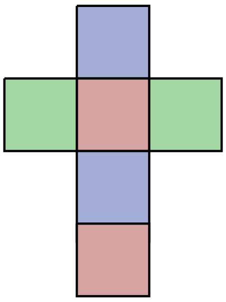
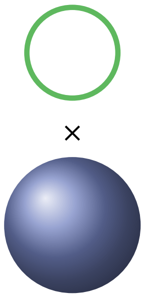
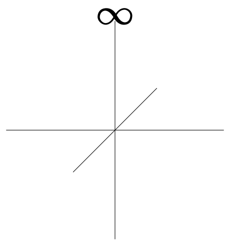
Spectral properties of Spin$^{\mathbb{C}}$ Dirac operators on

T^3 $S^1 \times S^2$ S^3







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Contents

Abstract

$\mathrm{Spin}^{\mathbb{C}}$ Dirac operators on 3-manifolds mainly arise in the study of 4-manifolds with boundary. Especially for TQFT-like theories it is necessary to obtain information about $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators on nice 3-manifolds.

For a given Riemannian manifold M a $\mathrm{Spin}^{\mathbb{C}}$ Dirac operator is determined by two pieces of data:

(i) A $\mathrm{Spin}^{\mathbb{C}}$ structure on M.

(ii) A $U(1)$ connection on the associated determinant bundle.

Since all 3-manifolds have Spin structures, we get a natural "zero" $\mathrm{Spin}^{\mathbb{C}}$ structure; therefore, we can parametrise all $\mathrm{Spin}^{\mathbb{C}}$ structures by elements $\hat{a} \in H^2(M;\mathbb{Z})$ in a natural way (we assume that there is no torsion in cohomology).

If we choose a line bundle K for $\hat{a}$ and a background connection ∇^K we can parametrise the $U(1)$ connections by one-forms $\alpha \in \Omega^1(M;\mathbb{R})$.

Let $\mathscr{L}$ be a subspace of the space of closed one-forms. Our aim is to investigate — for every $\hat{a}$ — the family of Dirac operators $\mathscr{D}$ parametrised by $\mathscr{L}/\ell$ (where we divide $\mathscr{L}$ by an appropriate lattice).

Particularly we will be interested in the following questions (which will be restated at the end of chapter 1 after we have given the necessary definitions):

1. Given a $\mathrm{Spin}^{\mathbb{C}}$ structure $\hat{a} = c_1(K)$ and a closed one-form α^c: What is the spectrum of $\mathscr{D}^K_{\alpha^c}$?
2. Under the same conditions: How can we explicitly calculate an orthogonal eigenbasis for $\mathscr{D}^K_{\alpha^c}$?
3. For which $\mathscr{L}$ do spectral sections (in the sense of Melrose-Piazza) of $\mathscr{D}$ exist?
4. If spectral sections exist: How can we explicitly construct one of them?
5. What does the set of *infinitesimal* spectral sections look like? What is its image in $K(\mathscr{L}/\ell)$?

The most interesting case we examine is $M = T^3$; here all questions have non-trivial answers. Especially interesting is the dichotomy between the case $\hat{a} = 0$ and $\hat{a} \neq 0$, which show completely different behaviour:

$\hat{a} \neq 0$ Here we attack the problem by a projection of T^3 onto an appropriate 2-torus, where we can use methods of [Almorox06]. Especially in the "boundary case", where $\mathscr{L}$ comes from the space of forms on a 4-manifold ([Melrose97]), we can explicitly describe (and improve) a theorem proved before in an abstract way. We can reduce everything to the phrase: Either the spectrum is constant or spectral sections do not exist at all.

$\hat{a} = 0$ The spectrum can be found by a direct calculation. It changes in every direction of $\mathscr{L}$; here we are particularly interested in classifying the infinitesimal spectral sections i.e. those which are very near to the spectral projection. This is mainly done by using homotopy groups.

As a by-product we look at $S^1 \times S^2$, which can be analysed along the lines of T^3 but with much less difficulties. The necessary examination of the Dirac operator on S^2 uses again parts of [Almorox06].

The space S^3 has a totally different flavour. Since the first and second cohomology vanishes, we have no parametrisation. At first we calculate the spectrum by translating methods of [Hitchin74] to the quaternions. This is mainly done by using representation theory of $\mathrm{Sp}(1) \cong S^3$ in the context of the Laplace-Beltrami operator Δ. Afterwards we improve the results of Hitchin by giving an explicit orthogonal eigenbasis. For that purpose we analyse operators coming from $\mathbb{C} \otimes \mathbf{sp}(1)$.

Chapter 0

Introduction

In this chapter we run through the whole thesis in fast motion to give the reader an impression of what to expect from the different parts of it. After giving some general information about the subject, we shortly introduce and describe each of the four chapters. Of course we cannot always state exact definitions here but the reader will find them at the appropriate places.

Spin Dirac operators (in contrast to $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators) were intensively studied on 3-manifolds. On some, like the 3-torus, you can completely describe their spectrum and eigenspaces (see [Friedrich84]), on others, bounds for special eigenvalues were established (see e.g. [Bär00]).

Manifolds of dimension 4 seem to be the natural habitat for $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators; reason for that are their importance in Seiberg-Witten theory and of course the fact that — in contrast to the 3-dimensional case — only the existence of $\mathrm{Spin}^{\mathbb{C}}$ structures is guaranteed while Spin 4-manifolds are rare.

The investigation of $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators on 3-manifolds is therefore mainly motivated by looking at non-closed compact 4-manifolds. Those manifolds induce $\mathrm{Spin}^{\mathbb{C}}$ structures on their boundary components, which, of course, carry Dirac operators on their associated bundles. Especially if we want to understand generalized Seiberg-Witten theory in the context of "gluing manifolds together", it should be necessary to understand the induced operators on 3-manifolds.

There are "much more" Dirac operators in the $\mathrm{Spin}^{\mathbb{C}}$ case. If we assume that there is no 2-torsion in cohomology then $\mathrm{Spin}^{\mathbb{C}}$ structures are in bijective correspondence with $H^2(M;\mathbb{Z})$, where the trivial $\mathrm{Spin}^{\mathbb{C}}$ structure corresponds to zero. Each of them has an associated bundle where Dirac operators can act on.

After fixing a $\mathrm{Spin}^{\mathbb{C}}$ structure $\hat{a} \in H^2(M;\mathbb{Z})$ and choosing an associated bundle with background connection, the Dirac operators are parametrised by real one-forms on M. In this thesis we will restrict ourselves to the parameter space of closed one-forms, although the methods of calculation may be generalised to more general (but not arbitrary) subspaces of the one-forms.

Our first aim then is to calculate spectrum and eigenbases for these Dirac operators on the three manifolds T^3, $S^1 \times S^2$ and S^3.

After that we want to investigate the behaviour of the spectrum viewed as "function" of the one-form. For that we choose an arbitrary linear subspace $\ell \subset H^1(M;\mathbb{Z})$ and consider the Dirac operator $\mathscr{D}$ parametrised by the torus $(\ell \otimes \mathbb{R})/\ell =: \mathscr{L}/\ell$.

If $\mathscr{L}/\ell \cong S^1$, the obvious way would be to analyse the spectral flow of $\mathscr{D}$ (which is the number of eigenvalues crossing zero during one "round", counted with sign).

In other cases we need a generalisation of this concept called *spectral section*. It is defined to be a family of projections continuously depending on $\mathscr{L}/\ell$ which is "nearly" the spectral projection (onto the positive eigenspaces), i.e. differs from it only in finitely many dimensions.

The existence of spectral sections is guaranteed in the *boundary case*: This is the case where all structures on M are induced by a 4-manifold.

As the existence proof does not tell you anything about the actual construction of spectral sections it is our aim to describe them concretely. We will also classify the infinitesimal ones, i.e. those which are very near to the spectral projection.

0.a Spin$^{\mathbb{C}}$ Dirac operators on 3-manifolds

Here we define Dirac operators on 2- and 3-manifolds, and also the relevant bundles and Clifford algebras. Furthermore we introduce spectral sections and give methods for finding them or disproving their existence.

The definitions of this chapter are used afterwards without reference but the reader is advised to look up the relevant terms in the notation appendix or the index at the end of this thesis.

0.b The space T^3

On the 3-torus we use two different methods to calculate spectrum and eigenbasis of the operator $\mathscr{D}$:

1. *The projection method*: We interpret a Spin$^{\mathbb{C}}$ structure $\hat{a} \neq 0$ as a vector in $\mathbb{R}^3$ ($H^1(M;\mathbb{Z}) \cong \mathbb{Z}^3$) and use it to define an orthonormal projection $\mathbb{R}^3 \to W$ which leads to a non-canonical trivial S^1 bundle $T^3 \to T_\Lambda$ over a 2-torus. Since T^3 is not the orthogonal product $S^1 \times T_\Lambda$, we cannot directly combine eigenvectors on

both spaces to get eigenvectors on T^3. But if we consider Dirac operators on T_Λ which depend on a basis over S^1 in a sensible way we can reduce the problem from T^3 to T_Λ. On T_Λ we choose an appropriate complex structure to solve the problem with complex geometry (using [Almorox06]).

Qualitatively we get the following statement (if we identify one-forms and two-forms by the Hodge star operator and interpret them as vectors in $\mathbb{R}^3$): Moving the parameter one-form α of $\mathscr{D}_\alpha$ in the direction of $\hat{a}$ changes the spectrum, but not the eigenbasis, while moving it orthogonally to $\hat{a}$ changes the eigenbasis but not the spectrum. Detailed results can be found in theorem 2.e(ii).

2. *The direct method*: For $\hat{a} = 0$ above method does not work. But here we can derive the eigenbasis from the one of the Spin case.

In the first case we can split the one-form α (on which $\mathscr{D}$ depends) into the parts parallel and orthogonal to T_Λ: $\alpha = \alpha_{\shortparallel} + \alpha_\perp$. Changing $\alpha_{\shortparallel}$ only changes the eigenbasis but fixes the spectrum while for $\alpha_\perp$ this is the other way round.

In the second case we see that a change of α always changes the spectrum.

Looking at spectral sections in the first case we get the result: Either the spectrum is constant over $\mathscr{L}/\ell$ or spectral sections do not exist. The argument uses spectral flows in special directions (see theorem 2.g(ii)).

In the second case there are "more" spectral sections; the ones very near to the spectral projection can be classified by using homotopy groups(see 2.h(iv)).

0.c The space $S^1 \times S^2$

The product space $S^1 \times S^2$ can be analysed along the lines of chapter 2. For finding appropriate eigenbases over S^2 we consider it as the complex projective space $\mathbb{P}^1$ and again use [Almorox06]. From there we easily get to eigenspaces over $S^1 \times S^2$; the question of spectral sections also is rather trivial.

0.d The space S^3

On S^3 the parametrisation plays no role at all. The calculation of the spectrum by means of representation theory of SU(2) goes back to [Hitchin74]. We discuss it in much more detail using the language of quaternions. For that, we interpret the Dirac operator as given by representations of $\mathbf{sp}(1)$ and Clifford multiplication and exploit the representation theory over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

Hitchin calculated the eigenbasis only on abstract representation spaces but his arguments do not give rise to calculating actual sections over S^3. We will do so by using the eigenspaces under special operations of the complexified Lie algebra and also some elementary combinatorics in the context of homogeneous polynomials (see 4.g(iv)).

0.e Acknowledgements

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Chapter 1

Spin$^{\mathbb{C}}$ Dirac operators on 3-manifolds

We want to describe the necessary definitions and prerequisites for studying Dirac operators on 3-manifolds. After that we state some natural questions; those will be answered for the 3-torus, the product $S^1 \times S^2$ and the 3-sphere in the following chapters.

1.a Spin$^{\mathbb{C}}$ structures on 2- and 3-manifolds

On connected compact oriented Riemannian 2- and 3-manifolds (denoted S for "surface" and M for "manifold" throughout this chapter) you can easily describe the Clifford bundle and similar bundles in terms of quaternions. In this section we want to give the relevant formulas and also list all the conventions we will then use throughout this thesis. All facts mentioned here without further reference can be found in [Morgan96].

1.a.1 The Clifford algebra

The Clifford algebra Cl_2 is generated by a 2-dimensional vector space: For reasons which are explained later, we call its standard basis e_2 and e_3. Their product $e_2 e_3$ will be denoted by e_1; the part of degree zero has one generator which we name e_0. In this way we identify Cl_2 with the quaternions $\mathbb{H}$ given by $\operatorname{span}\{e_0, e_1, e_2, e_3\}$. If we explicitly want to distinguish e_i from a (local) vector field which is e_i at every point, we print the latter one in bold face (only done in chapter 1 and 4).

The complexified Clifford algebra $\mathrm{Cl}_2 \otimes \mathbb{C}$, which is isomorphic to the algebra of 2×2 complex matrices, has a unique irreducible representation. If we also identify the representation space $\mathbb{C}^2$ with $\mathbb{H}$, we can write this representation as

$$\begin{array}{rcl} \left(\mathrm{Cl}_2 \otimes \mathbb{C}\right) \times \mathbb{H} & \stackrel{c}{\longrightarrow} & \mathbb{H} \\ (e_i \otimes z, e_j) & \mapsto & c_{e_i \otimes z} e_j = z \cdot e_j(-e_i) \quad \forall z \in \mathbb{C}, e_i, e_j \in \mathbb{H}, \end{array}$$

where the complex structure on $\mathbb{H}$ is defined by $\mathtt{i} \cdot e_0 = e_1$ and $\mathtt{i} \cdot e_2 = e_3$. Note that:

- Multiplication by $\mathtt{i}$ and the action of c_{e_i} commute.
- $\mathbb{H}$ splits into the two complex subspaces $\mathrm{span}\{e_0, e_1\}$ and $\mathrm{span}\{e_2, e_3\}$. The map c_{e_1} acts as $-\mathtt{i}$ on the first and as $\mathtt{i}$ on the second.

The algebra $\mathrm{Cl}_3 \otimes \mathbb{C}$ can be seen as the sum of two copies of $\mathrm{Cl}_2 \otimes \mathbb{C}$. If we consider $\mathrm{span}\{e_1, e_2, e_3\}$ as generating 3-dimensional space for Cl_3, then we can define the following isomorphism of algebras:

$$\begin{aligned} \mathrm{Cl}_3 \otimes_{\mathbb{R}} \mathbb{C} &\cong \left(\mathbb{C}^+ \oplus \mathbb{C}^-\right) \otimes_{\mathbb{C}} \left(\mathrm{Cl}_2 \otimes_{\mathbb{R}} \mathbb{C}\right) \\ e_1 &\mapsto (1,-1) \otimes e_1 \\ e_2 &\mapsto (1,1) \otimes e_2 \\ e_3 &\mapsto (1,1) \otimes e_3 \end{aligned}$$

The two irreducible representations of $\mathrm{Cl}_3 \otimes \mathbb{C}$ on $\mathbb{H}$ can be produced by projecting onto the plus or minus component of the above splitting and then using the representation of $\mathrm{Cl}_2 \otimes \mathbb{C}$. For the definition of the Dirac operator in the next section, we have to make a choice here: From now on we only consider the *plus representation.* It will also be denoted by c_{e_i}, since it is the same map as the in the former definition of c_{e_i} (if you use the isomorphism above to identify e_i with $1 \otimes e_i$).

1.a.2 Spin and Spin$^{\mathbb{C}}$ structures

All (compact, oriented) manifolds of dimension less or equal to 3 admit Spin structures. Each Spin structure induces a Spin$^{\mathbb{C}}$ structure, and all Spin$^{\mathbb{C}}$ structures which are created this way are equivalent (see [Nicolaescu00], p.45).

Let P_S and P_M denote the SO principal bundle associated to the tangent bundles of S and M respectively. Then $\widetilde{P}_S$ and $\widetilde{P}_M$ should be the Spin$^{\mathbb{C}}$ principal bundle we get by above construction. Using the representations described in the preceding section, we define the bundles

$$\mathscr{S}_{\mathbb{C}}\left(\widetilde{P}_S\right) = \widetilde{P}_S \times_{\mathrm{Spin}^{\mathbb{C}}(2)} \mathbb{H} \qquad \mathscr{S}_{\mathbb{C}}\left(\widetilde{P}_M\right) = \widetilde{P}_M \times_{\mathrm{Spin}^{\mathbb{C}}(3)} \mathbb{H}.$$

We take these two bundles as "zero point" in the space of all Spin$^{\mathbb{C}}$ structures. This means the following: Generally, the space of Spin$^{\mathbb{C}}$ structures is an affine space over the second integral cohomology group. If we identify the bundle $\widetilde{P}_M$ with the zero element of this group, we get an isomorphism. We then define the associated bundle to the Spin$^{\mathbb{C}}$ structure coming from an element $\hat{a} \in H^2(M;\mathbb{Z})$ as $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M) \otimes K$ where K is a chosen line bundle with $c_1(K) = \hat{a}$. For S instead of M we just do the same.

Over S and M we also have Clifford algebra bundles

$$\mathrm{Cl}(P_S) = \widetilde{P}_S \times_{\mathrm{Spin}^{\mathbb{C}}(2)} \mathrm{Cl}(2) \qquad \mathrm{Cl}(P_M) = \widetilde{P}_M \times_{\mathrm{Spin}^{\mathbb{C}}(3)} \mathrm{Cl}(3).$$

They act on $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_S) \otimes L$ and $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M) \otimes K$ (by the representations defined above), but only non-trivially on the $\mathbb{H}$-component of $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_S)$ and $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M)$. If we have a (local) basis of orthonormal sections $\{\mathbf{e_i}\}$ of the tangent bundle, then we can write $c_{\mathbf{e_i}}$ and use it in the same way as in the algebra case. Most of the calculations are done in trivialisations.

The bundles $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_S)$ and $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M)$ have natural connections $\tilde{\nabla}_S$ and $\tilde{\nabla}_M$ which come from the Levi-Civita connection of S and M: this is true because we could have equally well defined these bundles by using Spin instead of Spin$^{\mathbb{C}}$ representations since $\widetilde{P}_S$ comes from extending the structure group of a Spin bundle.

In the following chapters we will define for each class $\hat{a} \in H^2(M;\mathbb{Z})$ a line bundle K with a (unitary) background connection ∇^K. Combined with $\tilde{\nabla}_M$ we can define a connection $\widetilde{\nabla}^K$ on $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M) \otimes K$.

We will get a space of unitary connections $\nabla^{\alpha^c} = \widetilde{\nabla}^K + \mathtt{i}\alpha^c$ if we alter ∇^K by a closed one-form α^c. Our aim will be to understand Dirac operators depending on the parameters $\hat{a} = c_1(K)$ and α^c.

1.b The Dirac operator and the torus $\mathscr{L}/\ell$

The Dirac operator is a differential operator

$$\mathscr{D}^K_{\alpha^c} : \Gamma\big(\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M) \otimes K\big) \to \Gamma\big(\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M) \otimes K\big),$$

which is locally defined as

$$\mathscr{D}^K_{\alpha^c}(\sigma) = c_{\mathbf{e_1}} \nabla^{\alpha^c}_{\mathbf{e_1}} \sigma + c_{\mathbf{e_2}} \nabla^{\alpha^c}_{\mathbf{e_2}} \sigma + c_{\mathbf{e_3}} \nabla^{\alpha^c}_{\mathbf{e_3}} \sigma,$$

where $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ is an oriented (local) basis of sections of the tangent bundle. The Γ should indicate $\mathscr{C}^\infty$ sections.

We usually drop the K in the notation and also denote $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M)$ by $\mathscr{S}_{\mathbb{C}}$. In the same manner we can define a Dirac operator on S as

$$\widetilde{\mathscr{D}}^L_{\alpha_L}(\sigma) = c_{\mathbf{e_2}} \nabla^{\alpha_L}_{\mathbf{e_2}} \sigma + c_{\mathbf{e_3}} \nabla^{\alpha_L}_{\mathbf{e_3}} \sigma,$$

where $\mathbf{e_2}, \mathbf{e_3}$ is an oriented (local) basis as above.

In this thesis we are interested in families of Dirac operators constructed in the following way:

We fix a subgroup ℓ of the group $H^1(M;\mathbb{Z})$, which is free because of the universal coefficient theorem. Tensoring with $\mathbb{R}$ gives us a linear subspace $\mathscr{L} = \ell \otimes \mathbb{R}$ of $H^1(M;\mathbb{R})$.

Let $\mathscr{G} = \mathrm{Map}(M, S^1)$ be the gauge group. We get a group isomorphism from $[M, S^1]$ (homotopy classes) to $H^1(M;\mathbb{Z})$ by mapping $u : M \to S^1$ onto $-\mathtt{i}\, udu^{-1}$ (this is standard). Thus ℓ also defines a subgroup $\mathscr{G}_\ell$ of the gauge group (take only the maps which lie in the specified homotopy classes). Furthermore let $\ker(d)_{\mathscr{L}}$ be the space of closed one-forms which represents elements of $\mathscr{L}$. Then $\nabla^K + \mathtt{i}\ker(d)_{\mathscr{L}}$ forms a subspace of the space of connections on K.

The group $\mathscr{G}_\ell$ acts on $\Gamma(\mathscr{S}_{\mathbb{C}} \otimes K)$ by multiplication and on the space of connections by addition of udu^{-1}.

Lemma 1.b(i). *The space $\nabla^K + \mathtt{i}\ker(d)_{\mathscr{L}}$ is invariant under the action of $\mathscr{G}_\ell$*

Proof. We have to prove that $udu^{-1} \in \mathtt{i}\ker(d)_{\mathscr{L}}$ if u represents a homotopy class of maps $[M, S^1]$ given by ℓ. But this is clear from the definition of the isomorphism $[M, S^1] \cong H^1(M;\mathbb{Z})$. □

Knowing this, we can define an infinite dimensional bundle $\mathscr{S}_\ell$ over $(\nabla^k + \mathtt{i}\ker(d)_{\mathscr{L}})/\mathscr{G}_\ell$ as

$$\mathscr{S}_\ell = \Big((\nabla^K + \mathtt{i}\ker(d)_{\mathscr{L}}) \times \Gamma(\mathscr{S}_{\mathbb{C}} \otimes K) \Big) \Big/ _{\mathscr{G}_\ell},$$

using the action of $\mathscr{G}_\ell$ just defined.

Lemma 1.b(ii). *The base space of $\mathscr{S}_\ell$ is canonically isomorphic to $\mathscr{L}/\ell$.*

Proof. We split $\mathtt{i}\ker(d)_{\mathscr{L}}$ into the space of harmonic forms and the space of exact forms. The exact forms are all equivalent to zero under the action of $\mathscr{G}_\ell$. The harmonic forms represent $\mathscr{L}$. If we again look at the isomorphism $[M, S^1] \cong H^1(M;\mathbb{Z})$, the result becomes obvious. □

We now want to look at the Dirac operator as a bundle map of $\mathscr{S}_\ell$. Therefore, we have to check that it is compatible with the equivalence relation we have defined above.

Lemma 1.b(iii). *For every $u \in \mathscr{G}_\ell$, we have*

$$u \cdot \mathscr{D}_{\alpha^c}(\sigma) = \mathscr{D}_{\alpha^c - \mathtt{i}udu^{-1}}(u \cdot \sigma).$$

Proof. Direct calculation, using the derivational properties of the connection. □

For the purpose of above definitions, ℓ can be chosen arbitrarily. But there are choices which are especially nice and interesting:

Assume that M is the connected boundary of a connected 4-manifold X. The embedding $\iota : M \hookrightarrow X$ gives a map ι^* in cohomology. Now define ℓ as $\iota^*\big(H^1(X;\mathbb{Z})\big)$. For those spaces the following condition holds:

Lemma 1.b(iv). *Let K be the restriction of a line bundle over X to M. Then, for ℓ chosen as above, we have*

$$c_1(K) \cup \ell = 0. \tag{1.b-1}$$

Proof. Let $c_1(K) = \beta \in H^2(M;\mathbb{Z})$. Then we have to show

$$\iota^*\big(\beta \cup H^1(M;\mathbb{Z})\big) = 0.$$

We look at the following diagram (see also [Lück05], p.150):

$$\begin{array}{ccccc} H^3(X;\mathbb{Z}) & \xrightarrow{\iota^*} & H^3(M;\mathbb{Z}) & & \\ \Big\downarrow{\scriptstyle PD} & & \Big\downarrow{\scriptstyle PD} & & \\ H_1(X,M;\mathbb{Z}) & \xrightarrow{\delta} & H_0(M;\mathbb{Z}) & \xrightarrow{\cong} & H_0(X;\mathbb{Z}) \end{array}$$

Here PD stands for Poincaré duality. Since the map on the very right is an isomorphism, we know that δ is zero and thus also ι^* has to vanish. $\square$

We will not restrict ourselfs to spaces ℓ fulfilling (1.b-1), but this condition will play a prominent role in the construction of spectral sections.

Lemma 1.b(iii) has the following immediate consequence (for $u = \exp(2\pi \mathtt{i} f)$):

Corollary 1.b(v). *The spectra of $\mathscr{D}_{\alpha_1^c}$ and $\mathscr{D}_{\alpha_2^c}$ are equal if $\alpha_1^c - \alpha_2^c$ equals $2\pi df$ for some function f. The eigenbases differ from each other just by multiplication by* $\exp(2\pi\mathtt{i} f)$.

It is therefore enough to consider harmonic forms α instead of closed forms α^c.

1.c Spectral sections

We have now defined a family $\mathscr{D}$ of first order differential operators over the torus $\mathscr{L}/\ell$. To study this family as a whole the concept of a *spectral section* of Melrose and Piazza (see [Melrose97]) seems to be fruitful.

We need the following data:

- A fibre bundle $\psi : M' \to B$ with fibre (diffeomorphic to) M and an arbitrary base space B.

- A vector bundle E^0 over M'.
- A family $D \in \mathrm{Diff}^1_\psi(M'; E^0)$ of self-adjoint elliptic operators of order 1 on the fibres of ψ.

Definition 1.c(i). A *spectral section* of a family D defined as above is a family of self-adjoint projections $P \in \Psi^0_\psi(M'; E^0)$, such that there exists a smooth function $R : B \to \mathbb{R}$ (depending on P) with

$$D_x u = \nu u \Rightarrow \begin{cases} P_x u = u & \text{for } \nu > R(z) \\ P_x u = 0 & \text{for } \nu < -R(z) \end{cases}$$

for all $x \in B$, $\nu \in \mathbb{R}$ and $u \in \Gamma\big(E^0|_{\psi^{-1}(x)}\big)$.

$P \in \Psi^0_\psi(M'; E^0)$ means that P_x is a pseudodifferential operator of order 0 for all $x \in B$, continuously depending on $x \in B$ (compare [Booss-Bavnbek05], p.155).

Remark 1.c(ii). For compact B the following is true: If P is a spectral section for a function R, then P is also a spectral section for $R_{\max} := \max_x R(x)$. Therefore, it suffices to consider constant maps R.

Another (equivalent) definition is the following one (see [Nazajkinskij06, p.312])

Definition 1.c(iii). A *spectral section* of a family D is a family P of self-adjoint projections (in $P \in \Psi^0_\psi(M'; E^0)$), so that every P_x is equal to the spectral projection up to a compact operator. Differently stated (for $\Pi^+(D_x)$ the projection onto the positive eigenspaces):

$$P_x - \Pi^+(D_x) \quad \text{is a compact operator for every } x \in B.$$

A family D has a spectral section if and only if a special index in $K^1(B)$ vanishes. It is defined as follows: On p.308 of [Atiyah69] the authors show that the space of self-adjoint Fredholm operators of a separable Hilbert space is a classifying space for the functor K^1. If we associate to D the bounded family $Q = (D^2+1)^{-1/2} D$ and trivialise the Hilbert bundle of L^2-sections, we get an element in $K^1(B)$.

The index theorem gives us the opportunity to find abstract existence proofs for spectral sections as in the following case:

Theorem 1.c(iv). *For boundary families spectral sections always exist.*

[Melrose97] defines the term *boundary family* on p.101-103 by means of his b-calculus.

We will not repeat these three pages here but first describe the general ideas of b-calculus and then give an appropriate definition of what is meant by a boundary family in our situation.

In b-calculus we look at a manifold X with boundary M and Riemmanian metric g. For this thesis we furthermore assume that X is a connected compact 4-manifold with connected compact boundary M and that g has a fixed product structure near the boundary.

We are interested in all sections $\mathcal{V}_b$ of TX which are tangent to M at the boundary. The b-tangent bundle ${}^bTX$ is defined by $\mathcal{V}_b/_{\mathcal{I}_p\cdot\mathcal{V}_b}$ (with $\mathcal{I}_p$ the functions vanishing at p); it agrees with TX on the interior of X and its sections are exactly given by $\mathcal{V}_b$.

With this philosophy you can redevelop differential topology and geometry in the b-world (see [Melrose93]) and also define the Clifford bundle of ${}^bTX$ and b-connections on vector bundles E over X. So you can also define b-Dirac operators.

The picture probably becomes clearer when we look at the collar: We choose coordinates y_1, y_2, y_3 on M and use t for the orthogonal direction where $t = 0$ corresponds to the boundary.

The standard basis of ${}^bTX$ is then given by $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, t\frac{\partial}{\partial t}$, whereas in the cotangent bundle we have $dy_1, dy_2, dy_3, \frac{dt}{t}$. [Melrose97] only uses b-connections which come from ordinary connections; so the concept of a b-connections coincides with the concept of a usual connection restricted to elements of TX tangent to the boundary. This is equivalent to saying that

$$ {}^b\nabla^E_{t\frac{\partial}{\partial t}} \equiv 0 \quad \text{on } M. $$

Therefore, a b-Dirac operator defined in the basis above induces a Dirac operator on M.

A *boundary family* now is to be defined by a smooth fibration $\phi : X' \to B$ with fibres diffeomorphic to X, and furthermore a vector bundle E over X' with b-connection ${}^b\nabla^E$. The fibrewise Dirac operators restrict to operators on $\psi : M' \to B$ in the way described above. ψ is now called a boundary family.

There are two direct tools to detect whether there is a spectral section or not.

Lemma 1.c(v). *If there is a* spectral gap *(a continuous map $\tau : B \to \mathbb{R}$ so that $\tau(x)$ is not an eigenvalue of D_x for any x), then the family P defined by projecting on all eigenspaces for which the eigenvalue $\nu(x)$ is greater than $\tau(x)$ is a spectral section.*

Proof. See [Melrose97], p.106 at the bottom. □

Lemma 1.c(vi). *Assume there is an embedding $S^1 \subset B$ for which the subfamily D^{S^1} parametrised by S^1 has a non-vanishing spectral flow. Then there is no spectral section for D.*

Proof. From page 109 of [Melrose97] we know that the value of the analytical index of D^{S^1} equals its spectral flow. The non-vanishing of the spectral flow implies the non-existence of a spectral section for D^{S^1}.

If there were a spectral section for D, then its restriction to S^1 would be a spectral section of D^{S^1}, in contradiction to the last paragraph. This proves the assertion. □

The image of two spectral sections P and P' only differs from each other in finitely many dimensions. Therefore, $\mathtt{Im}\,P - \mathtt{Im}\,P'$ defines an element in the topological K-theory of B. We later want to determine *infinitesimal spectral sections*, which should be those which exist for $R \ll 1$. Formally, we define:

Definition 1.c(vii). Let R_{inf} be defined as the infimum of the set

$$\{R > 0 \,|\, \text{for } R \text{ exists at least one spectral section}\}.$$

Furthermore, choose a (small) positive number ε_P. Then a *system of infinitesimal spectral sections* is a map

$$\begin{aligned} \left]R_{\text{inf}}, R_{\text{inf}} + \varepsilon_P\right] \times I &\to \left\{\text{spectral sections for a fixed operator } D\right\} \\ (R, i) &\mapsto P_R^i, \end{aligned}$$

where

- I is an arbitrary index set,
- P_R^i is a spectral section for the constant map R,
- every $\left(P_R^i\right)_\alpha$, $\alpha \in B$, depends continuously on R (where we consider $\left(P_R^i\right)_\alpha$ as operator between L^2 spaces), and
- $\cup_{i\in I}\{P_R^i\}$ is a representation system for all spectral sections for R, i.e. for all P_R there is a P_R^i, so that $\mathtt{Im}\,P_R - \mathtt{Im}\,P_R^i$ is zero in K-theory.

1.c.1 The framework for $B = \mathscr{L}/\ell$

In this subsection we will interpret the concept of a spectral section in the context of $\mathscr{L}/\ell$ and $\mathscr{D}$. We will also come back to the term *boundary family*.

From now on we choose B to be $\mathscr{L}/\ell$ and M' to be the trivial bundle $M \times B$. ψ is therefore the obvious projection. We define E^0 in the following way: Consider the pull-back bundle $\mathrm{pr}_M^*\left(\mathscr{S}_{\mathbb{C}} \otimes K\right)$ defined through the map

$$\mathrm{pr}_M : M \times \left(\nabla^K + \mathtt{i}\ker(d)_{\mathscr{L}}\right) \to M.$$

If v is an element of the fibre over

$$(y, \nabla^K + \mathtt{i}\alpha^c) \in M \times \left(\nabla^K + \mathtt{i}\ker(d)_{\mathscr{L}}\right),$$

we can define the following action of $\mathscr{G}_\ell$:

$$\begin{aligned} \mathscr{G}_\ell \times \mathrm{pr}_M^*\left(\mathscr{S}_{\mathbb{C}} \otimes K\right) &\to \mathrm{pr}_M^*\left(\mathscr{S}_{\mathbb{C}} \otimes K\right) \\ \left(u, \left(v, y, \nabla^K + \mathrm{i}\alpha^c\right)\right) &\mapsto \left(u(y)\cdot v, y, \nabla^K + \mathrm{i}\alpha^c + u\,du^{-1}\right), \end{aligned} \tag{1.c-1}$$

which is similar to the action we used for the definition of $\mathscr{S}_\ell$. We now divide out by this action and call the resulting bundle E^0, which is obviously a bundle over $M \times \mathscr{L}/\ell$.

Its connection is given by the third term in the construction above.

Our next goal is to show that $\mathscr{D}$ gives an operator in $\mathrm{Diff}^1_\psi\left(M \times \mathscr{L}/\ell; E^0\right)$ in a natural way.

We fix an element $[\alpha] \in \mathscr{L}/\ell$. Since pr_M is just the projection we know that $E^0|_{M\times[\alpha]}$ is equal to $\mathscr{S}_{\mathbb{C}} \otimes K$. So we can use $\mathscr{D}_\alpha$ to define a differential operator on the sections $\Gamma\left(E^0|_{M\times[\alpha]}\right)$. We only have to check that this operator is independent of the choice of the one-form in $[\alpha]$. But if we add $u\,du^{-1}$ to α, the section changes by multiplication with u; looking at lemma 1.b(iii) we know that this is exactly what we want.

As a result we can consider $\mathscr{D}$ as an element of $\mathrm{Diff}^1_\psi\left(M \times \mathscr{L}/\ell; E^0\right)$. This allows us to search for spectral sections of $\mathscr{D}$. Their existence in the case $\ell = \iota^*\left(H^1(X;\mathbb{Z})\right)$ and $c_1(K) \in \iota^*\left(H^2(X;\mathbb{Z})\right)$ is guaranteed by the following result:

Lemma 1.c(viii). *The family $\mathscr{D}$ over $M \times \mathscr{L}/\ell$ defined above is a boundary family.*

Proof. We have a short exact sequence

$$0 \longrightarrow \ker \iota^* \longrightarrow H^1(X;\mathbb{Z}) \xrightarrow{\iota^*} \ell \longrightarrow 0$$

which splits since ℓ is free. Choose an arbitrary splitting morphism $\tau : \ell \to H^1(X;\mathbb{Z})$ and define $\ell_\tau := \tau(\ell) \subset H^1(X;\mathbb{Z})$.

Furthermore, let $\mathscr{L}_\tau := \ell_\tau \otimes \mathbb{R}$. Then ι^* induces an isomorphism $\mathscr{L}_\tau/\ell_\tau \cong B$.

If we choose an arbitrary Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on X with associated bundles S^+ and S^-, we get an associated bundle S for a Spin$^{\mathbb{C}}$ structure $\mathfrak{t}$ on M as $S := S^+|_M \cong S^-|_M$. Spin$^{\mathbb{C}}$ structures on M are canonically classified by $c_1(S) \in 2H^2(M;\mathbb{Z}) \subset H^2(M;\mathbb{Z})$. Since S is defined as a restriction, we furthermore must have $c_1(S) \in \iota^*\left(H^2(X;\mathbb{Z})\right)$. We choose an arbitrary element

$$y \in \left(\iota^*\right)^{-1}\left(-\frac{c_1(S)}{2}\right)$$

and take a line bundle F_y over X with $c_1(F_y) = y$.

If we now consider the Spin$^{\mathbb{C}}$ structure given by $\mathscr{S}^+_{\mathbb{C}} := S^+ \otimes F_y$, $\mathscr{S}^-_{\mathbb{C}} := S^- \otimes F_y$ on X, we know that $\mathscr{S}_{\mathbb{C}} := \mathscr{S}^+_{\mathbb{C}}|_M$ has Chern class zero and is therefore isomorphic to the "trivial" Spin$^{\mathbb{C}}$ structure coming from the Spin structure.

Now for every line bundle K with $c_1(K) \in \iota^*\big(H^2(X;\mathbb{Z})\big)$ choose a line bundle F over X with $\iota^*(F) = K$ and thus

$$\iota^*\big(\mathscr{S}_{\mathbb{C}}^+ \otimes F\big) = \mathscr{S}_{\mathbb{C}} \otimes K.$$

Take also a background connection ∇^F on F, of product form near the boundary, which coincides with ∇^K on the boundary (done by patching together one-forms by a partition of unity).

With these data we want to define a bundle E^1 over $X \times B$ in analogy to the definition of E^0. Take

$$\mathrm{pr}_X : X \times \big(\nabla^F + \mathrm{i}\ker(d)_{\mathscr{L}_\tau}\big) \to X$$

and consider $\mathrm{pr}_X^*\big(\mathscr{S}_{\mathbb{C}}^+ \otimes F\big)$. Let v be an element of the fibre over $(x, \nabla^F + \mathrm{i}\beta) \in X \times \big(\nabla^F + \mathrm{i}\ker(d)_{\mathscr{L}_\tau}\big)$. Define $\mathscr{G}_{\ell_\tau}$ as subgroup of $\mathscr{G}_X = \mathrm{Map}(X, S^1)$ and use the action

$$\begin{aligned}\mathscr{G}_{\ell_\tau} \times \mathrm{pr}_X^*\big(\mathscr{S}_{\mathbb{C}}^\pm \otimes F\big) &\to \mathrm{pr}_X^*\big(\mathscr{S}_{\mathbb{C}}^\pm \otimes K\big)\\ \Big(u, \big(v, x, \nabla^F + \mathrm{i}\beta\big)\Big) &\mapsto \big(u(x)\cdot v, x, \nabla^F + \mathrm{i}\beta + u\,du^{-1}\big).\end{aligned}$$

We now divide by the action and identify $\mathscr{L}_\tau/\ell_\tau$ with B to get a bundle E^1 over $X \times B$.

Now back to the b:

∇^F induces a b-connection ${}^b\nabla^F$ which restricts to the connection ∇^K on M. By the definition of b-connections, we also know that $\nabla^F + \mathrm{i}\beta$ restricts to $\nabla^K + \mathrm{i}\cdot\iota^*(\beta)$.

With the usual action of the Clifford bundle on $\mathscr{S}_{\mathbb{C}}^+$ we get a family $\mathscr{D}^X$ of Dirac operators parametrised by $\mathscr{L}_\tau/\ell_\tau$. Since $\iota^* : \mathscr{L}_\tau/\ell_\tau \to \mathscr{L}/\ell$ is an isomorphism, we see that $\mathscr{D}$ is just the boundary family for $\mathscr{D}^X$.

□

1.d Questions

The following questions will be answered in the next three chapters for M equal to T^3, $S^1 \times S^2$ and S^3. For this purpose we will choose the flat metric on T^3 and S^1 and the round metric on S^2 and S^3.

1. Given a $\mathrm{Spin}^{\mathbb{C}}$ structure $\hat{a} = c_1(K)$ and a closed one-form α^c: What is the spectrum of $\mathscr{D}^K_{\alpha^c}$?
2. Under the same conditions: How can we explicitly calculate an orthogonal eigenbasis for $\mathscr{D}^K_{\alpha^c}$?
3. For which ℓ do spectral sections of $\mathscr{D}$ exist?
4. If spectral sections exist: How can we explicitly construct one of them?
5. What does the set of *infinitesimal* spectral sections look like? What is its image in $K(\mathscr{L}/\ell)$?

Chapter 2

The space T^3

The case T^3 is the most interesting one because all five questions of 1.d have non-trivial answers. We will shortly introduce the following sections:

In **section 2.a** the specialities of the torus are stated and named. We look at T^3 as a quotient space; this point of view will be heavily used. From here on we will only look at Spin$^{\mathbb{C}}$ structures not equal to zero (The missing case will be postponed to section 2.f). **Section 2.b** uses the Spin$^{\mathbb{C}}$ structure to define a 2-torus T_Λ and tells us how to interpret T^3 as a S^1-bundle over T_Λ. **Section 2.c** solves question 1 and 2 on T_Λ. The one-form α_L can be translated into a holomorphic structure on a line bundle L. After moving the problem to the world of complex geometry, we use methods of [Almorox06] to find the relevant eigenspaces.

A family of eigenbases for 1- and 2-dimensional Dirac operators can be combined in a non-trivial way to define an eigenbasis for $\mathscr{D}^2$. This is done in **section 2.d**. Due to our explicit knowledge of eigenbases we can use this result to calculate an eigenbasis for $\mathscr{D}$ in **section 2.e**.

In **section 2.f** we look at the trivial Spin$^{\mathbb{C}}$ structure. Since the methods mentioned above do not apply here, we solve this case by direct calculation. The result is completely different from the result of 2.e as the spectrum changes in every direction in $\mathscr{L}$.

In **section 2.g** and **section 2.h** we can bring in the harvest. We distinguish the two cases of non-trivial and trivial Spin$^{\mathbb{C}}$ structures. In the first case the result is particularly nice: We can show that the spectrum is constant over $\mathscr{L}/\ell$. The latter case can be examined using homotopy groups. They are the tool to determine the set of *infinitesimal* spectral sections. Their image in K-theory will also be calculated.

In **section 2.i** we discuss possible extensions of the method: If we replace T_Λ by a surface of higher genus, what can be said about the existence of spectral sections?

2.a Definitions

We define $M = T^3$ as the flat space $\mathbb{R}^3$ divided by the lattice $\mathbb{Z}^3$ equipped with the usual orientation(Any other lattice would do the job equally well, but this one reduces the amount of necessary notation). The coordinates are called x_1, x_2, x_3 as usual; the differentials dx_1, dx_2, dx_3 form a basis of the cohomology ring $H^*(T^3;\mathbb{R})$. Taking the bases dx_1, dx_2, dx_3 for the first cohomology and $dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2$ for the second cohomology we have identifications $H^1(T^3;\mathbb{R}) \cong \mathbb{R}^3$ and $H^2(T^3;\mathbb{R}) \cong \mathbb{R}^3$. The same is true if we replace $\mathbb{R}$ by $\mathbb{Z}$. The integral cohomology is always considered as a discrete subset of the real one. We also identify vectors and their duals in the obvious way (defined by the flat metric).

As explained in the first chapter, we have a natural identification of the set of $\mathrm{Spin}^{\mathbb{C}}$ structures with the second cohomology. Using the identifications just made, we fix a class $\hat{a} = h \cdot (a_1, a_2, a_3)$. If $\hat{a}$ is not zero, we choose h to be a positive integer so that $\gcd(a_1, a_2, a_3) = 1$, otherwise we choose h to be zero. The vector (a_1, a_2, a_3) will often be abbreviated by a.

We will see in the following discussion that there is a fundamental difference between the cases $\hat{a} \neq 0$ and $\hat{a} = 0$. Sections 2.b to 2.e look at the first case while 2.f analyses the second one.

2.b Dimensional reduction

Let W be the 2-dimensional vector space $(a_1, a_2, a_3)^\perp$ in $\mathbb{R}^3$ and $\pi_a : \mathbb{R}^3 \to W$ the orthogonal projection. Furthermore let $\Lambda := \pi_a(\mathbb{Z}^3)$ be the image of the standard lattice. If we divide by $\mathbb{Z}^3$, we get a map

$$\pi_{\overline{a}} : T^3 \to W/_\Lambda =: T_\Lambda.$$

If we identify the tangent spaces of T^3 and T_Λ with $\mathbb{R}^3$ and W respectively, we see that $(\pi_{\overline{a}})_*$ equals π_a.

The following easy lemma will be useful for our purposes:

Lemma 2.b(i). *We can choose an oriented basis $w_1, w_2 \in W$ so that $\Lambda = \mathbb{Z}\{w_1, w_2\}$ and furthermore w_1 and w_2 form a fundamental parallelogram of T_Λ.*

Proof. Since $\mathbb{Z}^3 \subset \mathbb{R}^3$ is discrete and the projection has a rational direction, we know that every finite area of W can only be hit by finitely many lattice points of $\mathbb{Z}^3$. This implies that Λ is discrete. It also spans W since $\mathbb{Z}^3$ spans $\mathbb{R}^3$ and π_a is a projection. Choose an enumeration

$$w_{\mathrm{I}}, w_{\mathrm{II}}, w_{\mathrm{III}}, \ldots$$

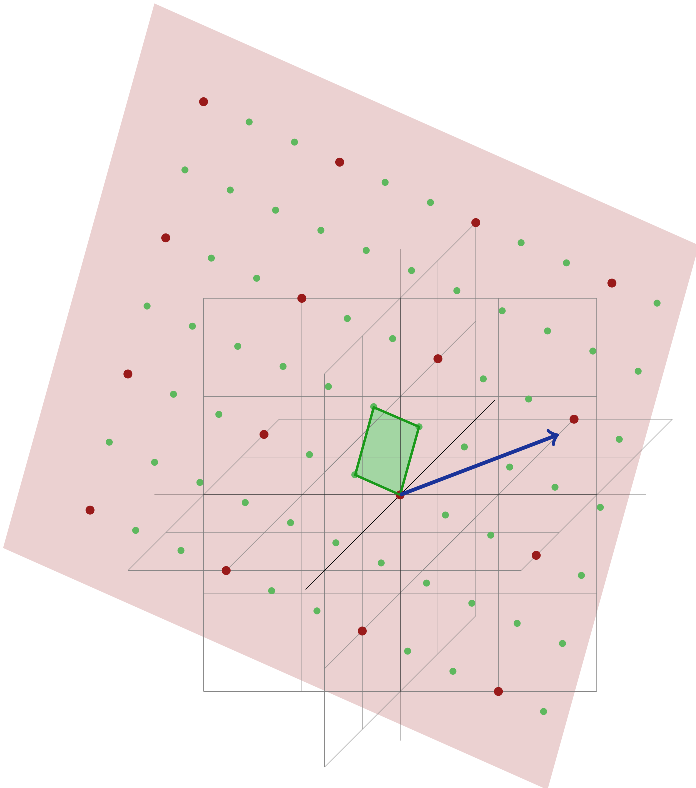


Figure 2.1: Projection in direction of $a = (2,1,1)$ (blue arrow): W is indicated by the light red plane, in which the red dots are lattice points in W and the green dots are the other points of Λ. The light green area indicates a fundamental parallelogram.

of Λ with $|w_{\mathrm{I}}| \leq |w_{\mathrm{II}}| \leq |w_{\mathrm{III}}| \leq \ldots$. Now let $w_1 = w_{\mathrm{I}}$ and let w_2 be the first vector which is linearly independent of w_1. If w_1, w_2 is not oriented, we replace w_2 by $-w_2$. We now want to show that $\Lambda \subset \mathbb{Z}\{w_1, w_2\}$:

Assume that $w \in \Lambda$ is not in $\mathbb{Z}\{w_1, w_2\}$. Then w lies in some parallelogram of the tiling induced by $\mathbb{Z}\{w_1, w_2\}$. The distance of the point w to at least one of the four corners is necessarily smaller than $|w_2|$. Now we subtract this corner vector from w and get a vector of shorter length than $|w_2|$ which contradicts our assumption. This proves the lemma. □

To avoid to much messing around with exp we introduce the following notation: Let S^1 be the unit circle in $\mathbb{C}$ and $S^{[1]}$ be the space $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$.

Lemma 2.b(ii). *$\pi_{\overline{a}} : T^3 \to T_\Lambda$ is a trivial $S^{[1]}$ bundle.*

Proof. We want to define a global trivialisation. Let $c^1, c^2 \in S^{[1]}$ be defined by the condition (which is independent of the choice of representatives)

$$w_1 - c^1 \cdot a \in \mathbb{Z}^3 \qquad\qquad w_2 - c^2 \cdot a \in \mathbb{Z}^3,$$

where uniqueness follows from the $\gcd(a_1, a_2, a_3) = 1$ condition. We define the trivialisation map by using the covering spaces $\mathbb{R}^3$ for T^3, W for T_Λ and $\mathbb{R}$ for $S^{[1]}$. The elements of quotient spaces are written in square brackets. Every element of $\mathbb{R}^3$ can be written as a unique linear combination $\chi_1 w_1 + \chi_2 w_2 + \chi a$. The map is defined as the following group morphism:

$$\begin{array}{ccc} T^3 & \xrightarrow{\pi_{\overline{a}} \times \mathrm{tri}} & T_\Lambda \times S^{[1]} \\ \Big[\chi_1 w_1 + \chi_2 w_2 + \chi a\Big] & \mapsto & \Big(\big[\chi_1 w_1 + \chi_2 w_2\big], \big[c^1 \chi_1 + c^2 \chi_2 + \chi\big]\Big) \end{array} \tag{2.b-1}$$

We have to check some things:

1. $\pi_{\overline{a}} \times \mathrm{tri}$ *is well-defined:* Let $\chi_1 w_1 + \chi_2 w_2 + \chi a \in \mathbb{Z}^3$. Since $\Lambda = \mathbb{Z}\{w_1, w_2\}$, we know that χ_1 and χ_2 are necessarily integers: hence $[\chi_1 w_1 + \chi_2 w_2]$ equals $[0]$. We know that
$$(-\chi_1)\big(w_1 - c^1 \cdot a\big) + (-\chi_2)\big(w_2 - c^2 \cdot a\big) \tag{2.b-2}$$
is forced to be in $\mathbb{Z}^3$ by the choice of c^1 and c^2. Adding $\chi_1 w_1 + \chi_2 w_2 + \chi a \in \mathbb{Z}^3$, we get
$$\big(c^1 \chi_1 + c^2 \chi_2 + \chi\big) \cdot a \in \mathbb{Z}^3.$$
Since a_1, a_2 and a_3 have not common divisor, the coefficient has to be an integer. The shows that the map is well-defined.

2. $\pi_{\overline{a}} \times \mathrm{tri}$ *is injective:* We just invert the argumentation of the preceding step: Assume we have
$$\Big(\big[\chi_1 w_1 + \chi_2 w_2\big], [t]\Big) \quad \chi_1, \chi_2, t \in \mathbb{Z}.$$
Then we subtract the term (2.b-2) from t to see that the inverse image of this point lies in $\mathbb{Z}^3$.

3. $\pi_{\overline{a}} \times \mathrm{tri}$ *is surjective:* $\Big(\big[\chi_1 w_1 + \chi_2 w_2\big], [t]\Big)$ for arbitrary χ_1, χ_2, t is the image of
$$\Big[\chi_1 w_1 + \chi_2 w_2 + \big(t - c^1 \chi_1 - c^2 \chi_2\big) \cdot a\Big].$$

This shows the bundle equivalence. □

We see that $[\chi_1 w_1 + \chi_2 w_2]$ in T^3 is *not* mapped onto the zero section in $T_\Lambda \times S^{[1]}$. This produces some extra terms in our calculations which have to be eliminated by a careful choice of holomorphic structure.

This rather complicated way of writing T^3 as a $S^{[1]}$ bundle over a 2-torus is necessary because we require the fibres to be *orthogonal to the base space*.

2.b.1 Line bundles over T_Λ and T^3

For answering the questions in 1.d it is necessary to define a line bundle K with $c_1(K) = h \cdot a$ and a (unitary) connection on it (all connections on line bundles are assumed to be $U(1)$ connections).

T_Λ has a natural metric given by W. Integration identifies the second real cohomology with $\mathbb{R}$; then $\mathbb{Z} \subset \mathbb{R}$ represents the integer cohomology. In 2.c we will define a line bundle L and a background connection ∇^L for every positive Chern class $h \in \mathbb{Z}$.

Definition 2.b(iii). $K := \pi_{\overline{a}}^*(L)$ and $\nabla^K = \pi_{\overline{a}}^*(\nabla^L)$.

For the following calculations, it will be helpful to have an oriented *orthonormal* basis of sections of the tangent space $\mathrm{T}(T^3) \cong T^3 \times \mathbb{R}^3$ of T^3. We choose:

$$e_1 = \frac{1}{\|a\|} \cdot a \qquad e_2 = \frac{1}{\left\|\begin{pmatrix} a_2 \\ -a_1 \\ 0 \end{pmatrix}\right\|} \begin{pmatrix} a_2 \\ -a_1 \\ 0 \end{pmatrix} \qquad e_3 = \frac{1}{\left\|\begin{pmatrix} a_1 a_3 \\ a_2 a_3 \\ -a_1^2 - a_2^2 \end{pmatrix}\right\|} \begin{pmatrix} a_1 a_3 \\ a_2 a_3 \\ -a_1^2 - a_2^2 \end{pmatrix}. \tag{2.b-3}$$

We use e_i instead of $\mathbf{e_i}$ for these vector fields to keep the notation simple.

Lemma 2.b(iv). *$c_1(K) = h \cdot a$.*

Proof. For the dual basis we have

$$\int_{T_3} e_1^* \wedge e_2^* \wedge e_3^* = 1.$$

Using $\pi_{\overline{a}} \times \mathrm{tri}$, we can rewrite this as an integral over $T_\Lambda \times S^{[1]}$. The map $(\pi_{\overline{a}})_*$ maps e_2 and e_3 to themselves (now viewed as an orthonormal basis of $\mathrm{T}(T_\Lambda)$). The vector e_1 is mapped onto $\|a\| \cdot \frac{\partial}{\partial t}$. Since we know that $\int_{S^{[1]}} \|a\| \frac{\partial}{\partial t} = \|a\|$, we have shown that

$$\mathrm{vol}_{T_\Lambda} = \int_{T_\Lambda} e_2^* \wedge e_3^* = \frac{1}{\|a\|}.$$

This means that the integral cohomology class h is represented by the one-form

$$h \|a\| \, e_2^* \wedge e_3^*.$$

Since Chern classes behave well under pull-back operations, we just have to show that that

$$\pi_{\overline{a}}^*\big(h\|a\|\, e_2^* \wedge e_3^*\big) = h \cdot a \quad \text{(both viewed as vector in } \mathbb{R}^3\text{)}.$$

The spaces $H^1(T^3;\mathbb{R})$ and $H^2(T^3;\mathbb{R})$ are both identified with $\mathbb{R}^3$; the mutual identification of these spaces caused by these two isomorphisms is just induced by the Hodge star operator. But we have

$$e_2^* \wedge e_3^* = * e_1^* = * \frac{1}{\|a\|} \cdot a^*.$$

In our identifications this is exactly what we wanted. □

2.b.2 The Spin$^{\mathbb{C}}$ bundle

Using the definitions of chapter 1, we will describe the special properties of $\mathscr{S}_{\mathbb{C}}$ and $\mathscr{D}$ on the 3-torus.

Since the tangent bundle of T^3 is trivial, the bundle $\mathscr{S}_{\mathbb{C}}$ is the trivial quaternionic bundle $\underline{\mathbb{H}}$. We take e_1, e_2, e_3 as a basis for $\mathrm{T}(T^3)$ (where e_1, e_2, e_3 are defined as in (2.b-3)); these vector also form an algebra basis of $\underline{\mathbb{H}}$.

The connection induced on $\underline{\mathbb{H}}$ is the trivial one. If we split $\underline{\mathbb{H}}$ as $\mathbb{C}\{e_0\} \oplus \mathbb{C}\{e_2\}$, we can identify the connection $\widetilde{\nabla}^K$ of $\underline{\mathbb{H}} \otimes K$ with $\nabla^K \oplus \nabla^K$.

On T_Λ the situation looks pretty much the same. If we trivialise the tangent bundle by e_2, e_3, we can identify the Spin$^{\mathbb{C}}$ bundle with the trivial quaternionic bundle $\underline{\mathbb{H}}^\Lambda$ tensored with L. By the definition of K we know that

$$\pi_{\overline{a}}^*\big(\underline{\mathbb{H}}^\Lambda \otimes L\big) = \underline{\mathbb{H}} \otimes K.$$

Also we have

$$\pi_{\overline{a}}^*\big(\widetilde{\nabla}^L\big) = \pi_{\overline{a}}^*\big(\nabla^L \oplus \nabla^L\big) = \nabla^K \oplus \nabla^K.$$

As explained in chapter 1 we can embed Cl(2) into Cl(3) so that their representations c_{e_i} for $i = 1, 2, 3$ on $\mathbb{H}$ are equal. In our case this embedding is induced by the embedding $\mathrm{T}\big(T_\Lambda\big) \subset \mathrm{T}\big(T^3\big)$ which is equal to $W \subset \mathbb{R}^3$. This of course induces trivial bundles $\underline{\mathrm{Cl}(2)}$ and $\underline{\mathrm{Cl}(3)}$ over T_Λ and T^3 respectively. The action of $\underline{\mathrm{Cl}}(2)$ on $\underline{\mathbb{H}}^\Lambda \otimes L$ and $\underline{\mathrm{Cl}}(3)$ on $\underline{\mathbb{H}} \otimes K$ will also be written as c_{e_i}, where e_i is the section defined in (2.b-3).

It is trivial to see that c_{e_i} commutes with $\pi_{\overline{a}}^*$.

2.c The 2-dimensional torus

In section 2.b we have defined a 2-dimensional torus T_Λ. Two things have to be done:

1. Explicitly define a line bundle L on T_Λ with a background connection ∇^L for every Chern class $h \in \mathbb{Z}$ (that was promised in 2.b.1).
2. Calculate the spectrum and an eigenbasis for $\widetilde{\mathscr{D}}_{\widetilde{\alpha}_L} := \widetilde{\mathscr{D}}^L_{\widetilde{\alpha}_L}$, where $\widetilde{\alpha}_L$ is any harmonic one-form (remember that on a flat torus every harmonic one-form is constant and vice versa).

For that, we choose an appropriate complex structure on T_Λ. After reviewing the prerequisites about complex abelian varieties, we translate the problem into the world of holomorphic line bundles. Here the elliptic chains of [Almorox06] can be used to give an explicit solution in terms of spaces of holomorphic sections. To make these spaces more explicit we furthermore discuss theta functions.

2.c.1 The choice of a complex structure (and scaling)

We consider $\mathbb{R}^3$ as split into $\mathrm{span}\{a\} \oplus W$, and use the induced metric and orientation on W as above. We furthermore again take the oriented basis w_1, w_2 of W. Now define $r_\Lambda := |w_1|$. We get an oriented isometry between W and the complex numbers by sending $r_\Lambda^{-1} w_1$ to 1. The image of $r_\Lambda^{-1} w_2$ under this map shall be called τ. From the orientation we know that $\mathtt{Im}\,\tau > 0$.

The resulting complex plane shall be called $\widetilde{\mathbb{C}}$ and the lattice spanned by $\{1, \tau\}$ will still be called Λ, although we scaled by $\frac{1}{r_\Lambda}$; we will rescale in 2.c.7 and use the fact that a global scaling by a constant factor just changes the spectrum by the same factor.

The choice of a complex structure allows us to look at the space of *holomorphic* line bundles on the torus $T_\Lambda = \widetilde{\mathbb{C}}/_\Lambda$.

Before we investigate this space by means of theta functions, we will have a short look at the cohomological properties of T_Λ.

2.c.2 The cohomology of T_Λ

We identify the tangent space of T_Λ with $\widetilde{\mathbb{C}}$; hence we can choose 1^* and τ^* as a basis for $\mathrm{T}^*(T_\Lambda)$. As linear independent harmonic one-forms they span $H^1(T_\Lambda;\mathbb{R})$ and their product spans $H^2(T_\Lambda;\mathbb{R})$. As we know, integration over T_Λ identifies $H^2(T_\Lambda;\mathbb{R})$ with $\mathbb{R}$ and the preimage of $\mathbb{Z}$ under this map equals $H^2(T_\Lambda;\mathbb{Z})$. Integrating $1^* \wedge \tau^*$ over T_Λ gives:

$$\int_{T_\Lambda} 1^* \wedge \tau^* = \int_{T_\Lambda} \mathtt{Im}\,\tau \mathrm{vol}_{T_\Lambda} = \left(\mathtt{Im}\,\tau\right)^2.$$

So $\left(\mathtt{Im}\,\tau\right)^{-2} 1^* \wedge \tau^*$ is the oriented generator of $H^2(T_\Lambda;\mathbb{Z})$.

2.c.3 Holomorphic line bundles on T_Λ

We want to explain the classification of holomorphic line bundle which is provided by the Appell-Humbert theorem. Our main source for this subsection will be the book by Lange and Birkenhake ([Birkenhake04]).

Factors of automorphy

The group Λ acts on $\widetilde{\mathbb{C}}$, therefore also on the sections $H^0\left(\mathcal{O}^*_{\widetilde{\mathbb{C}}}\right)$ of the sheaf of non-vanishing holomorphic functions. Hence $H^0\left(\mathcal{O}^*_{\widetilde{\mathbb{C}}}\right)$ is a Λ-module and we can consider the group cohomology group $H^1\left(\Lambda, H^0\left(\mathcal{O}^*_{\widetilde{\mathbb{C}}}\right)\right)$. It can be directly defined as follows:

Let $Z^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big)$ be the multiplicative group of functions $f : \Lambda \times \widetilde{\mathbb{C}}^* \to \widetilde{\mathbb{C}}^*$, holomorphic in the second variable, which satisfy the *cocycle condition*

$$f(s+t, v) = f(s, t+v) f(t, v).$$

The elements of $Z^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big)$ are called *factors of automorphy*.

Now let $B^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big) \subset Z^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big)$ be the subgroup of factors of automorphy of the form

$$(s, v) \mapsto h(s+v)h(v)^{-1}, \quad h \in H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big).$$

Then we define

$$H^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big) := \frac{Z^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big)}{B^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big)}$$

(see also p.571, [Birkenhake04]).

A factor of automorphy f defines a line bundle on T_Λ in the following way: Define an action of Λ on the trivial bundle $\widetilde{\mathbb{C}} \times \mathbb{C}$ by

$$s \circ (v, z) = \big(s+v, f(s, v)\, z\big).$$

If we now divide by this action we get a bundle

$$L_f = \big(\widetilde{\mathbb{C}} \times \mathbb{C}\big)/_\Lambda$$

over T_Λ.

Following p.24 of [Birkenhake04], the map $f \mapsto L_f$ is an isomorphism

$$H^1\Big(\Lambda,\, H^0\big(\mathscr{O}^*_{\widetilde{\mathbb{C}}}\big)\Big) \cong \operatorname{Pic}\big(T_\Lambda\big),$$

where Pic is the group of holomorphic line bundles. So every line bundle can be described by a factor of automorphy.

The first Chern class of L_f can be calculated from f in the following way:

Since f maps to $\mathbb{C}^*$ we can define an "logarithm" $g : \Lambda \times \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$, holomorphic in the second variable, with $f = \exp(2\pi \mathtt{i}\, g)$.

The following map identifies the alternating 2-forms with the second cohomology group (exercise 1, p.41, [Birkenhake04]):

$$\begin{aligned} \operatorname{Alt}^2\big(\Lambda, \mathbb{Z}\big) &\to H^2\big(T_\Lambda, \mathbb{Z}\big) \\ E &\mapsto \hat{E} := E(\tau, 1)\, dx \wedge dy. \end{aligned}$$

Then theorem 2.1.2 (p.24) of the same book implies:

$$c_1(L_f) = \hat{E}_{L_f} \tag{2.c-1}$$

where

$$E_{L_f}(s,t) := g(t, v+s) + g(s,v) - g(s, v+t) - g(t,v). \tag{2.c-2}$$

The Appell-Humbert theorem

We use the following ad hoc definition of the Néron-Severi group; the equivalence to the usual definition can be derived from the Appell-Humbert theorem (2.c(ii)).

Definition 2.c(i). The *Néron-Severi group* $\mathrm{NS}(T_\Lambda)$ is defined to be the additive group of alternating forms $E : \widetilde{\mathbb{C}} \times \widetilde{\mathbb{C}} \to \mathbb{R}$ with $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(\mathrm{i}v, \mathrm{i}w) = E(v, w)$.

If we define a hermitian form

$$H(v,w) := E(\mathrm{i}v, w) + \mathrm{i}E(v,w), \tag{2.c-3}$$

then we can also view $\mathrm{NS}(T_\Lambda)$ as the group of hermitian forms with $\mathrm{Im}\, H(\Lambda,\Lambda) \subset \mathbb{Z}$. A *semicharacter* of $H \in \mathrm{NS}(T_\Lambda)$ is a map

$$\chi : \Lambda \to S^1$$

which satisfies

$$\chi(s+t) = \chi(s)\chi(t)\exp\big(\pi\mathrm{i}\ \mathrm{Im}\, H(s,t)\big) \quad \forall s,t \in \Lambda.$$

For $H = 0$ the semicharacters become the characters of the group Λ.

Now let $\mathscr{P}(\Lambda)$ be set of all pairs (H,χ) with the group structure given by

$$(H_1, \chi_1) \cdot (H_2, \chi_2) = (H_1 + H_2, \chi_1\chi_2).$$

On $\mathscr{P}(\Lambda)$ we define $f = f_{(H,\chi)}$ as:

$$f(s,v) := \chi(s)\exp\left(\pi H(v,s) + \frac{\pi}{2} H(s,s)\right).$$

On p.30, [Birkenhake04] shows that $f \in Z^1\left(\Lambda, H^0(\mathscr{O}^*_{\widetilde{\mathbb{C}}})\right)$. Therefore, f defines a line bundle $L(H,\chi)$. By formula (2.c-2) we can show (see bottom of the last page):

$$c_1\big(L(H,\chi)\big) = \hat{E}, \tag{2.c-4}$$

where $\hat{E}$ is the map defined in the preceding subsection. The following theorem ([Birkenhake04]) gives us the desired tool to understand $\mathrm{Pic}\,(T_\Lambda)$:

Theorem 2.c(ii) (Appell-Humbert). *The map*

$$\mathscr{P}(\Lambda) \to \mathrm{Pic}(T_\Lambda)$$
$$(H,\chi) \mapsto L(H,\chi)$$

defined above is an isomorphism of groups.

We see that H (or equivalently, E) gives us the Chern class of the bundle; all line bundles with same H but different χ are isomorphic as *complex bundles*, but represent different holomorphic bundles.

Our next aim is to construct a "basis element" χ_0 for every H.

The characteristic of a line bundle

We will now heavily use the chosen basis $\{\tau,1\}$ of Λ. Since we know that $E \in \mathrm{NS}(T_\Lambda)$ has to be an alternating 2-form with values in $\mathbb{Z}$, E is uniquely determined by knowing the value $E(\tau,1) = h$ with $h \in \mathbb{Z}$. Looking at the calculation of the Chern class of $L(H,\chi)$, the integer h also represents the first Chern class after identifying $H^2(T_\Lambda;\mathbb{Z})$ with $\mathbb{Z}$.

Following [Birkenhake04], p.46, we choose a decomposition of Λ into $\Lambda_1 \oplus \Lambda_2$ with $\Lambda_1 = \mathbb{Z}\cdot\tau$ and $\Lambda_2 = \mathbb{Z}\cdot 1$ (The ordered basis $\tau, 1$ seems unnatural, because it is negatively oriented, but for defining Λ_1 and Λ_2 in the way usually used for theta functions it is necessary). In this decomposition, E is the matrix

$$E = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}.$$

The spaces in $\widetilde{\mathbb{C}}$ which are $\mathbb{R}$-spanned by Λ_1 and Λ_2 will be called $\widetilde{\mathbb{C}}_1$ and $\widetilde{\mathbb{C}}_2$ respectively. Every vector v in $\widetilde{\mathbb{C}}$ therefore splits into a sum $v_1 + v_2$.

For the sake of completeness we calculate the hermitian form H defined by E (see formula (2.c-3)):

$$\begin{aligned} H(v,w) &= E(\mathtt{i}\,v,w) + \mathtt{i}\,E(v,w) \\ &= h\left(\frac{|\tau|^2}{\mathtt{Im}\,\tau}\,v_1w_1 + \frac{\mathtt{Re}\,\tau}{\mathtt{Im}\,\tau}\,(v_1w_2+v_2w_1) + \frac{1}{\mathtt{Im}\,\tau}\,v_2w_2 + \mathtt{i}\big(v_1w_2 - v_2w_1\big)\right). \end{aligned}$$

The decomposition $\widetilde{\mathbb{C}} = \widetilde{\mathbb{C}}_1 \oplus \widetilde{\mathbb{C}}_2$ enables us to define χ_0:

$$\chi_0(v) := \exp\big(\pi\mathtt{i}\,E(v_1,v_2)\big).$$

It follows from p.46 of [Birkenhake04] that this is a semicharacter for H.

Furthermore, let $\xi_c(v) = \exp(2\pi\mathtt{i}\,E(c,v))$ for $c \in \widetilde{\mathbb{C}}$ (called characteristic). Then for every $c \in \widetilde{\mathbb{C}}$ we get a semicharacter

$$\chi_c(v) := \chi_0(v)\,\xi_c(v).$$

Definition 2.c(iii). Let $L^{h,c}$ for $h \in \mathbb{Z}$ and $c \in \widetilde{\mathbb{C}}$ be the line bundle $L(H, \chi_c)$, where H is the hermitian form corresponding to h and χ_c is the semicharacter just defined.

The bundles $L^{h,c}$ and $L^{h,c+s}$ for $s \in \Lambda$ are holomorphically equivalent since ξ_s is just a holomorphic function on T_Λ. Therefore it makes sense to choose c from a fundamental parallelogram of T_Λ.

On p.47, [Birkenhake04] shows by using the Appell-Humbert theorem that this gives us a bijection between $\mathbb{Z} \times T_\Lambda$ and the space of (isomorphism classes of) holomorphic line bundles.

Lemma 2.c(iv).

(i) $L^{h_1,c_1} \otimes L^{h_2,c_2} \cong L^{h_1+h_2,c_1+c_2}$, *for* $h_1, h_2 \in \mathbb{Z}$, $c_1, c_2 \in \widetilde{\mathbb{C}}$.

(ii) $c_1\big(L^{h,c}\big) = h \in \mathbb{Z}$.

Proof.

(i) Multiply the factors of automorphy.

(ii) Follows from (2.c-4).

□

Since our h here will finally be the same h used on T^3, we will restrict our further investigations to the case where h is positive.

Metric, Chern connection and curvature

To define a hermitian metric on $L^{h,c}$ we will write down a hermitian metric on $\widetilde{\mathbb{C}} \times \mathbb{C}$ which is compatible with the action of Λ. This can be done by (compare [Polishchuk03], p.10)

$$\langle z_1, z_2 \rangle_v := \exp\big(-\pi H(v,v)\big)\, z_1 \overline{z}_2.$$

The following observations are trivial but important; they will used implicitly during the rest of the calculation:

The covering space map $\widetilde{\mathbb{C}} \to T_\Lambda$ provides us with a family of charts for T_Λ, in which $[v]$ is identified with any of its representatives $v \in \widetilde{\mathbb{C}}$. For calculations we often choose one of these charts. Please note that the hermitian metric on $\widetilde{\mathbb{C}} \times \mathbb{C}$ is independent of the parameter c.

Before we look at connections, it seems reasonable to recapitulate the splitting of the complexified tangent space:

$$\mathrm{T}\big(T_\Lambda\big)\otimes\mathbb{C} = \Big(\mathrm{T}\big(T_\Lambda\big)\Big)^{10}\oplus\Big(\mathrm{T}\big(T_\Lambda\big)\Big)^{01}.$$

into the *holomorphic* and *anti-holomorphic* part. If we identify the left hand space with $\mathbb{C}\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\}$, they can be realised as $\mathbb{C}\{\frac{\partial}{\partial v}\}$ and $\mathbb{C}\{\frac{\partial}{\partial \overline{v}}\}$ where

$$\frac{\partial}{\partial v}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathtt{i}\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \overline{v}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathtt{i}\frac{\partial}{\partial y}\right).$$

Analogously, we can split the cotangent space, where we have the bases

$$dv = dx+\mathtt{i}dy \qquad d\overline{v}=dx-\mathtt{i}dy.$$

The replacement of the usual z by v comes from the fact that we are studying the trivial bundle $\widetilde{\mathbb{C}}\times\mathbb{C}$ over $\widetilde{\mathbb{C}}$, where elements of $\widetilde{\mathbb{C}}$ were always called v or w, while the notation z was reserved for $\mathbb{C}$. Since the tangent space belongs to $\widetilde{\mathbb{C}}$, it seems reasonable not to confuse both notations at this point but keep the v.

The operator d acting on alternating forms also splits into $\partial+\overline{\partial}$. Especially the operator $\overline{\partial}:\Omega^{00}\big(T_\Lambda\big)\to\Omega^{01}\big(T_\Lambda\big)$ will be of importance to us: Tensoring with a line bundle L gives us an operator $\overline{\partial}:\Omega^{00}(L)\to\Omega^{01}(L)$ of L-valued forms. A priori it is only defined locally, but an easy calculation (p.70, [Griffiths78]) shows that the pieces patch together.

A hermitian holomorphic vector bundle has a connection ∇ uniquely defined by the two conditions (see p.73 of [Griffiths78]):

1. If we split $\nabla:\Omega^0(L)\to\Omega^1(L)$ into ${}_{10}\nabla:\Omega^{00}(L)\to\Omega^{10}(L)$ and ${}_{01}\nabla:\Omega^{00}(L)\to\Omega^{01}$, we require that ${}_{01}\nabla=\overline{\partial}$ (compatibility with the complex structure).

2. Furthermore:

$$d\langle\sigma_1,\sigma_2\rangle_\bullet = \langle\nabla\sigma_1,\sigma_2\rangle_\bullet+\langle\sigma_1,\nabla\sigma_2\rangle_\bullet.$$

   (compatibility with the metric).

This connection will be called *Chern connection.*

Following [Huybrechts05], p.177, the Chern connection of $\nabla^{h,c}$ of $L^{h,c}$ can be calculated explicitly in the following way (using local coordinates):

$$\begin{aligned}\nabla^{h,c} &= d+\exp\big(\pi H(v,v)\big)\partial\exp\big(-\pi H(v,v)\big)\\ &= d+\exp\big(\pi H(v,v)\big)\cdot\frac{\partial}{\partial v}\exp\big(-\pi H(v,v)\big)dv\\ &= d-\pi\frac{\partial}{\partial v}H(v,v)\,dv.\end{aligned}$$

Thus, the curvature form is

$$F^{h,c} = -\pi d\left(\frac{\partial}{\partial v} H(v,v)\right) \wedge dv$$
$$= -\pi \frac{\partial}{\partial \overline{v}} \frac{\partial}{\partial v} H(v,v)\, d\overline{v} \wedge dv.$$

Direct calculation shows:

$$= -\pi \frac{h}{\mathtt{Im}\,\tau} 2\mathtt{i}\, dx \wedge dy$$
$$= -\frac{2\pi \mathtt{i} h}{\mathtt{Im}\,\tau} dx \wedge dy,$$

which is consistent with the calculation of $c_1\left(L^{h,c}\right)$ done before.

Defining L and ∇^L

As promised at the beginning of the section we will now define the bundles L and ∇^L. Any $L^{h,c}$ would do the job but the natural choice seems to be $L^{h,0}$ equipped with $\nabla^{h,0}$.

2.c.4 Translating $\widetilde{\alpha}_L$ into a holomorphic structure

We now come to the second problem. The given data are the bundle L and the connection $\nabla^{\widetilde{\alpha}_L} = \nabla^L + \mathtt{i}\widetilde{\alpha}_L$, depending on a constant one-form $\widetilde{\alpha}_L$. We now want to determine a characteristic c for which there is an isomorphism of *complex* bundles

$$L \xrightarrow{\mathscr{I}} L^{h,c}$$

identifying ∇^{α_L} with $\nabla^{h,c}$.

At first we will define this map on $\widetilde{\mathbb{C}} \times \mathbb{C} \to \widetilde{\mathbb{C}} \times \mathbb{C}$; after that we will show that our definition is compatible with taking quotients. If we view $\widetilde{\mathbb{C}}$ as the (uncomplexified) tangent space of T_Λ, we can plug a vector $v \in \widetilde{\mathbb{C}}$ into our one-form $\widetilde{\alpha}_L$. This allows us to define

$$\begin{array}{ccc} \widetilde{\mathbb{C}} & \xrightarrow{\gamma} & S^1 \\ v & \mapsto & \exp\left(\mathtt{i}\,\widetilde{\alpha}_L(v)\right). \end{array}$$

Direct calculation shows $\gamma d\gamma^{-1} = -\mathtt{i}\widetilde{\alpha}_L$ (please note that γ is a gauge transformation for $\widetilde{\mathbb{C}}$ but usually does not define a gauge transformation for T_Λ). Since E is a non-degenerate bilinear form (we assumed h to be positive), we can compute an element $c \in \widetilde{\mathbb{C}}$ with $2\pi E(c,v) = \widetilde{\alpha}_L(v)$ for all $v \in \widetilde{\mathbb{C}}$. This c will be fixed throughout the rest of the section. Therefore, we know that $\xi_c = \gamma$.

Now let $\mathscr{I}_{\xi_c} : \widetilde{\mathbb{C}} \times \mathbb{C} \to \widetilde{\mathbb{C}} \times \mathbb{C}$ be the multiplication with ξ_c. The map $\mathscr{I}$ is now defined by the following diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{C}} \times \mathbb{C} & \xrightarrow{\mathscr{I}_{\xi_c}} & \widetilde{\mathbb{C}} \times \mathbb{C} \\ {\scriptstyle \mathrm{pr}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}} \\ L & \xrightarrow{\mathscr{I}} & L^{h,c} \end{array}$$

Since the factors of automorphy of L and $L^{h,c}$ differ from each other just by the factor ξ_c, the diagram commutes and the map $\mathscr{I}$ is well-defined. We also get an inverse map by using $\mathscr{I}_{\xi_c^{-1}}$ instead of $\mathscr{I}_{\xi_c}$. Since ξ_c maps to S^1 and the metric on L and $L^{h,c}$ is defined through the projections, we know that $\mathscr{I}$ is compatible with the hermitian metric. Summarising this, we know that $\mathscr{I}$ is an isomorphism of hermitian line bundles.

The connection $\nabla^{\widetilde{\alpha}_L}$ defines a connection on $L^{h,c}$ by

$$\nabla^{\widetilde{\alpha}_L,c}\sigma = \mathscr{I}\left(\nabla^{\widetilde{\alpha}_L}\mathscr{I}^{-1}(\sigma)\right).$$

To calculate $\nabla^{\widetilde{\alpha}_L,c}$ explicitly, we again use the model space $\widetilde{\mathbb{C}} \times \mathbb{C}$, where $\mathscr{I}_{\xi_c}$ is just a multiplication. Hence, the right hand side equals (following 2.c.3):

$$d - \pi\frac{\partial}{\partial v}H(v,v)\,dv + \mathtt{i}\widetilde{\alpha}_L + \xi_c d\left(\xi_c^{-1}\right) = d - \pi\frac{\partial}{\partial v}H(v,v)\,dv.$$

But this is exactly the Chern connection of $L^{h,c}$ (after going to the quotient again).

This allows us to do all further computations on $L^{h,c}$ equipped with the Chern connection.

2.c.5 Explicit calculations

The holomorphic sections of $L^{h,c}$

We want to describe the set of holomorphic sections of $L^{h,c}$ in terms of theta functions.

Using the usual covering space, we can identify the set of holomorphic sections of $L^{h,c}$ with the set of holomorphic maps $\vartheta : \widetilde{\mathbb{C}} \to \mathbb{C}$ satisfying

$$\vartheta(v+s) = f_{(H,\chi_c)}\vartheta(v) \quad \forall s \in \Lambda.$$

We can also define a *bilinear* form B on $\widetilde{\mathbb{C}}$ which is the extension of $H|_{\widetilde{\mathbb{C}}_2 \times \widetilde{\mathbb{C}}_2}$ to the whole of $\widetilde{\mathbb{C}}$: In our case we have $B(v,w) = hvw$.

We need some further definitions from [Birkenhake04]:

$$\Lambda\left(L^{h,c}\right) = \left\{v \in \widetilde{\mathbb{C}} \mid E(v,\Lambda) \subset \mathbb{Z}\right\} \qquad K\left(L^{h,c}\right) = \Lambda\left(L^{h,c}\right)/_{\Lambda}$$

$$\Lambda\left(L^{h,c}\right)_1 = \widetilde{\mathbb{C}}_1 \cap \Lambda\left(L^{h,c}\right) \qquad \Lambda\left(L^{h,c}\right)_2 = \widetilde{\mathbb{C}}_2 \cap \Lambda\left(L^{h,c}\right)$$

$$K_1 = \Lambda\left(L^{h,c}\right)_1/_{\Lambda_1} \qquad K_2 = \Lambda\left(L^{h,c}\right)_2/_{\Lambda_2}.$$

Since we have $E = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}$, we conclude:

$$\Lambda\big(L^{h,c}\big) = \Lambda\,\frac{1}{h} \qquad\qquad K\big(L^{h,c}\big) = \Lambda\,\frac{1}{h}/\Lambda \cong \big(\mathbb{Z}/h\mathbb{Z}\big)^2$$

($\cong$ meaning group isomorphism)

$$\Lambda\big(L^{h,c}\big)_1 = \mathbb{Z}\,\frac{\tau}{h} \qquad\qquad \Lambda\big(L^{h,c}\big)_2 = \mathbb{Z}\,\frac{1}{h}$$

$$K_1 = \left\{\frac{0}{h}\,\tau,\dots,\frac{h-1}{h}\,\tau\right\} \qquad\qquad K_2 = \left\{\frac{0}{h}\,1,\dots,\frac{h-1}{h}\,1\right\}.$$

The following theorem (theorem 3.2.7 on p.53 in [Birkenhake04] with changed notation) defines a basis of the space of holomorphic sections:

Theorem 2.c(v) (Theta functions)**.** *Suppose $L(H,\chi)$ is a positive definite line bundle on T_Λ, and let c be a characteristic with respect to a decomposition $\widetilde{\mathbb{C}} = \widetilde{\mathbb{C}}_1 \oplus \widetilde{\mathbb{C}}_2$ for $L(H,\chi)$. Then the set*

$$\mathscr{B}_L := \left\{\vartheta^c_{\overline{w}} \,\middle|\, \overline{w} \in K_1\right\}.$$

is a basis of the vector space $H^0\big(L(H,\chi)\big)$ of canonical theta functions for $L(H,\chi)$.

$\vartheta^c_{\overline{w}}$ is defined as follows:

$$\vartheta^c_{\overline{w}} := \big(f_{(H,\chi_c)}(w,\bullet)\big)^{-1}\,\vartheta^c(\bullet + w)$$

where

$$\vartheta^c(v) := \exp\Big(-\pi H(v,c) - \frac{\pi}{2}\,H(c,c) + \frac{\pi}{2}\,B(v+c,v+c)\Big) \cdot \sum_{s\in\Lambda_1} \exp\Big(\pi\big(H-B\big)(v+c,s) - \frac{\pi}{2}\big(H-B\big)(s,s)\Big).$$

Here we use the extension of $f_{(H,\chi_c)}$ to $\widetilde{\mathbb{C}} \times \widetilde{\mathbb{C}}$ which is given by

$$f_{(H,\chi_c)}(v,w) = \chi_c(v)\exp\Big(2\pi\mathrm{i}\,E(c,v) + \pi H(w,v) + \frac{\pi}{2}\,H(v,v)\Big).$$

These theta functions can be derived from the classical theta functions, which is discussed in [Birkenhake04].

Following our calculation of K_1 above, we know that

$$\overline{w} \quad \text{is given by} \quad \frac{k}{h}\,\tau \quad \text{with} \quad k \in \{0,\dots,h-1\}.$$

Since $H^0\big(L^{h,c}\big)$ (as it is defined in the theorem above) is also the space of holomorphic sections of $L^{h,c}$, we have found a basis $\vartheta^{h,c}_k := \vartheta^c_{\frac{k}{h}\tau}$, $k = 0,\dots,h-1$ for $H^0\big(L^{h,c}\big)$.

The tangent bundle and its relatives

The tangent bundle $\mathrm{T}\big(T_\Lambda\big)$ is isomorphic to $\widetilde{\mathbb{C}} \times \widetilde{\mathbb{C}}$ with the induced flat hermitian metric $g_{\mathbb{C}}(v,w) = v\overline{w}$. The *real* metric $g_{\mathbb{R}} = \mathtt{Re}\, g_{\mathbb{C}}$ on $\mathrm{T}\big(T_\Lambda\big)$ induces by complex continuation a hermitian metric $g_\otimes$ on $\mathrm{T}\big(T_\Lambda\big) \otimes \mathbb{C}$. On the real space $\widetilde{\mathbb{C}}$ we can use orthonormal coordinates (x, y), which introduce bases $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and dx, dy on $\mathrm{T}\big(T_\Lambda\big)$ and $\mathrm{T}^*\big(T_\Lambda\big)$ respectively. Notice also that L_0^0 represents the trivial bundle $T_\Lambda \times \mathbb{C}$.

With this definitions we get the following *hermitian isomorphisms*:

$$\begin{array}{ccc} \Big(\mathrm{T}^*\big(T_\Lambda\big)\Big)^{10} & \xrightarrow{I_{10}} & \mathrm{T}^*\big(T_\Lambda\big) \\ dv & \mapsto & \sqrt{2}\,(dx)_{\mathbb{C}} \end{array} \qquad \begin{array}{ccc} \Big(\mathrm{T}^*\big(T_\Lambda\big)\Big)^{01} & \xrightarrow{I_{01}} & \mathrm{T}^*\big(T_\Lambda\big) \\ d\overline{v} & \mapsto & \sqrt{2}\,(dx)_{\mathbb{C}} \end{array}$$

$$\begin{array}{ccc} \mathrm{T}\big(T_\Lambda\big) & \xrightarrow{I_{\mathrm{T}}} & L_0^0 \\ \frac{\partial}{\partial x} & \mapsto & \vartheta_{0,0}^0 \end{array} \qquad \begin{array}{ccc} \mathrm{T}^*\big(T_\Lambda\big) & \xrightarrow{I_{\mathrm{T}^*}} & L_0^0 \\ (dx)_{\mathbb{C}} & \mapsto & \vartheta_{0,0}^0 \end{array} \tag{2.c-5}$$

Here $(dx)_{\mathbb{C}}$ is the complexified map dx in $\mathrm{T}^*\big(T_\Lambda\big)$.

2.c.6 Applying [Almorox06] to the Dirac operator

During this subsection we will use the notation of [Almorox06] (so far as it does not clash with our previous notation); nevertheless we will introduce the terms briefly to make reading easier. Everything we have done so far in this section will now be put into practise.

First of all, we have to choose an associated spin bundle $\mathbb{S}$ on T_Λ for the trivial spin structure, which comes with the usual splitting into $\mathbb{S}^+ \oplus \mathbb{S}^-$. This can be done by defining:

$$\mathbb{S}^+ := \Big(\mathrm{T}^*\big(T_\Lambda\big)\Big)^{00} \qquad\qquad \mathbb{S}^- := \Big(\mathrm{T}^*\big(T_\Lambda\big)\Big)^{01}.$$

Since we are now in complex geometry, we can compute the Clifford multiplication by the usual formula (see [Morgan96], p.51):

$$c_{e_i}\beta = \sqrt{2}\big(\pi^{01}(e_i^*) \wedge \beta - \pi^{01}(e_i^*) \angle \beta\big) \quad e_i \in \underline{\mathbb{H}},\ \beta \in \mathbb{S}.$$

We choose

$$e_2 = \frac{\partial}{\partial x} \qquad\qquad e_3 = \frac{\partial}{\partial y}.$$

Then we have:

$$e_2^* = \overline{(dx)_{\mathbb{C}}} \qquad\qquad e_3^* = \mathtt{i}\,\overline{(dx)_{\mathbb{C}}}.$$

Furthermore:

$$\pi^{01}(e_2^*) = \frac{1}{2}\, d\overline{v} \qquad\qquad \pi^{01}(e_3^*) = \frac{\mathtt{i}}{2}\, d\overline{v}.$$

Therefore, the Clifford multiplication becomes (Taking $1 \in \mathbb{S}^+$ and $d\overline{v} \in \mathbb{S}^-$ as bases):

$$\begin{aligned}
c_{e_2} \cdot 1 &= \sqrt{2}\left(\frac{1}{2}\, d\overline{v}\right) = \frac{\sqrt{2}}{2}\, d\overline{v} \\
c_{e_3} \cdot 1 &= \sqrt{2}\left(\frac{\mathtt{i}}{2}\, d\overline{v}\right) = \frac{\sqrt{2}}{2}\, \mathtt{i}\, d\overline{v} \\
c_{e_2} \cdot d\overline{v} &= -\sqrt{2}\, g_{\otimes}\left(d\overline{v}, \frac{1}{2}\, d\overline{v}\right) = -\sqrt{2} \\
c_{e_3} \cdot d\overline{v} &= -\sqrt{2}\, g_{\otimes}\left(d\overline{v}, \frac{\mathtt{i}}{2}\, d\overline{v}\right) = \sqrt{2}\mathtt{i}.
\end{aligned}$$

We now use I_{T^*} and $I_{\mathrm{T}^*} \circ I_{01}$ to identify $\mathbb{S}^+$ and $\mathbb{S}^-$ both with L_0^0. To distinguish both bundles afterwards we call them L_0^+ and L_0^- respectively. Although theta functions $\vartheta_k^{h,c}$ in the above sense do not exist on L_0^0 we will denote the constant sections spanning L_0^+ and L_0^- by ϑ_0^+ and ϑ_0^-. The Clifford multiplication now reads:

$$\begin{aligned}
c_{e_2}\, \vartheta_0^+ &= \vartheta_0^- & \qquad c_{e_3}\, \vartheta_0^- &= \mathtt{i}\vartheta_0^- \\
c_{e_2}\, \vartheta_0^- &= -\vartheta_0^+ & \qquad c_{e_3}\, \vartheta_0^- &= \mathtt{i}\vartheta_0^+.
\end{aligned}$$

Viewed as a cohomology class, h defines a Spin$^{\mathbb{C}}$ structure; the associated bundle can be represented by $\mathbb{S} \otimes L^{h,c}$. This splits into:

$$\mathbb{E}^+ = \left(L^{h,c}\right)^+ \qquad\qquad \mathbb{E}^- = \left(L^{h,c}\right)^-,$$

where $\left(L^{h,c}\right)^\pm$ represents $L_0^\pm \otimes L^{h,c} \cong L^{h,c}$.

The elliptic chain

The computation of the eigenspaces in [Almorox06] is done by using a so called *elliptic chain.* It is defined as $\mathscr{C}^q(\mathbb{E}^+) = K_{T_\Lambda}^q \otimes \mathbb{E}^+$. Here K_{T_Λ} is the square-root of L_0^0 and q runs through the integers. Since L_0^0 is canonically trivial, we consider $\mathscr{C}^q(\mathbb{E}^+)$ and $\mathbb{E}^+$ to be equal.

We now want to define maps ∂_+^q from $\mathscr{C}^q(\mathbb{E}^+)$ to $\mathscr{C}^{q+1}(\mathbb{E}^+)$ and $\overline{\partial}_+^q$ from $\mathscr{C}^q(\mathbb{E}^+)$ to $\mathscr{C}^{q-1}(\mathbb{E}^+)$. For that purpose we split $\nabla^{h,c}$ into ${}_{10}\nabla^{h,c} + {}_{01}\nabla^{h,c}$ as usual:

$$\begin{aligned}
\Gamma(\mathbb{E}^+) &\xrightarrow{{}_{10}\nabla^{h,c}} \Gamma(\mathbb{E}^+) \otimes \Omega^{10}(T_\Lambda) \\
\Gamma(\mathbb{E}^+) &\xrightarrow{{}_{01}\nabla^{h,c}} \Gamma(\mathbb{E}^+) \otimes \Omega^{01}(T_\Lambda).
\end{aligned}$$

If we use the isomorphisms $I_{T^*} \circ I_{10}$ and $I_{T^*} \circ I_{01}$ on forms we come back to $\Gamma(\mathbb{E}^+)$. Using the calculations of 2.c.3, we conclude

$$\partial^+ := \partial_+^q = \sqrt{2}\left(\frac{\partial}{\partial v} - \pi \frac{\partial}{\partial v} H(v,v)\right)$$
$$\overline{\partial}_+ := \overline{\partial}_+^q = \sqrt{2}\,\frac{\partial}{\partial \overline{v}}.$$

These data suffice to write down the eigenspaces.

The eigenspaces of $\widetilde{\mathscr{D}}^+_{h,c}\widetilde{\mathscr{D}}^-_{h,c}$

The Dirac operator $\widetilde{\mathscr{D}}_{h,c}$ splits (as usual on even-dimensional manifolds) into a direct sum $\widetilde{\mathscr{D}}^+_{h,c} \oplus \widetilde{\mathscr{D}}^-_{h,c}$.

We will use the following theorem (with adjusted notation):

Theorem 2.c(vi) (5.7 in [Almorox06]). *Given a hermitian line bundle $L^{h,c} \to T_\Lambda$ with a Chern connection $\nabla^{h,c}$ and $\deg L^{h,c} \neq 0$, the spectrum of the operator $\widetilde{\mathscr{D}}^+_{h,c}\widetilde{\mathscr{D}}^-_{h,c}$ is the set*

$$Spec(\widetilde{\mathscr{D}}^+_{h,c}\widetilde{\mathscr{D}}^-_{h,c}) = \left\{E_q = 4\pi \frac{|\deg L^{h,c}|}{\mathtt{Im}\,\tau}(q+a) \quad \forall q \in \mathbb{Z},\, q \geq 0\right\},$$

where $a = 0$ if $\deg L^{h,c} > 0$ and $a = 1$ if $\deg L^{h,c} < 0$.

If $\deg L^{h,c} > 0$, then the space of eigensections with eigenvalue E_q gets identified with $H^0(T_\Lambda, K_{T_\Lambda}^{-q} \otimes \mathbb{E}^+)$. In the same way, if $\deg L^{h,c} < 0$, then the space of eigensections with eigenvalue E_q gets identified with $H^0(T_\Lambda, K_{T_\Lambda}^{-q} \otimes (\mathbb{E}^+)^{-1})$. Therefore, the multiplicity of the eigenvalue E_q is $\left|\deg L^{h,c}\right|$.

The following remarks are important to us:

1. The degree of the line bundle $L^{h,c}$ is equal to h. Since we are only interested in line bundles with $h > 0$, we will omit the case $\deg L^{h,c} < 0$. Hence the eigenvalue E_q becomes $4\pi hq/(\mathtt{Im}\,\tau)$.

2. The proof of this theorem in [Almorox06] is more precise about the identification between the space $\widehat{\mathbb{E}}_q$ of eigensections with eigenvalue E_q and the space $H^0(T_\Lambda, K_{T_\Lambda}^{-q} \otimes \mathbb{E}^+)$: It is the image of that space under the injective (see [Almorox06]) map

$$\underbrace{\partial_+ \cdots \partial_+}_{q} : H^0(T_\Lambda, \mathbb{E}^+) \to \Omega^0(T_\Lambda, \mathbb{E}^+)$$

   in the space of (not necessarily holomorphic) sections of $\mathbb{E}^+$. Since $\mathbb{E}^+ \cong L^{h,c}$ in a canonical way we know that

$$\widehat{\mathbb{E}}_q = \operatorname{span}\left\{(\partial_+ \cdots \partial_+ \vartheta_0^{h,c}), \ldots, (\partial_+ \cdots \partial_+ \vartheta_h^{h,c})\right\}.$$

From $\widetilde{\mathscr{D}}^+_{h,c}\widetilde{\mathscr{D}}^-_{h,c}$ to $\widetilde{\mathscr{D}}_{h,c}$

For every $q > 0$ in $\mathbb{Z}$ we get two eigenspaces $\mathbb{E}_{-q}$ and $\mathbb{E}_q$ for the eigenvalues $-\sqrt{E_q}$ and $\sqrt{E_q}$ for $\widetilde{\mathscr{D}}_{h,c}$. They can be described by the isomorphism:

$$\begin{array}{ccc} \widehat{\mathbb{E}}_q & \xrightarrow{\pi^q_\pm} & \mathbb{E}_{\pm q} \\ \vartheta & \mapsto & \frac{1}{2\sqrt{E_q}}\left(\sqrt{E_q}\,\vartheta \pm \widetilde{\mathscr{D}}^+_{h,c}\vartheta\right). \end{array}$$

We also know that $\widetilde{\mathscr{D}}^+_{h,c} = \sqrt{2}\,\overline{\partial}_+ (= \sqrt{2}\,\frac{\partial}{\partial\overline{v}})$, which gives us:

$$\mathbb{E}_{\pm q} = \operatorname{span}\left\{ \frac{1}{2\sqrt{E_q}}\left(\sqrt{E_q} \pm \sqrt{2}\,\overline{\partial}_+\right)\partial_+ \cdots \partial_+ \vartheta^{h,c}_k \,\middle|\, k = 0, \ldots, h-1 \right\}.$$

For $q = 0$ we have

$$\ker \widetilde{\mathscr{D}}^+_{h,c} = H^0\left(T_\Lambda, \mathbb{E}^+\right) = \operatorname{span}\left\{\vartheta^{h,c}_k \mid k = 0, \ldots, h-1\right\} \qquad \ker \widetilde{\mathscr{D}}^-_{h,c} = 0.$$

Now we have, for every $q \in \mathbb{Z}$, an h-dimensional eigenspace with given basis.

2.c.7 Final computations (and rescaling)

First of all we now have to go back from $L^{h,c}$ to L. The map $\mathscr{I}$ translates the eigenspaces just found for $\widetilde{\mathscr{D}}_{h,c}$ back to sections of $L \oplus L$. To get eigenspaces for our original operator $\widetilde{\mathscr{D}}$, we also have to rescale by the factor r_Λ. For simplicity we still call the eigenspaces $\mathbb{E}_q$, $q \in \mathbb{Z}$. They are now eigenspaces for $\widetilde{\mathscr{D}}_{\tilde{\alpha}_L}$.

Since each $\mathbb{E}_q$ is h-dimensional, we can choose an orthonormal eigenbasis

$$\tilde{\sigma}_{qh}, \ldots, \tilde{\sigma}_{qh+h-1}$$

for it by applying Gram-Schmidt to the basis given in 2.c.6. The rescaling changes E_q by the factor $1/r_\Lambda$; the section σ_m is therefore a section with eigenvalue

$$\begin{aligned} \mu_m &= \frac{1}{r_\Lambda} \cdot \operatorname{sgn} m \sqrt{4\pi \frac{h}{\operatorname{Im}\tau} \left\lfloor \frac{|m|}{h} \right\rfloor} \\ &= \operatorname{sgn} m \sqrt{4\pi \frac{h}{r_\Lambda^2 \operatorname{Im}\tau} \left\lfloor \frac{|m|}{h} \right\rfloor}. \end{aligned}$$

Now notice that $r_\Lambda^2 \operatorname{Im}\tau$ is the area of T_Λ, which is equal to $\|a\|^{-1}$ (calculated in 2.b.1)

$$= \operatorname{sgn} m \sqrt{2\pi h \|a\| \left\lfloor \frac{|m|}{h} \right\rfloor}.$$

To summarise this section: From the input data

$$h \in \mathbb{Z} \qquad\qquad \tilde{\alpha}_L \in H^2\big(T_\Lambda;\mathbb{Z}\big),$$

we calculated the eigenbasis

$$\tilde{\sigma}_m \quad \text{for the eigenvalue} \quad \mu_m$$

for the Dirac operator $\widetilde{\mathscr{D}}_{\tilde{\alpha}_L}$.

2.d An eigenbasis for $\mathscr{D}^2$

Lemma 2.b(ii) described T^3 as a trivial bundle over T_Λ with fibre $S^{[1]}$. Using this, we will define an orthogonal eigenbasis $\hat{\sigma}_{l,m}$ for $l, m \in \mathbb{Z}$.

First of all, let $s_l : S^{[1]} \to S^1$ be given by $s_l(t) := \exp\big(2\pi \mathrm{i}\, l t\big)$ for $l \in \mathbb{Z}$. This can be considered as the eigenbasis for the 1-dimensional Dirac operator.

Now we look at the map $s_l \circ \mathrm{tri} : \mathbb{R}^3/\mathbb{Z}^3 \to S^1 \subset \mathbb{C}$ and want to calculate the complex one-form $d(s_l \circ \mathrm{tri})$. We choose coordinates χ_1, χ_2, χ for $\mathbb{R}^3$ as in 2.b(ii). Then we have:

$$\begin{aligned} d\big(s_l \circ \mathrm{tri}\big)_{(\chi_1,\chi_2,\chi)} &= \left(\frac{\partial}{\partial t}\, s_l\right)_{\mathrm{tri}(\chi_1,\chi_2,\chi)} \circ \left(\frac{\partial}{\partial \chi_1}\,\mathrm{tri},\ \frac{\partial}{\partial \chi_2}\,\mathrm{tri},\ \frac{\partial}{\partial \chi}\,\mathrm{tri}\right)_{(\chi_1,\chi_2,\chi)} \\ &= 2\pi \mathrm{i}\, l\, s_l\big(\mathrm{tri}(\chi_1,\chi_2,\chi)\big)\,\big(c^1, c^2, 1\big) \\ &= 2\pi \mathrm{i}\, l \Big[\big(s_l \circ \mathrm{tri}\big)(\chi_1,\chi_2,\chi)\Big] \Big(c^1, c^2, 1\Big), \end{aligned}$$

which means

$$d\big(s_l \circ \mathrm{tri}\big) = 2\pi \mathrm{i}\, l \big(s_l \circ \mathrm{tri}\big) \Big(c^1, c^2, 1\Big).$$

We now want to separate this form into its parallel and orthogonal part with respect to W:

$$d\big(s_l \circ \mathrm{tri}\big) = 2\pi \mathrm{i}\,(s_l \circ \mathrm{tri}) \cdot \big(\omega^l_{\parallel} + \omega^l_{\perp}\big),$$

where

$$\omega^l_{\parallel} = l c^1 w_1^* + l c^2 w_2^* \qquad\qquad \omega^l_{\perp} = l a = l \|a\|\, e_1^*.$$

In the same way we split α into $\alpha_{\parallel} + \alpha_{\perp}$. The parallel part of a harmonic form can be written as a pull-back:

$$\omega^l_{\parallel} = \pi^*_{\overline{a}}\big(\omega^l_L\big) \qquad\qquad \alpha_{\parallel} = \pi^*_{\overline{a}}\big(\alpha_L\big).$$

We set $\tilde{\alpha}^l_L := \alpha_L + 2\pi\omega^l_L$. This one-form will now be used as in 2.c: For every $l \in \mathbb{Z}$ we get an eigenbasis $\tilde{\sigma}^l_m$ with corresponding eigenvalues μ_m (remember that we saw in 2.c.7 that the eigenvalues are independent of the choice of $\tilde{\alpha}_L$).

Let furthermore λ_l be defined as $(2\pi l + x^a)\|a\|$ with x^a defined by $\alpha_{\perp} = x^a \cdot a$.

Definition 2.d(i). For $l, m \in \mathbb{Z}$ let

$$\hat{\sigma}_{l,m}(v) := \big(s_l \circ \mathrm{tri}\big)(v) \cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big)(v)$$

be a section of $\underline{\mathbb{H}} \otimes K$, where $\cdot$ means complex multiplication.

Theorem 2.d(ii). *The set $\{\hat{\sigma}_{l,m} \mid l, m, \in \mathbb{Z}\}$ forms an orthogonal eigenbasis for $\mathscr{D}_\alpha^2$ for the respective eigenvalues $\lambda_l^2 + \mu_m^2$. Furthermore, we have*

$$\mathscr{D}_\alpha \hat{\sigma}_{l,m} = \big(\lambda_l \mathtt{i} c_{e_1} + \mu_m\big)\hat{\sigma}_{l,m}. \tag{2.d-1}$$

Proof. We have to show three things:

1. Formula (2.d-1) is true. We use this to show that $\hat{\sigma}_{l,m}$ is an eigenvector for $\mathscr{D}_\alpha^2$.
2. The set of linear combinations of $\{\hat{\sigma}_{l,m} \mid l, m, \in \mathbb{Z}\}$ is dense in the set of all sections of $\underline{\mathbb{H}} \otimes K$.
3. The sections $\hat{\sigma}_{l,m}$ are pairwise orthogonal.

Proof of part 1. We calculate

$$\mathscr{D}_\alpha \hat{\sigma}_{l,m} = c_{e_1}\nabla^\alpha_{e_1}\hat{\sigma}_{l,m} + c_{e_2}\nabla^\alpha_{e_2}\hat{\sigma}_{l,m} + c_{e_3}\nabla^\alpha_{e_3}\hat{\sigma}_{l,m}$$

in local coordinates. For this remember that e_1 is the vector $a/\|a\|$, and e_2 and e_3 are parallel to W.

The Leibniz formula for connections reads:

$$\begin{aligned}\mathscr{D}_\alpha \hat{\sigma}_{l,m} = c_{e_1}&\Big(d(s_l\circ \mathrm{tri})(e_1)\cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big) + (s_l\circ \mathrm{tri})\cdot \nabla^\alpha_{e_1}\pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big)\Big)\\ + c_{e_2}&\Big(d(s_l\circ \mathrm{tri})(e_2)\cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big) + (s_l\circ \mathrm{tri})\cdot \nabla^\alpha_{e_2}\pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big)\Big)\\ + c_{e_3}&\Big(d(s_l\circ \mathrm{tri})(e_3)\cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big) + (s_l\circ \mathrm{tri})\cdot \nabla^\alpha_{e_3}\pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big)\Big)\end{aligned}$$

Now we split ∇^α into $\nabla^{\alpha_{\shortparallel}} + \mathtt{i}\alpha_\perp$:

$$\begin{aligned} = c_{e_1}&\Big(d(s_l\circ \mathrm{tri})(e_1)\cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big) + (s_l\circ \mathrm{tri})\cdot \nabla^{\alpha_{\shortparallel}}_{e_1}\pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big) + (s_l\circ \mathrm{tri})\cdot \mathtt{i}\alpha_\perp(e_1)\cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big)\Big)\\ + c_{e_2}&\Big(d(s_l\circ \mathrm{tri})(e_2)\cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big) + (s_l\circ \mathrm{tri})\cdot \nabla^{\alpha_{\shortparallel}}_{e_2}\pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big)\Big)\\ + c_{e_3}&\Big(d(s_l\circ \mathrm{tri})(e_3)\cdot \pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big) + (s_l\circ \mathrm{tri})\cdot \nabla^{\alpha_{\shortparallel}}_{e_3}\pi_{\overline{a}}^{*}\big(\tilde{\sigma}^{l}_{m}\big)\Big).\end{aligned}$$

We now list the green terms at the beginning and view $\nabla^{\alpha_{\shortparallel}}$ as pull-back of ∇^{α_L}:

$$
\begin{aligned}
&= \Big(d(s_l \circ \mathrm{tri})(e_1) c_{e_1} + (s_l \circ \mathrm{tri}) \cdot \mathtt{i} x^a \|a\| c_{e_1} \\
&\quad + d(s_l \circ \mathrm{tri})(e_2) c_{e_2} + d(s_l \circ \mathrm{tri})(e_3) c_{e_3} \Big) \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) \\
&\quad + (s_l \circ \mathrm{tri}) \Big(c_{e_1} \Big(\pi_{\overline{a}}^*(\nabla^{\alpha_L}_{\pi_{\overline{a}*}(e_1)} \tilde{\sigma}^l_m) \Big) + c_{e_2} \Big(\pi_{\overline{a}}^*(\nabla^{\alpha_L}_{\pi_{\overline{a}*}(e_2)} \tilde{\sigma}^l_m) \Big) + c_{e_3} \Big(\pi_{\overline{a}}^*(\nabla^{\alpha_L}_{\pi_{\overline{a}*}(e_3)} \tilde{\sigma}^l_m) \Big) \Big).
\end{aligned}
$$

We know that $\pi_{\overline{a}*}(e_1) = 0$, $\pi_{\overline{a}*}(e_2) = e_2$ and $\pi_{\overline{a}*}(e_3) = e_3$. Furthermore, we can replace $d(s_l \circ \mathrm{tri})$ by $2\pi\mathtt{i}(s_l \circ \mathrm{tri}) \cdot (\omega^l_{\shortparallel} + \omega^l_{\perp})$:

$$
\begin{aligned}
&= \Big(2\pi\mathtt{i}(s_l \circ \mathrm{tri}) \cdot (\omega^l_{\shortparallel} + \omega^l_{\perp})(e_1) c_{e_1} + (s_l \circ \mathrm{tri}) \cdot \mathtt{i} x^a \|a\| c_{e_1} \\
&\quad + 2\pi\mathtt{i}(s_l \circ \mathrm{tri}) \cdot (\omega^l_{\shortparallel} + \omega^l_{\perp})(e_2) c_{e_2} + 2\pi\mathtt{i}(s_l \circ \mathrm{tri}) \cdot (\omega^l_{\shortparallel} + \omega^l_{\perp})(e_3) c_{e_3} \Big) \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) \\
&\quad + (s_l \circ \mathrm{tri}) \Big(c_{e_2} \Big(\pi_{\overline{a}}^*(\nabla^{\alpha_L}_{e_2} \tilde{\sigma}^l_m) \Big) + c_{e_3} \Big(\pi_{\overline{a}}^*(\nabla^{\alpha_L}_{e_3} \tilde{\sigma}^l_m) \Big) \Big) \\
&= (s_l \circ \mathrm{tri}) \Big(2\pi\mathtt{i} \cdot (\omega^l_{\shortparallel} + \omega^l_{\perp})(e_1) c_{e_1} + \cdot \mathtt{i} x^a \|a\| c_{e_1} \\
&\quad + 2\pi\mathtt{i} \cdot (\omega^l_{\shortparallel} + \omega^l_{\perp})(e_2) c_{e_2} + 2\pi\mathtt{i} \cdot (\omega^l_{\shortparallel} + \omega^l_{\perp})(e_3) c_{e_3} \Big) \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) \\
&\quad + (s_l \circ \mathrm{tri}) \Big(c_{e_2} \Big(\pi_{\overline{a}}^*(\nabla^{\alpha_L}_{e_2} \tilde{\sigma}^l_m) \Big) + c_{e_3} \Big(\pi_{\overline{a}}^*(\nabla^{\alpha_L}_{e_3} \tilde{\sigma}^l_m) \Big) \Big).
\end{aligned}
$$

The blue terms can be reduced to $(2\pi\mathtt{i}l + \mathtt{i}x^a)\|a\| \cdot c_{e_1} = \lambda_l \mathtt{i} c_{e_1}$.

Since $\omega^l_{\perp}(e_2) = \omega^l_{\perp}(e_3) = 0$ and e_2, e_3 forms a basis of W, the red terms equal $2\pi\mathtt{i}\, c_{\omega^l_{\shortparallel}}$, where we do Clifford multiplication with one-forms in the usual manner.

We also defined c_w in a way that it can be exchanged with $\pi_{\overline{a}}^*$ if $w \in W = \mathrm{T}(T_\Lambda)$:

$$
\begin{aligned}
&= \lambda_l \mathtt{i} c_{e_1} (s_l \circ \mathrm{tri}) \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) + (s_l \circ \mathrm{tri})\, \pi_{\overline{a}}^*(2\pi\mathtt{i} c_{\omega^l_{\shortparallel}} \tilde{\sigma}^l_m) \\
&\quad + (s_l \circ \mathrm{tri}) \Big(\pi_{\overline{a}}^* \Big(c_{e_2} (\nabla^{\alpha_L}_{e_2} \tilde{\sigma}^l_m) \Big) + \pi_{\overline{a}}^* \Big(c_{e_3} (\nabla^{\alpha_L}_{e_3} \tilde{\sigma}^l_m) \Big) \Big) \\
&= \lambda_l \mathtt{i} c_{e_1} (s_l \circ \mathrm{tri}) \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) + \\
&\quad + (s_l \circ \mathrm{tri})\, \pi_{\overline{a}}^* \Big(c_{e_2} (\nabla^{\alpha_L}_{e_2} \tilde{\sigma}^l_m) + c_{e_3} (\nabla^{\alpha_L}_{e_3} \tilde{\sigma}^l_m) + 2\pi\mathtt{i} c_{\omega^l_{\shortparallel}} \tilde{\sigma}^l_m \Big) \\
&= \lambda_l \mathtt{i} c_{e_1} (s_l \circ \mathrm{tri}) \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) + \\
&\quad + (s_l \circ \mathrm{tri})\, \pi_{\overline{a}}^* \Big(\widetilde{\mathscr{D}}_{\tilde{\alpha}^l_L} \tilde{\sigma}^l_m \Big) \\
&= \Big(\lambda_l \mathtt{i} c_{e_1} + \mu_m \Big) \hat{\sigma}_{l,m}.
\end{aligned}
$$

We also know that

$$
\begin{aligned}
(\mathscr{D}_\alpha c_{e_1} + c_{e_1} \mathscr{D}_\alpha) \hat{\sigma}_{l,m} &= -2 \cdot \nabla^\alpha_{e_1} \hat{\sigma}_{l,m} \\
&= -2 \lambda_l \hat{\sigma}_{l,m}.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}\mathscr{D}_\alpha^2 \hat{\sigma}_{l,m} &= \mathscr{D}_\alpha\big(\lambda_l \mathtt{i} c_{e_1} + \mu_m\big)\hat{\sigma}_{l,m}\\ &= \big(-\lambda_l \mathtt{i} c_{e_1} + \mu_m\big)\mathscr{D}_\alpha \hat{\sigma}_{l,m} - 2(\lambda_l \mathtt{i})^2 \hat{\sigma}_{l,m}\\ &= \big(-\lambda_l \mathtt{i} c_{e_1} + \mu_m\big)\big(\lambda_l \mathtt{i} c_{e_1} + \mu_m\big)\hat{\sigma}_{l,m} + 2\lambda_l^2 \hat{\sigma}_{l,m}\\ &= \big(\lambda_l^2 + \mu_m^2\big)\,\hat{\sigma}_{l,m}.\end{aligned}$$

Since for any $l, m \in \mathbb{Z}$, $\hat{\sigma}_{l,m}$ is not the zero section, we know that all $\hat{\sigma}_{l,m}$ are eigenvectors. End of 1

Proof of part 2. We divide this proof into two steps:

1. Sections of the form $(s \circ \mathrm{tri}) \cdot \pi^*_{\overline{a}}(\sigma)$, with $s : S^{[1]} \to S^1$ an arbitrary smooth map and σ an arbitrary smooth section of $\underline{\mathbb{H}} \otimes L$, form a dense subset of all sections of $\underline{\mathbb{H}} \otimes K$.

   Using Stone-Weierstraß, we see that this is true for any local trivialisation of L. Now we use a (finite) partition of unity to patch those local approximations together to form a global one. The result transfers from supremum norm density to L^2 density.

2. For given s and σ as above and any $\varepsilon > 0$, we find $l_i, m_i^j \in \mathbb{Z}$ (for finitely many $i, j \in \mathbb{Z}$) and constants $p_i, q_i^j \in \mathbb{C}$ with

$$\Big\| (s \circ \mathrm{tri}) \cdot \pi^*_{\overline{a}}(\sigma) - \sum_{i,j} p_i (s_{l_i} \circ \mathrm{tri}) \cdot q_i^j \pi^*_{\overline{a}}\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) \Big\|_{\mathrm{L}^2} < \varepsilon.$$

   To show this, we use the fact that linear combinations of s_l and $\tilde{\sigma}^l_m$ are dense in their respective spaces of sections.

   First we choose for $i = 1, \ldots, N$ numbers $l_i \in \mathbb{Z}$ and $p_i \in \mathbb{C}$ so that

$$\|(s \circ \mathrm{tri}) - \sum_i p_i (s_{l_i} \circ \mathrm{tri})\| < \frac{\varepsilon}{4\|\pi^*_{\overline{a}}(\sigma)\|}, \tag{2.d-2}$$

   where we drop the L^2 subscript for the sake of readability. For every l_i we take finitely many $m_i^j \in \mathbb{Z}$ and $q_i^j \in \mathbb{C}$ so that

$$\begin{aligned}&\Big\| \frac{1}{N}\pi^*_{\overline{a}}(\sigma) - \sum_j q_i^j \pi^*_{\overline{a}}\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) \Big\|\\ &\qquad < \min\left\{ \frac{\varepsilon}{2N\|(s \circ \mathrm{tri})\|}, \frac{\varepsilon}{4N\big\|\frac{1}{N}(s \circ \mathrm{tri}) - p_i(s_{l_i} \circ \mathrm{tri})\big\|} \right\},\end{aligned} \tag{2.d-3}$$

where the min is well-defined since the first term is always finite. Now we have

$$
\begin{aligned}
&\Big\|(s\circ \mathrm{tri})\cdot \pi_{\overline{a}}^*(\sigma) - \sum_{i,j} p_i(s_{l_i}\circ \mathrm{tri})\cdot q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big)\Big\| \\
&= \Big\|(s\circ \mathrm{tri})\cdot \pi_{\overline{a}}^*(\sigma) - (s\circ \mathrm{tri})\cdot \sum_i\sum_j q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) \\
&\quad + (s\circ \mathrm{tri})\cdot \sum_i\sum_j q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) - \sum_{i,j} p_i(s_{l_i}\circ \mathrm{tri})\cdot q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big)\Big\| \\
&\le \Big\|(s\circ \mathrm{tri})\cdot \pi_{\overline{a}}^*(\sigma) - (s\circ \mathrm{tri})\cdot \sum_i\sum_j q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big)\Big\| \\
&\quad + \Big\|(s\circ \mathrm{tri})\cdot \sum_i\sum_j q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) - \sum_{i,j} p_i(s_{l_i}\circ \mathrm{tri})\cdot q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big)\Big\|.
\end{aligned}
$$

For the first summand, we use N times (2.d-3) and get $\varepsilon/2$. For the second one, we have:

$$
\begin{aligned}
&\Big\|(s\circ \mathrm{tri})\cdot \sum_i\sum_j q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) - \sum_{i,j} p_i(s_{l_i}\circ \mathrm{tri})\cdot q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big)\Big\| \\
&= \Big\|\sum_i \Big(\Big(\frac{1}{N}(s\circ \mathrm{tri}) - p_i(s_{l_i}\circ \mathrm{tri})\Big)\sum_j q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big)\Big)\Big\| \\
&= \Big\|\sum_i \Big(\Big(\frac{1}{N}(s\circ \mathrm{tri}) - p_i(s_{l_i}\circ \mathrm{tri})\Big)\sum_j \Big(q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) - \frac{1}{N}\pi_{\overline{a}}^*(\sigma)\Big)\Big) \\
&\quad + \sum_i \Big(\frac{1}{N}(s\circ \mathrm{tri}) - p_i(s_{l_i}\circ \mathrm{tri})\Big)\pi_{\overline{a}}^*(\sigma)\Big\| \\
&\le \sum_i \Big\|\frac{1}{N}(s\circ \mathrm{tri}) - p_i(s_{l_i}\circ \mathrm{tri})\Big\| \, \Big\|\sum_j \Big(q_i^j \pi_{\overline{a}}^*\Big(\tilde{\sigma}^{l_i}_{m_i^j}\Big) - \frac{1}{N}\pi_{\overline{a}}^*(\sigma)\Big)\Big\| \\
&\quad + \Big\|\sum_i \Big(\frac{1}{N}(s\circ \mathrm{tri}) - p_i(s_{l_i}\circ \mathrm{tri})\Big)\Big\| \, \Big\|\pi_{\overline{a}}^*(\sigma)\Big\|.
\end{aligned}
$$

Estimate (2.d-3) shows that the first sum is smaller than $\varepsilon/4$, whereas the second summand can be plugged into (2.d-2) to get the same result.

Adding all this up we get the required inequality.

Since density is transitive we get the desired result. End of 2

Proof of part 3. Since $\mathscr{D}_\alpha$ is self-adjoint, we know that eigensections for different eigenvalues are always orthogonal to each other. The following fact about $\hat{\sigma}_{l,m}$ is important: The change of $\alpha_\perp$ changes the eigenvalue (since λ_l depends on x^a), but not the eigensection, whereas the change of $\alpha_{\shortparallel}$ changes the eigensection but not the eigenvalue.

So if there is some x^a for which $\hat{\sigma}_{l',m'}$ and $\hat{\sigma}_{l,m}$ have different eigenvalues, they must be orthogonal to each other. So we write down the equation:

$$\lambda_{l'}^2 + \mu_{m'}^2 = \lambda_l^2 + \mu_m^2,$$

which is equivalent to

$$4\pi\|a\|^2(l'-l)x^a + (2\pi l'\|a\|)^2 - (2\pi l\|a\|)^2 + \mu_{m'}^2 - \mu_m^2 = 0.$$

Since all coefficients of x^a have to vanish, we get $l' = l$. So we look at

$$\left\langle (s_l \circ \mathrm{tri}) \cdot \pi_{\overline{a}}^*(\tilde{\sigma}^l_{m'}), (s_l \circ \mathrm{tri}) \cdot \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) \right\rangle_{\mathrm{L}^2}.$$

Since $s_l \circ \mathrm{tri}$ has norm 1, we get

$$\left\langle \pi_{\overline{a}}^*(\tilde{\sigma}^l_{m'}), \pi_{\overline{a}}^*(\tilde{\sigma}^l_m) \right\rangle_{\mathrm{L}^2}.$$

This is an integral over the fibre bundle $T_\Lambda \times S^{[1]}$ which is constant on every fibre. Therefore, we can calculate it on the base space and multiply it by the length of the fibre (which is $\|a\|$) afterwards.

But on T_Λ we just have the L^2 product of $\tilde{\sigma}^l_m$ and $\tilde{\sigma}^l_{m'}$, which is zero by the definition of the bases over T_Λ. End of 3

□

2.e An eigenbasis for $\mathscr{D}$

In this section we use the basis $\hat{\sigma}_{l,m}$ to produce a basis $\sigma_{l,m}^{-/0/+}$ (with eigenvalue $\nu_{l,m}^{-/0/+}$) for $\mathscr{D}_\alpha$. For this, we look again at 2.c.6 to write down some important properties of $\widetilde{\mathscr{D}}_{\tilde{\alpha}^l_L}$ and $\tilde{\sigma}^l_m$:

$\widetilde{\mathscr{D}}_{\tilde{\alpha}^l_L}$ is an odd operator on $\underline{\mathbb{H}} \otimes L \cong L_+ \oplus L_-$, where $\mathbb{H} \cong \mathbb{C}\{e_0\} \oplus \mathbb{C}\{e_2\}$. So $\widetilde{\mathscr{D}}_{\tilde{\alpha}^l_L} \tilde{\sigma}^l_m = \mu_m \tilde{\sigma}^l_m$ means that we can write $\tilde{\sigma}^l_m = \tilde{\sigma}^{l+}_m + \tilde{\sigma}^{l-}_m$ and we have

$$\widetilde{\mathscr{D}}_{\tilde{\alpha}^l_L} \tilde{\sigma}^{l\pm}_m = \mu_m \tilde{\sigma}^{l\mp}_m.$$

From the definition in 2.c.7, we see that $\mu_0 = \ldots \mu_{h-1} = 0$ and that the minus component of $\tilde{\sigma}^l_m$ vanishes for $m = 0, \ldots, h-1$.

Furthermore we know that the sections $\tilde{\sigma}^l_m$ are derived from eigensections of the squared Dirac operator. Therefore, they appear in pairs: For $\tilde{\sigma}^l_m = \tilde{\sigma}^{l+}_m + \tilde{\sigma}^{l-}_m$ with eigenvalue μ_m (assume $m \geq h$), we have $\tilde{\sigma}^l_{-m+h-1} = \sigma^{l+}_m - \sigma^{l-}_m$ with eigenvalue $\mu_{-m+h-1} = -\mu_m$.

Definition 2.e(i). Let

$$\sigma^{\pm}_{l,m} := (s_l \circ \mathrm{tri}) \cdot \Big(\Big(\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \Big) \pi^*_{\bar{a}} \big(\tilde{\sigma}^{l+}_m \big) + \Big(-\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \Big) \pi^*_{\bar{a}} \big(\tilde{\sigma}^{l-}_m \big) \Big)$$
$$\sigma^0_{l,m} := \hat{\sigma}_{l,m}$$

and

$$\nu^{\pm}_{l,m} := \pm \sqrt{\lambda_l^2 + \mu_m^2}$$
$$\nu^0_{l,m} := \begin{cases} \lambda_l & \text{for } 0 \le m \le h-1 \\ \mu_m & \text{otherwise} \end{cases}$$

Theorem 2.e(ii). *We get an orthogonal eigenbasis of* $\mathscr{D}_\alpha$ *by*

$$\Big\{ \sigma^{\pm}_{l,m} \,\Big|\, (l,m) \in \mathbb{Z}^2 \quad \textit{with } \lambda_l \neq 0 \textit{ and } m \ge h \Big\}$$
$$\cup \Big\{ \sigma^0_{l,m} \,\Big|\, (l,m) \in \mathbb{Z}^2 \quad \textit{with } \lambda_l = 0 \textit{ or } 0 \le m \le h-1 \Big\},$$

which will be written as $M^{\pm}_\alpha \cup M^0_\alpha$.

Proof. The sections $\sigma^-_{l,m}, \sigma^0_{l,m}, \sigma^+_{l,m}$ are produced by the usual method: We solve the "quadratic equation" given by $\big(\mathscr{D}^2_\alpha - (\lambda_l^2 + \mu_m^2)\big)\hat{\sigma}_{l,m} = 0$. The interesting part is the choice of a *dense* subset of *non-zero* vectors.

First of all, we check that all vectors of the given set $M^{\pm}_\alpha \cup M^0_\alpha$ are mapped to the specified multiple of themselves by $\mathscr{D}_\alpha$. For that we use the formula from 2.d(ii), but with a general section $\tilde{\sigma}$ instead of $\tilde{\sigma}^l_m$, which then reads

$$\mathscr{D}_\alpha (s_l \circ \mathrm{tri}) \cdot \pi^*_{\bar{a}}(\tilde{\sigma}) = \lambda_l (s_l \circ \mathrm{tri}) \mathtt{i} c_{e_1} \pi^*_{\bar{a}}(\tilde{\sigma}) + (s_l \circ \mathrm{tri}) \pi^*_{\bar{a}} \big(\widetilde{\mathscr{D}}_{\tilde{\alpha}^l_L} \tilde{\sigma} \big). \tag{2.e-1}$$

We choose $\tilde{\sigma}$ to be

$$\Big(\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \Big) \tilde{\sigma}^{l+}_m + \Big(-\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \Big) \tilde{\sigma}^{l-}_m,$$

as in the definition of $\sigma^{\pm}_{l,m}$. Also remember that $c_{e_1} \tilde{\sigma}^{l+}_m = -\mathtt{i} \tilde{\sigma}^{l+}_m$ and $c_{e_1} \tilde{\sigma}^{l-}_m = \mathtt{i} \tilde{\sigma}^{l-}_m$.

Therefore, we have

$$\begin{aligned}
&\mathscr{D}_\alpha\big((s_l\circ\mathrm{tri})\cdot\pi_{\overline{a}}^*(\tilde\sigma)\big)\\
&\quad=\lambda_l\mathtt{i}(s_l\circ\mathrm{tri})\,\pi_{\overline{a}}^*\Big(\Big(\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)(-\mathtt{i})\,\tilde\sigma_m^{l+}+\Big(-\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\mathtt{i}\,\tilde\sigma_m^{l-}\Big)\\
&\qquad+(s_l\circ\mathrm{tri})\,\pi_{\overline{a}}^*\Big(\mu_m\Big(-\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l+}+\mu_m\Big(\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l-}\Big)\\
&\quad=(s_l\circ\mathrm{tri})\,\pi_{\overline{a}}^*\Big(\lambda_l\Big(\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l+}-\lambda_l\Big(-\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l-}\Big)\\
&\qquad+(s_l\circ\mathrm{tri})\,\pi_{\overline{a}}^*\Big(\mu_m\Big(-\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l+}+\mu_m\Big(\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l-}\Big)\\
&\quad=\pm\sqrt{\lambda_l^2+\mu_m^2}\,(s_l\circ\mathrm{tri})\,\pi_{\overline{a}}^*\Big(\Big(\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l+}+\Big(-\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l-}\Big)\\
&\quad=\pm\sqrt{\lambda_l^2+\mu_m^2}\,(s_l\circ\mathrm{tri})\pi_{\overline{a}}^*(\tilde\sigma),
\end{aligned}$$

which proves the assertion for $M_\alpha^\pm$.

Now we look at M_α^0: For $0\le m\le h-1$ we know that $\mu_m=0$ and $\tilde\sigma_m^{l-}=0$. Therefore, by (2.e-1) we get the eigenvalue $\lambda_l=\nu_{l,m}^0$. In the other case we have $\lambda_l=0$, so that (2.e-1) reduces to multiplication by $\mu_m=\nu_{l,m}^0$. When both conditions are fulfilled, the eigenvalue becomes naught which is consistent with our definitions.

The next thing to check is that none of the sections vanishes. For M_α^0 this is clear from the definition. Therefore, assume that

$$(s_l\circ\mathrm{tri})\cdot\bigg(\Big(\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\pi_{\overline{a}}^*(\tilde\sigma_m^{l+})+\Big(-\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\pi_{\overline{a}}^*(\tilde\sigma_m^{l-})\bigg)=0.$$

Since $(s_l\circ\mathrm{tri})$ is never zero and $\pi_{\overline{a}}^*$ is injective, we drop them

$$\Big(\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l+}+\Big(-\lambda_l+\mu_m\pm\sqrt{\lambda_l^2+\mu_m^2}\Big)\tilde\sigma_m^{l-}=0.$$

The equation $\tilde\sigma_m^{l+}=0$ would imply $\tilde\sigma_m^{l-}=0$ (and the other way round) since $\widetilde{\mathscr{D}}_{\tilde\alpha_L^l}\tilde\sigma_m^{l\pm}=\mu_m\tilde\sigma_m^{l\mp}$. Thus none of them is zero and their coefficients have to vanish; but the difference of the coefficients is $2\lambda_l$ which is nonzero throughout $M_\alpha^\pm$. Hence our assumption can be rejected and we know that all elements in $M_\alpha^\pm\cup M_\alpha^0$ are eigensections.

At last we have to check that we have produced an orthogonal eigenbasis. The following assertion will be helpful:

$$\operatorname{span}\{\hat\sigma_{l,m},\mathscr{D}_\alpha\hat\sigma_{l,m}\}=\operatorname{span}\{\hat\sigma_{l,m},\hat\sigma_{l,-m+h-1}\}=\operatorname{span}\{\sigma_{l,m}^+,\sigma_{l,m}^-\}\quad\text{for every } l,m \text{ with } \lambda_l\neq 0 \text{ and } m\ge h. \tag{2.e-2}$$

Proof of assertion. First notice that all of these spaces are truly 2-dimensional, since $\hat\sigma_{l,m}$ is not an eigensection for $\mathscr{D}_\alpha$. Hence, it is enough to prove the inclusions

$$\operatorname{span}\{\hat\sigma_{l,m},\mathscr{D}_\alpha\hat\sigma_{l,m}\}\subset\operatorname{span}\{\hat\sigma_{l,m},\hat\sigma_{l,-m+h-1}\}\supset\operatorname{span}\{\sigma_{l,m}^+,\sigma_{l,m}^-\}\quad\text{for every } l,m \text{ with } \lambda_l\neq 0 \text{ and } n\ge h.$$

From (2.e-1) we know that $\mathscr{D}_\alpha \hat{\sigma}_{l,m}$ lies in the span of $\mathtt{i}c_{e_1}\hat{\sigma}_{l,m} = \hat{\sigma}_{l,-m+h-1}$ (the pairing mentioned at the beginning of this section) and $\hat{\sigma}_{l,m}$. This proves the first inclusion.

For the second inclusion notice that $\sigma^+_{l,m} - \sigma^-_{l,m}$ is a multiple of $\hat{\sigma}_{l,m}$. From there, we produce $(s_l \circ \mathrm{tri}) \cdot \pi^*_{\underline{a}}(\tilde{\sigma}^{l\pm}_m)$, which in consequence gives us $\hat{\sigma}_{l,-m+h-1}$. End of assertion

The assertion shows

- $\operatorname{span}\{\hat{\sigma}_{l,m} \mid (l,m) \in \mathbb{Z}^2\} \subset \operatorname{span} M^\pm_\alpha \cup M^0_\alpha$. Therefore, the second set is a dense subset of all sections.
- Since $\sigma^+_{l,m}, \sigma^-_{l,m}$ is an orthogonal eigenbasis for $\operatorname{span}\{\sigma^+_{l,m}, \sigma^-_{l,m}\}$ (eigensections for different eigenvalues) and all spaces of the form $\operatorname{span}\{\hat{\sigma}_{l,m}, \hat{\sigma}_{l,-m+h-1}\}$ are orthogonal to each other, we know that $M^\pm_\alpha \cup M^0_\alpha$ forms an orthogonal basis of the sections.

This proves the theorem. □

An important consequence of this theorem is the fact that the spectrum of $\mathscr{D}_\alpha$ only depends on $\alpha_\perp$, but not on $\alpha_{\shortparallel}$.

2.f The trivial Spin$^{\mathbb{C}}$ structure

In this section we look at the case $\hat{a} = 0$, i.e. at the Spin$^{\mathbb{C}}$ structure coming from a spin structure. The method of "projecting in direction of a" does not make sense in this context. We therefore make a direct calculation.

Let ∇^0 be the flat connection on $\underline{\mathbb{H}} = \underline{\mathbb{C}}_+ \oplus \underline{\mathbb{C}}_-$ and $\nabla^\alpha = \nabla^0 + \mathtt{i}\alpha$ for a harmonic one-form α. The basis e_1, e_2, e_3 of the trivialised tangent space will be defined by using the standard coordinate directions of x_1, x_2, x_3 on T^3. Now we want to calculate the spectrum and eigenbasis of $\mathscr{D}_\alpha$.

This problem is of course strongly related to the standard Laplace operator. We define

$$\sigma_b(x_1, x_2, x_3) := \exp\bigl(2\pi\mathtt{i}(b_1x_1 + b_2x_2 + b_3x_3)\bigr)$$

as a map from T^3 to $\mathbb{C}$ which is defined for every $b \in \mathbb{Z}^3$. It is easy to see (Stone-Weierstraß), that

$$\operatorname{span}\{\sigma^+_b = (\sigma_b, 0) \mid b \in \mathbb{Z}^3\} \cup \{\sigma^-_b = (0, \sigma_b) \mid b \in \mathbb{Z}^3\}$$

forms a dense subset of all sections of $\underline{\mathbb{H}}$, where the basis elements are orthonormal.

In our case, the odd and even part of $\mathscr{D}_\alpha$ (with respect to the splitting $\underline{\mathbb{C}}_1 \oplus \underline{\mathbb{C}}_2$) anti-commute; therefore, $\mathscr{D}_\alpha^2$ leaves $\underline{\mathbb{C}}_+$ and $\underline{\mathbb{C}}_-$ invariant. We get

$$\begin{aligned}\mathscr{D}_\alpha^2 \sigma_b^\pm &= \left(c_{e_1}\frac{\partial}{\partial x_1} + c_{e_2}\frac{\partial}{\partial x_2} + c_{e_3}\frac{\partial}{\partial x_3} + \mathrm{i} c_\alpha\right)^2 \sigma_b^\pm \\ &= -\left(\left(\frac{\partial}{\partial x_1} + \mathrm{i}\alpha_1\right)^2 + \left(\frac{\partial}{\partial x_2} + \mathrm{i}\alpha_2\right)^2 + \left(\frac{\partial}{\partial x_3} + \mathrm{i}\alpha_3\right)^2\right)\sigma_b^\pm\end{aligned}$$

Since $\frac{\partial}{\partial x_i}\sigma_b^\pm = 2\pi \mathrm{i} b_i \sigma_b^\pm$, we have:

$$\begin{aligned}&= -\left((2\pi \mathrm{i} b_1 + \mathrm{i}\alpha_1)^2 + (2\pi \mathrm{i} b_2 + \mathrm{i}\alpha_2)^2 + (2\pi \mathrm{i} b_3 + \mathrm{i}\alpha_3)^2\right)\sigma_b^\pm \\ &= \|2\pi b + \alpha\|^2 \sigma_b^\pm,\end{aligned}$$

where we consider b as a one-form. Now define $\beta = \beta(b,\alpha)$ to be the one-form $2\pi b + \alpha$. Therefore, we have found an orthonormal eigenbasis for $\mathscr{D}_\alpha^2$ for the respective eigenvalues $\|\beta\|$.

Theorem 2.f(i). *We get an orthogonal eigenbasis for $\mathscr{D}_\alpha$ as*

$$\begin{aligned}&\left\{\|\beta\|\sigma_b^+ - \mathscr{D}_\alpha\sigma_b^+ \,\middle|\, b \in \mathbb{Z}^3 \text{ with } \beta_2 \neq 0 \text{ or } \beta_3 \neq 0\right\} \\ \cup &\left\{\|\beta\|\sigma_b^+ + \mathscr{D}_\alpha\sigma_b^+ \,\middle|\, b \in \mathbb{Z}^3 \text{ with } \beta_2 \neq 0 \text{ or } \beta_3 \neq 0\right\} \\ \cup &\left\{\sigma_b^\pm \,\middle|\, \beta_2 = \beta_3 = 0\right\}.\end{aligned}$$

Furthermore, we have for $\beta_2 \neq 0$ or $\beta_3 \neq 0$:

$$\operatorname{span}\{\sigma_b^+, \sigma_b^-\} = \operatorname{span}\{\|\beta\|\sigma_b^+ - \mathscr{D}_\alpha\sigma_b^+, \|\beta\|\sigma_b^+ + \mathscr{D}_\alpha\sigma_b^+\}.$$

The spectrum consists of all numbers $\pm\|\beta(b,\alpha)\|$ for $b \in \mathbb{Z}^3$.

Proof. The structure of the proof is the same as in the last section: First we have to check that all vectors are mapped by $\mathscr{D}_\alpha$ to the specified multiples; this is immediately clear from the calculation above.

Then we have to check that none of the sections is zero. So assume that $\|\beta\|\sigma_b^+ \pm \mathscr{D}_\alpha\sigma_b^+$ vanishes. We know that

$$\mathscr{D}_\alpha\sigma_b^+ = \mathrm{i} c_\beta \sigma_b^+ = \left(\mathrm{i}\beta_1 c_{e_1} + \mathrm{i}\beta_2 c_{e_2} + \mathrm{i}\beta_3 c_{e_3}\right)\sigma_b^+.$$

The minus part of our section (which is zero) is given by $\pm\left(i\beta_2 c_{e_2} + \mathrm{i}\beta_3 c_{e_3}\right)\sigma_b^+$; the first summand of this sum is an imaginary multiple of σ_b^- whereas the second one is a real multiple of σ_b^-. Hence we have $\beta_2 = \beta_3 = 0$ which is impossible in this case.

As in 2.e(ii), the density follows from

$$\operatorname{span}\{\sigma_b^+, \sigma_b^-\} = \operatorname{span}\{\|\beta\|\sigma_b^+ - \mathscr{D}_\alpha\sigma_b^+, \|\beta\|\sigma_b^+ + \mathscr{D}_\alpha\sigma_b^+\},$$

which we now want to show. For that we have the following ad hoc formulas:

$$\left(\|\beta\|\sigma_b^+ - \mathtt{i}c_\beta\sigma_b^+\right)\left(\frac{\beta_3+\beta_2\mathtt{i}}{\|\beta\|-\beta_1}\right) = \|\beta\|\sigma_b^- - \mathtt{i}c_\beta\sigma_b^-$$
$$\left(\|\beta\|\sigma_b^+ + \mathtt{i}c_\beta\sigma_b^+\right)\left(-\frac{\beta_3+\beta_2\mathtt{i}}{\|\beta\|+\beta_1}\right) = \|\beta\|\sigma_b^- + \mathtt{i}c_\beta\sigma_b^-.$$

Therefore, we can construct σ_b^+ and also σ_b^- as linear combinations of $\|\beta\|\sigma_b^+ - \mathscr{D}_\alpha\sigma_b^+$ and $\|\beta\|\sigma_b^+ + \mathscr{D}_\alpha\sigma_b^+$. This proves the assertion and shows analogously to 2.e(ii) that the set defined in the theorem forms an orthogonal eigenbasis.

At last, we want to calculate the spectrum. For all b with $\beta_2 \neq 0$ or $\beta_3 \neq 0$ we obviously get one eigenvector for $\|\beta\|$ and one for $-\|\beta\|$. If $\beta_2 = \beta_3 = 0$ we get the eigenvalues $(2\pi b_1 + \alpha_1)$ for σ_b^+ and $-(2\pi b_1 + \alpha_1)$ for σ_b^-; so one of them coincides with $\|\beta\|$ and one with $-\|\beta\|$. So for each b with $\beta \neq 0$ we have exactly two eigenvalues. □

2.g Spectral sections

In 1.c.1 we explained the framework for spectral sections on $\mathscr{L}/\ell$ for $\mathscr{D}$. Using our calculations above we will pursue concrete constructions of them. For that purpose we will look explicitly (but not exclusively) at the case where T^3 is a boundary of a four-manifold X (and ℓ is induced by $H^1(X;\mathbb{Z})$). This case will be called *boundary case*.

We already know that in the boundary case we have (1.b-1). Interpreting $c_1(K) = \hat{a}$ and ℓ as elements or subsets of $\mathbb{R}^3$ we can equivalently say:

$$\langle \hat{a}, \ell\rangle = 0$$

or, since $\hat{a} = h \cdot a$:

$$\langle a, \ell\rangle = 0 \iff a \perp \ell. \tag{2.g-1}$$

On T^3 there is one further restriction coming from the boundary case ([Bauer09]):

Lemma 2.g(i). *Assume we have the boundary case for T^3. Then* $\dim \ell \leq 2$.

Proof. Assume the converse: $\iota^*\colon H^1(X;\mathbb{Z}) \to H^1(T^3;\mathbb{Z})$ is surjective. Since ι^* is an algebra homomorphism and $H^*(T^3;\mathbb{Z})$ is generated by $H^1(T^3;\mathbb{Z})$, we also know that

ι^* is surjective in dimension 2 and 3. We use the following commutative diagram ([Lück05], p.150)

$$\begin{array}{ccc} H^2(X;\mathbb{Z}) & \xrightarrow{\iota^*} & H^2(T^3;\mathbb{Z}) \\ \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} \\ H_2(X,T^3;\mathbb{Z}) & \xrightarrow{\delta} & H_1(T^3;\mathbb{Z}) \end{array}$$

where PD means Poincaré duality and δ is the connecting homomorphism from the long exact homology sequence. This shows that δ is surjective and therefore we get by $\mathrm{Hom}(\bullet,\mathbb{Z})$ an injective map

$$\overline{\delta} : H^1(T^3;\mathbb{Z}) \to H^2(X,T^3;\mathbb{Z})$$

by the universal coefficient theorem (see [Lück05], p.107). Now look at the long exact cohomology sequence

$$\ldots \to H^1(X;\mathbb{Z}) \xrightarrow{\iota^*} H^1(T^3;\mathbb{Z}) \xrightarrow{\overline{\delta}} H^2(X,T^3;\mathbb{Z}) \to \ldots$$

Since ι^* is surjective and $\overline{\delta}$ is injective, $H^1(T^3;\mathbb{Z})$ has to be zero: contradiction. □

Now we will give explicit constructions of spectral sections.

2.g.1 Spectral sections for $\hat{a} \neq 0$

Theorem 2.g(ii). *For $\hat{a} \neq 0$, the family $\mathscr{D}$ of first order operators over $\mathscr{L}/\ell$ has a spectral section if and only if (2.g-1) is fulfilled. In the case of existence, the spectral projection $\Pi^+(\mathscr{D})$ provides (the simplest) example.*

Proof. First assume that ℓ is not orthogonal to a. From all elements $\alpha \in \ell$ with $\alpha \not\perp a$, we choose one with minimal length and call it $\alpha_{\min}$ (remember that ℓ is discrete). We furthermore assume that the coefficient in $\alpha^{\perp}$ direction called $x^a{}_{\min}$ is positive (otherwise we replace $\alpha_{\min}$ by $-\alpha_{\min}$).

The interval $[-\alpha_{\min}, \alpha_{\min}]$ now forms an embedded $S^{[1]}$ in $\mathscr{L}/\ell$, for which we will calculate the spectral flow.

So we look at the eigenvalues of $\mathscr{D}_{t\alpha_{\min}}$ while t runs from -1 to 1 and count how many of them pass zero upwards and downwards. From 2.e(ii) we know that we have two sets of eigensections called $M^{\pm}_{t\alpha_{\min}}$ for $\pm\sqrt{\lambda_l^2 + \mu_m^2}$ and $M^0_{t\alpha_{\min}}$ for λ_l or μ_m. Notice that μ_m is independent of the choice of t while λ_l depends on it linearly (through its dependence on x^a).

For all elements of $M^{\pm}_{t\alpha_{\min}}$ we know that $|\nu^{\pm}_{l,m}| \geq \mu_h$, where μ_h is the lowest positive eigenvalue of the two-dimensional Dirac operator. This is true because in this set we assume μ_m to be non-zero. Hence eigenvalues coming from $M^{\pm}_{t\alpha_{\min}}$ cannot cross zero while t runs through $[-1,1]$. They are therefore unimportant for the spectral flow.

The set $M^0_{t\alpha_{\min}}$ decomposes into two classes: When $\mu_m \neq 0$, we can argue in the same way as above. For $\mu_m = 0$, we know that $\nu^0_{l,m} = \lambda_l = (2\pi l + t x^a{}_{\min})\|a\|$. Since this function is growing with t, the spectral flow has to be non-negative. It is even strictly positive, since for $l = 0$ we have h "flow-lines" which cross zero for $t = 0$.

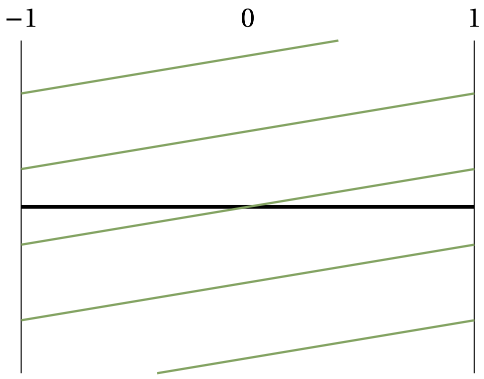


Figure 2.2: The spectral flow shown schematically.

Now 1.c(vi) tells us that $\mathscr{D}$ cannot have a spectral section.

At last consider the case where ℓ is orthogonal to a. Now x^a is always naught, so both μ_m and λ_l are independent of α. Therefore, we have a constant spectrum throughout $\mathscr{L}/\ell$, which is the trivial case of a spectral gap (see 1.c(v)). □

2.g.2 Spectral sections for $\hat{a} = 0$

From 2.f we know that the spectrum moves in every direction.

Lemma 2.g(iii). *There is no spectral gap for $\mathscr{D}$ for $\hat{a} = 0$.*

Proof. Assume we have a spectral gap $\tau : \mathscr{L}/\ell \to \mathbb{R}$. Then we would also have a spectral gap $\tau_{\mathscr{L}} : \mathscr{L} \to \mathbb{R}$ for the family of Dirac operators without the quotient relation. Now choose $\alpha \in \mathscr{L}$ so that $\|\beta(\alpha,0)\| = \|\alpha\| > \max_{\mathscr{L}/\ell} \|\tau\|$. The map

$$\begin{array}{ccc} \left\{t\alpha \middle| t \in \mathbb{R}\right\} & \xrightarrow{f} & \left\{\mathrm{Spec}(\mathscr{D}_{t\alpha}) \middle| t \in \mathbb{R}\right\} \\ t\alpha & \mapsto & \mathrm{sgn}(t)\|\beta(t\alpha, 0)\| \end{array}$$

is a continuous map from a line in $\mathscr{L}$ to the spectrum of $\mathscr{D}$. For $t = -1$, we know that $f((-1)\cdot\alpha)$ is smaller than $\tau_{\mathscr{L}}(-\alpha)$, whereas for $t = 1$ we have $f(1\cdot\alpha)$ is greater than $\tau_{\mathscr{L}}(\alpha)$. Hence both functions have to coincide at some point; therefore, $\tau_{\mathscr{L}}$ cannot be a spectral gap. Thus spectral gaps cannot exist in this case. □

This shows that there is a fundamental difference between the two cases $\hat{a} \neq 0$ and $\hat{a} = 0$. For the rest of the section we assume the dimensional restriction of 2.g(i).

Let $\Sigma_b := \operatorname{span}\{\sigma_b^+, \sigma_b^-\}$. Then the space of sections of $\underline{\mathbb{H}}$ is the orthogonal sum of all these spaces. We now want to define spectral projections P_α for a given spectral radius $R > 0$, which will be separately defined on each of the spaces Σ_b. For that purpose we define maps $\overline{P}_\beta : \Sigma_b \to \Sigma_b$ depending on the variable β as in 2.f(i) (we can do that since for fixed b the variables α and β are in bijective correspondence). Our construction will be the same for all b; we will identify $\Sigma_b \cong \mathbb{C}^2$ by above basis when necessary.

From 2.f(i), we know that for $\beta \neq 0$ the space Σ_b decomposes into two 1-dimensional spaces which have eigenvalues $\|\beta\|$ and $-\|\beta\|$. Therefore, for $\|\beta\| > R$ the map $\overline{P}_\beta$ is already defined as a projection with one-dimensional image.

We make a short digression to the space of orthogonal projections. Let $\mathscr{P}_2(\Sigma_b)$ be the space of orthogonal projections of $\Sigma_b = \operatorname{span}\{\sigma_b^+ \, \sigma_b^-\}$ to itself. The Grassmanian $\mathrm{G}_i(\Sigma_b)$ for $i = 0, 1, 2$ can be seen as the space of orthogonal projections onto an i-dimensional subspace of Σ_b and can therefore be continuously embedded into $\mathscr{P}_2(\Sigma_b)$. Since all three Grassmanians are compact, we know that they lie in different connected components of $\mathscr{P}_2(\Sigma_b)$.

For $\|\beta\| > R$ we know that $\overline{P}_\beta$ lies in $\mathrm{G}_1(\Sigma_b)$. Since $\overline{P}_\beta$ depends continuously on β, it cannot leave the component $\mathrm{G}_1(\Sigma_b)$ of $\mathscr{P}_2(\Sigma_b)$ while β moves.

Since $\mathrm{G}_1(\Sigma)$ is isomorphic to $\mathbb{P}^1$, our task is to find a map

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\overline{P}} & \mathbb{P}^1 \\ \beta & \mapsto & \overline{P}_\beta \end{array}$$

which coincides with the given projection for $\|\beta\| > R$ and is compatible with taking the quotient $\mathscr{L}/\ell$.

The first question is: What does the image of $\overline{P}$ look like for $\|\beta\| > R$? From 2.f(i) we know that (taking σ_b^+, σ_b^- as a basis of Σ_b) the map is

$$\begin{array}{ccc} \{\beta \in \mathscr{L} \mid \|\beta\| > R\} & \xrightarrow{\overline{P}} & \mathbb{P}^1 \\ \beta & \mapsto & \begin{cases} [\|\beta\| - \beta_1, \beta_3 - \beta_2 \mathtt{i}] & \text{for } \beta_3 \neq 0 \text{ or } \beta_2 \neq 0 \\ [1, 0] & \text{otherwise} \end{cases} \end{array} \tag{2.g-2}$$

which is continuous, since $(\beta_3 - \beta_2 \mathtt{i})/(\|\beta\| - \beta_1)$ goes to zero for $\beta_2, \beta_3 \to 0$. We see that the image of β does not change if we multiply it with a constant factor. If we fix the norm of β, say $\|\beta\| = R + 1$, we get an embedding of a $(\dim \mathscr{L} - 1)$-dimensional sphere into $\mathbb{P}^1$.

Our task is therefore reduced to the following topological problem: Continue a map $S^{\dim \mathscr{L} - 1} \to \mathbb{P}^1$ to the whole disk $D^{\dim \mathscr{L}}$. Since we assumed $\mathscr{L}$ to have dimension less or equal to two, this is always possible.

The next thing we have to check is the compatibility with taking quotients. For that, the following diagram has to commute (with $l \in \ell$):

$$\begin{array}{ccc} \Sigma_b & \xrightarrow{\overline{P}_{\beta(b,\alpha)}} & \Sigma_b \\ {\scriptstyle \cdot\exp(\langle l,x\rangle)}\downarrow & & \downarrow{\scriptstyle \cdot\exp(\langle l,x\rangle)} \\ \Sigma_{b+l} & \xrightarrow{\overline{P}_{\beta(b+l,\alpha-2\pi l)}} & \Sigma_{b+l} \end{array}$$

Since $\beta(b,\alpha) = \beta(b+l,\alpha-2\pi l)$ and $\exp(\langle l,x\rangle)$ identifies $\sigma_b^\pm$ with $\sigma_{b+l}^\pm$, this is true for all elements of the lattice.

Of course, we also have to check continuity with respect to α and that P_α is a self-adjoint projection and a pseudo-differential operator of degree 0. We postpone this to the more general discussion of subsection 2.h.1.

The following lemma summarises the results:

Lemma 2.g(iv). *For every $R > 0$, we can define a spectral projection P_α on $\mathscr{L}/\ell$ by separately defining one-dimensional projections on each of the 2-dimensional spaces Σ_b.*

2.h Spectral sections in K-theory

Now we consider the (formal) difference of spectral sections in K-theory. Two different spectral sections P and P' produce an element $\mathtt{Im}\,P - \mathtt{Im}\,P' \in K(\mathscr{L}/\ell)$ (For this to be well-defined in general you need to take an auxiliary bundle as explained in Lemma 7, [Melrose97]). For $\dim \mathscr{L} \le 1$, the space $K(\mathscr{L}/\ell)$ contains no information except for the dimension of the vector bundles involved. Therefore, we will only investigate the case $\dim \mathscr{L} = 2$.

We have $K(\mathscr{L}/\ell) \cong \mathbb{Z}^2$ (see [Cuntz07, p.69]). The isomorphism is given by the Chern character, which here reads (with addition in the Cohomology ring)

$$V \mapsto \dim(V) + c_1(V).$$

For further investigations (especially the construction of a system of infinitesimal spectral sections), we will distinguish the two main cases.

2.h.1 K-theory for $\hat{a} = 0$

Since we had imposed no assumptions on R when we constructed spectral sections in 2.g.2, we know that $R_{\text{inf}} = 0$ (see definition 1.c(vii)).

For a system of infinitesimal spectral sections, we have to choose a parameter ε_P (see 1.c(vii)). For that we define $\ell_{\mathbb{Z}} := \mathscr{L} \cap \mathbb{Z}^3$ and state the following lemma:

Lemma 2.h(i). *If we denote the Euclidean distance of $b \in \mathbb{Z}^3$ to $\mathscr{L}$ by $d(b, \mathscr{L})$, then the number*

$$d_{\min} := \min_{b \in \mathbb{Z}^3 \setminus \ell_{\mathbb{Z}}} d(b, \mathscr{L})$$

is well-defined and positive.

Proof. Take a $\mathbb{Z}$-basis ℓ_1, ℓ_2 of ℓ and add a vector $\ell^\perp \in \mathbb{Z}^3$ with $\ell \perp \ell^\perp$. Then every vector in $\mathbb{Z}^3$ can be written in this basis with coefficients from $\frac{1}{n}\mathbb{Z}$, where $n = \det(\ell_1, \ell_2, \ell^\perp)$. Since the distance of b to $\mathscr{L}$ is given by the norm of the coefficient of $\ell^\perp$, we know that this distance is at least $\frac{1}{n}$ if b is not in $\ell_{\mathbb{Z}}$. □

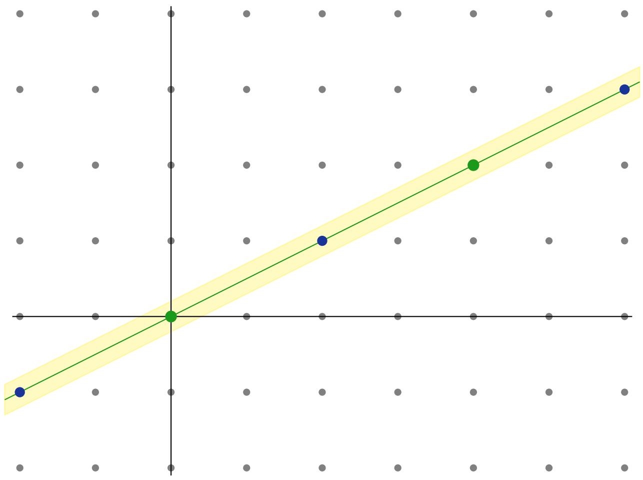


Figure 2.3: The green line represents $\mathscr{L}$ (which is, of course, two-dimensional in reality), the green dots indicate ℓ, while the green and blue dots together form $\ell_{\mathbb{Z}}$. The yellow area is the ε_P neighbourhood of $\mathscr{L}$

Now choose $\varepsilon_P < \min\{\frac{d_{\min}}{4\pi}, \frac{1}{1000}\}$. Furthermore, we choose the index set $I = \{g : \ell_{\mathbb{Z}}/\ell \to \mathbb{Z}\}$, which can be seen as the finite dimensional free $\mathbb{Z}$-module $\mathbb{Z}^{|\ell_{\mathbb{Z}}/\ell|}$. The spectral section P_R^g will be constructed by defining it separately for each $\alpha \in \mathscr{L}$ and $b \in \mathbb{Z}^3$ as map

$$(P_R^g)_\alpha|_{\Sigma_b} : \Sigma_b \to \Sigma_b.$$

After that we check

- that the map is well-defined for $\alpha \in \mathscr{L}/\ell$,
- that $(P_R^g)_\alpha$ depends continuously on α and R, and

- that that P_R^g is indeed a family of self-adjoint projections fulfilling the properties of a spectral section for the constant map R.

At the end we prove that this system represents every spectral section for $R < \varepsilon_P$ in the sense of condition 4 (infinitesimal spectral sections) and therefore forms a system of infinitesimal spectral sections.

Now to the actual construction: We will again consider the vector $\beta = \alpha + 2\pi b$ in $\mathbb{R}^3$ and define $(P_R^g)_\beta|_{\Sigma_b}$ depending on it. So we consider g and $R < \varepsilon_P$ to be fixed during the next paragraphs. First assume that $b \notin \ell_{\mathbb{Z}}$. Then $\|\beta\| > R$ because of the choice of ε_P, so $(P_R^g)_\beta|_{\Sigma_b}$ is already defined (as the projection onto the eigenspace for $\|\beta\|$). The picture 2.3 shows a 2-dimensional analogon to our 3-dimensional situation.

Now assume $b \in \ell_{\mathbb{Z}}/\ell$ and let $i = g(b)$.

Each Σ_b has a standard basis σ_b^+, σ_b^- which we use to identify it with $\mathbb{C}^2$. Therefore, the space of projections onto a one-dimensional subspace can be written as $\mathbb{P}^1$, which will be identified with the 2-sphere. For $\|\beta\| \geq R/2$, the projection $\left(P_R^g\right)_\alpha|_{\Sigma_b}$ is chosen to be the map given by (2.g-2) (blue area in figure 2.4). For $R/4 \leq \|\beta\| \leq R/2$ we take the usual map from an annulus to a disk (green area), which is given by mapping the circle $\|\beta\| = R/4$ onto the point $*$ (indicated by red colour), which is arbitrary fixed point not hit by the circle $\|\beta\| = R/2$.

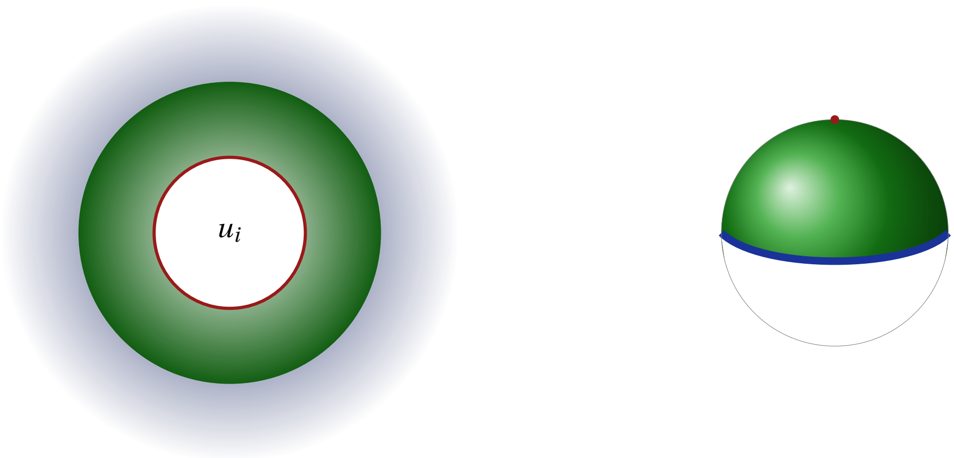


Figure 2.4: The colours indicate which area in the plane is mapped to which part of $\mathbb{P}^1 \cong S^2$

Let $u_i : S^2 \to S^2$, $i \in \mathbb{Z}$, be a map which represents the class of i in $\pi_2(S^2) \cong \mathbb{Z}$. If we identify $\mathbb{P}^1$ and S^2 and take $*$ as a base point, we can choose $\left(P_R^g\right)_\alpha|_{\Sigma_b}$ on $\|\beta\| \leq R/4$ to be u_i.

For $\left(P_R^g\right)_\alpha$ to be well-defined on $\mathscr{L}/\ell$, we have to check that our definition is consistent with the action of ℓ. But this is clear from the diagram in the previous section since g does not change under the action of ℓ.

It is also clear that each $\left(P_R^g\right)_\alpha$ is a self-adjoint projection since it consists of orthogonal projections from $\Sigma_b \to \Sigma_b$.

The next thing we want to check is the continuity:

Lemma 2.h(ii). *If we consider $\left(P_R^g\right)_\alpha$ as lying in the normed space of bounded linear maps from $\Gamma_{L^2}(\underline{\mathbb{H}})$ to itself, then the maps*

$$\begin{array}{rcl} \mathscr{L} & \to & \mathscr{B}\Big(\Gamma_{L^2}(\underline{\mathbb{H}}), \Gamma_{L^2}(\underline{\mathbb{H}})\Big) \\ \alpha & \mapsto & \left(P_R^g\right)_\alpha \end{array}$$

and

$$\begin{array}{rcl}]0, \varepsilon_P] & \to & \mathscr{B}\Big(\Gamma_{L^2}(\underline{\mathbb{H}}), \Gamma_{L^2}(\underline{\mathbb{H}})\Big) \\ R & \mapsto & \left(P_R^g\right)_\alpha \end{array}$$

are continuous.

Proof. We first show the continuity with respect to α. The assertion can be stated as follows:

$$\forall \varepsilon > 0 \, \exists \delta > 0 \quad \text{with} \quad \|\gamma\| < \delta \Rightarrow \left\| \left(P_R^g\right)_{\alpha+\gamma} - \left(P_R^g\right)_\alpha \right\| < \varepsilon.$$

For a fixed α there is at most one $b \in \mathbb{Z}^3$ (which we call b_α, if it exists) with $\|\beta(b,\alpha)\| < R$ (R is much smaller than the distance between lattice points). We will assume that δ is so small, that $b_\alpha = b_{\alpha+\gamma}$ for all $\|\gamma\| < \delta$.

At first we will prove the following assertion:

If $\delta < \varepsilon \frac{R}{2}$, then we have

$$\left\| \left(P_R^g\right)_{\alpha+\gamma}(\sigma_b^\pm) - \left(P_R^g\right)_\alpha(\sigma_b^\pm) \right\| < \varepsilon \quad \textit{for} \quad b \neq b_\alpha.$$

Proof of assertion. We use the fact that both maps are on Σ_b given as the projection onto the positive eigenspace. Looking at the eigenbasis calculated in 2.f(i) we get:

$$\begin{aligned} \left\| \left(P_R^g\right)_{\alpha+\gamma}(\sigma_b^\pm) - \left(P_R^g\right)_\alpha(\sigma_b^\pm) \right\| &= \left\| \sigma_b^+ - \frac{\mathtt{i} c_{\beta+\gamma}\sigma_b^+}{2\|\beta+\gamma\|} - \sigma_b^+ + \frac{\mathtt{i} c_\beta \sigma_b^+}{2\|\beta\|} \right\| \\ &= \left\| \mathtt{i}\Big(\frac{-1}{2\|\beta+\gamma\|} + \frac{1}{2\|\beta\|}\Big) c_\beta \sigma_b^+ - \frac{\mathtt{i}}{2\|\beta+\gamma\|} c_\gamma \sigma_b^+ \right\| \\ &\le \left\| \frac{\|\beta+\gamma\| - \|\beta\|}{2\|\beta\| \, \|\beta+\gamma\|} \right\| \|\beta\| + \frac{\|\gamma\|}{2\|\beta+\gamma\|} \\ &\le \frac{2\|\gamma\|}{2\|\beta+\gamma\|} \le 2\frac{\delta}{R} < \varepsilon. \end{aligned}$$

For σ_b^- this works analogously. End of assertion

Since $\left(P_R^g\right)_\alpha|_{\Sigma_{b_\alpha}}$ depends continuously on α (if b_α exists), we can choose δ so small, that the condition

$$\left\|\left(P_R^g\right)_{\alpha+\gamma}(\sigma_b^\pm)-\left(P_R^g\right)_\alpha(\sigma_b^\pm)\right\|<\varepsilon$$

is fulfilled for all $b\in\mathbb{Z}^3$.

We use the following standard lemma:

Lemma 2.h(iii). *Let A be a bounded linear operator from a separable Hilbert space H to itself and e_i, $i\in\mathbb{Z}$ an orthonormal Hilbert space basis. Then the following two conditions are equivalent:*

(i) $\|A(e_i)\|<\varepsilon$ *for all* $i\in\mathbb{Z}$.

(ii) $\|A(h)\|<\varepsilon$ *for all* $h\in H$ *with* $\|h\|=1$.

Since $\sigma_b^\pm$ forms an orthonormal Hilbert space basis, we know that

$$\left\|\left(P_R^g\right)_{\alpha+\gamma}(h)-\left(P_R^g\right)_\alpha(h)\right\|<\varepsilon$$

is fulfilled for every $h\in\Gamma_{\mathrm{L}^2}\left(\underline{\mathbb{H}}\right)$ with $\|h\|=1$. Using the definition of the operator norm we see that continuity with respect to α is shown.

For continuity in R we use the same kind of argument. We choose δ so small that $\|\beta\|<R+r$ with $|r|<\delta$ can only be fulfilled for at most one b (called b_α). If we plug $\sigma_b^\pm$ for $b\neq b_\alpha$ into

$$\left(P_{(R+r)}^g\right)_\alpha(\sigma_b^\pm)-\left(P_R^g\right)_\alpha(\sigma_b^\pm),$$

we get zero. Since the map is continuous on Σ_{b_α}, we get continuity on the whole space. □

As P_α is an orthogonal projection on each of the Σ_b, it is self-adjoint on the whole space. Since the spectral projection is a pseudo-differential operator of degree 0 (see [Booss-Bavnbek93], p.106) and P_α differs from it in only finitely many dimensions, it is also an operator of this kind.

Therefore, the only thing left to show is:

Lemma 2.h(iv). *P_R^g is a representation system of the spectral sections for the constant R.*

Proof. Let P_R be an arbitrary spectral section. We look at $(P_R)_\alpha|_{\Sigma_b}$: If $\|\beta(\alpha,b)\|>R$, then this map is already defined as the projection onto the one-dimensional subspace which belongs to the eigenvalue $+\|\beta\|$. Since b comes from a lattice of minimal distance 1, we know that for fixed α, the inequality $\|\beta(\alpha,b)\|\leq R$ can only be fulfilled for at most one lattice element b (which we again call b_α, if it exists). Since for every

$b \neq b_\alpha$, the projection $(P_R)_\alpha$ leaves the space Σ_b invariant, it must also fix the space Σ_{b_α}. Therefore, we can investigate the maps $(P_R)_\alpha|_{\Sigma_b} : \Sigma_b \to \Sigma_b$.

Now we want to determine the right g so that $\left(\mathtt{Im}\, P_R^g - \mathtt{Im}\, P_R\right)|_{\Sigma_b}$ is zero in K-theory for all b. At first we want to represent this class by a difference of two genuine (and easy to handle) line bundles over $\mathscr{L}/\ell$. Consider b as fixed variable during the rest of the proof, and remember that the only "relevant" b are those from $\ell_{\mathbb{Z}}$.

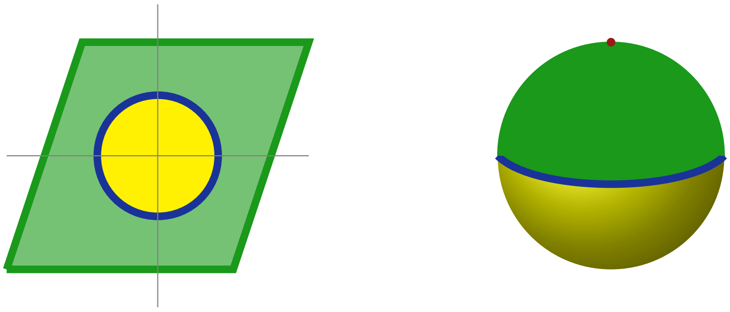


Figure 2.5: The map *kill* described by coloured areas.

Line bundles on $\mathscr{L}/\ell$ are classified by homotopy classes of maps $\mathscr{L}/\ell \to \mathbb{P}^1$ (higher dimensional cells of $\mathbb{P}^\infty$ are unimportant due to cellular approximation). If we again look at $P_R^g|_{\Sigma_b}$ and $P_R|_{\Sigma_b}$ as maps from $\mathscr{L}$ to $\mathbb{P}^1$, we see that they coincide for $\|\beta\| > R$ and map the set of β with $\|\beta\| = R + \varepsilon$ (for ε any reasonably small number) to a circle in $\mathbb{P}^1$. Let the map $\mathit{kill} : \mathscr{L}/\ell \to \mathbb{P}^1$ be given by killing the one-cells of $\mathscr{L}/\ell$; it should also map $\|\beta\| = R + \varepsilon$ to the same circle in $\mathbb{P}^1$ and the outer part of this circle to the disk in which $*$ lies. Now replace $P_R^g|_{\Sigma_b}$ and $P_R|_{\Sigma_b}$ outside of the $(R+\varepsilon)$-circle by *kill*. The resulting maps we receive shall be called $\widetilde{P}_R^g|_{\Sigma_b}$ and $\widetilde{P}_R|_{\Sigma_b}$.

Now $\left(\mathtt{Im}\, P_R^g - \mathtt{Im}\, P_R\right)|_{\Sigma_b}$ represents the same class in K-theory as $\left(\mathtt{Im}\, \widetilde{P}_R^g - \mathtt{Im}\, \widetilde{P}_R\right)|_{\Sigma_b}$. This is the difference of two line bundles over $\mathscr{L}/\ell$, represented as maps $\widetilde{P}_R^g|_{\Sigma_b}$ and $\widetilde{P}_R|_{\Sigma_b}$ from $\mathscr{L}/\ell$ to $\mathbb{P}^1$. This K-theory class is zero if and only if both line bundles have the same first Chern class.

The first Chern class can be determined by applying the pull-back map to the orientation class of $\mathbb{P}^1$ in $H^2(\mathbb{P}^1;\mathbb{Z})$. If we show that for every class in $H^2\left(\mathscr{L}/\ell;\mathbb{Z}\right)$ we find a g, so that $c_1(\mathtt{Im}\, \widetilde{P}_R^g|_{\Sigma_b})$ represents this class, we are finished.

This is equivalent to the following topological lemma: For that we replace $\mathbb{P}^1$ with the given circle by (S^2, S^1), where S^1 is the equator. $\square$

Lemma 2.h(v) ([Meier10])**.** *The homotopy classes of maps $f : (D^2, S^1) \to (S^2, S^1)$ with $f|_{S^1} = \mathrm{id}$ are in canonical one-to-one correspondence with $\pi_2(S^2) \cong \mathbb{Z}$. Here S^1 is considered as boundary of D^2 and an equator of S^2.*

Proof. We can exchange the condition $f|_{S^1} = \mathrm{id}$ by $f|_{S^1} \sim \mathrm{id}$ since we can pursue homotopies (relative to basepoints) near the equator. So we want to classify:

$$\mathrm{Map}_{S^2} := \{f : (D^2, S^1, *) \to (S^2, S^1, *) \,|\, f|_{S^1} \sim_* \mathrm{id}\}/\sim.$$

Generally we have:

$$\mathrm{Map}_{S^2} \subset \pi_2(S^2, S^1, *) = \{f : (D^2, S^1, *) \to (S^2, S^1, *)\}.$$

From [Hilton53], p.41, we know that $\pi_2(S^2, S^1) = \pi_2(S^2) \oplus \pi_1(S^1)$. In the long exact sequence of homotopy groups we get:

$$\begin{array}{ccccccccc} \pi_2(S^1) & \to & \pi_2(S^2) & \to & \pi_2(S^2, S^1) & \to & \pi_1(S^1) & \to & \pi_1(S^2) \\ \cong & & \cong & & \cong & & \cong & & \cong \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} & \to & \mathbb{Z} & \to & 0 \end{array}$$

Now Map_{S^2} can be described differently: It is the subspace of $\pi_2(S^2, S^1)$ which is mapped to 1 in $\pi_1(S^1)$ (this is clear from the definition of the sequence). Thus we have $\mathrm{Map}_{S^2} \cong \mathbb{Z} \oplus 1 \cong \mathbb{Z}$, parametrised by $\pi_2(S^2)$ through the standard map used in the sequence. □

Of course our system of infinitesimal spectral sections maybe replaced by a carefully chosen subset which also runs through all possible first Chern classes. But the general picture is that for small R, everything is fixed outside of very small neighbourhoods of lattice points. In these neighbourhoods, we have some freedom coming from $\pi_2(S^2)$, which allows us to represent different classes in K-theory.

2.h.2 K-theory for $\hat{a} \neq 0$

This case if very different from the preceding one. Over the only two dimensional space $\mathscr{L}$ with spectral sections (namely $\mathscr{L} = \hat{a}^{\perp}$) the spectrum of $\mathscr{D}$ is constant.

A system of infinitesimal spectral sections is therefore easily found: R_{inf} is obviously zero, and for ε smaller than the smallest eigenvalue $v^{+/0}_{l,m}$, there is just one spectral section (given by Π^+).

For greater R one might ask the question whether the eigenspaces for a fixed eigenvalue v form a bundle with non-trivial Chern class. The following lemma gives an answer.

Lemma 2.h(vi). *For fixed v the eigenspaces for v form a trivial bundle over $\mathscr{L}/\ell$.*

Proof. Instead of $\mathscr{L}/\ell$ we will consider $\mathscr{L}/2h\ell$ (we say at the end why this is enough).

For every $v = v^{\pm}_{l,m}$ or $v = v^0_{l,m}$ we consider the h-dimensional space $E^{\alpha}_{l,m}$ of eigenvectors for $\mathscr{D}_{\alpha}$ we have defined in 2.e; this may not be the whole eigenspace since several of the $v^{\pm/0}$ maybe equal, but if we show that any of the $E^{\alpha}_{l,m}$ forms a trivial bundle, we are done.

Over $\mathscr{L}$ the space $E^{\alpha}_{l,m}$ forms a trivial h-dimensional bundle. To show that it is still trivial after taking the quotient $\mathscr{L}/2h\ell$, we will at first replace $E^{\alpha}_{l,m}$ in several steps by simpler bundles; after that we will describe non-vanishing sections coming from the theta functions.

The steps are:

1. *If* $EE^{\alpha}_{l,m} = \operatorname{span}\{\hat{\sigma}^{\alpha}_{l,mh}, \dots, \hat{\sigma}^{\alpha}_{l,mh+h-1}\}$ *is a trivial bundle (over the quotient space), then also* $E^{\alpha}_{m,n}$*:* This is trivial if $\lambda_l = 0$ or $\mu_m = 0$. So we consider the other case. For fixed α the spaces $EE^{\alpha}_{l,m}$ and $E^{\alpha}_{l,m}$ are subspaces of $\Gamma\big(K \oplus K\big)$. We define the fibre preserving map

$$\begin{pmatrix} \lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} & 0 \\ 0 & -\lambda_l + \mu_m \pm \sqrt{\lambda_l^2 + \mu_m^2} \end{pmatrix}$$

   from $EE^{\alpha}_{l,m}$ to $E^{\alpha}_{l,m}$ for all α. This gives us a bundle equivalence (since none of the λ_l and μ_m vanish).

2. *If* $EEE^{\alpha}_{l,m} = \operatorname{span}\{(\sigma^{mh}_l)^{\alpha}, \dots, (\sigma^{mh+h-1}_l)^{\alpha}\}$ *forms a trivial bundle, then also* $EE^{\alpha}_{m,n}$*:* From the definition we know

$$\hat{\sigma}^{\alpha}_{l,\hat{m}}(v) := (s_l \circ \operatorname{tri}_2)(v) \cdot \pi^*_{\bar{a}}\big((\sigma^{\hat{m}}_l)^{\alpha}\big)(v) \quad \text{for all } \hat{m}.$$

   Since l is fixed, we get a map $\Gamma(L \oplus L) \to \Gamma(K \oplus K)$, which maps $EEE^{\alpha}_{m,n}$ to $EE^{\alpha}_{m,n}$ injectively (and therefore bijectively). This again induces a bundle equivalence.

3. *If* $EEE^{\alpha}_{0,0}$ *is trivial, then also* $EEE^{\alpha}_{m,n}$*:* At first we transform $EEE^{\alpha}_{m,n}$ into $EEE^{\alpha}_{m,0}$. For that purpose we look at 2.c.6. The space $\mathscr{I}^{-1}EEE^{\alpha}_{m,0}$ can be mapped by $\frac{1}{2\sqrt{E_q}}\big(\sqrt{E_q} + \sqrt{2}\bar{\partial}_+\big)\partial_+ \dots \partial_+$ into $\mathscr{I}^{-1}EEE^{\alpha}_{m,n}$. Following [Almorox06], this is injective and induces a bundle isomorphism between $EEE^{\alpha}_{m,n}$ and $EEE^{\alpha}_{m,0}$. From the definition of σ^n_m we also know that $EEE^{\alpha}_{m,0} = EEE^{\alpha+2\pi \mathrm{i}\omega^m_L}_{0,0}$. This is just a shift; if one of these bundles is trivial, then the other one has be trivial as well.

4. $EEE^{\alpha}_{0,0}$ *is trivial:* The bundle $\mathscr{I}^{-1}EEE^{\alpha}_{0,0}$ is the bundle of holomorphic sections of L^c_h, where c is defined as in 2.c.4.

   If we scale $\vartheta^c_{k,h}$ by an appropriate (non-vanishing holomorphic) function on $\mathscr{L}/2h\ell$, we get a non-vanishing section of $\mathscr{I}^{-1}EEE^{\alpha}_{0,0}$: To see this take an element $\alpha_{2h\ell}$ from the lattice $2h\ell$ and define $c_{2h\ell}$ to be the corresponding characteristic. Then we have to compare $\mathscr{I}^{-1}\vartheta^{c+c_{2h\ell}}_{k,h}$ with $\mathscr{I}^{-1}\vartheta^c_{k,h}$ which is an easy calculation.

   Since the $\vartheta^c_{k,h}$ are linear independent for different k, we get a basis on non-vanishing sections.

Hence we have shown that the bundle $E^{\alpha}_{l,m}$ is trivial over $\mathscr{L}/2h\ell$. We now take the canonical covering space $\pi_{2h} : \mathscr{L}/2h\ell \to \mathscr{L}/\ell$. The induced map π^*_{2h} in cohomology is just multiplication by $2h$. So the pull-back of any bundle under π^*_{2h} is trivial if and only if the original bundle was trivial as well. This shows that $E^{\alpha}_{l,m}$ is also trivial over $\mathscr{L}/\ell$. □

2.i Directions for generalisation

One question one might ask is what happens to our eigenvalues if we look at more general (i.e. non-closed) one-forms α. Here we again look at the splitting $\alpha_\perp + \alpha_{\shortparallel}$. Theorems about holomorphic structures on 2-tori indicate that it should be possible to construct holomorphic structures for forms $\alpha_{\shortparallel}$ which are not closed. Then the same method as above would construct eigensections on T^3 from the ones given on T_Λ. Changing $\alpha_\perp$ would result in x^a being non-constant; this can sometimes be repaired by solving an appropriate ODE.

An other direction might be to look at other manifolds of the form $S^1 \times S$ with S an arbitrary surface. For $S = S^2$ this is easy (we explain this in chapter 3). For S having genus greater than 1, there are partial results of [Almorox06], determining the smallest eigenvalues. Therefore, it might be possible to determine spectral sections through a similar method as in the case T^3.

For $\hat{a} \in H^2(S^1 \times S; \mathbb{Z})$ we have to find a trivial bundle structure $S^1 \times S_\Lambda$, which seems increasingly difficult as the dimension of $H^2(S^1 \times S; \mathbb{Z})$ grows.

Chapter 3

The space $S^1 \times S^2$

The case $S^1 \times S^2$ is in many points analogous (but easier) than the case T^3. We nevertheless present it in a separate chapter because

- A joined presentation with T^3 would have caused an unreadable amount of notation,
- many arguments are unnecessary in the case of $S^1 \times S^2$ and
- the 2-sphere has to be treated separately (e.g. it has non-trivial tangent bundle).

3.a Definitions

We choose S^1 to have the flat Riemannian metric and give S^2 the metric of the unit sphere in $\mathbb{R}^3$ (with constant scalar curvature 2). For the purpose of our calculation we identify S^2 with $\mathbb{P}^1$, where we use a scaled Fubini-Study metric.

The real cohomology ring is generated by $H^1(S^1;\mathbb{R})$ and $H^2(S^2;\mathbb{R})$. If we identify $H^1(S^1;\mathbb{R})$, $H^2(S^2;\mathbb{R})$ and $H^3(S^1 \times S^2;\mathbb{R})$ with $\mathbb{R}$ in the usual manner, we see that the cup product is just multiplication in $\mathbb{R}$ (look e.g. at the intersection product to prove this).

Here the one-form α is therefore an element of $\mathbb{R}$, whereas $\hat{a} = h$ is an element of $\mathbb{Z}$. The trivial S^1 bundle is given by the obvious projection $\pi : S^1 \times S^2 \to S^2$.

We now have to solve the eigenbasis problem on S^2 by the methods of [Almorox06]; as S^2 has trivial first cohomology, we know that $\alpha_{\shortparallel}$ is always zero and so we do not have to change holomorphic structures.

3.b The 2-dimensional sphere

As mentioned in the last section, we will consider $\mathbb{P}^1$ as a model for S^2. For the explicit calculations it is necessary to fix charts, give descriptions of the line bundles and also describe the tangent space and the metric structure. This will be the topic of the next subsections.

After that we want to solve the following problems:

1. Explicitly define a line bundle L on $\mathbb{P}^1$ with a background connection ∇_L for every first Chern class $h \in \mathbb{Z}$.

2. Calculate the spectrum and an eigenbasis for $\widetilde{\mathscr{D}} = \widetilde{\mathscr{D}}_0$.

For that we will use methods of [Almorox06]. Unlike the torus case, here the elliptic chain has a non-trivial structure since the tangent bundle of $\mathbb{P}^1$ is non-trivial.

3.b.1 Charts

For $\mathbb{P}^1$ we choose an atlas $\{V_0 \xrightarrow{\varphi_0} U_0, V_1 \xrightarrow{\varphi_1} U_1\}$ with

$$\begin{aligned} V_0 &= \{[z_0, z_1] \mid |z_1| < 2|z_0|\} \\ V_1 &= \{[z_0, z_1] \mid |z_0| < 2|z_1|\} \\ U_0 = U_1 &= \{z \in \mathbb{C} \mid |z| < 2\} \end{aligned}$$

and

$$\begin{aligned} \varphi_0 : V_0 &\to U_0 & \varphi_1 : V_1 &\to U_1 \\ [z_0, z_1] &\mapsto \frac{z_1}{z_0} & [z_0, z_1] &\mapsto \frac{z_0}{z_1} \\ [1, z] &\leftarrow\!\shortmid z & [z, 1] &\leftarrow\!\shortmid z \end{aligned}$$

The change of charts is given by

$$\varphi_1 \circ \varphi_0^{-1}(z) = \frac{1}{z} \quad \text{for } \frac{1}{2} < |z| < 2.$$

3.b.2 Line bundles

For every integer n, we want to define a line bundle $\mathscr{O}(n)$. At first this will done by means of trivialisations and trivialisation changes.

Let $\mathscr{O}(n)$ be trivial over V_0 and V_1. We call the (1-dimensional) bases of the two trivial bundles o_0^n and o_1^n respectively. The change of trivialisation from V_0 to V_1 should be given by

$$\begin{aligned} g_{10}^n : V_0 \cap V_1 &\to \mathrm{Gl}_1(\mathbb{C}) \\ [z_0, z_1] &\mapsto \left(\frac{z_0}{z_1}\right)^n \end{aligned}$$

and in the same manner

$$\begin{aligned} g_{01}^n : V_0 \cap V_1 &\to \mathrm{Gl}_1(\mathbb{C}) \\ [z_0, z_1] &\mapsto \left(\frac{z_1}{z_0}\right)^n . \end{aligned}$$

The bundles $\mathscr{O}(n)$ can also be constructed differently:
Let $\mathscr{O}(-1)$ be the subbundle of $\mathscr{O} \times \mathbb{C}^2$, which consists of all elements of the form $\big([z_0, z_1], t(z_0, z_1)\big)$ for $t \in \mathbb{C}$. For positive n let $\mathscr{O}(-n)$ be defined as the n-fold tensor product of $\mathscr{O}(-1)$. Then $\mathscr{O}(n)$ can be defined as the dual bundle for $\mathscr{O}(-n)$ $(n \in \mathbb{Z}^+)$.

To compare the new and old definition we write down the following trivialisations:

A trivialisation for $\mathscr{O}(-n)$ over V_i is given by

$$\big([z_0, z_1],\ t_1(z_0, z_1) \otimes t_2(z_0, z_1) \otimes \cdots \otimes t_n(z_0, z_1)\big) \mapsto \big([z_0, z_1],\ t_1 t_2 \cdots t_n\, z_i^n\big).$$

A dual trivialisation for $\mathscr{O}(n)$ can be calculated as

$$\big([z_0, z_1],\ \beta_1^* \otimes \cdots \otimes \beta_n^*\big) \mapsto \left([z_0, z_1],\ \beta_1^*(z_0, z_1) \cdots \beta_n^*(z_0, z_1)\, \frac{1}{z_i^n}\right).$$

Obviously, we have the same trivialisation changing morphisms as before.

Furthermore, we have the following isomorphism:

$$\mathscr{O}(n) \otimes \mathscr{O}(m) \xrightarrow{\cong} \mathscr{O}(n+m)$$

which is locally given by

$$o_i^n \otimes o_i^m \mapsto o_i^{n+m},$$

which is compatible with the change of trivialisation because of the equation $g_{10}^n \cdot g_{10}^m = g_{10}^{n+m}$.

3.b.3 Hermitian metric

Now we want to define a hermitian metric on $\mathcal{O}(n)$. For that purpose let

$$g^i_{\mathbb{C}}\big(o^n_i, o^n_i\big)_{[z_0,z_1]} = H^n_i\big([z_0,z_1]\big)$$

with

$$H_i\big([z_0,z_1]\big) = \frac{2|z_i|^2}{\big(|z_0|^2+|z_1|^2\big)},$$

which we define to be complex linear in the first component. This defines a hermitian metric $g_{\mathbb{C}}$ on $\mathcal{O}(n)$, since

$$\begin{aligned} g^0_{\mathbb{C}}\big(o^n_0, o^n_0\big)_{[z_0,z_1]} &= H^n_0\big([z_0,z_1]\big) \\ &= \left|\frac{z_0}{z_1}\right|^{2n} H^n_1\big([z_0,z_1]\big) \\ &= g^1_{\mathbb{C}}\left(\left(\frac{z_0}{z_1}\right)^n o^n_1, \left(\frac{z_0}{z_1}\right)^n o^n_1\right)_{[z_0,z_1]} \\ &= g^1_{\mathbb{C}}\big(o^n_0, o^n_0\big)_{[z_0,z_1]}. \end{aligned}$$

3.b.4 Tangent and cotangent space

Following [Osborn82], p.130, (generalized to $\mathbb{C}$) we have

$$\begin{aligned} g^{T\mathbb{P}^1}_{10} &= \left(\frac{\partial\big(\varphi_1\circ\varphi_0^{-1}\big)}{\partial z}\circ\varphi_0\big([z_0,z_1]\big)\right) \\ &= \left(\left(z\mapsto -\frac{1}{z^2}\right)\left(\frac{z_1}{z_0}\right)\right) = -\left(\frac{z_0}{z_1}\right)^2. \end{aligned}$$

We get an isomorphism from $\mathcal{O}(2)$ to $\mathrm{T}\mathbb{P}^1$ by

$$\begin{aligned} V_i\times\mathbb{C} &\xrightarrow{I_{\mathrm{T}}} V_i\times\mathbb{C} \\ \big([z_0,z_1],\alpha o^2_i\big) &\mapsto \left(\big([z_0,z_1],(-1)^i\left(\alpha_1\frac{\partial}{\partial x}+\alpha_2\mathtt{I}\frac{\partial}{\partial x}\right)\right), \end{aligned}$$

where $\mathtt{I}$ is the almost complex structure on $\mathrm{T}\mathbb{P}^1$ which is defined by $\mathtt{I}(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$ (coming from the complex structure on $\mathbb{P}^1$).

We scale the Fubini-Study metric of [Huybrechts05] by the factor 8π to force I_{T} to be an isometry (compare p.178).

In the same way, we get for $\mathrm{T}^*\mathbb{P}^1$:

$$g_{10}^{T\mathbb{P}^{1*}} = -\left(\frac{z_1}{z_0}\right)^2$$

and we have the isomorphism

$$\begin{aligned} V_i \times \mathbb{C} &\xrightarrow{I_{\mathrm{T}^*}} V_i \times \mathbb{C} \\ \left([z_0, z_1], \alpha o_i^{-2}\right) &\mapsto \left([z_0, z_1], (-1)^i\left(\alpha_1 dx + \alpha_2 dy\right)\right). \end{aligned}$$

We also consider the complexified tangent space $\mathrm{T}\mathbb{P}^1 \otimes_{\mathbb{R}} \mathbb{C}$ and cotangent space $\mathrm{T}\mathbb{P}^1 \otimes_{\mathbb{R}} \mathbb{C}$, which decompose as usual in the direct sums ([Huybrechts05], p.25/26):

$$\begin{aligned} \left(\mathrm{T}\mathbb{P}^1\right)^{10} &= \left\{v \,|\, \mathtt{I}v = \mathtt{i}v\right\} & \left(\mathrm{T}^*\mathbb{P}^1\right)^{10} &= \left\{f \in \mathrm{Hom}(T\mathbb{P}^1, \mathbb{C}) \,|\, f(\mathtt{I}(v)) = \mathtt{i}f(v)\right\} \\ \left(\mathrm{T}\mathbb{P}^1\right)^{01} &= \left\{v \,|\, \mathtt{I}v = -\mathtt{i}v\right\} & \left(\mathrm{T}^*\mathbb{P}^1\right)^{01} &= \left\{f \in \mathrm{Hom}(T\mathbb{P}^1, \mathbb{C}) \,|\, f(\mathtt{I}(v)) = -\mathtt{i}f(v)\right\}. \end{aligned}$$

Locally we get the following basis elements:

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathtt{i}\frac{\partial}{\partial y}\right) & dz &= dx + \mathtt{i}dy \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathtt{i}\frac{\partial}{\partial y}\right) & d\bar{z} &= dx - \mathtt{i}dy. \end{aligned}$$

For that we trivialise the tangent bundle with the help of φ_i over U_i. Sections over V_i can therefore be considered as maps $\mathbb{C} \to \mathbb{C}$, which can be differentiated in direction $\frac{\partial}{\partial z}$ or $\frac{\partial}{\partial \bar{z}}$.

$\mathrm{T}\mathbb{P}^1$ shares its metric structure with $\mathcal{O}(2)$. Explicitly we have

$$g_{\mathbb{C}}^i\left((\alpha_1 + \alpha_2\mathtt{I})\frac{\partial}{\partial x}, (\beta_1 + \beta_2\mathtt{I})\frac{\partial}{\partial x}\right) = \alpha\bar{\beta}H_i^2([z_0, z_1]).$$

Now let $(g_{\mathbb{R}}^i)_{[z_0,z_1]} = \mathtt{Re}(g_{\mathbb{R}}^i)_{[z_0,z_1]}$ be a Riemannian metric on $\mathrm{T}\mathbb{P}^1$. It has a complex continuation $g_{\otimes}^i$ on $\mathrm{T}\mathbb{P}^1 \otimes_{\mathbb{R}} \mathbb{C}$.

We will now define isomorphisms from $\left(\mathrm{T}^*\mathbb{P}^1\right)^{10}$ to $\mathrm{T}^*\mathbb{P}^1$ and from $\left(\mathrm{T}^*\mathbb{P}^1\right)^{01}$ to $\mathrm{T}\mathbb{P}^1$ which are isometries in the respective metrics.

$$
\begin{aligned}
\overline{(\mathrm{T}\mathbb{P}^1)^{10}} &\to \left(\mathrm{T}^*\mathbb{P}^1\right)^{10} \\
\frac{\partial}{\partial z} &\mapsto g^i_\otimes\left(\cdot, \frac{\partial}{\partial z}\right) = \frac{1}{2} H_i^2 dz \\
\overline{(\mathrm{T}\mathbb{P}^1)^{01}} &\to \left(\mathrm{T}^*\mathbb{P}^1\right)^{01} \\
\frac{\partial}{\partial \overline{z}} &\mapsto g^i_\otimes\left(\cdot, \frac{\partial}{\partial \overline{z}}\right) = \frac{1}{2} H_i^2 d\overline{z} \\
\overline{\mathrm{T}\mathbb{P}^1} &\to \overline{\left(\mathrm{T}\mathbb{P}^1\right)^{10}} \\
\frac{\partial}{\partial x} &\mapsto \sqrt{2}\,\frac{\partial}{\partial z} \\
\mathrm{T}\mathbb{P}^1 &\to \overline{\left(\mathrm{T}\mathbb{P}^1\right)^{01}} \\
\frac{\partial}{\partial x} &\mapsto \sqrt{2}\,\frac{\partial}{\partial \overline{z}} \\
\overline{\mathrm{T}\mathbb{P}^1} &\to \mathrm{T}^*\mathbb{P}^1 \\
\frac{\partial}{\partial x} &\mapsto g^i_{\mathbb{C}}\left(\cdot, \frac{\partial}{\partial x}\right) = H_i^2 (dx)_{\mathbb{C}}
\end{aligned}
$$

Here $(dx)_{\mathbb{C}}$ is the complex linear map which has the value 1 on $\frac{\partial}{\partial x}$.

By composition, we get the isomorphisms

$$
\begin{array}{ccc}
\left(\mathrm{T}^*\mathbb{P}^1\right)^{10} & \xrightarrow{I_{10}} & \mathrm{T}^*\mathbb{P}^1 \\
dz & \mapsto & \sqrt{2}(dx)_{\mathbb{C}} \\
\left(\mathrm{T}^*\mathbb{P}^1\right)^{01} & \xrightarrow{I_{01}} & \mathrm{T}\mathbb{P}^1 \\
d\overline{z} & \mapsto & \sqrt{2} H_i^{-2} \dfrac{\partial}{\partial x}
\end{array}
$$

By means of I_{T} and I_{T^*}, we can map these elements to $\mathscr{O}(2)$ and $\mathscr{O}(-2)$ respectively.

3.b.5 Chern connection and Curvature

The Kähler form ω_{FS} for the (unscaled) Fubini-Study metric ([Huybrechts05], p.117) is (locally) given by

$$\begin{aligned}\omega_i &= \frac{\mathtt{i}}{2\pi}\partial\overline{\partial}\log\left(1+|z|^2\right)\\ &= \frac{\mathtt{i}}{2\pi}\partial\frac{z}{1+|z|^2}\,d\overline{z}\\ &= \frac{\mathtt{i}}{2\pi}\frac{(1+|z|^2)-z\overline{z}}{\left(1+|z|^2\right)^2}\,dz\wedge d\overline{z}\\ &= \frac{\mathtt{i}}{2\pi}\left(\frac{1}{1+|z|^2}\right)^2 dz\wedge d\overline{z}.\end{aligned}$$

Following [Huybrechts05], p.119, we know

$$\int_{\mathbb{P}^1}\omega_{\mathrm{FS}} = 1.$$

Using the charts given by φ_i, we calculate the Chern connection of $\mathcal{O}(n)$ by the following formula ([Huybrechts05, p.177]):

$$\begin{aligned}\nabla = \nabla^n &= d + H_i^{-n}\left(\varphi_i^{-1}(z)\right)\left(\partial H_i^n\left(\varphi_i^{-1}(z)\right)\right)\\ &= d+\left(1+|z|^2\right)^n\left(\frac{\partial}{\partial z}(1+z\overline{z})^{-n}\right)dzv\\ &= d+\left(1+|z|^2\right)^n\left((-n)\overline{z}\,(1+z\overline{z})^{-n-1}\right)dz\\ &= d-\frac{n\overline{z}}{1+|z|^2}\,dz.\end{aligned}$$

N.B.: As we did not use orthonormal chart changing morphisms, the coefficient of dz does not lie in so(2).

Following [Huybrechts05], p.186 (iii), the curvature of $\mathcal{O}(n)$ can be calculated as

$$\begin{aligned}F^n &= \overline{\partial}\partial\log\left(H_i^n\left(\varphi_i^{-1}(z)\right)\right)\\ &= (-n)\,\overline{\partial}\partial\log\left(1+|z|^2\right)\\ &= n\,\partial\overline{\partial}\log\left(1+|z|^2\right) = n\frac{2\pi}{\mathtt{i}}\omega_{\mathrm{FS}}.\end{aligned}$$

Since $T\mathbb{P}^1 \cong \mathcal{O}(2)$, we compute the curvature form of $\mathbb{P}^1$ as

$$\begin{aligned}2\cdot\frac{2\pi}{\mathtt{i}}\cdot\frac{\mathtt{i}}{2\pi}\cdot\frac{1}{\left(1+|z|^2\right)^2}dz\wedge d\overline{z} &= 2(-2\mathtt{i})\frac{1}{4}\cdot H_i^n\left(\varphi_i^{-1}(z)\right)dx\wedge H_i^n\left(\varphi_i^{-1}(z)\right)dy\\ &= -\mathtt{i}\,\mathrm{vol}(\mathbb{P}^1).\end{aligned}$$

The scalar curvature κ is given by (see also [Almorox06], section 5):

$$\kappa = -\frac{2}{\mathtt{i}} \cdot (-\mathtt{i}) = 2.$$

Therefore, we have chosen the correct metric of $\mathbb{P}^1$ to make it isometric to the standard round sphere.

From p.200, [Huybrechts05], we know that $\left[\frac{\mathtt{i}}{2\pi} F^n\right]$ describes the (first) Chern class of $\mathcal{O}(n)$ as integer in the deRham cohomology $H^2_{\mathrm{dR}}(\mathbb{P}^1, \mathbb{R})$. This cohomology group can be canonically identified with $\mathbb{R}$ by integration over $\mathbb{P}^1$:

$$\begin{aligned} H^2_{\mathrm{dR}}(\mathbb{P}^1, \mathbb{R}) &\to \mathbb{R} \\ \omega &\mapsto \int_{\mathbb{P}^1} \omega. \end{aligned}$$

Since $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 1$ (see above), we get

$$\int_{\mathbb{P}^1} \frac{\mathtt{i}}{2\pi} F^n = n.$$

As the first Chern class classifies the line bundles, $\mathcal{O}(n)$, $n \in \mathbb{Z}$ gives us a representatives for every complex line bundle over $\mathbb{P}^1$.

Furthermore the degree $\deg \mathcal{O}(n)$ can be written as evaluation of the Chern class on the fundamental class (see [Griffiths78], S. 144), so that we can also say

$$\deg \mathcal{O}(n) = n.$$

3.b.6 Calculations following [Almorox06]

We now want to calculate eigenvectors for the $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators on $\mathbb{P}^1$.

Following paragraph 2 of [Almorox06], we look at the canonical bundle

$$K_{\mathbb{P}^1} = \left(\mathrm{T}^* \mathbb{P}^1\right)^{10} \cong \mathcal{O}(-2).$$

and take $K_{\mathbb{P}^1}^{\frac{1}{2}} = \mathcal{O}(-1)$ as its square-root (i.e. as a bundle whose square is isomorphic to $K_{\mathbb{P}^1}$). The associated bundle to the Spin structure is as usual given by

$$\begin{aligned} \mathbb{S} &= \mathbb{S}^+ \oplus \mathbb{S}^- \quad \text{with} \\ \mathbb{S}^+ &= \left(\mathrm{T}^* \mathbb{P}^1\right)^{00} \otimes K_{\mathbb{P}^1}^{\frac{1}{2}} \cong \left(\mathrm{T}^* \mathbb{P}^1\right)^{00} \otimes \mathcal{O}(-1) \\ \mathbb{S}^- &= \left(\mathrm{T}^* \mathbb{P}^1\right)^{01} \otimes K_{\mathbb{P}^1}^{\frac{1}{2}} \cong \left(\mathrm{T}^* \mathbb{P}^1\right)^{01} \otimes \mathcal{O}(-1). \end{aligned}$$

In the usual manner, we can define a Clifford multiplication on $\mathbb{S}$ by the formula (see e.g. [Morgan96]):

$$c_v \gamma = \sqrt{2}\left(\pi^{01}(v^*) \wedge \gamma - \pi^{01}(v^*) \angle \gamma\right) \quad v \in \mathrm{T}\mathbb{P}^1, \gamma \in \Omega^{0,q}(\mathbb{P}^1).$$

We choose the orthonormal basis $e_2 = H^{-1}\frac{\partial}{\partial x}$, $e_3 = H^{-1}\frac{\partial}{\partial y}$ for the tangent space ($H := H_i(\varphi_i^{-1}(z))$). Furthermore, we identify $\mathbb{S}^+$ with $\mathscr{O}(-1)$ and $\mathbb{S}^-$ with $\mathscr{O}(1)$ by the isomorphisms from the preceding section. Using the local bases o_i^n, $i = 0,1$, for $\mathscr{O}(n)$ as above, we get:

$$c_{e_2} \cdot o_i^{-1} = H^{-1} o_i^1 \qquad c_{e_3} \cdot o_i^{-1} = H^{-1}\mathtt{i}\, o_i^1$$
$$c_{e_2} \cdot o_i^{1} = -H^{1} o_i^{-1} \qquad c_{e_3} \cdot o_i^{1} = H^{1}\mathtt{i}\, o_i^1.$$

The Spin$^{\mathbb{C}}$ structure is now generated by tensoring $\mathbb{S}$ with $L = \mathscr{O}(l)$ (for the class $l \in \mathbb{Z} \cong H^2(\mathbb{P}^1;\mathbb{Z})$). We therefore get

$$\mathbb{E}^+ = \mathbb{S}^+ \otimes L = \mathscr{O}(l-1) \qquad \mathbb{E}^- = \mathbb{S}^- \otimes L = \mathscr{O}(l+1).$$

As usual, the Dirac operator $\widetilde{\mathscr{D}}$ decomposes as $\widetilde{\mathscr{D}}^+ \oplus \widetilde{\mathscr{D}}^-$ on $\mathbb{E}^+ \oplus \mathbb{E}^-$.

The elliptic chain

We define the chain $\mathscr{C}^\bullet(\mathbb{E}^+)$ as follows (see [Almorox06]):

$$\mathscr{C}^q(\mathbb{E}^+) = K^q_{\mathbb{P}^1} \otimes \mathbb{E}^+ = \mathscr{O}(l-1-2q) \quad \text{for all } q \in \mathbb{Z}.$$

The connection ∇^{l-1-2q} on $\mathscr{C}^q(\mathbb{E}^+)$ can be decomposed as

$$_{10}\nabla^{l-1-2q} : \Omega^0\big(\mathbb{P}^1, \mathscr{O}(l-1-2q)\big) \to \Omega^{10}\big(\mathbb{P}^1, \mathscr{O}(l-1-2q)\big)$$
$$_{01}\nabla^{l-1-2q} : \Omega^0\big(\mathbb{P}^1, \mathscr{O}(l-1-2q)\big) \to \Omega^{01}\big(\mathbb{P}^1, \mathscr{O}(l-1-2q)\big).$$

We have the following isomorphisms:

$$\Lambda^{10}\big(\mathscr{O}(l-1-2q)\big) \cong \big(\mathrm{T}^*\mathbb{P}^1\big)^{10} \otimes \mathscr{O}(l-1-2q) \overset{I_{10}\times \mathrm{id}}{\cong} \mathrm{T}^*\mathbb{P}^1 \otimes \mathscr{O}(l-1-2q)$$
$$\overset{I_{\mathrm{T}^*}\times \mathrm{id}}{\cong} \mathscr{O}(-2) \otimes \mathscr{O}(l-1-2q) = \mathscr{O}(l-1-2q-2).$$

Acting on sections we call this isomorphism $\widehat{I}_{10}$. Equivalently we get

$$\Lambda^{01}\big(\mathscr{O}(l-1-2q)\big) \cong \big(\mathrm{T}^*\mathbb{P}^1\big)^{01} \otimes \mathscr{O}(l-1-2q) \overset{I_{01}\times \mathrm{id}}{\cong} \mathrm{T}\mathbb{P}^1 \otimes \mathscr{O}(l-1-2q)$$
$$\overset{I_{\mathrm{T}}\times \mathrm{id}}{\cong} \mathscr{O}(2) \otimes \mathscr{O}(l-1-2q) = \mathscr{O}(l-1-2q+2).$$

It should be called $\widehat{I}_{01}$. Now we define the chain maps on $\mathscr{C}^q(\mathbb{E}^+)$ as

$$\partial_+^q = \widehat{I}_{10} \circ {}_{10}\nabla^{l-1-2q} : \Omega^0\big(\mathbb{P}^1, \mathscr{O}(l-1-2q)\big) \to \Omega^0\big(\mathbb{P}^1, \mathscr{O}(l-1-2(q+1))\big)$$
$$\overline{\partial}_+^q = \widehat{I}_{01} \circ {}_{01}\nabla^{l-1-2q} : \Omega^0\big(\mathbb{P}^1, \mathscr{O}(l-1-2q)\big) \to \Omega^0\big(\mathbb{P}^1, \mathscr{O}(l-1-2(q-1))\big).$$

The eigenspaces of $\mathscr{D}^-\mathscr{D}^+$

We use theorem 5.1 from [Almorox06]:

Theorem 3.b(i). *Let $L \to \mathbb{P}^1$ be a Hermitian line bundle with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -\mathrm{i}B\omega$, then*

1. *The spectrum of the operator $\mathscr{D}^-\mathscr{D}^+$ on $\mathbb{P}^1$, for the metric of constant scalar curvature κ, is the set*

$$\operatorname{Spec}\left(\mathscr{D}^-\mathscr{D}^+\right) = \left\{E_q = \frac{\kappa}{2}\left((q+a)^2 + (q+a)|\deg L|\right) \quad \forall q \in \mathbb{Z},\, q \geq 0\right\}.$$

   where $a = 0$ if $\deg L \geq 1$ and $a = 1$ if $\deg L < 1$.

2. *If $\deg L \geq 1$, then the space of eigensections of $\mathscr{D}^-\mathscr{D}^+$ with eigenvalue E_q gets identified with $H^0\left(\mathbb{P}^1, K^{-q} \otimes \mathbb{E}^+\right)$ in the same way, if $\deg L < 1$, then the space of eigensections with eigenvalue E_q gets identified with $H^0\left(\mathbb{P}^1, K_{\mathbb{P}^1}^{-q} \otimes \left(\mathbb{E}^+\right)^{-1}\right)$. Therefore, the multiplicity of E_q is*

$$m(E_q) = 1 + |\deg L - 1| + 2q.$$

The term "gets identified" is clarified during the proof of the theorem above. In the case $\deg L \geq 1$ the eigenspace $\widehat{\mathbb{E}}_q$ for the eigenvalue $E_q = \frac{1}{2}(q+a)(q+a+h)$ is given by

$$\widehat{\mathbb{E}}_q = \left\{\partial_+^{-1} \circ \ldots \circ \partial_+^{-q} s^{-q} \mid s^{-q} \in H^0\left(\mathbb{P}^1, K_{\mathbb{P}^1}^{-q} \otimes \mathbb{E}^+\right)\right\}$$

and for $\deg L < 1$ we get

$$\widehat{\mathbb{E}}_q = \left\{\overline{\partial^{-1}}_+ \circ \ldots \circ \overline{\partial^{-q}}_+ \overline{s^{-q}} \mid \overline{s^{-q}} \in \overline{H^0\left(\mathbb{P}^1, K_{\mathbb{P}^1}^{-q} \otimes \mathbb{E}^+\right)}\right\},$$

where H^0 and $\overline{H^0}$ denote holomorphic and anti-holomorphic sections respectively.

From [Almorox06] we know that the operator $\partial_+^{-1} \circ \ldots \circ \partial_+^{-q}$ (or $\overline{\partial^{-1}}_+ \circ \ldots \circ \overline{\partial^{-q}}_+$) is injective. The further calculations will be done in the first case; the second one works analogously.

To construct a basis for $\widehat{\mathbb{E}}_q$, it suffices to define a basis for

$$H^0\left(\mathbb{P}^1, K_{\mathbb{P}^1}^{-q} \otimes \mathbb{E}^+\right) = H^0\left(\mathbb{P}^1, \mathscr{O}(l-1+2q)\right)$$

because of injectively. We use Proposition 2.4.1 (p.91) in [Huybrechts05]:

Theorem 3.b(ii). *For $k \geq 0$ the space $H^0\left(\mathbb{P}^n, \mathscr{O}(k)\right)$ is canonically isomorphic to the space $\mathbb{C}\left[z_0, \ldots, z_n\right]_k$ of all homogeneous polynomials of degree k.*

The space above is therefore isomorphic to $\mathbb{C}[z_0, z_1]_{l-1+2q}$. What does the isomorphism look like?

$\mathcal{O}\big(-(l-1+2q)\big)$ can be embedded in $\mathbb{P}^1 \times \big(\mathbb{C}^2\big)^{l-1+2q}$. Hence, every element of $\mathbb{P}^1 \times \Big(\big(\mathbb{C}^2\big)^*\Big)^{l-1+2q}$ can be applied to $\mathcal{O}\big(-(l-1+2q)\big)$, and therefore represents an element of $\mathcal{O}(l-1+2q)$. $\big(\mathbb{C}^2\big)^*$ has two canonical projections π_0 and π_1. The monomial $z_0^k z_1^{l-1+2q-k}$ will now be identified with

$$\underbrace{\pi_0 \otimes \ldots \otimes \pi_0}_{k} \otimes \underbrace{\pi_1 \otimes \ldots \otimes \pi_1}_{l-1+2q-k}$$

(from a more systematic point of view we should have used a symmetric tensor product here, but it does not matter). For $\mathcal{O}$ we calculated trivialisations in 3.b.2. For $\pi_0^k \pi_1^{l-1+2q-k}$, we get:

$$\begin{aligned}\big([z_0, z_1], \pi_0^k \pi_1^{l-1+2q-k}\big) &\overset{\psi_0}{\mapsto} \left([z_0, z_1], \pi_0(z_0, z_1) \cdots \pi_1(z_0, z_1) \frac{1}{z_0^{l-1+2q-k}}\right) \\ &= \left([z_0, z_1], \frac{z_1^{l-1+2q-k}}{z_0^{l-1+2q-k}}\right) = \big([z_0, z_1], z^{l-1+2q-k}\big) \quad \text{using } \varphi_0.\end{aligned}$$

Using φ_1, we get in the same way the result z^k.

Hence, we can define a basis for $\widehat{\mathbb{E}}_q$ by

$$\hat{e}_q^l(k) := \partial_+^{-1} \ldots \partial_+^{-q}\big(\pi_0^k \pi_1^{l-1+2q-k}\big). \tag{3.b-1}$$

Over U_0 the term $\pi_0^k \pi_1^{l-1+2q-k}$ corresponds to $z^{l-1+2q-k}$ and the operator ${}_{10}\nabla^{l-1+2q}\sigma$ is given by $\left(\frac{\partial \sigma}{\partial z} - \frac{(l-1+2q)\overline{z}}{1+|z|^2}\sigma\right) dz$, which will be translated by $\widehat{I}_{10}$ into $\left(\frac{\partial \sigma}{\partial z} - \frac{(l-1+2q)\overline{z}}{1+|z|^2}\sigma\right)\sqrt{2}o_0^2$. Locally $\hat{e}_q^l(k)$ is therefore also given by

$$\prod_{p=-q}^{-1}\left(\frac{\partial}{\partial z} - \frac{(l-1+2p)\overline{z}}{1+|z|^2}\right) z^{l-1+2q-k}.$$

From $\mathscr{D}^-\mathscr{D}^+$ to $\mathscr{D}$

We use proposition 4.4 from [Almorox06]:

Theorem 3.b(iii). *The spectral resolution of $\mathscr{D}$ is completely determined by its kernel and the spectral resolution of $\mathscr{D}^-\mathscr{D}^+$ or $\mathscr{D}^+\mathscr{D}^-$. In particular, for any $\lambda^2 \in \operatorname{Spec}\big(\mathscr{D}^2\big)\setminus\{0\}$, we have the isomorphisms*

$$\begin{aligned}\pi_{\pm\lambda} &: \mathbb{E}_{\lambda^2}\big(\mathscr{D}^-\mathscr{D}^+\big) \to \mathbb{E}_{\pm\lambda}\big(\mathscr{D}\big) \\ \pi_{\pm\lambda} &: \mathbb{E}_{\lambda^2}\big(\mathscr{D}^+\mathscr{D}^-\big) \to \mathbb{E}_{\pm\lambda}\big(\mathscr{D}\big)\end{aligned}$$

defined by $\pi_{\pm\lambda}\big(s^\pm\big) = \frac{1}{2\lambda}\big[\lambda s^\pm \pm \mathscr{D}^\pm\big(s^\pm\big)\big]$, where $s^+ \in \mathbb{E}_{\lambda^2}\big(\mathscr{D}^-\mathscr{D}^+\big)$ and $s^- \in \mathbb{E}_{\lambda^2}\big(\mathscr{D}^+\mathscr{D}^-\big)$.

Hence, the eigenspace $\pm\sqrt{E_q}$ for $E_q \neq 0$ of $\mathscr{D}$ is given by

$$\mathbb{E}_{\pm q} = \left\{ {}^{+}e_q^l(k) := \frac{1}{2\sqrt{E_q}}\left(\sqrt{E_q}\,\hat{e}_q^l(k) \pm \mathscr{D}^+\left(\hat{e}_q^l(k)\right)\right) \middle| \hat{e}_q^l(k) \in \widehat{\mathbb{E}_q} \right\}$$

To compute this concretely, we note that, by remark 3.3, [Almorox06], we have:

$$\mathscr{D}^+ = \sqrt{2}\partial_+^0$$

As seen above, this can be (locally) calculated.

For $E_q = 0$, 5.3 of [Almorox06] implies:

$\ker\mathscr{D}^+ = H^0\left(\mathbb{P}^1, \mathbb{E}^+\right)$	$\ker\mathscr{D}^- = 0$	for $\deg L \geq 1$
$\ker\mathscr{D}^+ = 0$	$\ker\mathscr{D}^- \cong H^1\left(\mathbb{P}^1, \mathbb{E}^+\right)$	otherwise.

The last identification follows by using Serre duality.

As in the case of T_Λ, we define an orthonormal basis called $\tilde{\sigma}_m$, running through the eigenspaces and call the respective eigenvalue μ_m.

3.c An eigenbasis for $\mathscr{D}^2$ and $\mathscr{D}$

Similar to the case of T^3, we define a map $s_l : S^1 \to S^1$, $\chi \mapsto \chi^l$. By using the projection π_1 onto the first factor of $S^1 \times S^2$, we can define elements

$$\hat{\sigma}_{l,m}(v) := (s_l \circ \pi_1)(v) \cdot \pi_2^*\left(\tilde{\sigma}_m\right)(v)$$

as sections of the associated bundle for the Spin$^{\mathbb{C}}$ structure given by $K = \pi_2^*(L)$, $L = \mathscr{O}(h)$, where h is our class in $H^2(S^1 \times S^2; \mathbb{Z})$.

At first we consider the operator $\mathscr{D}_\alpha^2$ for α being a harmonic one-form. To use the same notation as above, we shall call α (viewed as element in $\mathbb{R}$) x^a. λ_l will be defined as $2\pi l + x^a$.

Then theorem 2.d(ii) can be restated here without change:

Theorem 3.c(i). *The set $\left\{\hat{\sigma}_{l,m} \mid l, m, \in \mathbb{Z}\right\}$ forms an orthogonal eigenbasis for $\mathscr{D}_\alpha^2$ for the respective eigenvalues $\lambda_l^2 + \mu_m^2$. Furthermore we have*

$$\mathscr{D}_\alpha \hat{\sigma}_{l,m} = \left(\lambda_l \mathrm{i} c_{e_1} + \mu_m\right)\hat{\sigma}_{l,m}. \tag{3.c-1}$$

Proof. The proof of 2.d(ii) can be repeated nearly verbatim. The only differences are:

- We can use $\widetilde{\mathscr{D}}_0$ all the time.

- We have to do our calculation in two charts which arise by using the standard charts of $\mathbb{P}^1$.

□

For the rest of the section, we assume h to be positive; the case of $h \le 0$ can be treated analogously by using anti-holomorphic sections in 3.b.6.

Since subsection 2.e uses only formal properties of $\tilde{\sigma}_m$, which are fulfilled in the case $\mathbb{P}^1$ as well, the definitions, theorem and proof can be copied word by word. For completeness, we state the theorem again:

Theorem 3.c(ii). *We get an orthogonal eigenbasis of $\mathscr{D}_\alpha$ by*

$$\left\{\sigma^{\pm}_{l,m} \,\middle|\, (l,m) \in \mathbb{Z}^2 \quad \text{with } \lambda_l \neq 0 \text{ and } m \geq h\right\} \cup \left\{\sigma^{0}_{l,m} \,\middle|\, (l,m) \in \mathbb{Z}^2 \quad \text{with } \lambda_l = 0 \text{ or } 0 \leq m \leq h-1\right\},$$

which will be written as $M^{\pm}_\alpha \cup M^0_\alpha$.

Notice that there is no "exceptional case" for $h = 0$ since the theorem for $\mathbb{P}^1$ hold for all h.

3.d Spectral sections

Spectral sections for $S^1 \times S^2$ are pretty boring since the parameter space $\mathscr{L}$ can be at most one-dimensional (we should therefore speak of spectral flows).

We easily see that spectral sections exist if and only if $\mathscr{L}$ and h are orthogonal which means that one of them is zero. If $\mathscr{L} = 0$ this is clear. If $\mathscr{L} \neq 0$, then (as in the case of T^3) vectors coming from $\mu_m = 0$ move upwards through zero, so that spectral sections can only exist if $\widetilde{\mathscr{D}}_0$ on $\mathbb{P}^1$ has trivial kernel. This is equivalent to $h = 0$; in the case $h = 0$ their existence is obvious as we have a lower bound for the norm of the eigenvalues ν of $\mathscr{D}_\alpha$.

Chapter 4

The space S^3

For the case S^3 we use completely different methods. The following properties of S^3 are especially important:

- There is just one Spin$^{\mathbb{C}}$ structure on S^3 (the one coming from a Spin structure). Therefore, we have always $\hat{a} = 0$.
- The space $\mathscr{L}/\ell$ is always zero as all harmonic one-forms vanish. Therefore (up to "gauging away" closed one-forms), we only have to consider one Dirac operator $\mathscr{D}$.
- As a manifold, S^3 is equivalent to the Lie group Sp(1). The representation theory of Sp(1) and **sp**(1) will be essential for our investigations.
- As in the case of T^3, the tangent bundle is trivial, so that we can use global trivialising sections.

We consider $S^3 = \mathrm{Sp}(1)$ as canonically embedded into $\mathbb{H}$. The tangent space $T_{e_0}\,\mathrm{Sp}(1)$ at the neutral element e_0 will be identified as usual with the Lie algebra **sp**(1). As a vector space, **sp**(1) can be seen as the imaginary quaternions in $\mathbb{H}$.

Our first goal is to calculate the spectrum of $\mathscr{D}$. We will follow the path which has been paved by [Hitchin74], but translate it into the language of quaternions.

Hitchin used a metric which depends on three parameters $\lambda_1, \lambda_2, \lambda_3$. We will only use the standard metric ($\lambda_1 = \lambda_2 = \lambda_3 = 1$); everything can be calculated in the same way using the more general metric but we want to avoid an unnecessary amount of notation.

Unlike Hitchin, we are also interested in the calculation of a concrete basis of eigensections. For that purpose we have to transform the results of Hitchin, given in an abstract representation space, to the "world of sections". The required isomorphism will be constructed using the complexified Lie algebra of Sp(1) and some elementary combinatorics.

4.a Definitions

Sp(1) will be equipped with the canonical metric, coming from the embedding into $\mathbb{H} \cong \mathbb{R}^4$. The Lie algebra **sp**(1) is spanned by $\{e_1, e_2, e_3\}$. They can be considered as left-invariant vector fields, which will be called $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$.

In contrast to the case T^3 we have a non-trivial Levi-Civita connection, which can be computed by the standard formula (see e.g. [Carmo92], p.55):

$$\begin{aligned}\langle X, \nabla_Z Y\rangle = {} & Z\cdot\langle X, Y\rangle + \langle Z, [X,Y]\rangle \\ & + Y\cdot\langle X, Z\rangle + \langle Y, [X,Z]\rangle \\ & - X\cdot\langle Y, Z\rangle - \langle X, [Y,Z]\rangle .\end{aligned}$$

For left-invariants vector fields X, Y, Z and a left-invariant metric, the red terms vanish, so we get:

$$2\langle X, \nabla_Z Y\rangle = \langle Z, [X,Y]\rangle + \langle Y, [X,Z]\rangle - \langle X, [Y,Z]\rangle .$$

Since $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ fulfil the relations $[\mathbf{e_i}, \mathbf{e_j}] = 2\mathbf{e_i}\mathbf{e_j}$ (Clifford multiplication understood), we get the following lemma:

Lemma 4.a(i). *We have*

$$\begin{aligned}&\nabla_{\mathbf{e_i}}\mathbf{e_i} = 0\\ &\nabla_{\mathbf{e_i}}\mathbf{e_j} = \mathbf{e_i}\mathbf{e_j} \quad \textit{for } i \neq j.\end{aligned}$$

For ∇ we have a uniquely defined Spin connection $\tilde{\nabla}$. Using this we can define $\mathscr{D}$ on a section $\check{\sigma}$ as follows:

$$\mathscr{D}\check{\sigma} = c_{\mathbf{e_1}}\tilde{\nabla}_{\mathbf{e_1}}\check{\sigma} + c_{\mathbf{e_2}}\tilde{\nabla}_{\mathbf{e_2}}\check{\sigma} + c_{\mathbf{e_3}}\tilde{\nabla}_{\mathbf{e_3}}\check{\sigma}.$$

As we have a trivial tangent bundle, we also have a trivial Spin bundle and Clifford bundle. Using a section named $\mathbf{e_0}$, we identify the Spin bundle $\mathscr{S}_{\mathbb{C}}(\tilde{P}_{\mathrm{Sp}(1)})$ with Sp(1) × $\mathbb{H}$. Therefore, every section $\check{\sigma}$ can be written as $\sigma \cdot \mathbf{e_0}$, where $\sigma \in \mathscr{C}^\infty(\mathrm{Sp}(1); \mathbb{H})$ and the $\cdot$ means $\mathbb{H}$-multiplication.

With the help of lemma 4.a(i) we want to examine the structure of $\tilde{\nabla}$. Since $\mathbf{e_0}$ is constant in the chosen global trivialisation, we know that

$$\tilde{\nabla}_{\mathbf{e_h}}\mathbf{e_0} = \frac{1}{4}\sum_{i,j}\omega_{ji}(\mathbf{e_h})c_{\mathbf{e_i}}c_{\mathbf{e_j}}\mathbf{e_0},$$

where ω_{ji} is the matrix of one-forms representing ∇ (see 4.a(i)). In detail, we get:

$$\tilde{\nabla}_{\mathbf{e_1}}\mathbf{e_0} = \frac{1}{2}c_{\mathbf{e_2}}c_{\mathbf{e_3}}\mathbf{e_0} = -\frac{1}{2}\mathbf{e_1}$$

and in the same way

$$\tilde{\nabla}_{\mathbf{e_2}}\mathbf{e_0} = -\frac{1}{2}\mathbf{e_2} \qquad\qquad \tilde{\nabla}_{\mathbf{e_3}}\mathbf{e_0} = -\frac{1}{2}\mathbf{e_3}.$$

Hence, for $\mathscr{D}\mathbf{e_0}$ we know

$$\mathscr{D}\mathbf{e_0} = c_{\mathbf{e_1}}\left(-\frac{1}{2}\mathbf{e_1}\right) + \ldots + c_{\mathbf{e_3}}\left(-\frac{1}{2}\mathbf{e_3}\right) = -\frac{3}{2}\mathbf{e_0}.$$

If we again represent a section $\check{\sigma}$ by $\sigma \cdot \mathbf{e_0}$, we get the formula (easy generalisation of the real formula [Carmo92, p.50] to the case $\mathbb{H}$):

$$\mathscr{D}\big(\sigma \cdot \mathbf{e_0}\big) = \sum_i c_{\mathbf{e_i}}\big(d\sigma(\mathbf{e_i})\,\mathbf{e_0}\big) + c_{\mathbf{e_i}}\Big(\sigma\big(\tilde{\nabla}_{\mathbf{e_i}}\mathbf{e_0}\big)\Big). \tag{4.a-1}$$

Consider the left action L_s of Sp(1) on $\mathscr{C}^\infty\big(\mathrm{Sp}(1);\mathbb{H}\big)$ given by

$$L_s\sigma(x) = \sigma(xs) \quad x, s \in \mathrm{Sp}(1),$$

which induces an infinitesimal action l_S for $S \in \mathbf{sp}(1)$ (this is not a right action although it might look like it was one). To describe the Dirac operator by representation theory we use the following lemma:

Lemma 4.a(ii). *We have*

$$d\sigma(\mathbf{e_i}) = l_{\mathbf{e_i}}\sigma.$$

Proof. Let $\tau : (-1/1000, 1/1000) \to \mathrm{Sp}(1)$ be a curve with $\tau(0) = e_0$ and $\dot{\tau}(0) = \mathbf{e_i}$.

From the definition of infinitesimal representations we know that

$$\big(l_{\mathbf{e_i}}\sigma\big)(x) = \frac{\partial}{\partial t}\Big|_0\Big(\sigma\big(x\tau(t)\big)\Big).$$

Now $\tau_2(t) := x\tau(t)$ represents a curve with $\tau_2(0) = x$ and $\dot{\tau}_2(0) = x \cdot \mathbf{e_i} = \mathbf{e_i}$ since $\mathbf{e_i}$ is left-invariant. As $\sigma(x\tau(t)) = \sigma(\tau_2(x))$ and

$$\frac{\partial}{\partial t}\Big|_0\Big(\sigma\big(\tau_2(t)\big)\Big) = d\sigma_x(\mathbf{e_i}),$$

the assertion is proved. □

We plug this into (4.a-1) and get:

$$\begin{aligned}\mathscr{D}\big(\sigma \cdot \mathbf{e_0}\big) &= \sum_i c_{\mathbf{e_i}}\big((l_{\mathbf{e_i}}\sigma)\mathbf{e_0}\big) + \sum_i c_{\mathbf{e_i}}\Big(\sigma\big(\tilde{\nabla}_{\mathbf{e_i}}\mathbf{e_0}\big)\Big)\\ &= \sum_i (l_{\mathbf{e_i}}\sigma)\cdot\mathbf{e_0}\cdot(-\mathbf{e_i}) + \sigma\sum_i c_{\mathbf{e_i}}\big(\tilde{\nabla}_{\mathbf{e_i}}\mathbf{e_0}\big)\\ &= -\Big(\sum_i (l_{\mathbf{e_i}}\sigma)\cdot\mathbf{e_i} + \frac{3}{2}\sigma\Big).\end{aligned}$$

Notice that $c_{\mathbf{e_i}}$ and $\sigma\cdot$ commute since they act on the right and on the left respectively.

From now on, we consider the above trivialisation of $\mathscr{S}_{\mathbb{C}}(\widetilde{P}_{\mathrm{Sp}(1)})$ as implicitly chosen and write

$$\mathscr{D}(\sigma) = \overline{\mathscr{D}}(\sigma) - \frac{3}{2}\sigma \tag{4.a-2}$$

with

$$\overline{\mathscr{D}}(\sigma) = -\sum_i (l_{\mathbf{e_i}}\sigma)\cdot\mathbf{e_i} \quad \text{for } \sigma \in \mathscr{C}^\infty\big(\mathrm{Sp}(1);\mathbb{H}\big).$$

In the next section we will investigate the connection between the Dirac operator and the Laplace-Beltrami operator.

4.b Δ vs. $\mathscr{D}$

We consider the Laplace-Beltrami operator Δ (just called Laplace operator in the following discussion) given by the metric. In our sign convention it will be locally defined as follows:

Definition 4.b(i) (Laplace-Beltrami operator)**.** Let X_j be a local basis of vector fields. Then define

$$H(\sigma)_{\alpha\beta} = \nabla_{X_\alpha}\nabla_{X_\beta}\sigma - \nabla_{\nabla_{X_\alpha}X_\beta}\sigma$$
$$\Delta\sigma = \sum_{\alpha,\beta}\langle,\rangle^{\alpha\beta}H(\sigma)_{\alpha\beta},$$

where $\langle,\rangle^{\alpha\beta}$ represents the (α,β)-entry of the metric in the given basis.

For the definition of Δ we view σ componentwise (as four functions to $\mathbb{R}$). By standard methods you can show that this definition is independent of the choice of basis.

Lemma 4.b(ii). *The following two assertions hold:*

(i) Δ *is* $\mathbb{H}$*-linear, i.e.* $\Delta(\sigma\cdot\mathbf{e_i}) = \big(\Delta\sigma\big)\cdot\mathbf{e_i}$

(ii) $\Delta(l_{\mathbf{e_i}}\sigma) = l_{\mathbf{e_i}}\big(\Delta\sigma\big)$

Proof. (i) Clear, since Δ is linear over the reals and acts componentwise.

(ii) We use $\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}$ as basis of vector fields. As they are orthonormal, we have $\langle,\rangle^{\alpha\beta} = \delta_{\alpha\beta}$ (Kronecker symbol).

Therefore, we have

$$\Delta(l_{e_i}) = \sum_\alpha \nabla_{\mathbf{e}_\alpha}\nabla_{\mathbf{e}_\alpha}\big(l_{\mathbf{e_i}}\sigma\big)$$

Since $l_{\mathbf{e_i}}\sigma = d\sigma(\mathbf{e_i}) = \nabla_{\mathbf{e_i}}\sigma$:

$$= \sum_{\alpha} \nabla_{\mathbf{e}_\alpha}\nabla_{\mathbf{e}_\alpha}\nabla_{\mathbf{e_i}}\sigma.$$

Now we have to exchange $\nabla_{\mathbf{e_i}}$ and $\nabla_{\mathbf{e}_\alpha}\nabla_{\mathbf{e}_\alpha}$ to get $l_{\mathbf{e_i}}\Delta$. This produces some Lie brackets which cancel each other so that the assertion is true.

□

Corollary 4.b(iii). *$\mathscr{D}\Delta = \Delta\mathscr{D}$ on $\mathscr{C}^\infty(\mathrm{Sp}(1);\mathbb{H})$.*

Hence, $\mathscr{D}$ leaves the eigenspaces of Δ invariant. They are determined in the next section.

4.c The eigenspaces of Δ

The *real* eigenspaces of $\Delta^{\mathbb{R}}$ can be found in [Sakai96] (inverting the signs). Theorem 3.13 on page 272 states:

Theorem 4.c(i). *The eigenvalues of $\Delta^{\mathbb{R}}$ on $(S^3,\langle,\rangle)$ are given by $\lambda_k = 1-(k+1)^2$, $k \in \mathbb{Z}^+$. The dimension of the eigenspaces is $(k+1)^2$. They are given by the homogeneous harmonic polynomials of degree k on $\mathbb{R}^4$ (which shall be called H^4_k).*

To carry this over to $\mathbb{H}$ we make the following statement:

Lemma 4.c(ii). *On $\mathscr{C}^\infty(\mathrm{Sp}(1);\mathbb{H})$, the operator Δ has only real eigenvalues.*

Proof. From [Sakai96, lemma 3.5, p.266] we know:

$$\left\langle \Delta^{\mathbb{R}} f_1, f_2 \right\rangle_{\mathrm{L}^2} = \left\langle f_1, \Delta^{\mathbb{R}} f_2 \right\rangle_{\mathrm{L}^2},$$

where $\langle,\rangle_{\mathrm{L}^2}$ denotes the L^2 scalar product.

On $\mathscr{C}^\infty(\mathrm{Sp}(1);\mathbb{H})$ the L^2 scalar product is defined as

$$\left\langle \sigma_1, \sigma_2 \right\rangle_{\mathrm{L}^2} := \int_{\mathrm{Sp}(1)} \overline{\sigma_1}\,\sigma_2\, dv_{\langle\rangle} \quad \in \mathbb{H}.$$

Since Δ act componentwise and Δ is linear we also have

$$\left\langle \Delta\sigma_1, \sigma_2 \right\rangle_{\mathrm{L}^2} = \left\langle \sigma_1, \Delta\sigma_2 \right\rangle_{\mathrm{L}^2}.$$

Now if $\lambda_{\mathbb{H}} \in \mathbb{H}$ an eigenvalue of Δ with eigensection $\sigma_{\mathbb{H}}$. A direct calculation shows

$$\overline{\lambda_{\mathbb{H}}}\int |\sigma|^2 = \lambda_{\mathbb{H}}\int |\sigma|^2.$$

Therefore, $\lambda_{\mathbb{H}}$ has to be real. □

Hence, we know that every $\lambda_{\mathbb{H}}$ is equal to λ_k for one $k \in \mathbb{Z}^+$. The eigenspaces of Δ are given by $V_k := \mathbb{H} \otimes_{\mathbb{R}} H_k^4$. They are right $\mathbb{H}$ vector spaces (by multiplication on the first factor).

Now we know that $\mathscr{D}|_{V_k}$ maps V_k to V_k. To understand this operation more precisely, we have to dive into the representation theory of $\mathrm{Sp(1)} \times \mathrm{Sp(1)}$.

4.d The operation of $\mathrm{Sp(1)} \times \mathrm{Sp(1)}$ on $\mathscr{C}^\infty(\mathrm{Sp(1)}; \mathbb{H})$

The standard metric of Sp(1) is left- and right-invariant, so that left and right multiplication act isometrically.

Particularly, the left operation

$$\beta(s,t)\sigma(x) = \sigma\big(b(s,t)x\big)$$
$$\text{with } b(s,t)x = t^{-1}xs$$

is a unitary operation (concerning the L^2-structure) of $\mathrm{Sp(1)} \times \mathrm{Sp(1)}$ on $\mathscr{C}^\infty(\mathrm{Sp(1)}; \mathbb{H})$.

Lemma 4.d(i). *β and Δ commute: $\beta(s,t)\Delta = \Delta\beta(s,t)$.*

Proof. Since $\beta(s,t)$ and Δ act separately on each component, we only have to look at the real case. Here the assertion is true because $\beta(s,t)$ is an isometry (see [Helgason94, prop 2.4, p.246]) □

So $\beta(s,t)$ acts on each of the eigenspaces V_k of Δ.

For the rest of the section, V should denote an arbitrary finite dimensional β-invariant subspace of $\mathscr{C}^\infty(\mathrm{Sp(1)}; \mathbb{H})$. Furthermore, we define

$$K_b = \big\{(s,t) \in \mathrm{Sp(1)} \times \mathrm{Sp(1)} \mid b(s,t)e_0 = e_0\big\}.$$

The space of *zone functions* ζ_b is now defined to be

$$\zeta_b(V) = \big\{\sigma \in V \mid \beta(s,s)\sigma = \sigma \,\forall (s,s) \in K_b\big\}.$$

Theorem 4.d(ii). *If $V \neq 0$, then there exists an element $\sigma_\zeta \in \zeta_b(V)$ with $\sigma_\zeta(e_0) \neq 0$.*

Proof. Define $V^* = \{\sigma^* : V \to \mathbb{H}, \text{right-linear}\}$ to be the dual space of V (as left $\mathbb{H}$ vector space). Now define $\xi \in V^*$ to be the map $\xi(\sigma) = \sigma(e_0)$.

Since $V \neq 0$ and β acts transitively on Sp(1), we know that there is an element $\sigma \in V$ with $\sigma(e_0) \neq 0$. Therefore, $(\ker\xi)$ has (quaternionic) codimension 1.

As K_b fixes the point e_0, we know that its induced action leaves the subspace $(\ker\xi)$ invariant. But then it also has to fix its orthogonal complement $(\ker\xi)^\perp$.

So we have an operation of K_b on the one-dimensional space $(\ker\xi)^\perp$. Let σ_ζ be an arbitrary basis of this space. Then we have a function $f : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathbb{H}$ with

$$\beta(s,s)\sigma_\zeta = \sigma_\zeta \cdot f(s,s) \quad \forall (s,s) \in K_b.$$

At e_0 we have the following chain of equations:

$$\sigma_\zeta(e_0) = \big(\beta(s,s)\sigma_\zeta\big)(e_0) = \sigma_\zeta(e_0) \cdot f(s,s).$$

We know that $\sigma_\zeta \neq 0$ (since it comes from $(\ker\xi)^\perp$) and therefore see

$$f(s,s) = 1 \quad \forall (s,s) \in K_b.$$

This shows that the action of K_b on σ_ζ is trivial which proves the theorem. □

We furthermore need the following lemma:

Lemma 4.d(iii). *K_b acts transitively on the unit tangent vectors in $T_{e_0}\mathrm{Sp}(1)$.*

Proof. Following [Morgan96, p.13], we have $\mathrm{Sp}(1) = \mathrm{Spin}(3)$ and the diagonal action of $\mathrm{Spin}(3)$ on $T_{e_0}\mathrm{Sp}(1)$ coincides with the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$. But this is known to be transitive on the unit sphere. □

Theorem 4.d(iv). *The action β of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on the eigenspaces V_k of Δ is irreducible (Idea: [Taylor86], p.119).*

Proof. Assume that V_k splits into the direct sum $V_k^1 \oplus V_k^2$ with respect to β. Following 4.d(ii), there are elements $\sigma^1 \in \zeta(V_k^1)$ and $\sigma^2 \in \zeta(V_k^2)$ with $\sigma^i(e_0) \neq 0$. Furthermore, notice that $\sigma^1, \sigma^2 \in \zeta(V_k)$.

Assertion: There is a neighbourhood U of e_0, so that we have for all $\sigma \in \zeta(V_k)$:

$$\mathrm{dist}(e_0, x) = \mathrm{dist}(e_0, y) \quad \Rightarrow \quad \sigma(x) = \sigma(y).$$

Proof of Assertion. Let $\overline{\delta} > 0$ be smaller than the radius for which $\exp : \mathbf{sp}(1) \to \mathrm{Sp}(1)$ is bijective and let U be the image of the ball $B(\overline{\delta})$ under this map. x and y shall be any points of distant $\delta < \overline{\delta}$ to e_0 in $\mathrm{Sp}(1)$. We denote by X and Y the corresponding points in $\mathbf{sp}(1)$ (under the bijection exp).

Since K_b acts transitively on the unit tangent vectors and preserves distances, it also operates transitively on the surface of $B(\delta)$. If $\overline{b}$ denotes the induced action of b on $\mathbf{sp}(1)$, we find an element $(s,s) \in K_b$, so that $\overline{b}(s,s)X = Y$. This implies $b(s,s)x = y$.

As $\sigma \in \zeta(V_k)$, we directly see $\sigma(x) = \beta(s,s)\,\sigma(x) = \sigma(y)$. Assertion

Choose $U(\varepsilon_0) = \{x \in \mathrm{Sp}(1) \,\big|\, \mathrm{dist}(x, e_0) < \varepsilon_0\} \subset U$ with $\sigma^1(x) \neq 0$ and $\sigma^2(x) \neq 0$ for all $x \in U(\varepsilon_0)$. We now look at neighbourhoods $U(\varepsilon)$ with $\varepsilon < \varepsilon_0$. We also define the quaternionic constants $c^1_\varepsilon = \sigma^1(x_\varepsilon)$ and $c^2_\varepsilon = \sigma^2(x_\varepsilon)$ for an arbitrary element $x_\varepsilon \in \partial U(\varepsilon)$. Now define

$$\sigma^\varepsilon(x) := \sigma^1(x)\left(c^1_\varepsilon\right)^{-1} - \sigma^2(x)\left(c^2_\varepsilon\right)^{-1} \in V_k.$$

Being in V_k implies

$$\left(\Delta - \lambda_k\right)\sigma^\varepsilon \equiv 0 \quad \text{on } U(\varepsilon)$$

and we also know:

$$\sigma^\varepsilon \equiv 0 \quad \text{on } \partial U(\varepsilon).$$

Digression: In [Bers57], p.152, they consider problems of the following form (Dirichlet problem):

$$Lu = \left(\sum_{\alpha,\beta}^{n} l_{\alpha\beta}\frac{\partial^2}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{n} l_\alpha \frac{\partial}{\partial x_\alpha} + l\right) u = f \quad \text{on } \Omega^{\mathrm{o}}$$

$$u = \phi \quad \text{on } \partial\Omega.$$

For that they assume that $l_{\alpha\beta}$, l_α and l are continuous and that L is conformally elliptic, i.e. there are constants $c > 0$ and C with

$$\sum_{\alpha,\beta=1}^{n} l_{\alpha\beta}\xi_\alpha\xi_\beta \geq c\sum_{\alpha=1}^{n}\xi_\alpha^2 \qquad |l_{\alpha\alpha}| \leq C \qquad |l_\alpha| \leq C.$$

I cite (with changed notation) the second theorem on page 153 ([Bers57]):

Theorem 4.d(v). *Assume that $l \leq M$, where M is a positive number and furthermore that the diameter d of Ω is so small that we have*

$$e^{\hat{\alpha} d} - 1 < \frac{1}{M} \quad \text{with } \hat{\alpha} := \left(\frac{1}{2}c\right)\left(C + (C^2 + 4c)^{\frac{1}{2}}\right).$$

Then every solution of the Dirichlet problem fulfils the inequality

$$|u| \leq \frac{\max|\phi| + \left(e^{\hat{\alpha} d} - 1\right)\max|f|}{1 - M\left(e^{\hat{\alpha} d} - 1\right)}.$$

Now we want to use this theorem. Let L be the map $\Delta^{\mathbb{R}} - \lambda_k$ on $U(\varepsilon)$, applied to the components σ^ε_i of σ^ε. For the conformal ellipticity, we choose $c = 1$ and C big enough to restrict the coefficient functions on $U(\varepsilon_0)$ (and thus also on every $U(\varepsilon)$ with $\varepsilon < \varepsilon_0$). The functions f and ϕ are chosen to be zero.

Let furthermore M be defined as $-\lambda_k$ and ε be so small that the diameter condition is fulfilled. Then the inequality implies $\sigma_i^\varepsilon = 0$ for all components. Therefore, we have on $U(\varepsilon)$:

$$\sigma^1(x)\big(c_\varepsilon^1\big)^{-1} - \sigma^2(x)\big(c_\varepsilon^2\big)^{-1} = 0.$$

Since σ^1 and σ^2 are polynomials and therefore analytic, this equation is true on the whole of Sp(1). But then σ^1 and σ^2 are linearly independent over $\mathbb{H}$, which contradicts our assumption. □

Our next aim is to classify the $\mathbb{H}$-representations of Sp(1) × Sp(1). We will use this to determine which of the "abstract" representations of our list corresponds to the representations on V_k just found.

4.e Representations of Sp(1) × Sp(1)

The groups Sp(1) and SU(2) are isomorphic, so we can use [Bröcker03, p.76] to give a classification of the *complex* (left) representations of Sp(1):

For that, let $\mathbb{H}_1$ be the canonical inverse right representation of Sp(1) on $\mathbb{H}$ (coming from the quaternionic multiplication). As before we consider $\mathbb{H}$ as $\mathbb{C}^2$ using the basis e_0, e_2 and note that the (left) action of $\mathtt{i}$ commutes with the representation of Sp(1).

Now let $\mathbb{H}_k$ be the k-fold complex symmetric tensor product $\mathbb{H}_1 \circledcirc \ldots \circledcirc \mathbb{H}_1$ with the induced representation of Sp(1). We choose the basis

$$\big|d\big\rangle = e_0 \circledcirc \ldots \circledcirc e_0 \circledcirc e_2 \circledcirc \ldots \circledcirc e_2 \quad d = 0, \ldots, k,$$

where d is the number of e_2-terms. The spaces $\mathbb{H}_k$ with the standard metric from $\mathbb{C}^2$ form exactly the irreducible complex representations of Sp(1) (following [Bröcker03]).

Hence, the irreducible complex representations of Sp(1) × Sp(1) are given by $\mathbb{H}_i \otimes_{\mathbb{C}} \mathbb{H}_j$, $i, j \in \mathbb{Z}^+$ (see e.g. [Adams69, theorem 3.65, p.70]).

From theorem 3.57, p.66 of [Adams69] we know

Theorem 4.e(i). *For a compact Lie group we can find families of real representations $[\mathbb{R}]_m$, complex representations $[\mathbb{C}]_n$ and quaternionic representations $[\mathbb{H}]_p$, so that we have:*

The non-equivalent irreducible representations are exactly

(i) $[\mathbb{R}]_m, [r^{\downarrow}\mathbb{C}]_n, [r^{\downarrow}c^{\downarrow}\mathbb{H}]_p$ *over* $\mathbb{R}$.

(ii) $[c^{\uparrow}\mathbb{R}]_m, [\mathbb{C}]_n, \overline{[\mathbb{C}]}_n, [c^{\downarrow}\mathbb{H}]_p$ *over* $\mathbb{C}$.

(iii) $\left[h^{\uparrow}c^{\uparrow}\mathbb{R}\right]_m$, $\left[h^{\uparrow}\mathbb{C}\right]_n$, $\left[\mathbb{H}\right]_p$ *over* $\mathbb{H}$.

The term $\overline{\left[\mathbb{C}\right]}_n$ *means* $\left[\mathbb{C}\right]_n$ *with conjugated scalar multiplication,* $c^{\downarrow}$ *and* $r^{\downarrow}$ *view the respective spaces as complex or real space (i.e. forget some structure), whereas* $c^{\uparrow}$ *and* $h^{\uparrow}$ *tensor by* $\mathbb{C}\otimes_{\mathbb{R}}$ *or* $\mathbb{H}\otimes_{\mathbb{C}}$ *respectively.*

Adams also outlined how to find the three classes of representations: Decompose the irreducible complex representations into those which are not self-conjugate (they form $\left[\mathbb{C}\right]_n$) and those which are self-conjugate, to be divided further in real and quaternionic ones.

At first we will analyse this for Sp(1): Since in every complex dimension there is exactly one representation, we know that all representations have to be self-conjugate. We distinguish two cases:

k is even Here $\mathbb{H}_k$ has odd complex dimension, so it cannot be quaternionic and must be real.

k is odd To show that $\mathbb{H}_k$ is quaternionic, we have to find a complex antilinear map $J : \mathbb{H}_k \to \mathbb{H}_k$ with $J^2 = -\mathrm{id}$. We define

$$J\big(|d\rangle\big) = (-1)^d\,|k-d\rangle,$$

which obviously fulfils the condition when we continue it in a complex anti-linear fashion.

Hence, we know that

$$\left[c^{\uparrow}\mathbb{R}\right]_m = \mathbb{H}_{2m} \qquad\qquad \left[c^{\downarrow}\mathbb{H}\right]_p = \mathbb{H}_{2p+1}.$$

After considering Sp(1), we move to Sp(1) × Sp(1). To distinguish the representation spaces from the ones above, we call them $\left[\mathbb{R}^{\times}\right]_m$, $\left[\mathbb{C}^{\times}\right]_n$, $\left[\mathbb{H}^{\times}\right]_p$; they appear as products of the representations of Sp(1).

The product of two self-conjugate spaces is always self-conjugated: This rules out spaces of the form $\left[\mathbb{C}^{\times}\right]_n$. Furthermore the real representations are the product of two representations of the same kind (i.e. same division algebra), while the quaternionic ones are created by two representations of different kind.

All in all the representations of Sp(1) × Sp(1) over $\mathbb{H}$ are:

$$\mathbb{H}\otimes_{\mathbb{C}}\Big(\left[c^{\uparrow}\mathbb{R}\right]_{m_1}\otimes_{\mathbb{C}}\left[c^{\uparrow}\mathbb{R}\right]_{m_2}\Big) \qquad\qquad \left[\mathbb{R}\right]_{m_1}\otimes_{\mathbb{R}}\left[\mathbb{H}\right]_{p_2}$$

$$\mathbb{H}\otimes_{\mathbb{C}}\Big(\left[c^{\downarrow}\mathbb{H}\right]_{p_1}\otimes_{\mathbb{C}}\left[c^{\downarrow}\mathbb{H}\right]_{p_2}\Big) \qquad\qquad \left[\mathbb{H}\right]_{p_1}\otimes_{\mathbb{R}}\left[\mathbb{R}\right]_{m_2}.$$

Now we want to detect which of these representations is equivalent to the one given by V_k.

We look at the transposition map

$$\begin{aligned} \mathrm{Sp}(1) \times \mathrm{Sp}(1) &\xrightarrow{\circlearrowleft} \mathrm{Sp}(1) \times \mathrm{Sp}(1) \\ (s,t) &\mapsto (t,s). \end{aligned}$$

Furthermore, let $\mathrm{inv} : \mathrm{Sp}(1) \to \mathrm{Sp}(1)$ be the isometry given by $x \mapsto x^{-1}$. This induces an isometry $\mathrm{inv}^\infty : \mathscr{C}^\infty(\mathrm{Sp}(1); \mathbb{H}) \to \mathscr{C}^\infty(\mathrm{Sp}(1); \mathbb{H})$. Since isometries commute with Δ (see above) we get a map

$$\mathrm{inv}^\infty|_{V_k} : V_k \to V_k.$$

$\mathrm{inv}^\infty|_{V_k}$ is therefore a self-inverse isomorphism of V_k. Look at the diagram:

$$\begin{array}{ccc} \big(\mathrm{Sp}(1) \times \mathrm{Sp}(1)\big) \times V_k & \xrightarrow{\quad \beta \quad} & V_k \\ \Big\downarrow {\scriptstyle \mathrm{id} \times \mathrm{inv}^\infty} & & \Big\downarrow {\scriptstyle \mathrm{inv}^\infty} \\ (\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \times V_k & \xrightarrow{\beta \circ (\circlearrowleft \times \mathrm{id})} & V_k \end{array}$$

This diagram commutes since

$$\mathrm{inv}^\infty\big(\beta(s,t)\sigma\big)(x) = \big(\beta(s,t)\sigma\big)\big(x^{-1}\big) = \sigma\big(t^{-1}x^{-1}s\big)$$

and

$$\beta \circ (\circlearrowleft \times \mathrm{id}) \circ (\mathrm{id} \times \mathrm{inv}^\infty)(s,t,\sigma)(x) = \beta(t,s)\big(\mathrm{inv}^\infty(\sigma)\big)(x) = \mathrm{inv}^\infty(\sigma)\big(s^{-1}xt\big) = \sigma\big(t^{-1}x^{-1}s\big).$$

Hence, the representations β and $\beta \circ (\circlearrowleft \times \mathrm{id})$ are equivalent. The representation γ from the list which corresponds to β must have the same property: Since $\circlearrowleft$ exchanges the arguments of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, γ can only be one of the symmetric representations

$$\begin{aligned} &\mathbb{H} \otimes_{\mathbb{C}} \Big([c^{\uparrow}\mathbb{R}]_m \otimes_{\mathbb{C}} [c^{\uparrow}\mathbb{R}]_m \Big) \\ &\mathbb{H} \otimes_{\mathbb{C}} \Big([c^{\downarrow}\mathbb{H}]_p \otimes_{\mathbb{C}} [c^{\downarrow}\mathbb{H}]_p \Big). \end{aligned}$$

This family can also be expressed as

$$\mathbb{H} \otimes_{\mathbb{C}} (\mathbb{H}_k \otimes_{\mathbb{C}} \mathbb{H}_k) \quad \forall k \geq 1.$$

These spaces have the respective $\mathbb{H}$-dimension $(k+1)^2$. So we must have

$$V_k \cong \mathbb{H} \otimes_{\mathbb{C}} (\mathbb{H}_k \otimes_{\mathbb{C}} \mathbb{H}_k) \quad \forall k \geq 1,$$

where by $\cong$ we mean an isometry of right $\mathbb{H}$ vector spaces which commutes with the respective representations β and γ. This abstract isomorphism will be calculated concretely in 4.g. Until then we use the abstract isomorphism to calculate the eigenvalues of $\mathscr{D}$.

4.f The spectrum of $\mathscr{D}$

As $\overline{\mathscr{D}}(\sigma) = -\sum_i (l_{\mathbf{e_i}}\sigma)\cdot\mathbf{e_i}$ acts on V_k only be means of scalar multiplication and representation of $\mathbf{sp}(1)$, we can replace V_k by $\mathbb{H}\otimes_{\mathbb{C}}(\mathbb{H}_k\otimes_{\mathbb{C}}\mathbb{H}_k)$ for the calculation of the eigenvalues (Notice that L can be written as $L_s = \beta(s, e_0)$).

As representation space for Sp(1), the space $\mathbb{H}\otimes_{\mathbb{C}}(\mathbb{H}_k\otimes_{\mathbb{C}}\mathbb{H}_k)$ decomposes into the direct sum of the irreducible representations

$$\mathbb{H}_k^q := \mathbb{H}\otimes_{\mathbb{C}}\left(\mathbb{H}_k\otimes_{\mathbb{C}}\operatorname{span}\{|q\rangle\}\right).$$

Here we want to explicitly calculate $\overline{\mathscr{D}}$. For that purpose we choose a complex basis $e_r\otimes|p\rangle$, $r = 0, 2$.

The space $\mathbb{H}_k^q \cong \mathbb{H}\otimes_{\mathbb{C}}\mathbb{H}_k$ is not only a complex vector space, but is a right $\mathbb{H}$ vector space by multiplication on the first component and a quaternionic representation space for Sp(1) by inverse right multiplication on the second component. The representation $\gamma(s, e_0)$, which corresponds to $\beta(s, e_0) = L_s$, will still be denoted by L.

We now want to examine the action of $l_{\mathbf{e_i}}$ on $e_r\otimes|p\rangle$. We will leave out the e_r-part since it is unimportant for the representation:

$$l_{\mathbf{e_1}}(|p\rangle) = \left(\frac{\partial}{\partial t}\bigg|_0 \exp(t\mathbf{e_1})e_0 \circledcirc \ldots \circledcirc \exp(t\mathbf{e_1})e_2\right).$$

Since Sp(1) is a matrix group, we can calculate its exponential in the usual fashion:

$$\exp(t\mathbf{e_i}) = 1 + te_i + \frac{(te_i)^2}{2} + \ldots$$

Algebraically $\frac{\partial}{\partial t}\big|_0$ means that we only look at the linear part in t:

$$\begin{aligned} l_{\mathbf{e_1}}(|p\rangle) &= -(k-p)\,e_1\circledcirc e_0\ldots\circledcirc e_2 + p\,e_0\circledcirc\ldots\circledcirc e_3\circledcirc e_2\circledcirc\ldots\circledcirc e_2 \\ &= -(k-p)\mathtt{i}\,|p\rangle + p\mathtt{i}|p\rangle. \end{aligned} \tag{4.f-1}$$

In the same way, we get:

$$l_{\mathbf{e_2}}|p\rangle = (p-k)\,|p+1\rangle + p\,|p-1\rangle \tag{4.f-2}$$
$$l_{\mathbf{e_3}}|p\rangle = (p-1)\mathtt{i}\,|p+1\rangle - p\mathtt{i}\,|p-1\rangle. \tag{4.f-3}$$

Notice: In these and the following formulas $|-1\rangle$ and $|k+1\rangle$ are assumed to be zero. Now we calculate $\overline{\mathscr{D}}^2$:

$$\begin{aligned} \overline{\mathscr{D}}^2 &= \sum_{i,j} l_{\mathbf{e_j}} l_{\mathbf{e_i}}\sigma\cdot\mathbf{e_i}\cdot\mathbf{e_j} \\ &= -l_{\mathbf{e_1}}^2\sigma - l_{\mathbf{e_2}}^2\sigma - l_{\mathbf{e_3}}^2\sigma \\ &\quad + \left(l_{\mathbf{e_3}} l_{\mathbf{e_2}} - l_{\mathbf{e_2}} l_{\mathbf{e_3}}\right)\sigma\cdot\mathbf{e_1} + \left(l_{\mathbf{e_1}} l_{\mathbf{e_3}} - l_{\mathbf{e_3}} l_{\mathbf{e_1}}\right)\sigma\cdot\mathbf{e_2} + \left(l_{\mathbf{e_2}} l_{\mathbf{e_1}} - l_{\mathbf{e_1}} l_{\mathbf{e_2}}\right)\sigma\cdot\mathbf{e_3}. \end{aligned}$$

We have

$$l_{\mathbf{e_i}} l_{\mathbf{e_j}} - l_{\mathbf{e_j}} l_{\mathbf{e_i}} = l_{[\mathbf{e_i},\mathbf{e_j}]} = l_{2\mathbf{e_i}\mathbf{e_j}}.$$

If we plug this in, we get:

$$\begin{aligned}\overline{\mathscr{D}}^2 &= -l_{\mathbf{e_1}}^2\sigma - l_{\mathbf{e_2}}^2\sigma - l_{\mathbf{e_3}}^2\sigma \\ &\quad -2l_{\mathbf{e_1}}\sigma\mathbf{e_1} - -2l_{\mathbf{e_2}}\sigma\mathbf{e_2} - -2l_{\mathbf{e_3}}\sigma\mathbf{e_3},\end{aligned}$$

which means, that

$$\left(\overline{\mathscr{D}}^2 - 2\overline{\mathscr{D}}\right)\sigma = -l_{\mathbf{e_1}}^2\sigma - l_{\mathbf{e_2}}^2\sigma - l_{\mathbf{e_3}}^2\sigma.$$

Lemma 4.f(i). *On $\mathbb{H}_k^q$, we have*

$$\left(-l_{\mathbf{e_1}}^2 - l_{\mathbf{e_2}}^2 - l_{\mathbf{e_3}}^2\right) = k(k+2)\,\mathrm{id}.$$

Proof. We take the basis $e_r \otimes |p\rangle$ and use the formulas (4.f-1), (4.f-2) and (4.f-3) twice. □

As a result we have the quadratic equation

$$\left(\overline{\mathscr{D}} + k\right)\left(\overline{\mathscr{D}} - (k+2)\right)\sigma = 0 \quad \forall \sigma \in \mathbb{H}_k^q. \tag{4.f-4}$$

Therefore, every element $\sigma \in \mathbb{H}_k^q$ generates a one- or two-dimensional complex $\overline{\mathscr{D}}$-invariant subspace.

On our basis we have the following two formulas for $\overline{\mathscr{D}}$:

$$\begin{aligned}\overline{\mathscr{D}}\left(e_0 \otimes |p\rangle\right) &= (2p-k)\,e_0 \otimes |p\rangle - 2pe_2 \otimes |p-1\rangle \\ \overline{\mathscr{D}}\left(e_2 \otimes |p\rangle\right) &= -(2p-k)\,e_2 \otimes |p\rangle - 2(k-p)e_0 \otimes |p-1\rangle.\end{aligned}$$

Hence we get the invariant subspaces

$$\left\{e_0 \otimes |p\rangle,\, e_2 \otimes |p-1\rangle\right\}$$

for $p = 0,\ldots,k+1$, where the first and last one are one-dimensional. (4.f-4) gives us two eigenvectors in every of the two-dimensional subspaces, one for $-k$ and one for $k+2$.

If we subtract $\frac{3}{2}$ to get from $\overline{\mathscr{D}}$ to $\mathscr{D}$ and go back to the whole space $\mathbb{H}\otimes_{\mathbb{C}}\left(\mathbb{H}_k \otimes_{\mathbb{C}} \mathbb{H}_k\right)$, we get the following complex orthogonal basis (leaving out the $\otimes$ in the usual ket-manner):

For $k+\frac{1}{2}$:

$$e_0 \otimes |p\rangle\,|q\rangle - e_2 \otimes |p-1\rangle\,|q\rangle \quad p = 1,\ldots,k \quad q = 0,\ldots,k$$

For $-k-\frac{3}{2}$:

$$(p-k-1)e_0 \otimes |p\rangle\,|q\rangle - pe_2 \otimes |p-1\rangle\,|q\rangle \quad p=1,\dots,k \quad q=0,\dots k$$
$$e_0 \otimes |0\rangle\,|q\rangle,\ e_2 \otimes |k\rangle\,|q\rangle \quad q=0,\dots,k$$

Now we want to translate these abstract basis vectors into concrete ones; for that purpose we need a computable isomorphism from V_k to $\mathbb{H} \otimes_{\mathbb{C}} (\mathbb{H}_k \otimes_{\mathbb{C}} \mathbb{H}_k)$.

4.g An eigenbasis for $\mathscr{D}$

Since V_k and $\mathbb{H} \otimes_{\mathbb{C}} (\mathbb{H}_k \otimes_{\mathbb{C}} \mathbb{H}_k)$ are both quaternionic representations which come from real representations, we can compare them equally well on the real or complex level; for simplicity we choose the latter one: The aim of this section will be to find an isomorphism between the $\mathbb{C}$-representations β on $W_k := H_k^4 \otimes_{\mathbb{R}} \mathbb{C}$ and γ on $\mathbb{H}_k \otimes_{\mathbb{C}} \mathbb{H}_k$.

Both of them induce representations of the Lie algebra $\mathbf{sp}(1) \oplus \mathbf{sp}(1)$. Since we look at complex representations, we get a canonical representation of the complexified Lie algebra $\mathbb{C} \otimes (\mathbf{sp}(1) \oplus \mathbf{sp}(1)) = \mathbf{sl}(2,\mathbb{C}) \oplus \mathbf{sl}(2,\mathbb{C})$ (compare [Hall04, p.102]); those complexified representations shall be called $\overline{\beta}$ and $\overline{\gamma}$.

As a basis for $\mathbf{sl}(2,\mathbb{C})$, we choose

$$H^{\mathrm{sl}} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X^{\mathrm{sl}} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y^{\mathrm{sl}} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

As described in the proof of theorem D.1 on page 322 [Hall04], there exists a basis

$$v_{-k}, v_{-k+2}, \dots, v_{k-2}, v_k$$

for $\mathbb{H}_k$ so that

$$\overline{\gamma}(H^{\mathrm{sl}}, H^{\mathrm{sl}})(v_a \otimes v_b) = (a+b)\, v_a \otimes v_b.$$

In this decomposition of $\mathbb{H}_k$ into eigenspaces of $\overline{\gamma}(H^{\mathrm{sl}}, H^{\mathrm{sl}})$, we see that the one for the eigenvalue $2k$ is 1-dimensional.

We now want to determine the eigenspace for $2k$ of $\overline{\gamma}(H^{\mathrm{sl}}, H^{\mathrm{sl}})$ and $\overline{\beta}(H^{\mathrm{sl}}, H^{\mathrm{sl}})$ and use this for defining an isomorphism.

Notice that H^{sl} corresponds to $\mathtt{i} \otimes e_1$, using the isomorphism $\mathbf{sl}(2,\mathbb{C}) \cong \mathbb{C} \otimes \mathbf{sp}(1)$. We have

$$\begin{aligned}\overline{\gamma}(H^{\mathrm{sl}}, H^{\mathrm{sl}})\,|0\rangle\,|0\rangle &= \mathtt{i}(e_1 \circledcirc e_0 \circledcirc \dots \circledcirc e_0 \dots) \\ &= 2k\,|0\rangle\,|0\rangle.\end{aligned}$$

This shows that $|0\rangle\,|0\rangle$ is a normed basis vector of the $2k$-eigenspace.

Now we look at W_k. It consists of complex harmonic homogeneous polynomials of degree k in x_0, x_1, x_2, x_3. We write $z = (z_1, z_2) = (x_0, x_1, x_2, x_3)$ with $z_1 = x_0 + x_1\mathtt{i}$ and $z_2 = x_2 + x_3\mathtt{i}$.

Our strategy is to solve the problem combinatorically by using the functions

$$g_2(z) = z_2 \qquad \overline{g_2}(z) = \overline{z}_2$$
$$g_{-1}(z) = -z_1 \qquad \overline{g_1}(z) = \overline{z}_1.$$

We compute

$$\begin{aligned}\overline{\beta}(\mathbf{e_1},\mathbf{e_1})\,g_2(z) &= \frac{\partial}{\partial t}\Big|_0 g_2\big(\exp(-t\mathbf{e_1})z\exp(t\mathbf{e_1})\big)\\ &= \frac{\partial}{\partial t}\Big|_0 g_2\big(z + t(-\mathbf{e_1}z + z\mathbf{e_1}) + t^2\cdot\ldots\big)\\ &= g_2\left(\begin{pmatrix} -\mathbf{e_1}z_1 + z_1\mathbf{e_1}\\ -\mathbf{e_1}z_2 + z_2(-\mathbf{e_1})\end{pmatrix}\right)\\ &= -2\mathbf{e_1}\,z_2.\end{aligned}$$

Since $\mathtt{i}\cdot\overline{\beta}(\mathbf{e_1},\mathbf{e_1}) = \overline{\beta}(H^{\mathrm{sl}}, H^{\mathrm{sl}})$, we know that

$$\overline{\beta}\big(H^{\mathrm{sl}}, H^{\mathrm{sl}}\big)\,g_2(z) = 2g_2(z).$$

Furthermore, you can see explicitly that $g_2^k(z)$ is a harmonic polynomial of degree k, which means $g_2^k \in W_k$. Following the usual formula for products we have

$$\overline{\beta}\big(H^{\mathrm{sl}}, H^{\mathrm{sl}}\big)\,g_2^k = 2k\cdot g_2^k.$$

Therefore, we know that g_2^k generates the $2k$-eigenspace of $\overline{\beta}\big(H^{\mathrm{sl}}, H^{\mathrm{sl}}\big)$. To calculate the norm of g_2^k we use the following integral formula:

Lemma 4.g(i). *For $l_1, l_2, l_3, l_4 \in \mathbb{Z}_{\geq 0}$, we have*

$$\int_{S^3} g_2^{l_1}\,\overline{g_2}^{l_2}\,g_{-1}^{l_3}\,\overline{g_1}^{l_4}\,dv_{\langle\rangle} = \begin{cases} 2\pi^2(-1)^{l_4}\,\frac{l_1!\,l_3!}{(l_1+l_3+1)!} & \text{for } l_1 = l_2 \text{ and } l_3 = l_4\\ 0 & \text{otherwise}\end{cases}$$

Proof. Let $S^3 = \big\{(z_1, z_2)\,\big|\,|z_1|^2 + |z_2|^2 = 1\big\}$ and let

$$N := \big\{(z_1, z_2) \in S^3\,|\,z_1 = 0 \vee z_2 = 0\big\}.$$

As N has measure zero, we have

$$\int_{S^3} f * 1 = \int_{S^3\setminus N} f * 1$$

for all functions $f : S^3 \to \mathbb{C}$ which are integrable.

We consider the map

$$\begin{array}{rcl} \mathbb{R}/_{2\pi\mathbb{Z}} \times \mathbb{R}/_{2\pi\mathbb{Z}} \times]0,1[& \xrightarrow{\eta} & S^3 \setminus N \\ (t \quad , \quad s \quad , \quad \rho) & \longmapsto & \left(e^{t\mathtt{i}} \cdot \sqrt{\rho}, e^{s\mathtt{i}} \cdot \sqrt{1-\rho}\right) \\ \left(-\mathtt{i}\log^{-1}\left(\frac{z_1}{|z_1|}\right), -\mathtt{i}\log^{-1}\left(\frac{z_2}{|z_2|}\right), |z_1|^2\right) & \longleftarrow & (z_1, z_2) \end{array}$$

This is a diffeomorphism with given inverse. Thus we have:

$$\int_{\mathbb{R}/_{2\pi\mathbb{Z}} \times \mathbb{R}/_{2\pi\mathbb{Z}} \times]0,1[} (f \circ \eta) \cdot \eta^*(*1_{S^3}) = \int_{S^3 \setminus N} f * 1.$$

A direct calculation shows

$$\eta^*(*1_{S^3}) = \eta^*(\mathbf{e_1}^* \wedge \mathbf{e_2}^* \wedge \mathbf{e_3}^*) = \frac{1}{2} dt \wedge ds \wedge d\rho.$$

Using this, we can reduce everything to a standard integral which is computable by the gamma function (see [Friedman71, p.294]). □

Using this formula we have

$$\langle g_2^k, g_2^k \rangle_{\mathrm{L}^2} = \int_{S^3} g_2^k \overline{g_2}^k \, dv_{\langle\rangle} = 2\pi^2 \cdot \frac{k!}{(k+1)!},$$

which means $\|g_2^k\|_{\mathrm{L}^2} = \sqrt{\frac{2\pi^2}{k+1}}$.

As $\overline{\beta}$ and $\overline{\gamma}$ are isomorphic as representations, our isomorphism has to map

$$|0\rangle\,|0\rangle \quad \text{onto} \quad u\sqrt{\frac{k+1}{2\pi^2}}\, g_2^k,$$

where $u \in S^1$. But since multiplication with u is an isomorphism of complex representations, we can choose u to be anything we like (and take $u = 1$).

Definition 4.g(ii). Let $\mathscr{I}$ be the isomorphism of complex $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ representations which maps $|0\rangle\,|0\rangle$ onto $u\sqrt{\frac{k+1}{2\pi^2}}\, g_2^k$.

The map $\mathscr{I}$ is uniquely defined because it commutes with the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$; this action creates a generating system out of every non-zero vector due to irreducibility.

Now want to compute $\mathscr{I}$ on the basis $|p\rangle\,|q\rangle$. For that purpose we use the action of $(Y^{\mathrm{sl}}, 0)$ and $(0, Y^{\mathrm{sl}})$. In $\mathbb{C} \otimes \mathbf{sp}(1)$ we have

$$Y^{\mathrm{sl}} = \frac{1}{2}\big((-1) \otimes \mathbf{e_2} + \mathtt{i} \otimes \mathbf{e_3}\big).$$

Therefore, we get

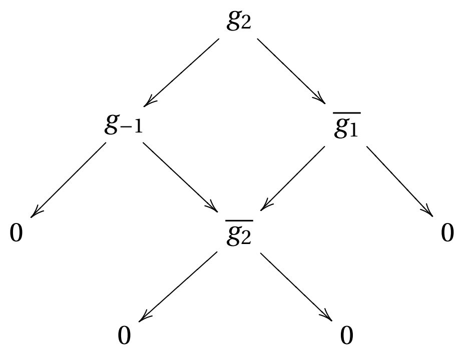
$$\begin{aligned}\overline{\gamma}\big(Y^{\mathrm{sl}},0\big)\,|p\rangle\,|q\rangle &= -\frac{1}{2}\overline{\gamma}(\mathbf{e_2},0)|p\rangle\,|q\rangle - \frac{\mathrm{i}}{2}\,\overline{\gamma}(\mathbf{e_3},0)|p\rangle\,|q\rangle \\ &= -\frac{1}{2}(p-k)|p+1\rangle\,|q\rangle - \frac{1}{2}p|p-1\rangle\,|q\rangle \\ &\quad -\frac{1}{2}(p-k)|p+1\rangle\,|q\rangle + \frac{1}{2}p|p-1\rangle\,|q\rangle \\ &= (k-p)\,|p+1\rangle\,|q\rangle.\end{aligned}$$

In the same matter $\overline{\gamma}(0,Y^{\mathrm{sl}})\,|p\rangle\,|q\rangle = (k-q)\,|p\rangle\,|q+1\rangle$.

This gives us a method to calculate $|p\rangle\,|q\rangle$ out of $|0\rangle\,|0\rangle$ in $p+q$ steps.

To compute $\mathscr{I}\big(|p\rangle\,|q\rangle\big)$ we need to understand $\overline{\beta}\big(Y^{\mathrm{sl}},0\big)$ and $\overline{\beta}\big(0,Y^{\mathrm{sl}}\big)$:

Lemma 4.g(iii). *If we describe $\overline{\beta}\big(Y^{sl},0\big)$ by a left-down arrow and $\overline{\beta}\big(0,Y^{sl}\big)$ by a right-down arrow, we get the following diagram*



Proof. This is a direct calculation. □

With the help of the lemma above we can show

Theorem 4.g(iv). *Let $\mathscr{I}$ be the map defined above. Then we have for $p,q\in\{0,\dots,k\}$:*

$$\begin{array}{ccc}\mathbb{H}_k\otimes\mathbb{H}_k & \xrightarrow{\mathscr{I}} & W_k \\ |p\rangle\,|q\rangle & \mapsto & \binom{k}{p}^{-1}\binom{k}{q}^{-1}\sqrt{\frac{k+1}{2\pi^2}}\sum_{i=0}^{k}\binom{k}{k-q-i,\,p-i,\,i}\,g_2^{\,k-q-i}\,\overline{g_2}^{\,p-i}\,g_{-1}^{\,i}\,\overline{g_1}^{\,q-p+i}\end{array}$$

Proof. By induction. □

With the help of the theorem, we can translate the basis in $\mathbb{H}\otimes_{\mathbb{C}}\big(\mathbb{H}_k\otimes_{\mathbb{C}}\mathbb{H}_k\big)$ into a basis in V_k.

The terms in W_k can be further examined by the rich theory of harmonic polynomials (see e.g. [Axler01]).

Appendix A

Notation

This appendix gives an overview over the notation used throughout this thesis and indicates which variables belong to which space. For more information please look up the symbol in the index at the end of the thesis.

1.a Notation for the whole thesis

$\hat{a} = h \cdot a$	An element of the second integral cohomology group, used to describe the present $\mathrm{Spin}^{\mathbb{C}}$ structure. $h \in \mathbb{Z}$ is chosen to be the maximal integer which "divides" $h \cdot a \in H^2(M;\mathbb{Z})$.
α^c	A closed one-form
α	The harmonic part of α^c
b	An element of $\mathbb{Z}^3$
β	A one-form depending on b and α
$\mathscr{D}_\alpha$	The Dirac operator, depending on the chosen one-form
e_i	Basis vector of the quaternions $\mathbb{H}$
$\mathtt{i}$	The complex unit
$\mathscr{C}^\infty(\mathrm{Sp}(1);\mathbb{H})$	The space of smooth maps from the manifold $\mathrm{Sp}(1)$ to $\mathbb{H}$
K	A line bundle over the present 3-dimensional manifold
λ_l	Eigenvalue on a 1-dimensional space, indexed by $l \in \mathbb{Z}$
ℓ	A discrete subspace of $H^1(M;\mathbb{Z})$
L	A line bundle over a 2-dimensional manifold
$\mathscr{L}$	Defined as $l \otimes \mathbb{R}$ in $H^1(M;\mathbb{R})$
μ_m	Eigenvalue on a 2-dimensional space, indexed by $m \in \mathbb{Z}$

∇^K	The background connection of the bundle K
ν_n	Eigenvalue on M, indexed by $n \in \mathbb{Z}$
$\widetilde{P}_M$	Spin$^{\mathbb{C}}$ bundle coming from a Spin structure
$\mathscr{S}_{\mathbb{C}}(\widetilde{P}_M)$	Associated vector bundle to $\widetilde{P}_M$
S^1	The group S^1 considered as unit circle in $\mathbb{C}$
$S^{[1]}$	The group S^1 considered as $[0,1]/\sim$
z	An element of $\mathbb{C}$
$\langle , \rangle$	The Riemannian metric

1.b Notation for chapter 2 and 3

$\alpha_{\parallel}$, $\alpha_{\perp}$	Parallel and orthogonal part of α with respect to W
α_L, $\tilde{\alpha}^l_L$	One-form on T_Λ
$\widetilde{\mathbb{C}}$	The complexified version of W
c^1, c^2	Constants in $S^{[1]}$ depending on w_i and a
$\underline{\mathrm{Cl}}$	Trivial Clifford algebra bundle
c	Characteristic for the line bundle $L(H, \chi_c)$
$\tilde{\mathscr{D}}$	Dirac operator on a two-dimensional space
E	An alternating form on $\widetilde{\mathbb{C}}$
$\mathbb{E}$	Associated Spin$^{\mathbb{C}}$ bundle on T_Λ
f	An factor of automorphy
H	A hermitian form on $\widetilde{\mathbb{C}}$
$\underline{\mathbb{H}}$, $\underline{\mathbb{H}}^\Lambda$	The trivial quaternionic bundle over M or S (respectively)
I_{10}, I_{01}	Isometries used for the 10- and 01-part of the complexified tangent space
K_1, K_2	Defined to be $\Lambda(L^{h,c})_i/_{\Lambda_i}$ with $i = 1,2$
Λ	Lattice in W of dimension 2
$\ell_{\mathbb{Z}}$	Defined as $\mathscr{L} \cap \mathbb{Z}^3$
$M^{\pm}_\alpha \cup M^0_\alpha$	A basis of eigensections for $\mathscr{D}_\alpha$
$\nu^0_{l,m}$, $\nu^{\pm}_{l,m}$	Set of eigenvalues for $\mathscr{D}_\alpha$
$\mathrm{NS}(T_\Lambda)$	The Néron-Severi group (alternating forms which respect Λ)
ω^l	One-form used in the definition of $\hat{\sigma}_{l,m}$, depending on tri

o_i^n	Local basis of $\mathscr{O}(n)$ on $\mathbb{P}^1$
$\mathscr{O}(n)$	Line bundle on $\mathbb{P}^1$ of Chern class n
π_a	The orthogonal projection onto W
$\pi_{\overline{a}}$	The map from T^3 to T_Λ induced by π_a
Pic	The space of holomorphic line bundles on T_Λ
r_Λ	Scaling factor for the complex structure on T_Λ
$\sigma_b^\pm$	Basis of eigensections for $\mathscr{D}_\alpha$ in the case $\hat{a}=0$
Σ_b	2-dimensional space spanned by σ_b^+ and σ_b^-
$\tilde{\sigma}_m$	Basis of eigensections for $\tilde{\mathscr{D}}$
$\mathbb{S}$	Associated spin bundle on T_Λ
$\hat{\sigma}_{l,m}$	Basis of eigensections for $\mathscr{D}_\alpha^2$
tri	Map from T^3 to $S^{[1]}$ depending on a, w_1 and w_2
$\vartheta_k^{h,c}$	kth theta function for given characteristic c and Chern class h
w_1, w_2	"Shortest" basis for the lattice Λ
W	The orthogonal complement of a in $\mathbb{R}^3$
χ_c	The semicharacter for c
ξ_c	Quotient of semicharacters

1.c Notation for chapter 4

$\beta(s,t)$	Representation of Sp(1) × Sp(1) on $\mathscr{C}^\infty(\mathrm{Sp}(1);\mathbb{H})$ coming from $b(s,t)$
$b(s,t)$	Canonical right representation of Sp(1) × Sp(1) on Sp(1)
$\overline{\beta}, \overline{\gamma}$	Representations of the complexified Lie algebra induced by β and γ
$\gamma(s,t)$	Representation on abstract representation space, equivalent to $\beta(s,t)$
Δ	The quaternionic Laplace-Beltrami operator for the chosen metric
$\Delta^\mathbb{R}$	The real Laplace-Beltrami operator
$\overline{\mathscr{D}}$	Dirac operator reduced by the constant term $3/2$
$\mathbf{e_i}$	Left-invariant vector field on Sp(1)

H_k^4	Space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^4$
$\mathbb{H}_1$	Canonical left representation space of Sp(1) (by inverse right quaternionic multiplication)
$\mathbb{H}_k$	k-fold symmetric product of $\mathbb{H}_1$ with induced representation
$\mathbb{H}_k^q$	Representation of Sp(1) given by setting the second basis vector to $\lvert q\rangle$
$\lvert p\rangle$	Standard basis for $\mathbb{H}_k$
λ_k	Eigenvalues of Δ (given by $1-(k+1)^2$)
L_s	Left action of Sp(1) on $\mathscr{C}^\infty(\mathrm{Sp}(1);\mathbb{H})$ given by right multiplication in the argument
l_S	Derived representation of **sp**(1) coming from L
$\tilde{\nabla}$	Spin connection on Sp(1)
Sp(1)	The symplectic group in $\mathbb{H}$ (elements of unit length)
sp(1)	The Lie Algebra of Sp(1)
$\mathbf{sl}(2,\mathbb{C})$	The Lie algebra of the special linear group
V_k	Quaternionified version of H_k^4

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