \eta^1 \setminus \zeta^1} \left(e^{-\beta E^\phi(x', \sigma^2)} - 1\right) \\
\times & \prod_{y \in \zeta^2} e^{-\beta E^\phi(y, \sigma^1)} \prod_{y' \in \eta^2 \setminus \zeta^2} \left(e^{-\beta E^\phi(y', \sigma^1)} - 1\right) \\
\times & G(\zeta^1 \cup \sigma^1, \zeta^2 \cup \sigma^2) \lambda(d\sigma^1) \lambda(d\sigma^2),
\end{aligned} \tag{5.10}$$

for $G \in \mathcal{L}_C$. Notice, that for every $(\eta^1, \eta^2) \in \Gamma_0^2$, $\hat{P}_\delta G(\eta^1, \eta^2) < +\infty$.

We will now prove a series of Lemmas, which we will later use in the proof of the Theorem 5.1.

Lemma 5.2. *Let $\varkappa$, β and C satisfy*

$$\varkappa e^{CC(\beta)} \leq C. \tag{5.11}$$

Then $\hat{P}_\delta$ is a $\mathcal{L}_C$-contraction, i.e. for $G \in \mathcal{L}_C$ we have

$$\|\hat{P}_\delta G\|_C \leq \|G\|_C. \tag{5.12}$$

Proof. Let $G \in \mathcal{L}_C$, then

$$\begin{aligned}\|\hat{P}_\delta G\|_C = \int\limits_{\Gamma_0^2} \Bigg| \sum_{\zeta^1\subset\eta^1} \sum_{\zeta^2\subset\eta^2} (1-\delta)^{|\zeta^1|+|\zeta^2|} \int\limits_{\Gamma_0} \int\limits_{\Gamma_0} (\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|} \\ \times \prod_{x\in\zeta^1} e^{-\beta E^\phi(x,\sigma^2)} \prod_{x'\in\eta^1\setminus\zeta^1} \left(e^{-\beta E^\phi(x',\sigma^2)}-1\right) \\ \times \prod_{y\in\zeta^2} e^{-\beta E^\phi(y,\sigma^1)} \prod_{y'\in\eta^2\setminus\zeta^2} \left(e^{-\beta E^\phi(y',\sigma^1)}-1\right) \\ \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2)\Bigg| C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2).\end{aligned}$$

Using modulus properties and the fact, that $\phi > 0$ the latter can be estimated by

$$\begin{aligned}&\int\limits_{\Gamma_0^2} \sum_{\zeta^1\subset\eta^1} \sum_{\zeta^2\subset\eta^2} (1-\delta)^{|\zeta^1|+|\zeta^2|} \int\limits_{\Gamma_0} \int\limits_{\Gamma_0} (\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|} \\ &\times \prod_{x'\in\eta^1\setminus\zeta^1} \left|e^{-\beta E^\phi(x',\sigma^2)}-1\right| \prod_{y'\in\eta^2\setminus\zeta^2} \left|e^{-\beta E^\phi(y',\sigma^1)}-1\right| \\ &\times \left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right| \lambda(d\sigma^1)\lambda(d\sigma^2) C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2).\end{aligned}$$

Using Minlos lemma, this is equal to

$$\begin{aligned}&\int\limits_{\Gamma_0^2} \int\limits_{\Gamma_0} \int\limits_{\Gamma_0} (1-\delta)^{|\zeta^1|+|\zeta^2|} \int\limits_{\Gamma_0} \int\limits_{\Gamma_0} (\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|} \\ &\times \prod_{x'\in\eta^1} \left|e^{-\beta E^\phi(x',\sigma^2)}-1\right| \prod_{y'\in\eta^2} \left|e^{-\beta E^\phi(y',\sigma^1)}-1\right| C^{|\eta^1\cup\zeta^1|+|\eta^2\cup\zeta^2|} \\ &\times \left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right| \lambda(d\sigma^1)\lambda(d\sigma^2)\lambda(d\zeta^1)\lambda(d\zeta^2)\lambda(d\eta^1)\lambda(d\eta^2),\end{aligned}$$

thus

$$\begin{aligned}\|\hat{P}_\delta G\|_C \le \quad & \int\limits_{\Gamma_0^2} \int\limits_{\Gamma_0} \int\limits_{\Gamma_0} (1-\delta)^{|\zeta^1|+|\zeta^2|} \int\limits_{\Gamma_0} \int\limits_{\Gamma_0} (\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|} \\ &\times \prod_{x'\in\eta^1} \left|e^{-\beta E^\phi(x',\sigma^2)}-1\right| \prod_{y'\in\eta^2} \left|e^{-\beta E^\phi(y',\sigma^1)}-1\right| \\ &\times C^{|\eta^1|+|\zeta^1|+|\eta^2|+|\zeta^2|} \\ &\times \left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right| \lambda(d\sigma^1)\lambda(d\sigma^2)\lambda(d\zeta^1)\lambda(d\zeta^2)\lambda(d\eta^1)\lambda(d\eta^2).\end{aligned}$$

But this is equal to

$$\int_{\Gamma_0}\int_{\Gamma_0}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times \exp\left\{C\int_{\mathbb{R}^d}\left(1-e^{-\beta E^\phi(x',\sigma^2)}\right)dx'\right\}$$
$$\times \exp\left\{C\int_{\mathbb{R}^d}\left(1-e^{-\beta E^\phi(y',\sigma^1)}\right)dy'\right\}$$
$$\times C^{|\zeta^1|+|\zeta^2|}\left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right|\lambda(d\sigma^1)\lambda(d\sigma^2)\lambda(d\zeta^1)\lambda(d\zeta^2).$$

It can be shown (see e.g. [FKKZ10]), that

$$\left(1-e^{-\beta E^\phi(x,\sigma)}\right)\leq\sum_{y\in\sigma}\left(1-e^{-\beta\phi(x-y)}\right)$$

for $\phi>0$, $\beta>0$ and $\sigma\in\Gamma_0$, $x\notin\sigma$, hence

$$\begin{aligned}
\|\hat{P}_\delta G\|_C \leq\; & \int_{\Gamma_0}\int_{\Gamma_0}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}e^{|\sigma^1|CC(\beta)}(\varkappa\delta)^{|\sigma^2|}e^{|\sigma^2|CC(\beta)}\\
& \times C^{|\zeta^1|+|\zeta^2|}\left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right|\lambda(d\sigma^1)\lambda(d\sigma^2)\lambda(d\zeta^1)\lambda(d\zeta^2)\\
=\; & \int_{\Gamma_0}\int_{\Gamma_0}\sum_{\sigma^1\subset\zeta^1}\Big((1-\delta)C\Big)^{|\zeta^1\setminus\sigma^1|}\left(\varkappa\delta e^{CC(\beta)}\right)^{|\sigma^1|}\\
& \times\sum_{\sigma^2\subset\zeta^2}\Big((1-\delta)C\Big)^{|\zeta^2\setminus\sigma^2|}\left(\varkappa\delta e^{CC(\beta)}\right)^{|\sigma^2|}\\
& \times\left|G(\zeta^1,\zeta^2)\right|\lambda(d\zeta^1)\lambda(d\zeta^2)\\
=\; & \int_{\Gamma_0}\int_{\Gamma_0}\left((1-\delta)C+\varkappa\delta e^{CC(\beta)}\right)^{|\zeta^1|}\left((1-\delta)C+\varkappa\delta e^{CC(\beta)}\right)^{|\zeta^2|}\\
& \times\left|G(\zeta^1,\zeta^2)\right|\lambda(d\zeta^1)\lambda(d\zeta^2).
\end{aligned}$$

Using the assumptions we finally obtain the contraction property:

$$\|\hat{P}_\delta G\|_C\leq\int_{\Gamma_0}\int_{\Gamma_0}\left|G(\zeta^1,\zeta^2)\right|C^{|\zeta^1|+|\zeta^2|}\lambda(d\zeta^1)\lambda(d\zeta^2)=\|G\|_C.$$

□

Next proposition shows, that the infinitesimal generator of $\hat{P}_\delta$ approximates $\hat{L}$ in $\mathcal{L}_C$.

Proposition 5.5. *Let* $\hat{L}_\delta := \frac{1}{\delta}\left(\hat{P}_\delta - \mathbb{1}\right)$ *and let the assumptions of Lemma 5.2 be fulfilled. Then for* $G \in \mathcal{L}_C$ *and every* $\delta \in (0,1)$ *the following inequality holds:*

$$\left\| \left(\hat{L}_\delta - \hat{L}\right) G \right\|_C \leq 4\delta \|G\|_{2C}. \tag{5.13}$$

Proof. Let $G \in \mathcal{L}_C$ and recall that $\hat{L} = L_0 + L_1$, where

$$\begin{aligned} L_0 G(\eta^1, \eta^2) =& -(|\eta^1| + |\eta^2|) G(\eta^1, \eta^2), \\ L_1 G(\eta^1, \eta^2) =& \varkappa \sum_{\xi^2 \subset \eta^2} \int_{\mathbb{R}^d} G(\eta^1 \cup x, \xi^2) e_\lambda(e^{-\beta\phi(x-\cdot)} - 1, \eta^2 \setminus \xi^2) e^{-\beta E^\phi(x, \xi^2)} dx \\ &+ \varkappa \sum_{\xi^1 \subset \eta^1} \int_{\mathbb{R}^d} G(\xi^1, \eta^2 \cup y) e_\lambda(e^{-\beta\phi(y-\cdot)} - 1, \eta^1 \setminus \xi^1) e^{-\beta E^\phi(y, \xi^1)} dy. \end{aligned}$$

We can also rewrite the operator $\hat{P}_\delta$ as the sum of the following operators:

$$\begin{aligned} \hat{P}_\delta^{(0)} G(\eta^1, \eta^2) &= (1-\delta)^{|\eta^1| + |\eta^2|} G(\eta^1, \eta^2), \\ \hat{P}_\delta^{(1)} G(\eta^1, \eta^2) &= \varkappa\delta \sum_{\zeta^1 \subset \eta^1} (1-\delta)^{|\zeta^1| + |\eta^2|} \int_{\mathbb{R}^d} \prod_{x \in \zeta^1} e^{-\beta\phi(x-y)} \\ &\qquad \times \prod_{x' \in \eta^1 \setminus \zeta^1} \left(e^{-\beta\phi(x'-y)} - 1\right) G(\zeta^1, \eta^2 \cup y) dy \\ &\quad + \varkappa\delta \sum_{\zeta^2 \subset \eta^2} (1-\delta)^{|\eta^1| + |\zeta^2|} \int_{\mathbb{R}^d} \prod_{y \in \zeta^2} e^{-\beta\phi(y-x)} \\ &\qquad \times \prod_{y' \in \eta^2 \setminus \zeta^2} \left(e^{-\beta\phi(y'-x)} - 1\right) G(\eta^1 \cup x, \zeta^2) dx, \end{aligned}$$

and

$$\begin{aligned} \hat{P}_\delta^{(\geq 2)} G(\eta^1, \eta^2) &= \hat{P}_\delta - \left(\hat{P}_\delta^{(0)} + \hat{P}_\delta^{(1)}\right) \\ &= \sum_{\zeta^1 \subset \eta^1} \sum_{\zeta^2 \subset \eta^2} (1-\delta)^{|\zeta^1| + |\zeta^2|} \int_{\Gamma_0} \int_{\Gamma_0} \mathbb{1}_{\{|\sigma^1| + |\sigma^2| \geq 2\}} (\varkappa\delta)^{|\sigma^1|} (\varkappa\delta)^{|\sigma^2|} \\ &\qquad \times \prod_{x \in \zeta^1} e^{-\beta E^\phi(x, \sigma^2)} \prod_{x' \in \eta^1 \setminus \zeta^1} \left(e^{-\beta E^\phi(x', \sigma^2)} - 1\right) \\ &\qquad \times \prod_{y \in \zeta^2} e^{-\beta E^\phi(y, \sigma^1)} \prod_{y' \in \eta^2 \setminus \zeta^2} \left(e^{-\beta E^\phi(y', \sigma^1)} - 1\right) \\ &\qquad \times G(\zeta^1 \cup \sigma^1, \zeta^2 \cup \sigma^2) \lambda(d\sigma^1) \lambda(d\sigma^2). \end{aligned}$$

Using this notation the inequality (5.13) becomes

$$\left\|\left(\hat{L}_\delta - \hat{L}\right)G\right\|_C = \left\|\frac{1}{\delta}\left(\hat{P}_\delta - \mathbb{1}\right)G - \hat{L}G\right\|_C \tag{5.14}$$
$$\leq \left\|\frac{1}{\delta}\left(\hat{P}_\delta^{(0)}G - G\right) - L_0 G\right\|_C + \left\|\frac{1}{\delta}\hat{P}_\delta^{(1)}G - L_1 G\right\|_C + \frac{1}{\delta}\left\|\hat{P}_\delta^{(\geq 2)}G\right\|_C .$$

We begin with the first part of the latter inequality which is equal to

$$\left\|\frac{1}{\delta}\left(\hat{P}_\delta^{(0)}G - G\right) - L_0 G\right\|_C = \int_{\Gamma_0}\int_{\Gamma_0}\left|\left[\frac{1}{\delta}\Big((1-\delta)^{|\eta^1|+|\eta^2|} - 1\Big) + |\eta^1| + |\eta^2|\right]\right.$$
$$\left. \times\, G(\eta^1,\eta^2)\right| C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2),$$

but for $(\eta^1, \eta^2) \in \Gamma_0 \times \Gamma_0$ we have

$$\begin{aligned}
&\left[\frac{1}{\delta}\Big((1-\delta)^{|\eta^1|+|\eta^2|} - 1\Big) + |\eta^1| + |\eta^2|\right] \\
&\qquad = \frac{1}{\delta}\left[\sum_{k=0}^{|\eta^1|+|\eta^2|}\binom{|\eta^1|+|\eta^2|}{k}(-1)^k\delta^k - 1 + \delta\left(|\eta^1|+|\eta^2|\right)\right] \\
&\qquad = \frac{1}{\delta}\sum_{k=2}^{|\eta^1|+|\eta^2|}\binom{|\eta^1|+|\eta^2|}{k}(-1)^k\delta^k \\
&\qquad = \delta\sum_{k=2}^{|\eta^1|+|\eta^2|}\binom{|\eta^1|+|\eta^2|}{k}(-1)^k\delta^{k-2}
\end{aligned}$$

which can be estimated from above by

$$\delta\sum_{k=2}^{|\eta^1|+|\eta^2|}\binom{|\eta^1|+|\eta^2|}{k} < \delta 2^{|\eta^1|+|\eta^2|}.$$

From this, we immediately get

$$\left\|\frac{1}{\delta}\left(\hat{P}_\delta^{(0)}G - G\right) - L_0 G\right\|_C \leq \delta\|G\|_{2C}. \tag{5.15}$$

Using the properties of modulus function and Minlos lemma we can estimate the second term of the right hand side of (5.14) in the following way:

$$\begin{aligned}
&\left\|\frac{1}{\delta}\hat{P}_\delta^{(1)}G - L_1 G\right\|_C \\
&\qquad \leq \varkappa \int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta^1\subset\eta^1}\left|1-(1-\delta)^{|\zeta^1|+|\eta^2|}\right|\int_{\mathbb{R}^d}\prod_{x\in\zeta^1}e^{-\beta\phi(x-y)} \\
&\qquad \times \prod_{x'\in\eta^1\setminus\zeta^1}\left|e^{-\beta\phi(x'-y)}-1\right||G(\zeta^1,\eta^2\cup y)|dyC^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2) \\
&\qquad + \quad \varkappa \int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta^2\subset\eta^2}\left|1-(1-\delta)^{|\zeta^2|+|\eta^1|}\right|\int_{\mathbb{R}^d}\prod_{y\in\zeta^2}e^{-\beta\phi(y-x)} \\
&\qquad \times \prod_{y'\in\eta^2\setminus\zeta^2}\left|e^{-\beta\phi(y'-x)}-1\right||G(\eta^1\cup x,\zeta^2)|dxC^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2).
\end{aligned}$$

Because of the structure of the expression above, we estimate only the first term (two first lines). Thus

$$\begin{aligned}
&\varkappa \int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta^1\subset\eta^1}\left|1-(1-\delta)^{|\zeta^1|+|\eta^2|}\right|\int_{\mathbb{R}^d}\prod_{x\in\zeta^1}e^{-\beta\phi(x-y)} \\
&\qquad \times \prod_{x'\in\eta^1\setminus\zeta^1}\left|e^{-\beta\phi(x'-y)}-1\right||G(\zeta^1,\eta^2\cup y)|dyC^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)
\end{aligned}$$

can be estimated from above by

$$\begin{aligned}
&\varkappa \int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}\left|1-(1-\delta)^{|\zeta^1|+|\eta^2|}\right|\int_{\mathbb{R}^d}\prod_{x'\in\eta^1}\left|e^{-\beta\phi(x'-y)}-1\right||G(\zeta^1,\eta^2\cup y)|dy \\
&\qquad\qquad \times C^{|\eta^1|+|\eta^2|+|\zeta^1|}\lambda(d\zeta^1)\lambda(d\eta^1)\lambda(d\eta^2)
\end{aligned}$$

which is, by Minlos lemma equal to:

$$\frac{\varkappa}{C}e^{CC(\beta)}\int_{\Gamma_0}\int_{\Gamma_0}|\eta^2|\left|1-(1-\delta)^{|\zeta^1|+|\eta^2|-1}\right||G(\zeta^1,\eta^2)|C^{|\eta^2|+|\zeta^1|}\lambda(d\zeta^1)\lambda(d\eta^2).$$

Now notice, that for $\delta\in(0,1)$, $n,m\in\mathbb{N}$, $n,m\geq 1$

$$1-(1-\delta)^n = \delta\sum_{k=0}^{n-1}(1-\delta)^k \leq \delta n,$$

and

$$(n+m-1)m \leq 2^{n+m},$$

hence we obtain that the latter expression can be estimated from above by

$$\delta \frac{\varkappa}{C} e^{CC(\beta)} \int_{\Gamma_0} \int_{\Gamma_0} 2^{|\eta^2|+|\zeta^1|} |G(\zeta^1, \eta^2)| C^{|\eta^2|+|\zeta^1|} \lambda(d\zeta^1) \lambda(d\eta^2).$$

We can proceed in the same way with the second part of $\hat{P}_\delta^{(1)}$, and using the assumption (5.11) we finally get

$$\left\| \frac{1}{\delta} \hat{P}_\delta^{(1)} G - L_1 G \right\|_C \leq 2\delta \|G\|_{2C}. \tag{5.16}$$

It remains to show that $\frac{1}{\delta} \left\| \hat{P}_\delta^{(\geq 2)} G \right\|_C \leq \delta \|G\|_{2C}$, but

$$\begin{aligned}
\frac{1}{\delta} \left\| \hat{P}_\delta^{(\geq 2)} G \right\|_C = & \int_{\Gamma_0} \int_{\Gamma_0} \Bigg| \sum_{\zeta^1 \subset \eta^1} \sum_{\zeta^2 \subset \eta^2} (1-\delta)^{|\zeta^1|+|\zeta^2|} \int_{\Gamma^0} \int_{\Gamma^0} \mathbb{1}_{\{|\sigma^1|+|\sigma^2| \geq 2\}} \\
& \times \delta^{|\sigma^1|+|\sigma^2|-1} \prod_{x \in \zeta^1} e^{-\beta E^\phi(x, \sigma^2)} \prod_{x' \in \eta^1 \setminus \zeta^1} \left(e^{-\beta E^\phi(x', \sigma^2)} - 1 \right) \\
& \times \prod_{y \in \zeta^2} e^{-\beta E^\phi(y, \sigma^1)} \prod_{y' \in \eta^2 \setminus \zeta^2} \left(e^{-\beta E^\phi(y', \sigma^1)} - 1 \right) \varkappa^{|\sigma^1|} \varkappa^{|\sigma^2|} \\
& \times G(\zeta^1 \cup \sigma^1, \zeta^2 \cup \sigma^2) C^{|\eta^1|+|\eta^2|} \Bigg| \lambda(d\sigma^1) \lambda(d\sigma^2) \lambda(d\eta^1) \lambda(d\eta^2) \\
\leq \quad & \delta \int_{\Gamma_0} \int_{\Gamma_0} \int_{\Gamma_0} \int_{\Gamma_0} |1-\delta|^{|\zeta^1|+|\zeta^2|} \int_{\Gamma^0} \int_{\Gamma^0} \varkappa^{|\sigma^1|} \varkappa^{|\sigma^2|} \\
& \times \prod_{x \in \zeta^1} e^{-\beta E^\phi(x, \sigma^2)} \prod_{x' \in \eta^1} \left| e^{-\beta E^\phi(x', \sigma^2)} - 1 \right| \\
& \times \prod_{y \in \zeta^2} e^{-\beta E^\phi(y, \sigma^1)} \prod_{y' \in \eta^2} \left| e^{-\beta E^\phi(y', \sigma^1)} - 1 \right| \left| G(\zeta^1 \cup \sigma^1, \zeta^2 \cup \sigma^2) \right| \\
& \times C^{|\eta^1|+|\eta^2|+|\zeta^1|+|\zeta^2|} \lambda(d\sigma^1) \lambda(d\eta^1) \lambda(d\eta^2) \lambda(d\sigma^2) \lambda(d\zeta^1) \lambda(d\zeta^2)
\end{aligned}$$

which is equal to

$$\begin{aligned}
\delta \int_{\Gamma_0} \int_{\Gamma_0} |1-\delta|^{|\zeta^1|+|\zeta^2|} & \int_{\Gamma^0} \int_{\Gamma^0} \left[\varkappa e^{CC(\beta)} \right]^{|\sigma^1|} \left[\varkappa e^{CC(\beta)} \right]^{|\sigma^2|} \\
& \times \prod_{x \in \zeta^1} e^{-\beta E^\phi(x, \sigma^2)} \prod_{y \in \zeta^2} e^{-\beta E^\phi(y, \sigma^1)} \left| G(\zeta^1 \cup \sigma^1, \zeta^2 \cup \sigma^2) \right| \\
& \times C^{|\zeta^1|+|\zeta^2|} \lambda(d\sigma^1) \lambda(d\sigma^2) \lambda(d\zeta^1) \lambda(d\zeta^2).
\end{aligned}$$

Using Minlos lemma and the fact that for $\phi > 0$, $e^{-\beta E^{\phi}(x,\sigma)} < 1$ for all $x \in \mathbb{R}^d, \sigma \in \Gamma_0$ we can estimate the latter from above by

$$\delta \int_{\Gamma_0} \int_{\Gamma_0} \sum_{\sigma^1 \subset \zeta^1} [C(1-\delta)]^{|\zeta^1 \setminus \sigma^1|} \left[\varkappa e^{CC(\beta)}\right]^{|\sigma^1|} \times \sum_{\sigma^2 \subset \zeta^2} [C(1-\delta)]^{\zeta^2 \setminus \sigma^2} \left[\varkappa e^{CC(\beta)}\right]^{|\sigma^2|} \left|G(\zeta^1, \zeta^2)\right| \lambda(d\zeta^1)\lambda(d\zeta^2)$$

but this is the same as

$$\delta \int_{\Gamma_0} \int_{\Gamma_0} \left[C(1-\delta) + \varkappa e^{CC(\beta)}\right]^{|\zeta^1|} \times \left[C(1-\delta) + \varkappa e^{CC(\beta)}\right]^{|\zeta^2|} \left|G(\zeta^1, \zeta^2)\right| \lambda(d\zeta^1)\lambda(d\zeta^2)$$

and because of the assumption (5.11) we finally obtain

$$\begin{aligned} \frac{1}{\delta} \left\| \hat{P}_\delta^{(\geq 2)} G \right\|_C &\leq \delta \int_{\Gamma_0} \int_{\Gamma_0} [(2-\delta)\, C]^{|\zeta^1|+|\zeta^2|} \left|G(\zeta^1, \zeta^2)\right| \lambda(d\zeta^1)\lambda(d\zeta^2) \\ &\leq \delta \int_{\Gamma_0} \int_{\Gamma_0} [2C]^{|\zeta^1|+|\zeta^2|} \left|G(\zeta^1, \zeta^2)\right| \lambda(d\zeta^1)\lambda(d\zeta^2) \qquad (5.17) \\ &= \delta \|G\|_{2C}. \end{aligned}$$

Putting together the last inequality with the inequalities (5.15) and (5.16) we obtain (5.13) and the Proposition is proven. □

Now we proceed to the proof of Theorem 5.1.

Proof of Theorem 5.1. As we mentioned before, we will use Lemma 5.1 to show that the operator $\hat{L}$ generates a strongly continuous contraction semigroup on $\mathcal{L}_C$. To check the assumptions of Lemma 5.1, set $A := \hat{L}$, $L := \mathcal{L}_C$, $D(A) := \mathcal{L}_{2C}$, $n = \frac{1}{\delta}$, $T_n := \hat{P}_\delta$ (which gives us $A_n = \hat{L}_\delta$) and $\varepsilon_n = \frac{1}{4n} = \frac{\delta}{4}$.

First, using Proposition 5.2 and Lemma 5.2 we see, that $(\hat{L}, \mathcal{L}_{2C})$ defines a linear operator on $\mathcal{L}_C$ and that $\hat{P}_\delta$ is a $\mathcal{L}_C$-contraction. Moreover, putting $2C$ instead of C in the Lemma 5.2, we obtain that for $G \in \mathcal{L}_{2C}$, $\hat{P}_\delta G \in \mathcal{L}_{2C}$ and additionally (5.9) is fulfilled with $\omega = 0$. Finally, using Proposition 5.5 we obtain (5.8), thus all the assumptions of Lemma 5.1 are satisfied. □

Using additionally [EK05, Theorem 6.5] we can obtain the following

Corollary 5.1. *Assume that the conditions of the Theorem 5.1 hold, then*

$$\left(\hat{P}_{\frac{1}{n}}\right)^{[nt]} G \to \hat{U}_t G, \quad n \to \infty$$

for every $G \in \mathcal{L}_C$ and all $t \geq 0$ uniformly on bounded intervals.

5.3 Vlasov-type scaling

We devote this section to the Vlasov type scaling of the pre-generator L associated to the Glauber-Potts model. As in previous chapter, we show that the scaled operator is also related to a semigroup on Γ_0^2 but in this case we mainly focus on the evolution of correlation functions for the associated state directly, i.e. we show that the respective semigroups converge in the space of correlation functions which we will define later. This is stronger compared to the previous chapter where the evolution of the system of correlation functions was defined only in the weak sense (i.e. with respect to the duality between $\mathcal{L}_C$ and $\mathcal{Q}_C$).

5.3.1 Scaled operator $\hat{L}_{\varepsilon,ren}$ and Vlasov operator $\hat{L}^V$

Let $\varepsilon > 0$, then the proper scaling of the generator L defined in (5.2) is as follows:

$$\begin{aligned} L_\varepsilon F(\gamma^1,\gamma^2) := \quad & \sum_{x\in\gamma^1} D_x^{1-}F(\gamma^1,\gamma^2) + \frac{\varkappa}{\varepsilon}\int_{\mathbb{R}^d} e^{-\varepsilon\beta E^\phi(x,\gamma^2)} D_x^{1+}F(\gamma^1,\gamma^2)dx \\ & + \sum_{y\in\gamma^2} D_y^{2-}F(\gamma^1,\gamma^2) + \frac{\varkappa}{\varepsilon}\int_{\mathbb{R}^d} e^{-\varepsilon\beta E^\phi(y,\gamma^1)} D_y^{2+}F(\gamma^1,\gamma^2)dy. \end{aligned} \tag{5.18}$$

The symbol $\hat{L}_\varepsilon$ can be calculated in the exactly same way as $\hat{L}$ so we omit its calculation here. Recall the following renormalization mapping of functions on $\Gamma_0 \times \Gamma_0$:

$$\left(R_\varepsilon G\right)(\eta^1,\eta^2) := \varepsilon^{|\eta^1|+|\eta^2|}G(\eta^1,\eta^2), \quad \varepsilon > 0, \tag{5.19}$$

with $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$, and the definition of the renormalized operator:

$$\hat{L}_{\varepsilon,ren}G(\eta^1,\eta^2) := R_{\varepsilon^{-1}}\mathcal{K}^{-1}L_\varepsilon\mathcal{K}R_\varepsilon \tag{5.20}$$

be the renormalized symbol of the operator L_ε. For $\hat{L}_{\varepsilon,ren}$ we have the following:

Lemma 5.3. *For every $G \in B_{bs}(\Gamma_0^2)$ the operator*

$$\begin{aligned} \hat{L}_{\varepsilon,ren}G(\eta^1,\eta^2) &= -\left(|\eta^1|+|\eta^2|\right)G(\eta^1,\eta^2) \\ &+ \varkappa\sum_{\xi^2\subset\eta^2}\int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2)\prod_{y\in\xi^2} e^{-\varepsilon\beta\phi(x-y)}\prod_{y'\in\eta^2\setminus\xi^2}\left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right)dx \\ &+ \varkappa\sum_{\xi^1\subset\eta^1}\int_{\mathbb{R}^d} G(\xi^1,\eta^2\cup y)\prod_{x\in\xi^1} e^{-\varepsilon\beta\phi(y-x)}\prod_{x'\in\eta^1\setminus\xi^1}\left(\frac{e^{-\varepsilon\beta\phi(y-x')}-1}{\varepsilon}\right)dy \end{aligned}$$

together with the domain $D(\hat{L}_{\varepsilon,ren}) := \mathcal{L}_{2C}$ (dense in $\mathcal{L}_C$) defines a linear operator in $\mathcal{L}_C$.

Before we prove the Lemma let us introduce the following notation:

$$\begin{aligned}
L_0 G(\eta^1,\eta^2) &:= -\left(|\eta^1|+|\eta^2|\right) G(\eta^1,\eta^2),\\
L_{1,\varepsilon} G(\eta^1,\eta^2) &:= \varkappa \sum_{\xi^2\subset\eta^2} \int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2) \prod_{y\in\xi^2} e^{-\varepsilon\beta\phi(x-y)}\\
&\qquad\qquad \times \prod_{y'\in\eta^2\setminus\xi^2} \left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right) dx,\\
L_{2,\varepsilon} G(\eta^1,\eta^2) &:= \varkappa \sum_{\xi^1\subset\eta^1} \int_{\mathbb{R}^d} G(\xi^1,\eta^2\cup y) \prod_{x\in\xi^1} e^{-\varepsilon\beta\phi(y-x)}\\
&\qquad\qquad \times \prod_{x'\in\eta^1\setminus\xi^1} \left(\frac{e^{-\varepsilon\beta\phi(y-x')}-1}{\varepsilon}\right) dy.
\end{aligned}$$

It is clear that $\hat{L}_{\varepsilon,ren} = L_0 + L_{1,\varepsilon} + L_{2,\varepsilon}$ and we will use this notation henceforth.

Proof. Take $G \in \mathcal{L}_{2C}$, then clearly

$$\|\hat{L}_{\varepsilon,ren}\|_C \le \|L_0 G\|_C + \|L_{1,\varepsilon} G\|_C + \|L_{2,\varepsilon} G\|_C. \tag{5.21}$$

Similarly as in the proof of the Proposition 5.2, we can calculate the $\mathcal{L}_C$-norm of $\hat{L}_{\varepsilon,ren}$, and because $\|L_0 G\|_C < \|G\|_{2C}$ (see the proof of the abovementioned Proposition) it is sufficient to estimate the norm of $L_{1,\varepsilon} G$ (also, because of the symmetry, $L_{2,\varepsilon} G$) hence:

$$\begin{aligned}
\|L_{1,\varepsilon} G\|_C &= \int_{\Gamma_0}\int_{\Gamma_0} \Bigg| \varkappa \sum_{\xi^2\subset\eta^2} \int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2) \prod_{y\in\xi^2} e^{-\varepsilon\beta\phi(x-y)}\\
&\qquad \times \prod_{y'\in\eta^2\setminus\xi^2} \left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right) dx \Bigg| C^{|\eta^1|+|\eta^2|} \lambda(d\eta^1)\lambda(d\eta^2)\\
&\le \varkappa \int_{\Gamma_0}\int_{\Gamma_0} \sum_{\xi^2\subset\eta^2} \int_{\mathbb{R}^d} |G(\eta^1\cup x,\xi^2)| \prod_{y\in\eta^2\setminus\xi^2} |\beta\phi(x-y)|\, dx\\
&\qquad\qquad \times C^{|\eta^1|+|\eta^2|} \lambda(d\eta^1)\lambda(d\eta^2),
\end{aligned}$$

where we have used the fact that $1-e^{-\phi} \leq \phi$ for $\phi \geq 0$. Using Minlos lemma, we can estimate the latter by

$$\varkappa \int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d} |G(\eta^1\cup x,\xi^2)| \prod_{y\in\eta^2} |\beta\phi(x-y)|\,dx \\ \times C^{|\eta^1|+|\eta^2|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1)\lambda(d\eta^2)$$

and this is equal to

$$\varkappa e^{\beta C\Phi}\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d} |G(\eta^1\cup x,\xi^2)|dx C^{|\eta^1|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1).$$

Using Minlos lemma again, we finally obtain

$$\begin{aligned}\|L_{1,\varepsilon}G\|_C &\leq \frac{\varkappa}{C}e^{\beta C\Phi}\int_{\Gamma_0}\int_{\Gamma_0} |\eta^1||G(\eta^1,\xi^2)|C^{|\eta^1|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1)\\ &\leq \frac{\varkappa}{C}e^{\beta C\Phi}\int_{\Gamma_0}\int_{\Gamma_0} 2^{|\eta^1|}|G(\eta^1,\xi^2)|C^{|\eta^1|+|\xi^2|}\lambda(d\xi^2)\lambda(d\eta^1)\\ &\leq \frac{\varkappa}{C}e^{\beta C\Phi}\|G\|_{2C} < \infty.\end{aligned}$$

As result we get

$$\|\hat{L}_{\varepsilon,ren}\|_C \leq \left(1+\frac{\varkappa}{C}e^{\beta C\Phi}+\frac{\varkappa}{C}e^{\beta C\Phi}\right)\|G\|_{2C},$$

and as it was mentioned before, $\mathcal{L}_{2C}$ is densely embedded in $\mathcal{L}_C$. □

The natural candidate for the Vlasov generator is the pointwise limit of $\hat{L}_{\varepsilon,ren}$ as ε tends to 0. We will denote it with $\hat{L}^V$ and it is easy to see that

$$\begin{aligned}\hat{L}^V G(\eta^1,\eta^2) =& -\left(|\eta^1|+|\eta^2|\right)G(\eta^1,\eta^2)\\ &+\varkappa\sum_{\xi^2\subset\eta^2}\int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2)(-\beta)^{|\eta^2\setminus\xi^2|}\prod_{y\in\eta^2\setminus\xi^2}\phi(x-y)dx\\ &+\varkappa\sum_{\xi^1\subset\eta^1}\int_{\mathbb{R}^d} G(\xi^1,\eta^2\cup y)(-\beta)^{|\eta^1\setminus\xi^1|}\prod_{x2\in\eta^1\setminus\xi^1}\phi(y-x)dy.\end{aligned} \tag{5.22}$$

5.3.2 Semigroups associated with $\hat{L}_{\varepsilon,ren}$ and $\hat{L}^V$

In this part we prove, that both $\hat{L}_{\varepsilon,ren}$ and $\hat{L}^V$ are closable and their closures generate strongly continuous contraction semigroups, which will be denoted $\hat{U}_{\varepsilon,ren}(t)$ and $\hat{U}^V(t)$ respectively. To do this, we apply the similar method to the one used in the previous section, namely the approximation by linear contraction operators and the use of the Lemma 5.1. Let now $\delta \in \mathbb{R}_+$ be fixed and define the approximation operators:

$$\begin{aligned}\hat{P}_{\varepsilon,\delta}G(\eta^1,\eta^2) = \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\\ \times\prod_{x\in\zeta^1}e^{-\varepsilon\beta E^\phi(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left(\frac{e^{-\varepsilon\beta E^\phi(x',\sigma^2)}-1}{\varepsilon}\right)\\ \times\prod_{y\in\zeta^2}e^{-\varepsilon\beta E^\phi(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left(\frac{e^{-\varepsilon\beta E^\phi(y',\sigma^1)}-1}{\varepsilon}\right)\\ \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2)\end{aligned}$$

and

$$\begin{aligned}\hat{Q}_\delta G(\eta^1,\eta^2) = \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\\ \times\prod_{x\in\zeta^1}\left(-\beta E^\phi(x,\sigma^2)\right)\prod_{y\in\zeta^2}\left(-\beta E^\phi(y,\sigma^1)\right)\\ \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2).\end{aligned}$$

Next we derive some properties of the two operators defined above.

Lemma 5.4. *Assume, that $\varkappa e^{\beta C\Phi} \leq C$. Then both $\hat{P}_{\varepsilon,\delta}$ and $\hat{Q}_\delta$ are linear $\mathcal{L}_C$-contractions.*

Proof. Let $G \in \mathcal{L}_C$, then

$$\begin{aligned}\|\hat{P}_{\varepsilon,\delta}G\|_C = \int_{\Gamma_0}\int_{\Gamma_0}\Bigg|\sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\\ \times\prod_{x\in\zeta^1}e^{-\beta E^\phi(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left(\frac{e^{-\varepsilon\beta E^\phi(x',\sigma^2)}-1}{\varepsilon}\right)\\ \times\prod_{y\in\zeta^2}e^{-\beta E^\phi(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left(\frac{e^{-\varepsilon\beta E^\phi(y',\sigma^1)}-1}{\varepsilon}\right)\\ \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2)\Bigg|C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2).\end{aligned}$$

This can be estimated by

$$\int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times \prod_{x\in\zeta^1} e^{-\beta E^{\phi}(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left|\frac{e^{-\varepsilon\beta E^{\phi}(x',\sigma^2)}-1}{\varepsilon}\right|$$
$$\times \prod_{y\in\zeta^2} e^{-\beta E^{\phi}(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left|\frac{e^{-\varepsilon\beta E^{\phi}(y',\sigma^1)}-1}{\varepsilon}\right|$$
$$\times \left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right|\lambda(d\sigma^1)\lambda(d\sigma^2)C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)$$

and further by

$$\int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times \prod_{x'\in\eta^1\setminus\zeta^1}\left|\beta E^{\phi}(x',\sigma^2)\right|\prod_{y'\in\eta^2\setminus\zeta^2}\left|\beta E^{\phi}(y',\sigma^1)\right|$$
$$\times \left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right|\lambda(d\sigma^1)\lambda(d\sigma^2)C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2).$$

The latter is equal to

$$\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\prod_{x'\in\eta^1}\left|\beta E^{\phi}(x',\sigma^2)\right|\prod_{y'\in\eta^2}\left|\beta E^{\phi}(y',\sigma^1)\right|\left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right|$$
$$\times\lambda(d\sigma^1)\lambda(d\sigma^2)C^{|\eta^1\cup\zeta^1|+|\eta^2\cup\zeta^2|}\lambda(d\zeta^1)\lambda(d\zeta^2)\lambda(d\eta^1)\lambda(d\eta^2)$$

and it can be estimated from above by

$$\int_{\Gamma_0}\int_{\Gamma_0}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}e^{\beta C\Phi|\sigma^1|}e^{\beta C\Phi|\sigma^2|}$$
$$\times\left|G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\right|C^{|\zeta^1|+|\zeta^2|}\lambda(d\sigma^1)\lambda(d\sigma^2)\lambda(d\zeta^1)\lambda(d\zeta^2).$$

Using Minlos lemma together with the assumptions we obtain:

$$\begin{aligned}\|\hat{P}_{\varepsilon,\delta}G\|_C &\le \int_{\Gamma_0}\int_{\Gamma_0}\left[C(1-\delta)+\varkappa\delta e^{\beta C\Phi}\right]^{|\zeta^1|}\left[C(1-\delta)+\varkappa\delta e^{\beta C\Phi}\right]^{|\zeta^2|}\\ &\qquad\qquad\times\left|G(\zeta^1,\zeta^2)\right|\lambda(d\zeta^1)\lambda(d\zeta^2)\\ &\le \int_{\Gamma_0}\int_{\Gamma_0}\left|G(\zeta^1,\zeta^2)\right|C^{|\zeta^1|+|\zeta^2|}\lambda(d\zeta^1)\lambda(d\zeta^2)=\|G\|_C.\end{aligned}$$

But because for every $\sigma^1, \sigma^2, \zeta^1, \zeta^2 \in \Gamma_0$ we have

$$\left| \prod_{x\in\zeta^1} \left(-\beta E^{\phi}(x,\sigma^2)\right) \prod_{y\in\zeta^2} \left(-\beta E^{\phi}(y,\sigma^1)\right) \right| \leq \prod_{x\in\zeta^1} \left|\beta E^{\phi}(x,\sigma^2)\right| \prod_{y\in\zeta^2} \left|\beta E^{\phi}(y,\sigma^1)\right|,$$

we can conclude that also

$$\|\hat{Q}_\delta G\|_C \leq \|G\|_C$$

for all $G \in \mathcal{L}_C$. □

Now define for $\delta \in (0,1)$

$$\hat{L}^{\varepsilon}_{\delta} := \frac{1}{\delta}\left(\hat{P}_{\varepsilon,\delta} - 1\!\!1\right), \qquad \hat{L}^{V}_{\delta} := \frac{1}{\delta}\left(\hat{Q}_{\delta} - 1\!\!1\right).$$

We will now show that $\hat{L}^{\varepsilon}_{\delta}$ and $\hat{L}^{V}_{\delta}$ approximate operators $\hat{L}_{\varepsilon,ren}$ and $\hat{L}^V$ respectively.

Proposition 5.6. *Under the assumptions of the previous Lemma and for every $\delta \in (0,1)$, $G \in \mathcal{L}_C$ the following inequalities hold:*

$$\|\left(\hat{L}^{\varepsilon}_{\delta} - \hat{L}_{\varepsilon,ren}\right) G\|_C \leq 4\delta \|G\|_{2C} \tag{5.23}$$

and

$$\|\left(\hat{L}^{V}_{\delta} - \hat{L}^{V}\right) G\|_C \leq 4\delta \|G\|_{2C}. \tag{5.24}$$

Proof. We will omit the proof for it is analogue to the proof of Proposition 5.5. □

The next theorem follows from Theorem 5.1. Its proof is, with small modifications of notation, the same hence we omit it here.

Theorem 5.2. *Let*

$$\varkappa \leq \min\left\{2Ce^{-2\beta C\Phi}, Ce^{-\beta C\Phi}\right\}, \tag{5.25}$$

Then the operators $\left(\hat{L}_{\varepsilon,ren}, \mathcal{L}_{2C})\right)$ and $\left(\hat{L}^V, \mathcal{L}_{2C})\right)$ are closable. Their closures $\left(\hat{L}_{\varepsilon,ren}, D(\hat{L}_\varepsilon)\right)$ and $\left(\hat{L}^V, D(\hat{L}^V)\right)$ resp. generate strongly continuous contraction semigroups $\hat{U}_{\varepsilon,ren}(t)$ and $\hat{U}^V(t)$ (resp.) on $\mathcal{L}_C$ and for $G \in \mathcal{L}_C$, $\varepsilon > 0$ we have:

$$\left(\hat{P}^{\varepsilon}_{\frac{1}{n}} G\right)^{[nt]} \to \hat{U}_{\varepsilon,ren}(t)G, \qquad \hat{Q}^{[nt]}_{\frac{1}{n}} G \to \hat{U}^V(t)G \tag{5.26}$$

as $n \to \infty$ for all $t \geq 0$ uniformly on any bounded interval.

Now one can ask, whether the associated semigroups converge provided the convergence of their generators. Given previous results we could expect that the answer is positive. However, the main point of interest for us is the evolution of the system of correlation functions for the Glauber-Potts type dynamic, which is governed by the "dual" operators to the ones defined in this section. In the next part we will give a proper meaning to the notion of "duality" we have in mind.

5.3.3 Dual semigroups

Denote with $\lambda_C(d\eta^1, d\eta^2) := C^{|\eta^1|+|\eta^2|}\lambda(d\eta^1)\lambda(d\eta^2)$ and consider the dual space to the space $\mathcal{L}_C$, namely $(\mathcal{L}_C)' = L^\infty(\Gamma_0 \times \Gamma_0, \lambda_C)$. Recall the definition of the space

$$\mathcal{Q}_C := \left\{ k : \Gamma_0^2 \to \mathbb{R} : \quad \text{esssup}_{(\eta^1,\eta^2)\in\Gamma_0\times\Gamma_0} \left| k(\eta^1,\eta^2) C^{-\left(|\eta^1|+|\eta^2|\right)} \right| < \infty \right\}$$

equipped with the norm

$$\|k\|_{\mathcal{Q}_C} := \left\| C^{-\left(|\cdot^1|+|\cdot^2|\right)} k(\cdot^1, \cdot^2) \right\|_{L^\infty(\Gamma_0\times\Gamma_0, \lambda\otimes\lambda)}.$$

Recall also, that for every $k \in \mathcal{Q}_C$ we have $|k(\eta^1,\eta^2)| \le \|k\|_{\mathcal{Q}_C} C^{|\eta^1|+|\eta^2|}$ for $\lambda\otimes\lambda$-a.a. $(\eta^1,\eta^2) \in \Gamma_0\times\Gamma_0$. Furthermore, the space $\mathcal{Q}_C$ is isometrically isomorphic to $(\mathcal{L}_C)'$ given the isomophism

$$R_C k(\eta^1,\eta^2) := C^{|\eta^1|+|\eta^2|} k(\eta^1,\eta^2), \quad k \in (\mathcal{L}_C)'. \tag{5.27}$$

Hence, we can define the duality between $(\mathcal{L}_C)'$ and $\mathcal{Q}_C$ with help of the following relation:

$$\langle\langle G, k\rangle\rangle = \int_{\Gamma_0}\int_{\Gamma_0} k(\eta^1,\eta^2) G(\eta^1,\eta^2) \lambda(d\eta^1)\lambda(d\eta^2) \tag{5.28}$$

where $G \in \mathcal{L}_C$ and $k \in \mathcal{Q}_C$, and

$$\left|\left\langle\left\langle G, k\right\rangle\right\rangle\right| \le \|G\|_C \|k\|_{\mathcal{Q}_C}. \tag{5.29}$$

Notice that for every function $k \in \mathcal{Q}_C$, we have

$$\left|k(\eta^1,\eta^2)\right| \le \|k\|_{\mathcal{Q}_C} C^{|\eta^1|+|\eta^2|} \tag{5.30}$$

for $\lambda\otimes\lambda$-a.e. $(\eta^1,\eta^2) \in \Gamma_0\times\Gamma_0$.

Let $\left(\hat{L}^*_{\varepsilon,ren}, D(\hat{L}^*_\varepsilon)\right)$ and $\left(\hat{L}^*_V, D(\hat{L}^*_V)\right)$ be the images of the duals (in the standard sense) of the operators $\left(\hat{L}_{\varepsilon,ren}, D(\hat{L}^\varepsilon)\right)$ and $\left(\hat{L}^V, D(\hat{L}^V)\right)$, resp.

under the isometry R_C. As it was shown in [FKK10b], for a given operator $\hat{L}$, the $\hat{L}^*$ is the dual of $\hat{L}$ with respect to the duality defined in (5.28). We will use this fact to prove the following

Proposition 5.7. *Let $k \in \mathcal{Q}_C$, then*

$$\begin{aligned}\hat{L}^*_{\varepsilon,ren}k(\eta^1,\eta^2) &= -\left(|\eta^1|+|\eta^2|\right)k(\eta^1,\eta^2)\\ &+\varkappa\sum_{x\in\eta^1}\int_{\Gamma_0}k(\eta^1\setminus x,\eta^2\cup\xi^2)\prod_{y\in\eta^2}e^{-\varepsilon\beta\phi(x-y)}\prod_{y'\in\xi^2}\left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right)\lambda(d\xi^2)\\ &+\varkappa\sum_{y\in\eta^2}\int_{\Gamma_0}k(\eta^1\cup\xi^1,\eta^2\setminus y)\prod_{x\in\eta^1}e^{-\varepsilon\beta\phi(y-x)}\prod_{x'\in\xi^1}\left(\frac{e^{-\varepsilon\beta\phi(y-x')}-1}{\varepsilon}\right)\lambda(d\xi^1)\end{aligned}$$

and

$$\begin{aligned}\hat{L}^*_V k(\eta^1,\eta^2) = &-\left(|\eta^1|+|\eta^2|\right)k(\eta^1,\eta^2)\\ &+\varkappa\sum_{x\in\eta^1}\int_{\Gamma_0}k(\eta^1\setminus x,\eta^2\cup\xi^2)\prod_{y\in\xi^2}\left(-\beta\phi(x-y)\right)\lambda(d\xi^2)\\ &+\varkappa\sum_{y\in\eta^2}\int_{\Gamma_0}k(\eta^1\cup\xi^1,\eta^2\setminus y)\prod_{x\in\xi^1}\left(-\beta\phi(y-x)\right)\lambda(d\xi^1).\end{aligned}$$

Moreover, for any $\alpha\in(0,1)$, $\varepsilon>0$ and for all $k\in\mathcal{Q}_{\alpha C}$:

$$\hat{L}^*_{\varepsilon,ren}k\in\mathcal{Q}_C,\quad \hat{L}^*_V k\in\mathcal{Q}_C. \tag{5.31}$$

Proof. As we mentioned before, we will use the duality (5.28) to calculate the dual operators. Recall, that $\hat{L}_{\varepsilon,ren}=L_0+L_{1,\varepsilon}+L_{2,\varepsilon}$ (cf. Lemma 5.3). Let $G\in\mathcal{L}_C$, $k\in\mathcal{Q}_C$, then

$$\left\langle\!\left\langle\hat{L}_{\varepsilon,ren}G,k\right\rangle\!\right\rangle=\left\langle\!\left\langle\left(L_0+L_{1,\varepsilon}+L_{2,\varepsilon}\right)G,k\right\rangle\!\right\rangle=\left\langle\!\left\langle G,\left(L_0^*+L_{1,\varepsilon}^*+L_{2,\varepsilon}^*\right)k\right\rangle\!\right\rangle.$$

It is clear, that $L_0^*=L_0$, so we proceed to $L_{1,\varepsilon}^*$. We have

$$\begin{aligned}\left\langle\!\left\langle L_{1,\varepsilon}G,k\right\rangle\!\right\rangle=\varkappa\int_{\Gamma_0}\int_{\Gamma_0}\sum_{\xi^2\subset\eta^2}\int_{\mathbb{R}^d}&G(\eta^1\cup x,\xi^2)\prod_{y\in\xi^2}e^{-\varepsilon\beta\phi(x-y)}\\ \times\prod_{y'\in\eta^2\setminus\xi^2}&\left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right)dxk(\eta^1,\eta^2)\lambda(d\eta^1)\lambda(d\eta^2)\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\varkappa \int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta^1\cup x,\xi^2)\prod_{y\in\xi^2} e^{-\varepsilon\beta\phi(x-y)} \\
&\qquad\times \prod_{y'\in\eta^2}\left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right) dx k(\eta^1,\eta^2\cup\xi^2)\lambda(d\xi^2)\lambda(d\eta^1)\lambda(d\eta^2) \\
&\qquad= \varkappa\int_{\Gamma_0}\int_{\Gamma_0} G(\eta^1,\xi^2)\sum_{x\in\eta^1}\int_{\Gamma_0} k(\eta^1\setminus x,\eta^2\cup\xi^2) \\
&\qquad\times \prod_{y\in\xi^2} e^{-\varepsilon\beta\phi(x-y)}\prod_{y'\in\eta^2}\left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right)\lambda(d\eta^2)\lambda(d\xi^2)\lambda(d\eta^1) \\
&\qquad= \left\langle\left\langle G, L^*_{1,\varepsilon}k\right\rangle\right\rangle.
\end{aligned}$$

In the case of $L_{2,\varepsilon}$ we proceed in the same way. Hence we have obtained the form of $\hat{L}^*_{\varepsilon,ren}$. Sending $\varepsilon\to 0$, we also get $\hat{L}^*_V$.

Using the fact that for $a\in(0,1)$ and all $x\in\mathbb{R}$ we have $xa^x\le -\frac{1}{\ln a}$, we obtain

$$\left|C^{-\left(|\eta^1|+|\eta^2|\right)}L^*_0k(\eta^1,\eta^2)\right| \le -\frac{1}{\ln\alpha}\|k\|_{\mathcal{Q}_{\alpha C}}.$$

Similarly,

$$\begin{aligned}
\left|C^{-\left(|\eta^1|+|\eta^2|\right)}L^*_{1,\varepsilon}k(\eta^1,\eta^2)\right| = &\left|C^{-\left(|\eta^1|+|\eta^2|\right)}\varkappa\sum_{x\in\eta^1}\int_{\Gamma_0} k(\eta^1\setminus x,\eta^2\cup\xi^2)\right. \\
&\left.\times\prod_{y\in\eta^2} e^{-\varepsilon\beta\phi(x-y)}\prod_{y'\in\xi^2}\left(\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right)\lambda(d\xi^2)\right|
\end{aligned}$$

can be estimated from above by

$$\frac{\varkappa}{C^{|\eta^1|+|\eta^2|}}\|k\|_{\mathcal{Q}_{\alpha C}}\sum_{x\in\eta^1}\int_{\Gamma_0}(\alpha C)^{|\eta^1|-1+|\eta^2|+|\xi^2|}\prod_{y'\in\xi^2}\left|\frac{e^{-\varepsilon\beta\phi(x-y')}-1}{\varepsilon}\right|\lambda(d\xi^2),$$

which can be further estimated by

$$\frac{\varkappa\|k\|_{\mathcal{Q}_{\alpha C}}}{\alpha C}\alpha^{|\eta^1|}\sum_{x\in\eta^1}\int_{\Gamma_0}(\alpha C)^{|\xi^2|}\prod_{y'\in\xi^2}|\beta\phi(x-y')|\,\lambda(d\xi^2).$$

The last expression is equal to

$$\frac{\varkappa\|k\|_{\mathcal{Q}_{\alpha C}}}{\alpha C}|\eta^1|\alpha^{|\eta^1|}e^{\alpha\beta C\Phi}$$

thus we obtain

$$\left| C^{-\left(|\eta^1|+|\eta^2|\right)} L^*_{1,\varepsilon} k(\eta^1,\eta^2) \right| \le -\frac{\varkappa \|k\|_{\mathcal{Q}_{\alpha C}}}{\alpha C \ln \alpha} e^{\alpha\beta C\Phi}. \tag{5.32}$$

In the same manner we can estimate

$$\left| C^{-\left(|\eta^1|+|\eta^2|\right)} L^*_{2,\varepsilon} k(\eta^1,\eta^2) \right| \le -\frac{\varkappa \|k\|_{\mathcal{Q}_{\alpha C}}}{\alpha C \ln \alpha} e^{\alpha\beta C\Phi}, \tag{5.33}$$

thus (5.31) is proven for $\hat{L}^*_{\varepsilon,ren}$. Because it holds for all non-negative ε, we can let $\varepsilon \to 0$ obtaining the similar result for $\hat{L}^*_V$, which concludes the proof. □

Recall that if (5.25) hold then $\left(\hat{L}_{\varepsilon,ren}, D(\hat{L}_\varepsilon)\right)$ and $\left(\hat{L}^V, D(\hat{L}^V)\right)$ generate strongly continuous contraction semigroups $\hat{U}_{\varepsilon,ren}(t)$ and $\hat{U}^V(t)$ (resp.) on $\mathcal{L}_C$ respectively (cf. Theorem 5.2). Let now $\hat{U}'_{\varepsilon,ren}(t)$ and $\hat{U}'_V(t)$ be the respective dual semigroups and denote with $\hat{U}^*_{\varepsilon,ren}(t)$ and $\hat{U}^*_V(t)$ their images under the isometry R_C and $\left(\hat{L^*}_{\varepsilon,ren}, D(\hat{L}^*_\varepsilon)\right)$ and $\left(\hat{L}^V, D(\hat{L}^*_V)\right)$ are their weak*-generators (in the $weak^*$- lim sense), see e.g. [Nee92].

Unfortunately, the strong continuity of $\hat{U}_{\varepsilon,ren}(t)$ and $\hat{U}^V(t)$ (resp.) on $\mathcal{L}_C$ is not sufficient to assure that the corresponding *-semigroups will be strongly continuous, they are weak*-continuous though.

The short procedure which we will introduce now is valid for both $\hat{U}^*_{\varepsilon,ren}(t)$ and $\hat{U}^*_V(t)$, hence for the simplicity we will use an abstract C_0-semigroup $T(t)$ on some space X. For more details, see [Nee92, Section 1.3].

Define the semigroup dual of X w.r.t. $T(t)$ as

$$X^\odot := \left\{ x^* \in X^* : \quad \lim_{t\downarrow 0} \|T^*(t)x^* - x^*\| = 0 \right\}. \tag{5.34}$$

It is $T^*(t)$-invariant and if A^* is the weak*-generator of $T^*(t)$, then $X^\odot = \overline{D(A^*)}$. Now consider $T^\odot(t)$, the restriction of $T^*(t)$ to the subspace $X^\odot$, then from the definition (5.34) it is clear that $T^\odot(t)$ is a strongly continuous semigroup on $X^\odot$. Let $A^\odot$ be its generator. We have the following description of $A^\odot$ in terms of A^*:

Theorem 5.3 ([Nee92],Theorem 1.3.3). *$A^\odot$ is a part of A^* in $X^\odot$, that is*

$$D(A^\odot) := \left\{ x \in D(A^*) : \quad A^* x \in X^\odot \right\}$$
$$A^\odot x := A^* x, \quad x \in D(A^\odot).$$

Applying this scheme to the semigroups $\hat{U}_{\varepsilon,ren}(t)$ and $\hat{U}^V(t)$ and their corresponding duals, we obtain strongly continuous semigroups $\hat{U}^{\odot}_{\varepsilon,ren}(t)$ and $\hat{U}^{\odot}_V(t)$ acting on the (respectively invariant) subspaces of $\mathcal{Q}_C$. Let $\alpha \in (0,1)$, then

$$\overline{\mathcal{Q}_{\alpha C}} \subset \left(\bigcap_{\varepsilon>0} \overline{D(\hat{L}^*_{\varepsilon,ren})} \right) \cap \overline{D(\hat{L}^*_V)}. \tag{5.35}$$

Next proposition shows that the space $\overline{\mathcal{Q}_{\alpha C}}$ is a good candidate to work with when considering Vlasov scaling.

Proposition 5.8 ([FKK10b]). *Assume that* (5.25) *holds, then there exists a constant* $\alpha_0 := \alpha_0(\varkappa, \phi, C)$, $\alpha_0 \in (0,1)$ *such that for all* $\alpha \in (\alpha_0, 1)$ *the space* $\overline{\mathcal{Q}_{\alpha C}}$ *is* $\hat{U}^{\odot}_V(t)$ *and* $\hat{U}^{\odot}_{\varepsilon,ren}(t)$*-invariant for all* $\varepsilon > 0$.

Proof. By (5.25) we have

$$\varkappa\beta\Phi \le \min\left\{C\beta\Phi e^{-C\beta\Phi}, 2C\beta\Phi e^{-2C\beta\Phi}\right\},$$

using this and the fact that the function $f(x) = xe^{-x}$ increases on $(0,1)$ from 0 to e^{-1} we have $\varkappa\beta\Phi < e^{-1}$. This implies, that the equation $f(x) = \varkappa\beta\Phi$ has exactly two roots, say $0 \le x_1 < 1 < x_2 < +\infty$. Using (5.25) again, we obtain $x_1 < C\beta\Phi < 2C\beta\Phi < x_2$.

If $C\beta\Phi > 1$, we set $\alpha_0 := \max\left\{\frac{1}{2}, \frac{1}{C\beta\Phi}, \frac{1}{C}\right\} < 1$ which gives us $C\beta\Phi < 2\alpha C\beta\Phi$ and $x_1 < 1 < \alpha C\beta\Phi$. In the case when $x_1 < C\beta\Phi \le 1$ we set $\alpha_0 := \max\left\{\frac{1}{2}, \frac{x_1}{C\beta\Phi}, \frac{1}{C}\right\} < 1$. That gives $C\beta\Phi < 2\alpha C\beta\Phi$ and $x_1 < \alpha C\beta\Phi$. Then, for all $\alpha \in (\alpha_0, 1)$ we have

$$x_1 < \alpha C\beta\Phi < C\beta\Phi < 2\alpha C\beta\Phi < 2C\beta\Phi < x_2 \tag{5.36}$$

and $1 < \alpha C < C < 2\alpha C < 2C$, thus the following inclusion hold:

$$\mathcal{L}_{2C} \subset \mathcal{L}_{2\alpha C} \subset \mathcal{L}_C \subset \mathcal{L}_{\alpha C}. \tag{5.37}$$

Setting αC in the place of C in Theorem 5.2 we obtain, that $(\hat{L}_{\varepsilon,ren}, \mathcal{L}_{2\alpha C})$ and $(\hat{L}^V, \mathcal{L}_{2\alpha C})$ are closable in $\mathcal{L}_{\alpha C}$ and their closures generate strongly continuous contraction semigroups on $\mathcal{L}_{\alpha C}$. We will denote these semigroups $\hat{U}_{\alpha,\varepsilon,ren}(t)$ and $\hat{U}_{\alpha,V}(t)$ respectively. First observe that, for $G \in \mathcal{L}_C$, $\hat{U}_V(t)G, \hat{U}_{\alpha,V}(t)G \in \mathcal{L}_{\alpha C}$ and

$$\begin{aligned}\left\|\hat{U}_V(t)G - \hat{U}_{\alpha,V}G\right\|_{\alpha C} &\le \left\|\hat{U}_V(t)G - \hat{Q}_\delta^{[\frac{t}{\delta}]}G\right\|_{\alpha C} + \left\|\hat{U}_{\alpha,V}(t)G - \hat{Q}_\delta^{[\frac{t}{\delta}]}G\right\|_{\alpha C} \\ &\le \left\|\hat{U}_V(t)G - \hat{Q}_\delta^{[\frac{t}{\delta}]}G\right\|_{C} + \left\|\hat{U}_{\alpha,V}(t)G - \hat{Q}_\delta^{[\frac{t}{\delta}]}G\right\|_{\alpha C} \longrightarrow 0,\end{aligned}$$

as $\delta \downarrow 0$, hence $\hat{U}_V(t)G(\eta) = \hat{U}_{\alpha,V}G(\eta)$ for $\lambda \otimes \lambda$-a.a. $\eta \in \Gamma_0$ which also means, that $\hat{U}_V(t)G = \hat{U}_{\alpha,V}G$ in $\mathcal{L}_C$. Now let $k \in \mathcal{Q}_{\alpha C}$, then

$$\left\langle\left\langle \hat{U}_{\alpha,V}(t)G, k \right\rangle\right\rangle = \left\langle\left\langle G, \hat{U}^*_{\alpha,V}(t)k \right\rangle\right\rangle$$

where by construction $\hat{U}^*_{\alpha,V}k \in \mathcal{Q}_{\alpha C}$. But $G \in \mathcal{L}_C$, $k \in \mathcal{Q}_C$ implies

$$\left\langle\left\langle \hat{U}_{\alpha,V}(t)G, k \right\rangle\right\rangle = \left\langle\left\langle \hat{U}_V(t)G, k \right\rangle\right\rangle = \left\langle\left\langle G, \hat{U}^*_V(t)k \right\rangle\right\rangle$$

thus $\hat{U}^*_{\alpha,V}(t)k = \hat{U}^*_V(t)k \in \mathcal{Q}_{\alpha C}$. We can proceed similarly with $\hat{U}^*_{\varepsilon,ren}(t)$.

We can conclude the proof using (5.35) and the fact, that $\hat{U}^{\odot}_V(t)$ and $\hat{U}^{\odot}_{\varepsilon,ren}(t)$ are restrictions of the corresponding *-semigroups. □

Let us summarize. From now on we will fix $\alpha \in (\alpha_0, 1)$; then by the previous considerations the restrictions $\hat{U}^{\odot\alpha}_{\varepsilon,ren}(t)$, $\hat{U}^{\odot\alpha}_V(t)$ of the corresponding strongly continuous $^{\odot}$-semigroups onto the (invariant) space $\overline{\mathcal{Q}_{\alpha C}}$ are themselves strongly continuous contraction semigroups. Obviously we have $\hat{U}^{\odot\alpha}_{\varepsilon,ren}(t)k = \hat{U}^*_{\varepsilon,ren}(t)k$, $\hat{U}^{\odot\alpha}_V(t)k = \hat{U}^*_V(t)k$ for all $k \in \overline{\mathcal{Q}_{\alpha C}}$ and $\varepsilon > 0$. Moreover, their corresponding generators $\hat{L}^{\odot\alpha}_{\varepsilon,ren}$, $\hat{L}^{\odot\alpha}_V$ can be described in terms of *-generators as follows:

$$D(\hat{L}^{\odot\alpha}_{\varepsilon,ren}) = \left\{ k \in \overline{\mathcal{Q}_{\alpha C}} | \hat{L}^*_{\varepsilon,ren}k \in \overline{\mathcal{Q}_{\alpha C}} \right\}, \tag{5.38}$$

$$D(\hat{L}^{\odot\alpha}_V) = \left\{ k \in \overline{\mathcal{Q}_{\alpha C}} | \hat{L}^*_V k \in \overline{\mathcal{Q}_{\alpha C}} \right\}, \tag{5.39}$$

and it holds:

$$\hat{L}^{\odot\alpha}_{\varepsilon,ren}k = \hat{L}^*_{\varepsilon,ren}k, \text{ for } k \in D(\hat{L}^{\odot\alpha}_{\varepsilon,ren}) \tag{5.40}$$

and

$$\hat{L}^{\odot\alpha}_V k = \hat{L}^*_V k, \text{ for } k \in D(\hat{L}^{\odot\alpha}_V), \tag{5.41}$$

for every $\varepsilon > 0$.

5.3.4 Convergence of the semigroups

In what follows we prove several useful lemmas which we will use in the proof of the main result of this part. Recall the approximation operators

$$\begin{aligned}\hat{P}_{\varepsilon,\delta}G(\eta^1,\eta^2) = \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\\ \times\prod_{x\in\zeta^1}e^{-\varepsilon\beta E^{\phi}(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left(\frac{e^{-\varepsilon\beta E^{\phi}(x',\sigma^2)}-1}{\varepsilon}\right)\\ \times\prod_{y\in\zeta^2}e^{-\varepsilon\beta E^{\phi}(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left(\frac{e^{-\varepsilon\beta E^{\phi}(y',\sigma^1)}-1}{\varepsilon}\right)\\ \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2)\end{aligned}$$

and

$$\begin{aligned}\hat{Q}_{\delta}G(\eta^1,\eta^2) = \sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\\ \times\prod_{x\in\zeta^1}\left(-\beta E^{\phi}(x,\sigma^2)\right)\prod_{y\in\zeta^2}\left(-\beta E^{\phi}(y,\sigma^1)\right)\\ \times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2).\end{aligned}$$

Denote with $\hat{P}^*_{\varepsilon,\delta}$, $\hat{Q}^*_{\delta}$ their respective duals (with respect to the duality (5.28)). Because the duality preserves norm, the *-operators are also linear contractions in $\mathcal{Q}_C$. In the following we will show some properties of these operators. First we calculate the explicit form of $\hat{P}^*_{\varepsilon,\delta}$ and $\hat{Q}^*_{\delta}$.

Lemma 5.5. *For $\delta > 0$ and $\varepsilon > 0$ the dual operators $\hat{P}^*_{\varepsilon,\delta}$, $\hat{Q}^*_{\delta}$ have the following form:*

$$\begin{aligned}\hat{P}^*_{\varepsilon,\delta}k(\eta^1,\eta^2) = \sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|+|\eta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\\ \times\prod_{x'\in\eta^1\setminus\sigma^1}e^{-\varepsilon\beta E^{\phi}(x',\sigma^2)}\prod_{y'\in\eta^2\setminus\sigma^2}e^{-\varepsilon\beta E^{\phi}(y',\sigma^1)}\\ \times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}\left(\frac{e^{-\varepsilon\beta E^{\phi}(x,\sigma^2)}-1}{\varepsilon}\right)\prod_{y\in\zeta^2}\left(\frac{e^{-\varepsilon\beta E^{\phi}(y,\sigma^1)}-1}{\varepsilon}\right)\\ \times k\left(\left(\eta^1\setminus\sigma^1\right)\cup\zeta^1,\left(\eta^2\setminus\sigma^2\right)\cup\zeta^2\right)\lambda(d\zeta^1)\lambda(d\zeta^2)\end{aligned}$$

and

$$\hat{Q}^*_\delta k(\eta^1,\eta^2) = \sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|+|\eta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}\left(-\beta E^\phi(x,\sigma^2)\right)\prod_{y\in\zeta^2}\left(-\beta E^\phi(y,\sigma^1)\right)$$
$$\times k\left(\left(\eta^1\setminus\sigma^1\right)\cup\zeta^1,\left(\eta^2\setminus\sigma^2\right)\cup\zeta^2\right)\lambda(d\zeta^1)\lambda(d\zeta^2)$$

for $k\in\mathcal{Q}_{\alpha C}$.

Proof. Let $G\in\mathcal{L}_{\alpha C}$ and $k\in\mathcal{Q}_{\alpha C}$, then standard calculation yields:

$$\left\langle\left\langle\hat{P}_{\varepsilon,\delta}G,k\right\rangle\right\rangle = \int_{\Gamma_0}\int_{\Gamma_0}\sum_{\zeta^1\subset\eta^1}\sum_{\zeta^2\subset\eta^2}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\prod_{x\in\zeta^1}e^{-\varepsilon\beta E^\phi(x,\sigma^2)}\prod_{x'\in\eta^1\setminus\zeta^1}\left(\frac{e^{-\varepsilon\beta E^\phi(x',\sigma^2)}-1}{\varepsilon}\right)$$
$$\times\prod_{y\in\zeta^2}e^{-\varepsilon\beta E^\phi(y,\sigma^1)}\prod_{y'\in\eta^2\setminus\zeta^2}\left(\frac{e^{-\varepsilon\beta E^\phi(y',\sigma^1)}-1}{\varepsilon}\right)$$
$$\times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)\lambda(d\sigma^1)\lambda(d\sigma^2)k(\eta^1,\eta^2)\lambda(d\eta^1)\lambda(d\eta^2).$$

But using Lemma 1.3 we obtain

$$\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}(1-\delta)^{|\zeta^1|+|\zeta^2|}\int_{\Gamma_0}\int_{\Gamma_0}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\prod_{x\in\zeta^1}e^{-\varepsilon\beta E^\phi(x,\sigma^2)}\prod_{x'\in\eta^1}\left(\frac{e^{-\varepsilon\beta E^\phi(x',\sigma^2)}-1}{\varepsilon}\right)$$
$$\times\prod_{y\in\zeta^2}e^{-\varepsilon\beta E^\phi(y,\sigma^1)}\prod_{y'\in\eta^2}\left(\frac{e^{-\varepsilon\beta E^\phi(y',\sigma^1)}-1}{\varepsilon}\right)$$
$$\times G(\zeta^1\cup\sigma^1,\zeta^2\cup\sigma^2)k(\eta^1\cup\zeta^1,\eta^2\cup\zeta^2)$$
$$\times\lambda(d\sigma^1)\lambda(d\sigma^2)\lambda(d\zeta^1)\lambda(d\zeta^2)\lambda(d\eta^1)\lambda(d\eta^2),$$

and this is equal to

$$\int_{\Gamma_0}\int_{\Gamma_0} G\left(\zeta^1,\zeta^2\right) \sum_{\sigma^1\subset\zeta^1}\sum_{\sigma^2\subset\zeta^2} \int_{\Gamma_0}\int_{\Gamma_0} (1-\delta)^{|\zeta^1\setminus\sigma^1|+|\zeta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times \prod_{x\in\zeta^1\setminus\sigma^1} e^{-\varepsilon\beta E^{\phi}(x,\sigma^2)} \prod_{x'\in\eta^1}\left(\frac{e^{-\varepsilon\beta E^{\phi}(x',\sigma^2)}-1}{\varepsilon}\right)$$
$$\times \prod_{y\in\zeta^2\setminus\sigma^2} e^{-\varepsilon\beta E^{\phi}(y,\sigma^1)} \prod_{y'\in\eta^2}\left(\frac{e^{-\varepsilon\beta E^{\phi}(y',\sigma^1)}-1}{\varepsilon}\right)$$
$$\times k\left(\left(\zeta^1\setminus\sigma^1\right)\cup\eta^1,\left(\zeta^2\setminus\sigma^2\right)\cup\eta^2\right)\lambda(d\eta^1)\lambda(d\eta^2)\lambda(d\zeta^1)\lambda(d\zeta^2).$$

One can easily deduce the form of $\hat{P}^*_{\varepsilon,\delta}$ from the expression above. We can obtain $\hat{Q}^*_{\delta}$ by letting $\varepsilon\to 0$. □

Next we show, that the space $\overline{\mathcal{Q}_{\alpha C}}$ is $\hat{P}^*_{\varepsilon,\delta}$ and $\hat{Q}^*_{\delta}$-invariant.

Lemma 5.6. *Let $k\in\overline{\mathcal{Q}_{\alpha C}}$, then for all $\alpha\in(\alpha_0,1)$*

$$\hat{P}^*_{\varepsilon,\delta}k\in\overline{\mathcal{Q}_{\alpha C}} \tag{5.42}$$

and

$$\hat{Q}^*_{\delta}k\in\overline{\mathcal{Q}_{\alpha C}}. \tag{5.43}$$

Proof. We will start by proving (5.43). Given $k\in\mathcal{Q}_{\alpha C}$ we can do the following estimation:

$$(\alpha C)^{-\left(|\eta^1|+|\eta^2|\right)}\left|\hat{Q}^*_{\delta}k(\eta^1,\eta^2)\right|$$
$$\leq (\alpha C)^{|\eta^1|+|\eta^2|}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|+|\eta^2\setminus\sigma^2|}$$
$$\times(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}\left(\beta E^{\phi}(x,\sigma^2)\right)\prod_{y\in\zeta^2}\left(\beta E^{\phi}(y,\sigma^1)\right)$$
$$\times\left|k\left(\left(\eta^1\setminus\sigma^1\right)\cup\zeta^1,\left(\eta^2\setminus\sigma^2\right)\cup\zeta^2\right)\right|\lambda(d\zeta^1)\lambda(d\zeta^2).$$

By (5.30), this is less or equal than

$$\|k\|_{\mathcal{Q}_{\alpha C}}(\alpha C)^{-\left(|\eta^1|+|\eta^2|\right)}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|+|\eta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}\left(\beta E^{\phi}(x,\sigma^2)\right)\prod_{y\in\zeta^2}\left(\beta E^{\phi}(y,\sigma^1)\right)$$
$$\times(\alpha C)^{\left|\eta^1\setminus\sigma^1\cup\zeta^1\right|+\left|\eta^2\setminus\sigma^2\cup\zeta^2\right|}\lambda(d\zeta^1)\lambda(d\zeta^2)$$

and this can be further estimated from above by

$$\|k\|_{\mathcal{Q}_{\alpha C}}(\alpha C)^{-\left(|\eta^1|+|\eta^2|\right)}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}[\alpha C(1-\delta]^{|\eta^1\setminus\sigma^1|+|\eta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}\left(\alpha C\beta E^{\phi}(x,\sigma^2)\right)\prod_{y\in\zeta^2}\left(\alpha C\beta E^{\phi}(y,\sigma^1)\right)\lambda(d\zeta^1)\lambda(d\zeta^2)$$

which is equal to

$$\|k\|_{\mathcal{Q}_{\alpha C}}(\alpha C)^{-\left(|\eta^1|+|\eta^2|\right)}\sum_{\sigma^1\subset\eta^1}[\alpha C(1-\delta]^{|\eta^1\setminus\sigma^1|}(\varkappa\delta)^{|\sigma^1|}e^{\alpha C\beta\Phi|\sigma^1|}$$
$$\times\sum_{\sigma^2\subset\eta^2}[\alpha C(1-\delta]^{|\eta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^2|}e^{\alpha C\beta\Phi|\sigma^2|}.$$

Next, (5.36) yields

$$\left[\frac{\alpha C(1-\delta)+\varkappa\delta e^{\alpha C\beta\Phi}}{\alpha C}\right]\leq 1,$$

thus

$$(\alpha C)^{-\left(|\eta^1|+|\eta^2|\right)}\left|\hat{Q}^*_{\delta}k(\eta^1,\eta^2)\right|\leq\|k\|_{\mathcal{Q}_{\alpha C}}.$$

Because of the continuity of $\hat{Q}^*_{\delta}$, the latter holds also for $k\in\overline{\mathcal{Q}_{\alpha C}}$, hence $\overline{\mathcal{Q}_{\alpha C}}$ is $\hat{Q}^*_{\delta}$-invariant. We can proceed in exactly the same way with $\hat{P}^*_{\varepsilon,\delta}$ and conclude the proof. □

From now on, we will consider the restrictions of $\hat{P}^*_{\varepsilon,\delta}$ and $\hat{Q}^*_{\delta}$ onto the subspace $\overline{\mathcal{Q}_{\alpha C}}$, while preserving the latter notation. Denote D_ε the core of the generator $\hat{L}^{\odot\alpha}_{\varepsilon,ren}$, i.e. $D_\varepsilon:=\left\{k\in\mathcal{Q}_{\alpha C}|\hat{L}^*_{\varepsilon,ren}k\in\overline{\mathcal{Q}_{\alpha C}}\right\}$ and with D_V, the core of the operator $\hat{L}^{\odot\alpha}_V$, that is $D_V:=\left\{k\in\mathcal{Q}_{\alpha C}|\hat{L}^*_Vk\in\overline{\mathcal{Q}_{\alpha C}}\right\}$.

Proposition 5.9. *For every $\varepsilon>0$ the following holds:*

$$\lim_{\delta\downarrow 0}\left\|\frac{1}{\delta}\left(\hat{P}^*_{\varepsilon,\delta}-\mathbb{1}\right)k-\hat{L}^{\odot\alpha}_{\varepsilon,ren}k\right\|_{\mathcal{Q}_C}=0,\quad k\in D_\varepsilon \tag{5.44}$$

and

$$\lim_{\delta\downarrow 0}\left\|\frac{1}{\delta}\left(\hat{Q}^*_{\delta}-\mathbb{1}\right)k-\hat{L}^{\odot\alpha}_{V}k\right\|_{\mathcal{Q}_C}=0,\quad k\in D_V. \tag{5.45}$$

Proof. The proof is similar to the one of Proposition 5.5, but for the completeness of this thesis we will present it here in details.

To prove (5.44), let $k \in D_\varepsilon$ and set

$$\begin{aligned}
\hat{P}^{*,(0)}_{\varepsilon,\delta} k(\eta^1,\eta^2) :=& \quad (1-\delta)^{|\eta^1|+|\eta^2|} k(\eta^1,\eta^2), \\
\hat{P}^{*,(1)}_{\varepsilon,\delta} k(\eta^1,\eta^2) :=& \quad \varkappa\delta \sum_{x\in\eta^1} (1-\delta)^{|\eta^1|-1} \prod_{y'\in\eta^2} e^{-\varepsilon\beta\phi(y'-x)} \\
& \quad \times \int_{\Gamma_0} \prod_{y\in\zeta^2} \left(\frac{e^{-\varepsilon\beta\phi(y-x)}-1}{\varepsilon} \right) k(\eta^1 \setminus x, \eta^2 \cup \zeta^2) \lambda(d\zeta^2) \\
+& \quad \varkappa\delta \sum_{y\in\eta^2} (1-\delta)^{|\eta^2|-1} \prod_{x'\in\eta^1} e^{-\varepsilon\beta\phi(x'-y)} \\
& \quad \times \int_{\Gamma_0} \prod_{x\in\zeta^1} \left(\frac{e^{-\varepsilon\beta\phi(y-x)}-1}{\varepsilon} \right) k(\eta^1 \cup \zeta^1, \eta^2 \setminus y) \lambda(d\zeta^1)
\end{aligned}$$

and let

$$\hat{P}^{*,(2)}_{\varepsilon,\delta} k(\eta^1,\eta^2) := \quad \left(\hat{P}^{*}_{\varepsilon,\delta} - \hat{P}^{*,(0)}_{\varepsilon,\delta} - \hat{P}^{*,(1)}_{\varepsilon,\delta} \right) k(\eta^1,\eta^2).$$

then

$$\begin{aligned}
&\left\| \frac{1}{\delta} \left(\hat{P}^{*}_{\varepsilon,\delta} - 1\!\!1 - \hat{L}^{\odot\alpha}_{\varepsilon,ren} \right) k \right\|_{\mathcal{Q}_C} \qquad (5.46) \\
&\qquad = \quad \left\| \frac{1}{\delta} \left(\hat{P}^{*,(0)}_{\varepsilon,\delta} + \hat{P}^{*,(1)}_{\varepsilon,\delta} + \hat{P}^{*,(2)}_{\varepsilon,\delta} - \hat{L}^{\odot\alpha}_{\varepsilon,ren} \right) k \right\|_{\mathcal{Q}_C}.
\end{aligned}$$

First note, that for $n \geq 0$ and $\delta \in (0,1)$ we have ([FKK10d])

$$0 \leq n - \frac{1-(1-\delta)^n}{\delta} \leq \delta \frac{n(n-1)}{2},$$

and thus

$$\begin{aligned}
C^{-(|\eta^1|+|\eta^2|)} & \left| \frac{1}{\delta} \left((1-\delta)^{|\eta^1|+|\eta^2|} - 1 \right) k(\eta^1,\eta^2) + (|\eta^1|+|\eta^2|) k(\eta^1,\eta^2) \right| \\
&\leq \|k\|_{\mathcal{Q}_{\alpha C}} \alpha^{|\eta^1|+|\eta^2|} \left| \frac{(1-\delta)^{|\eta^1|+|\eta^2|}-1}{\delta} + |\eta^1| + |\eta^2| \right| \\
&\leq \frac{\delta}{2} \|k\|_{\mathcal{Q}_{\alpha C}} \alpha^{|\eta^1|+|\eta^2|} \left(|\eta^1| + |\eta^2| \right) \left(|\eta^1| + |\eta^2| - 1 \right),
\end{aligned}$$

and the function $\alpha^x x(x-1)$ is bounded for $x>0$. Next

$$
\begin{aligned}
C^{-(|\eta^1|+|\eta^2|)} & \left| \frac{1}{\delta} \hat{P}^{*,(1)}_{\varepsilon,\delta} k(\eta^1,\eta^2) - \left(\hat{L}^{\odot\alpha}_{\varepsilon,ren} + (|\eta^1|+|\eta^2|) \right) k(\eta^1,\eta^2) \right| \\
\leq \quad & C^{-(|\eta^1|+|\eta^2|)} \varkappa \sum_{x\in\eta^1} \left| 1-(1-\delta)^{|\eta^1|-1} \right| \prod_{y'\in\eta^2} e^{-\varepsilon\beta\phi(y'-x)} \\
& \times \int_{\Gamma_0} \prod_{y\in\zeta^2} \left| \frac{e^{-\varepsilon\beta\phi(y-x)}-1}{\varepsilon} \right| |k(\eta^1\setminus x, \eta^2\cup\zeta^2)| \lambda(d\zeta^2) \\
+ \quad & C^{-(|\eta^1|+|\eta^2|)} \varkappa \sum_{y\in\eta^2} \left| 1-(1-\delta)^{|\eta^2|-1} \right| \prod_{x'\in\eta^1} e^{-\varepsilon\beta\phi(x'-y)} \\
& \times \int_{\Gamma_0} \prod_{x\in\zeta^1} \left| \frac{e^{-\varepsilon\beta\phi(x-y)}-1}{\varepsilon} \right| |k(\eta^1\cup\zeta^1, \eta^2\setminus y)| \lambda(d\zeta^1)
\end{aligned}
$$

which, using similar arguments as before, can be estimated by

$$
\begin{aligned}
& \delta \|k\|_{\mathcal{Q}_{\alpha C}} C^{-(|\eta^1|+|\eta^2|)} \varkappa \sum_{x\in\eta^1} \left| \frac{1-(1-\delta)^{|\eta^1|-1}}{\delta} \right| \\
& \qquad \times \int_{\Gamma_0} \prod_{y\in\zeta^2} |\beta\phi(y-x)| \, (\alpha C)^{|\eta^1|-|1|+|\eta^2|+|\zeta^2|} \lambda(d\zeta^2) \\
& + \delta \|k\|_{\mathcal{Q}_{\alpha C}} C^{-(|\eta^1|+|\eta^2|)} \varkappa \sum_{y\in\eta^2} \left| \frac{1-(1-\delta)^{|\eta^2|-1}}{\delta} \right| \\
& \qquad \times \int_{\Gamma_0} \prod_{x\in\zeta^1} |\beta\phi(x-y)| \, (\alpha C)^{|\eta^1|+|\zeta^1|+|\eta^2|-1} \lambda(d\zeta^1).
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
& \frac{\delta}{2\alpha C} \|k\|_{\mathcal{Q}_{\alpha C}} \varkappa \alpha^{|\eta^1|+|\eta^2|} |\eta^1| \left(|\eta^1|-1\right) \left(|\eta^1|-2\right) e^{\alpha C\beta\Phi} \\
& \qquad + \frac{\delta}{2\alpha C} \|k\|_{\mathcal{Q}_{\alpha C}} \varkappa \alpha^{|\eta^1|+|\eta^2|} |\eta^2| \left(|\eta^2|-1\right) \left(|\eta^2|-2\right) e^{\alpha C\beta\Phi}
\end{aligned}
$$

which tends to 0 as $\delta \downarrow 0$. It remains to estimate

$$C^{-(|\eta^1|+|\eta^2|)}\frac{1}{\delta}\Big|\sum_{\substack{\sigma^1\subset\eta^1\\|\sigma^1|\geq 2}}\sum_{\substack{\sigma^2\subset\eta^2\\|\sigma^2|\geq 2}}(1-\delta)^{|\eta^1\setminus\sigma^1|+|\eta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\prod_{x'\in\eta^1\setminus\sigma^1}e^{-\varepsilon\beta E^\phi(x',\sigma^2)}\prod_{y'\in\eta^2\setminus\sigma^2}e^{-\varepsilon\beta E^\phi(y',\sigma^1)}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}\left(\frac{e^{-\varepsilon\beta E^\phi(x,\sigma^2)}-1}{\varepsilon}\right)\prod_{y\in\zeta^2}\left(\frac{e^{-\varepsilon\beta E^\phi(y,\sigma^1)}-1}{\varepsilon}\right)$$
$$\times k\left((\eta^1\setminus\sigma^1)\cup\zeta^1,(\eta^2\setminus\sigma^2)\cup\zeta^2\right)\lambda(d\zeta^1)\lambda(d\zeta^2)\Big|.$$

But this less or equal than

$$\|k\|_{\mathcal{Q}_{\alpha C}}C^{-(|\eta^1|+|\eta^2|)}\frac{1}{\delta}\sum_{\substack{\sigma^1\subset\eta^1\\|\sigma^1|\geq 2}}\sum_{\substack{\sigma^2\subset\eta^2\\|\sigma^2|\geq 2}}(1-\delta)^{|\eta^1\setminus\sigma^1|+|\eta^2\setminus\sigma^2|}(\varkappa\delta)^{|\sigma^1|}(\varkappa\delta)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}\left|\beta E^\phi(x,\sigma^2)\right|\prod_{y\in\zeta^2}\left|\beta E^\phi(y,\sigma^1)\right|$$
$$\times(\alpha C)^{|\eta^1|-|\sigma^1|+|\zeta^1|+|\eta^2|-|\sigma^2|+|\zeta^2|}\lambda(d\zeta^1)\lambda(d\zeta^2),$$

and further (w.l.o.g. assuming that $\delta<1$) we can estimate the latter by

$$\|k\|_{\mathcal{Q}_{\alpha C}}\alpha^{|\eta^1|+|\eta^2|}\frac{1}{\delta^2}\sum_{\substack{\sigma^1\subset\eta^1\\|\sigma^1|\geq 2}}(1-\delta)^{|\eta^1\setminus\sigma^1|}\left(\frac{\varkappa\delta}{\alpha C}e^{\alpha C\beta\Phi}\right)^{|\sigma^1|}$$
$$\times\sum_{\substack{\sigma^2\subset\eta^2\\|\sigma^2|\geq 2}}(1-\delta)^{|\eta^2\setminus\sigma^2|}\left(\frac{\varkappa\delta}{\alpha C}e^{\alpha C\beta\Phi}\right)^{|\sigma^2|}.$$

But as in [FKK10d], this is less or equal to

$$\delta\|k\|_{\mathcal{Q}_{\alpha C}}\alpha^{|\eta^1|}|\eta^1|\left(|\eta^1|-1\right)\alpha^{|\eta^2|}|\eta^2|\left(|\eta^2|-1\right).$$

Summing up previous considerations, we obtain (5.44). Because the proof of (5.45) is completely analogous, we will omit it here. □

The proof of next lemma can be found e.g. in [FKK10d].

Lemma 5.7. *Let X be a Banach space with a norm $\|\cdot\|_X$; A and B be linear contraction mappings on X. Let Y with a norm $\|\cdot\|_Y$ be a Banach subspace of X such that Y is invariant w.r.t. B. Suppose also that there exists $c > 0$ such that for any $f \in Y$*

$$\|Af - Bf\|_X \leq c\|f\|_Y. \tag{5.47}$$

Then, for any $m \in \mathbb{N}$ and for any $f \in Y$

$$\|A^m f - B^m f\|_X \leq cm\|f\|_Y. \tag{5.48}$$

Now we proceed to the main result of this part, namely we show the convergence in $\mathcal{Q}_C$ norm of the scaled semigroup to the corresponding Vlasov semigroup.

Theorem 5.4. *Let* (5.25) *hold and let*

$$\overline{\phi} := \sup_{x\in\mathbb{R}^d} \phi(x) < +\infty, \tag{5.49}$$

then for any $\alpha \in (\alpha_0, 1)$ and $k \in \mathcal{Q}_{\alpha C}$

$$\lim_{\varepsilon\to 0} \left\| \hat{U}^{\odot\alpha}_{\varepsilon,ren}(t)k - \hat{U}^{\odot\alpha}_V(t)k \right\|_{\mathcal{Q}_C} = 0. \tag{5.50}$$

Proof. Let $k \in \mathcal{Q}_{\alpha C}$ and recall the approximation operators $\hat{P}^*_{\varepsilon,\delta}$ and $\hat{Q}^*_\delta$. Using previous reslts and Corollary 5.1, we have

$$\left(\hat{P}^*_{\varepsilon,\delta}\right)^{[\frac{t}{\delta}]} k \to \hat{U}^{\odot\alpha}_{\varepsilon,ren}(t)k \quad \text{and} \quad \left(\hat{Q}^*_\delta\right)^{[\frac{t}{\delta}]} k \to \hat{U}^{\odot\alpha}_V(t)k \tag{5.51}$$

in the space $\overline{\mathcal{Q}_{\alpha C}}$ with the $\|\cdot\|_{\mathcal{Q}_C}$ norm. Hence, using the triangle inequality we can write

$$\begin{aligned}\left\| \hat{U}^{\odot\alpha}_{\varepsilon,ren}(t)k - \hat{U}^{\odot\alpha}_V(t)k \right\|_{\mathcal{Q}_C} \leq & \left\| \hat{U}^{\odot\alpha}_{\varepsilon,ren}(t)k - \left(\hat{P}^*_{\varepsilon,\delta}\right)^{[\frac{t}{\delta}]} k \right\|_{\mathcal{Q}_C} \\ & + \left\| \left(\hat{Q}^*_\delta\right)^{[\frac{t}{\delta}]} k - \hat{U}^{\odot\alpha}_V(t)k \right\|_{\mathcal{Q}_C} \\ & + \left\| \left(\hat{P}^*_{\varepsilon,\delta}\right)^{[\frac{t}{\delta}]} k - \left(\hat{Q}^*_\delta\right)^{[\frac{t}{\delta}]} k \right\|_{\mathcal{Q}_C}\end{aligned}$$

and because the two first terms on the right hand side tend to 0 as $\delta \downarrow 0$, it remains only to show, that

$$\left\| \left(\hat{P}^*_{\varepsilon,\delta}\right)^{[\frac{t}{\delta}]} k - \left(\hat{Q}^*_\delta\right)^{[\frac{t}{\delta}]} k \right\|_{\mathcal{Q}_C} \to 0 \tag{5.52}$$

as $\varepsilon \to 0$. Using Lemma 5.7 the latter can be deduced from the fact that for every ε, $\delta > 0$ there exists $c > 0$, such that

$$\left\| \hat{P}^*_{\varepsilon,\delta} k - \hat{Q}^*_\delta k \right\|_{\mathcal{Q}_C} \leq \varepsilon \delta c \|k\|_{\mathcal{Q}_{\alpha C}}. \tag{5.53}$$

But for any $\eta^1, \eta^2, \zeta^1, \zeta^2 \in \Gamma_0$

$$\begin{aligned}
&\Bigg| \prod_{x' \in \eta^1 \setminus \sigma^1} e^{-\varepsilon\beta E^\phi(x',\sigma^2)} \prod_{y' \in \eta^2 \setminus \sigma^2} e^{-\varepsilon\beta E^\phi(y',\sigma^1)} \\
&\qquad \times \prod_{x \in \zeta^1} \left(\frac{e^{-\varepsilon\beta E^\phi(x,\sigma^2)} - 1}{\varepsilon} \right) \prod_{y \in \zeta^2} \left(\frac{e^{-\varepsilon\beta E^\phi(y,\sigma^1)} - 1}{\varepsilon} \right) \\
&\qquad\qquad - \prod_{x \in \zeta^1} (-\beta E^\phi(x,\sigma^2)) \prod_{y \in \zeta^2} (-\beta E^\phi(y,\sigma^1)) \Bigg| \\
&= \prod_{x \in \zeta^1} (\beta E^\phi(x,\sigma^2)) \prod_{y \in \zeta^2} (\beta E^\phi(y,\sigma^1)) \\
&\times \Bigg| \prod_{x' \in \eta^1 \setminus \sigma^1} e^{-\varepsilon\beta E^\phi(x',\sigma^2)} \prod_{y' \in \eta^2 \setminus \sigma^2} e^{-\varepsilon\beta E^\phi(y',\sigma^1)} \\
&\qquad \times \prod_{x \in \zeta^1} \left(\frac{1 - e^{-\varepsilon\beta E^\phi(x,\sigma^2)}}{\varepsilon\beta E^\phi(x,\sigma^2)} \right) \prod_{y \in \zeta^2} \left(\frac{1 - e^{-\varepsilon\beta E^\phi(y,\sigma^1)}}{\varepsilon\beta E^\phi(y,\sigma^1)} \right) - 1 \Bigg|.
\end{aligned}$$

Using the fact that for $a_k \in [0,1]$ we have

$$1 - \prod_k a_k \leq \sum_k (1 - a_k)$$

we can estimate the latter expression by:

$$\begin{aligned}
&\prod_{x \in \zeta^1} (\beta E^\phi(x,\sigma^2)) \prod_{y \in \zeta^2} (\beta E^\phi(y,\sigma^1)) \\
&\quad \times \Bigg[\sum_{x' \in \eta^1 \setminus \sigma^1} \left(1 - e^{-\varepsilon\beta E^\phi(x',\sigma^2)}\right) + \sum_{y' \in \eta^2 \setminus \sigma^2} \left(1 - e^{-\varepsilon\beta E^\phi(y',\sigma^1)}\right) \\
&\qquad + \sum_{x \in \zeta^1} \left(1 - \frac{1 - e^{-\varepsilon\beta E^\phi(x,\sigma^2)}}{\varepsilon\beta E^\phi(x,\sigma^2)} \right) + \sum_{y \in \zeta^2} \left(1 - \frac{1 - e^{-\varepsilon\beta E^\phi(y,\sigma^1)}}{\varepsilon\beta E^\phi(y,\sigma^1)} \right) \Bigg].
\end{aligned}$$

Next, because $1 - e^{-a} < a$, $a > 0$, this is less or equal than

$$\prod_{x\in\zeta^1} (\beta E^\phi(x,\sigma^2)) \prod_{y\in\zeta^2} (\beta E^\phi(y,\sigma^1))$$
$$\times \Bigg[\underbrace{\sum_{x'\in\eta^1\setminus\sigma^1} \left(\varepsilon\beta E^\phi(x',\sigma^2)\right)}_{:=S_1} + \underbrace{\sum_{y'\in\eta^2\setminus\sigma^2} \left(\varepsilon\beta E^\phi(y',\sigma^1)\right)}_{:=S_2}$$
$$+ \underbrace{\sum_{x\in\zeta^1} \left(1 - \frac{1-e^{-\varepsilon\beta E^\phi(x,\sigma^2)}}{\varepsilon\beta E^\phi(x,\sigma^2)}\right)}_{:=S_3} + \underbrace{\sum_{y\in\zeta^2} \left(1 - \frac{1-e^{-\varepsilon\beta E^\phi(y,\sigma^1)}}{\varepsilon\beta E^\phi(y,\sigma^1)}\right)}_{:=S_4} \Bigg].$$

Hence the norm

$$\left\| \hat{P}^*_{\varepsilon,\delta} k - \hat{Q}^*_\delta k \right\|_{\mathcal{Q}_C}$$

can be estimated by

$$C^{-\left(|\eta^1|+|\eta^2|\right)} \sum_{\sigma^1\subset\eta^1} \sum_{\sigma^2\subset\eta^2} (1-\delta)^{|\eta^1\setminus\sigma^1|} (\varkappa\delta)^{|\sigma^1|} (1-\delta)^{|\eta^2\setminus\sigma^2|} (\varkappa\delta)^{|\sigma^2|}$$
$$\times \int_{\Gamma_0}\int_{\Gamma_0} \prod_{x\in\zeta^1} (\beta E^\phi(x,\sigma^2)) \prod_{y\in\zeta^2} (\beta E^\phi(y,\sigma^1)) \left[S_1+S_2+S_3+S_4\right]$$
$$\times \left| k\left((\eta^1\setminus\sigma^1)\cup\zeta^1, (\eta^2\setminus\sigma^2)\cup\zeta^2\right)\right| \lambda(d\zeta^1)\lambda(d\zeta^2).$$

and further by

$$\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C} \sum_{\sigma^1\subset\eta^1} \sum_{\sigma^2\subset\eta^2} (1-\delta)^{|\eta^1\setminus\sigma^1|} \left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^1|} (1-\delta)^{|\eta^2\setminus\sigma^2|} \left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^2|}$$
$$\times \int_{\Gamma_0}\int_{\Gamma_0} \prod_{x\in\zeta^1} (\alpha\beta C E^\phi(x,\sigma^2)) \prod_{y\in\zeta^2} (\alpha\beta C E^\phi(y,\sigma^1)) \sum_{i=1}^4 S_i \lambda(d\zeta^1)\lambda(d\zeta^2). \tag{5.54}$$

Using linearity of integral it is enough to estimate each of the four terms above.

Thus, starting with S_1 we get:

$$\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^1|}(1-\delta)^{|\eta^2\setminus\sigma^2|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}(\alpha\beta CE^{\phi}(x,\sigma^2))\prod_{y\in\zeta^2}(\alpha\beta CE^{\phi}(y,\sigma^1))$$
$$\times\sum_{x'\in\eta^1\setminus\sigma^1}\left(\varepsilon\beta E^{\phi}(x',\sigma^2)\right)\lambda(d\zeta^1)\lambda(d\zeta^2),$$

but this is equal to

$$\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^1|}(1-\delta)^{|\eta^2\setminus\sigma^2|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^2|}$$
$$\times e^{\alpha\beta C\Phi|\sigma^2|}e^{\alpha\beta C\Phi|\sigma^1|}\sum_{x'\in\eta^1\setminus\sigma^1}\left(\varepsilon\beta E^{\phi}(x',\sigma^2)\right).$$

Using (5.25) and (5.36) we can estimate the latter by

$$\varepsilon\beta\overline{\phi}\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|}\delta^{|\sigma^1|}(1-\delta)^{|\eta^2\setminus\sigma^2|}\delta^{|\sigma^2|}|\eta^1\setminus\sigma^1||\sigma^2|.$$

Let $|\eta^1|=n$ and $|\eta^2|=m$, then

$$\begin{aligned}
&\sum_{\sigma^1\subset\eta^1}|\eta^1\setminus\sigma^1|(1-\delta)^{|\eta^1\setminus\sigma^1|}\delta^{|\sigma^1|}\sum_{\sigma^2\subset\eta^2}|\sigma^2|(1-\delta)^{|\eta^2\setminus\sigma^2|}\delta^{|\sigma^2|}\\
&\quad=\sum_{k=1}^{n-1}\binom{n}{n-k}(n-k)(1-\delta)^{n-k}\delta^k\sum_{l=1}^{m}\binom{m}{l}l(1-\delta)^{m-l}\delta^l\\
&\quad=\sum_{k=1}^{n-1}\frac{n!}{(n-k-1)!k!}(1-\delta)^{n-k}\delta^k\sum_{l=1}^{m}\frac{m!}{(m-l)!(l-1)!}(1-\delta)^{m-l}\delta^l\\
&\quad=(1-\delta)\delta n(n-2)\sum_{k=0}^{n-2}\frac{(n-2)!}{(n-2-k)!(k+1)!}(1-\delta)^{n-2-k}\delta^k\\
&\qquad\qquad\times\delta m\sum_{l=0}^{m-1}\frac{(m-1)!}{(m-1-l)!l!}(1-\delta)^{m-1-l}\delta^l\\
&\quad\le(1-\delta)\delta^2n(n-2)m<\infty.
\end{aligned}$$

Note also, that the case $i=2$ (i.e. with S_2) can be estimated in the same manner. Thus the terms containing S_1 and S_2 in (5.54) can be estimated by

$$\varepsilon(1-\delta)\delta^2\beta\overline{\phi}\|k\|_{\alpha C}\alpha^{|\eta^1|+|\eta^2|}\left(|\eta^1|^2|\eta^2|+|\eta^1||\eta^2|^2\right)<\varepsilon\delta A_1\|k\|_{\alpha C}\qquad(5.55)$$

with $A_1 := \beta\overline{\phi}\sup_{n,m\in\mathbb{N}}\left[\alpha^{n+m}\left(n^2m+nm^2\right)\right]$. We proceed now to the estimation of terms which include S_3 and S_4. For the case $i=3$ we have:

$$\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^1|}(1-\delta)^{|\eta^2\setminus\sigma^2|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\prod_{x\in\zeta^1}(\beta E^\phi(x,\sigma^2))\prod_{y\in\zeta^2}(\alpha\beta C E^\phi(y,\sigma^1))$$
$$\times\sum_{x\in\zeta^1}\left(1-\frac{1-e^{-\varepsilon\beta E^\phi(x,\sigma^2)}}{\varepsilon\beta E^\phi(x,\sigma^2)}\right)(\alpha C)^{|\zeta^1|}\lambda(d\zeta^1)\lambda(d\zeta^2).$$

By Minlos lemma this is equal to

$$\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^1|}(1-\delta)^{|\eta^2\setminus\sigma^2|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}\prod_{x'\in\zeta^1\cup x}(\beta E^\phi(x',\sigma^2))\prod_{y\in\zeta^2}(\alpha\beta C E^\phi(y,\sigma^1))$$
$$\times\left(1-\frac{1-e^{-\varepsilon\beta E^\phi(x,\sigma^2)}}{\varepsilon\beta E^\phi(x,\sigma^2)}\right)(\alpha C)^{|\zeta^1|}(\alpha C)dx\lambda(d\zeta^1)\lambda(d\zeta^2)$$

and because (see [FKK10d])

$$\beta E^\phi(x,\sigma^2)\left(1-\frac{1-e^{-\varepsilon\beta E^\phi(x,\sigma^2)}}{\varepsilon\beta E^\phi(x,\sigma^2)}\right)\le\varepsilon\left(\beta E^\phi(x,\sigma^2)\right)^2,$$

the latter expression can be estimated by:

$$\varepsilon\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^1|}(1-\delta)^{|\eta^2\setminus\sigma^2|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^2|}$$
$$\times\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}\prod_{x'\in\zeta^1}(\beta E^\phi(x',\sigma^2))\prod_{y\in\zeta^2}(\alpha\beta C E^\phi(y,\sigma^1))$$
$$\times\left(\beta E^\phi(x,\sigma^2)\right)^2(\alpha C)^{|\zeta^1|}(\alpha C)dx\lambda(d\zeta^1)\lambda(d\zeta^2)$$

and further by

$$\varepsilon\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^1\setminus\sigma^1|}\delta^{|\sigma^1|}(1-\delta)^{|\eta^2\setminus\sigma^2|}\left(\frac{\varkappa\delta}{\alpha C}\right)^{|\sigma^2|}\tag{5.56}$$
$$\times\int_{\Gamma_0}\int_{\mathbb{R}^d}\prod_{x'\in\zeta^1}(\beta E^\phi(x',\sigma^2))\left(\beta E^\phi(x,\sigma^2)\right)^2(\alpha C)^{|\zeta^1|}(\alpha C)dx\lambda(d\zeta^1).$$

The integral in the last expression can be calculated:

$$\begin{aligned}
&\int_{\Gamma_0}\int_{\mathbb{R}^d}\prod_{x'\in\zeta^1}\left(\beta E^\phi(x',\sigma^2)\right)\left(\beta E^\phi(x,\sigma^2)\right)^2(\alpha C)^{|\zeta^1|}(\alpha C)dx\lambda(d\zeta^1)\\
&\quad=\int_{\Gamma_0}\sum_{x\in\zeta^1}\left(\beta E^\phi(x,\sigma^2)\right)^2\prod_{x'\in\zeta^1\backslash x}\left(\beta E^\phi(x',\sigma^2)\right)(\alpha C)^{|\zeta^1|}\lambda(d\zeta^1)\\
&\quad\le\beta\overline{\phi}|\sigma^2|\int_{\Gamma_0}\sum_{x\in\zeta^1}\left(\beta E^\phi(x,\sigma^2)\right)\prod_{x'\in\zeta^1\backslash x}\left(\beta E^\phi(x',\sigma^2)\right)(\alpha C)^{|\zeta^1|}\lambda(d\zeta^1)\\
&\quad=\alpha\beta C\overline{\phi}|\sigma^2|\int_{\Gamma_0}\int_{\mathbb{R}^d}\left(\beta E^\phi(x,\sigma^2)\right)dx\prod_{x'\in\zeta^1}\left(\beta E^\phi(x',\sigma^2)\right)(\alpha C)^{|\zeta^1|}\lambda(d\zeta^1)\\
&\quad=\alpha\beta^2C\overline{\phi}|\sigma^2|^2e^{\alpha\beta C\Phi|\sigma^2|}
\end{aligned}$$

thus the value of (5.56) is less or equal to

$$\varepsilon\alpha\beta^2C\overline{\phi}\alpha^{|\eta^1|+|\eta^2|}\|k\|_{\alpha C}\sum_{\sigma^1\subset\eta^1}(1-\delta)^{|\eta^1\backslash\sigma^1|}\delta^{|\sigma^1|}\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^2\backslash\sigma^2|}\delta^{|\sigma^2|}|\sigma^2|^2. \tag{5.57}$$

Finally, let us estimate the second sum in the latter expression (let $|\eta^2|=n$):

$$\begin{aligned}
\sum_{\sigma^2\subset\eta^2}(1-\delta)^{|\eta^2\backslash\sigma^2|}\delta^{|\sigma^2|}|\sigma^2|^2 &= \sum_{k=1}^{n}\frac{n!}{(n-k)!k!}k^2(1-\delta)^{n-k}\delta^k\\
&=\sum_{k=1}^{n}\frac{n!}{(n-k)!(k-1)!}k(1-\delta)^{n-k}\delta^k\\
&=\delta n\sum_{k=1}^{n}\frac{n!}{((n-1)-(k-1))!(k-1)!}(1-\delta)^{(n-1)-(k-1)}\delta^{k-1}\\
&=\delta n^2\sum_{k=0}^{n-1}\frac{(n-1)!}{((n-1)-k)!k!}(1-\delta)^{(n-1)-k}\delta^k\\
&\le\delta n^2<\infty.
\end{aligned}$$

The similar calculation for the part with S_4 allows us to estimate the S_3+S_4 by:

$$\varepsilon\delta\alpha\beta^2C\overline{\phi}\|k\|_{\alpha C}\alpha^{|\eta^1|+|\eta^2|}\left(|\eta^1|^2+|\eta^2|^2\right)\le\varepsilon\delta A_2\|k\|_{\alpha C} \tag{5.58}$$

with $A_2:=\alpha\beta^2C\overline{\phi}\sup_{n,m\in\mathbb{N}}[\alpha^{n+m}(n^2+m^2)]$.

Summing up the previous considerations, we obtain:

$$\left\| \hat{P}^*_{\varepsilon,\delta} k - \hat{Q}^*_{\delta} k \right\|_{\mathcal{Q}_C} \leq \varepsilon\delta \left(A_1 \vee A_2\right) \|k\|_{\mathcal{Q}_{\alpha C}} \tag{5.59}$$

and thus (5.53) is fulfilled. Using Lemma 5.7 we get (5.52) and (5.50) follows, and the corresponding scaled semigroup converges in $\mathcal{Q}_C$ to the semigroup associated to the virtual system, $\hat{U}_V^{\odot\alpha}(t)$.

□

5.3.5 Vlasov-type equation for the model

To conclude this chapter, we will derive the Vlasov-type equation for the Glauber-Potts model.

Theorem 5.5. *Assume* (5.25) *and let functions* ρ_0^1 *and* $\rho_0^2 \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ *be such that there exists some* $\alpha \in (\alpha_0, 1)$ *for which the following holds:*

$$ess\ sup_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} |\rho_0^i(x,y)| \leq \alpha C,\ i = 1,2$$

and assume that

$$k_0(\eta^1, \eta^2) = e_\lambda(\rho_0^1, \eta^1) \cdot e_\lambda(\rho_0^2, \eta^2).$$

Then the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} k_t = & \hat{L}_V^* k_t, \\ \\ k_0 = & e_\lambda(\rho_0) \end{cases} \tag{5.60}$$

is well defined in $\overline{\mathcal{Q}_{\alpha C}}$ *and its mild solution* $k_t = \hat{U}_V^{\odot\alpha}(t) k_0 \in \mathcal{Q}_{\alpha C}$ *has the form* $k_t(\eta^1, \eta^2) = e_\lambda(\rho_t^1, \eta^1) e_\lambda(\rho_t^2, \eta^2)$ *where* ρ_t^1, ρ_t^2 *satisfy the following equations:*

$$\begin{cases} \frac{\partial}{\partial t}\rho_t^1(x) & = & -\rho_t^1(x) + \varkappa e^{-\beta\left(\rho_t^2 * \phi\right)(x)} \\ \\ \rho_t^1(x)|_{t=0} & = & \rho_0^1(x), \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial t}\rho_t^2(y) & = & -\rho_t^2(y) + \varkappa e^{-\beta\left(\rho_t^1 * \phi\right)(y)} \\ \\ \rho_t^2(y)|_{t=0} & = & \rho_0^2(y), \end{cases}$$

Proof. Using the assumptions and properties of $\hat{U}_V^{\odot\alpha}(t)$ it is obvious that $k_t = \hat{U}_V^{\odot\alpha}(t) \in \mathcal{Q}_{\alpha C}$ and that it is strongly differentiable with respect to the norm in $\mathcal{Q}_{\alpha C}$. The equations for ρ_t^1 and ρ_t^2 can be deduced by inserting $k_t(\eta^1, \eta^2) = e_\lambda(\rho_t^1, \eta^1)e_\lambda(\rho_t^2, \eta^2)$ into the equation (5.60), similarly as in Section 4.4. Note also, that $k_t \in \mathcal{Q}_{\alpha C}$ means that

$$\text{ess sup}_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} |\rho_t(x, y)| \leq \alpha C.$$

$\square$

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