Semilinear perturbations of harmonic spaces and Martin-Orlicz capacities: An approach to the trace of moderate $U$-functions

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Khalifa El Mabrouk

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Referenten:
Prof. Dr. Wolfhard Hansen
Prof. Dr. Michael Röckner

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Gratefully Dedicated to
my father and my mother

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Semilinear perturbations of harmonic spaces and Martin-Orlicz capacities: An approach to the trace of moderate $U$-functions

By
Khalifa El Mabrouk
Abstract

Let \((X, \mathcal{H})\) be a harmonic space in the sense of H. Bauer [7] which has a Green function \(G_X\). It is known [31] that to every reference measure \(r\) there corresponds a suitable integral representation of functions in

\[\mathcal{H}_r^+(X) := \mathcal{H}^+(X) \cap L^1(X, r)\]

Let \(Y\) be the minimal Martin boundary, \(P\) the Martin kernel, and denote by \(\mathcal{M}(Y)\) the set of all signed Borel measures on \(Y\) with bounded variation. In this work we consider the perturbed (semilinear) structure \((X, \mathcal{U})\) obtained from \((X, \mathcal{H})\) by means of \((\gamma, \Psi)\) where \(\gamma\) is a local Kato measure on \(X\) and \(\Psi\) belongs to a class of real-valued functions on \(X \times \mathbb{R}\) containing, in particular,

\[\Psi_\alpha : (x, t) \mapsto t|t|^\alpha - 1\]

where \(\alpha\) is a real > 1.

We show that for every function \(u\) belonging to

\[\mathcal{U}_r(X) := \{u \in \mathcal{U}(X) : |u| \leq h \text{ for some } h \in \mathcal{H}_r^+(X)\}\]

there corresponds a unique signed measure \(\nu \in \mathcal{M}(Y)\) such that

\[u + \int_X G_X(\cdot, \zeta)\Psi(\zeta, u(\zeta)) d\gamma(\zeta) = \int_Y P(\cdot, y) d\nu(y)\]

Conversely, we prove that this integral equation admits a solution \(u \in \mathcal{U}_r(X)\) whenever \(\nu\) does not charge compact sets \(K \subset Y\) of zero Martin-Orlicz capacity, that is, \(|\nu|(K) = 0\) for every compact set \(K \subset Y\) with the property that the integral

\[\int_X \int_X G_X(x, \zeta)\Psi(\zeta, \int_Y P(\zeta, y) d\mu(y)) d\gamma(\zeta) dr(x)\]

is equal to 0 or \(\infty\) for every \(\mu \in \mathcal{M}^+(Y)\) such that \(\mu(Y \setminus K) = 0\).

In Section 6, we use our approach to investigate the trace of moderate solutions to some semilinear equations.
Contents

Abstract iv

1 Introduction 1

2 Preliminaries 5
  2.1 Basic notations 5
  2.2 Harmonic kernels 5
  2.3 Superharmonic functions, potentials 6
  2.4 Potential kernels 6
  2.5 Admissible pairs 7
  2.6 Young functions 7

3 First tools 9
  3.1 Semilinear perturbations 9
  3.2 Operators $L$ and $Q$ 12
  3.3 Martin type representation 16

4 The notion of the trace 17
  4.1 An existence lemma 17
  4.2 Properties of the trace 18
  4.3 Removable singularities 20

5 Polar sets 22
  5.1 Orlicz type spaces 22
  5.2 The Martin-Orlicz capacity 24
  5.3 Sufficient conditions for $\nu$ to be in $Q_\Psi(Y)$ 26

6 Applications to semilinear PDEs 30
  6.1 Continuous solutions to (6.2) 30
  6.2 Examples of $\Psi$ 31
  6.3 Examples of $\gamma$ 32
  6.4 Removable singularities 33
  6.5 A semilinear Dirichlet problem 34
  6.6 Solutions to problem (6.1) 35
  6.7 Parabolic setting 36

Appendix 38

References 41
1 Introduction

Let \( r \) be a reference measure relative to a given harmonic space \((X, \mathcal{H})\) in the sense of H. Bauer [7], and let \( \mathcal{H}_r^+(X) \) be the set of all positive harmonic functions on \( X \) (i.e., which belong to \( \mathcal{H}(X) \)) which are \( r \)-integrable. Developing an integral representation of functions in \( \mathcal{H}_r^+(X) \), K. Janssen determined in [31] a Polish space \( Y \) (minimal Martin boundary) and a function \( P : X \times Y \to \mathbb{R}_+ \) (Martin kernel) such that:

**Theorem 1.1 ([31]).** Every harmonic function \( h \in \mathcal{H}_r(X) := \mathcal{H}_r^+(X) - \mathcal{H}_r^+(X) \) has a unique representation

\[
h(x) = P\nu(x) := \int_Y P(x,y) \, d\nu(y) \quad (x \in X)
\]

where \( \nu \) belongs to the set \( \mathcal{M}(Y) \) of all signed Borel measures on \( Y \) with bounded variation. Conversely, \( P\nu \in \mathcal{H}_r(X) \) for any \( \nu \in \mathcal{M}(Y) \).

In this work we are interested in the analogous representation problem in a non-linear setting. To simplify the presentation of our approach let us suppose that the harmonic space \((X, \mathcal{H})\) possesses a Green function \( G_X \) (see [13, Sect. 4]), and assume that \( 1 \in \mathcal{H}(X) \). Standard examples of \((X, \mathcal{H})\) are:

1. (Elliptic case) \( X \) is a Greenian domain of \( \mathbb{R}^d \) and \( \mathcal{H} \) is the sheaf of classical harmonic functions (i.e., solutions to the Laplace equation).

2. (Parabolic case) \( X \) is a domain of \( \mathbb{R}^d \times \mathbb{R} \) and \( \mathcal{H} \) is the sheaf of parabolic functions in the terminology of [20] (i.e., solutions to the heat equation).

Any probability measure can serve as reference measure in Example 1, while this is not true in Example 2. However, a probability measure whose support is the whole space \( X \) is always a reference measure relative to \((X, \mathcal{H})\).

Let \( \Psi \) be a function in \( \mathcal{Y}(X) \) having the doubling property (see Subsection 2.6, for instance \( \Psi(x, t) = t|t|^\alpha - 1 \) where \( \alpha > 1 \)), and consider a positive Radon measure \( \gamma \) on \( X \) in the local Kato class \( K_{loc}^+(X) \), i.e., such that \( \int_K G_X(\cdot, \zeta) \, d\gamma(\zeta) \) is a bounded continuous potential on \( X \) for every compact set \( K \subset X \). A continuous function \( u \) on \( X \) is called a \( \mathcal{U} \)-function if, for every open relatively compact subset \( D \) of \( X \), the function

\[
u + \int_D G_D(\cdot, \zeta)\Psi(\zeta, u(\zeta)) \, d\gamma(\zeta)
\]

is harmonic on \( D \). If moreover \(|u| \leq h \) for some \( h \in \mathcal{H}_r^+(X) \), we say that \( u \) is moderate. We denote by \( \mathcal{U}(X) \) the set of all \( \mathcal{U} \)-functions on \( X \) and by \( \mathcal{U}_r(X) \) the set of all moderate functions in \( \mathcal{U}(X) \). First, we establish the following existence result:
Proposition 1.2. For every moderate \( U \)-function \( u \) on \( X \), there exists a unique measure \( \nu \in \mathcal{M}(Y) \), which will be denoted by \( \text{tr}(u) \) and called the trace of \( u \) on \( Y \), such that
\[
    u(x) + \int_X G_X(x, \zeta) \Psi(\zeta, u(\zeta)) \, d\gamma(\zeta) = P\nu(x) \quad (x \in X).
\]
Moreover, for all \( u, v \in \mathcal{U}_r(X) \), \( u \geq v \) if and only if \( \text{tr}(u) \geq \text{tr}(v) \).

We then extend the first part of Theorem 1.1 to the perturbed semilinear structure \((X, U)\) (observe that for \( \gamma = 0 \), \( \mathcal{U}_r(X) = \mathcal{H}_r(X) \) and \( \nu = \text{tr}(u) \) means that \( u = P\nu \)). Furthermore, although it may happen that (1.2) is not solvable for a given \( \nu \in \mathcal{M}(Y) \) (see [26]), the last part of the above proposition assures that (1.2) admits at most one solution \( u \in \mathcal{U}_r(X) \). This function \( u \) is interpreted as the solution of the (boundary value) problem
\[
    u \in \mathcal{U}_r(X) \quad \text{and} \quad u = \nu \quad \text{on} \ Y.
\]
In other words, (1.3) is considered to be equivalent to the integral equation (1.2).

The main purpose of this work is to investigate the set \( Q_{\Psi}(Y) \) consisting of all \( \nu \in \mathcal{M}(Y) \) for which (1.3) possesses a solution \( u \in \mathcal{U}_r(X) \).

Remark 1.3. [Details are in Subsection 6] Let \( \gamma \in K^+_{\text{loc}}(\mathbb{R}^d) \), \( \Psi \in Y(\mathbb{R}^d) \), and consider Example 1 where \( X = B \) is the unit open ball of \( \mathbb{R}^d \). Then \( Y = \partial B \) and a continuous function \( u \) on \( B \) is a solution of (1.3) if and only if it is a solution of the boundary value problem
\[
    \Delta u = \Psi(\cdot, u) \gamma \quad \text{in} \ B, \\
    u = \nu \quad \text{on} \ \partial B.
\]
In particular, (1.4) is solvable for every \( \nu = f\sigma \) where \( f \) is a continuous function on \( \partial B \) and \( \sigma \) is the surface area measure on \( \partial B \). Furthermore, the boundary condition \( u = \nu \) means, in this case, that \( \lim_{x \to y} u(x) = f(y) \) for all \( y \in \partial B \).

By means of minimal thin subsets of \( X \), we established in [25] necessary and sufficient conditions under which a given positive finite measure \( \nu \) on \( Y \) is a trace of some moderate \( U \)-function on \( X \). In the present work, we discuss the solvability of problem (1.3) by investigating some exceptional subsets of \( Y \).

Definitions. A Borel set \( E \subset Y \) is called removable if for every \( \nu \in \mathcal{M}^+(E) \) (i.e., \( \nu \in \mathcal{M}^+(Y) \) such that \( \nu(Y \setminus E) = 0 \)) the following holds:
\[
    u \in \mathcal{U}(X) \quad \text{and} \quad 0 \leq u \leq P\nu \quad \Rightarrow \quad u \equiv 0 \ \text{on} \ X.
\]
We say that \( E \) is \( c_\Psi \)-polar if for every \( \nu \in \mathcal{M}^+(E) \) the following holds:
\[
    \int_X \int_X G_X(x, \zeta) \Psi(\zeta, P\nu(\zeta)) \, d\gamma(\zeta) \, dr(x) < \infty \quad \Rightarrow \quad \nu = 0.
\]
In the situation of Example 1 and assuming that $X$ is bounded and Lipschitz, it will be shown (see Subsection 6.4) that a Borel subset $E$ of $\partial X$ ($Y = \partial X$) is removable if and only if for every $u \in U^+_r(X)$,

$$u = 0 \text{ on } \partial X \setminus E \Rightarrow u \equiv 0 \text{ on } X.$$  

A tool of vital importance in our study (especially in the proof of Theorem 1.5 below) is the Martin-Orlicz capacity $c_\Psi$ defined for every Borel subset $E$ $\subset Y$ by

$$c_\Psi(E) = \sup \{ \nu(E) : \nu \in M^+(E) \text{ and } \|P\nu\|_\Psi \leq 1 \}$$

where $\| \cdot \|_\Psi$ is the Orlicz norm in the Orlicz type space $L_\Psi(X)$ consisting of all (classes of equivalent) Borel measurable functions $f$ on $X$ such that

$$\int_X \int_X G_\lambda(x,\zeta)\Psi(\zeta,|f(\zeta)|) \, d\gamma(\zeta) \, dx < \infty$$

(for this characterization of $L_\Psi(X)$ the doubling property of $\Psi$ is used). Notice that $c_\Psi$-polar sets are subsets $E$ of $Y$ such that $c_\Psi(E) = 0$.

Among the important properties of $Q_\Psi(Y)$, we shall prove that $\nu \in Q_\Psi(Y)$ if and only if $|\nu| \in Q_\Psi(Y)$. This allows us to restrict our study of the solvability of problem (1.3) to the case when $\nu$ is positive. In particular, it will be not difficult to prove:

**Theorem 1.4.** If $\nu \in Q_\Psi(Y)$ then all removable subsets of $Y$ are $\nu$-null sets.

Imposing some additional assumptions on $\gamma$, we give sufficient conditions for (1.3) to be solvable. More precisely, we obtain the following result:

**Theorem 1.5.** If all $c_\Psi$-polar subsets of $Y$ are $\nu$-null sets then $\nu \in Q_\Psi(Y)$.

Consider once again Example 1 where $X$ is assumed to be bounded and sufficiently smooth. Then, for $r = \delta_{x_0}$ ($x_0 \in X$), $Y$ can be identified with the Euclidean boundary $\partial X$ of $X$, and $P$ is the normalized ($P(x_0, \cdot) \equiv 1$) Martin kernel on $X$ (here a possible choice for $\gamma$ is the restriction of the $d$-dimensional Lebesgue measure $\lambda$ to $X$, but $\gamma$ might as well be singular with respect to $\lambda$).

Let $\gamma = \lambda|_X$ and $\Psi(x,t) = t|t|^{\alpha-1}$, $\alpha > 1$. Then, for every $\nu \in M^+(\partial X)$, (1.3) is equivalent to the boundary value problem

$$\begin{align*}
\Delta u &= u^\alpha \text{ in } X, \\
u &= \nu \text{ on } \partial X,
\end{align*}$$

which has been investigated by various techniques (see [26, 37, 23, 22, 42]). In this setting, $L_\Psi(X)$ is a classical Lebesgue space and $c_\Psi$ coincides with the Martin
capacity $c_\alpha$ introduced in [22]. It is shown (Le Gall [37] for $\alpha = 2$, Dynkin and Kuznetsov [23] for $\alpha \leq 2$, Marcus and Véron [42] for $\alpha > 2$) that for every Borel subset $E$ of $\partial X$, $E$ is removable if and only if $c_\alpha(E) = 0$. Consequently, (1.5) has a solution if and only if $\nu$ does not charge $c_\alpha$-polar subsets of $\partial X$. It will be shown that, in general, this condition does not characterize the class $\mathcal{Q}_\Psi(Y)$. In fact, we shall give an example (see Remark 6.5) for which the converse statement in Theorem 1.5 does not hold.

After recalling in Section 2 the basic notions and facts on harmonic spaces, we study in Section 3 semilinear perturbations of harmonic spaces. In Section 4, we introduce the trace of a moderate $U$-function and give its first properties. In the last part of the same section, we investigate removable sets and prove Theorem 1.4 (Proposition 4.4). Section 5 deals with the Martin-Orlicz capacity $c_\Psi$ and the proof of Theorem 1.5 (Theorem 5.7). Finally, as application of our work, Section 6 is devoted to a study of semilinear problems of the type (1.4).
2 Preliminaries

In the following \((X, \mathcal{H})\) will always denote a harmonic space in the sense of H. Bauer [7] such that the constant functions are harmonic on \(X\). We shall recall in this section the basic notions and facts on harmonic spaces that we need (for more details see [5, 7, 11, 14, 18, 20, 29]). The reader who is not familiar with these notions and is mainly interested in boundary value problems of the kind (1.4) may simply restrict himself to Example 1 already mentioned in the introduction. Section 6 will deal explicitly with this situation.

2.1 Basic notations

Given a set \(\mathcal{F}\) of numerical functions, \(\mathcal{F}_b\) (\(\mathcal{F}^+\) resp.) will denote the set of all functions in \(\mathcal{F}\) which are bounded (positive resp.). For every open subset \(\Omega\) of \(X\) let \(\mathcal{B}(\Omega)\) (\(\mathcal{C}(\Omega)\) resp.) be the set of all Borel measurable numerical (continuous real resp.) functions on \(\Omega\). By \(\mathcal{B}_{bc}(\Omega)\) we shall denote the set of all functions in \(\mathcal{B}_b(\Omega)\) with compact support in \(\Omega\).

For \(A \subset X\) we denote by \(A^c\) the complement of \(A\) in \(X\) and define \(1_A\) to be the characteristic function of \(A\): \(1_A(x) = 1\) if \(x \in A\) and \(1_A(x) = 0\) if \(x \in A^c\).

Given a topological space \(T\), \(\mathcal{M}(T)\) will denote the set of all signed Borel measures \(\mu\) on \(T\) such that \(\|\mu\| = |\mu|(T)\) is finite. Recall that \(|\mu| = \mu^+ + \mu^-\) where \(\mu^+ = \sup(\mu, 0)\) and \(\mu^- = \sup(-\mu, 0)\). For any Borel set \(E \subset T\), we denote by \(\mu_E\) the restriction of \(\mu\) to \(E\) and by \(\mathcal{M}(E)\) the set of all \(\mu \in \mathcal{M}(T)\) which are supported by \(E\) (i.e., \(\mu(T \setminus E) = 0\)). Finally, by a kernel on \(T\) we shall mean a family \((k(\tau, \cdot))_{\tau \in T}\) of Borel measures on \(T\) such that \(\int f(t)k(\cdot, dt) = kf \in \mathcal{B}^+(T)\) for every \(f \in \mathcal{B}^+(T)\).

2.2 Harmonic kernels

Let \(\mathcal{O}\) be the set of all open relatively compact subsets of \(X\) and let \(\Omega \in \mathcal{O}\). A Borel measurable function \(f\) on \(\partial\Omega\) is resolutive if and only if \(f\) is \(\mu_x^\Omega\)–integrable for all \(x \in \Omega\) where \(\mu_x^\Omega\) is the harmonic measure of \(x\) with respect to \(\Omega\) (see [7]). To each resolutive function \(f \in \mathcal{B}(\partial\Omega)\) we associate the harmonic function \(H_{\Omega}f\) on \(\Omega\) given by

\[
H_{\Omega}f(x) = \int_{\partial\Omega} f(y) d\mu_x^\Omega(y).
\]

If \(f \in \mathcal{B}(X)\) such that the restriction of \(f\) to \(\partial\Omega\) is resolutive we define

\[
H_{\Omega}f(x) = \begin{cases} 
H_{\Omega}(f|_{\partial\Omega})(x) & \text{if } x \in \Omega, \\
fv(x) & \text{if } x \in X \setminus \Omega.
\end{cases}
\]
We call $H_{\Omega}$ the harmonic kernel associated to $\Omega$. A point $z \in \partial \Omega$ is called regular provided
\[ f(z) = \lim_{x \in \Omega, x \to z} H_{\Omega} f(x) \]
for every $f \in C(\partial \Omega)$, and we say that $\Omega$ is regular if all points $z \in \partial \Omega$ are regular.

### 2.3 Superharmonic functions, potentials

For every open subset $\Omega$ of $X$ let $S(\Omega)$ be the set of all lower semicontinuous (l.s.c) functions $s > -\infty$ on $\Omega$ such that for every $D \in \mathcal{O}$ with $\overline{D} \subset \Omega$,
\[ H_D s \in H(D) \quad \text{and} \quad H_D s \leq s. \]
Functions in $S(\Omega)$ ($-S(\Omega)$ resp.) are called superharmonic (subharmonic resp.) on $\Omega$. A potential on $\Omega$ is a function $p \in S^+(\Omega)$ such that the constant zero is the greatest harmonic minorant of $p$ on $\Omega$. Let $\mathcal{P}(\Omega)$ denote the set of all potentials on $\Omega$.

We suppose that $\mathcal{P}(X)$ contains a strictly positive function on $X$.

### 2.4 Potential kernels

Throughout this work we fix a potential kernel $V_X$ on $X$, that is, $V_X$ is a kernel on $X$ such that for every $f \in B_b^+(X)$
\[ V_X f \in \mathcal{P}(X) \cap C_b(X) \cap \mathcal{H}\left(X \backslash \{f \neq 0\}\right). \quad (2.1) \]
If moreover $V_X(1_D) \neq 0$ on $X$ for every nonempty open subset $D$ of $X$ we shall say that the potential kernel $V_X$ is strictly positive. For each $\Omega \in \mathcal{O}$ (open and relatively compact) we define
\[ V_\Omega := V_X - H_{\Omega} V_X. \quad (2.2) \]
Then $V_\Omega$ is a potential kernel on $\Omega$ and $V_\Omega(\mathcal{B}_b^+(\Omega)) \subset \mathcal{P}(\Omega) \cap C_b(\Omega)$. Furthermore, it is not hard to verify that the family $(V_\Omega)_{\Omega \in \mathcal{O}}$ is compatible, in the sense that for any $\Omega_1, \Omega_2 \in \mathcal{O}$ and any $f \in \mathcal{B}_b(\Omega_1 \cup \Omega_2)$
\[ V_{\Omega_1} f - V_{\Omega_2} f \in \mathcal{H}(\Omega_1 \cap \Omega_2). \]

**Remark 2.1.** Suppose that for every $\Omega \in \mathcal{O}$, $W_{\Omega}$ is a potential kernel on $\Omega$ so that $(W_\Omega)_{\Omega \in \mathcal{O}}$ is compatible. Then, in view of [7, Satz 5.3.6] there exists a unique potential kernel $W_X$ on $X$ such that $W_\Omega = W_X - H_{\Omega} W_X$ for every $\Omega \in \mathcal{O}$. More on potential kernels (also for balayage spaces) can be found in [28, Sect.2].
Assuming that \( X \) has a (continuous) Green function \( G_X \) (see [13] for the definition of \( G_X \)), a positive Radon measure \( \gamma \) on \( X \) is called a \textit{local Kato measure} on \( X \) if \( V^\gamma_X \) defined by

\[
V^\gamma_X f := \int_X G_X(\cdot, \zeta)f(\zeta) \, d\gamma(\zeta) \quad (2.3)
\]

is a potential kernel on \( X \). Notice that \( V^\gamma_X \) is strictly positive if and only if \( \gamma \) charges every nonempty subset of \( X \).

### 2.5 Admissible pairs

A closed subset \( A \) of \( X \) is called an \textit{absorbing set} if it contains the support of every harmonic measure \( \mu^D_x \) for any \( x \in A \) and any regular open relatively compact set \( D \) containing \( x \). We say that a probability measure on \( X \) is a \textit{reference measure} if the only absorbing set containing its support is the whole space \( X \). A pair \( (V, r) \) of a potential kernel \( V \) on \( X \) and a reference measure \( r \) on \( X \) will be said to be \textit{admissible} if the following conditions are fulfilled:

\begin{enumerate}[(AP1)]
  \item \( V \) is strictly positive.
  \item For every compact subset \( K \subset X \), there are \( \Omega \in \mathcal{O} \) and \( c > 0 \) such that \( K \subset \Omega \) and the inequality
    \[
    \sup_{x \in K} |h(x)| \leq c \int_\Omega |h| \, dr \quad (2.4)
    \]
    holds for all \( h \in H_0(\Omega) \).
\end{enumerate}

We say that \( (\gamma, r) \) is an admissible pair provided \( \gamma \) is a local Kato measure on \( X \) and conditions (AP1)-(AP2) hold for \( V = V^\gamma_X \) given by (2.3). See Section 6 for some examples of admissible pairs.

### 2.6 Young functions

An odd strictly increasing function \( Y : \mathbb{R} \to \mathbb{R} \) will be called a \textit{Young function} if it is convex on \( \mathbb{R}_+ \), \( \lim_{t \to 0} Y(t)/t = 0 \) and \( \lim_{t \to \infty} Y(t)/t = \infty \). Let \( Y_0 \) be the set of all Young functions and define \( \mathcal{Y}(X) \) to be the class of all Borel measurable functions \( \Psi : X \times \mathbb{R} \to \mathbb{R} \) satisfying the following properties:

\begin{enumerate}[(i)]
  \item The functions \( \Psi(x, \cdot) \) are in \( Y_0 \) for all \( x \in X \).
  \item For every compact subset \( K \) of \( X \) there exist \( M_K, N_K \in Y_0 \) such that
    \[
    M_K(t) \leq \Psi(x, t) \leq N_K(t) \quad \text{for all } (x, t) \in K \times \mathbb{R}_+ .
    \]
\end{enumerate}
Clearly $\mathcal{Y}_0 \subset \mathcal{Y}(X)$ and for any $\Psi \in \mathcal{Y}(X)$ the following holds:

$$(A_1)$$ For every $x \in X$, $\Psi(x, \cdot)$ is continuous, odd, and increasing on $\mathbb{R}$.

$$(A_2)$$ The function $\Psi$ is locally bounded on $X \times \mathbb{R}$.

$$(A_3)$$ $\Psi(x, t + s) \geq \Psi(x, t) + \Psi(x, s)$ for all $x \in X$ and all $t, s \geq 0$.

$$(A_4)$$ For every $x \in X$, $\Psi(x, \cdot)$ is convex on $\mathbb{R}_+$. 

To each $\Psi \in \mathcal{Y}(X)$ we associate the function $\Psi^*$ defined on $X \times \mathbb{R}$ by

$$\Psi^*(x, t) = \text{sgn}(t) \sup_{s \geq 0} (s|t| - \Psi(x, s)). \quad (2.5)$$

It is well known (see, e.g., [33, 34]) that $\Psi^* \in \mathcal{Y}_0$ for any $\Psi \in \mathcal{Y}_0$. Analogously, it is easy to remark that $\Psi^* \in \mathcal{Y}(X)$ and $(\Psi^*)^* = \Psi$ if $\Psi \in \mathcal{Y}(X)$.

We shall say that a real function $\Psi$ on $X \times \mathbb{R}$ has the \textit{doubling property} if there exists a constant $\kappa > 0$ such that

$$\Psi(x, 2t) \leq \kappa \Psi(x, t) \text{ for all } (x, t) \in X \times \mathbb{R}_+. \quad (2.6)$$

In the theory of Orlicz spaces, this property is known as $\Delta_2$-\textit{condition}.

If $\Psi \in \mathcal{Y}(X)$, it can be shown that $\Psi^*$ possesses the doubling property if and only if the function $\Psi$ satisfies the $\nabla_2$-\textit{condition}: There exists $\ell > 1$ such that

$$\Psi(x, \ell t) \geq 2\ell \Psi(x, t) \text{ for all } (x, t) \in X \times \mathbb{R}_+. \quad (2.7)$$
3 First tools

Assumptions of this section: Ψ is a Borel measurable real function on $X \times \mathbb{R}$ which satisfies $(A_1)$ and $(A_2)$.

3.1 Semilinear perturbations

For every $\Omega \in \mathcal{O}$ (or $\Omega = X$) we define

$$V_{\Omega}^\Psi f := V_{\Omega} \Psi(\cdot, f)$$

whenever the right side in (3.1) has a sense. Then, for any open set $D$ such that $\overline{D} \subset \Omega$ we easily see, in view of (2.2), that

$$V_{\Omega}^\Psi = V_{D}^\Psi + H_{D}V_{\Omega}^\Psi. \quad (3.2)$$

Notice that for $\Omega = X$ we may write $V$ instead of $V_{X}$ and $V_{\Omega}^\Psi$ instead of $V_{X}^\Psi$.

Proposition 3.1. (Comparison principle) Let $\Omega \in \mathcal{O} \cup \{X\}$ and let $f, g$ be two real Borel measurable functions on $\Omega$ such that $V_{\Omega}^\Psi |f|$ and $V_{\Omega}^\Psi |g|$ are finite potentials on $\Omega$ and the function $f - g + V_{\Omega}^\Psi f - V_{\Omega}^\Psi g$ is superharmonic on $\Omega$. Then

$$f \geq g \text{ if and only if } f + V_{\Omega}^\Psi f \geq g + V_{\Omega}^\Psi g.$$  

Proof. Since $\Psi(x, \cdot)$ is increasing for any $x \in X$ we easily see that

$$f + V_{\Omega}^\Psi f \geq g + V_{\Omega}^\Psi g$$

whenever $f \geq g$ on $\Omega$. To prove the converse statement let

$$\phi = \Psi(\cdot, f) - \Psi(\cdot, g)$$

and suppose that $f + V_{\Omega}^\Psi f \geq g + V_{\Omega}^\Psi g$ on $\Omega$. Then $s := f - g + V_{\Omega}\phi^+$ is a positive superharmonic function on $\Omega$ and

$$s \geq V_{\Omega}\phi^+ \text{ on } \{\phi^+ > 0\}. \quad (3.3)$$

Therefore, by the same arguments as in the proof of Proposition 2.4 of [13], it follows from (3.3) that $s$ dominates $V_{\Omega}\phi^+$ on $\Omega$. Thus $f \geq g$ on $\Omega$. \hfill \Box

Corollary 3.2. Let $\Omega \in \mathcal{O}$, $f, g$ as in the previous proposition and assume moreover that $\liminf_{x \to z}[f(x) - g(x)] \geq 0$ for all $z \in \partial \Omega$. Then $f \geq g$ on $\Omega$.  

**First tools**

**Proof.** We only need to prove that \( s = f + V_\Omega^\Psi f - g - V_\Omega^\Psi g \) is positive on \( \Omega \). Let again \( \phi = \Psi(\cdot, f) - \Psi(\cdot, g) \) then

\[
s + V_\Omega^\phi = f - g + V_\Omega^\phi^+.
\]

Since \( s + V_\Omega^\phi^- \) is superharmonic on \( \Omega \) and \( \liminf_{x \to z} s(x) \geq 0 \) for every \( z \in \partial \Omega \), the minimum principle relative to the harmonic space \( (X, \mathcal{H}) \) implies that

\[
s + V_\Omega^\phi^- \geq 0 \quad \text{on} \quad \Omega.
\]

This in turn yields that \( s \geq 0 \) on \( \Omega \).

The following theorem is recently shown in [6] for a general setting. We give here the proof for the sake of completeness.

**Theorem 3.3.** For every \( \Omega \in \mathcal{O} \) and every \( f \in \mathcal{B}_b(\partial \Omega) \), there exists a unique bounded continuous function \( u \) on \( \Omega \), which will be denoted by \( U_\Omega f \), satisfying

\[
u + V_\Omega^\Psi u = H_\Omega f.
\]

**Proof.** We only have to prove the existence of \( u \). In fact, the uniqueness of \( u \) satisfying (3.4) is assured by the comparison principle.

Take \( \Omega \in \mathcal{O} \), \( f \in \mathcal{B}_b(\Omega) \) and let \( a = \sup_{\partial \Omega} |f| \). The function \( \Psi_a \) defined on \( X \times \mathbb{R} \) by

\[
\Psi_a(x, t) = \text{sgn}(t)\Psi(x, \min(|t|, a))
\]

satisfies the assumptions \((A_1)\) and \((A_2)\). For every \( v \in \mathcal{B}_b(\Omega) \) consider

\[
\Lambda(v) := H_\Omega f - V_\Omega^{\Psi_a} v.
\]

It is easy verified that \( V_\Omega^{\Psi_a}(\mathcal{B}_b(\Omega)) \) is a bounded subset of \( \mathcal{B}_b(\Omega) \). So, since \( V_\Omega \) is a compact operator on \( \mathcal{B}_b(\Omega) \) (see [27, Proposition 3.1]), it follows from Schauder’s fixed point theorem that \( \Lambda(u) = u \) for some \( u \in \mathcal{B}_b(\Omega) \). Remark now that \( |u| \leq a \) by Proposition 3.1, which yields that \( V_\Omega^{\Psi_u} u = V_\Omega^\Psi u \). Consequently, (3.4) holds and the proof is finished.

If \( \Omega \in \mathcal{O} \) and \( f \) is a Borel measurable function on a set containing \( \overline{\Omega} \) such that \( f \) is bounded on \( \partial \Omega \) we shall denote by \( U_\Omega f \) the function which equals \( U_\Omega(f|_{\partial \Omega}) \) on \( \Omega \) and equals \( f \) elsewhere. Clearly, the mapping \( U_\Omega \) is odd and increasing.

For every open subset \( \Omega \subset X \) we define \( \mathcal{U}^*(\Omega) \) to be the set of all l.s.c locally bounded functions \( u \) on \( \Omega \) such that

\[
U_D u \leq u \quad \text{for all} \quad D \in \mathcal{O} \text{ with } \overline{D} \subset \Omega.
\]
We also define
\[ U_*(\Omega) := -U^*(\Omega), \ U(\Omega) := U^*(\Omega) \cap U_*(\Omega), \]
and we call \( U \)-function \( (U^* \)-function, \( U_* \)-function resp.) on \( \Omega \) every element of \( U(\Omega) \) \( (U^*(\Omega), U_*(\Omega) \) resp.).

**Remark 3.4.** Using (3.2) and (3.4) it is easy verified that for all \( D, \Omega \in \mathcal{O} \) such that \( D \subset \Omega \) we have
\[ U_D \circ U_\Omega = U_\Omega. \quad (3.5) \]

Therefore, \( U_\Omega f \) is a \( U \)-function on \( \Omega \) for every \( \Omega \in \mathcal{O} \) and every \( f \in \mathcal{B}_b(\partial \Omega) \). If, moreover, \( \Omega \) is regular and \( f \) is continuous on \( \partial \Omega \) then \( U_\Omega f \) is the unique continuous extension of \( f \) to \( \Omega \) which is a \( U \)-function on \( \Omega \).

**Theorem 3.5.** If \( \Omega \in \mathcal{O} \) and \( u \in \mathcal{B}_b(\Omega) \) then \( u \in U(\Omega) \) \( (U^*(\Omega) \) resp.) if and only if \( u + V_\Omega^\psi u \in \mathcal{H}(\Omega) \) \( (\mathcal{S}(\Omega) \) resp.). In particular, if \( u \in \mathcal{B}(\Omega) \) is locally bounded on \( \Omega \) where \( \Omega \) is an arbitrary open subset of \( X \), then \( u \in U(\Omega) \) \( (U^*(\Omega) \) resp.) if and only if \( u + V_D^\psi u \in \mathcal{H}(D) \) \( (\mathcal{S}(D) \) resp.) for every \( D \in \mathcal{O} \) such that \( \overline{D} \subset \Omega \).

**Proof.** Let \( u \in \mathcal{B}_b(\Omega) \) and let \( D \in \mathcal{O} \) such that \( \overline{D} \subset \Omega \). From (3.2) and (3.4) we get that
\[
  u + V_\Omega^\psi u - H_D V_\Omega^\psi u = u + V_D^\psi u,
  H_D(u + V_\Omega^\psi u) - H_D V_\Omega^\psi u = U_D u + V_D^\psi U_D u.
\]
Therefore Proposition 3.1 completes the proof. \( \Box \)

Combining the above theorem and Corollary 3.2 we obtain:

**Corollary 3.6.** Let \( \Omega \in \mathcal{O} \) and let \( u, v \in \mathcal{B}_b(\Omega) \) such that \( \liminf_{x \to z} [u(x) - v(x)] \geq 0 \) for all \( z \in \partial \Omega \). If \( u \in U^*(\Omega) \) and \( v \in U_*(\Omega) \) then \( u \geq v \) on \( \Omega \).

We deduce from Theorem 3.5 that \( U(\Omega) \) is closed under uniform convergence on compact subsets of \( \Omega \). Note also that all positive \( U_* \)-function on \( \Omega \) are subharmonic on \( \Omega \).

**Theorem 3.7.** Let \( \Omega \subset X \) be an open subset and let \( (u_n) \) be a sequence of \( U \)-functions on \( \Omega \) which are locally uniformly bounded on \( \Omega \). The following holds:

(a) If \( (u_n) \) increases to \( u \) then \( u \) is a \( U \)-function on \( \Omega \).

(b) There exists a subsequence of \( (u_n) \) which converges locally uniformly on \( \Omega \). In particular, if \( (u_n) \) converges pointwise to a function \( u \) then \( u \in U(\Omega) \) and \( (u_n) \) converges uniformly to \( u \) on every compact subset of \( \Omega \).
Proof. Take $D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$. For every $n \geq 1$ let

$$h_n = u_n + V_D^\Psi u_n.$$  

(a) Since $(h_n)$ is an increasing sequence of harmonic functions on $D$ and is uniformly bounded, we conclude that $h = \sup_{n \geq 1} h_n$ is harmonic on $D$. Passing to the limit in the above formula we obtain that $u + V_D^\Psi u = h$. So, by Theorem 2.3, statement (a) is proved.

(b) Let $K \subset D$ be a compact subset and choose a subsequence $(h_{n_k})$ of $(h_n)$ which converges uniformly on $K$. Since the family $\{V_D^\Psi u_n : k \geq 1\}$ is equicontinuous [27, Proposition 3.1], by Ascoli’s theorem there exists a subsequence $(v_k)$ of $(u_{n_k})$ such that $(V_D^\Psi v_k)$ converges uniformly on $K$. Consequently, $(v_k)$ is uniformly convergent on $K$. Now, in order to show the first statement of (b) it will be enough to use an exhaustion $(\Omega_n)$ of $X$ and apply the diagonal procedure. The second statement in (b) is obvious.

To finish this subsection, let us note that various kinds of perturbations of harmonic spaces were investigated by several authors. The reader is referred to [13, 32] for the linear setting and to [39, 45, 9, 10, 12, 6] for nonlinear cases.

### 3.2 Operators $L$ and $Q$

In the following, we fix an exhaustion $(\Omega_n)$ of $X$, that is, $\Omega_n \in \mathcal{O}$, $\overline{\Omega}_n \subset \Omega_{n+1}$ for every $n \geq 1$, and $X = \bigcup_{n \geq 1} \Omega_n$. Clearly, for every $f \in \mathcal{B}^+(X)$

$$Vf = \lim_{n \to \infty} V_{\Omega_n} f.$$  

The following convergence lemma follows easily from the fact that $V$ and $V_{\Omega_n}$ are kernels.

**Lemma 3.8.** Let $f,f_n \in \mathcal{B}(X)$ and let $g,g_n \in \mathcal{B}^+(X)$. The following holds:

(a) $V(\liminf_{n \to \infty} g_n) \leq \liminf_{n \to \infty} V_{\Omega_n} g_n$.

(b) Assume that $|f_n| \leq g_n$ for all $n \geq 1$, and $(f_n),(g_n),(V_{\Omega_n} g_n)$ converge pointwise to $f,g,Vg$ respectively. If $Vg < \infty$ then $\lim_{n \to \infty} V_{\Omega_n} f_n = Vf$.

We shall use the operators $L$ and $Q$ which are introduced in [25] in order to study a Liouville property related to equations of the type $\Delta u = \Psi(\cdot,u)\gamma$. For every positive harmonic function $h$ on $X$ we consider

$$Lh := \inf_{\Omega \in \mathcal{O}} U_\Omega h \quad \text{and} \quad Qh := \sup_{\Omega \in \mathcal{O}} H_\Omega Lh.$$
Lemma 3.9. Let $\Omega, D \in \mathcal{O}$ such that $\overline{D} \subset \Omega$ and let $s$ be a positive, locally bounded, superharmonic function on a neighborhood of $\overline{\Omega}$. Then $U_D s \geq U_\Omega s$.

Proof. From the formula $U_\Omega s + V_{\Omega}^\Psi s = H_\Omega s$ we have $0 \leq U_\Omega s \leq H_\Omega s$ and consequently $0 \leq U_\Omega s \leq s$. So the monotonicity of $U_D$ and (3.5) imply that $U_\Omega s \leq U_D s$.

\[ U_D s \leq U_\Omega s \leq s. \]

Theorem 3.10. Let $h \in \mathcal{H}^+(X)$. The following holds:

(a) $Lh \in \mathcal{U}^+(X)$, $Qh \in \mathcal{H}^+(X)$, and we have

\[ Lh \leq Qh \leq h, \] \[ Lh + V^\Psi Lh = Qh. \]

(b) If $V^\Psi h < \infty$ then $Qh = h$.

(c) $L$ and $Q$ are monotone increasing on $\mathcal{H}^+(X)$.

(d) $Lh$ and $Qh$ can be characterized as follows:

\[ Lh = \max \{ u \in \mathcal{U}^+(X) : u \leq h \} \] \[ = \max \{ u \in \mathcal{U}(X) : |u| \leq h \}. \]

\[ Qh = \min \{ g \in \mathcal{H}^+(X) : g \geq Lh \} \] \[ = \max \{ g \in \mathcal{H}^+(X) : g \leq h; Qg = g \}. \]

(e) $L \circ Q = L$ and $Q \circ Q = Q$.

Proof. (a) By Lemma 3.9, the sequence $(U_{\Omega_n}h)$ is decreasing and

\[ Lh = \lim_{n \to \infty} U_{\Omega_n}h. \] \[ Qh = \lim_{n \to \infty} H_{\Omega_n}Lh. \]

Because $0 \leq U_{\Omega_n}h \leq h$ for every $n \geq 1$, Theorem 3.7.b assures that $Lh$ is a $\mathcal{U}$-function on $X$. Now, since $0 \leq Lh \leq h$ and $Lh$ is subharmonic on $X$ we conclude that the sequence $(H_{\Omega_n}Lh)$ is increasing and

\[ \lim_{n \to \infty} H_{\Omega_n}Lh. \] \[ \text{Whence, the fact that } Lh \leq H_{\Omega_n}Lh \leq h \text{ yields that } Qh \in \mathcal{H}^+(X) \text{ and the inequality (3.6) holds. To get (3.7) it suffices to pass to the limit in the formula } \]

\[ Lh + V_{\Omega_n}^\Psi Lh = H_{\Omega_n}Lh. \]

(b) Since $0 \leq U_{\Omega_n}h \leq h$ and

\[ U_{\Omega_n}h + V_{\Omega_n}^\Psi U_{\Omega_n}h = h. \]
by Lemma 3.8 we obtain that $Lh + V^\Psi Lh = h$. Therefore, $h = Qh$ in virtue of (3.7).

c) Trivial.

d) To justify (3.8) and (3.9) it is enough to show that $Lh \geq |u|$ for every $u \in U(X)$ satisfying $|u| \leq h$. So, if $u$ is a such function then for all $n \geq 1$

$$|u| = |U_{\Omega_n} u| \leq U_{\Omega_n} h,$$

and therefore $|u| \leq Lh$.

The equality (3.10) is a consequence of (3.6) and the monotonicity of the harmonic kernel $H_\Omega$ for any $\Omega \in \mathcal{O}$. To obtain (3.11) it suffices to use the fact that $Q(Qh) = Qh$ which is given by the statement (e).

e) Since $Lh \leq Qh \in \mathcal{H}^+(X)$, we conclude by (3.8) that $Lh \leq LQh$ and therefore $Qh = Lh + V^\Psi Lh \leq LQh + V^\Psi LQh = Q(Qh) \leq Qh$.

Thus $Q(Qh) = h$ and, by comparison principle, $L(Qh) = Lh$.

\begin{proof}

\textbf{Lemma 3.11.} Let $\Omega \in \mathcal{O}$, and let $\alpha, \beta \geq 0$ such that

$$\Psi(x, \alpha t + \beta s) \geq \alpha \Psi(x, t) + \beta \Psi(x, s) \quad \text{for all } x \in X, t, s \geq 0. \quad (3.15)$$

Then

$$U_\Omega(\alpha f + \beta g) \leq \alpha U_\Omega f + \beta U_\Omega g \quad \text{for all } f, g \in \mathcal{B}_b^+(\partial \Omega). \quad (3.16)$$

Furthermore, the converse inequality in (3.15) implies the converse one in (3.16).

\begin{proof}

Let $f, g \in \mathcal{B}_b^+(\partial \Omega)$ and denote by $u = U_\Omega f$, $v = U_\Omega g$ and $w = U_\Omega (\alpha f + \beta g)$. Then

$$\phi := \Psi(\cdot, \alpha u + \beta v) - \alpha \Psi(\cdot, u) - \beta \Psi(\cdot, v) \in \mathcal{B}_b^+(\Omega)$$

which implies that

$$V^\Psi_\Omega (\alpha u + \beta v) - \alpha V^\Psi_\Omega u - \beta V^\Psi_\Omega v = V^\Psi_\Omega \phi \in \mathcal{P}(\Omega) \cap \mathcal{C}_b(\Omega).$$

From (3.4) it follows that

$$\alpha u + \beta v + V^\Psi_\Omega (\alpha u + \beta v) = H_\Omega(\alpha f + \beta g) + V^\Psi_\Omega \phi,$$

$$w + V^\Psi_\Omega w = H_\Omega(\alpha f + \beta g).$$

Therefore, applying Proposition 3.1 we get that $\alpha u + \beta v \geq w$ which finishes the proof. Clearly the second statement can be proved in a similar way. \qed
\end{proof}

\end{proof}
Corollary 3.12. (a) If $(A_3)$ holds then $L$ and $Q$ are subadditive on $\mathcal{H}^+(X)$.
(b) If $(A_4)$ holds then $L$ and $Q$ are concave (and also subadditive) on $\mathcal{H}^+(X)$.

Proof. (a) Assumption $(A_3)$ means that \((3.15)\) holds true for $\alpha = \beta = 1$. Hence, by the previous lemma, $U_\Omega$ is subadditive on $\mathcal{B}(\partial \Omega)$ for every $\Omega \in \mathcal{O}$. This, $(3.12)$ and $(3.13)$ prove statement (a).

(b) To see that $L$ and $Q$ are concave it is enough to apply again Lemma 3.11 for all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. It is not hard to see that under $(A_4)$, assumption $(A_4)$ yields $(A_3)$. So, if $(A_4)$ holds we conclude by statement (a) that $L$ and $Q$ are subadditive on $\mathcal{H}^+(X)$.

Corollary 3.13. Suppose that $(A_3)$ is satisfied and let $(h_n)$ be an increasing sequence in $\mathcal{H}^+(X)$ such that $h := \sup_{n \geq 1} h_n \in \mathcal{H}^+(X)$. Then

$$\sup_{n \geq 1} Lh_n = Lh \text{ and } \sup_{n \geq 1} Qh_n = Qh.$$  

Proof. By $(3.6)$ and Corollary 3.12, we obtain for every $n \geq 1$ that

$$0 \leq Lh - Lh_n \leq h - h_n \text{ and } 0 \leq Qh - Qh_n \leq h - h_n.$$  

This completes the proof.

Proposition 3.14. Suppose that $(A_3)$ holds and the function $\Psi$ has the doubling property. Then $Q$ is "linear" on $\mathcal{H}^+(X)$, i.e., for all functions $g, h \in \mathcal{H}^+(X)$ and every $\alpha \geq 0$,

$$Q(\alpha g + h) = \alpha Qg + Qh. \quad (3.17)$$

Proof. Let $g, h \in \mathcal{H}^+(X)$, $u_n = U_{\Omega_n}(Qg)$, $v_n = U_{\Omega_n}(Qh)$ and $w_n = U_{\Omega_n}(Qg + Qh)$. By Lemma 3.11, we have $w_n \leq u_n + v_n$ and hence

$$0 \leq \Psi(\cdot, w_n) \leq \Psi(\cdot, u_n + v_n) \leq \kappa(\Psi(\cdot, u_n) + \Psi(\cdot, v_n)) =: \phi_n$$

where $\kappa$ is the constant given in (2.6). On the other hand, the continuity of $\Psi(x, \cdot)$ and statement (e) of Theorem 3.10 imply that

$$\lim_{n \to \infty} \phi_n = \kappa(\Psi(\cdot, Lg) + \Psi(\cdot, Lh)) =: \phi, \quad \text{and} \quad \lim_{n \to \infty} V_{\Omega_n} \phi_n = V\phi = \kappa(V^\Psi Lg + V^\Psi Lh) < \infty.$$  

Then Lemma 3.8.b shows that $(V_{\Omega_n}^\Psi w_n)$ converges to $V^\Psi L(Qg + Qh)$. So, letting $n$ tend to infinity in the formula $w_n + V_{\Omega_n}^\Psi w_n = Qg + Qh$ we obtain that

$$L(Qg + Qh) + V^\Psi L(Qg + Qh) = Qg + Qh.$$
This means that $Q(Qg + Qh) = Qg + Qh$ and consequently $Qg + Qh \leq Q(g + h)$ by monotonicity of $Q$ on $\mathcal{H}^+(X)$. Therefore, according to Corollary 3.12.a we get that

$$Q(g + h) = Qg + Qh.$$ 

Finally, this additivity property of $Q$, Corollary 3.13 and the density of $Q_+$ in $\mathbb{R}_+$ yield that $Q$ is positively homogeneous on $\mathcal{H}^+(X)$. \qed

### 3.3 Martin type representation

From now on $r$ is a fixed reference measure on $X$. Define $\mathcal{H}_r^+(X)$ to be the set of all positive harmonic functions which are integrable on $X$ with respect to $r$ and let

$$\mathcal{H}_r(X) := \mathcal{H}_r^+(X) - \mathcal{H}_r^+(X).$$

We know [31] that there exist a Polish space $Y$ and a family $(P(\cdot, y))_{y \in Y}$ of positive harmonic functions on $X$ such that:

1. The map $y \mapsto P(\cdot, y)$ is one-to-one from $Y$ to the set of all minimal harmonic functions $h$ on $X$ satisfying $\int_X h \, dr = 1$. (Recall that a function $h \in \mathcal{H}_r^+(X)$ is called minimal if $h \not\equiv 0$ and if every harmonic function $g$ satisfying the inequality $0 \leq g \leq h$ is a constant multiple of $h$.)

2. For every $x \in X$, the function $P(x, \cdot) : y \mapsto P(x, y)$ is continuous on $Y$.

3. The formula

$$h = P\nu := \int_Y P(\cdot, y) \, d\nu(y) \quad (3.18)$$

defines a one-to-one correspondence between $h \in \mathcal{H}_r(X)$ and $\nu \in \mathcal{M}(Y)$. Furthermore for any $\nu \in \mathcal{M}(Y)$,

$$|\nu|(Y) = \int_X P|\nu| \, dr;$$

and $\nu \geq 0$ if and only if $P\nu \geq 0$.

**Remark 3.15.** If $X$ is a Greenian domain of $\mathbb{R}^d$ and $\mathcal{H}$ is the classical sheaf of harmonic functions, $(Y, P)$ can be chosen so that $Y$ is the minimal part of the Martin boundary and $P(\cdot, y)$ is the Martin function with pole at $y \in Y$. 

4 The notion of the trace

Assumptions of this section: \( \Psi \) is a Borel measurable real-valued function on \( X \times \mathbb{R} \) which satisfies \((A_1),(A_2)\) and \((A_3)\).

4.1 An existence lemma

We consider the subset \( \mathcal{U}_r(X) \) of \( \mathcal{U}(X) \) given by

\[
\mathcal{U}_r(X) := \{ u \in \mathcal{U}(X) : |u| \leq h \text{ for some } h \in \mathcal{H}_r^+(X) \}.
\]

A function \( u \in \mathcal{U}_r(X) \) will be called a moderate \( \mathcal{U} \)-function on \( X \). It is clear that a function \( u \in \mathcal{U}(X) \) is moderate if and only if \( |u| \leq v \) for some \( v \in \mathcal{U}_r^+(X) \).

Lemma 4.1. If \( u \in \mathcal{U}_r(X) \), then \( V^\Psi|u| \in \mathcal{P}(X) \cap \mathcal{C}(X) \) and \( u + V^\Psi u \in \mathcal{H}_r(X) \).

Proof. Take \( u \in \mathcal{U}_r(X) \) and choose \( g \in \mathcal{H}_r^+(X) \) such that \( |u| \leq g \). Then \( |u| \leq Lg \) by (3.9). On the other hand, in view of formula (3.7),

\[
V^\Psi Lg \in \mathcal{P}(X) \cap \mathcal{C}(X).
\]

Therefore \( V^\Psi|u| \) is a continuous potential on \( X \). Put \( h = u + V^\Psi u \). Combining (3.2) and (3.4) we see that \( H_D h = h \) for every \( D \in \mathcal{O} \) which implies that \( h \) is harmonic on \( X \). Finally, since

\[
|h| \leq |u| + V^\Psi|u| \leq Lg + V^\Psi Lg \leq g
\]

we conclude that \( h \in \mathcal{H}_r(X) \). \( \square \)

From the above lemma it follows that the formula

\[
u + V^\Psi u = P\mu
\]

(4.1)

assigns to each moderate \( \mathcal{U} \)-function \( u \) on \( X \) a unique signed measure \( \mu \in \mathcal{M}(Y) \). Conversely, the comparison principle assures that for each \( \mu \in \mathcal{M}(Y) \) there is at most one function \( u \in \mathcal{U}_r(X) \) which satisfies (4.1). We call the measure \( \mu \) given by (4.1) the trace of \( u \) on \( Y \) and we write

\[
\mu = tr(u).
\]

We shall denote by \( \mathcal{Q}_\Psi(Y) \) the set of all \( \mu \in \mathcal{M}(Y) \) such that \( \mu \) is the trace of some moderate \( \mathcal{U} \)-function on \( X \). In other words, \( \mu \in \mathcal{Q}_\Psi(Y) \) means that the equation (4.1) is solvable in \( \mathcal{U}_r(X) \).
4.2 Properties of the trace

Let \( \mu \in \mathcal{M}^+(Y) \) and \( h = P\mu \). Then (3.7) yields that the measure \( \nu \in \mathcal{M}^+(Y) \) satisfying \( Qh = P\nu \) belongs to the class \( Q_\psi^+(Y) \). Defining

\[
Q\mu := \nu
\]

we obtain an increasing subadditive operator \( Q \) from \( \mathcal{M}^+(Y) \) into \( Q_\psi^+(Y) \). Furthermore,

\[
Q_\psi^+(Y) = \{ \mu \in \mathcal{M}^+(Y) : Q\mu = \mu \}.
\] (4.2)

In the sequel, we may write \( L\mu \) to mean \( L(P\mu) \).

**Theorem 4.2.** Let \( \mu, \nu, \mu_1, \mu_2, \cdots \in \mathcal{M}(Y) \). The following holds:

(a) If \( |\mu| \leq \nu \) and \( \nu \in Q_\psi^+(Y) \) then \( \mu \in Q_\psi^+(Y) \).

(b) \( \mu \in Q_\psi(Y) \) if and only if \( |\mu| \in Q_\psi^+(Y) \).

(c) If \( \mu_n \in Q_\psi^+(Y) \) for all \( n \geq 1 \) and \( (\mu_n) \) increases to \( \mu \), then \( \mu \in Q_\psi^+(Y) \).

(d) If \( \Psi \) satisfies (A4) then \( Q_\psi(Y) \) is convex.

(e) If \( \Psi \) has the doubling property then \( Q_\psi(Y) \) is a linear subspace of \( \mathcal{M}(Y) \).

In this case, \( f\mu \in Q_\psi(Y) \) whenever \( \mu \in Q_\psi^+(Y) \) and \( f \in L^1(Y, \mu) \).

**Proof.** (a) Let \( h = P\mu \) and \( g = P\nu \). For every \( n \geq 1 \) we have

\[
|U_{\Omega_n}h| \leq U_{\Omega_n}g \leq g.
\]

Then, by Theorem 3.7, there exists a subsequence \( (u_k) \) of \( (U_{\Omega_n}h) \) which is uniformly convergent on every compact subset of \( X \). So

\[
u := \lim_{k \to \infty} u_k
\]

is a moderate \( \mathcal{U} \)-function on \( X \). Using the monotonicity and the continuity of \( \Psi(x, \cdot) \), we obtain that

\[
|\Psi(\cdot, u_k)| \leq \Psi(\cdot, U_{\Omega_k}g),
\]

\[
\lim_{k \to \infty} \Psi(\cdot, u_k) = \Psi(\cdot, u),
\]

\[
\lim_{k \to \infty} \Psi(\cdot, U_{\Omega_k}g) = \Psi(\cdot, Lg).
\]

On the other hand, the fact that \( \nu \in Q_\psi^+(Y) \) implies that

\[
\lim_{k \to \infty} V_{\Omega_k}\Psi U_{\Omega_k}g = V^\Psi Lg < \infty.
\]

Therefore, by Lemma 3.8 we conclude that

\[
\lim_{k \to \infty} V_{\Omega_k}\Psi u_k = V^\Psi u
\]
The notion of the trace

and consequently

\[ u + V^\Psi u = h. \]

This means that \( \mu \in \mathcal{Q}_\Psi(Y) \) and \( \text{tr}(\mu) = u \).

(b) If \( |\mu| \in \mathcal{Q}_\Psi^+(Y) \) then \( \mu \in \mathcal{Q}_\Psi(Y) \) by statement (a). Suppose now that \( \mu \in \mathcal{Q}_\Psi(Y) \) and let \( u \) be the moderate \( \mathcal{U} \)-function on \( X \) satisfying \( \mu = \text{tr}(u) \). Choose \( \nu \in \mathcal{M}^+(Y) \) such that \( |u| \leq P\nu \). Then \( |u| \leq L\nu \) by (3.9) and thereby

\[ |P\mu| \leq P(Q\nu). \]

This yields that \( |\mu| \leq Q\nu \) (recall that \( P|\mu| \) is the least harmonic majorant of \( |P\mu| \)). So \( |\mu| \in \mathcal{Q}_\Psi^+(Y) \) by statement (a).

(c) follows trivially from Corollary 3.13.

(d) Since, by Corollary 3.12, \( Q \) is a concave operator on \( \mathcal{M}^+(Y) \) we easily deduce from (4.2) that \( \mathcal{Q}_\Psi^+(Y) \) is a convex subset of \( \mathcal{M}^+(Y) \). So statement (b) proves that \( \mathcal{Q}_\Psi(Y) \) is also convex.

(e) By Proposition 3.14, \( \mathcal{Q}_\Psi^+(Y) \) is a cone. In fact, for every \( \mu, \nu \in \mathcal{M}^+(Y) \) and every \( \alpha \geq 0 \) we have

\[ Q(\alpha\mu + \nu) = \alpha Q\mu + Q\nu. \]

So from (b) it follows that

\[ \mathcal{Q}_\Psi(Y) = \mathcal{Q}_\Psi^+(Y) - \mathcal{Q}_\Psi^+(Y) \]  \hspace{1cm} (4.3)

which proves that \( \mathcal{Q}_\Psi(Y) \) is a linear space. The second part of (e) is a consequence of statements (b) and (c).

Studying equations \( \Delta u = u|u|^\alpha - 1, \alpha > 1, \) on bounded domains \( \Omega \subset \mathbb{R}^d \), analogous results as in the previous theorem are obtained in [42]. To see the interest of introducing the operators \( L \) and \( Q \), the reader may compare our proof to the proof given by M. Marcus and L. Véron [42, Proof of Proposition A] who used a result of H. Brézis concerning the boundary value problem

\[ \Delta u = f \text{ in } \Omega \quad \text{and} \quad u = \phi \in L^1(\partial\Omega) \text{ on } \partial\Omega. \]

We also notice that, using probabilistic tools, E. B. Dynkin and S. E. Kuznetsov proved a result [24, Theorem 4.3] similar as assertion (c) of the preceding theorem.
4.3 Removable singularities

Let $E$ be a Borel subset of $Y$. We shall say that $E$ is removable if the function $\vartheta_E$ which is defined at every point $x \in X$ by

$$\vartheta_E(x) := \sup_{\mu \in \mathcal{M}^+(E)} L\mu(x) \quad (4.4)$$

is identically zero. Since $\{L\mu : \mu \in \mathcal{M}^+(E)\}$ is an upward filtering family of continuous functions, we may find an increasing sequence $(\mu_n) \in \mathcal{M}^+(E)$ such that

$$\vartheta_E = \sup_{n \geq 1} L\mu_n,$$

which yields, in particular, that $\vartheta_E \in \mathcal{U}^+(X)$ if it is locally bounded on $X$. In the following proposition, we have collected basic properties of the map $E \mapsto \vartheta_E$.

**Proposition 4.3.** Let $E, F, E_1, E_2, \ldots \subset Y$ be Borel sets. Then:

(a) If $E \subset F$ then $\vartheta_E \leq \vartheta_F$.

(b) If $(E_n)$ increases to $E$ then $\vartheta_E = \sup_{n \geq 1} \vartheta_{E_n}$.

(c) If $E = \bigcup_{n=1}^{\infty} E_n$ then $\vartheta_E \leq \sum_{n=1}^{\infty} \vartheta_{E_n}$.

**Proof.** (a) Obvious.

(b) Let $u = \sup_{n \geq 1} \vartheta_{E_n}$ and let $\mu \in \mathcal{M}^+(E)$. Seeing that $\mu_{E_n} \in \mathcal{M}^+(E_n)$ for all $n \geq 1$ and $(\mu_{E_n})$ increases to $\mu$, we conclude that

$$L\mu = \sup_{n \geq 1} L\mu_{E_n} \leq u.$$

Whence $\vartheta_E \leq u$. Therefore $u = \vartheta_E$ since $u \leq \vartheta_E$ by (a).

(c) For every $k \geq 1$ let

$$F_k := \bigcup_{n=1}^{k} E_n$$

and choose $\mu \in \mathcal{M}^+(F_k)$. Because $L$ is subadditive and $\mu \leq \sum_{n=1}^{k} \mu_{E_n}$, it follows that $L\mu \leq \sum_{n=1}^{k} L\mu_n$ and consequently

$$L\mu \leq \sum_{n=1}^{k} \vartheta_{E_n}.$$

Thus, for all $k \geq 1$

$$\vartheta_{F_k} \leq \sum_{n=1}^{k} \vartheta_{E_n},$$

which yields the desired inequality remarking that $\vartheta_E = \sup_{k \geq 1} \vartheta_{F_k}$. \qed
As immediate consequences of the previous proposition, we see that every Borel subset of a removable set of $Y$ is also removable, and $\bigcup_{n=1}^{\infty} E_n$ is removable whenever $(E_n)$ is a sequence of removable subsets of $Y$.

**Proposition 4.4.** Let $E$ be a Borel subset of $Y$. The following statements are equivalent:

(a) $E$ is removable.

(b) $\nu(E) = 0$ for all $\nu \in Q_+^+(Y)$.

(c) Every compact subset $K \subset E$ is removable.

**Proof.** From the fact that $Q\mu \in Q_+^+(Y)$ and $L\mu = L(Q\mu)$ for every $\mu \in M^+(Y)$ we obtain that

$$\vartheta_E = \sup_{\nu \in M^+(E) \cap Q_+^+(Y)} \nu(L\mu).$$

(4.5)

This yields the equivalence between (a) and (b). To finish the proof it suffices to recall that every $\mu \in M^+(Y)$ is inner regular (see, e.g., [8]).
5 Polar sets

Assumption of this section: $\Psi \in \mathcal{Y}(X)$.

5.1 Orlicz type spaces

For our purpose it will be convenient to identify all Borel measurable functions $f, g$ on $X$ satisfying
\[ \int_X V(|f - g|) \, dr = 0. \]

We define $L_{\Psi}(X)$ (Orlicz class) to be the set of all $f \in B(X)$ such that
\[ \varrho_{\Psi}(f) := \int_X V^\Psi|f| \, dr < \infty. \]

Let $L_{\Psi}(X)$ (Orlicz space) be the smallest linear space containing $L_{\Psi}(X)$, and let $E_{\Psi}(X)$ be the largest linear space contained in $L_{\Psi}(X)$. Classical analogous definitions, for $X \subset \mathbb{R}^d$ and $\Psi \in \mathcal{Y}_0$, are well known (see, e.g., [33]). An alternative approach to the theory of Orlicz spaces can be found in [19]. Notice that if $\Psi$ has the doubling property then
\[ E_{\Psi}(X) = L_{\Psi}(X) = L_{\Psi}(X). \]

Notation. Here and in the following, $\Phi$ denotes the function $\Psi^*$ given by (2.5) (of course $\Phi \in \mathcal{Y}(X)$ and $\Phi^* = \Psi$).

For every Borel measurable function $f$ on $X$ we consider
\[ \|f\|_{\Psi} = \sup \left\{ \int_X V|fg| \, dr : g \in B(X), \varrho_{\Phi}(g) \leq 1 \right\}, \quad (5.1) \]
\[ \|f\|_{(\Psi)} = \inf \left\{ \alpha > 0 : \varrho_{\Psi}(\alpha^{-1}f) \leq 1 \right\}. \quad (5.2) \]

Obviously, $\| \cdot \|_{\Psi}$ and $\| \cdot \|_{(\Psi)}$ are increasing on $B^+(X)$. Furthermore,
\[ \|f\|_{\Psi} \leq 1 \Rightarrow \varrho_{\Psi}(f) \leq \|f\|_{\Psi}, \quad (5.3) \]
\[ \|f\|_{(\Psi)} \leq 1 \Leftrightarrow \varrho_{\Psi}(f) \leq 1. \quad (5.4) \]

We also need the following kind of Hölder inequality which follows from (5.4):
\[ \int_X V|fg| \, dr \leq \|f\|_{\Psi} \|g\|_{(\Psi)}. \quad (5.5) \]

From (5.3) and (5.4) we deduce that
\[ \|f\|_{(\Psi)} \leq \|f\|_{\Psi} \leq 2\|f\|_{(\Psi)}. \]
Polar sets

Therefore

\[ L_\Psi(X) = \{ f \in B(X) : \| f \|_\Psi < \infty \} \]

and \( \| \cdot \|_\Psi \) and \( \| \cdot \|_{\Psi} \) define two equivalent norms on \( L_\Psi(X) \). Moreover, it is not difficult to verify that \( L_\Psi(X) \) endowed with \( \| \cdot \|_\Psi \) is a Banach space. We call \( \| \cdot \|_{\Psi} \) (\( \| \cdot \|_{\Psi} \) resp.) the Orlicz (Luxemburg resp.) norm.

Let \( f \in E_\Psi(X) \) and consider the sequence \((f_n)\) given for every \( n \geq 1 \) by

\[ f_n = 1_{\Omega_n} \inf(\sup(f, -n), n). \quad (5.6) \]

Seeing that

\[ f_n \in B_{bc}(X), |f_n| \leq |f|, \text{ and } \lim_{n \to \infty} f_n = f, \]

it follows that for every \( \alpha > 0 \)

\[ \lim_{n \to \infty} \varrho_{\Psi}(\alpha|f - f_n|)) = 0. \]

Therefore, \( E_\Psi(X) \) coincides with the closure (relative to the convergence in norm) of \( B_{bc}(X) \) in \( L_\Psi(X) \). Define \( B_{(\Phi)} \) to be the closed unit ball in \( L_\Phi(X) \) with respect to the Luxemburg norm and let

\[ E B_{(\Phi)} := E_\Phi(X) \cap B_{(\Phi)}. \]

Clearly (5.4) means that \( B_{(\Phi)} = \{ f \in B(X) : \varrho_{\Phi}(f) \leq 1 \} \). Using sequences defined by (5.6) it is not difficult to see that

\[ \| f \|_\Psi = \sup_{g \in E B_{(\Phi)}^+} \int_X V(|f|g) \, dr. \quad (5.7) \]

Now, slightly modifying the proof of Theorem 14.2 in [33] we get the following result which characterizes the topological dual of \( E_\Psi(X) \).

**Theorem 5.1.** For every continuous linear form \( T \) on \( E_\Psi(X) \), endowed with the Luxemburg norm, there exists a unique function \( g \in L_\Phi(X) \) such that for all \( f \in E_\Psi(X) \)

\[ T(f) = \int_X V(fg) \, dr. \quad (5.8) \]

Moreover:

(a) \( \| T \| := \sup_{f \in E B_{(\Phi)}} |T(f)| = \| g \|_{\Phi}. \)

(b) If \( T \geq 0 \) (i.e., \( T(f) \geq 0 \) for all \( f \in E_\Psi^+(X) \)) then \( g \in L_\Phi^+(X) \).
5.2 The Martin-Orlicz capacity

We call Martin-Orlicz capacity the set function \( c_\Psi \) defined for every Borel subset \( E \) of \( Y \) by
\[
c_\Psi(E) := \sup \{ \nu(Y) : \nu \in \mathcal{M}^+(E), \| P\nu \|_\Psi \leq 1 \}
\]
and extended to any (arbitrary) subset \( F \) of \( Y \) by
\[
c_\Psi(F) = \inf \{ c_\Psi(E) : E \supset F, E \text{ Borel} \}.
\]
Then \( c_\Psi \) is a capacity in the terminology of N. G. Meyers [44]. In other words,
\[
c_\Psi(\emptyset) = 0
\]
and for any sequence \((F_n)\) of subsets of \( Y \) the following properties hold:
\[
F_1 \subset F_2 \Rightarrow c_\Psi(F_1) \leq c_\Psi(F_2), \tag{5.9}
\]
\[
c_\Psi(\bigcup_{n=1}^\infty F_n) \leq \sum_{n=1}^\infty c_\Psi(F_n). \tag{5.10}
\]

A set \( F \subset Y \) will be called \( c_\Psi \)-polar if \( c_\Psi(F) = 0 \), and we shall say that a property \( P \) holds \( c_\Psi \)-quasi-everywhere (abb., \( c_\Psi \)-q.e) provided \( P \) is valid on \( Y \setminus F \) for some \( c_\Psi \)-polar subset \( F \subset Y \).

From (5.9) it follows that every subset of a \( c_\Psi \)-polar set is also \( c_\Psi \)-polar, and by (5.10) it is clear that the union of any countable family of \( c_\Psi \)-polar sets of \( Y \) is again \( c_\Psi \)-polar.

Using the fact that
\[
\mu(E) = \sup \{ \mu(K) : K \subset E, K \text{ compact} \}
\]
for any Borel subset \( E \) of \( Y \) and any \( \mu \in \mathcal{M}^+(Y) \), we easily obtain the following proposition.

Proposition 5.2. For every Borel set \( E \subset X \) we have
\[
c_\Psi(E) = \sup \{ c_\Psi(K) : K \subset E, K \text{ compact} \}. \tag{5.11}
\]

For \( f \in \mathcal{B}(X) \) we consider the function \( \hat{P}f \) defined at every \( y \in Y \) by
\[
\hat{P}f(y) = \int_X V(P_y f) \, dr
\]
provided the integral makes sense. Recall that \( P_y = P(\cdot, y) \) is the (Martin) function given by (J.1). If \( f \in \mathcal{B}^+(X) \) and \( \nu \in \mathcal{M}^+(Y) \), it is obvious that
\[
\int_Y \hat{P}f \, d\nu = \int_X V(fP\nu) \, dr. \tag{5.12}
\]
Proposition 5.3. For every compact subset $K$ of $Y$ we have
\[ c_\Psi(K) = \inf \left\{ \| f \|_{\Phi} : f \in E_\Phi^+(X) \text{ and } \hat{P} f \geq 1 \text{ on } K \right\}. \] (5.13)
Moreover, (5.13) holds also true if $E_\Phi^+(X)$ is replaced by $L_\Phi^+(X)$.

Proof. Let $K$ be a compact subset of $Y$ and denote by $\alpha$ the right side in (5.13)(1).
Let
\[ \mathcal{W} := \{ \nu \in M^+(K) : \nu(Y) = 1 \} \]
and endow it with the weak* topology. Then $\mathcal{W}$ is a compact Hausdorff space. On the other hand, by (J.2) the mapping $\nu \mapsto P\nu(x)$ is continuous on $\mathcal{W}$ for any fixed $x \in X$. Consequently the function $\nu \mapsto \int_Y \hat{P} f \, d\nu$ is lower semicontinuous on $\mathcal{W}$ for every fixed function $f \in E_\Phi^B(X)^+$. Then, in view of (5.7) and (5.12), the minimax theorem (see, e.g., [1]) yields that
\[ \inf_{\nu \in \mathcal{W}} \| P\nu \|_{\Psi} = \sup_{f \in E_\Phi^B(X)^+} \inf_{\nu \in \mathcal{W}} \int_Y \hat{P} f \, d\nu = \sup_{f \in E_\Phi^B(X)^+} \inf_{y \in K} \hat{P} f(y). \] (5.14)

Remark first that by the definition of $c_\Psi(K)$ it is not difficult to obtain (5.13) in the case of
\[ \{ \alpha, c_\Psi(K) \} \cap \{ 0, \infty \} \neq \emptyset. \]
So suppose that $0 < c_\Psi(K), \alpha < \infty$. Then
\[ \frac{1}{c_\Psi(K)} = \inf \left\{ \frac{1}{\nu(K)} : \nu \in M^+(K), \nu \neq 0, \| P\nu \|_{\Psi} \leq 1 \right\} \]
\[ = \inf \left\{ \frac{\| P\nu \|_{\Psi}}{\nu(K)} : \nu \in M^+(K), \nu \neq 0 \right\} \]
\[ = \inf_{\nu \in \mathcal{W}} \| P\nu \|_{\Psi}, \]
and
\[ \frac{1}{\alpha} = \sup \left\{ \frac{1}{\| f \|_{\Phi}} : f \in E_\Phi^+(X), f \neq 0, \hat{P} f \geq 1 \text{ on } K \right\} \]
\[ = \sup \left\{ \inf_{y \in K} \frac{\hat{P} f(y)}{\| f \|_{\Phi}} : f \in E_\Phi^+(X), f \neq 0 \right\} \]
\[ = \sup_{f \in E_\Phi^B(X)^+} \inf_{y \in K} \hat{P} f(y). \]
\[ ^1 \text{If there is no } f \in E_\Phi^+(X) \text{ such that } \hat{P} f \geq 1 \text{ on } K \text{ then, by convention, } \alpha = \infty. \]
So the proof of equality (5.13) is finished in view of (5.14). Finally, using (5.1) instead of (5.7), the second statement of the proposition can be shown by the same reasoning.

5.3 Sufficient conditions for \( \nu \) to be in \( Q_\Psi(Y) \)

In addition to the fact that \( \Psi \) is a function in \( Y(X) \), we also suppose in the present subsection that:

(†) \( \Psi \) has the doubling property, and
‡(V, r) is an admissible pair (see subsection 2.5).

Let us consider the duality \( \langle \cdot, \cdot \rangle \) between \( E_\Phi(X) \) and \( L_\Psi(X) \) given by
\[
\langle f, g \rangle = \int_X V(fg) \, dr
\]
for every \( f \in E_\Phi(X) \) and \( g \in L_\Psi(X) \). If \( F \subset E_\Phi(X) \), we denote by \( F^\perp \) the (closed) subspace of \( L_\Psi(X) \) consisting of all \( g \in L_\Psi(X) \) such that \( \langle f, g \rangle = 0 \) for all \( f \in F \). For a set \( G \subset L_\Psi(X) \), \( G^\perp \) is the subspace of \( E_\Phi(X) \) defined in the same way.

We define
\[
\mathcal{H}_\Psi^\perp(X) := \mathcal{H}_\Psi^\perp(X) \cap L_\Psi(X),
\]
\[
\mathcal{H}_\Psi(X) := \mathcal{H}_\Psi^\perp(X) - \mathcal{H}_\Psi^\perp(X),
\]
\[
\mathcal{M}_\Psi(Y) := \{ \nu \in \mathcal{M}(Y) : P\nu \in \mathcal{H}_\Psi(X) \}.
\]

By Theorems 4.2.b and 3.10.b we have \( \mathcal{M}_\Psi(Y) \subset Q_\Psi(Y) \). (Notice that assumption (†) above implies that \( E_\Psi(X) = L_\Psi(X) = L_\Psi(X) \))

**Lemma 5.4.** Let \( E \subset Y \) be a Borel set. The following holds:

(a) \( E \) is \( c_\Psi \)-polar if and only if \( \nu(E) = 0 \) for all \( \nu \in \mathcal{M}_\Psi^+(Y) \).

(b) \( \mathcal{H}_\Psi(X)^\perp = \{ f \in E_\Phi(X) : Pf = 0 \text{ c.e on } Y \} \)

(c) \( \mathcal{H}(X) \cap L_\Psi(X) \) is a closed subspace of \( L_\Psi(X) \).

**Proof.** (a) Trivial.

(b) This follows from (5.12) and assertion (a).

(c) Let \( K \) be a compact subset of \( X \) and choose \( \Omega \in \mathcal{O}, c > 0 \) as in (2.4).

Applying the Hölder inequality we obtain that
\[
\sup_K |h| \leq c \int_X V|h1_\Omega| \, dr \leq c \|1_\Omega\|_{(\Phi)} \|h\|_\Psi
\]
for every \( h \in \mathcal{H}(X) \). Therefore, any sequence in \( \mathcal{H}(X) \cap L_\Psi(X) \) converges locally uniformly on \( X \) whenever it converges in \( L_\Psi(X) \) relative to the Orlicz norm. This finishes the proof of (c). □
**Remark 5.5.** From the first part of Lemma A.2 (see Appendix) we conclude that the set
\[ \{ y \in Y : \hat{P}|f|(y) = \infty \} \]
is \( c_\Phi \)-polar for every \( f \in E_\Phi(X) \). The second statement of the same lemma yields that every sequence \( (f_n) \subset E_\Phi(X) \) convergent (in norm) to some function \( f \) admits a subsequence \( (g_n) \) with the property that \( (\hat{P}g_n) \) converges \( c_\Phi \)-q.e to \( \hat{P}f \).

**Remark 5.6.** If \( f \in C(X) \) such that for all \( g \in B_\Phi^+(X) \)
\[ \int_X V(fg) \, dr \geq 0, \]
then \( f(x) \geq 0 \) for all \( x \in X \). In fact, it suffices to remark that the measure \( m \) defined for every Borel subset \( A \subset X \) by
\[ m(A) = \int_X V1_A \, dr \]
charges all open nonempty subsets of \( X \). To see this, let \( D \in \mathcal{O} \) and suppose that \( m(D) = 0 \). Seeing that
\[ \{ x \in X : V1_D(x) = 0 \} \]
is an absorbing set (see, [7, Satz 1.4.1]) and recalling the definition of a reference measure (see Subsection 2.5) we conclude that \( V1_D \) is identically zero on \( X \). Consequently, \( D = \emptyset \) by (AP1).

**Theorem 5.7.** Every \( \nu \in \mathcal{M}(Y) \) which does not charge any compact \( c_\Phi \)-polar subset of \( Y \) is a trace of some moderate \( U \)-function on \( X \).

**Proof.** In virtue of Theorem 4.2.b we consider only the case when \( \nu \) is positive. Let \( \nu \in \mathcal{M}^+(Y) \) not charging compact \( c_\Phi \)-polar subsets of \( Y \) and define for every \( f \in E_\Phi(X) \)
\[ \Lambda(f) := \int_Y [\hat{P}f]^+ \, d\nu. \]
Then \( \Lambda \) is a positively homogeneous subadditive map from \( E_\Phi(X) \) into \( \mathbb{R}_+ \). Furthermore, \( \Lambda \) is lower semicontinuous on \( E_\Phi(X) \) (see Remark 5.5) and thereby
\[ \text{epi} \, \Lambda := \{(f, t) \in E_\Phi(X) \times \mathbb{R} : \Lambda(f) \leq t\} \]
is a closed convex cone of \( E_\Phi(X) \times \mathbb{R} \) (see, e.g., [15]). Considering \( \varphi := \sum_{n=1}^{\infty} \alpha_n 1_{\Omega_n} \), where
\[ \alpha_n = \frac{2^{-n}}{(1 + \langle 1_{\Omega_n}, \nu \rangle)(1 + \|1_{\Omega_n} \|_\Phi)}, \]

it is not difficult to see that

\[ \varphi \geq \alpha_n > 0 \text{ on } \Omega_n, \quad \varphi \in E^+_{\Phi}(X), \text{ and } \Lambda(\varphi) < \infty. \]

Then Theorem 5.1 and the Hahn-Banach theorem (see, e.g., [15, Théorème I.7]) imply that there exist \( g_n \in L_{\Psi}(X) \) and \( a_n \in \mathbb{R} \) such that

\[ \langle \varphi, g_n \rangle > a_n(\Lambda(\varphi) - 1/n) \quad (5.15) \]

and

\[ \langle f, g_n \rangle \leq a_n t \quad \text{for all } (f, t) \in \text{epi } \Lambda. \quad (5.16) \]

Taking \( f = 0 \) and \( t = 1 \) in (5.16) we get that \( a_n \geq 0 \). Assuming that \( a_n = 0 \) we obtain that \( \langle \varphi, g_n \rangle > 0 \) by (5.15), and \( \langle \varphi, g_n \rangle \leq 0 \) by (5.16), which yields a contradiction. So we suppose without loss of generality that \( a_n = 1 \) (otherwise we replace \( g_n \) by \( a_n^{-1}g_n \)).

We claim that \( g_n \in \mathcal{H}^+ (X) \). In fact, using the characterization of \( \mathcal{H}_{\Psi}(X)^\perp \) given by Lemma 5.4.b, we deduce from (5.16) that

\[ g_n \in (\mathcal{H}_{\Psi}(X)^\perp)^\perp. \]

On the other hand, Lemma 5.4.c and [15, Proposition II.12] prove that

\[ (\mathcal{H}_{\Psi}(X)^\perp)^\perp \subset L_{\Psi}(X) \cap \mathcal{H}(X). \]

Now, applying (5.16) to \((-f, 0)\) we get that \( \langle f, g_n \rangle \geq 0 \) for every \( f \in B_{bc}^+(X) \), which implies that \( g_n(x) \geq 0 \) for all \( x \in X \) (see Remark 5.6 above). The claim is proved.

Put \( h = P\nu \) and apply again (5.16) for \( f \in B_{bc}^+(X) \) and \( t = \Lambda(f) \), we obtain in view of (5.12) that

\[ \int_X V(f(h - g_n)) \, dr \geq 0 \]

for every \( f \in B_{bc}^+(X) \), which yields that \( h \geq g_n \) on \( X \). Define now

\[ h_n = \lim_{k \to \infty} H_{\Omega_k} \sup_{1 \leq i \leq n} g_i, \]

i.e., \( h_n \) is the least harmonic majorant of \( \{g_i : 1 \leq i \leq n\} \). Then \((h_n)\) is an increasing sequence of positive harmonic functions on \( X \) satisfying

\[ \int_X V(\varphi(h - h_n)) \, dr \leq \frac{1}{n} \quad (n \geq 1). \quad (5.17) \]
Recalling that $\varphi > 0$ on $X$ we conclude from (5.17) that $h = \sup_{n \geq 1} h_n$, and consequently

$$\nu = \sup_{n \geq 1} \nu_n$$

where $\nu_n \in \mathcal{M}^+(Y)$ satisfying $P\nu_n = h_n$ for all $n \geq 1$. The fact that

$$h_n \leq \sum_{i=1}^{n} g_i$$

and $g_i \in \mathcal{H}_\varphi^+(X)$ for all $i \geq 1$, proves that all measures $\nu_n$ belong to the class $\mathcal{Q}_\varphi^+(Y)$. Whence, $\nu \in \mathcal{Q}_\varphi^+(Y)$ by Theorem 4.2.c.

We notice that, in general, the converse statement in the above theorem does not hold. A counterexample will be given in subsection 6.6.
6 Applications to semilinear PDEs

We call Greenian domain every open and connected set $D \subset \mathbb{R}^d$ which has a Green function $G_D$ ($-\Delta G_D(\cdot, \zeta) = \delta_\zeta$ for every $\zeta \in D$). As usual, $\Delta$ denotes the Laplace operator on $\mathbb{R}^d$, $d \geq 2$. Let $X$ be a Greenian domain of $\mathbb{R}^d$ and let $\mathcal{H}$ be the classical sheaf of harmonic functions on $X$. Fix a point $x_0$ in $X$ and consider, as reference measure on $X$, the Dirac measure $r = \delta_{x_0}$ concentrated at the point $x_0$ (here $X$ and the empty set are the only absorbing subsets of $X$; see, e.g., [7]). So, trivially

$$\mathcal{H}_r(X) = \mathcal{H}^+(X) - \mathcal{H}^+(X).$$

We choose $Y$ and $P$ so that $Y$ is the set of all minimal Martin boundary points of $X$ and $P$ is the Martin kernel satisfying $P(x_0, y) = 1$ for every $y \in Y$.

Let $\Psi \in \mathcal{Y}(X)$ and denote by $\Phi$ the function $\Psi^*$. Consider also a local Kato measure $\gamma$ on $X$, i.e., $V = V_X^\gamma$ given by (2.3) is a potential kernel on $X$. Then it is not difficult to see that, for every $D \in \mathcal{O}$, the kernel $V_D$ is given by the formula

$$V_D f = \int_D G_D(\cdot, \zeta) f(\zeta) d\gamma(\zeta).$$

Our goal here is to apply the general study presented in the preceding sections in order to investigate the boundary value problem:

$$\begin{align*}
\Delta u &= \Psi(\cdot, u)\gamma \quad \text{in } X, \\
u &= \nu \quad \text{on } Y,
\end{align*}$$

(6.1)

where $\nu$ is a signed Borel measure with bounded variation on $Y$.

6.1 Continuous solutions to (6.2)

A solution to the equation

$$\Delta u = \Psi(\cdot, u)\gamma$$

(6.2)

on an open subset $\Omega \subset X$ has to be understood as a continuous function $u$ on $\Omega$ which satisfies (6.2) in the distributional sense, i.e.,

$$\int_\Omega u(x) \Delta \varphi(x) \, dx = \int_\Omega \Psi(x, u(x)) \varphi(x) \, d\gamma(x)$$

(6.3)

for every $\varphi$ in the space $C^\infty_c(\Omega)$ of all infinitely differentiable functions on $\Omega$ with compact support in $\Omega$. 
Proposition 6.1. Let \( \Omega \) be an open subset of \( X \) and let \( u \in C(\Omega) \). Then \( u \) is a solution to (6.2) in \( \Omega \) if and only if \( u \) is a \( U \)-function on \( \Omega \).

Proof. Suppose first that \( u \) is a \( U \)-function on \( \Omega \). Let \( \varphi \in C_c^\infty(\Omega) \) and choose \( D \in \mathcal{O} \) such that \( \text{supp}(\varphi) \subset \overline{D} \subset \Omega \). By Theorem 3.5, the function

\[
h := u + \int_D G_D(\cdot, \zeta) \Psi(\zeta, u(\zeta)) \, d\gamma(\zeta)
\]

(6.4)

is harmonic and bounded on \( D \). Therefore, multiplying (6.4) by \( \Delta \varphi \) and integrating, we obtain (6.3) which means that \( u \) is a solution to (6.2) in \( \Omega \).

Conversely, assume that (6.3) holds true for every \( \varphi \in C_c^\infty(\Omega) \). A similar computation proves that for any \( D \in \mathcal{O} \) with \( \overline{D} \subset \Omega \), the function \( h \) given by (6.4) is harmonic on \( D \). So, again by Theorem 3.5, this yields that

\[
U_D u = u
\]

for all \( D \in \mathcal{O} \) such that \( \overline{D} \subset \Omega \). Whence \( u \in U(\Omega) \). \( \square \)

6.2 Examples of \( \Psi \)

The class \( Y(X) \) contains every function of the form

\[
\Psi(x, t) = \xi(x) M(t)
\]

where \( M \) is a Young function (see Subsection 2.6) and \( \xi \) is a Borel measurable positive function on \( X \) such that \( \xi \) and \( 1/\xi \) are bounded on \( X \). Furthermore, \( \Psi \) has the doubling property if and only if \( M \) possesses the same property.

We quote as first example the function

\[
\Psi(x, t) = t |t|^{\alpha - 1}, \quad x \in X, \quad t \in \mathbb{R},
\]

(6.5)

where \( \alpha \) is a real \( > 1 \). In this case, \( L_\Psi(X) \) is the classical Lebesgue space \( L^\alpha(X, m) \) where

\[
m = G_X(x_0, \cdot) \gamma;
\]

hence trivially

\[
L_\Psi(X) = L^{\alpha'}(X, m) \quad (\alpha' := \alpha/(\alpha - 1)).
\]

In this example, clearly both functions \( \Psi \) and \( \Phi \) possess the doubling property.

As second example of \( \Psi \), we consider

\[
\Psi(x, t) = \text{sgn}(t)[-|t| + (1 + |t|) \ln(1 + |t|)], \quad x \in X, \quad t \in \mathbb{R}.
\]

(6.6)
In this example, the function $\Psi$ has the doubling property but it is not the case for $\Phi$. In fact, by elementary calculations we may show that

$$\Phi(x,t) = \text{sgn}(t)[-1 - |t| + \exp |t|].$$

The reader has certainly noticed that our results (especially Theorem 5.7) hold without assuming that $\Phi$ possesses the doubling property.

### 6.3 Examples of $\gamma$

Obviously the $d$-dimensional Lebesgue measure $\lambda$ and any Radon measure on $X$ with a locally bounded density with respect to $\lambda$ are local Kato measures on $X$. A further example of $\gamma$ can be constructed as follows: Suppose that

$$X = B := B(0,1)$$

is the open unit ball of $\mathbb{R}^d$ and let $x_0 = 0$. From the definition of the Green function $G_B$ (see [20]) we know that for every $0 < \rho < 1$ there exists $a_\rho > 0$ such that

$$\{\zeta \in B : G_B(0,\zeta) > a_\rho\} = B_\rho := B(0,\rho).$$

Denote by $\sigma_\rho$ the normalized surface area measure on $\partial B_\rho$ and let $I$ be the set of all rational numbers $0 < \rho < 1$. For each $\rho \in I$ choose $\eta_\rho > 0$ so that

$$\sum_{\rho \in I} \eta_\rho a_\rho < \infty,$$

and define

$$\gamma := \sum_{\rho \in I} \eta_\rho \sigma_\rho. \quad (6.7)$$

Then $\gamma$ is a (local) Kato measure on $B$ which is singular with respect to $\lambda$ and it charges all nonempty open subsets of $B$.

**Proposition 6.2.** For $r = \delta_{x_0}$, the pair $(\gamma, r)$ is admissible in each of the following cases:

(a) $\gamma$ is the restriction of the Lebesgue measure $\lambda$ to $X$.

(b) $\gamma$ is given by (6.7) (where $X = B$ and $x_0 = 0$).

**Proof.** In both cases the measure $\gamma$ charges all nonempty open subsets of $X$. So it only remains to prove that (AP2) is satisfied. Let $K$ be a compact subset of $X$. 
(a) Take \( \Omega, D \in \mathcal{O} \) such that \( K \cup \{x_0\} \subset D \subset \overline{D} \subset \Omega \) and let \( h \in \mathcal{H}_b(\Omega) \). From the mean-value property of \( h \) it follows that

\[
\sup_K |h| \leq a \int_D |h| d\lambda
\]

where \( a \) is a strictly positive constant not depending on \( h \). Consequently, remarking that

\[
\inf_{\zeta \in \overline{D}} G_{\Omega}(x_0, \zeta) := \alpha > 0
\]

we obtain that

\[
V_\Omega |h|(x_0) \geq \int_D G_{\Omega}(x_0, \zeta)|h(\zeta)| d\lambda(\zeta) \geq \alpha \int_D |h| d\lambda \geq \frac{\alpha}{a} \sup_K |h|.
\]

This finishes the proof in the case of \( \gamma = \lambda|_X \).

(b) Let \( \rho \in I \) such that \( K \cup \{0\} \subset B_{\rho} \). Seeing that \( \sigma_\rho = \mu_0^{B_\rho} \), it follows from the Harnack inequality that there exists a constant \( a > 0 \) such that the inequality

\[
\mu_x^{B_\rho} \leq a \sigma_\rho
\]

is valid for all \( x \in K \). Choose \( \tau \in I \) such that \( \tau > \rho \) and put

\[
\alpha := \inf_{\zeta \in \partial B_{\rho}} G_{B_\tau}(0, \zeta).
\]

Since \( \alpha > 0 \) we get that

\[
|h(x)| \leq \int_{\partial B_{\rho}} |h| d\mu_x^{B_\rho} \leq a \int_{\partial B_{\rho}} |h| d\sigma_\rho \leq \frac{a}{\alpha \eta_{\rho}} V_{B_\tau} |h|(0)
\]

for every \( x \in K \) and every \( h \in \mathcal{H}_b(B_\tau) \). Thus, the proof is complete. \( \square \)

### 6.4 Removable singularities

We suppose in this subsection that \( X \) is a bounded Lipschitz domain. Consequently, the boundary Harnack principle holds for \( X \) and we may choose \( Y \) to be the Euclidean boundary \( \partial X \) of \( X \) (see, e.g., [5, Sect. 8.7]).

Given \( u \in \mathcal{B}^+(X), u = 0 \) on \( \Gamma \subset \partial X \) will mean that for all \( z \in \Gamma \)

\[
\lim_{x \in X, x \to z} u(x) = 0.
\]

**Proposition 6.3.** Let \( E \subset \partial X \) be a Borel set. The following statements are equivalent:

(a) \( E \) is a removable set.

(b) Equation (6.2) has no nontrivial continuous solution \( u \) in \( X \) such that

\[
u = 0 \text{ on } \partial X \setminus E \text{ and } 0 \leq u \leq g \text{ for some } g \in \mathcal{H}_b^+(X).
\]
Proof. Take $u$ as in (b). By Lemma 4.1,

$$h := u + \int_X G_X(\cdot, \zeta) \Psi(\zeta, u(\zeta)) d\gamma(\zeta)$$

is a harmonic function on $X$. Moreover, $u = L\mu$ where $\mu$ is the measure in $\mathcal{M}^+(\partial X)$ satisfying $h = P\mu$. We claim that $\mu$ is supported by $E$. Indeed, let $O$ be a relatively open subset of $\partial X$ such that $E \subset O$ and let $\nu$ be the restriction of $\mu$ to $\partial X \setminus O$. Then, in view of the boundary Harnack principle, we see that $P\nu$ vanishes on $O$ and thereby $L\nu = 0$ on $O$. On the other hand, since

$$L\nu \leq L\mu = u$$

it follows that $L\nu = 0$ on $\partial X \setminus E$. Therefore, $L\nu \equiv 0$ on $X$ which in turn implies that

$$\nu = Q\nu = 0.$$ 

Notice that $\nu \in Q^+\Psi(\partial X)$ by Theorem 4.2.a. We then conclude that

$$\mu(O) = \mu(\partial X)$$

for every open subset $O$ of $\partial X$ containing $E$ which means that $\mu \in \mathcal{M}^+(E)$.

(a)$\Rightarrow$(b) If $E$ is removable then $u = L\mu = 0$ on $X$ by definition (see (4.4)).

(b)$\Rightarrow$(a) Suppose that $E$ is not removable. By Proposition 4.4, there exists a compact subset $K \subset E$ which is not removable. Therefore, we may find a measure $\tau \in \mathcal{M}(K)$ such that

$$u := L\tau$$

is not identically zero on $X$. This contradicts (b). \hfill $\Box$

Remark 6.4. Assume that all positive solutions to the equation (6.2) are locally uniformly bounded. (For instance, in the case of $\gamma = \lambda_X$ and $\Psi(x, t) \geq t^\alpha$ for some $\alpha > 1$; see [12].) Then, a compact set $K \subset \partial X$ is removable if and only if every positive solution to (6.2) vanishing on $\partial X \setminus K$ belongs to $\mathcal{L}_\Psi(X)$. In fact, in this setting, $\vartheta_K$ is a non-moderate solution to (6.2) in $X$ satisfying $\vartheta_K = 0$ on $\partial X \setminus K$.

6.5 A semilinear Dirichlet problem

Suppose that $\Psi \in \mathcal{Y}(\mathbb{R}^d)$ and $\gamma$ is a local Kato measure on $\mathbb{R}^d$. Consider the case when $X = B$ is an open ball of $\mathbb{R}^d$, $Y$ is the sphere $\partial B$ and the formula (3.18) is the Poisson integral. According to Theorem 3.3, for every $f \in \mathcal{C}(\partial B)$ the semilinear Dirichlet problem

\begin{align*}
\Delta u &= \Psi(\cdot, u)\gamma \quad \text{in } B, \\
u &= f \quad \text{on } \partial B
\end{align*} 

(6.8)
has a unique continuous solution $u$. It is the only continuous extension of $f$ to $\bar{B}$ which belongs to $U(B)$. Furthermore, $u$ is a solution to (6.8) if and only if $u$ solves the following integral equation:

$$u + \int_B G_B(\cdot, \zeta)\Psi(\zeta, u(\zeta)) \, d\gamma(\zeta) = \int_{\partial B} P(\cdot, y) f(y) \, d\sigma(y),$$

(6.9)

where $\sigma$ denotes the surface area measure on $\partial B$. Here, $P$ is chosen so that $P\sigma \equiv 1$.

### 6.6 Solutions to problem (6.1)

The boundary value problem (6.1) is interpreted as the natural generalization of (6.8). In other words, a continuous function $u$ on $X$ is a solution to (6.1) means that $|u|$ is dominated by some harmonic function on $X$ and that

$$u + \int_X G_X(\cdot, \zeta)\Psi(\zeta, u(\zeta)) \, d\gamma(\zeta) = \int_Y P(\cdot, y) \, d\nu(y).$$

(6.10)

So the class $Q_\Psi(Y)$ is the set of all $\nu \in M(Y)$ for which (6.1) has a solution. In particular, by Proposition 4.4,

- **(NC) $|\nu|(E) = 0$ for every removable set $E \subset Y$**

whenever (6.1) has a solution, and if $\Psi$ possesses the doubling property then Theorem 5.7 assures that the condition

- **(SC) $|\nu|(\Gamma) = 0$ for every compact $c_\Psi$-polar set $\Gamma \subset Y$**

is sufficient for (6.1) to be solvable.

Let $\gamma = \lambda$ and $\Psi$ as in (6.5). For $1 < \alpha \leq 2$ and if $X$ is bounded and sufficiently smooth, Dynkin and Kuznetsov [23, 22] (see also Le Gall [37] for $\alpha = 2$) showed using probabilistic methods that removable sets are the $c_\Psi$-polar sets (which claims a conjecture of Dynkin [21]). Consequently, (6.1) is solvable if and only if $\nu$ does not charge any $c_\Psi$-polar set. Similar results are given by Marcus and Véron [41, 42] for $\alpha > 2$.

Analogous parabolic problems were also investigated by similar techniques in [38, 36, 35, 43, 40].

**Remark 6.5.** In virtue of Theorem 3.10.b, if $\Psi$ has the doubling property then all removable sets are $c_\Psi$-polar. However, in general a $c_\Psi$-polar subset of $Y$ is not necessarily removable. In fact, let again $X, Y, P$ be as in Subsection 6.5 and suppose that $\gamma = \lambda_X$. Take a ball $B'$ internally tangent to $\partial B$ at a point $z \in \partial B$. Then

$$A := B \backslash B'$$
is minimal thin at \( z \) (see, e.g., [20]). Put \( h = P\delta_z \). Choose

\[
1 < \alpha < (d + 1)/(d - 1)
\]

and a locally bounded Borel measurable function \( \theta \geq 1 \) on \( B \) such that

\[
\int_A G_B(x_0, \zeta)[h(\zeta)]^{\alpha}\theta(\zeta) d\zeta = \infty
\]  
(6.11)

where \( x_0 \) is a fixed point of \( B \) (here \( r := \delta_{x_0} \)). Let

\[
\Psi(x, t) = [1_{B^c}(x) + \theta(x)1_A(x)] t|t|^{\alpha - 1}, \quad (x, t) \in B \times \mathbb{R}.
\]

Seeing that

\[
\int_{B^c} G_B(x_0, \zeta)\Psi(\zeta, h(\zeta)) d\zeta < \infty
\]

and applying [25, Theorem 5.1] we conclude that the problem (6.1) is solvable for \( \nu = \delta_z \). This implies that the set \( \{z\} \) is not removable. However, by (6.11) it is clear that \( \{z\} \) is a \( c_\psi \)-polar subset of \( \partial B \).

**Remark 6.6.** Let \( X_0 \) be an open subset of \( \mathbb{R}^d, d \geq 3 \), and consider a uniformly elliptic second order differential operator of the kind

\[
\mathcal{L}u = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( \sum_{j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i}
\]  
(6.12)

where \( a_{ij} \) are Borel measurable bounded functions on \( X_0 \) and \( b_i \) are in the Lebesgue space \( L^p(X_0, \lambda) \) for some \( p > d \). If \( X \) is an \( \mathcal{L} \)-adapted domain of \( X_0 \) in the sense of R. M. and M. Hervé [30], we get the same results replacing the Laplacian by the operator \( \mathcal{L} \).

### 6.7 Parabolic setting

As application of our abstract study we may suppose that the harmonic space \( (X, \mathcal{H}) \) is given by a domain \( X \) of \( \mathbb{R}^d \times \mathbb{R}, d \geq 1 \), endowed with the sheaf \( \mathcal{H} \) of the solutions to the heat equation on \( X^{(2)} \). Consider the semilinear problem

\[
\Delta u - \frac{\partial u}{\partial t} = \Psi(\cdot, u)\gamma \text{ in } X,
\]  
(6.13)

\[
u = \nu \text{ on } Y,
\]  
(6.14)

\[\text{Since in this case there are nontrivial absorbing subsets of } X, \text{ we cannot choose } r \text{ to be a Dirac measure.}\]
where \( \nu \in \mathcal{M}(Y) \), \((\gamma, r)\) is an admissible pair, and \( \Psi \in \mathcal{Y}(X) \) admitting the doubling property. Similar to the previous elliptic case, \( U(X) \) coincides with the set of all continuous solutions (in the distributional sense) to (6.13). Therefore, for any \( \nu \in \mathcal{M}(Y) \)

\[(SC) \implies (6.13)-(6.14) \text{ has a solution in } U_r(X) \implies (NC).\]
Appendix

Let $\Psi \in \mathcal{Y}(X)$ and put $\Phi = \Psi^*$. For every subset $F$ of $Y$ we define

$$C_{\Phi}(F) := \inf \{ \| f \|_{(\Phi)} : f \in L_{\Phi}^+(X), \partial f(y) \geq 1 \text{ for all } y \in F \},$$

(6.15)

and $C_{\Phi}'(F)$ by the same formula where $L_{\Phi}^+(X)$ is replaced by $E_{\Phi}^+(X)$. It is not difficult to see that for any arbitrary subset $F$ of $Y$

$$c_{\Phi}(F) \leq C_{\Phi}(F) \leq C_{\Phi}'(F).$$

(6.16)

We have already proved in Proposition 5.3 that $c_{\Phi}, C_{\Phi}$, and $C_{\Phi}'$ coincide on compact subsets of $Y$. So, according to Choquet’s Theorem [17], one immediately concludes that

$$c_{\Phi}(E) = C_{\Phi}(E) = C_{\Phi}'(E)$$

for every $\mathcal{K}$-Suslin subset $E$ of $Y$ (see [16]) provided $C_{\Phi}'$ defines a capacity in the sense of G. Choquet [17] (see also [2] and [11, p. 27]).

**Assumption:** We suppose that both functions $\Psi$ and $\Phi$ possess the doubling property (so that $C_{\Phi} = C_{\Phi}'$ by assumption).

Using the same techniques as in Chapter 2 of [1] (see also [4]) we obtain the following properties of $C_{\Phi}$:

1. $C_{\Phi}$ is a capacity on $Y$ (in the sense of Section 5).
2. $C_{\Phi}$ is an outer capacity, that is, for every $F \subset Y$, $C_{\Phi}(F) = \inf C_{\Phi}(O)$ where the infimum is taken over all open subsets $O$ containing $E$.
3. $C_{\Phi}(\bigcap_{n=1}^{\infty} \Gamma_n) = \inf_{n \geq 1} C_{\Phi}(\Gamma_n)$ for every decreasing sequence $(\Gamma_n)$ of compact subsets of $Y$. (This is a consequence of the previous property.)

We notice that properties (1)-(3) hold, for every function $\Phi \in \mathcal{Y}(X)$, even if both functions $\Phi$ and $\Psi$ do not satisfy the $\Delta_2$-condition.

**Proposition A.1.** $C_{\Phi}$ is a Choquet capacity.

To prove the proposition we shall proceed as in the proof of [3, Théorème 2]. Let us first note that for every subset $E \subset Y$,

$$C_{\Phi}(E) = \inf_{f \in \mathcal{F}_E} \| f \|_{(\Phi)} \quad \text{where} \quad \mathcal{F}_E := \{ f \in L_{\Phi}^+(X) : \partial f \geq 1 \text{ $C_{\Phi}$ - q.e on } E \}.$$
Lemma A.2. Let \( f, f_n \in L_\Phi(X) \) such that \((f_n)\) converges (in norm) to \( f \).

(a) The set \( \{ \hat{P} | f | = \infty \} \) is \( C_\Phi \)-polar.

(b) There exists a subsequence \((g_n)\) of \((f_n)\) such that \((\hat{P} g_n)\) converges \( C_\Phi \)-q.e to \( \hat{P} f \).

Proof. (a) For every \( j \geq 1 \),

\[
C_\Phi(\{ \hat{P} | f | = \infty \}) \leq C_\Phi(\{ \hat{P} | f | \geq j \}) \leq j^{-1} \| f \|_{(\Phi)}.
\]

(b) Choose a subsequence \((g_j)\) of \((f_n)\) such that \( \| f - g_j \|_\Phi \leq 2^{-j}/j \) for every \( j \geq 1 \), and let

\[
E_j = \{ j \hat{P} | f - g_j | > 1 \}, \quad F_j = \bigcup_{n \geq j} E_n, \quad \text{and} \quad E = \bigcap_{j \geq 1} F_j.
\]

Then

\[
C_\Phi(E) \leq C_\Phi(F_j) \leq \sum_{n=j}^{\infty} C_\Phi(E_n) \leq 2^{1-j}
\]

which yields that \( E \) is \( C_\Phi \)-polar. Thus the proof of (b) is finished seeing that \( \hat{P} g_j(y) \) converges to \( \hat{P} f(y) \) for every \( y \in Y \setminus E \). \( \square \)

Proof of Proposition A.1. By Theorem 5.1,

\[
L_\Phi(X)^* = L_\Psi(X) \quad \text{and} \quad L_\Psi(X)^* = L_\Phi(X)
\]

which implies, in particular, that \( L_\Phi(X) \) is reflexive. Let \((E_n)\) be an increasing sequence of subsets of \( Y \) and let \( E = \bigcup_{n=1}^{\infty} E_n \). We claim that

\[
C_\Phi(E) = \sup_{n \geq 1} C_\Phi(E_n).
\]

To prove this fact it is sufficient to check that

\[
\alpha := \sup_{n \geq 1} C_\Phi(E_n) \geq C_\Phi(E).
\]

So, without loss of generality we assume that \( \alpha < \infty \). Fix \( \varepsilon > 0 \). Then the convex subset

\[
\mathcal{A}_n := \{ f \in \mathcal{F}_E : \| f \|_{(\Phi)} \leq \alpha + \varepsilon \}
\]

is nonempty for every \( n \geq 1 \). Besides, by statement (b) of the above lemma, \( \mathcal{A}_n \) is closed in \( L_\Phi(X) \). So, \( \mathcal{A}_n \) is compact with respect to the topology \( \sigma(L_\Phi(X), L_\Psi(X)) \) (see, e.g., [15]). Therefore, since \( \mathcal{A}_n \) is decreasing we deduce that there exists

\[
f \in \bigcap_{n=1}^{\infty} \mathcal{A}_n.
\]

Now, seeing that \( f \in \mathcal{F}_E \) and \( \| f \|_{(\Phi)} \leq \alpha + \varepsilon \) it follows that \( C_\Phi(E) \leq \alpha + \varepsilon \) for every \( \varepsilon > 0 \). Whence \( C_\Phi(E) \leq \alpha \). \( \square \)
Corollary A.3. $C_\Phi$ and $c_\Psi$ coincide on $K$-Suslin subsets of $Y$. In particular, if the Borel subsets of $Y$ are $K$-Suslin (for instance, if $Y$ is locally compact) then $c_\Psi(F) = C_\Phi(F)$ for every subset $F$ of $Y$. 
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References


References


Lebenslauf

Persönliche Daten:
Name, Vorname: El Mabrouk, Khalifa
Geburtsdatum: 24. 02. 1969
Geburtsort: Ouled Chamekh (Tunesien)
Staatsangehörigkeit: tunesisch
Familienstand: ledig

Ausbildung und Tätigkeiten:
1976–1981 École Primaire d’Ouled Chamekh
1981 Concours National de Fin d’Études Primaires
1981–1984 Lycée Secondaire de Souassi
1984–1988 Lycée Technique de Mahdia
1988 Baccalauréat “Abitur” (Mathematik und Technik)
1988–1991 Studium der Mathematik und Physik an der Faculté des Sciences de Monastir
1991 Diplôme Universitaire d’Études Scientifiques in Mathematik und Physik
1991–1993 Studium der Mathematik an der Faculté des Sciences de Monastir
1993 Maîtrise in Mathematik
1993–1995 Studium der angewandten Mathematik an der École Nationale d’Ingénieur de Tunis
April 1996 Diplôme d’Études Approfondies (angewandten Mathematik)
1996–1998 Assistent an der Institut Supérieur des Études Technologique de Sousse
1998–2002 Doktorand an der Fakultät für Mathematik der Universität Bielefeld
10/00-09/01 Wissenschaftler Angestellter an der Fakultät für Mathematik der Universität Bielefeld
seit 10/01 Wissenschaftler Angestellter an der Mathematisches Institut der Universität Düsseldorf.