On Singular Control Games - With Applications to Capital Accumulation

Inauguraldissertation zur Erlangung des Grades eines Doktors der Wirtschaftswissenschaften (Dr. rer. pol.) an der Fakultät für Wirtschaftswissenschaften der Universität Bielefeld

vorgelegt von

Diplom-Wirtschaftsingenieur Jan-Henrik Steg

Bielefeld, April 2010

Erstgutachter
Professor Dr. Frank Riedel
Institut für Mathematische
Wirtschaftsforschung (IMW)
Universität Bielefeld

Zweitgutachter
Professor Dr. Herbert Dawid
Institut für Mathematische
Wirtschaftsforschung (IMW)
Universität Bielefeld

Gedruckt auf alterungsbeständigem Papier nach DIN-ISO 9706

Contents

Chapter 1

Introduction

The aim of this work is to establish a mathematically precise framework for studying games of capital accumulation under uncertainty. Such games arise as a natural extension from different perspectives that all lead to *singular control* exercised by the agents, which induces some essential formalization problems.

Capital accumulation as a game in continuous time originates from the work of Spence [33], where firms make dynamic investment decisions to expand their production capacities irreversibly. Spence analyses the strategic effect of capital commitment, but in a deterministic world. We add uncertainty to the model — as he suggests — to account for an important further aspect of investment. Uncertain returns induce a reluctance to invest and thus allow to abolish the artificial bound on investment rates, resulting in singular control.

In a rather general formulation, this intention has only been achieved before for the limiting case of perfect competition, where an individual firm's action does not influence other players' payoffs and decisions, see [6]. The perfectly competitive equilibrium is linked via a social planner to the other extreme, monopoly, which benefits similarly from the lack of interaction. There is considerable work on the single agent's problem of sequential irreversible investment, see e.g. [12, 30, 31], and all instances involve singular control. In our game, the number of players is finite and actions have a strategic effect, so this is the second line of research we extend.

With irreversible investment, the firm's opportunity to freely choose the time of investment is a perpetual real option. It is intuitive that the value of the option is strongly affected when competitors can influence the value of the underlying by their actions. The classical option value of waiting [15, 29] is threatened under competition and the need arises to model option exercise games.

While typical formulations [23, 28] assume fixed investment sizes and pose only the question how to schedule a single action, we determine investment sizes endogenously. Our framework is also the limiting case for repeated investment opportunities of arbitrarily small size. Since investment is allowed to take the form of singular control, its rate need not be defined even where it occurs continuously.

An early instance of such a game is the model by Grenadier [22]. It received much attention because it connects the mentioned different lines of research, but it became also clear that one has to be very careful with the formulation of strategies. As Back and Paulsen [4] show, it is exactly the singular nature of investment which poses the difficulties. They explain that Grenadier's results hold only for open loop strategies, which are investment plans merely contingent on exogenous shocks. Even to specify sensible feedback strategies poses severe conceptual problems.

We also begin with open loop strategies, which condition investment only on the information concerning exogenous uncertainty. Technically, this is the multi-agent version of the sequential irreversible investment problem, since determining a best reply to open loop strategies in a rather general formulation is a monotone follower problem. The main new mathematical problem is then consistency in equilibrium. We show that it suffices to focus on the instantaneous strategic properties of capital to obtain quite concise statements about equilibrium existence and characteristics, without a need to specify the model or the underlying uncertainty in detail. Nevertheless, the scope for strategic interaction is rather limited when modelling open loop strategies.

With our subsequent account of closed loop strategies, we enter completely new terrain. While formulating the game with open loop strategies is a quite clear extension of monopoly, we now have to propose classes of strategies that can be handled, and conceive of an appropriate (subgame perfect) equilibrium definition. To achieve this, we can borrow only very little from the differential games literature.

After establishing the formal framework in a first effort, we encounter new control problems in equilibrium determination. Since the methods used for open loop strategies are not applicable, we take a dynamic programming approach and develop a suitable verification theorem. It is applied to construct different classes of Markov perfect equilibria for the Grenadier model [22] to study the effect of preemption on the value of the option to delay investment. In fact, there are Markov perfect equilibria with positive option values despite perfect circumstances for preemption.

1.1 Capital accumulation

Capital accumulation games have become classical instances of differential games since the work by Spence [33]. In these games[1], firms typically compete on some output good market in continuous time and obtain instantaneous equilibrium profits depending on the firms' current capital stocks, which act as strategic substitutes. The firms can control their investment rates at any time to adjust their capital stocks.

By irreversibility, undertaken investment has commitment power and we can observe the effect of preemption. However, as Spence elaborated, this depends on the type of strategies that firms are presumed to use. The issue is discussed in the now common terminology by Fudenberg and Tirole [21], who take up his model.

If firms commit themselves at the beginning of the game to investment paths such that the rates are functions of time only, one speaks of open loop strategies. In this case, the originally dynamic game becomes in fact static in the sense that there is a single instance of decision making and there are no reactions during the implementation of the chosen investment plans. In equilibrium, the firms build up capital levels that are — as a steady state — mutual best replies.

However, if one firm can reach its open loop equilibrium capital level earlier than the opponent, it may be advantageous to keep investing further ahead. Then, the lagging firm has to adapt to the larger firm's capital stock and its best reply may be to stop before reaching the open loop equilibrium target, resulting in an improvement for the quicker firm. The laggard cannot credibly threaten to expand more than the best reply to the larger opponent's capital level in order to induce the latter to invest less in the first place. So, we observe preemption with asymmetric payoffs.

Commitments like to an open loop investment profile should only be allowed if they are a clear choice in the model setup. Whenever a revision of the investment policy is deemed possible, an optimal continuation of the game from that point on should be required in equilibrium. Strategies involving commitment in general do not form such *subgame perfect* equilibria. To model dynamic decision making, at least state-dependent strategies have to be considered, termed closed loop or feedback strategies[2].

In capital accumulation games, the natural (minimal) state to condition instantaneous investment decisions on are the current capital levels. They comprise all influence of past actions on current and future payoffs. Closed

[1] See also [16].
[2] This terminology is adapted from control theory.

loop strategies of this type are called Markovian strategies, and with a properly defined state, subgame perfect equilibria in these strategies persist also with richer strategy spaces.

In order to observe any dynamic interaction and preemption in the deterministic model, one has to impose an upper bound on the investment rates. Since the optimal Markovian strategies are typically “bang-bang” (i.e., whenever there is an incentive to invest, it should occur at the maximally feasible rate), an unlimited rate would result in immediate jumps, terminating all dynamics in the model. The ability to expand faster is a strategic advantage by the commitment effect and no new investment incentives arise in the game.

Introducing uncertainty adds a fundamental aspect to investment, fostering endogenous reluctance and more dynamic decisions. With stochastically evolving returns, it is generally not optimal to invest up to capital levels that imply a mutual lock-in for the rest of time. Although investment may occur infinitely fast, the firms prefer a stepwise expansion under uncertainty, because the option to wait is valuable with irreversible investment.

1.2 Irreversible investment and singular control

The value of the option to wait is an important factor in the problem of sequential irreversible investment under uncertainty (e.g. [1, 30]). When the firm can arbitrarily divide investments, it owns de facto a family of real options on installing marginal capital units. The exercise of these options depends on the gradual revelation of information regarding the uncertain returns, analogously to single real options. It is valuable to reduce the probability of low returns by investing only when the net present value is sufficiently positive.

The relation between implementing a monotone capital process with unrestricted investment rate but conditional on dynamic information about exogenous uncertainty and timing the exercise of growth options based on the same information is in mathematical terms that between singular control and optimal stopping.

For all degrees of competition discussed in the literature — monopoly, perfect competition [27], and oligopoly [5, 22] — optimal investment takes the form of singular control. This means that investment occurs only at singular events, though usually not in lumps but nevertheless at undefined rates.

Typically only initial investment is a lump. In most models, subsequent investment is triggered by the output good price reaching a critical threshold and the additional output dynamically prevents the price from exceeding this boundary. This happens in a minimal way so that the control paths needed for the "reflection" are continuous. While the location of the reflection boundary incorporates positive option premia for the monopolist, it coincides with the zero net present value threshold in the case of perfect competition, which eliminates any positive (expected) profits derived from delaying investment. The results for oligopoly depend on the strategy types, see Section 1.4 below.

The relation between singular control and optimal stopping holds at a quite abstract level, which permits to study irreversible investment more generally than for continuous Markov processes and also in absence of explicit solutions, see [31] for monopoly and [6] regarding perfect competition. Such a general approach in fact turns out particularly beneficial for studying oligopoly.

Here, the presence of opponent capital processes increases the complexity of the optimization problems and consistency in equilibrium is another issue. Consequently, one has to be very careful to transfer popular option valuation methods or otherwise acknowledged principles on the one hand, while the chance to obtain closed form solutions shrinks correspondingly on the other hand.

The singular control problems of the monopolist and of the social planner introduced for equilibrium determination under perfect competition are of the monotone follower type. For these control problems there exists a quite general theory built on their connection to optimal stopping, see [7, 19]. This theory facilitates part of our study of oligopoly, too. It is a quite straightforward extension of the polar cases to *formalize* a general game of irreversible investment with a finite number of players using open loop strategies. In this case, the individual optimization problems are of the monotone follower type as well. The main new problem becomes to ensure consistency in equilibrium.

A crucial facet for us is the characterization of optimal controls by a first order condition in terms of discounted marginal revenue, used by Bertola [12] and introduced to the general theory of singular control by Bank and Riedel [10, 7]. Note that given some investment plan, it is feasible to schedule additional investments at *stopping times*. With an optimal plan, the additional expected profit from marginal investment at any stopping time cannot be positive. Contrarily, at any stopping time such that capital increases by optimal investment, marginal profit cannot be negative since reducing the corresponding investment is feasible.

Based on this intuitive characterization, which is actually sufficient for optimal investment, we show that equilibrium determination can be reduced to solving a single monotone follower problem. However, the final step requires some work on the utilized methods, to which we dedicate a separate discourse.

The actual equilibrium capital processes are derived in terms of a signal process by tracking the running supremum of the latter. Riedel and Su call the signal "base capacity"[31], because it is the minimal capital level that a firm would ever want. Using the base capacity as investment signal corresponds to the mentioned price threshold to trigger investment insofar as adding capacity is always profitable for current levels below the base capacity (resp. when the current output price exceeds the trigger price), but never when the capital stock exceeds the base capacity (resp. when the output price is below the threshold). Tracking the — unique — base capacity is the optimal policy for any starting state or time, similar to a stationary trigger price for a Markovian price process.

Under certain conditions, the signal process can be obtained as the solution to a particular backward equation, where existence is guaranteed by a corresponding stochastic representation theorem (for a detailed presentation, see [8], for further applications [7, 9]).

When the necessary condition for this method is violated, which is typical for oligopoly, one can still resort to the related optimal control approach via stopping time problems. Here, the optimal times to install each marginal capital unit are determined independently, like exercising a real option. The right criterion therefor is the opportunity cost of waiting.

These optimal stopping (resp. option exercise) problems form a family which allows a unified treatment by monotonicity and continuity. Indeed, at each point in time, there exists a maximal capital level for which the option to delay (marginal) investment is worthless. This is exactly the base capacity described above and the same corresponding investment rule is optimal.

As a consequence, irreversible investment is optimal not when the *net present value* of the additional investment is greater or equal zero, but when the *opportunity cost of delaying* the investment is greater or equal zero.

1.3 Strategic option exercise

The incentives of delaying investment due to dynamic uncertainty on the one hand and of strategic preemption on the other hand contradict each other. Therefore, when the considered real option is not exclusive, it is necessary to study games of option exercise. The usual setting in the existing literature is a two-player game, where each player possesses a single real option, e.g. to enter or exit some market or to increase capacity by an investment of fixed size. One then tries to determine an equilibrium in exercise times.

Depending on the relative strengths of the involved incentives, generally two types of Markov perfect equilibria arise. For instance Mason and Weeds [28] formulate a "reduced form" model to study the influence of varying degrees of uncertainty on the strategic effects that appear in option exercise games. Two firms can decide in continuous time when to make an irreversible investment, which starts a payoff stream that is affected by an exogenous stochastic shock process. The latter is an observable geometric Brownian motion, so a Markovian strategy takes the simple form of a trigger value for the shock process.

Of course, the payoff streams depend also on the exercise decisions of both firms, for which two effects are important. First, if there is a considerable advantage to being the leader — for instance due to subsequent entry deterrence allowing temporary monopoly profits — one speaks of the incentive to preempt the opponent. The (ubiquitous) equilibrium driven by *preemption* is sequential exercise. Second, the *externalities* from simultaneous exercise may under circumstances be sufficiently favourable to allow for such a second class of equilibria.

These two types of equilibria have also been obtained by Fudenberg and Tirole in their study of a deterministic timing game [20], to which the typical option exercise games are analogies. Markov perfect equilibria with sequential exercise are determined in the latter according to the same principle. By backward induction, the second mover exercises optimally "in isolation", conditional on the first mover having exercised before. Then, since the firms need to be indifferent regarding their roles in equilibrium, the leader has to exercise such that the expected payoff equals that of the follower, i.e. earlier than in isolation. We necessarily observe *rent equalization* in the respectively unique sequential equilibrium.

Simultaneous exercise is usually conceived to be inferior to sequential exercise and corresponds to a coordination error. However, the particular model may also allow a second class of Markov perfect equilibria with simultaneous exercise, which then even forms a continuum with Pareto ranked elements. Under uncertainty, the real option benefits are qualitatively unmitigated in these shared equilibria, as Mason and Weeds' comparative statics demonstrate. This result contrasts the observation that the leader's exercise trigger increases much less with uncertainty in sequential equilibrium than in isolation and is even bounded. Preemption strongly limits the option value of waiting.

The sequential equilibria with Markovian strategies as proposed by Mason and Weeds are arguably unsatisfactory because ex ante identical firms are supposed to coordinate on the roles of leader and follower. While the payoffs are unaffected due to rent equalization, the conceptual problem is rather how to exclude the possibility of coordination failure or that one would prefer the outcome to result from symmetric strategies. This issue is addressed by the authors Huisman, Kort, Pawlina and Thijssen in a number of works, summarized in [24]. Again, they build on the deterministic case [20]. Fudenberg and Tirole provide a concept to treat coordination problems endogenously by extending the strategy spaces. If the players regret having stopped simultaneously, they play a repeated auxiliary game (which costs no time) to determine the winner. The stage probabilities of playing "invest" in this auxiliary game are part of the extended strategies in the timing game.

Huisman et al. transfer this concept to formulate mixed (symmetric) strategies for the stochastic option exercise game. Then, when the initial state of the game does not induce immediate exercise, one observes the common preemption outcome where the leader is endogenously determined with equal probabilities. Otherwise the repeated game decides who invests immediately. When the model admits the Pareto improving simultaneous exercise equilibria, they also exist with those extended strategies that represent the corresponding pure strategies.

The simultaneous equilibria in [24] arise in the case of capacity expansion by two already active firms. In this dynamic situation, it is quite a limitation to dictate the investment sizes and to endow each firm with a single action only. Boyer et al. aim to analyse the effect of repeated investment opportunities [13, 14]. However, in order to be able to conduct backward induction, it is necessary to assume a finite market size, resp. that capacity increments are always large relative to the market, which induces an (endogenous) end of the game by enforcing situations such that no further investment ever becomes profitable. Since the authors also adapt the concept of Fudenberg and Tirole (though in a different way than Huisman et al.) to define mixed strategies, tractability limits the analysis to few steps.

1.4 Grenadier's model

Grenadier has formulated a model with investment opportunities coming closer to capital accumulation. In [22], he proposes a game between a finite number of firms that can increase their output of a homogeneous good at any time by arbitrarily divisible investment, where inverse demand is affected by an exogenous diffusion process. Since the firms are identical, he aims at a symmetric equilibrium (in strategies *and* outcome) from the out-set. Furthermore, the model is designed such that investment takes the form of "infinitesimal increments" when the shock process reaches certain thresholds, intending continuous output paths. This means the singular controls described above. A crucial ingredient therefor is the purely proportional investment cost and that marginal revenue is increasing in the exogenous shock. The only further structural assumption made is that marginal revenue decreases in the individual firm's own output.

Unfortunately, Grenadier does not provide a strictly formal definition of the strategies that players may use. On the one hand, he states quite clearly that each firm i chooses an output process $q_i(t)$ and recognizes that opponent output $Q_{-i}(t)$ is beyond its control, but taken account of. Consequently, firms use open loop strategies as defined above. On the other hand, Grenadier defines the state of the game comprising the current shock value $X(t)$ and the output quantities and sets out to determine investment trigger functions $X^i(q_i, Q_{-i})$ for each firm i with current output q_i and facing opponent output Q_{-i}, in order to generate the output paths. This approach strongly resembles Markovian strategies.

Grenadier also provides conditions for optimal behaviour in terms of the trigger functions. For concreteness, he heuristically derives a set of value matching and smooth pasting conditions at the level of Dumas [17] to identify the value function and best reply trigger for an individual player. He then applies the argument of Leahy [27] to claim that the myopic triggers are optimal, which are the optimal triggers for marginal investment if the current output levels are fixed forever. While the steps up to this point involve general individual output processes, Grenadier now resorts to the symmetric expansion path to observe that one ends up with a standard problem with a single control variable, aggregate output.

Equilibrium aggregate output Q^* is then determined by the "standard" real option methods, again with the help of a trigger function $X^*(Q)$. The actual symmetric equilibrium output processes are simply $\frac{1}{n}Q^*(t)$. One can indeed define proper open loop output processes in this way, which Grenadier should have emphasized more clearly.

His model and results obtain considerable attention and works building upon the formulation and methodology emerge, see for instance [2]. However, it is also recognized that further clarification is necessary, to prevent misunderstanding regarding the employed strategies and implied potential subgame perfection, as well as to address the interdependent new optimization problems rigorously.

These issues are the focus of Back and Paulsen [4]. They start with a formal definition of open loop strategies as the same controls available to the monopolist in the sequential irreversible investment problem. Then a set of optimality conditions similar to Grenadier's is given to verify a symmetric equilibrium, but accompanied by a formal sufficiency proof (also involving myopic triggers). Consequently, the symmetric equilibrium of Grenadier does exist in open loop strategies. However, it does not exist if one interprets the trigger functions as Markovian strategies, for which the authors construct a counterexample with profitable preemption. Consequently, the equilibrium is not subgame perfect.

Back and Paulsen also discuss the conceptual problems in even defining the game with closed loop strategies and conclude they do not know how to do so. If one wants to preserve singular control, the undertaken investments are no proper actions to be assigned by strategies, because investment is neither discrete nor does it occur at well defined (bounded) rates. Consequently, strategies have to relate to the absolute levels of capital. In principle, this can be managed also with the restrictions of monotonicity and of conditioning on current information in a mathematically feasible way, for instance precisely by trigger functions. However, the additional problem prevailing in any continuous-time game that plausible strategies do not uniquely define the course of the game can hardly be excluded.

We propose a formal framework for singular control games with feedback strategies in Chapter 3 and prove that there exist further Markov perfect equilibria in Grenadier's model besides the one argued by Back and Paulsen, who suggest that firms invest in perfect competition quantities under complete dissipation of option values.

Chapter 2

Open loop strategies

In this chapter we study a game of capital accumulation under uncertainty, which is the multi-agent version of the sequential irreversible investment problem. Like the monopolist planning to expand capacity by arbitrarily divisible investment, the players choose adapted, nondecreasing, left-continuous stochastic processes for the expansion of their capital stocks. Such are *open loop* strategies.

Back and Paulsen [4] clarify that the symmetric equilibrium proposed by Grenadier for his model [22] only exists with open loop strategies, so it is an instance of the game we formulate. However, their rigorous proof provides only a set of sufficient conditions for very similar frameworks and general existence is not clear. Our aim is to propose an unambiguous framework, in which no investment trigger functions are involved (which could cause misunderstanding) and with a quite abstract notion of the underlying uncertainty.

We fully concentrate on the strategic properties of capital to study commitment power and characterize existence and nature of equilibria in the general game formulation. Allowing for non-Markovian shock processes (possibly with jumps) and heterogeneous initial capital stocks, we show that one can determine existence of and behaviour in equilibrium by focusing on the strategic properties of instantaneous profit.

The employed assumptions strongly resemble common ones from the classical Cournot literature. With them, we can reduce the determination of any equilibrium to solving a single agent problem, which is of the same structure as any firm's best reply problem. When all players use open loop strategies, these are classical monotone follower problems. While we can make use of the related literature, we tailor the methods to our needs, where consistency in equilibrium is the main new issue. A very helpful concept for us will be the characterization of optimal controls by a first order condition as already employed by Bertola [12] and developed more generally by Bank [7].

We introduce this central first order condition and illustrate its role for equilibrium by the simpler case of perfect competition, where we can reproduce the results of Baldursson and Karatzas [6] more directly. Then, we move to oligopoly and transfer the singular control approach developed by Bank and Riedel [10] to determining a symmetric equilibrium. In this case it is possible to construct the equilibrium capital processes by applying a stochastic representation theorem.

To be able to handle the case of asymmetric initial capital, we need to elaborate on the methods from the general theory of monotone follower problems. With this preparation, we will obtain very concise results and give a full characterisation of open loop equilibria as solutions of a single monotone follower problem.

After these general results, we explicitly derive a symmetric equilibrium for the same model as in [22, 4], but allowing for a shock process with jumps. Based on the solution we demonstrate the expected effect of dissolving option value and incorporate the limit of perfect competition as in [6].

2.1 Perfect competition

We would like to illustrate the role and meaning of the first order condition by a "degenerate" case of the game. Baldursson and Karatzas [6] study a perfectly competitive setting, where an individual firm's action does not influence the revenue opportunities of any other firm in the industry. Their approach is to solve a social planner's problem and then show under which conditions it has the properties of an equilibrium capital process. As we will see, the defining properties of equilibrium coincide with our first order condition, so that we can address equilibrium determination more directly. Afterwards, the connection to the social planner's singular control problem will be highlighted, but which is not necessary anymore.

So, consider a non-atomic continuum $[0, \infty)$ of homogeneous investors, all owning a perpetual option to enter a common market. Exercising such an option starts a non-callable stochastic profit stream. To model the underlying uncertainty, let $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness.

An entering strategy is then a stopping time τ with respect to the given filtration, i.e. the decision whether to exercise the option at any point in time t has to be based on the information reflected by the σ-algebra $\mathscr{F}_t$. Formally, the strategy space of each individual firm is $\mathscr{T}$, the set of all stopping times that take values in $[0, \infty]$.

Although investors are negligibly small so that individual entry does not affect the level of capital in the industry, the entering firms collectively generate an aggregate investment process. We identify the current capital stock, denoted Q_t, by the measure of firms having entered so far. An individual firm will take this capital process as given and obtains as expected payoff from entering at random time τ

$$j(\tau|Q) \triangleq \mathbf{E}\left[\int_{\tau}^{\infty} \pi(t, Q_t)\,dt - k_\tau\right]. \tag{2.1}$$

Since exit is not allowed and depreciation abstracted from for expositional simplicity,[1] the process $(Q_t)_{t\geq 0}$ belongs to the following class of feasible aggregate investment processes:

$$\mathscr{A}(q_0) \triangleq \{Q \text{ adapted, nondecreasing, left-continuous, with } Q_0 = q_0 \ \mathbf{P}\text{-a.s.}\}.$$

Here, $q_0 > 0$ is some given incumbent capital[2]. The capital stock clearly is assumed to influence any active firm's instantaneous profit, which is for this purpose modelled as a random field $\pi(\omega, t, q) : \Omega \times [0, \infty) \times \mathbb{R}_+ \to \mathbb{R}$, where we suppress the argument ω as above when taking expectations. Dependence on time t incorporates possible discounting and q is some capital level. Here, we chose an infinite horizon, but note that one might as well consider a finite horizon together with some scrap value function as terminal payoff, conditional on having entered the market before.

Finally, the (investment) cost for exercising any of the options may be random as well and is thus formalized by the stochastic process k.

In order for (2.1) to be well defined, we make the following

Assumption 1.

i. For any $(\omega, t) \in \Omega \times [0, \infty)$, the mapping $q \mapsto \pi(\omega, t, q)$ is continuously decreasing from $\pi(\omega, t, 0) = +\infty$ to $\pi(\omega, t, +\infty) \leq 0$.

ii. For $q \in \mathbb{R}_+$ fixed, $(\omega, t) \mapsto \pi(\omega, t, q)$ is progressively measurable.

iii. For any $Q \in \mathscr{A}(q_0)$ with $q_0 > 0$, $\pi(\omega, t, Q_t(\omega))$ is $\mathbf{P} \otimes dt$-integrable.

Furthermore, we assign the following properties to the investment cost process k.

[1] Depreciation would effectively only change the discount rate.

[2] We only consider strictly positive incumbent capital for convenience. It eases the reconciliation of integrability with some necessary conditions for applying a stochastic representation theorem below without elaborating a modification to avoid appearance of negative auxiliary capital stocks.

Assumption 2. The optional process $(k_t)_{t\geq 0}$ is a right-continuous supermartingale, strictly positive for $t \in \mathbb{R}_+$ and $k_\infty = 0$, $\mathbf{P}$-a.s.

Assumption 2 is satisfied in the common case where the investment cost is constant but discounted at a nonnegative optional or deterministic rate.

2.1.1 Characterization of equilibrium

We will not worry to solve the investors' optimal stopping problems for arbitrary capital stock processes. In an equilibrium, optimal behaviour on behalf of *all* firms — entering and refraining — limits the observational variety of outcomes. In fact, because staying outside the market gives zero profit and in our model there is always a positive measure of option holders not having exercised yet, further exercise at any stopping time cannot yield positive expected payoff in equilibrium. On the other hand, a positive measure of firms enter at any time when aggregate investment increases and this decision needs to be optimal. Consequently, an equilibrium is characterized by the corresponding capital stock process as follows.

Definition 2.1. $Q^* \in \mathscr{A}(q_0)$ is a *perfect competition equilibrium* process for incumbent capital $q_0 \in \mathbb{R}_+$ if $\sup_{\tau\in\mathscr{T}} j(\tau|Q^*)$ — the option value given Q^* — is zero, and exercising is optimal whenever Q^* increases, i.e. at all stopping times $\tau^*(x) \triangleq \inf\{t \geq 0 | Q^*_t > x\}$, $x \geq q_0$.

Note that at the times when equilibrium investment increases, all option holders are indifferent whether to exercise immediately or to keep waiting, possibly forever. Thus we may conclude that there is an equilibrium in individual strategies where just enough firms enter at any such time to support the aggregate equilibrium investment. The reasoning up to this point will be formalized stronger when we determine the — as we will see unique — equilibrium investment process in the following.

We directly start with the observation that the defining properties of an equilibrium process are in fact a first order condition for an optimal process in the stochastic control problem which we formulate now. For an economic interpretation let us take the perspective of a fictitious *social planner*, like it is common practice in finding perfectly competitive equilibria. Consider that this authority can control how many firms enter at each moment, but still without foresight. Its objective is to pursue an efficient irreversible investment process in the sense of maximizing the aggregate expected profit, net of investment cost. If the firm level profit flow π is inverse demand minus variable production cost, the planner is benevolent in the classical meaning that consumer surplus shall be maximal while taking account of all incurred costs. Formally, this leads to the stochastic control problem of maximizing

$$J(Q) \triangleq \mathbf{E}\left[\int_0^\infty \Pi(t, Q_t)\,dt - \int_0^\infty k_t\,dQ_t\right] \tag{2.2}$$

over all $Q \in \mathscr{A}(q_0)$, where the random field $\Pi : \Omega \times [0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ relates to π by

$$\Pi(\omega, t, q) = \int_0^q \pi(\omega, t, y)\,dy. \tag{2.3}$$

Consequently, it inherits measurability from Assumption 1 and is concave in capital with continuous partial derivative $\Pi_q \triangleq \partial\Pi/\partial q$. Furthermore, by the integrability assumption, the negative part of Π is also $\mathbf{P} \otimes dt$-integrable for fixed $q \in \mathbb{R}_+$.

By (2.3), attainable revenue is nonnegative, but to have a meaningful stochastic control problem, we impose the additional

Assumption 3. The process $(\omega, t) \mapsto \sup_{q \in \mathbb{R}_+} \Pi(\omega, t, q)^+$ is $\mathbf{P}\otimes dt$-integrable.

In combination with Assumption 2, the value of the problem is finite and it suffices to consider *admissible* controls with bounded expected cost.

Since the problem is of the monotone follower type with concave objective functional J, one can solve it by the approach developed by Bank and Riedel [10]. The starting point is to formulate a first order condition for potential solutions, which turns out to be very illustrative for our purpose, because the relation between the social planner's control problem and equilibrium determination becomes immediate.

The first order condition for an optimal control policy is based on the following gradient, which has also been used by Bertola [12] for a more specific single-agent problem. Let $\nabla J(Q)$ denote for any $Q \in \mathscr{A}(q)$ the unique optional process such that

$$\nabla J(Q)_\tau = \mathbf{E}\left[\int_\tau^\infty \Pi_q(t, Q_t)\,dt \,\middle|\, \mathscr{F}_\tau\right] - k_\tau \quad \text{for all stopping times } \tau \in \mathscr{T}. \tag{2.4}$$

Heuristically, it describes the marginal profit from *irreversible* investment at any stopping time, see the discussion below. The first order condition in terms of this gradient in fact coincides with our given definition of an equilibrium investment process since $\Pi_q = \pi$.

Proposition 2.2. *If Assumptions 1,2, and 3 are satisfied, a control policy* $Q^* \in \mathscr{A}(q_0)$ *maximizes the social planner's objective* (2.2) *if it is a perfect* *competition equilibrium process according to Definition 2.1, because then*

$$\nabla J(Q^*) \le 0 \quad \textit{and} \quad \int_0^\infty \nabla J(Q^*)_s \, dQ^*_s = 0 \quad \mathbf{P}\text{-}\textit{a.s.} \tag{2.5}$$

Proof. We first show the claimed optimality, so let $Q^* \in \mathscr{A}(q_0)$ satisfy (2.5) and assume it causes finite investment cost. Further consider an arbitrary $Q \in \mathscr{A}(q_0)$ with $J(Q) > -\infty$. Since Π is concave in q by its definition (2.3) and Assumption 1, we can estimate

$$\begin{aligned}
J(Q) - J(Q^*) &= \mathbf{E}\left[\int_0^\infty \Pi(t, Q_t) - \Pi(t, Q^*_t)\, dt - \int_0^\infty k_t \, d(Q_t - Q^*_t)\right] \\
&\le \mathbf{E}\left[\int_0^\infty \Pi_q(t, Q^*_t)\big(Q_t - Q^*_t\big)\, dt - \int_0^\infty k_t \, d(Q_t - Q^*_t)\right] \\
&= \mathbf{E}\left[\int_0^\infty \Pi_q(t, Q^*_t)\Big(\int_0^t d(Q_s - Q^*_s)\Big)\, dt - \int_0^\infty k_t \, d(Q_t - Q^*_t)\right] \\
&= \mathbf{E}\left[\int_0^\infty \int_s^\infty \Pi_q(t, Q^*_t)\, dt \, d(Q_s - Q^*_s) - \int_0^\infty k_s \, d(Q_s - Q^*_s)\right] \\
&= \mathbf{E}\left[\int_0^\infty \nabla J(Q^*)_s \, d(Q_s - Q^*_s)\right].
\end{aligned}$$

In the second last line, we use Fubini's theorem to change the order of integration. By the first order condition (2.5), the last expression above is nonpositive. So we conclude $J(Q) \le J(Q^*)$.

Regarding the assumed admissibility of $Q^* \in \mathscr{A}(q_0)$ satisfying (2.5), go through the steps above backward with $Q \equiv q_0$ to obtain that $J(Q^*) \ge 0$ and that consequently the expected investment cost of Q^* is finite, so the control is admissible.

Now we show that if $Q^* \in \mathscr{A}(q_0)$ is a perfect competition equilibrium process according to Definition 2.1, it satisfies (2.5). Remember $\Pi_q = \pi$ and an individual firm's objective (2.1). So, the Definition 2.1 of an equilibrium investment process translates into (i) $\mathbf{E}[\nabla J(Q^*)_\tau] \le 0$ for all stopping times $\tau \in \mathscr{T}$, which implies the inequality in (2.5), and (ii) $\mathbf{E}[\nabla J(Q^*)_{\tau^*(x)}] = 0$ for all $x \in \mathbb{R}_+$ and $\tau^*(x)$ as in Definition 2.1. To deduce the required equality, note that τ^* is the right-continuous inverse of the monotone Q^* (see also (2.8) below). This permits to use the change-of-variable formula

$$\int_0^\infty \nabla J(Q^*)_s \, dQ^*_s = \int_0^\infty \nabla J(Q^*)_{\tau^*(x)} \, dx \quad \mathbf{P}\text{-a.s.},$$

cf. [6]. The integrand on the right-hand side is zero $\mathbf{P}$-a.s. by the equilibrium definition, which completes the proof. ο.ε.δ.

The intuition conveyed by the first order condition reveals the connection between the optimal control problem and equilibrium determination. Given some capital stock process, the social planner may consider a further marginal investment at any stopping time. Irreversibility then induces a flow of marginal profit from that moment onward. If the capital process is optimal, marginal investment cannot be profitable, corresponding to entry of additional small investors. On the other hand, optimality requires that marginal profit is not negative at any time when investment occurs, because reducing investment would then be beneficial by continuity of Π_q, resp. π. This principle corresponds to consistency of the small investors' optimal entry times with increases of aggregate capital.

The first order condition is sufficient for an optimal control in the social planner's problem. One could now continue by solving the aggregate investment problem and verifying whether the equilibrium properties are also necessary. However, when marginal instantaneous revenue satisfies the Inada condition of Assumption 1, one can pursue the approach of Bank and Riedel and construct a control process satisfying the first order condition by means of a stochastic representation theorem. This way is much more direct for equilibrium determination, since the fact that the identified capital process will also maximize the social planner's objective is a helpful interpretation, but not needed as an intermediate result.

2.1.2 Construction of equilibrium investment

The first order condition is not constructive because the inequality will only be binding when investment occurs. One *does* obtain equality at an arbitrary point in time by considering to start with zero capital at that instant and determining an "initial" investment size such that equality holds, conditional on equality also holding at all subsequent investments. The resulting family of equations can actually be summarized by a single backward equation, which the process of such "initial capital levels" has to satisfy. Specifically, the stochastic representation problem arises to find the (unique) optional process L satisfying

$$\mathbf{E}\left[\int_\tau^\infty \pi(t, \sup_{\tau\le u<t} L_u)\, dt \,\middle|\, \mathscr{F}_\tau\right] - k_\tau = 0 \quad \text{for all stopping times } \tau \in \mathscr{T}. \tag{2.6}$$

Our assumptions ensure the existence of a solution to this problem, from which we can directly deduce a capital process satisfying the first order condition. In comparison to (2.5), we replaced Q^* by the running supremum (starting at τ) of the process L to be determined, while enforcing equality to hold **P**-a.s. Bank and El Karoui [8] discuss this representation problem in detail and we can use their central existence result [8, Theorem 3]. Under some quite common specifications of π and k, one can also derive closed-form solutions. We will discuss these in the oligopoly case below, the limit of which turning out to be the present perfectly competitive equilibrium.

Once we derived the seemingly abstract process L, we obtain the social planner's optimal control policy — resp. perfect competition equilibrium process — as follows.

Proposition 2.3. *Under Assumptions 1,2, and 3, the unique perfect competition equilibrium process for incumbent capital $q_0 \in \mathbb{R}_+$ is given by*

$$Q^* \triangleq q_0 \vee \Big(\sup_{0 \le u < t} L_u \Big)_{t \ge 0}, \tag{2.7}$$

where L is the unique optional process solving representation problem (2.6).

Proof. Q^* belongs to $\mathscr{A}(q_0)$, so it is feasible. Use the definition of Q^* and the representation (2.6) of k_τ to obtain for any $\tau \in \mathscr{T}$

$$\begin{aligned} j(\tau|Q^*) &= \mathbf{E}\left[\int_\tau^\infty \pi(t, Q_t^*)\, dt - k_\tau \right] \\ &= \mathbf{E}\left[\int_\tau^\infty \pi(t, q_0 \vee \sup_{0 \le u < t} L_u)\, dt - \int_\tau^\infty \pi(t, \sup_{\tau \le u < t} L_u)\, dt \right] \end{aligned}$$

As π is decreasing in q, the last expectation is nonpositive.

Now, fix an $x \ge q_0$ and the corresponding $\tau^*(x) \in \mathscr{T}$. Then, for any $t > \tau^*(x)$, $q_0 \vee \sup_{0 \le u < t} L_u = \sup_{\tau^*(x) \le u < t} L_u$ by the definition of Q^*, so the two integrands cancel. Thus, $\tau^*(x)$ yields zero payoff and is consequently optimal.

Uniqueness follows from optimality of Q^* for the strictly concave objective functional of the social planner. Optimality for the social planner's problem follows from Proposition 2.2. ο.ε.δ.

Combining our results up to this point, we have a quite direct, rigorous proof for existence and uniqueness of a perfect competition equilibrium. In equilibrium, optimal entry timing merely yields zero expected net profit, implying that we may expect consistency of individual with aggregate behaviour.

2.1.3 Myopic optimal stopping

We close the discussion of perfect competition by illustrating the familiar connection between singular control problems and optimal stopping. Since the social planner's problem is of the monotone follower type, it can also be solved by determining the optimal stopping time to install each capital unit $x \in \mathbb{R}_+$. The optimal control process is then given by the right-continuous inverse of this family of stopping times:

$$Q_t^* = \sup\{x \in [0,\infty) : \tau^*(x) < t\} \quad t \in [0,\infty). \tag{2.8}$$

This is actually how Baldursson and Karatzas [6] determine the equilibrium investment process. In the current context, the stopping problems are given the interpretation of optimal entry by *myopic* investors.

These hypothetical agents solve similar stopping problems as rational ones, they only assume that aggregate capital remains fixed forever at some level, say $x \geq 0$. In all other respects, they have the same knowledge as the rational agents. Formally, the myopic agents evaluate any stopping time $\tau \in \mathscr{T}$ by

$$j^m(\tau|x) \triangleq \mathbf{E}\left[\int_\tau^\infty \pi(t,x)\,dt - k_\tau\right].$$

By the proof of 2.3 it is easy to see that $\tau^*(x)$ is optimal for a myopic firm facing the specific capital level $x \geq q_0$.

With the help of the representation theorem, one can determine the myopic stopping times simultaneously, which would otherwise have to be calculated by a continuum of Snell envelopes. However, they become now a by-product, since the equilibrium properties follow in our case directly from the representation theorem.

2.2 The game

We now model a continuous time capital accumulation game between a fixed number of firms, indexed by $i = 1, \ldots, n$. The firms may repeatedly make investments of arbitrary size (we do not require the investment rates to be defined) to *increase* their respective capital stocks. We assume a continuous revelation of uncertainty regarding future payoffs and allow the firms to condition their investment decisions on the accumulated information. Formally, let $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness. Then, based on its given initial capital stock $q^i \in \mathbb{R}_+$, every firm has to choose a strategy Q^i from $\mathscr{A}(q^i)$, its class of feasible investment processes, where for any $q \in \mathbb{R}_+$,

$$\mathscr{A}(q) \triangleq \{Q \text{ adapted, left-continuous, nondecreasing, with } Q_0 = q,\ \mathbf{P}\text{-a.s.}\}.$$

Thus, the firms are restricted to using *open loop* strategies, since the investment decisions during the run of the game only depend on information regarding the exogenous uncertainty, but not on deviations in the opponents' capital stocks. Strategic interaction then only affects the initial choices of investment processes[3].

The strategies are mapped to payoffs via profit flows that depend on the current capital levels of all firms, net of a linear investment cost. Denote the aggregate capital by $Q \triangleq \sum_{j=1,\ldots,n} Q^j$, such that firm i faces opponent capital $Q^{-i} \triangleq Q - Q^i$. Given a combination of strategies from $\prod_{j=1}^n \mathscr{A}(q^j)$, firm i then receives the payoff

$$J(Q^i|Q^{-i}) \triangleq \mathbf{E}\left[\int_0^\infty \Pi(t, Q_t^i, Q_t^{-i})\,dt - \int_0^\infty k_t\,dQ_t^i\right]. \tag{2.9}$$

The profit flow is specified by the random field $\Pi : \Omega \times [0,\infty) \times \mathbb{R}_+^2 \to \mathbb{R}$, where the argument ω will typically be suppressed. Assume for example that it arises from spot competition between the firms in a common market. We give Π some more structure to ensure that the payoffs are well defined.

Assumption 4.

i. For any $(\omega, t) \in \Omega \times [0,\infty)$, the mapping $(q^i, q^{-i}) \mapsto \Pi(\omega, t, q^i, q^{-i})$ is twice continuously differentiable. For $q^{-i} \in \mathbb{R}_+$ fixed, the partial derivative Π_{q^i} strictly decreases in q^i.

ii. For $(q^i, q^{-i}) \in \mathbb{R}_+^2$ fixed, $(\omega, t) \mapsto \Pi(\omega, t, q^i, q^{-i})$ is progressively measurable.

iii. For any $(Q^i, Q^{-i}) \in \mathscr{A}(0)^2$, $\Pi(\omega, t, Q_t^i(\omega), Q_t^{-i}(\omega))$ is $\mathbf{P} \otimes dt$-integrable.

Since we will make frequent use of the partial derivative Π_{q^i} in different contexts, we define the function $\pi \equiv \Pi_{q^i}$ for notational simplicity. Note that since Π is concave in q^i, $\pi(\omega, t, Q_t^i(\omega), Q_t^{-i}(\omega))$ is $\mathbf{P} \otimes dt$-integrable for any $(Q^i, Q^{-i}) \in \mathscr{A}(q^i) \times \mathscr{A}(q^{-i})$ with $q^i > 0$, especially for $Q^i \equiv q^i$.

Concerning the investment cost process k, Assumption 2 is maintained. Consequently, apart from individual initial capital levels q^i, the firms are homogeneous.

With these assumptions, the payoffs are well defined, taking values in $\mathbb{R} \cup \{-\infty\}$. Determining firm i's optimal choice of a strategy against a given process $Q^{-i} \in \mathscr{A}(\sum_{j=1}^n q^j - q^i)$ specifying opponent capital amounts

[3]Besides considerable technical difficulties, the choice of open loop strategies can be justified at the modelling stage if firms are not able to observe the opponents' capital stocks.

to solving a stochastic control problem of the monotone follower type (see Section 2.4) with value function

$$V(q^i, Q^{-i}) \triangleq \sup_{Q \in \mathscr{A}(q^i)} J(Q|Q^{-i}). \tag{2.10}$$

So we can close the description of the game by a standard Nash equilibrium concept.

Definition 2.4. $(Q^{*1}, \ldots, Q^{*n})$ is an *open loop* equilibrium if for all $i \in \{1, \ldots, n\}$, $Q^{*i} \in \mathscr{A}(q^i)$ and $J(Q^{*i}|Q^{*-i}) = V(q^i, Q^{*-i})$.

To solve the individual control problems arising with open loop strategies and to determine if there is a best reply for firm i to a given set of strategies $(Q^j)_{j\in\{1,\ldots,n\}, j\neq i}$, we can use results from the literature on monotone follower problems (cf. [19, 6, 7]). However, the presence of the exogenous process Q^{-i} induces that the verification is limited to a case-by-case basis. In order to actually determine an equilibrium, where consistency is the main new issue, we need to adapt the related techniques to our purposes.

Before considering equilibria for arbitrary initial capital levels, we present the case of homogeneous initial capital separately, where the stochastic representation theorem already used for perfect competition can be applied.

2.3 Symmetric equilibrium

To be sure that any equilibrium exists, we have to specify the strategic properties of capital some more, about which we have said nothing so far. For instance, it is common in the oligopoly literature to distinguish between strategic complements and strategic substitutes. In capital accumulation games with individual capital stocks typically the latter is assumed, i.e. opponent capital has a negative influence on the profitability of own capital. For obtaining a symmetric equilibrium, we do not need to require strategic substitutes, but at least that the capital stocks are not too strong complements.

Assumption 5. $\Pi_{q^iq^i} + (n-1)\Pi_{q^iq^{-i}} < 0$

Since instantaneous revenue is by assumption concave in own capital, Assumption 5 means that marginal revenue must not increase too strongly in opponent capital. In particular, if we increase the capital stocks on the symmetric expansion path, marginal revenue of an individual firm should still decrease. This condition also appears in the literature on Cournot competition [34, Sec. 4.2]. It is among the weakest known requirements to guarantee uniqueness of equilibrium in the static Cournot game with payoff Π. If the capital stocks are indeed strategic substitutes, i.e. $\Pi_{q^i q^{-i}} \leq 0$, this implies Assumption 5 and also existence of the static game's equilibrium.

For a symmetric open loop equilibrium, we now have to determine a capital stock process which is optimal for any firm if the opponents follow the same investment policy. Since the individual control problems are of the monotone follower type, optimal behaviour in equilibrium can be verified by the first order condition which was introduced for the social planner's problem above. Consequently, we look for a process $Q^{*1}(= \frac{1}{n}Q^*)$ such that

$$\nabla J(Q^{*1}|(n-1)Q^{*1}) \tag{2.11}$$

satisfies (2.5). Note that the gradient is still based on the partial derivative Π_{q^i} only, since opponent capital just *happens* to coincide in the symmetric equilibrium. Then, since the required concavity and integrability conditions are satisfied, one can repeat the proof of Proposition 2.2 to infer optimal investment.

Again, we try to determine a capital process satisfying the first order condition with the help of an auxiliary optional process L, which solves the following modified stochastic representation problem.

$$\mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, \sup_{\tau\leq u<t} L_u, (n-1)\cdot \sup_{\tau\leq u<t} L_u)\, dt \,\middle|\, \mathscr{F}_\tau\right] - k_\tau = 0 \quad \text{for all } \tau \in \mathscr{T}. \tag{2.12}$$

Under our monotonicity Assumption 5, we can once more apply the representation theorem of Bank and El Karoui to assure existence of a solution L, provided the range of marginal revenue is sufficiently large:

Proposition 2.5. *Let Assumptions 2, 4 and 5 hold. Suppose further for any* $(\omega, t) \in \Omega \times [0, \infty)$, $\lim_{q\to 0} \Pi_{q^i}(\omega, t, q, q) = \infty$ *and* $\lim_{q\to\infty} \Pi_{q^i}(\omega, t, q, q) = 0$. *Set* $q^i = q_0 \in \mathbb{R}_+$, $i = 1, \dots, n$.

Then, there is a symmetric open loop equilibrium where

$$Q^{*i} \triangleq q_0 \vee \left(\sup_{0\leq u<t} L_u\right)_{t\geq 0} \tag{2.13}$$

for all $i = 1, \dots, n$, *and* L *is the unique optional process solving representation problem* (2.12).

Remark 2.6. We do not prove uniqueness of the symmetric equilibrium here, since it would involve construction of an additional auxiliary optimization problem. This is done in the general case below, which covers the present setting.

Proof. The process Q^{*i} defined as above belongs to $\mathscr{A}(q^i)$. We only need to show that (2.11) satisfies the first order condition (2.5), because we can then repeat the proof of Proposition 2.2.

Indeed, for any stopping time $\tau \in \mathscr{T}$ we have due to the monotonicity Assumption 5 and the definition of Q^{*i}

$$\begin{aligned}\nabla J^i(Q^{*i}|Q^{*-i})_\tau &= \mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, Q_t^{*i}, Q_t^{*-i})\,dt\middle|\mathscr{F}_\tau\right] - k_\tau \\ &\leq \mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, \sup_{\tau\leq u<t} L_u, (n-1)\cdot \sup_{\tau\leq u<t} L_u)\,dt\middle|\mathscr{F}_\tau\right] - k_\tau,\end{aligned}$$

where the last expression is zero exactly by representation (2.12). To verify that the equality in (2.5) holds true $\mathbf{P}$-a.s., consider $dQ_s^{*i} > 0$. Then, L reaches an all-time high at s and $Q_t^{*i} = \sup_{0\leq u<t} L_u = \sup_{s\leq u<t} L_u > 0$ for all $t > s$. ο.ε.δ.

With Proposition 2.5, we obtained a quite short, rigorous proof for an equilibrium as in [22], but for a conceivably general formulation of the game. In particular, we did not use any Markov assumption and there may also be jumps of an exogenous shock. Below, we will explicitly solve Grenadier's model in presence of a Lévy process and conduct some further analysis relating to option premia.

Continuing with the general model, we ask about the influence of heterogeneous initial capital on equilibrium existence and characteristics, since we are particularly interested in the strategic effect of capital stocks. However, while the type of individual optimization problems remains the same, we cannot apply the representation theorem with heterogeneous capital levels anymore. For instance in the basic Grenadier model, the Inada conditions on marginal revenue are violated for any given opponent capital.

In the following, we will still characterize equilibrium by the first order condition prevailing above. It requires nevertheless some work to adapt the available methods for solving monotone follower problems to our needs. Therefore, the next section is entirely devoted to such control problems. While they can be interpreted as determining a best reply on behalf of firm i, we will later formulate further auxiliary instances, so the treatment is kept more general in notation.

2.4 Monotone follower problems

Consider now the problem of a single agent to choose a process $Q \in \mathscr{A}(q)$, where $q \in \mathbb{R}_+$ is given, in order to maximize the objective function

$$J^{\mathrm{M}}(Q) \triangleq \mathbf{E}\left[\int_0^\infty F(t, Q_t)\,dt - \int_0^\infty k_t\,dQ_t\right]. \tag{2.14}$$

The superscript "M" stands for monopolist. Concerning the random field F, we make similar assumptions as on Π above, only neglecting the exogenous process Q^{-i}.

Assumption 1'.

i. For any $(\omega, t) \in \Omega \times [0, \infty)$, the mapping $q \mapsto F(\omega, t, q)$ is continuously differentiable. The partial derivative F_q strictly decreases in q.

ii. For $q \in \mathbb{R}_+$ fixed, $(\omega, t) \mapsto F(\omega, t, q)$ is progressively measurable.

iii. For any $Q \in \mathscr{A}(0)$, $F(\omega, t, Q_t(\omega))$ is $\mathbf{P} \otimes dt$-integrable.

$F_q(t, Q_t)$ is analogously $\mathbf{P} \otimes dt$-integrable for any given $Q \in \mathscr{A}(q)$ with $q > 0$ by concavity of F in q.

Given Assumption 1', we can define the value function of the monopolist

$$V^{\mathrm{M}}(q) \triangleq \sup_{Q \in \mathscr{A}(q)} J^{\mathrm{M}}(Q). \tag{2.15}$$

In the following we try to find conditions under which there exists an optimal strategy $Q^* \in \mathscr{A}(q)$, i.e. which attains $J^{\mathrm{M}}(Q^*) = V^{\mathrm{M}}(q)$, and to determine it.

2.4.1 First order condition

It will turn out very helpful for our purposes to characterize potential solutions to monotone follower problems by a first order condition as above. However, contrary to its use by Bank and Riedel in [10] and in the further applications to singular control [7, 9] as a sufficient condition only, we are interested in *necessity*, too.

For stating the first order condition, we again need to define the gradient $\nabla J^{\mathrm{M}}(Q)$, which is for any $Q \in \mathscr{A}(q)$ the unique optional process satisfying $\nabla J^{\mathrm{M}}(Q)_S = \mathbf{E}[\int_S^\infty F_q(t, Q_t)\,dt \mid \mathscr{F}_S] - k_S$ for all stopping times $S \in \mathscr{T}$.

Proposition 2.7. *If Assumptions 1' and 2 are satisfied, there exists a control policy $Q^* \in \mathscr{A}(q)$ which attains $V^M(q)$ iff*

$$\nabla J^M(Q^*) \leq 0 \quad \textit{and} \quad \int_0^\infty \nabla J^M(Q^*)_s \, dQ^*_s = 0 \quad \mathbf{P}\text{-}a.s. \tag{2.16}$$

Proof. For sufficiency, see e.g. [7].

For necessity, assume there exists an optimal process $Q^* \in \mathscr{A}(q)$ with $J^{\mathrm{M}}(Q^*) = V^{\mathrm{M}}(q)$. Then in particular the value of the problem is finite. To see that there cannot exist a stopping time $S \in \mathscr{T}$ such that $\nabla J^{\mathrm{M}}(Q^*)_S > 0$, note that due to the continuity of F_q and linear investment cost, a sufficiently small extra investment at S would be profitable.

It remains to show $\nabla J^{\mathrm{M}}(Q^*)_S = 0$ for all points of increase S of Q^*, since these carry the measure dQ^*. A point of increase is a stopping time $S \in \mathscr{T}$ such that $Q^*_t > Q^*_S$ for all $t > S$ almost surely. Suppose $S \in \mathscr{T}$ is a point of increase and fix $\epsilon > 0$. Define the stopping time $S^\epsilon \triangleq \inf\{t \geq S | Q^*_t \geq Q^*_S + \epsilon\}$. Furthermore define the control process Q^ϵ by

$$Q^\epsilon_t = \begin{cases} Q^*_{t\wedge S} & \text{if } t \leq S^\epsilon \\ Q^*_t - \epsilon & \text{else.} \end{cases}$$

Then $Q^\epsilon \in \mathscr{A}(q)$ and is a feasible continuation policy from Q^*_S. Let $J^{\mathrm{M}}(Q)_S$ denote the conditional continuation value of control Q from S on. From the definition of Q^ϵ, we obtain

$$\frac{J^{\mathrm{M}}(Q^*)_S - J^{\mathrm{M}}(Q^\epsilon)_S}{\epsilon} =$$
$$\frac{1}{\epsilon}\mathbf{E}\left[\int_S^{S^\epsilon} F(t, Q^*_t) - F(t, Q^*_S)\, dt + \int_{S^\epsilon}^\infty F(t, Q^*_t) - F(t, Q^*_t - \epsilon)\, dt \middle| \mathscr{F}_S\right]$$
$$- \frac{1}{\epsilon}\mathbf{E}\left[\int_{[S,S^\epsilon]} k_t \, d(Q^* - Q^\epsilon)_t \middle| \mathscr{F}_S\right].$$

For any ϵ, this expression is nonnegative due to the hypothesized optimality of Q^*. Since it approaches $\nabla J^{\mathrm{M}}(Q^*)_S$ in the limit, the gradient must be zero at S.

To see that the claimed limit is true, note that S^ϵ tends to S almost surely, because the latter is a point of increase. Moreover$R^\epsilon \triangleq Q^* - Q^\epsilon \in \mathscr{A}(0)$ and satisfies $R^\epsilon_{S^\epsilon} = \epsilon$, almost surely. With the level passage times $\tau^{R^\epsilon}(l) \triangleq \inf\{t \geq 0 | R^\epsilon_t > l\}$ we can make the following change of variable as in [6].

$$\int_{[S,S^\epsilon]} k_t \, dR^\epsilon_t = \int_{[0,\infty)} k_t \mathbf{1}_{\{S,S^\epsilon\}}(t)\, dR^\epsilon_t = \int_0^\infty k_{\tau^{R^\epsilon}(l)} \mathbf{1}_{\{S,S^\epsilon\}}(\tau^{R^\epsilon}(l))\, dl$$

Note that

$$\tau^{R^\epsilon}(l) \geq S \Leftrightarrow l \geq 0 \text{ and } \tau^{R^\epsilon}(l) \leq S^\epsilon \Leftrightarrow l < \epsilon.$$

So, the cost difference equals $\int_{[0,\epsilon)} k_{\tau^{R^\epsilon}(l)}\, dl$. Consequently, the limit above is

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbf{E}\left[\int_{[S,S^\epsilon]} k_t \, d(Q^* - Q^\epsilon)_t \middle| \mathscr{F}_S \right] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{[0,\epsilon)} \mathbf{E}\big[k_{\tau^{R^\epsilon}(l)} \big| \mathscr{F}_S \big] \, dl = k_S,$$

since k is right-continuous. ο.ε.δ.

Note that because of the strict concavity assumption, there is at most one feasible control process that is optimal, resp. satisfies the first order condition. The gradient $\nabla J^{\mathrm{M}}(Q)_S$ can be interpreted as the conditional marginal profit of adding an extra capital unit at the stopping time S, when capital follows the process Q. The intuition of the first order condition is that when the gradient is positive at some stopping time, a small extra investment is profitable. On the other hand, investment should not occur in the region where it is negative, since similarly reducing such an investment would be beneficial.

With Proposition 2.7, we can now focus on identifying feasible control processes that satisfy the first order condition. In the given references [10, 9, 7], the existence of such a policy is achieved via a representation theorem, which confirms that there exists a solution of a stochastic representation problem (see [8]) inspired by the first order conditions. The running supremum of the solution to the representation problem is the sought optimal control process.

However, in order to apply the representation theorem, an Inada condition on F_q is required, which poses a problem in our oligopoly application. In typical instances of the revenue stream, installed capital by the opponents of a firm prevents the marginal revenue of this firm from approaching infinity for the first capital unit.

2.4.2 Base capacity

We adapt the general approach to solving monotone follower problems like (2.14) via their connection to optimal stopping (cf. [19]), which has been briefly illustrated in terms of the myopic investors under perfect competition. The approach consists of solving a family of auxiliary stopping problems (entry of myopic investors) and deriving the optimal control process as the inverse of the family of optimized stopping times.

By means of the social planner's monotone follower problem, we showed that if one can solve the proposed stochastic representation problem, the passage times of the solution provide the family of stopping times needed to construct the optimal control process. However, this connection was not even needed because of resorting to the first order condition.

In case the representation problem is not applicable due to the mentioned Inada condition, the principle is nevertheless valid and we will construct the optional signal process by adapting the existence proof of Bank and El Karoui [8]. Moreover, we care again for *necessity* of existence of such a signal process, too. Its passage times will be optimal for the auxiliary myopic stopping problems and tracking its running supremum will be an optimal policy for our monotone follower problem as well. We also attempt an economic interpretation of the principle.

Let us first state the result and then go through the proof and involved concepts step by step. To begin, define for any stopping time $S \in \mathscr{T}$ also the set of stopping times

$$\mathscr{T}(S) \triangleq \{T \in \mathscr{T} | T \geq S,\ \mathbf{P}\text{-a.s.}\}.$$

The signal process to be constructed now is a *base capacity* L satisfying

$$L_s = \sup\left\{ l \in \mathbb{R}_+ \middle| \operatorname*{ess\,inf}_{T \in \mathscr{T}(s)} \mathbf{E}\left[\int_s^T F_q(t,l)\,dt + k_T \middle| \mathscr{F}_s \right] = k_s \right\} \vee 0 \tag{2.17}$$

for all $s \in [0,\infty)$. Formally, it yields the maximal (capital level) l for which stopping immediately is optimal in the auxiliary stopping problem parametrized with l.

Concerning its economic interpretation, we adapt the term "base capacity" from Riedel and Su [31], since it turns out that a firm will never want to operate with less capital, even if it were given the one-time chance to sell some capital and continue expanding. In terms of optimal stopping, it is for any capital level $l < L_s$ not profitable to *delay* marginal investment to any future stopping time. Contrarily, for capital $l > L_s$, there exists a stopping time $T \geq s$ such that the *opportunity cost* of delaying marginal investment until T is negative.

Existence of this base capacity is connected to our optimization problem as follows.

Proposition 2.8. *If Assumptions 1' and 2 are satisfied, there exists a control policy* $Q^* \in \mathscr{A}(q)$ *which attains* $V^M(q)$ *iff there exists an optional process* L*, taking values in* $[0,\infty)$ *and satisfying* (2.17) *for all* $s \in [0,\infty)$ *almost surely.* Q^* *and* L *are then related by* $Q^*_t = q \vee \sup_{0 \leq u < t} L_u$ *for all* $t \in [0,\infty)$*.*

L *takes values in* $[0,+\infty)$ *almost surely if for all* $(\omega,t) \in \Omega \times [0,\infty)$*,* $\lim_{q\to\infty} F_q(\omega,t,q) \leq 0$.

We prove the proposition in a series of lemmata, collecting also the relevant results from the literature.

As a first step, we recall the well-known relation between this control problem and optimal stopping problems. One can write the payoff from any $Q \in \mathscr{A}(q)$ in terms of its inverse, i.e. the level passage times

$$\tau^Q(l) \triangleq \inf\{t \geq 0 | Q_t > l\} \in \mathscr{T} \text{ for any } l \in [q, +\infty) \tag{2.18}$$

by a change-of-variable formula.

Lemma 2.9. *Under Assumptions 1' and 2, the following holds for any $Q \in \mathscr{A}(q)$:*

$$\begin{aligned} J^M(Q) - J^M(q) &= \int_q^\infty \mathbf{E}\left[\int_{\tau^Q(l)}^\infty F_q(t,l)\,dt - k_{\tau^Q(l)}\right] dl \\ &\leq \int_q^\infty \operatorname*{ess\,sup}_{\tau \in \mathscr{T}} \mathbf{E}\left[\int_\tau^\infty F_q(t,l)\,dt - k_\tau\right] dl \\ &= \int_q^\infty \mathbf{E}\left[\int_0^\infty F_q(t,l)\,dt\right] dl \\ &\quad - \int_q^\infty \operatorname*{ess\,inf}_{\tau \in \mathscr{T}} \mathbf{E}\left[\int_0^\tau F_q(t,l)\,dt + k_\tau\right] dl \end{aligned}$$

Proof. See e.g. [6], Lemma 2. ο.ε.δ.

The proof is based on Fubini's theorem and relies on the monotonicity of Q. Once the l^{th} capital unit is installed at cost $k_{\tau^Q(l)}$, the marginal revenue $F_q(t,l)$ accrues at all future times t, independent of subsequent investments. Thus, one can decide at which (random) time to install the l^{th} capital unit optimally. On the other hand, if one can find a process $Q^* \in \mathscr{A}(q)$ whose passage times $\tau^{Q^*}(l)$ actually attain the values of the optimal stopping problems, this control must be optimal.

The last equality in the lemma states that instead of maximizing the profit from installing the l^{th} capital unit, one can equivalently minimize the associated *opportunity cost.*

The stopping problems in Lemma 2.9 with fixed $l \in \mathbb{R}_+$ are classically solved with the help of the Snell envelopes

$$Z^l(s) = \operatorname*{ess\,inf}_{T \in \mathscr{T}(s)} \mathbf{E}\left[\int_0^T F_q(t,l)\,dt + k_T \middle| \mathscr{F}_s\right], \quad s \in [0,\infty). \tag{2.19}$$

Z^l coincides almost surely with the largest submartingale dominated by the reward-upon-stopping $\int_0^s F_q(t,l)\,dt + k_s$, and it is optimal to stop as soon as Z^l touches this bound, cf. [18].

Of course, only future payments influence the decision whether to stop or to keep waiting, so we look at the following related family of stopping problems.

$$Y^l(S) = \operatorname*{ess\,inf}_{T \in \mathscr{T}(S)} \mathbf{E}\left[\int_S^T F_q(t,l)\,dt + k_T \middle| \mathscr{F}_S \right], \quad S \in \mathscr{T},\ l \in \mathbb{R}_+. \tag{2.20}$$

Note that $Y^l(S) \leq k_S$.

In the present setting, one can treat this family of stopping problems in a unified way because of the continuity and monotonicity in l. We cite the following results from [8], which are only slightly modified as we restrict ourselves to nonnegative values of l and a positive cost process k.

Lemma 2.10. *Under Assumptions 1' and 2, there is a jointly measurable mapping* $Y : \Omega \times [0,\infty] \times \mathbb{R}_+ \to \mathbb{R}$, $(\omega, t, l) \mapsto Y^l(\omega, t)$ *with the following properties:*

i. For $l \in \mathbb{R}_+$ *fixed,* $Y^l : \Omega \times [0,\infty] \to \mathbb{R}$ *is an optional process such that*

$$Y^l(S) = \operatorname*{ess\,inf}_{T \in \mathscr{T}(S)} \mathbf{E}\left[\int_S^T F_q(t,l)\,dt + k_T \middle| \mathscr{F}_S \right], \quad \mathbf{P}\text{-}a.s. \tag{2.21}$$

for every stopping time $S \in \mathscr{T}$.

ii. For any $l \in \mathbb{R}_+$, $S \in \mathscr{T}$, *the stopping time*

$$T_S^l \triangleq \inf\{t \geq S | Y^l(t) = k_t\}$$

is optimal in (2.20), *that is,*

$$Y^l(S) = \mathbf{E}\left[\int_S^{T_S^l} F_q(t,l)\,dt + k_{T_S^l} \middle| \mathscr{F}_S \right].$$

iii. For fixed $(\omega, s) \in \Omega \times [0,\infty]$, *the mapping* $l \mapsto Y^l(\omega, s)$ *is continuously decreasing.*

Proof. See [8], Lemma 4.12. o.ε.δ.

Consequently, there is a well-behaved version of the collection of value functions (2.20), which induces that our base capacity is well-defined. For Y as in Lemma 2.10, define the process L by

$$L(\omega, s) = \sup\{l \in \mathbb{R}_+ | Y^l(\omega, s) = k(\omega, s)\} \vee 0, \tag{2.22}$$

for $(\omega, s) \in \Omega \times [0, \infty)$.

Bank and El Karoui compare L to a "Gittins index", since it determines the maximal value of l, for which stopping immediately is optimal. Due to the derived monotonicity, such a supremum is well defined. A Gittins index represents the current profitability of a payoff stream by offering a fixed reward-upon-stopping, such that it is better to cash in on the reward immediately than to postpone the decision.

This is in fact an opportunity cost decision, which we propose as interpretation, since here l is neither a reward nor needed as a measure for profitability.

Suppose $l > L_s$. Then, by definition, there exists a stopping time $T \in \mathscr{T}(s)$, such that

$$\begin{aligned} & \mathbf{E}\left[\int_s^T F_q(t,l)\,dt + k_T \middle| \mathscr{F}_s\right] - k_s < 0 \\ \Leftrightarrow \qquad & \mathbf{E}\left[\int_s^T F_q(t,l)\,dt \middle| \mathscr{F}_s\right] < k_s - \mathbf{E}\left[k_T | \mathscr{F}_s\right]. \end{aligned} \tag{2.23}$$

The expected additional revenue from installing the l^{th} capital unit immediately instead of at the future (random) time T on the left hand side is less than the additional cost of investing immediately on the right hand side. Considering only these two dates, the situation is identical[4] afterwards, whether investment has happened at s or at T.

Put differently, the first line is the opportunity cost of *delaying* investment until T, which is a feasible plan since the latter is a stopping time. Whenever the opportunity cost is negative, delaying investment is the better choice.

When contrarily $l < L_s$, the value of the option to delay investment is zero, because then there is no feasible plan to delay which yields a positive net reward (resp. negative opportunity cost). Due to the monotonicity, there is at any state a maximal capital level such that immediate investment is optimal, which is exactly the base capacity.

L is indeed a feasible signal process for investment, i.e. its running supremum is a feasible control.

Lemma 2.11. *The process L defined by* (2.22) *is optional. It takes values in* $[0, +\infty)$ *almost surely if for all* $(\omega, t) \in \Omega \times [0, \infty)$, $\lim_{q\to\infty} F_q(\omega, t, q) \leq 0$.

Proof. See [8], Lemma 4.13. ο.ε.δ.

Suppose now that L takes values in $[0, +\infty)$ almost surely, for instance because the sufficient condition in Lemma 2.11 (resp. Proposition 2.8) is

[4]This fact is crucial for the current principle and we will refer to it in Chapter 3.

satisfied. Then the process Q^* given by $Q_t^* = q \vee \sup_{0 \le u < t} L_u$ for all $t \in [0, \infty)$ is a feasible control from $\mathscr{A}(q)$. In order to prove optimality by Lemma 2.9, it remains to show that the level passage times $\tau^{Q^*}(l)$ as defined above are optimal for all $l \in [q, \infty)$.

However, the optimal stopping times T_0^l from Lemma 2.10 are equivalent to $\tau_{Q^*}(l) \triangleq \inf\{t \ge 0 | Q_t^* \ge l\}$. These are the smallest optimal stopping times. One could now switch to the right-continuous modification of L like in [19] and use a slightly different change-of-variable formula, but we show that this is not necessary. In fact, the required stopping times $\tau^{Q^*}(l)$ are the largest optimal ones.

Lemma 2.12. *For any $l \in \mathbb{R}_+$, the stopping time $\tau^{Q^*}(l)$ is also optimal in* (2.20) *for $S = 0$, that is,*

$$Y^l(0) = \mathbf{E}\left[\int_0^{\tau^{Q^*}(l)} F_q(t, l)\, dt + k_{\tau^{Q^*}(l)}\right].$$

Proof. First we prove the identity $\tau_{Q^*}(l) = T_0^l$ for any $l \in \mathbb{R}_+$. Let $t \in [0, \infty)$, then

$$\begin{aligned} T_0^l = \inf\{t \ge 0 | Y^l(t) = k_t\} \le t &\Leftrightarrow \forall t' > t, \exists s \le t' : Y^l(s) = k_s \\ &\Leftrightarrow \forall t' > t : \sup_{0 \le u < t'} L_u \ge l \Leftrightarrow \tau_{Q^*}(l) \le t. \end{aligned}$$

Similarly,

$$\begin{aligned} T_0^l = \inf\{t \ge 0 | Y^l(t) = k_t\} \ge t &\Leftrightarrow \forall s < t : Y^l(s) < k_s \\ &\Leftrightarrow \sup_{0 \le u < t} L_u < l \Leftrightarrow \tau_{Q^*}(l) \ge t. \end{aligned}$$

Next, for any $l \in \mathbb{R}_+$ and $l' \in (l, \infty)$, $\tau_{Q^*}(l') \ge \tau^{Q^*}(l)$, since

$$\begin{aligned} &\tau^{Q^*}(l) \ge t \Rightarrow \forall u < t : L_u \le l \Rightarrow \forall u < t : Y^{l'}(u) < k_u \text{ and} \\ &\tau_{Q^*}(l') < t \Rightarrow \exists u < t : Y^{l'}(u) = k_u, \end{aligned}$$

a contradiction. Consequently, we also have for all $l \in \mathbb{R}_+$ and $n \in \mathbb{N}$, $\tau_{Q^*}(l + \frac{1}{n}) \ge \tau^{Q^*}(l)$, where the latter is the path-wise limit if we let $n \to \infty$. Thus, we can take the limit

$$\begin{aligned} &\lim_{n \to \infty} \mathbf{E}\left[\int_0^{\tau_{Q^*}(l + \frac{1}{n})} F_q(t, l + \frac{1}{n})\, dt + k_{\tau_{Q^*}(l + \frac{1}{n})}\right] \\ &= \mathbf{E}\left[\int_0^{\tau^{Q^*}(l)} F_q(t, l)\, dt + k_{\tau^{Q^*}(l)}\right] = Y^l(0). \end{aligned}$$

The last equality holds true because of optimality of $\tau_{Q^*}(l+\frac{1}{n})$ and continuity of Y^l. Thus, $\tau^{Q^*}(l)$ is optimal. ο.ε.δ.

Lemma 2.12 completes the proof of sufficiency in Proposition 2.8.

Now suppose to the contrary that there exists an optimal $Q^* \in \mathscr{A}(q)$, such that $\infty > J^{\mathrm{M}}(Q^*) = V^{\mathrm{M}}(q)$.

Lemma 2.13. *If there exists $Q^* \in \mathscr{A}(q)$ attaining $J^M(Q^*) = V^M(q)$, then its level passage times $\tau^{Q^*}(l)$ for any $l \in \mathbb{R}_+$ solve the optimal stopping problems appearing in Lemma 2.9.*

Proof. Q^* satisfies the first order conditions in Proposition 2.7. Since the $\tau^{Q^*}(l)$ are points of increase, $\nabla J^{\mathrm{M}}(Q^*)_{\tau^{Q^*}(l)} = 0$ for all $l \in \mathbb{R}_+$. Thus, for any $\tau \in \mathscr{T}$,

$$\mathbf{E}\left[\int_{\tau^{Q^*}(l)}^{\infty} F_q(t, l \vee Q^*_t)\, dt - k_{\tau^{Q^*}(l)}\right] = \mathbf{E}\left[\int_{\tau^{Q^*}(l)}^{\infty} F_q(t, Q^*_t)\, dt - k_{\tau^{Q^*}(l)}\right] \geq$$
$$\geq \mathbf{E}\left[\int_{\tau}^{\infty} F_q(t, Q^*_t)\, dt - k_{\tau}\right] \geq \mathbf{E}\left[\int_{\tau}^{\infty} F_q(t, l \vee Q^*_t)\, dt - k_{\tau}\right].$$

This implies

$$\mathbf{E}\left[\int_{0}^{\tau^{Q^*}(l)} F_q(t, l)\, dt + k_{\tau^{Q^*}(l)}\right] = \mathbf{E}\left[\int_{0}^{\tau^{Q^*}(l)} F_q(t, l \vee Q^*_t)\, dt + k_{\tau^{Q^*}(l)}\right] \leq$$
$$\leq \mathbf{E}\left[\int_{0}^{\tau} F_q(t, l \vee Q^*_t)\, dt + k_{\tau}\right] \leq \mathbf{E}\left[\int_{0}^{\tau} F_q(t, l)\, dt + k_{\tau}\right].$$

ο.ε.δ.

Consequently, the existence of an optimal stopping time for each $l \in \mathbb{R}_+$ is a necessary condition for the existence of an optimal control process. If we determine the family of optimal stopping times alternatively by Lemma 2.10 and consider $(\sup_{0 \leq u < t} L_u)_{t \in [0,\infty)}$ of the resulting base capacity (2.22), this must coincide with the left-continuous, optional process Q^* because of uniqueness and thus $L < \infty$, almost surely. ο.ε.δ.

Proposition 2.8 implies that any optimal control for a monotone follower problem of this section and thus any best reply of some player $i \in \{1, \ldots, n\}$ to given strategies $(Q^j)_{j \in \{1,\ldots,n\}, j \neq i}$ — whenever it exists — is of the form $(q^i \vee \sup_{0 \leq u < t} L^i_u)_{t \in [0,\infty)}$ for some optional signal process L^i.

With these results fixed, we can now turn to the determination of equilibria of our game.

2.5 Asymmetric equilibria

Without loss of generality, we order the firms in the following such that $q^1 \leq \cdots \leq q^n$. For identifying an equilibrium, we now have to find a consistent set of mutual best replies by each firm. In order to succeed with completely arbitrary initial capital levels, we need to make a further assumption on the strategic effect of capital, which is of course highly important for the existence and characteristics of any equilibria. Like the maintained Assumption 5, the following one is related to the literature on oligopoly theory and will be shown to have very clear implications for equilibrium determination.

Assumption 6. $\Pi_{q^i q^i} - \Pi_{q^i q^{-i}} < 0$

This assumption is for instance automatically satisfied by Cournot-type spot competition, because then it follows from inverse demand (resp. price) decreasing in aggregate supply, see the explicit example below. If the capital stocks are strategic substitutes, we assume that the influence of increasing own capital on marginal revenue is stronger than increasing opponent capital by the same amount. Assumption 6 actually implies that for a fixed level of *aggregate* capital, instantaneous marginal revenue decreases in own installed capital.

With this additional assumption, we can exploit the results of Section 2.4 to characterize existence conditions and actually provide a construction for all equilibria of the following type.

Definition 2.14. An open loop equilibrium $(Q^{*_1}, \ldots, Q^{*_n})$ is an *equalizing equilibrium* if $Q^{*_i} = q^i \vee Q^{*_1}$ for all $i \in \{1, \ldots, n\}$.

In these equilibria, there is only investment by the currently smallest firms. As Theorem 2.18 will show, the class is not restrictive under Assumption 6, e.g. whenever we consider Cournot-type spot competition, independent of the specific shock process and initial capital dispersion.

Our first result is that any equalizing equilibrium is exactly determined by the solution of a single monotone follower problem.

Theorem 2.15. *Under Assumptions 2, 4, 5, and 6 there exists for any $(q^1, \ldots, q^n) \in \mathbb{R}^n_+$ with $0 < q^1 \leq \cdots \leq q^n$ an equalizing equilibrium of the game iff there exists a control $\hat{Q} \in \mathscr{A}(q^1)$ attaining $\hat{J}(\hat{Q}) = \hat{V}(q^1) \triangleq \sup_{Q \in \mathscr{A}(q^1)} \hat{J}(Q)$, where*

$$\hat{J}(Q) = \mathbf{E}\left[\int_0^\infty \hat{\Pi}(t, Q_t)\, dt - \int_0^\infty k_t\, dQ_t\right] \tag{2.24}$$

and

$$\hat{\Pi}(\omega, t, q) \triangleq \mathbf{1}_{\{q \geq q^1\}} \int_{q^1}^{q} \pi(\omega, t, l, \sum_{j>1} q^j \vee l)\, dl \tag{2.25}$$

for all $(\omega, t, q) \in \Omega \times [0, \infty) \times [0, \infty)$*; so maximization of* $\hat{J}(Q)$ *over* $\mathscr{A}(q^1)$ *is a monotone follower problem as in Section 2.4. Then,* $Q^{*_1} = \hat{Q}$.

A sufficient condition for existence of an optimal control process is

$$\lim_{l \to \infty} \pi(\omega, t, l, \sum_{j>1} q^j \vee l) \leq 0 \text{ for all } (\omega, t) \in \Omega \times [0, \infty).$$

We suppose strictly positive initial capital only for saving a further integrability condition on the positive part of $\hat{\Pi}$. To prove the theorem, we need the following lemma.

Lemma 2.16. *Suppose Assumptions 2, 4, and 6 hold. Then, for any* $Q \in \mathscr{A}(0)$ *and* $i \in \{1, \ldots, n\}$, $\nabla J(q^i \vee Q | \sum_{j \neq i} q^j \vee Q)$ *satisfies the first order condition* (2.16) *if* $\nabla J(q^1 \vee Q | \sum_{j>1} q^j \vee Q)$ *does.*

Proof. Suppose, $\nabla J(q^1 \vee Q | \sum_{j>1} q^j \vee Q)$ satisfies (2.16). By the monotonicity Assumption 6 and $q^i \geq q^1$,

$$\nabla J(q^i \vee Q | \sum_{j \neq i} q^j \vee Q) \leq \nabla J(q^1 \vee Q | \sum_{j>1} q^j \vee Q),$$

which proves the first part. Further, if $S \in \mathscr{T}$ is a point of increase for $q^i \vee Q$, then it is a point of increase for $q^{i'} \vee Q$, $i' = 1, \ldots, i$. Thus, for any of these i' and $t > S$ we have $Q_t^{i'} = Q_t^1 > q^i$ and $\sum_{j \neq i'} (q^j \vee Q_t) = \sum_{j>1} (q^j \vee Q_t)$. This implies $\nabla J(q^{i'} \vee Q | \sum_{j \neq i'} (q^j \vee Q))_S = \nabla J(q^1 \vee Q | \sum_{j>1} (q^j \vee Q))_S = 0$ and proves the second part. ο.ε.δ.

Proof of Theorem 2.15. We show in the appendix that under Assumption 5, $\pi(\omega, t, q, \sum_{j>1} q^j \vee q)$ is monotonically decreasing in $q \in \mathbb{R}_+$. Thus, $\hat{\Pi}$ is strictly concave on $\{q \geq q^1\}$ and integrable due to integrability of $\pi(\cdot)$ for $q > 0$ following Assumption 4.

Note that for any $Q \in \mathscr{A}(q^1)$ and $(q^2, \ldots, q^n) \in \mathbb{R}_+^{n-1}$, $\nabla J(Q | \sum_{j>1} q^j \vee Q) = \nabla \hat{J}(Q)$ by definition. This, Proposition 2.7, and Lemma 2.16 imply:

$$\begin{aligned}
&(q^1 \vee Q^{*_1}, \ldots, q^n \vee Q^{*_1}) \text{ is an equalizing equilibrium} \\
\Leftrightarrow &\nabla J(q^i \vee Q^{*_1} | \sum_{j \neq i} q^j \vee Q^{*_1}) \text{ satisfies (2.16) for all } i \in \{1, \ldots, n\} \\
\Leftrightarrow &\nabla J(q^1 \vee Q^{*_1} | \sum_{j>1} q^j \vee Q^{*_1}) \text{ satisfies (2.16)} \\
\Leftrightarrow &\nabla \hat{J}(Q^{*_1}) \text{ satisfies (2.16)} \\
\Leftrightarrow &\exists Q^{*_1} \in \mathscr{A}(q^1) : \quad \hat{J}(Q^{*_1}) = \hat{V}(q^1)
\end{aligned}$$

With our assumptions on k and the stated sufficient condition we can adapt the proof of Lemma 4.13 in [8] to infer that $L < \infty$, almost surely. o.ε.δ.

Remark 2.17. If $q^1 = \cdots = q^n$, Assumption 6 is not needed in the proof and Assumption 5 needs only to hold on the symmetric path $q^{-i} = (n-1)q^i$.

If there exists an equalizing equilibrium process, for instance when the easy-to-check sufficient condition in Theorem 2.15 is satisfied, it is *unique* and can be determined by solving the auxiliary monotone follower problem as in Section 2.4.

Consider the associated signal process $\hat{L} = L^{*1}$.

$$\hat{L}_s = \sup\left\{ l \geq q^1 \,\middle|\, \operatorname*{ess\,inf}_{T \in \mathscr{T}(s)} \mathbf{E}\left[\int_s^T \pi(\omega, t, l, \sum_{j>1} q^j \vee l)\, dt + k_T \,\middle|\, \mathscr{F}_s \right] = k_s \right\} \vee q^1 \tag{2.26}$$

Here, we restrict ourselves again to the relevant capital levels (not less than $q^1 > 0$) because then integrability of π is ensured, cf. Assumption 4.

We observe that if firm i invests at time s, then in accordance with the opportunity-cost-of-delaying principle derived in Section 2.4, since $\pi = \Pi_{q^i}$. Investment occurs only if $\hat{L}$ sets a new record, and then the investment decision would be completely identical, were the capital levels of the opponents fixed forever at $q^j \vee \hat{L}_s = Q_s^{*j}$, $j \neq i$.

Similarly where $Q_s^i > \hat{L}_s$, firm i would also have excess capital if the opponents' capital stocks always remained at the current levels. Thus, the investment behaviour in equilibrium is equivalent to *myopic* investment.

The firms invest up to the best reply to the current capital stocks of the opponents, conditional on simultaneous investment by equally-sized firms. Subsequent investment occurs only if the exogenous conditions improve. For larger firms, investment is ceteris paribus less profitable due to decreasing marginal revenue.

Note that the base capacity $\hat{L}$ will depend on $(q^2, \ldots, q^n)$. Thus, the equilibrium is in general not subgame perfect.

Using a similar proof, we further obtain an even stronger *uniqueness* result: In common settings, the class of equalizing equilibria is not restrictive, so there are no other open loop equilibria. The argument relies on the concept of *cumulative best replies* following Selten, which can be helpful in equilibrium determination when any player's payoff depends only on the sum of the opponent actions. Cumulative best replies are strategies that are optimal and consistent with a given aggregate capital level. Taking this perspective, we can show that under Assumption 6 there is for *any* hypothetical equilibrium aggregate capital process a *unique* consistent signal process. All firms want to track this process once it exceeds their respective initial capital.

Theorem 2.18. *Under Assumptions 2,4, and 6, any open loop equilibrium is an equalizing equilibrium.*

Proof. There is an open loop equilibrium with aggregate capital Q^* iff each $Q^{*_i} \leq Q^*$, $i = 1, \ldots, n$, is a cumulative best reply to Q^*, i.e. iff each $\nabla J(Q^{*_i}|Q^* - Q^{*_i})$ satisfies the first order conditions (2.16). Then, the individual equilibrium capital processes also solve the following auxiliary monotone follower problems.

Define $\tilde{\Pi}(\omega, t, q) = \mathbf{1}_{\{q \geq q^1\}} \int_{q^1}^{q \wedge Q^*} \pi(\omega, t, l, Q^*(\omega, t) - l)\, dl$. Assumption 6 implies that $\tilde{\Pi}_q(\omega, t, q) = \pi(\omega, t, q, Q^*(\omega, t) - q)$ is monotonically decreasing in q on $(q^1, Q^*(\omega, t))$.

Let now $\tilde{J}(Q) \triangleq \mathbf{E}\left[\int_0^\infty \tilde{\Pi}(t, Q_t)\, dt - \int_0^\infty k_t\, dQ_t\right]$. Then, the gradient $\nabla \tilde{J}(Q^{*_1}) = \nabla J(Q^{*_1}|Q^* - Q^{*_1})$ satisfies the first order conditions (2.16), and Q^{*_1} is the unique maximizer of $\tilde{J}$ over all $Q \in \mathscr{A}(q^1)$ with $Q \leq Q^*$. This in turn implies that Q^{*_1} is the unique process from that subset such that $\nabla J(Q^{*_1}|Q^* - Q^{*_1})$ satisfies the first order conditions.

It follows easily that for each q^j, $j = 2, \ldots, n$, $\nabla \tilde{J}(q^j \vee Q^{*_1})$ satisfies the first order conditions and we can conclude similarly that $q^j \vee Q^{*_1}$ is the respectively unique process from $\mathscr{A}(q^j)$ not exceeding Q^* such that $\nabla J(Q|Q^* - Q)$ satisfies the first order conditions, yielding $Q^{*_j} = q^j \vee Q^{*_1}$ as claimed. o.ε.δ.

With Theorems 2.15 and 2.18, we have completely characterized any open loop equilibrium for a possibly general formulation of the game, only through the strategic properties of capital stocks concerning instantaneous revenue. In particular the nature of exogenous uncertainty is quite irrelevant for the strategic value of capital and instantaneous Cournot mechanics extend perfectly to the "dynamic" game. Investment follows the opportunity-cost-of-delay principle, which coincides with myopic investment also in our general formulation.

We conclude that although a high capital stock deters opponents from investing, it is not sufficiently profitable to build up capital by early, preemptive investment. The implicit reduction of opponent investment is too low to compensate for *premature* option exercise.

In order to enable a stronger, explicit influence on opponent investment, we need to allow for feedback strategies.

2.6 Explicit solutions

We now formulate an instance of the game to derive explicit solutions with the help of the stochastic representation theorem as in [7]. Based on the solution we discuss the effect of option value dispersion and show that perfect competition as presented above is the limiting case. The specification of the revenue stream and cost functions is that of Grenadier [22], except that we allow for Lévy processes with jumps for the exogenous shock process.

Suppose the firms obtain revenue from Cournot type spot competition, where inverse demand is influenced by exogenous shocks. Given aggregate capital Q, the revenue flow is of the form

$$e^{X_t}P(Q_t)Q^i_t.$$

Typically, inverse demand is decreasing in capital, which already induces Assumption 4 to hold. For the shock process X, we will allow any Lévy process without negative jumps. The firms discount revenue at a fixed rate $r > 0$, also applying to the cost of adding capital, which is normalized to one. Assume further inverse demand with constant elasticity $\alpha > 0$, i.e.

$$P(q) = q^{-\frac{1}{\alpha}}.$$

Then, in the notation of our game,

$$\Pi(\omega, t, q^i, q^{-i}) = e^{-rt}e^{X(\omega,t)}(q^i + q^{-i})^{-\frac{1}{\alpha}}q^i.$$

Suppose the integrability condition of Assumption 1 is satisfied. The concavity requirement is now equivalent to $\alpha > 1$ and Assumption 3 to $\alpha > n$.

We fix homogeneous initial capital levels $q^1 = \cdots = q^n = q$. Then, the relaxation allowed by Remark 2.17 translates into $\alpha > \frac{1}{n}$, which is weaker than the concavity condition.

To identify the unique open loop equilibrium, we now have to find a process L such that we can set $Q^{*i}_t = q \vee \sup_{0\leq u<t} L_u$ for every $i \in \{1, \ldots, n\}$ and $\nabla J(Q^{*i}|(n-1)Q^{*i})$ satisfies the first order conditions (2.16). For the current specification, it is easy to check that if $\alpha > \frac{1}{n}$, $\pi(\omega, t, q, (n-1)q)$ monotonically decreases in q and has the Inada properties $\lim_{q\to 0} \pi(\omega, t, q, (n-1)q) = \infty$ and $\lim_{q\to\infty} \pi(\omega, t, q, (n-1)q) = 0$. This allows to apply the method of [7]. We try to find the unique optional process L which solves the stochastic representation problem

$$\mathbf{E}[\int_S^\infty \pi(t, \sup_{S\leq u<t} L_u, (n-1) \sup_{S\leq u<t} L_u)\, dt|\mathscr{F}_S] - k_S = 0 \text{ for any } S \in \mathscr{T}. \quad (2.27)$$

Given the Inada properties, existence of a solution is guaranteed by the representation theorem in [8], which can be easily adapted. Starting from any q and tracking the running supremum of L will then immediately satisfy the first order conditions for equilibrium.

For the current specification, we can derive an explicit solution to (2.27). Begin by guessing

$$L_t = n^{-1}\kappa^\alpha e^{\alpha X_t} \text{ for } t \in [0,\infty), \tag{2.28}$$

with some constant parameter κ (for fixed n). Consequently, investment in equilibrium will occur whenever the factor X sets a new record, as one expects for Markovian processes positively influencing revenue.

Using the Markov property of L given by (2.28), we can eventually eliminate S in (2.27) and by some manipulations like in [31] get to

$$\kappa\Big(\frac{\alpha n}{\alpha n - 1}\Big) = \mathbf{E}\left[\int_0^\infty e^{-rt} e^{-\sup_{0\le u<t} -X_u}\, dt\right] = \frac{1}{r}\mathbf{E}\left[e^{-\sup_{0\le u<\tau(r)} -X_u}\right],$$

where $\tau(r)$ is an independent exponentially distributed time with rate r.

The distribution of the running supremum of a Lévy process $-X$ stopped at an independent exponential time is for instance known in the following case. If X has no negative jumps, Bertoin shows in [11, ch. VII] that the distribution is again exponential with rate $\Phi^{-X}(r)$, the Laplace exponent of $-X$ at r. Then, we obtain

$$\kappa\Big(\frac{\alpha n}{\alpha n - 1}\Big) = \frac{\Phi^{-X}(r)}{r\big(1+\Phi^{-X}(r)\big)} \triangleq \kappa_\infty. \tag{2.29}$$

Here we can reproduce the results of Grenadier [22] and Back and Paulsen [4], since if $X_t = X_0 + \mu t + \sigma B_t$ for standard Brownian motion B and constants μ and σ, the Laplace exponent is given by

$$\Phi^{-X}(r) = \frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}.$$

Regarding jumps in both directions, one could exploit the results of Kuo and Wang [26] for diffusions with jumps having a double exponential distribution to obtain a similar formula.

The right hand side of (2.29) is constant, so κ is increasing in n, and so is aggregate capital $Q^*_t = \sup_{0\le u<t} n \cdot L_u$. Thus increasing competition speeds up investment in equilibrium and reduces the option value of waiting. In the limit, κ converges to κ_∞ as the number of firms becomes infinite.

This allows us to pass to the case of perfect competition. In fact, Q^∞ given by $Q^\infty_t = \sup_{0\le u<t} \kappa^\alpha_\infty e^{\alpha X_t}$ solves the first order condition for a single agent problem with spot revenue Π^∞, where marginal revenue is given by

$$\Pi_q^\infty(\omega, t, q) = e^{-rt}e^{X(\omega,t)}P(q) = \lim_{n\to\infty} \pi\big(\omega, t, \tfrac{1}{n}q, \tfrac{n-1}{n}q\big). \tag{2.30}$$

The last equality is easily checked in the current specification. This is the problem solved by the social planner in presence of marginal investors, as discussed in Section 2.1.

Thus, Q^∞ is the aggregate capital process in a perfectly competitive equilibrium, with zero option value for the infinitesimal firms waiting to enter the market.

The driving force for the option value dispersion is the decreasing importance of the investment externality $q^iP'(q)$ (where $q = \sum_{j=1}^n q^j$), which disappears in the limit (2.30). This is the classical Cournot observation, but not a preemption effect like in the following chapter.

Chapter 3

Closed loop strategies

We saw in the previous chapter that the implicit strategic effects of investment with open loop strategies are quite limited, independent of the particular formulation of the game. Since investment does not depend on the actual capital stocks, each firm makes full use of the option value of waiting. The latter is only reduced by increasing competition because the externality of investment on own installed capital looses importance.

However, if we want to allow for explicit dynamic strategic interaction, by means of feedback strategies, there are some severe technical difficulties to be solved. These are explained by Back and Paulsen [4], who clarify that the equilibrium of Grenadier, which is formulated in terms of thresholds for the state to trigger investment, has to be understood in open loop strategies.

Optimal investment is typically singular, i.e. where it occurs, its rate is undefined. Consequently, one cannot formulate the game with feedback strategies relating to investment directly, if singular controls are not to be excluded a priori. Second, when the previous issue is circumvented by choosing capital *levels*, for instance by sufficiently well behaved threshold functions, a general problem with continuous-time games persists. The combination of "natural" strategies might not uniquely define the evolution of the (state of the) game. For these reasons, Back and Paulsen admit not to know how to formulate the game with feedback strategies.

Nevertheless, the authors *argue* that there exists an equilibrium with investment thresholds, where firms always invest at the Bertrand price. In this case, comparable to perfect competition, the option value of waiting is zero and investment occurs at the zero net present value threshold. It is a frequent conjecture that preemption incentives completely eliminate the option values.

Our aim in this chapter is to provide a mathematically rigorous framework for a capital accumulation game between two firms with closed loop strategies that resolves the fundamental conceptual problems and allows to derive subgame perfect equilibria formally.

In particular, we consider strategies of the Markovian type, which map the state of the game to desirable capital levels. The state contains all payoff relevant information of past play and is composed of an exogenous Markovian shock and the current capital stocks. Consequently, we also define *Markov perfect equilibria*.

Since the control problems arising in equilibrium verification are new, we establish a verification theorem for the case when the strategies form appropriate investment thresholds for the state. It enables us to examine the setting of Grenadier [22] and formally prove existence of the "Bertrand" equilibrium. While the Markovian strategies generating the open loop equilibrium outcome of Chapter 2 do not form a Markov perfect equilibrium, we actually determine a class of equilibria with similar investment behaviour and with *positive* option values.

3.1 The game

We want to formulate a stochastic game in continuous time, in which the players (two firms) strategically accumulate capital, by irreversible investment. Their respective objective is to maximize the value of a profit flow depending on both capital levels and exogenous uncertainty, net of investment costs. Formally, when the capital stock processes of player i and the opponent are Q^i and Q^{-i}, the payoff of player i is given by

$$J(Q^i, Q^{-i}) \triangleq \mathbf{E}\left[\int_0^\infty e^{-rt}\Pi(X_t, Q^i_t, Q^{-i}_t)\,dt - \int_0^\infty e^{-rt}\,dQ^i_t\right], \tag{3.1}$$

with a constant positive discount rate r. Since we focus on the pure strategic effect of capital commitment, the payoffs to the players differ only through the capital stock processes. The *instantaneous revenue* function Π is further affected by an exogenous stochastic process X, which is defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Assume, the latter satisfies the usual conditions of right-continuity and completeness and $\mathscr{F}_\infty = \mathscr{F}$. Concerning the stochastic capital stock processes Q^i and Q^{-i}, we allow the same class of processes available to the monopolist in the related irreversible investment problem. For given initial capital $q \in \mathbb{R}_+$, any feasible capital stock process has to belong to the class

$$\mathscr{A}(q) \triangleq \{Q \text{ adapted, right-continuous, nondecreasing, and } Q_0 \geq q,\ \mathbf{P}\text{-a.s.}\}.$$

Thus, investment decisions have to be conditioned on current accumulated information only and capital is installed without delay. In contrast to the monopolist, who chooses a *control policy* from $\mathscr{A}(q)$, the capital stock processes here will result from the strategies of the players.

For the payoffs (3.1) to be well defined, we make the following

Assumption 7.

i. $\Pi : \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}$, $(x, q^i, q^{-i}) \mapsto \Pi(x, q^i, q^{-i})$ is continuous and continuously differentiable in q^i. The partial derivative Π_{q^i} increases in x and decreases in q^i, respectively.

ii. $e^{-rt}\Pi(X_t(\omega), Q_t^i(\omega), Q_t^{-i}(\omega))$ is $\mathbf{P} \otimes dt$-integrable for any $(Q^i, Q^{-i}) \in \mathscr{A}(q^i) \times \mathscr{A}(q^{-i})$.

Our assumption on marginal instantaneous revenue relates to the local investment incentives of each firm, and gives some structure to the state space. The profitability of investment decreases for fixed competitive output q^{-i}, which together with the monotonicity in the exogenous shock will be helpful for the emergence of action and inaction regions. If there is furthermore an adverse influence of opponent capital on marginal revenue, i.e. if Π_{q^i} decreases in q^{-i} as well, the capital stocks are *strategic substitutes*. This will frequently be true, but we do not assume it for the entire state space a priori.

This leads us to the *strategies* of the players. While the processes in $\mathscr{A}(q)$ reflect the continuous revelation of uncertainty, we would like to enable the players to condition their investment decisions explicitly on the evolution of the capital stocks, too. Allowing reactions to deviating investment is necessary to obtain subgame perfect equilibria. Instead of considering investment processes adapted to a broadened filtration including the capital stock histories, we take a Markovian state space approach.

However, if one tries to define investment as a function of the state of the game, the following difficulty arises. Since the investment cost is linear, we know from the monopolistic and open loop cases that investment is likely to occur not in lumps, but continuously if the shock does not jump. Nevertheless is instantaneous investment in terms of the growth rate dQ^i unsuitable as an *action variable* if we do not artificially bound the rate. Although typical control paths in similar optimization problems are continuous, all exercise of control occurs at singular events and with an undefined rate. This phenomenon arises when one tries to keep a diffusion X off some barrier at minimal effort, here corresponding to a price trigger strategy, for instance.

Since we do not want to exclude such policies, the increments dQ^i are only meaningful in integrals as for the investment cost in (3.1).

Our treatment of the open loop case hints at a possibility to reconcile dynamic strategic decision making with the required properties of the resulting capital stock processes and consistency across subgames. There we showed that for any starting state the optimal investment policy is given by tracking the running supremum of a certain signal process L^i, once it exceeds currently installed capital. Formally,

$$Q^i_t = q^i_0 \vee \sup_{0 \le s \le t} L^i_s, \quad t \in [0, \infty), \tag{3.2}$$

with fixed initial capital q^i_0. L^i is the *base capacity* below which the firm never wants to operate. Such base capacities, which have to be optional processes, are in principle suitable to be determined by strategies or functions of the state of the game. The decisions of the players will then be related to the base capacity. If it exceeds installed capital, the latter is adjusted by investment, otherwise the signal is ignored. This idea is formalized as follows.

Strategies prescribe *actions*. In the related theory of differential games [16], the possible actions of a player at a particular moment are given by the space of instantaneous control. In our case, accounting for the exogenous uncertainty, the time-t *action set* of each player is defined as U_t, the set of $\mathscr{F}_t$-measurable random variables, taking values in $\mathbb{R}_+$ almost surely. A dynamic choice of actions $\{u^i_t \in U_t | t \in [0, \infty)\}$ by player i is *feasible* if the collection forms an optional process. Then, the capital stock process with the "law of motion"

$$Q^i_t = q^i_0 \vee \sup_{0 \le s \le t} u^i_s, \quad t \in [0, \infty), \tag{3.3}$$

is well defined and belongs to $\mathscr{A}(q^i_0)$.

With this concept, we can now define *strategies*, which are assignments of actions for all points in time t, conditional on the information available to the players. We begin with the definition of open loop strategies in this framework.

3.2 Open loop equilibrium

Open loop strategies are commitments to a particular control path, independent of the actions by any opponent during the run of the game. The revelation of information regarding the exogenous uncertainty is however taken into account. Formally, with U_∞ denoting the set of $\mathscr{F}_\infty$-measurable random variables, an open loop strategy for player i is a mapping $\phi^i : [0,\infty) \to U_\infty$ with $\phi^i(t) = u_t^i \in U_t$ for all $t \in [0,\infty)$. The open loop strategy is feasible if $(\phi^i(t))_{t\geq 0}$ belongs to $\mathscr{A}(q_0^i)$.

Here it is important to emphasize the difference between strategies and actions. A feasible open loop strategy according to our definition ϕ^i seems to be the same object as any feasible choice of actions $\{u_t^i | t \in [0,\infty)\}$. The crucial difference lies in the interpretation. ϕ^i is a *plan* of actions to be taken, valid for the entire run of the game, and fixed before the game starts. Once we begin to consider subgames, with a deviating capital stock of the opponent, the control given by any open loop strategy is unaffected. Generally, the *taken* sequence of actions $\{u_t^i | t \in [0,\infty)\}$ is decided upon dynamically and depends on the evolution of the game. It is a challenge to classify strategies that mimic such behaviour and generate well-defined outcomes.

We allow player i to optimize in the present setup against a given feasible open loop strategy ϕ^{-i} used by the opponent, by solving the following optimization problem.

$$\max_{u_t^i \in U_t, t\geq 0} J(Q^i, Q^{-i}) \tag{3.4}$$

s.t.

$$Q_t^i = q_0^i \vee \sup_{0\leq s\leq t} u_s^i \in \mathscr{A}(q_0^i)$$
$$Q_t^{-i} = q_0^{-i} \vee \sup_{0\leq s\leq t} \phi^{-i}(s)$$

In this problem, Q^{-i} is a fixed given process from $\mathscr{A}(q_0^{-i})$. Thus, if there is a best response by player i, it can be identified by the generated capital process Q^i itself and expressed by an open loop strategy as well. Consequently, this class is not restrictive for a best reply. An equilibrium of the game with open loop strategies is then defined as follows.

Definition 3.1. The pair (ϕ^1, ϕ^2) of open loop strategies is an open loop Nash equilibrium if for each $i \in \{1,2\}$ an optimal control $(u_t^i)_{t\geq 0}$ of the optimization problem (3.4) exists and satisfies $\phi^i(t) = u_t^i$ for all $t \in [0,\infty)$.

We observe that this equilibrium concept is equivalent to the open loop case as presented in Chapter 2, apart from the right-continuity, which will be important below. In this case all interaction is resolved at the beginning of the game.

3.3 Markov perfect equilibrium

To allow dynamic strategic interaction, we have to enable the players to base their investment decisions on past actions. These strategies are called *closed* *loop* strategies in general. Open loop strategies are degenerate closed loop strategies in the sense that they are independent of any history of actions. Another important subset of closed loop strategies are *Markovian* strategies that are conditioned only on the current state of the game. Such strategies are particularly appropriate if the state represents all payoff-relevant information concerning past play (e.g. [21]). In our case the payoff functions (3.1) imply that at any moment all payoff-relevant influence of past investment decisions is contained in the current capital levels (Q^1_t, Q^2_t).

While one could conceive of accounting for the exogenous shock X separately[1], the presentation becomes clearer if we suppose it is a Markov process and include its current value X_t in the state. Then, we may focus on stationary strategies since the horizon is infinite.

A stationary Markovian strategy assigns an action for any possible state (x, q^1, q^2) of the game, independent of time t. Formally, we define a stationary Markovian strategy for player i as a measurable function $\phi^i : \mathbb{R} \times \mathbb{R}^2_+ \to \mathbb{R}_+$. Thus, player i's action at time t given by the Markovian strategy ϕ^i is

$$u^i_t = \phi^i(X_t, Q^i_t, Q^{-i}_t), \tag{3.5}$$

an $\mathscr{F}_t$-measurable random variable for given $(Q^i, Q^{-i}) \in \mathscr{A}(q^i) \times \mathscr{A}(q^{-i})$. Note that it is not clear at all at this point, whether there exist feasible capital stock processes Q^i, Q^{-i} satisfying (3.5) and (3.3) for $i = 1, 2$ simultaneously. On the other hand, there may also be a multitude of solutions.

This is a key issue in any continuous-time game, independent of the added uncertainty, and will become clear in our examples. Anderson addresses the multiplicity problem in [3], while it is not explicitly mentioned in [16]. We propose to resolve it by the equilibrium definition, rather than by restricting the strategy spaces. Since it depends on the particular hypothesized equilibrium whether the outcome of the game might not be uniquely defined, it would seem to require a very strong restriction to exclude all such cases a priori.

We are looking for subgame perfect equilibria in Markovian strategies. Consequently, we identify a *subgame* by a starting time $t_0 \in [0, \infty)$ and an initial state $(x, q^1, q^2) \in \mathbb{R} \times \mathbb{R}^2_+$ only. From t_0 onwards, the game evolves according to (3.3) with (3.5) and payoffs follow (3.1) with time $t = 0$ shifted to t_0 and the initial state moved to (x, q^1, q^2), since X is by assumption Markovian.

Our notion of optimizing behaviour on behalf of the players is as follows. In equilibrium, given the Markovian strategy of the opponent, player i should not be able to increase the payoff in any subgame by any feasible control

[1] $\phi^i : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, \infty) \to U_\infty$ with $\phi^i(q^i, q^{-i}, t) \in U_t$ for all $(q^i, q^{-i}) \in \mathbb{R}^2_+$ and $t \in [0, \infty)$

path. Consider first the "subgame" starting at $t_0 = 0$ with $x \in \mathbb{R}$ fixed and let player i solve the following *verification problem*

$$\max_{u_t^i \in U_t, t \geq 0} J(Q^i, Q^{-i}) \tag{3.6}$$

s.t.

$$\begin{aligned} X_0 &= x \\ Q_t^i &= q^i \vee \sup_{0 \leq s \leq t} u_s^i \in \mathscr{A}(q^i) \\ Q_t^{-i} &= q^{-i} \vee \sup_{0 \leq s \leq t} \phi^{-i}(X_s, Q_s^{-i}, Q_s^i) \end{aligned}$$

In this problem, we can identify any control $\{u_t^i | t \in [0, \infty)\}$ satisfying the second constraint by the generated capital process Q^i itself. A *feasible* control for problem (3.6) is one that satisfies all constraints and by which $Q^{-i} \in \mathscr{A}(q^{-i})$ is uniquely ($\mathbf{P}$-a.s.) determined. Otherwise, the value of this problem would not be clear. The existence of feasible controls depends of course on the particular function ϕ^{-i}. A control is *optimal* for this problem if it is maximal among all feasible controls.

Note that in this optimization problem, player i has the same controls available as in the open loop problem (3.4). This is however again not a restriction but in fact gives the player the greatest conceivable power. With the reactions of the opponent specified by ϕ^{-i}, player i can in (3.6) perfectly control the entire evolution of the "subgame", without having to worry how to implement the desired outcome by a Markovian strategy. Player i can for instance perfectly preempt the opponent in (3.6), as will be illustrated below, without even an ϵ-margin.

Also note that because of our Markovian assumption on X and the stationarity of the strategies ϕ^i, a subgame starting at any time $t_0 \in [0, \infty)$ is fully characterized by its initial state (x, q^1, q^2). So, the verification problem analogous to (3.6) for the subgame beginning in t_0 is in fact of the same form as (3.6) with the appropriate initial state. Then, if we allow player i to optimize in *any* subgame by solving the related problem (3.6), this endows the player also with the greatest conceivable flexibility.

Summing up, if we require for a subgame perfect equilibrium in Markov strategies, called *Markov perfect equilibrium*, that each player i fares in any subgame as well as in the related verification problem (3.6), the players could not improve by any other closed loop strategy. The equilibrium would still persist with richer strategy spaces.

Definition 3.2. The pair (ϕ^1, ϕ^2) of Markovian strategies is a Markov perfect equilibrium for initial capital stocks $(q_0^1, q_0^2) \in \mathbb{R}_+^2$ if for each state $(x, q^1, q^2) \in \mathbb{R} \times \mathbb{R}_+^2$ with $q^i \geq q_0^i$ there exist a solution $(\tilde{Q}^1, \tilde{Q}^2) \in \mathscr{A}(q^1) \times \mathscr{A}(q^2)$ to

$$\begin{aligned} Q_t^1 &= q^1 \vee \sup_{0 \leq s \leq t} \phi^1(X_s, Q_s^1, Q_s^2) \\ Q_t^2 &= q^2 \vee \sup_{0 \leq s \leq t} \phi^2(X_s, Q_s^2, Q_s^1), \quad t \in [0, \infty) \end{aligned} \tag{3.7}$$

where $X_0 = x$, $\mathbf{P}$-a.s., and a pair of optimal controls $\{u_t^1 | t \in [0, \infty)\}$ and $\{u_t^2 | t \in [0, \infty)\}$ for problem (3.6) with initial state (x, q^1, q^2) yielding payoffs $J(\tilde{Q}^1, \tilde{Q}^2)$ and $J(\tilde{Q}^2, \tilde{Q}^1)$, respectively.

In the equilibrium definition, we do not ask for a unique solution of the combination of Markov strategies (3.7). But the chosen solution $(\tilde{Q}^1, \tilde{Q}^2)$ has to be optimal for both players simultaneously in the strong notion above. None of the players has at any moment or state of the game an incentive to employ any different control.

To identify any equilibria, we need to address the central optimization problems (3.6). These are singular control problems, but not of the monotone follower type as in the open loop case. Owing to the argument of the Markovian strategy ϕ^{-i}, there is a stronger path dependence and the methods employed in Chapter 2 are not applicable. In particular, we cannot switch from singular control to optimal stopping by the Fubini theorem anymore. Now, for planning ahead, not only the capital stock at a certain future time matters from that point onwards, but also how capital has evolved until then. The decision to delay investment thus obtains a new aspect and we have to take an alternative approach to account for it.

3.4 A verification theorem

We aim to establish a verification theorem for the optimization problems (3.6). We already argued that for a given Markovian strategy ϕ^{-i} and a Markov process X, the value of the problem depends only on the initial state (x, q^1, q^2) and not on the starting time, set to zero in (3.6). With our Assumption 7, the value function

$$V^*(x, q^i, q^{-i}) \triangleq \sup_{Q^i \in \mathscr{A}(q^i)} J(Q^i, Q^{-i}) \tag{3.8}$$

s.t.

$$\begin{aligned} X_0 &= x \\ Q_t^{-i} &= q^{-i} \vee \sup_{0 \leq s \leq t} \phi^{-i}(X_s, Q_s^{-i}, Q_s^i) \end{aligned}$$

is well defined, provided ϕ^{-i} is sufficiently regular such that the additional feasibility constraint for the "controls" Q^i to uniquely determine Q^{-i} is satisfied. As suggested above, we here replaced the control sequences $\{u_t^i | t \in [0,\infty)\}$ by the generated capital processes.

The stationarity of the value function V^* motivates a dynamic programming approach. Specifically, it holds for any $Q^i \in \mathscr{A}(q_0^i)$, with $Q^{-i} \in \mathscr{A}(q_0^{-i})$ generated by ϕ^{-i} and any almost surely finite $(\mathscr{F}_t)_{t\geq 0}$-stopping time τ

$$\begin{aligned} V^*(X_0, q_0^i, q_0^{-i}) \geq \mathbf{E}\bigg[\int_0^\tau e^{-rt}\Pi(X_t, Q_t^i, Q_t^{-i})\, dt - \int_0^\tau e^{-rt}\, dQ_t^i \\ + V^*(X_\tau, Q_\tau^i, Q_\tau^{-i}) \bigg]. \end{aligned} \tag{3.9}$$

Consequently, the argument of the expectation is a super-martingale for any feasible Q^i and the route is to identify such a process.

Suppose now that X is an Itō process, i.e. solves the stochastic differential equation

$$dX_t = \mu_t\, dt + \sigma_t\, dB_t, \quad t \in [0,\infty) \tag{3.10}$$

with $X_0 = x_0 \in \mathbb{R}$, $\mathbf{P}$-a.s., for a Brownian motion B on our filtered probability space and appropriate[2] drift and variance processes μ, σ. Then, for all feasible capital stock processes $(Q^1, Q^2) \in \mathscr{A}(q^1) \times \mathscr{A}(q^2)$, the state process is a semi-martingale, because the components Q^1 and Q^2 are monotone, adapted processes, i.e. of finite variation. Consequently, we may attempt to "construct" sufficiently smooth functions V and verify by Itō's lemma whether they coincide with the value function V^* in (3.8).

However, to identify a Markov perfect equilibrium, we need to solve (3.8) for all possible states and check whether there exists a solution $(\tilde{Q}^1, \tilde{Q}^2)$ to the combination of equilibrium strategies (ϕ^1, ϕ^2) as formalized in Definition 3.2, which actually attains the respective value. This procedure would be strongly simplified if we could apply the sought verification theorem somehow to Markovian strategies directly. To facilitate such an approach, we exploit the properties of local investment incentives following Assumption 7. They help to identify appropriate classes of best replies, that will eventually admit a Markovian representation as well.

Specifically, note that player i can undertake an initial discrete investment of size $\xi_0 > 0$. Thus, $V^*(x, q^i, q^{-i}) \geq V^*(x, q^i + \xi_0, q^{-i}) - \xi_0$, where the player behaves optimally after the investment. The optimal investment policy from state $(x, q^i + \xi_0, q^{-i})$ may require a further discrete investment, then the estimate holds with equality, but the jump ξ_0 may also have been

[2] μ and σ predictable, $\mu \in L^1(\mathbf{P} \otimes dt), \sigma \in L^2(\mathbf{P} \otimes dt)$

too large, i.e. unprofitable. Since we assumed that instantaneous marginal revenue decreases in q^i, it might be optimal for fixed x and q^{-i} to make an initial investment whenever q^i is below a certain value, and otherwise not. This critical value might further depend on x and q^{-i} in the same direction as marginal revenue, i.e. increase in x and decrease in q^{-i}. Formally, this hypothesis corresponds to a Markovian strategy with the properties $\phi^i_{q^i} = 0$, $\phi^i_x > 0$, and $\phi^i_{q^{-i}} \leq 0$ in case of differentiability. Then, the inverse

$$\bar{X}^i(q^i, q^{-i}) \triangleq \sup\{x \in \mathbb{R} |\, q^i \geq \phi^i(x, q^i, q^{-i})\} \tag{3.11}$$

is well defined and satisfies

$$\begin{aligned} &\bar{X}^i_{q^i} > 0, \\ &\bar{X}^i_{q^{-i}} \geq 0 \end{aligned} \tag{3.12}$$

(where the set is non-empty) and

$$\lim_{q^i \to \infty} \bar{X}^i(q^i, q^{-i}) = \infty \quad (q^{-i} \in \mathbb{R}_+). \tag{3.13}$$

Changing perspective, we can for any $\mathcal{C}^1$ function $\bar{X}^i : \mathbb{R}^2_+ \to \mathbb{R}$ with the properties (3.12) and (3.13) define a corresponding Markov strategy

$$\phi^i(x, q^{-i}) \triangleq \sup\{q \in \mathbb{R}_+ |\, x \geq \bar{X}^i(q, q^{-i})\} \vee 0 \tag{3.14}$$

with the argued properties. Here, we neglected the irrelevant argument of ϕ^i and dominated the supremum of the empty set, $-\infty$, by 0 for later use.

3.4.1 Reflection strategies

We will call strategies of the type (3.14) with the properties (3.12) and (3.13) *reflection strategies*. They prescribe to keep the state outside the "forbidden" region $\{x > \bar{X}^i(q^i, q^{-i})\}$ with minimal effort, like controls in obstacle problems. Since X has almost surely continuous paths, this policy involves only an initial discrete investment to bring the state onto the boundary of the forbidden region if necessary. Afterwards, the mentioned continuous singular control is exercised to reflect the state whenever X approaches the boundary $\bar{X}^i$.

Now suppose the opponent of player i uses a reflection strategy with $\bar{X}^{-i}$ satisfying (3.12) and (3.13). Our verification theorem will specify conditions under which a particular reflection strategy $\bar{X}^i$ is a best reply for player i. At this point, we face the problem that the two considered strategies might not uniquely define the capital processes Q^1 and Q^2 for any initial state, for instance if the boundaries are functions of the sum of the capital stocks. Thus, we have to pick a particular solution Q^i to be used as control in the verification problem (3.6), resp. (3.8). The solution we generally select is where player i acts as the *leader*.

Consider first the discrete initial investments for the state $(x_0, q_0^1, q_0^2) \in \mathbb{R} \times \mathbb{R}_+^2$. As the leader, player i first adjusts the capital stock to

$$Q_0^i = q_0^i \vee \phi^i(x_0, q_0^{-i}). \tag{3.15}$$

Then, the opponent's capital stock moves to

$$Q_0^{-i} = q_0^{-i} \vee \phi^{-i}(x_0, q_0^i \vee \phi^i(x_0, q_0^{-i})), \tag{3.16}$$

which is well defined. Now note that with these initial investments, the state is no longer in any of the forbidden regions, i.e.

$$x_0 \leq \bar{X}^i(Q_0^i, Q_0^{-i}) \wedge \bar{X}^{-i}(Q_0^{-i}, Q_0^i). \tag{3.17}$$

The only critical step here is when player $-i$ does invest, i.e. when

$$dQ_0^{-i} > 0 \Leftrightarrow \bar{X}^{-i}(q_0^{-i}, Q_0^i) < x_0 = \bar{X}^i(Q_0^i, q_0^{-i}). \tag{3.18}$$

But then the investment induces $\bar{X}^{-i}(Q_0^{-i}, Q_0^i) = x_0$ and $\bar{X}^i(Q_0^i, Q_0^{-i}) \geq x_0$ because of (3.12).

Outside the joint forbidden region, we still assume that player i acts as the leader in the sense that if player i chooses to invest at the same boundary $\bar{X}^i(q^i, q^{-i}) = \bar{X}^{-i}(q^{-i}, q^i)$ in a certain part of the state space, there usually is a strict incentive for preemption. Such perfect preemption with zero margin is feasible in problem (3.6) by the solution of the following *Skorohod*-type problem.

Problem 3.3. Given $i \in \{1, 2\}$, two reflection boundaries $\bar{X}^1$ and $\bar{X}^2$ which satisfy (3.12) and (3.13) and a starting state $(x_0, q_0^1, q_0^2) \in \mathbb{R} \times \mathbb{R}_+^2$, find two processes $Q^1 \in \mathscr{A}(q_0^1)$ and $Q^2 \in \mathscr{A}(q_0^2)$ such that

$$\left.\begin{aligned} & Q_0^i = q_0^i \vee \phi^i(x_0, q_0^{-i}) \\ & Q_0^{-i} = q_0^{-i} \vee \phi^{-i}(x_0, q_0^i \vee \phi^i(x_0, q_0^{-i})) \\ & X_t \leq \bar{X}^1(Q_t^1, Q_t^2) \wedge \bar{X}^2(Q_t^2, Q_t^1), \quad t \in [0, \infty) \\ & \int_0^\infty \left(1 - \mathbf{1}_{\{X_t \geq \bar{X}^i(Q_t^i, Q_t^{-i})\}}\right) dQ_t^i = 0 \\ & \int_0^\infty \left(1 - \mathbf{1}_{\{\bar{X}^{-i}(Q_t^{-i}, Q_t^i) \leq X_t < \bar{X}^i(Q_t^i, Q_t^{-i})\}}\right) dQ_t^{-i} = 0 \end{aligned}\right\} \quad \mathbf{P}\text{-a.s.} \tag{3.19}$$

The sought capital processes keep the state outside the joint forbidden region over the entire time interval, almost surely. The investment needed to do so is minimal, since it only occurs *on* the boundary.

Assume there exists a unique solution to this problem[3], which will be verified in the particular cases discussed below. Then, player i undertakes all the investments to reflect the state from the joint forbidden region, except where the boundary $\bar{X}^i$ *strictly* exceeds the minimum of the two. Where the boundaries coincide, player $-i$'s investment is completely preempted by the "leader" i.

Now we are in a position to state our verification theorem for reflection strategies with assignment of a leader.

3.4.2 Verification theorem

Assume now X is a geometric Brownian motion, i.e. solves the stochastic differential equation

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t, \quad t \in [0, \infty)$$

with *constant*, real μ and σ. We will only consider initial values $X_0 = x_0 \geq 0$, so for any $t \in [0, \infty)$, $X_t \in \mathbb{R}_+$, $\mathbf{P}$-a.s.

For the usual notation, introduce the *infinitesimal generator* $\mathcal{L}_x$ of the process X, which applied to any $\mathcal{C}^2$ function $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$ yields

$$\mathcal{L}_x f = \mu x f_x + \frac{1}{2} \sigma^2 x^2 f_{xx}.$$

In the following verification theorem, we want to identify a function V of the state that equals the payoff from a particular solution to Problem 3.3 at the given initial state, $J(Q^i, Q^{-i})$. Outside the joint forbidden region, i.e. absent any investment, the payoff evolves like an asset whose price is a function of X and which generates a dividend flow Π. At starting states inside the forbidden region, player i is the leader of initial investment, bringing the state on the boundary $\bar{X}^i$, at a cost equal to the size of the jump. Only if this still exceeds $\bar{X}^{-i}$ will the opponent make an *anticipated* investment, which will not affect the value of player i's payoff.

Theorem 3.4. *Let* $\bar{X}^1$ *and* $\bar{X}^2$ *satisfying* (3.12) *and* (3.13) *be given and assume* (Q^i, Q^{-i}) *solve Problem 3.3 for initial state* (x_0, q_0^1, q_0^2).

Suppose there exists a function $V : \mathbb{R}_+^3 \to \mathbb{R}, (x, q^i, q^{-i}) \mapsto V(x, q^i, q^{-i})$ *that is of class* $\mathcal{C}^{1,1,1}$ *and satisfies*

$$\lim_{T \to \infty} \mathbf{E}\big[e^{-rT} V(X_T, Q_T^i, Q_T^{-i})\big] = 0. \tag{3.20}$$

[3] For the classical Skorohod problem, see [19, 25, 32].

If

$\cdot$ *on* $\{x \le \bar{X}^i(q^i, q^{-i}) \wedge \bar{X}^{-i}(q^{-i}, q^i)\} : V$ *is class* $\mathcal{C}^{2,1,1}, -rV + \Pi + \mathcal{L}_x V = 0$

$\cdot$ *on* $\{x > \bar{X}^i(q^i, q^{-i})\} :\ V(x, q^i, q^{-i}) = V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i}) + q^i$

$\cdot$ *on* $\{\bar{X}^{-i}(q^{-i}, q^i) < x \le \bar{X}^i(q^i, q^{-i})\} :\ V(x, q^i, q^{-i}) = V(x, q^i, \phi^{-i}(x, q^i))$

then $V(x_0, q_0^1, q_0^2) = J(Q^i, Q^{-i})$.

If furthermore

$\cdot\ V$ *is class* $\mathcal{C}^{2,1,1}$ *on* $\{x \le \bar{X}^{-i}(q^{-i}, q^i)\}$

$\cdot\ V_{q^i} \le 1$ *on* $\{x \le \bar{X}^i(q^i, q^{-i}) \wedge \bar{X}^{-i}(q^{-i}, q^i)\}$

$\cdot\ V_{q^{-i}}(\bar{X}^i(q^i, q^{-i}), q^i, q^{-i}) \le 0$ *on* $\{\bar{X}^i(q^i, q^{-i}) \le \bar{X}^{-i}(q^{-i}, q^i)\}$

$\cdot\ -rV + \Pi + \mathcal{L}_x V \le 0$ *on* $\{\bar{X}^i(q^i, q^{-i}) < x \le \bar{X}^{-i}(q^{-i}, q^i)\}$

then $V(x_0, q_0^1, q_0^2) \ge J(Q^i, Q^{-i})$ *for any feasible* $(Q^i, Q^{-i}) \in \mathscr{A}(q_0^i) \times \mathscr{A}(q_0^{-i})$ *in* (3.6) *for which* (3.20) *holds.*

$V(x_0, q_0^1, q_0^2) = V^*(x_0, q_0^1, q_0^2)$ *only if*

$\cdot\ V_{q^i} \le 1$ *on* $\{x \le \bar{X}^i(q^i, q^{-i})\}$

$\cdot\ -rV + \Pi + \mathcal{L}_x V \le 0$ *on* $\{x \le \bar{X}^{-i}(q^{-i}, q^i)\}$

Proof. Suppose there exists V satisfying the first set of sufficient conditions.

Note the frequently used equivalence

$$x = \bar{X}^i(q^i, q^{-i}) \Leftrightarrow q^i = \phi^i(x, q^{-i}).$$

Note also that ϕ^i and ϕ^{-i} are continuously differentiable in x and q^{-i} and q^i, respectively, since $\bar{X}^i$ and $\bar{X}^{-i}$ are. We calculate the following partial derivatives.

On $\{x > \bar{X}^i(q^i, q^{-i})\}$:

$$\begin{aligned} V_{q^i} &= 1 = V_{q^i}(x, \phi^i(x, q^{-i}), q^{-i}) \quad (V_{q^i} \text{ continuous}) \\ V_{q^{-i}} &= \partial_{q^{-i}}\big(V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i})\big) \\ &= V_{q^{-i}}(x, \phi^i(x, q^{-i}), q^{-i}) \end{aligned} \tag{3.21}$$

To obtain the last line, we already used the first result.

On $\{\bar{X}^{-i}(q^{-i}, q^i) < x \leq \bar{X}^i(q^i, q^{-i})\}$:

$$\begin{aligned} V_{q^{-i}} &= 0 = V_{q^{-i}}(x, q^i, \phi^{-i}(x, q^i)) \quad (V_{q^{-i}} \text{ continuous}) \\ V_{q^i} &= \partial_{q^i}\big(V(x, q^i, \phi^{-i}(x, q^i))\big) \\ &= V_{q^i}(x, q^i, \phi^{-i}(x, q^i)) \end{aligned} \tag{3.22}$$

Again the first line implies the last.

Now consider an initial state (x_0, q_0^1, q_0^2) such that $X_0 = x_0 \leq \bar{X}^1(q_0^1, q_0^2) \wedge \bar{X}^2(q_0^2, q_0^1)$. Then, the paths of the semi-martingale (X, Q^1, Q^2), where the capital processes are the hypothesized solution to Problem 3.3, satisfy $X_t \leq \bar{X}^1(Q_t^1, Q_t^2) \wedge \bar{X}^2(Q_t^2, Q_t^1)$ for all $t \in [0, \infty)$ $\mathbf{P}$-a.s.

Since V is class $\mathcal{C}^{2,1,1}$ on $\{x \leq \bar{X}^i(q^i, q^{-i}) \wedge \bar{X}^{-i}(q^{-i}, q^i)\}$, we can apply Itō's lemma to obtain for arbitrary $T \in [0, \infty)$

$$\begin{aligned} e^{-rT} V(X_T, Q_T^i, Q_T^{-i}) - V(X_0, Q_0^i, Q_0^{-i}) = \\ \int_0^T e^{-rt}\big(-rV(X_t, Q_t^i, Q_t^{-i}) + \mathcal{L}_x V(X_t, Q_t^i, Q_t^{-i})\big)\, dt \\ + \int_0^T e^{-rt} \mu X_t V_x(X_t, Q_t^i, Q_t^{-i})\, dB_t \\ + \int_0^T e^{-rt} V_{q^i}(X_t, Q_t^i, Q_t^{-i})\, dQ_t^{i_c} \\ + \int_0^T e^{-rt} V_{q^{-i}}(X_t, Q_t^i, Q_t^{-i})\, dQ_t^{-i} \\ + \sum_{t \leq T} e^{-rt} \triangle V(X_t, Q_t^i, Q_t^{-i}), \qquad \mathbf{P}\text{-a.s.}, \end{aligned} \tag{3.23}$$

where Q^{i_c} is the continuous part of Q^i and the sum is over the jumps of Q^i up to T. Note that the presently discussed Q^i and Q^{-i} are continuous, but we will later allow for jumps of Q^i alone.

The second integral is a martingale, since for all $(Q^i, Q^{-i}) \in \mathscr{A}(q^i) \times \mathscr{A}(q^{-i})$ and $T \in \mathbb{R}_+$, $\big(\mathbf{1}_{\{t \le T\}} e^{-rt} \mu X_t V_x(X_t, Q_t^i, Q_t^{-i})\big)_{t \ge 0} \in L^2(\mathbf{P} \otimes dt)$ by continuity of V_x. It disappears when we now take expectations. We also rearrange and subtract the payoff stream including investment costs up to T on both sides.

$$
\begin{aligned}
&V(X_0, Q_0^i, Q_0^{-i}) - \mathbf{E}\left[\int_0^T e^{-rt}\Pi(X_t, Q_t^i, Q_t^{-i})\,dt - \int_0^T e^{-rt}\,dQ_t^i\right] = \\
&\quad \mathbf{E}\Bigg[-\int_0^T e^{-rt}\big(-rV(X_t, Q_t^i, Q_t^{-i}) + \Pi(X_t, Q_t^i, Q_t^{-i}) + \mathcal{L}_x V(X_t, Q_t^i, Q_t^{-i})\big)\,dt \\
&\qquad - \int_0^T e^{-rt}\big(V_{q^i}(X_t, Q_t^i, Q_t^{-i}) - 1\big)\,dQ_t^{i_c} \\
&\qquad - \sum_{t \le T} e^{-rt}\big(\triangle V(X_t, Q_t^i, Q_t^{-i}) - \triangle Q_t^i\big) \\
&\qquad - \int_0^T e^{-rt} V_{q^{-i}}(X_t, Q_t^i, Q_t^{-i})\,dQ_t^{-i}\Bigg] \\
&\qquad + \mathbf{E}\Big[e^{-rT} V(X_T, Q_T^i, Q_T^{-i})\Big]
\end{aligned}
\tag{3.24}
$$

All integrals and the sum on the right hand side are zero by the sufficient conditions. $dQ^i > 0$ only if $X_t \ge \bar{X}^i(Q_t^i, Q_t^{-i})$ and then $V_{q^i} = 1$. Similarly, $dQ^{-i} > 0$ only if $\bar{X}^{-i}(Q_t^{-i}, Q_t^i) \le X_t < \bar{X}^i(Q_t^i, Q_t^{-i})$ and then $V_{q^{-i}} = 0$. Finally, the last term goes to zero by hypothesis if we let T go to ∞, so

$$
V(x_0, q_0^i, q_0^{-i}) - J(Q^i, Q^{-i}) = 0, \tag{3.25}
$$

since there was no initial jump. However, for initial states (x_0, q_0^1, q_0^2) inside the forbidden region, the equality still holds since the jumps occur while $V_{q^i} = 1$ and $V_{q^{-i}} = 0$.

For the next claim, consider the additional set of sufficient conditions. Note that for any $Q^i \in \mathscr{A}(q_0^i)$, Q^{-i} given by ϕ^{-i} solves the Skorohod problem

$$
\begin{aligned}
X_t &\le \bar{X}^{-i}(Q_t^{-i}, Q_t^i), \quad t \in [0, \infty) \\
\int_0^\infty &\big(1 - \mathbf{1}_{\{X_t \ge \bar{X}^{-i}(Q_t^{-i}, Q_t^i)\}}\big)\,dQ_t^{-i} = 0,
\end{aligned}
\tag{3.26}
$$

$\mathbf{P}$-a.s., since no jump of Q^i can move the state into $-i$'s forbidden region, see (3.12). Thus, it is sufficient that V is class $\mathcal{C}^{2,1,1}$ on $\{x \le \bar{X}^{-i}(q^{-i}, q^i)\}$ to apply Itō's lemma on that region. Equation (3.24) remains valid for arbitrary $Q^i \in \mathscr{A}(q_0^i)$ after the initial jumps, where $X_0 \leq \bar{X}^{-i}(Q_0^{-i}, Q_0^i)$ follows for all dQ_0^i.

The second given condition implies that $V_{q^i} \leq 1$ on $\mathbb{R}_+^3$, since in the region $\{\bar{X}^{-i}(q^{-i}, q^i) < x \leq \bar{X}^i(q^i, q^{-i})\}$ it is evaluated at the lower boundary, see (3.22).

The third condition similarly implies $V_{q^{-i}} \leq 0$ on $\{x \geq \bar{X}^{-i}(q^{-i}, q^i)\}$ (the only region where dQ^{-i} can be strictly positive), since on $\{x \geq \bar{X}^i(q^i, q^{-i})\}$, $V_{q^{-i}}$ it is evaluated at the boundary, see (3.21).

The last condition implies $-rV + \Pi + \mathcal{L}_x V \leq 0$ on $\{x \leq \bar{X}^{-i}(q^{-i}, q^i)\}$, to which the state is constrained after initial jumps.

Together, the conditions imply that the integrals and sum on the right hand side of (3.24) are nonnegative. Letting again T go to ∞, we obtain

$$V(x_0, q_0^i, q_0^{-i}) - J(Q^i, Q^{-i}) \geq 0. \tag{3.27}$$

A similar remark regarding initial states inside the forbidden region applies and we conclude that there is no feasible capital stock process for player i with a payoff dominating V.

Now suppose the first necessary condition is violated, i.e. there exists a state such that $x < \bar{X}^i(q^i, q^{-i})$, where the strategy prescribes no investment, but where $V_{q^i} > 1$. Then, by continuity of the derivative, there exists $\epsilon > 0$ such that $V_{q^i}(x, q, q^{-i}) > 1$ for all $q \in [q^i, q^i + \epsilon]$. Then, the payoff from an ϵ-investment, followed by pursuing the given reflection strategy, is $V(x, q^i + \epsilon, q^{-i}) - \epsilon > V(x, q^i, q^{-i})$.

Similarly, if the second necessary condition is violated, there exists a state $x \leq \bar{X}^{-i}(q^{-i}, q^i)$ where $-rV + \Pi + \mathcal{L}_x V > 0$, which can only happen where $\bar{X}^{-i}(q^{-i}, q^i) < \bar{X}^i(q^i, q^{-i})$. Since V there is twice continuously differentiable in x, there exists $\epsilon > 0$ such that $-rV + \Pi + \mathcal{L}_x V > 0$ for all $x' \in [x - \epsilon, x]$. Then, for all initial states (x', q^i, q^{-i}), the capital process $(q^i \vee \mathbf{1}_{\{t \geq \tau_\epsilon\}} Q^i)$, where $\tau_\epsilon = \inf\{t \geq 0 | X_t \notin (x - \epsilon, x)\}$ yields a higher payoff then Q^i by the Itō-formula, since $dQ^{-i} = 0$ before τ_ϵ.

ο.ε.δ.

The sufficient conditions given in the verification theorem are quite constructive. For a given pair of reflection strategies, resp. boundaries $\bar{X}^1$ and $\bar{X}^2$, we can try to construct the associated V, by solving the partial differential equation, subject to the constraint that the extension to the other regions happens in a differentiable way. If the optimality conditions are satisfied, the verification problems (3.6) are solved for all initial states for which the Skorohod Problem 3.3 with $\bar{X}^1$ and $\bar{X}^2$ has a unique solution. This comes very close to our equilibrium definition and enables us to determine Markov perfect equilibria quite systematically in the following.

3.5 Bertrand equilibrium

The example that we discuss from now on is the revenue specification of Grenadier [22]. Suppose, the firms produce a homogeneous good at full capacity and sell it on a common market, facing inverse demand with constant elasticity. The price is multiplicatively affected by the exogenous shock X, our geometric Brownian motion defined above. With zero variable cost, the revenue function for firm i is then

$$\Pi(x, q^i, q^{-i}) = xP(q^i + q^{-i})q^i = x(q^i + q^{-i})^{-\frac{1}{\alpha}} q^i. \tag{3.28}$$

Assume $\alpha > 1$ to conform to Assumption 7. Regarding the integrability requirement, we anticipate a result of the subsequent section, where we will see that the monopolist's optimal payoff is finite iff $\alpha < \beta$. The latter is a function of the remaining parameters and will be presented soon. The expected *revenue* of any player in the game is now nonnegative and dominated by that of the monopolist, since competitive output can only decrease the price.

In this section, we begin with a simple type of reflection strategies, where a firm invests whenever the price $X_t P(Q_t^i + Q_t^{-i})$ rises above a certain constant threshold. Such policies will lead to the most commonly conjectured closed loop equilibrium, Bertrand quantities. Its elaboration is useful to illustrate some concepts and to derive some general results employed in the following, more involved cases.

Consequently, assume player i's opponent uses such a fixed price level to trigger investment, i.e.

$$\bar{X}^{-i}(q^{-i}, q^i) = \frac{p^{-i}}{P(q^i + q^{-i})} \tag{3.29}$$

with $p^{-i} \in \mathbb{R}_+$. From player i's point of view this means that independently of $Q^i \in \mathscr{A}(q_0^i)$, the price $X_t P(Q_t^i + Q_t^{-i})$ will never exceed p^{-i} for any $t > 0$. If i does not invest, the price will be reflected at this barrier. The problem is now how to preempt the opponent optimally, if at all.

Whenever player i invests, the additional net revenue is of course countered by the externality of lowering the price for existing output, as always in Cournot competition. This becomes obvious in marginal revenue $x\big(P(q^i + q^{-i}) + q^i P'(q^i + q^{-i})\big)$, where the derivative is negative.

In the cases of monopoly, oligopoly with open loop strategies, and perfect competition, the optimal investment strategies can be determined by deciding when to install marginal capital units in order to start the associated marginal revenue flow. For each unit, there is an option to delay investment and the optimal exercise time can be determined independently by optimal stopping. In particular we showed in Chapter 2 that the value of the option to delay marginal investment is closely related to the opportunity cost.

In the present case, player i's investment also influences the capital stock of the opponent and due to the running suprema in (3.6), resp. (3.8), some path-dependence arises and we cannot treat the marginal capital units independently, anymore. The opportunity cost principle is now only applicable subject to *full* preemption (when $Q^{-i} \equiv q_0^{-i}$), respectively over intervals in which the opponent's investment boundary is not reached. This will be illustrated below.

We begin the study of our present example with the question when it is profitable for player i to cause the price reflection by own investment. In fact, since the price barrier here is constant, the decision will always be the same whenever the boundary is reached. Thus, we aim to determine the values of always preempting, as well as never investing, with the help of Theorem 3.4.

First, we can solve the partial differential equation which V has to satisfy off the forbidden region. The general solution is polynomial.

$$V(x, q^i, q^{-i}) = A(q^i, q^{-i})x + B(q^i, q^{-i})x^{\beta} \tag{3.30}$$

Here, β is the positive root of the typical quadratic equation[4] and given by

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \tag{3.31}$$

Note that $\beta > 1 \Leftrightarrow r > \mu$, which is necessary for our assumption $\beta > \alpha$ to hold.

Furthermore must the first coefficient satisfy

$$A(q^i, q^{-i}) = \frac{1}{r - \mu} P(q^i + q^{-i})q^i. \tag{3.32}$$

[4] We neglected the corresponding negative root as further exponent in V, which would otherwise diverge to positive or negative infinity when x approaches zero.

As a consequence, the first term of V is necessarily equal to the net present value of the revenue flow, were the current capacities *fixed forever*. This observation holds independently of the considered boundaries $\bar{X}^1$ and $\bar{X}^2$.

The latter come into play by the boundary conditions for the partial derivatives of V in Theorem 3.4. We have to match the coefficient function B to $\bar{X}^1$ and $\bar{X}^2$ by these boundary conditions.

Begin in the current example with the case of never investing, i.e. $\bar{X}^i \equiv \infty$. So, since $\bar{X}^i(q^i, q^{-i}) > \bar{X}^{-i}(q^{-i}, q^i)$, V must satisfy (cf. (3.22))

$$\begin{aligned} & V_{q^{-i}}\left(\frac{p^{-i}}{P(q^i+q^{-i})}, q^i, q^{-i}\right) = 0 \\ \Leftrightarrow\ & B_{q^{-i}} = - A_{q^{-i}}(p^{-i})^{1-\beta}P(q^i+q^{-i})^{\beta-1} \\ & \quad = \frac{(p^{-i})^{1-\beta}}{\alpha(r-\mu)}q^i(q^i+q^{-i})^{-\frac{\beta}{\alpha}-1}. \end{aligned} \tag{3.33}$$

The last expression can be integrated to obtain

$$\begin{aligned} B(q^i, q^{-i}) &= -\frac{(p^{-i})^{1-\beta}}{\beta(r-\mu)}q^i(q^i+q^{-i})^{-\frac{\beta}{\alpha}} + C(q^i) \\ &= -\frac{p^{-i}}{\beta(r-\mu)}q^i\left(\frac{P(q^i+q^{-i})}{p^{-i}}\right)^{\beta} + C(q^i). \end{aligned} \tag{3.34}$$

Using this coefficient B, we can define by

$$\begin{aligned} V^\infty(x, q^i, q^{-i}) \triangleq & \frac{p^{-i}}{r-\mu}q^i\frac{xP(q^i+q^{-i}\vee\phi^{-i}(x,q^i))}{p^{-i}} \\ & -\frac{p^{-i}}{\beta(r-\mu)}q^i\left(\frac{xP(q^i+q^{-i}\vee\phi^{-i}(x,q^i))}{p^{-i}}\right)^{\beta} \end{aligned} \tag{3.35}$$

a function satisfying the first set of sufficient conditions and hypothesis of Theorem 3.4. The latter is true because the prices xP here are bounded above by p^{-i} and xV_x is of the same order as V. The corresponding solutions to Problem 3.3 are

$$Q^i \equiv q_0^i \text{ and } Q^{-i} = q_0^{-i} \vee \left(\sup_{0\le s\le t}(X_s/p^{-i})^\alpha - q_0^i\right)_{t\ge 0}.$$

The integration constant $C(q^i)$ has been set to zero, since V^∞ represents the net present value of selling the constant output flow q^i at a diffusion price, reflected at the barrier p^{-i}, which cannot indefinitely increase nor decrease in the initial shock value x. Also note that generally any constant coefficient component

$$C \cdot x^\beta$$

of V will drop out when we apply Itō's lemma in (3.23).

Now consider the other case and let

$$\bar{X}^i = \frac{p^{-i}}{P(q^i + q^{-i})},$$

i.e. player i preempts the investment of $-i$ at the identical boundary by implementing the capital stock process

$$Q^i = q_0^i \vee \Big(\sup_{0 \le s \le t} (X_s/p^{-i})^\alpha - q_0^{-i} \Big)_{t \ge 0}$$

so that $Q^{-i} \equiv q_0^{-i}$, which together again solve Problem 3.3.

Then, since $\bar{X}^i(q^i, q^{-i}) \le \bar{X}^{-i}(q^{-i}, q^i)$, the relevant boundary condition for V is (cf. (3.21))

$$V_{q^i}\big(\frac{p^{-i}}{P(q^i + q^{-i})}, q^i, q^{-i}\big) = 1$$

$$\Leftrightarrow B_{q^i} = -(p^{-i})^{-\beta}\Big(q^i\Big(\tfrac{p^{-i}(\alpha-1)}{(r-\mu)\alpha} - 1\Big) + q^{-i}\Big(\tfrac{p^{-i}}{r-\mu} - 1\Big)\Big)(q^i + q^{-i})^{-\frac{\beta}{\alpha}-1}.$$

This is again one of the rare cases in which one can explicitly integrate for B. Neglecting the integration constant for the same reason as before, we arrive at

$$\frac{B^{p^{-i}}(q^i, q^{-i})}{\big(P(q^i + q^{-i})\big)^\beta} = \tfrac{\alpha}{\beta-\alpha}(p^{-i})^{-\beta}\Big(q^i\Big(\tfrac{p^{-i}(\alpha-1)}{(r-\mu)\alpha} - 1\Big) + q^{-i}\Big(\tfrac{p^{-i}(\beta-1)}{(r-\mu)\beta} - 1\Big)\Big).$$

With this particular coefficient function, the value of preempting at p^{-i} from a given initial state is completely determined by[5]

$$V^{p^{-i}}(x, q^i, q^{-i}) \triangleq \begin{cases} A(q^i, q^{-i})x + B^{p^{-i}}(q^i, q^{-i})x^\beta & \text{if } x \le p^{-i}/P(q^i + q^{-i}) \\ V^{p^{-i}}(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i}) + q^i & \text{else.} \end{cases}$$

Now we can compare the values of both policies. Preempting is more profitable than never investing iff $V^{p^{-i}} \ge V^\infty$. On $\{q^i \ge \phi^i(x, q^{-i})\}$ this is equivalent to

$$\tfrac{\alpha}{\beta-\alpha}\Big(q^i\Big(\tfrac{p^{-i}(\alpha-1)}{(r-\mu)\alpha} - 1\Big) + q^{-i}\Big(\tfrac{p^{-i}(\beta-1)}{(r-\mu)\beta} - 1\Big)\Big) + q^i \tfrac{p^{-i}}{\beta(r-\mu)} \ge 0$$

$$\Leftrightarrow \Big(\tfrac{p^{-i}(\beta-1)}{(r-\mu)\beta} - 1\Big)\Big(\tfrac{\alpha}{\beta-\alpha}q^i + q^{-i}\Big) \ge 0$$

[5]At the price boundary, $V^{p^{-i}}$ is linear in q^i. We clarify below that the boundedness condition is satisfied.

$$\Leftrightarrow p^{-i} \geq \frac{\beta(r-\mu)}{(\beta-1)} \triangleq p^*.$$

Then, $V^{p^{-i}} \geq V^\infty$ also on $\{q^i < \phi^i(x, q^{-i})\}$, since in this region, $V_{q^i}^{p^{-i}} = 1 \leq V_{q^i}^\infty = p^{-i}/p^*$.

p^* is a quite important quantity, it is precisely the *Bertrand price*, which we know from the case of perfect competition. If the price is reflected at this barrier, the net present value of a marginal capital unit equals one at the boundary, its cost. Consequently, the option to delay investment is valueless.

In principle, we already know now that in the only Markov perfect equilibrium with a constant reflection price both players invest at the Bertrand price, where each is just indifferent. However, we want to formally prove this finding by completing the consistent application of Theorem 3.4 and checking our equilibrium definition.

Specifically, we have only determined when preemption at p^{-i} is *superior* to remaining passive. For optimality, we need to verify the further sufficient conditions.

In particular, for concluding that our candidate function V is really the value function defined in (3.8), we need to verify that it satisfies the boundedness condition for all relevant controls. This first problem can be easily tackled because the expected revenue from any capital process is finite by assumption and we only need to consider processes with finite investment cost. In this case integration by parts yields

$$\mathbf{E}\Big[\int_0^\infty e^{-rt}\, dQ_t\Big] < \infty \Rightarrow \lim_{T\to\infty} \mathbf{E}\Big[e^{-rT} Q_T\Big] = 0 \tag{3.36}$$

and it suffices to establish a linear bound on V for arbitrary capital processes. Then,

Lemma 3.5. *For any $Q^i \in \mathscr{A}(q^i)$ with finite investment cost and $q^{-i} \in \mathbb{R}_+$,*

$$\lim_{T\to\infty} \mathbf{E}\Big[e^{-rT} V^{p^{-i}}(X_T, Q_T^i, q^{-i})\Big] = 0.$$

The investment cost of reflecting the price at any constant barrier is finite.

The proof is given in the Appendix.

By Lemma 3.5, we may now check the second set of sufficient conditions in Theorem 3.4 to verify when either strategy is optimal[6] in (3.6).

[6]One could also restrict the search to the class of processes never exceeding the Bertrand quantity. Lemma 3.11 in the Appendix states that such a cap is profitable and is proven by the optimal stopping approach of Chapter 2.

Let us again begin with $\bar{X}^i \equiv \infty$. All but the second sufficient condition hold by construction. Since $V^{\infty}_{q^i}$ is increasing in x, the condition is satisfied iff

$$\begin{aligned} & V^{\infty}_{q^i}(\frac{p^{-i}}{P(q^i+q^{-i})}, q^i, q^{-i}) \leq 1 \\ \Leftrightarrow & \frac{p^{-i}}{(r-\mu)}\frac{(\beta-1)}{\beta} \leq 1 \\ \Leftrightarrow & \, p^{-i} \leq p^*. \end{aligned} \tag{3.37}$$

We may as expected conclude that at any constant reflection barrier lower than Bertrand it is optimal for player i to abstain from investment.

The corresponding condition for $\bar{X}^i = p^{-i}/P(q^i+q^{-i})$ can be verified by an important general result. For this, note that whenever $\bar{X}^i \leq \bar{X}^{-i}$, the coefficient function B needs to satisfy the boundary condition

$$\begin{aligned} & V_{q^i}(\bar{X}^i(q^i,q^{-i}), q^i, q^{-i}) = 1 \\ \Leftrightarrow & \, B_{q^i} = \big(1 - A_{q^i}\bar{X}^i(q^i,q^{-i})\big)\bar{X}^i(q^i,q^{-i})^{-\beta}. \end{aligned} \tag{3.38}$$

We want to answer the question in which cases the necessary optimality condition for V_{q^i} is compatible with (3.38).

$$\begin{aligned} & V_{q^i} \leq 1, \quad \forall x \in [0, \bar{X}^i(q^i,q^{-i})] \\ \Leftrightarrow & B_{q^i} \leq (1 - A_{q^i}x)x^{-\beta}, \quad \forall x \in [0, \bar{X}^i(q^i,q^{-i})] \end{aligned} \tag{3.39}$$

The last condition can only hold if the right hand side is not increasing in x (in the given interval), since equality is attained at the upper bound by (3.38). It is nonincreasing iff

$$x \leq \frac{\beta}{\beta-1}\big(A_{q^i}\big)^{-1} = p^* \frac{q^i+q^{-i}}{\frac{\alpha-1}{\alpha}q^i+q^{-i}}\big(P(q^i+q^{-i})\big)^{-1} \triangleq \frac{\bar{p}(q^i,q^{-i})}{P(q^i+q^{-i})}. \tag{3.40}$$

Thus, when $\bar{X}^i \leq \bar{X}^{-i}$, it satisfies the optimality condition if and only if $\bar{X}^i \leq \bar{p}(q^i,q^{-i})/P(q^i+q^{-i})$. The latter function is not only important because of this property, it also happens to be the *myopic* price trigger discussed in detail below. For future reference note also

$$\bar{X}^i \leq \frac{\bar{p}(q^i,q^{-i})}{P(q^i+q^{-i})} \Leftrightarrow V_{q^i x}(\bar{X}^i, q^i, q^{-i}) \geq 0. \tag{3.41}$$

Concerning the present example, we thus only have to verify whether $p^{-i} \leq \bar{p}(q^i,q^{-i})$, which is satisfied iff

$$q^i \geq q^{-i}\frac{p^{-i}-p^*}{p^* - \frac{\alpha-1}{\alpha}p^{-i}} \geq 0. \tag{3.42}$$

Both inequalities must hold, so if $p^{-i} \geq p^* \cdot \alpha/(\alpha-1)$, it cannot be the optimal boundary (which will be lower). Furthermore, we also consider subgames off the equilibrium path — at least with capital stocks not strictly below the initial levels —, so (3.42) has to be satisfied by all $q^i \geq q_0^i$ and $q^{-i} \geq q_0^{-i}$, and does so only if $p^{-i} \leq p^*$.

It remains to check under which conditions $V_{q^{-i}}^{p^{-i}} \leq 0$ at the investment boundary. A short calculation shows

$$V_{q^{-i}}^{p^{-i}}(x, \phi^i(x, q^{-i}), q^{-i}) = \frac{x}{p^{-i}}\Big(1 - \tfrac{p^{-i}}{p^*}\Big)P(q^i + q^{-i}) \leq 0 \Leftrightarrow p^{-i} \geq p^*.$$

Consequently, preempting at the fixed price level is optimal in all subgames iff[7] $p^{-i} = p^*$, i.e. when both players are just indifferent to invest at the boundary.

In fact, if both players use $\bar{X}^i = p^*/P(q^i + q^{-i})$, we can select any pair of processes $(Q^1, Q^2) \in \mathscr{A}(q_0^1) \times \mathscr{A}(q_0^2)$ that jointly reflect the price at p^* to comply with our equilibrium Definition 3.2. Let us summarize this result.

Proposition 3.6. *For all initial capital levels $(q_0^1, q_0^2) \in \mathbb{R}_+^2$, the pair of Markovian strategies (ϕ^B, ϕ^B), where*

$$\phi^B(x, q) \triangleq \left(\frac{x}{p^*}\right)^{\alpha} - q,$$

is a Markov perfect equilibrium for the game with revenue function (3.28). The equilibrium value of firm i at state $(x, q^i, q^{-i}) \in \mathbb{R}_+^3$ is given by the function

$$V^B(x, q^i, q^{-i}) \triangleq \begin{cases} q^i & \text{if} \quad xP(q^i + q^{-i}) \geq p^*, \\ \frac{\beta}{\beta-1} q^i \frac{xP(q^i+q^{-i})}{p^*} - \frac{1}{\beta-1} q^i \left(\frac{xP(q^i+q^{-i})}{p^*}\right)^{\beta} & \text{else.} \end{cases}$$

We observe that the value of each firm equals its current capital stock in the forbidden region. The additional revenue flow from any investment in this region has present value one, identical to its cost, and thus does not affect firm value. Consequently, the option value of waiting completely disappears. Any profitable investment is immediately exploited, like under perfect competition.

[7] We did not prove that $V_{q^{-i}}^{p^{-i}} \leq 0$ is necessary, but we showed $V^\infty > V^{p^{-i}} \Leftrightarrow p^{-i} < p^*$.

3.6 Myopic investment

We saw in the previous section that it might not be an optimal reply to wait until the price reaches the opponent's investment boundary. In this section we will take a look at earlier investment and elaborate on the associated optimality conditions. The results will have important implications for the existence of further equilibria.

3.6.1 The myopic investor

For concreteness, begin with the assumption that player i's opponent uses the "reflection" strategy $X^{-i} \equiv \infty$, which was shown to be optimal when the price never exceeds p^*. In this case, player i can act like a monopolist, taking the fixed competitive output as given, so the optimal strategy is not very difficult to determine. Nevertheless, the situation is of intrinsic interest, since we showed in Chapter 2 that the best reply to the current, fixed capital levels is the optimal investment policy in any open loop equilibrium under very general conditions. Such investment behaviour is called *myopic* and was already discussed by Leahy in his derivation of a perfectly competitive equilibrium [27]. The principle is widely known since then and we would like to know which role it is playing in our setting.

Let us try to identify an optimal myopic reflection boundary $X^m(q^i, q^{-i})$ with the help of Theorem 3.4, by constructing the myopic value function V^m. From the previous section, we know that necessarily on $\{x \leq X^m(q^i, q^{-i})\}$

$$V^m(x, q^i, q^{-i}) = \frac{1}{r - \mu} q^i x P(q^i + q^{-i}) + B^m(q^i, q^{-i}) x^\beta,$$

where we have to determine B^m by the boundary conditions. Furthermore, we know from (3.41) that the necessary optimality conditions can only hold if $\bar{X}^m \leq \bar{p}(q^i, q^{-i})/P(q^i + q^{-i})$.

Now consider the second necessary condition of the theorem for optimality, which we only need to check for $\{\bar{X}^i(q^i, q^{-i}) < x \leq \bar{X}^{-i}(q^{-i}, q^i)\}$, since it will hold by construction at smaller x. The notation is kept general for the moment since the intended result is, too.

In the given region, $V(x, q^i, q^{-i}) = V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i}) + q^i$. This implies for the first partial derivative required to evaluate $\mathcal{L}_x V$:

$$\begin{aligned} V_x(x, q^i, q^{-i}) &= \partial_x \big(V(x, \phi^i(x, q^{-i}), q^{-i}) - \phi^i(x, q^{-i}) \big) \\ &= V_x(x, \phi^i(x, q^{-i}), q^{-i}), \end{aligned} \tag{3.43}$$

where the last line follows from (3.21). Differentiating once more yields

$$\begin{aligned} V_{xx}(x, q^i, q^{-i}) =& \partial_x\big(V_x(x, \phi^i(x, q^{-i}), q^{-i})\big) \\ =& V_{xx}(x, \phi^i(x, q^{-i}), q^{-i}) + V_{xq^i}(x, \phi^i(x, q^{-i}), q^{-i})\phi^i_x(x, q^{-i}). \end{aligned} \tag{3.44}$$

By these formulae, V,V_x, and V_{xx} in the given region are — apart from some correction terms — all evaluated at $(x, \phi^i(x, q^{-i}), q^{-i})$, i.e. at an argument for which the partial differential equation is satisfied by construction. This observation admits the following simplification on $\{\bar{X}^i(q^i, q^{-i}) < x \leq \bar{X}^{-i}(q^{-i}, q^i)\}$:

$$\begin{aligned} -rV + \Pi + \mathcal{L}_x V =& r\phi^i(x, q^{-i}) - rq^i \\ &+ \Pi(x, q^i, q^{-i}) - \Pi(x, \phi^i(x, q^{-i}), q^{-i}) \\ &+ \frac{1}{2}\sigma^2 x^2 V_{xq^i}(x, \phi^i(x, q^{-i}), q^{-i})\phi^i_x(x, q^{-i}). \end{aligned} \tag{3.45}$$

Then, if q^i approaches $\phi^i(x, q^{-i})$, the only term on the right hand side that remains is the last. Since $\phi^i_x > 0$ (corresponding to $\bar{X}^i_{q^i} > 0$ and necessary for a well defined reflection strategy), we conclude that the necessary optimality condition can only be satisfied if

$$V_{xq^i}(x, \phi^i(x, q^{-i}), q^{-i}) \leq 0. \tag{3.46}$$

Combined with the first necessary condition (3.41), we must have equality. Equivalently, the optimal myopic investment boundary[8] is completely determined by

$$\bar{X}^m(q^i, q^{-i}) = \frac{\bar{p}(q^i, q^{-i})}{P(q^i + q^{-i})}, \tag{3.47}$$

as claimed in the previous section. Requiring $V_{xq^i} = 0$ at the investment boundary is actually the "smooth pasting condition" which is often treated like an abstract, universal optimality condition.

As the general conclusion, whenever player i considers to invest strictly before the opponent, at a boundary admitting a sufficiently smooth and bounded V, it must happen at the myopic boundary, and this is the only boundary at which we will encounter smooth pasting.

Continuing with the particular case $\bar{X}^{-i} \equiv \infty$ and $\bar{X}^i = \bar{X}^m$, we still need to verify for optimality that (3.45) is nonpositive in the entire given region, not only near the boundary. But this follows in fact from the equivalence

$$\partial_{q^i}\big(\Pi(x, q^i, q^{-i}) - rq^i\big) \geq 0 \tag{3.48}$$

[8] It is indeed a proper reflection boundary with $\bar{X}^m_{q^i} > 0$ and $\lim_{q^i \to \infty} \bar{X}^m = \infty$.

$$\Leftrightarrow x \geq r\big(P(q^i + q^{-i}) + q^i P'(q^i + q^{-i})\big)^{-1} = \frac{r}{p^*}\bar{X}^m,$$

which holds on $\{x \geq \bar{X}^m\}$ since our assumption $r > \mu$ implies $r < p^*$. The only sufficient condition for optimality left unanswered is the third item, which is however irrelevant since $dQ^{-i} \equiv 0$.

While the necessary conditions have already fixed X^m and simultaneously ensure that the associated value function will satisfy all sufficient conditions, we need of course the coefficient B^m to see that such V^m indeed exists, and to check the boundedness condition for V^m in the theorem's hypothesis.

Unfortunately, though the present case seems even simpler because i acts like a monopolist, it is not possible to integrate explicitly for B^m as in the previous section. As in later instances, we have to cope with its definition via an integral.

The determining boundary condition (3.21) simplifies in this case to

$$\begin{aligned} B^m_{q^i}(q^i, q^{-i}) &= -\tfrac{1}{\beta-1}\big(\bar{X}^m(q^i, q^{-i})\big)^{-\beta} \\ &= -\tfrac{1}{\beta-1}(p^*)^{-\beta}\big(P(q^i + q^{-i}) + q^i P'(q^i + q^{-i})\big)^{\beta} \\ &= -\tfrac{1}{\beta-1}(p^*)^{-\beta}\big(\tfrac{\alpha-1}{\alpha}q^i + q^{-i}\big)^{\beta}\big(q^i + q^{-i}\big)^{-\frac{\beta}{\alpha}-\beta}. \end{aligned} \tag{3.49}$$

Based on this partial derivative, we obtain the following result.

Proposition 3.7. *Let* $\bar{X}^i = \bar{X}^m$ *given by* (3.47), $\bar{X}^{-i} = \infty$, *and define* $\phi^m(x, q^{-i}) \triangleq \sup\{q \in \mathbb{R}_+ | x \geq \bar{X}^m(q, q^{-i})\} \vee 0$. *Then, for any* $(x, q^i, q^{-i}) \in \mathbb{R}^3_+$,

$$V^m(x, q^i, q^{-i}) \triangleq \begin{cases} \frac{\beta}{\beta-1}q^i \frac{xP(q^i+q^{-i})}{p^*} + B^m(q^i, q^{-i})x^\beta & \text{if } x \leq \bar{X}^m(q^i, q^{-i}) \\ V^m(x, \phi^m(x, q^{-i}), q^{-i}) - \phi^m(x, q^{-i}) + q^i & \text{else,} \end{cases}$$

with

$$B^m(q^i, q^{-i}) \triangleq -\int_{q^i}^{\infty} B^m_{q^i}(q, q^{-i})\, dq$$

satisfies the hypothesis of Theorem 3.4.

$Q^i_t = q^i \vee \sup_{0 \leq s \leq t} \phi^m(X_s, q^{-i})$ *is optimal for any* $(x, q^i, q^{-i}) \in \mathbb{R}^3_+$ *in problem* (3.6) *with* $\phi^{-i} \equiv 0$.

Proof. Concerning the boundedness condition for arbitrary $Q^i \in \mathscr{A}(q^i)$ with finite cost, one can repeat the proof of Lemma 3.5 with the following estimates:

$$\frac{xP(q^i \vee \phi^m + q^{-i})}{p^*} \leq \frac{\bar{p}(q^i \vee \phi^m, q^{-i})}{p^*} \leq \frac{\alpha}{\alpha - 1}$$

and

$$-B^m_{q^i} \le \frac{1}{\beta-1}(p^*)^{-\beta}\big(P(q^i+q^{-i})\big)^\beta$$

$$\Rightarrow 0 \le B^m(q^i \vee \phi^m, q^{-i})x^\beta \le \frac{\alpha}{\beta-\alpha}\frac{1}{\beta-1}\big(q^i \vee \phi^m + q^{-i}\big)\Big(\frac{\alpha}{\alpha-1}\Big)^\beta.$$

ο.ε.δ.

B^m actually can be calculated explicitly in the special case $q^{-i}=0$, true monopoly. Then,

$$V^m(x,q^i,0) = \begin{cases} q^i + \frac{\beta}{(\alpha-1)(\beta-\alpha)}\Big(\frac{\alpha-1}{\alpha}\frac{x}{p^*}\Big)^\alpha & \text{if } x > \bar{X}^m(q^i,0) \\ \frac{\beta}{\beta-1}q^i\frac{xP(q^i)}{p^*} + \frac{1}{\beta-1}\frac{\alpha}{\beta-\alpha}q^i\Big(\frac{\alpha-1}{\alpha}\Big)^\beta\Big(\frac{xP(q^i)}{p^*}\Big)^\beta & \text{else,} \end{cases}$$

which is well defined and finite for $\alpha < \beta$ as claimed before.

3.6.2 Playing against a myopic investor

In the preceding, we identified the central importance of the myopic investment boundary in any potential equilibrium with two differing reflection strategies, so the natural next step to take is the complementary point of view. Which reflection strategies can be best replies to a myopically investing firm? Thus, suppose the opponent of player i uses the myopic boundary in part of the state space, $\bar{X}^{-i} = \bar{X}^m$, and this is indeed strictly smaller than $\bar{X}^i$.

Then, the relevant boundary condition to construct player i's value function V is

$$V_{q^{-i}}\big(\frac{\bar{p}(q^{-i},q^i)}{P(q^i+q^{-i})}, q^i, q^{-i}\big) = 0$$

$$\begin{aligned} \Leftrightarrow B_{q^{-i}}(q^i,q^{-i}) &= \tfrac{\beta}{\alpha(\beta-1)}(p^*)^{-\beta}q^i\big(q^i+q^{-i}\big)^{-\frac{\beta}{\alpha}-\beta}\big(q^i+\tfrac{\alpha-1}{\alpha}q^{-i}\big)^{\beta-1} \\ &\triangleq B^{\text{tm}}_{q^{-i}}(q^i,q^{-i}). \end{aligned} \tag{3.50}$$

We denote the right hand side by $B^{\text{tm}}_{q^{-i}}$, since player i *tolerates* myopic investment at these points.

If we are able to determine the coefficient B from the preceding equation, the necessary optimality condition is

$$V_{q^i}\big(\frac{\bar{p}(q^{-i},q^i)}{P(q^i+q^{-i})}, q^i, q^{-i}\big) \le 1$$

$$
\begin{aligned}
\Leftrightarrow B_{q^i}(q^i,q^{-i}) \leq & \tfrac{1}{\alpha(\beta-1)}(p^*)^{-\beta}\big(q^i+q^{-i}\big)^{-\frac{\beta}{\alpha}-\beta}\big(q^i+\tfrac{\alpha-1}{\alpha}q^{-i}\big)^{\beta-1}\cdot \\
& \cdot\big((\beta-\alpha)q^i-(\beta+\alpha-1)q^{-i}\big) \\
\triangleq & B^{\mathrm{pm}}_{q^i}(q^i,q^{-i}).
\end{aligned} \tag{3.51}
$$

If equality holds, this is the relevant boundary condition if player i *preempts* the opponent's myopic investment.

In fact, if player i considered that it might only be optimal in part of the state space not to intervene when the opponent invests at $\bar{X}^{-i}$, the transition to the preemption regime has to occur continuously for a proper reflection strategy. At those points, equality must hold in (3.51).

Since the myopic boundary is strictly increasing in the first argument, the preempting decision for player i with q^i fixed is likely to be monotone in q^{-i}. The smaller the latter, the smaller is the expected return from a preemption investment. On the other hand, we saw that player i optimally has to invest no later than at the *own* myopic boundary. From $\bar{p}(q^i,q^{-i}) \leq \bar{p}(q^{-i},q^i) \Leftrightarrow q^i \leq q^{-i}$ it is clear that player i can only await and tolerate the opponent's investment when having a higher capital stock. Consequently, suppose there exists

$$
\bar{q}(q^i) \triangleq \inf\{q\in\mathbb{R}_+ | \bar{X}^i(q^i,q) \leq \bar{p}(q,q^i)/P(q+q^i)\} \leq q^i.
$$

The corresponding V can only be continuously differentiable at the transition if $B_{q^i}(q^i,\bar{q}(q^i)) = B^{\mathrm{pm}}_{q^i}(q^i,\bar{q}(q^i))$. This enables us to consider the optimality condition (3.51), although we only know the partial derivative $B^{\mathrm{tm}}_{q^{-i}}$ for $q^{-i} < \bar{q}(q^i)$.

With

$$
B_{q^i}(q^i,q^{-i}) = B^{\mathrm{pm}}_{q^i}(q^i,\bar{q}(q^i)) - \int_{q^{-i}}^{\bar{q}(q^i)} B^{\mathrm{tm}}_{q^iq^{-i}}(q^i,q)\,dq,
$$

and

$$
B^{\mathrm{pm}}_{q^i}(q^i,q^{-i}) = B^{\mathrm{pm}}_{q^i}(q^i,\bar{q}(q^i)) - \int_{q^{-i}}^{\bar{q}(q^i)} B^{\mathrm{pm}}_{q^iq^{-i}}(q^i,q)\,dq,
$$

(3.51) is on $\{q^{-i} < \bar{q}(q^i)\}$ equivalent to

$$
\int_{q^{-i}}^{\bar{q}(q^i)} B^{\mathrm{pm}}_{q^iq^{-i}}(q^i,q)\,dq \leq \int_{q^{-i}}^{\bar{q}(q^i)} B^{\mathrm{tm}}_{q^iq^{-i}}(q^i,q)\,dq.
$$

One can show[9] that the relation between the two integrands is very clear cut,

$$B^{\mathrm{pm}}_{q^iq^{-i}}(q^i,q^{-i}) \le B^{\mathrm{tm}}_{q^iq^{-i}}(q^i,q^{-i}) \Leftrightarrow \frac{q^i}{q^{-i}} \ge \left(\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\alpha-1}{\alpha}}\right) \triangleq D > 1.$$

This implies that if $\bar{q}(q^i)$ really is the upper boundary of an interval for q^{-i}, in which $\bar{X}^i(q^i,q^{-i}) > \bar{X}^m(q^{-i},q^i)$, this can only be optimal if $\bar{q}(q^i) \le D^{-1}\cdot q^i$. Put differently, not preempting the myopic investor can only be optimal for player i on $\{q^i \ge D\cdot q^{-i}\}$, when having *sufficiently more* capital than the opponent.

By a similar argument, we can derive a complementary condition for optimally preempting a myopic investor. In a preemption region, the boundary condition for V is equality in (3.51), i.e.

$$B_{q^i}(q^i,q^{-i}) = B^{\mathrm{pm}}_{q^i}(q^i,q^{-i}).$$

Correspondingly, (3.50) turns into the (sufficient) optimality condition

$$B_{q^{-i}}(q^i,q^{-i}) \le B^{\mathrm{tm}}_{q^{-i}}(q^i,q^{-i}). \tag{3.52}$$

We know that player i optimally invests at the own myopic boundary if this is below the opponent's, i.e. whenever $q^i \le q^{-i}$. So suppose that for given q^{-i}, player i stops preempting the opponent at

$$\hat{q}(q^{-i}) \triangleq \inf\{q\in\mathbb{R}_+ | \bar{X}^i(q,q^{-i}) > \bar{p}(q^{-i},q)/P(q+q^{-i})\} \ge q^{-i},$$

and becomes passive. Analogous to the above, (3.52) is then equivalent to

$$\int_{q^i}^{\hat{q}(q^{-i})} B^{\mathrm{tm}}_{q^{-i}q^i}(q,q^{-i})\,dq \le \int_{q^i}^{\hat{q}(q^{-i})} B^{\mathrm{pm}}_{q^{-i}q^i}(q,q^{-i})\,dq$$

on $\{q^i < \hat{q}(q^{-i})\}$. It can only be satisfied if $\hat{q}(q^{-i}) \le D\cdot q^{-i}$, respectively on $\{q^i \le D\cdot q^{-i}\}$, when player i's capital is not *too much* larger.

9

$$\begin{aligned}
B^{\mathrm{tm}}_{q^{-i}q^i} =& \frac{\beta}{\alpha^2(\beta-1)}(p^*)^{-\beta}\left(q^i+q^{-i}\right)^{-\frac{\beta}{\alpha}-\beta-1}\left(q^i+\frac{\alpha-1}{\alpha}q^{-i}\right)^{\beta-2}\\
&\cdot\left(-\beta(q^i)^2+\left(\beta/\alpha+\alpha-1\right)q^iq^{-i}+(\alpha-1)(q^{-i})^2\right)\\
B^{\mathrm{pm}}_{q^iq^{-i}} =& \frac{\beta}{\alpha^2(\beta-1)}(p^*)^{-\beta}\left(q^i+q^{-i}\right)^{-\frac{\beta}{\alpha}-\beta-1}\left(q^i+\frac{\alpha-1}{\alpha}q^{-i}\right)^{\beta-2}\\
&\cdot\left(-(2\beta-1)(q^i)^2+\left(\beta/\alpha+\alpha+\beta-2\right)q^iq^{-i}+(\alpha-1)\frac{(\beta+\alpha-1)}{\alpha}(q^{-i})^2\right)\\
B^{\mathrm{tm}}_{q^{-i}q^i} \ge& B^{\mathrm{pm}}_{q^iq^{-i}} \Leftrightarrow (q^i)^2 - q^iq^{-i} - \frac{\alpha-1}{\alpha}(q^{-i})^2 \ge 0
\end{aligned}$$

3.6.3 Equilibrium failure

In Chapter 2 on open loop strategies, we observed that in any equilibrium the investment behaviour is as follows. The smaller firm invests myopically until having caught up to the other firm. If jumps occur, either only the smaller firm jumps to a capital level not exceeding the opponent's, or both jump to the same level of mutual best myopic replies. We will now determine Markovian strategies that generate exactly these capital processes in any subgame.

In the previous section, we have already exploited the fact that the smaller firm's myopic investment boundary is strictly lower than the larger firm's, $\bar{p}(q^i, q^{-i}) \leq \bar{p}(q^{-i}, q^i) \Leftrightarrow q^i \leq q^{-i}$. But $\bar{X}^m_{q^{-i}} \geq 0$ iff $\alpha \cdot q^{-i} \geq q^i$, so we have to adjust the strategies to ensure that the initial jumps are clearly resolved as in Subsection 3.4.1. Therefore, note that the *symmetric* myopic capital level is well defined by

$$q^s(x) \triangleq \sup\{q \in \mathbb{R}_+ | x \geq \frac{\bar{p}(q,q)}{P(2q)}\} \vee 0, \tag{3.53}$$

since $\bar{p}(q,q)/P(2q)$ is strictly increasing in q. Whenever there is a simultaneous jump, both firms have to settle at this value. The appropriate myopic Markov strategies are then given by

$$\tilde{\phi}^m(x, q^{-i}) \triangleq \phi^m(x, q^{-i} \vee q^s(x)) = \phi^m(x, q^{-i}) \wedge q^s(x) \tag{3.54}$$

and have the required properties $\tilde{\phi}^m_{q^{-i}} \leq 0$ and $\tilde{\phi}^m_x > 0$ for all $(x, q^{-i}) \in \mathbb{R}^2_+$. In fact, if both players use these strategies, there is a unique solution to the state equation. For any initial state $(x_0, q^1_0, q^2_0) \in \mathbb{R}^3_+$, $Q^i_0 = q^i_0 \vee \tilde{\phi}^m(x_0, q^{-i}_0)$ is uniquely determined for both players, independent of assigning a leader. The state is then outside the forbidden region, $Q^i_0 \geq \tilde{\phi}^m(x_0, Q^{-i}_0)$, while no player *jumped* across $q^s(x_0)$. However, $Q^1_0 \vee Q^2_0 \geq q^s(x_0)$, and either the smaller firm now expands using the myopic signal ϕ^m, or both are equally sized and simultaneously track $q^s(x)$.

In order to construct a smooth function $\tilde{V}^m$ when both players use $\tilde{\phi}^m$, we again have to find the coefficient $\tilde{B}^m$ which ensures that all boundary conditions are satisfied. For this, we have to distinguish the regimes in which player i's capital stock is smaller or larger than the opponent's.

Begin with $q^i \leq q^{-i}$. Then, player i invests at the myopic boundary and the related boundary condition is the same as (3.49). Thus, on $\{0 \leq q^i \leq q^{-i}\}$,

$$\tilde{B}^m(q^i, q^{-i}) = \tilde{B}^m(q^{-i}, q^{-i}) - \int_{q^i}^{q^{-i}} B^m_{q^i}(q, q^{-i})\, dq. \tag{3.55}$$

This implies for the partial derivatives at the regime boundary

$$\begin{aligned} \partial^-_{q^i}\tilde{B}^m(q,q) &= B^m_{q^i}(q,q) \qquad \text{and} \\ \partial^+_{q^{-i}}\tilde{B}^m(q,q) &= \partial_q\tilde{B}^m(q,q) - B^m_{q^i}(q,q). \end{aligned} \tag{3.56}$$

Now consider the other regime, when i tolerates the myopic investment by the opponent. The related boundary condition here is the same as (3.50). Thus, on $\{0 \le q^{-i} < q^i\}$,

$$\tilde{B}^m(q^i,q^{-i}) = \tilde{B}^m(q^i,q^i) - \int_{q^{-i}}^{q^i} B^{tm}_{q^{-i}}(q^i,q)\,dq. \tag{3.57}$$

This implies similarly for the partial derivatives at the regime boundary

$$\begin{aligned} \partial^-_{q^{-i}}\tilde{B}^m(q,q) &= B^{tm}_{q^{-i}}(q,q) \qquad \text{and} \\ \partial^+_{q^i}\tilde{B}^m(q,q) &= \partial_q\tilde{B}^m(q,q) - B^{tm}_{q^{-i}}(q,q). \end{aligned} \tag{3.58}$$

Compare (3.56) and (3.58). For $\tilde{B}^m$ to be continuously differentiable, we must have

$$\partial_q\tilde{B}^m(q,q) = B^m_{q^i}(q,q) + B^{tm}_{q^{-i}}(q,q).$$

This equation in one variable can be integrated to find that

$$\tilde{B}^m(q,q) = -\frac{1}{\beta-1}(p^*)^{-\beta}q^{-\frac{\beta}{\alpha}+1}2^{-\frac{\beta}{\alpha}-\beta}\Big(\frac{\beta+1-2\alpha}{\beta-\alpha}\Big)\Big(\frac{2\alpha-1}{\alpha}\Big)^{\beta-1}, \tag{3.59}$$

which together with (3.55) and (3.57) determines $\tilde{B}^m$ on all of $\mathbb{R}^2_+$. Now,

$$\tilde{V}^m(x,q^i,q^{-i}) \triangleq \tag{3.60}$$

$$\begin{cases} \frac{\beta}{\beta-1}q^i\frac{xP(q^i+q^{-i})}{p^*} + \tilde{B}^m(q^i,q^{-i})x^\beta & \text{if } \begin{cases} q^i \ge \tilde{\phi}^m(x,q^{-i}) \\ q^{-i} \ge \tilde{\phi}^m(x,q^i) \end{cases} \\ \tilde{V}^m(x,\tilde{\phi}^m(x,q^{-i}),q^{-i}) - \tilde{\phi}^m(x,q^{-i}) + q^i & \text{if } \begin{cases} q^i < \tilde{\phi}^m(x,q^{-i}) \\ q^{-i} \ge \tilde{\phi}^m(x,q^i) \end{cases} \\ \tilde{V}^m(x,q^i,\tilde{\phi}^m(x,q^i)) & \text{if } \begin{cases} q^i \ge \tilde{\phi}^m(x,q^{-i}) \\ q^{-i} < \tilde{\phi}^m(x,q^i) \end{cases} \\ \tilde{V}^m(x,q^s(x),q^s(x)) - q^s(x) + q^i & \text{if } \begin{cases} q^i < \tilde{\phi}^m(x,q^{-i}) \\ q^{-i} < \tilde{\phi}^m(x,q^i) \end{cases} \end{cases}$$

satisfies the first set of sufficient conditions in Theorem 3.4. Note that in particular, because we have matched (3.56) and (3.58),

$$\begin{aligned} \tilde{V}^m_{q^i}(\frac{\bar{p}(q,q)}{P(2q)}, q, q) &= 1 \\ \tilde{V}^m_{q^{-i}}(\frac{\bar{p}(q,q)}{P(2q)}, q, q) &= 0 \end{aligned} \tag{3.61}$$

when simultaneous investment occurs.

Proposition 3.8. *Define* $\tilde{X}^m(q^i, q^{-i}) \triangleq \sup\{x \in \mathbb{R} | q^i \geq \tilde{\phi}^m(x, q^{-i})\}$ *and let* (Q^1, Q^2) *be the solution of Problem 3.3 if* $\bar{X}^1 = \bar{X}^2 = \tilde{X}^m$*, with initial state* $(x_0, q_0^1, q_0^2) \in \mathbb{R}^3_+$*. Then,*

$$\tilde{V}^m(x_0, q_0^i, q_0^{-i}) = J(Q^i, Q^{-i}).$$

Yet, $(\tilde{\phi}^m, \tilde{\phi}^m)$ *are not a Markov perfect equilibrium.*

$\tilde{V}^m$ satisfies the hypothesis of Theorem 3.4 by a very similar argument as in the proof of Proposition 3.7. However, $\tilde{V}^m$ does not satisfy the necessary conditions for optimality, because the strategies allow the smaller opponent to catch up gradually. We saw in the previous subsection that this cannot be optimal on $\{q^i < D \cdot q^{-i}\}$, with D strictly greater than one. Once the capital levels are within this distance, tolerating further investment is suboptimal. Consequently, the Markovian strategies generating the open loop equilibrium processes are *not* a Markov perfect equilibrium.

3.7 Collusive equilibria

The previous section tells us that there exists no Markov perfect equilibrium in which the players have different investment boundaries whenever their capital stocks differ. In these cases, the smaller firm necessarily invests myopically, but the larger firm has a strict incentive to preempt before the two have equal capital stocks. Consequently, in any equilibrium, the players must use the same investment boundary over part of the state space, also for some heterogeneous capital levels. An example is of course the earlier obtained Bertrand equilibrium with a shared price trigger.

On the other hand, it is clear from the equilibrium definition that at least one of the firms must be indifferent to invest at the mutual boundary. We can express this requirement in terms of the value function V.

So, suppose the players use the same reflection boundary $\bar{X}$ over part of the state space. At the respective own reflection boundary, V always has to satisfy the boundary condition (3.38), $V_{q^i}(\bar{X}, q^i, q^{-i}) = 1$. Expanded, it is equivalent to

$$B_{q^i} = \bar{X}^{-\beta} - \tfrac{\beta}{\beta-1}(p^*)^{-1}\Big(\tfrac{\alpha-1}{\alpha}q^i + q^{-i}\Big)\big(q^i + q^{-i}\big)^{-\frac{1}{\alpha}-1}\bar{X}^{1-\beta} \triangleq B^{\mathrm{p}}_{q^i}. \tag{3.62}$$

We denote the right hand side by $B^{\mathrm{p}}_{q^i}$, since this is the condition associated with investment, resp. preemption.

The intended indifference condition is then

$$V_{q^{-i}}(\bar{X}, q^i, q^{-i}) = 0, \tag{3.63}$$

i.e. the value of the strategy is not affected if the opponent invests at the same boundary. In the previous case, myopic investment, these two conditions only held simultaneously for each player when both have equal capital stocks, cf. (3.61). We can also expand the latter condition to

$$B_{q^{-i}} = \tfrac{1}{\beta-1}(p^*)^{-1}q^i\tfrac{\beta}{\alpha}\big(q^i + q^{-i}\big)^{-\frac{1}{\alpha}-1}\bar{X}^{1-\beta} \triangleq B^{\mathrm{t}}_{q^{-i}}. \tag{3.64}$$

Similarly as before, $B^{\mathrm{t}}_{q^{-i}}$ stands for tolerating investment.

Now, we can turn the approach taken so far around and try to identify a reflection boundary $\bar{X}$ that is consistent with (3.62) and (3.64) simultaneously.

Specifically, since $B_{q^iq^{-i}} = B_{q^{-i}q^i}$ must hold, we obtain[10] a partial differential equation for $\bar{X}$:

$$\bar{X}^{-1}p^*(q^i + q^{-i})^{\frac{1}{\alpha}+1}\bar{X}_{q^{-i}} - \big(\tfrac{\alpha-1}{\alpha}q^i + q^{-i}\big)\bar{X}_{q^{-i}} - \tfrac{1}{\alpha}q^i\bar{X}_{q^i} = 0. \tag{3.65}$$

If we require that both player be indifferent at the sought reflection boundary, it is easy to see[11] that the only solution of this PDE is the Bertrand price trigger $\bar{X}(q^i, q^{-i}) = p^*(q^i + q^{-i})^{\frac{1}{\alpha}}$.

However, it is sufficient for an equilibrium that only one player is indifferent, say who has more capital installed. One can narrow down further solutions of (3.65) by reflecting that a firm might only care about the observed price in the investment decision, and that the price of indifference

[10]

$$\begin{aligned}
\partial_{q^i}B^{\mathrm{t}}_{q^{-i}} =& \tfrac{1}{\beta-1}(p^*)^{-1}\tfrac{\beta}{\alpha}\bar{X}^{-\beta}(q^i + q^{-i})^{-\frac{1}{\alpha}-2}\Big(\big(-\tfrac{1}{\alpha}q^i + q^{-i}\big)\bar{X} - (\beta-1)q^i(q^i + q^{-i})\bar{X}_{q^i}\Big)\\
\partial_{q^{-i}}B^{\mathrm{p}}_{q^i} =& -\beta\bar{X}^{-\beta-1}\bar{X}_{q^{-i}} - \tfrac{\beta}{\beta-1}(p^*)^{-1}\bar{X}^{-\beta}\big(q^i + q^{-i}\big)^{-\frac{1}{\alpha}-2}\cdot\\
&\cdot\Big(-\tfrac{1}{\alpha}\big(-\tfrac{1}{\alpha}q^i + q^{-i}\big)\bar{X} - (\beta-1)\big(\tfrac{\alpha-1}{\alpha}q^i + q^{-i}\big)\big(q^i + q^{-i}\big)\bar{X}_{q^{-i}}\Big)
\end{aligned}$$

[11]Switch roles by swapping $i, -i$ and take the difference of the two equations.

depends only on the own installed capital. Correspondingly, we look for solutions of the functional form

$$\bar{X}(q^i, q^{-i}) = f(q^i)(q^i + q^{-i})^{\frac{1}{\alpha}}. \tag{3.66}$$

In this case, we calculate

$$\bar{X}_{q^i} = (q^i + q^{-i})^{\frac{1}{\alpha}-1}\big((q^i + q^{-i})f' + \tfrac{1}{\alpha}f\big) \qquad \text{and}$$
$$\bar{X}_{q^{-i}} = (q^i + q^{-i})^{\frac{1}{\alpha}-1}\tfrac{1}{\alpha}f.$$

(3.65) now dramatically simplifies to

$$f + q^i f' = p^*,$$

the general solution of which is

$$f(q^i) = p^* + c \cdot (q^i)^{-1}. \tag{3.67}$$

Since investment below the Bertrand price cannot be optimal, we only admit constants $c \in [0, \infty)$.

With all candidates other than the Bertrand equilibrium, i.e. whenever $c > 0$, indeed only player i can be indifferent. The question which player to make indifferent is easily answered, if we check whether (3.66) defines proper reflection strategies. In fact, $\bar{X}_{q^{-i}}$ is nonnegative since f is, but $\bar{X}_{q^i} > 0$ iff

$$(q^i + q^{-i})f' + \tfrac{1}{\alpha}f > 0$$

$$\Leftrightarrow p^*(q^i)^2 - (\alpha - 1)cq^i - \alpha c q^{-i} > 0.$$

This only holds for a wide range of capital levels if $q^i \geq q^{-i}$ and is then implied by

$$q^i > c\frac{2\alpha - 1}{p^*}. \tag{3.68}$$

Consequently, if we define the symmetric investment boundary

$$\bar{X}^c(q^i, q^{-i}) \triangleq \big(p^* + c \cdot (q^i \vee q^{-i})^{-1}\big)(q^i + q^{-i})^{\frac{1}{\alpha}}, \tag{3.69}$$

it is a proper reflection boundary on $\big(c\frac{2\alpha-1}{p^*}, \infty\big)^2$ with $\bar{X}^c_{q^i} > 0$ and $\bar{X}^c_{q^{-i}} > 0$.

Furthermore, we can apply Theorem 3.4 to show that $\bar{X}^c$ is a mutual best reply.

Proposition 3.9. *Let* $\bar{X}^1 = \bar{X}^2 = \bar{X}^c$ *given by* (3.69) *with fixed* $c \in \mathbb{R}_+$*, and define* $\phi^c(x, q^{-i}) \triangleq \sup\{q \in \mathbb{R}_+ | x \geq \bar{X}^c(q, q^{-i})\} \vee 0$*. Further fix* $(x_0, q_0^1, q_0^2) \in \mathbb{R}_+ \times \big(c\frac{2\alpha-1}{p^*}, \infty\big)^2$ *and set* $Q^i = \big(q_0^i \vee \sup_{0\leq s\leq t} \phi^c(X_s, q_0^{-i})\big)_{t\geq 0}$ *and* $Q^{-i} \equiv q_0^{-i}$.

Define for any $(x, q^1, q^2) \in \mathbb{R}_+ \times \big(c\frac{2\alpha-1}{p^*}, \infty\big)^2$

$$V^c(x, q^i, q^{-i}) \triangleq \begin{cases} \frac{\beta}{\beta-1} q^i \frac{xP(q^i+q^{-i})}{p^*} + B^c(q^i, q^{-i})x^\beta & \text{if } x \leq \bar{X}^c(q^i, q^{-i}) \\ V^c(x, \phi^c(x, q^{-i}), q^{-i}) - \phi^c(x, q^{-i}) + q^i & \text{else,} \end{cases}$$

where

$$B^c(q^i, q^{-i}) \triangleq -\int_{q^i}^{\infty} B^c_{q^i}(q, q^{-i})\, dq, \tag{3.70}$$

and

$$B^c_{q^i}(q^i, q^{-i}) \triangleq \Big(1 - \tfrac{\beta}{\beta-1}(p^*)^{-1}\big(\tfrac{\alpha-1}{\alpha}q^i + q^{-i}\big)\big(q^i + q^{-i}\big)^{-\frac{1}{\alpha}-1}\bar{X}^c\Big)\big(\bar{X}^c\big)^{-\beta}. \tag{3.71}$$

Then, $V^c(x_0, q_0^i, q_0^{-i}) = J(Q^i, Q^{-i}) = V^*(x_0, q_0^i, q_0^{-i})$.

Proof. Since $\bar{X}^1 = \bar{X}^2$, it is easy to see that Q^i, Q^{-i} solve Problem 3.3 for initial state (x_0, q_0^1, q_0^2). It remains to show that V^c satisfies the hypothesis and sufficient conditions of Theorem 3.4.

Note that B^c in (3.70) is well defined, since for fixed q^{-i}, $B^c_{q^i}$ goes to zero at speed $(q^i)^{-\frac{\beta}{\alpha}}$ when q^i gets large. One can indeed show by repeating the proof of Lemma 3.5 that V^c also satisfies the boundedness condition, just using different bounds for B^c and the reflection price:

$$\frac{xP(q^i \vee \phi^c(x, q^{-i}) + q^{-i})}{p^*} \leq \frac{2\alpha}{2\alpha - 1}$$

and

$$-\frac{\beta}{\beta-1}\Big(\frac{P(q^i + q^{-i})}{p^*}\Big)^\beta \leq B^c_{q^i} \leq \Big(\frac{P(q^i + q^{-i})}{p^*}\Big)^\beta$$

$$\Rightarrow \Big|B^c(q^i \vee \phi^c, q^{-i})x^\beta\Big| \leq \frac{\alpha}{\beta - \alpha}\frac{\beta}{\beta - 1}\big(q^i \vee \phi^c + q^{-i}\big)\Big(\frac{2\alpha}{2\alpha - 1}\Big)^\beta.$$

Note finally that the process Q^{-i}, which results from the Markovian strategy ϕ^c given arbitrary $Q^i \in \mathscr{A}(0)$, is dominated by the Bertrand quantity and thus has finite investment cost.

All sufficient conditions are actually satisfied by construction, except for the two sufficient conditions relating to the partial derivatives of V^c. The easier one, $V^c_{q^i} \leq 1$ is equivalent to (3.41) as we have shown. Suppose wlog $q^{-i} \geq q^i$. Then,

$$p^* + c(q^{-i})^{-1} \leq \bar{p}(q^i, q^{-i})$$

is satisfied for all $q^{-i} \geq q^i$ iff $q^i \geq c\frac{2\alpha-1}{p^*}$, the restriction we already encountered. Note that $\bar{p}(q^i, q^{-i}) \leq \bar{p}(q^{-i}, q^i)$.

The other sufficient condition is $V_{q^{-i}}(\bar{X}^c, q^i, q^{-i}) \leq 0$. It is equivalent to $B^c_{q^{-i}} \leq B^{\mathrm{t}}_{q^{-i}}$, cf. (3.64) using $\bar{X}^c$. For the player with larger capital stock, it holds with equality by construction. Still supposing $q^{-i} \geq q^i$, we can thus also write for player i

$$B^c_{q^{-i}}(q^i, q^{-i}) = B^{\mathrm{t}}_{q^{-i}}(q^{-i}, q^{-i}) - \int_{q^i}^{q^{-i}} B^c_{q^i q^{-i}}(q, q^{-i})\, dq,$$

and compare to

$$B^{\mathrm{t}}_{q^{-i}}(q^i, q^{-i}) = B^{\mathrm{t}}_{q^{-i}}(q^{-i}, q^{-i}) - \int_{q^i}^{q^{-i}} B^{\mathrm{t}}_{q^{-i} q^i}(q, q^{-i})\, dq.$$

For concreteness,

$$B^{\mathrm{t}}_{q^{-i}}(q^i, q^{-i}) = \tfrac{1}{\beta-1}(p^*)^{-1} q^i \tfrac{\beta}{\alpha}\big(q^i + q^{-i}\big)^{-\frac{\beta}{\alpha}-1}\big(p^* + c(q^{-i})^{-1}\big)^{1-\beta}$$

and

$$\begin{aligned} B^c_{q^i}(q^i, q^{-i}) =& \tfrac{1}{\beta-1}(p^*)^{-1}\big(q^i + q^{-i}\big)^{-\frac{\beta}{\alpha}-1}\big(p^* + c(q^{-i})^{-1}\big)^{-\beta} \cdot \\ & \cdot \Big(p^*(\beta-1)(q^i + q^{-i}) - \beta\big(\tfrac{\alpha-1}{\alpha} q^i + q^{-i}\big)\big(p^* + c(q^{-i})^{-1}\big)\Big). \end{aligned}$$

Then, a lengthy calculation yields that

$$\begin{aligned} & B^c_{q^i q^{-i}} \geq B^{\mathrm{t}}_{q^{-i} q^i} \\ \Leftrightarrow\ & q^{-i} \geq c\frac{\alpha-1}{p^*}. \end{aligned}$$

This is a weaker restriction than already imposed and thus $B^c_{q^{-i}} \leq B^{\mathrm{t}}_{q^{-i}}$ for all $q^{-i} \geq q^i \geq c\frac{2\alpha-1}{p^*}$. Consequently, $V_{q^{-i}}(\bar{X}^c, q^i, q^{-i}) \leq 0$. ο.ε.δ.

In the proposition, we selected the processes Q^i and Q^{-i} by solving Problem 3.3 and determined the solution to the verification problems (3.6). The outcome depends on which player is the leader, because there is full preemption. As the involved investment occurs at higher prices than Bertrand, the payoffs to the players also differ. Nevertheless, (ϕ^c, ϕ^c) are a Markov perfect equilibrium, because we can select feasible capital stock processes that are a Pareto improvement compared to full preemption as in the proposition. We exploit the indifference of the respective larger firm.

Theorem 3.10. *For any $c \in \mathbb{R}_+$, (ϕ^c, ϕ^c) as defined in Proposition 3.9 is a Markov perfect equilibrium for initial capital levels $(q_0^1, q_0^2) \in \big(c\frac{2\alpha-1}{p^*}, \infty\big)^2$.*

Proof. Introduce the symmetric capital levels which are just on the investment boundary by

$$Q^s(x) \triangleq \sup\{q \in \mathbb{R}_+ |\, x \geq \bar{X}^c(q,q)\} \vee 0, \tag{3.72}$$

which is well defined for $x \in \mathbb{R}_+$ because

$$\partial_q \bar{X}^c(q,q) = 2^{\frac{1}{\alpha}} q^{\frac{1}{\alpha}-1}\Big(\tfrac{1}{\alpha}p^* - \tfrac{\alpha-1}{\alpha}cq^{-1}\Big) \tag{3.73}$$

is strictly positive for all $q > c\frac{\alpha-1}{p^*}$.

Q^s will generate the capital processes where the firms grow jointly in equilibrium. We have to show that this is not to the disadvantage of any firm.

Begin with an initial state in the forbidden region, i.e. $x_0 > \bar{X}^c(q^i, q^{-i})$, which requires a jump.

On $\{x > \bar{X}^c(q^i, q^{-i})\}$, player i jumps when being the leader and by our definition,

$$V^c(x, q^i, q^{-i}) = V^c(x, \phi^c(x, q^{-i}), q^{-i}) - \phi^c(x, q^{-i}) + q^i.$$

Consequently, in this region, $V^c_{q^i} = 1$ and

$$\begin{aligned} V^c_{q^{-i}}(x, q^i, q^{-i}) &= V^c_{q^{-i}}(x, \phi^c(x, q^{-i}), q^{-i}) \\ &= 0 \qquad \text{if } \phi^c(x, q^{-i}) \geq q^{-i}, \end{aligned} \tag{3.74}$$

where the latter holds by construction, cf. (3.63).

Now we dictate a different investment for player i in two parts of the forbidden region.

If, on the one hand, $\{\bar{X}^c(q^i, q^{-i}) < x \leq \bar{X}^c(q^i, q^i)\}$, this is equivalent to $Q^s(x) \leq q^i < \phi^c(x, q^{-i})$.

The monotonicity of $\bar{X}^c$ then implies $q^i \geq \phi^c(x, q^i)$. Further, by symmetry of $\bar{X}^c$, we always have

$$q^i = \phi^c(x, \phi^c(x, q^i)).$$

So, for all $q \in [q^{-i}, \phi^c(x, q^i)]$, (3.74) holds, including the second line. It follows

$$V^c(x, q^i, q^{-i}) = V^c(x, q^i, \phi^c(x, q^i)),$$

i.e. player i is indifferent if we let $-i$ jump to bring the state onto the boundary of the forbidden region.

If, on the other hand, $\{x \geq \bar{X}^c(q^i, q^i) \vee \bar{X}^c(q^{-i}, q^{-i})\}$, this is equivalent to $q^i \vee q^{-i} \leq Q^s(x)$.

This time, the monotonicity of $\bar{X}^c$ implies that for all $q^{-i} \leq Q^s(x)$, $\phi^c(x, q^{-i}) \geq q^{-i}$. We apply the second line of (3.74) once more to obtain

$$V^c(x, q^i, q^{-i}) = V^c(x, q^i, Q^s(x)) = V^c(x, Q^s(x), Q^s(x)) - Q^s(x) + q^i$$

in this region. Thus, player i is indifferent if we allow both to jump to $Q^s(x)$.

Now consider reflection investment at the boundary. Then, by definition, $V^c_{q^i}(\bar{X}^c, q^i, q^{-i}) = 1$. We also constructed $\bar{X}^c$ such that a player is indifferent to invest at the boundary if the opponent does not have strictly more capital installed, i.e. if $q^i \geq q^{-i}$, $V^c_{q^{-i}}(\bar{X}^c, q^i, q^{-i}) = 0$. Consequently, we can choose capital processes such that only the smaller firm invests, or both invest simultaneously.

Such processes are indeed feasible.

Select $Q^{c,i} \in \mathscr{A}(q^i_0)$ that satisfies for all $t \in [0, \infty)$ $\mathbf{P}$-a.s.,

$$Q^{c,i}_t = q^i_0 \vee \Big(\mathbf{1}_{\{q^i_0 \vee \sup_{0\leq s\leq t} \phi^c(X_s, q^{-i}_0) < q^{-i}_0\}} \sup_{0\leq s\leq t} \phi^c(X_s, q^{-i}_0)$$
$$+ \mathbf{1}_{\{q^i_0 \vee \sup_{0\leq s\leq t} \phi^c(X_s, q^{-i}_0) \geq q^{-i}_0\}} \sup_{0\leq s\leq t} Q^s(X_s) \Big).$$

The larger firm starts tracking $Q^s(X_t)$ from the beginning and the smaller firm, i.e. with $q^i_0 < q^{-i}_0$, switches when

$$\phi^c(X_s, q^{-i}_0) \geq q^{-i}_0 \Leftrightarrow X_s \geq \bar{X}^c(q^{-i}_0, q^{-i}_0)$$
$$\Leftrightarrow Q^s(X_s) \geq q^{-i}_0.$$

This also implies that the processes solve (3.7):

$$Q^{c,i}_t = q^i_0 \vee \sup_{0\leq s\leq t} \phi^c(X_s, Q^{c,-i}_s).$$

Now we can perform the estimation in the proof of Theorem 3.4 with equality holding, to find that

$$V^c(x_0, q^i_0, q^{-i}_0) = J(Q^{c,i}, Q^{c,-i}) = J(Q^i, Q^{-i}).$$

ο.ε.δ.

Theorem 3.10 answers the "open question" (K. Back, [4]) whether there exist any other subgame perfect equilibria of the game with reflection strategies besides the Bertrand equilibrium.

Our additional class of equilibria is driven by the fact that preemptive investment of the larger firm lowers the *price* at which the smaller firm invests. The smaller firm does not face this externality and it also has the greater local investment incentive, based on marginal revenue. Thus, it is able to set a *dynamic* investment price boundary above Bertrand which leaves the opponent indifferent. Note that the investment price boundary has to decline gradually in equilibrium, since we saw in Section 3.5 that preemption is otherwise the dominant strategy.

With the present example we falsified the frequent conjecture that preemption concerns completely eliminate option values under arbitrarily divisible investment. The friction implied by uncertain returns and the decreasing marginal revenue effect do enable more collusive outcomes with quantities less than those enforcing the Bertrand price. We consequently opt for calling these *collusive* equilibria.

3.8 Conclusion

In this work we focused on the strategic effect of capital commitment with arbitrarily divisible, irreversible investment under dynamic uncertainty. We established frameworks for games with two crucially different classes of strategies, open loop vs. closed loop.

When there is no *explicit* feedback between the opponents' investment decisions, each player chooses an optimal capital stock process by solving a stochastic control problem of the monotone follower type. Our presentation is — owing to its generality — unambiguous in the sense that we operate with the capital processes directly without resorting to trigger functions. By this approach we can isolate the individual investment incentives in the interrelated optimization problems. Specifically, investment always occurs following an opportunity-cost-of-delay principle, which results from the classical real option effect. The value of the option to delay investment is reduced by competition because of the standard Cournot effect of diminishing investment externalities on own capital.

Consequently, similar existence conditions for equilibrium arise as in a one-shot Cournot game, independent of the underlying uncertainty model. The prevailing class of equilibria is that in which capital stocks are ultimately equalized by investment of the currently smallest firms only. For an arbitrary number of players with heterogeneous initial capital levels, these equilibria can be determined by solving a single monotone follower problem. At each state there is a well defined capital level which is the myopic best reply to the current capital stocks of the opponents, and all firms invest up to this "base capacity" if it exceeds their respective stock. We conclude that without explicit feedback, the option to delay investment is generally too valuable to admit preemptive investment.

As a natural conjecture, the opposite is true when closed loop strategies are modelled and one considers subgame perfect equilibria. In order to be able to verify formally whether ideal circumstances for preemption necessarily eliminate all option values, we propose a framework for the game between two players with closed loop strategies, too. Using a state space representation, we enable the players to make investment decisions by choosing desired capital levels, which are relevant only when exceeding installed capital.

This approach is inspired by the signal processes prevailing in the earlier monotone follower problems and allows to avoid artificial bounds on the speed of capital adjustment — so singular control is admissible —, while ensuring consistency across subgames. Since one encounters the typical difficulty of continuous-time games, that a priori natural strategies do not uniquely determine the course of the state, particular attention has to be paid to an appropriate equilibrium concept. We introduce a strong optimality notion by requiring that there exists a solution in capital processes, such that no player has an incentive to choose any other control process at any state, even if unilateral perfect preemption is made feasible by hypothesis.

Our equilibrium definition is complemented by establishing a verification theorem. It serves to solve the optimal control problems arising in equilibrium verification in the presence of Markovian strategies and geometric Brownian motion as exogenous shock process. We subsequently apply the theorem to the example in which firms face an inverse demand with constant elasticity, in order to derive Markov perfect equilibria.

The simplest instance is the arguably expected equilibrium, showing perfectly competitive investment. It results when firms use a constant price to trigger investment. Then, implied by stationarity, the decision whether to preempt at the threshold is always the same. By repeated investment opportunities and rent equalization, firms are always indifferent when investing and make zero expected profits.

The observation that firms invest (or threaten to) at an identical threshold is an important aspect of any equilibrium. Whenever the thresholds differ, we can show that one firm necessarily invests myopically to behave optimally. Then, it can only be optimal for the opponent to refrain from preemption when having sufficiently more capital already installed. On the other hand, when having too much capital, preemption is definitely unprofitable.

Consequently, there is potential for equilibria without full preemption. While those Markovian strategies generating the open loop equilibrium processes for any starting state are not eligible, we identify a particular class of Markov perfect equilibria with positive option values.

As the crucial component, a dynamic output good price to trigger investment allows for collusive behaviour. The respective larger firm is kept indifferent in these equilibria when the opponent invests. By refraining from preemption, the larger firm allows both players to obtain the highest possible returns, given the equilibrium Markovian strategies, since any investment by the larger firm reduces the (common) investment boundary. Neither player has an incentive to deviate from the Pareto optimal solution in capital processes. When they have equal capital stocks, simultaneous investment occurs. We observe collusion similar to the simultaneous investment equilibria in the real option exercise games discussed in the introduction.

As a future research question, it would be interesting to enquire in detail the relation of the identified equilibria to those with larger investment sizes.

But there are more general topics to address, since we proposed the first framework for games with singular control and feedback strategies in the literature. Besides broadening the present analysis, possibly to different applications, a fundamental question is for instance whether our equilibria can be represented as limits of discrete time games with frequent actions, in order to establish a more solid foundation in (classical) game theory.

Appendix

Lemma 3.11. *Let* $(q^i, q^{-i}) \in \mathbb{R}^2_+$ *and* $\bar{X}^{-i} \geq p^*/P(q^i + q^{-i})$ *satisfying* (3.12) *be given. Suppose* $Q^i \in \mathscr{A}(q^i)$ *and* $Q^{-i} = q^{-i} \vee \big(\sup_{0\leq s\leq t} \phi^{-i}(X_s, Q^i_s)\big)_{t\geq 0} \in \mathscr{A}(q^{-i})$.

Define the cumulative Bertrand quantity

$$Q^B \triangleq q^i \vee \Big(\sup_{0\leq s\leq t} \big(X_s/p^*\big)^\alpha - q^{-i}\Big)_{t\geq 0},$$

as well as the capped capital process

$$\hat{Q}^i \triangleq Q^i \wedge Q^B$$

and the resulting

$$\hat{Q}^{-i} \triangleq q^{-i} \vee \big(\sup_{0\leq s\leq t} \phi^{-i}(X_s, \hat{Q}^i_s)\big)_{t\geq 0}.$$

Then,

$$J(Q^i, Q^{-i}) - J(\hat{Q}^i, \hat{Q}^{-i}) \leq 0.$$

Proof. Define the stopping times

$$\tau^B \triangleq \inf\{t \geq 0 | Q^i_t > Q^B_t\} \quad \text{and} \quad \hat{\tau}^B \triangleq \inf\{t \geq \tau^B | Q^B_t \geq Q^i_t\}$$

and note that $Q^{-i}_t = \hat{Q}^{-i}_t = Q^{-i}_{\tau^B}$ for $t \in [\tau^B, \hat{\tau}^B]$.

This allows to use Fubini's theorem as in Chapter 2 to obtain

$$\begin{aligned}
&\mathbf{E}\Big[\int_{\tau^B}^{\hat{\tau}^B} e^{-rt}\Pi(X_t, Q^i_t, Q^{-i}_t)\,dt - \int_{\tau^B}^{\hat{\tau}^B} e^{-rt}\,dQ^i_t\Big] \\
&-\mathbf{E}\Big[\int_{\tau^B}^{\hat{\tau}^B} e^{-rt}\Pi(X_t, Q^B_t, \hat{Q}^{-i}_t)\,dt - \int_{\tau^B}^{\hat{\tau}^B} e^{-rt}\,dQ^B_t\Big] \\
&= \int_0^\infty \mathbf{E}\Big[\mathbf{1}_{\{Q^i_{\tau^B} \leq l \leq Q^i_{\hat{\tau}^B}\}} \int_{\tau^{Q^i}(l)}^{\tau^{Q^B}(l)} e^{-rt}\big(\Pi_{q^i}(X_t, l, Q^{-i}_{\tau^B}) - r\big)\,dt\Big]\,dl,
\end{aligned}$$

where $\tau^{Q^i}(l) \triangleq \inf\{t \geq 0 | Q^i_t \geq l\}$ and $\tau^{Q^B}(l)$ analogously.

Now note that in the random interval $\left[\tau^{Q^i}(l), \tau^{Q^B}(l)\right]$,

$$\Pi_{q^i}(X_t, l, Q^{-i}_{\tau^B}) \leq X_t P(l + Q^{-i}_{\tau^B}) \leq p^* X_t,$$

so that for any l, stopping at $\tau^{Q^i}(l)$ immediately is optimal for maximizing the expectation with respect to stopping times $\tau \geq \tau^{Q^i}(l)$. Consequently, for any l, the expectation is nonpositive and this implies the same for the payoff difference over $[\tau^B, \hat{\tau}^B]$. ο.ε.δ.

Proof of Lemma 3.5

We want to prove that

$$\lim_{T\to\infty} \mathbf{E}\left[e^{-rT} V(X_T, Q^i_T, q^{-i})\right] = 0$$

for arbitrary $Q^i \in \mathscr{A}(0)$ with finite cost.
The value function candidate is given by

$$V(x, q^i, q^{-i}) = \begin{cases} \frac{\beta}{\beta-1} q^i \frac{xP(q^i+q^{-i})}{p^*} + B(q^i, q^{-i}) x^\beta & \text{if } q^i \geq \phi^i(x, q^{-i}) \\ V(x, \phi^i, q^{-i}) - \phi^i + q^i & \text{else.} \end{cases}$$

We hide the arguments of ϕ^i in the following.

First, we derive joint bounds for all terms but Bx^β. Since the reflection price is not less than the Bertrand price,

$$xP(\phi^i + q^{-i}) \geq p^*$$

$$\Rightarrow \frac{\beta}{\beta-1} \phi^i \frac{xP(\phi^i + q^{-i})}{p^*} - \phi^i \geq 0.$$

For an upper bound, note that the price term is in both cases bounded by a constant:

$$\frac{xP(q^i \vee \phi^i + q^{-i})}{p^*} \leq \frac{p^{-i}}{p^*}. \tag{3.75}$$

Thus, we can estimate both cases of V simultaneously by

$$V(x, q^i, q^{-i}) \in \left(0, \frac{\beta}{\beta-1} \frac{p^{-i}}{p^*} \left(q^i \vee \phi^i\right)\right) + B(q^i \vee \phi^i, q^{-i}) x^\beta \tag{3.76}$$

The remaining term can also be estimated by using the price bound (3.75). In the present case:

$$\left|B^{p^{-i}}(q^i \vee \phi^i, q^{-i}) x^\beta\right| \leq \frac{\alpha}{\beta-\alpha} \left| (q^i \vee \phi^i) \left(\frac{p^{-i}(\alpha-1)}{(r-\mu)\alpha} - 1 \right) + q^{-i} \left(\frac{p^{-i}}{p^*} - 1 \right) \right|$$

Thus, V is bounded by a linear function of $q^i \vee \phi^i$.

Since ϕ^i never pushes the price below the Bertrand price, the Bertrand quantity (particularly neglecting competitive output) is an upper bound for it:

$$xP(\phi^i + q^{-i}) \geq p^* \Rightarrow \phi^i \leq \left(\frac{x}{p^*}\right)^\alpha - q^{-i} \leq \left(\frac{x}{p^*}\right)^\alpha.$$

This proves the first claim, since $\lim_{T\to\infty} \mathbf{E}\left[e^{-rT} X_T^\alpha\right] = 0$ for $\alpha < \beta$ and $\lim_{T\to\infty} \mathbf{E}\left[e^{-rT} Q_T^i\right] = 0$ by hypothesis.

For the second claim, note that the capital process resulting from the Bertrand reflection strategy $\bar{X}^i = p^*/P(q^i + q^{-i})$ is $Q_t^i = q_0^i \vee (X_t^*/p^*)^\alpha - q_0^{-i}$ with $X_t^* \triangleq \sup_{0\leq s\leq t} X_s$. Consequently, the investment cost is bounded if the following holds

$$\mathbf{E}\left[\int_0^\infty e^{-rt}(X_t^*)^\alpha \, dt\right] = \frac{\beta}{\beta - \alpha} \in \mathbb{R}_+ \Leftrightarrow \alpha < \beta, \tag{3.77}$$

cf. [31]; the left hand side equals

$$\mathbf{E}\left[(X_{\tau(r)}^*)^\alpha\right] = \frac{\Psi^{\alpha Y}(r)}{\Psi^{\alpha Y}(r) - 1},$$

where $\tau(r)$ is an independent, exponentially distributed time, and $\Psi^{\alpha Y}(r)$ is the Laplace exponent of the process αY at r. In our case, $X_t^\alpha = x_0^\alpha e^{\alpha Y_t}$, i.e. $Y_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \Rightarrow \Psi^{\alpha Y}(r) = \frac{\beta}{\alpha}$.

Proof of Theorem 2.15

Let $l', l \in \mathbb{R}_+$, with $l' > l$ and $q^{k-1} \leq l \leq q^k$, $q^{k'-1} \leq l' \leq q^{k'}$. Then,

$$\begin{aligned}
&\pi(t, l', \sum_{j>1} q^j \vee l') - \pi(t, l, \sum_{j>1} q^j \vee l) \\
&= \pi(t, l', (k'-1)l' + \sum_{j\geq k'} q^j) - \pi(t, l, (k-1)l + \sum_{j\geq k} q^j) \\
&= \pi(t, l', (k'-1)l' + \sum_{j\geq k'} q^j) - \pi(t, q^{k'-1}, (k'-1)q^{k'-1} + \sum_{j\geq k'} q^j) \\
&+ \pi(t, q^{k'-1}, (k'-2)q^{k'-1} + \sum_{j\geq k'-1} q^j) - \pi(t, q^{k'-2}, (k'-2)q^{k'-2} + \sum_{j\geq k'-1} q^j) \\
&\vdots \\
&+ \pi(t, q^k, (k-1)q^k + \sum_{j\geq k} q^j) - \pi(t, l, (k-1)l + \sum_{j\geq k} q^j) =
\end{aligned}$$

$$=$$
$$\int_{q^{k'-1}}^{l'} \left[\Pi_{q^iq^i}(t,y,(k'-1)y+\sum_{j\geq k'} q^j)+(k'-1)\Pi_{q^iq^{-i}}(t,y,(k'-1)y+\sum_{j\geq k'} q^j)\right] dy$$
$$+$$
$$\vdots$$
$$+\int_{l}^{q^k} \left[\Pi_{q^iq^i}(t,y,(k-1)y+\sum_{j\geq k} q^j)+(k-1)\Pi_{q^iq^{-i}}(t,y,(k-1)y+\sum_{j\geq k} q^j)\right] dy$$
$$<0$$

Bibliography

[1] A. B. Abel and J. C. Eberly. Optimal investment with costly reversibility. *Review of Economic Studies*, 63:581–593, 1996.

[2] F. Aguerrevere. Real options, product market competition, and asset returns. *Journal of Finance*, 64:957–983, 2009.

[3] R. M. Anderson. Quick-response equilibrium. Working Paper, 1984.

[4] K. Back and D. Paulsen. Open loop equilibria and perfect competition in option exercise games. *Review of Financial Studies*, 22(11):4531–4552, 2009.

[5] F. M. Baldursson. Irreversible investment under uncertainty in oligopoly. *Journal of Economic Dynamics and Control*, 22:627–644, 1998.

[6] F. M. Baldursson and I. Karatzas. Irreversible investment and industry equilibrium. *Finance Stochast.*, 1:69–89, 1997.

[7] P. Bank. Optimal control under a dynamic fuel constraint. *SIAM Journal on Control and Optimization*, 44(4):1529–1541, 2005.

[8] P. Bank and N. El Karoui. A stochastic representation theorem with applications to optimization and obstacle problems. *Ann. Probab.*, 32:1030–1067, 2004.

[9] P. Bank and H. Föllmer. American options, multi-armed bandits, and optimal consumption plans: A unifying view. In *Paris-Princeton Lectures on Mathematical Finance*, volume 1814 of *Lecture Notes in Math.*, pages 1–42. Springer-Verlag, Berlin, 2002.

[10] P. Bank and F. Riedel. Optimal consumption choice with intertemporal substitution. *Ann. Appl. Probab.*, 11:750–788, 2001.

[11] J. Bertoin. *Lévy Processes.* Cambridge University Press, Cambridge, 1996.

[12] G. Bertola. Irreversible investment. *Research in Economics*, 52:3–37, 1998.

[13] M. Boyer, P. Lasserre, T. Mariotti, and M. Moreaux. Preemption and rent dissipation under price competition. *International Journal of Industrial Organization*, 22:309–328, 2004.

[14] M. Boyer, P. Lasserre, and M. Moreaux. The dynamics of industry investments. CIRANO Working Paper 2007s-09, CIRANO, Montréal, 2007.

[15] A. K. Dixit and R. S. Pindyck. *Investment under uncertainty*. Princeton University Press, Princeton, N.J., 1994.

[16] E. J. Dockner, S. Jørgensen, N. V. Long, and G. Sorger. *Differential Games in Economics and Management Science*. Cambridge University Press, Cambridge, 2000.

[17] B. Dumas. Super contact and related optimality conditions. *Journal of Economic Dynamics and Control*, 15:675–695, 1991.

[18] N. El Karoui. Les aspects probabilistes du contrôle stochastique. *Ecole d'Eté de Probabilités de Saint-Flour IX. Lecture Notes in Math.*, 876:73–238, 1981.

[19] N. El Karoui and I. Karatzas. A new approach to the skorohod problem, and its applications. *Stochastics and Stochastics Reports*, 34:57–82, 1991.

[20] D. Fudenberg and J. Tirole. Preemption and rent equalization in the adoption of new technology. *Review of Economic Studies*, 52:383–401, 1985.

[21] D. Fudenberg and J. Tirole. *Game Theory*. MIT Press, Cambridge, MA, 1991.

[22] S. R. Grenadier. Option exercise games: An application to the equilibrium investment strategies of firms. *Review of Financial Studies*, 15(3):691–721, 2002.

[23] K. J. M. Huisman and P. M. Kort. Effects of strategic interactions on the option value of waiting. CentER Discussion Paper 9992, CentER, Tilburg University, 1999.

[24] K. J. M. Huisman, P. M. Kort, G. Pawlina, and J. J. J. Thijssen. Strategic investment under uncertainty: Merging real options with game theory. CentER Discussion Paper 2003-6, CentER, Tilburg University, 2003.

[25] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer Verlag, 1988.

[26] S. Kuo and H. Wang. First passage times of a jump diffusion process. *Advances in Applied Probability*, 35(2):504–531, 2003.

[27] J. V. Leahy. Investment in competitive equilibrium: The optimality of myopic behavior. *Quarterly Journal of Economics*, 108:1105–1133, 1993.

[28] R. Mason and H. Weeds. Investment, uncertainty and pre-emption. *International Journal of Industrial Organization*, 2010. forthcoming.

[29] R. McDonald and D. Siegel. The value of waiting to invest. *Quarterly Journal of Economics*, 101:707–727, 1986.

[30] R. S. Pindyck. Irreversible investment, capacity choice, and the value of the firm. *American Economic Review*, 78:969–985, 1988.

[31] F. Riedel and X. Su. On irreversible investment. *Finance Stochast.*, 2010. forthcoming.

[32] A. V. Skorohod. Stochastic equations for diffusion processes in a bounded region. *Theory Probab. Appl.*, 6:264–274, 1961.

[33] A. M. Spence. Investment strategy and growth in a new market. *Bell Journal of Economics*, 10(1):1–19, 1979.

[34] X. Vives. *Oligopoly Pricing: Old Ideas and New Tools.* MIT Press, Cambridge, Mass., 1999.

Kurzer Lebenslauf

Jan-Henrik Steg

Geboren am 11.08.1977 in Hameln

Abitur 1997 am Gymnasium Bad Nenndorf

Zivildienst in Bad Nenndorf

Studium des Wirtschaftsingenieurwesens an der Universität Karlsruhe (TH) mit einjährigem Auslandsaufenthalt an der HEC, Université de Lausanne

Diplom-Wirtschaftsingenieur 11/2005

Internationales Graduiertenkolleg EBIM an der Universität Bielefeld ab 04/2006 mit halbjährigem Aufenthalt an der Université Paris 1 Panthéon-Sorbonne

Promotion zum Dr. rer. pol. an der Fakultät für Wirtschaftswissenschaften, Universität Bielefeld 07/2010