Adjoining Roots of Unity to $E_\infty$ Ring Spectra in Good Cases - A Remark.

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Dedicated to Michael Boardman on the occasion of his 60th birthday.

Throughout this paper we work in categories of ring-, module-, and algebra spectra as constructed by Elmendorf, Kriz, Mandell, and May [2], who exploit a crucial observation by Hopkins [4].

We start by specifying what we mean by "adjoining roots of unity" to an $E_\infty$ ring spectrum or, more precisely, to a commutative S-algebra $E$. For a ring $R$ let $H(R)$ denote its associated Eilenberg-MacLane ring spectrum given as appropriate cell spectrum. If $E$ is connective there is a map of S-algebras $E \rightarrow H(\pi_0 E)$ realizing the identity on $\pi_0$ [2, IV.3.1]. Hence $H(\pi_0 E)$ is an $E$-algebra.

**Definition 1**: Let $E$ be a connective, commutative S-algebra, and $\pi_0(E) \subseteq R$ an extension of the commutative ring $\pi_0(E)$ in the usual algebraic sense. We say a map $E \rightarrow F$ of commutative S-algebras lifts the extension $\pi_0(E) \subseteq R$ if $\pi_0(F) \cong R$ and there is a weak equivalence of $H(\pi_0 E)$-algebras

$$F \wedge_E H(\pi_0 E) \rightarrow H(R)$$

**Motivation**: Let $A$ be an $S$-algebra and $B$ be an $A$-algebra. We can define topological Hochschild homology $THH^A(B)$ of $B$ over $A$ as the realization (in the category of $A$-module spectra) of the simplicial spectrum

$$[n] \mapsto B \wedge_A B \wedge_A \ldots \wedge_A B \quad (n + 1) \text{ factors}$$
with the well-known Hochschild structure maps. We have the following result from [6].

**Theorem 1:** Let $K$ be a classical commutative ring and $R$ a flat $K$-algebra. Assume there is a commutative $S$-algebra $A$ and an $A$-algebra $E$ such that
1. $HK$ is a commutative algebra over $A$.
2. there is a weak equivalence of $HK$-algebras $E \wedge_A HK \to HR$.
Then (modulo technical cofibrancy conditions)

$$THH_*^A(HR) \cong HH_*^K(R) \otimes_K^L THH_*^A(HK),$$

as graded $K$-modules, where $HH_*^K$ stands for the classical Hochschild homology over the ground ring $K$ and $\otimes^L$ for the total left derived of $\otimes$.

We want to investigate lifts of algebraic extensions by roots of unity. An investigation of more general “algebraic extensions” of algebra spectra is work in progress.

**Proposition 2:** In general there is no lift for extensions.

**Proof:** Take the sphere spectrum $S$. Suppose we could adjoin a fourth root of unity to $S$, then according to Theorem 1

$$THH_*^S(Z[i]) \cong HH_*^Z(Z[i]) \otimes_Z^L THH_*^S(Z)$$

In particular, $THH_*^S(Z)$ would be a direct summand of $THH_*^S(Z[i])$.

Calculations by Bökstedt and Lindenstrauss show that this is not the case: we use the following results from [1] and [5]

$$THH^S_k(Z) = \begin{cases} 
Z & \text{if } k = 0 \\
\mathbb{Z}/i & \text{if } k = 2i - 1 \\
0 & \text{if } k \text{ is even}
\end{cases}$$

and the induced map $THH^S_n(Z) \to THH^S_n(Z[i])$ for $n = 2j - 1$ comes from multiplication by 2 on $\mathbb{Z}/j \to \mathbb{Z}/2j$.

Our main result is
**Theorem 3:** Let $E$ be a connective commutative $S$-algebra spectrum and $p$ a prime which is invertible in $\pi_0(E)$. Suppose the cyclotomic polynomial

$$X^q(p-1) + X^q(p-2) + \ldots + X^q + 1$$

with $q = p^{n-1}$ is irreducible in $\pi_0(E)[X]$ and $\zeta$ is a $p^n$-th primitive root of unity. Then there is a commutative $E$-algebra spectrum $E(\zeta)$ lifting the extension $\pi_0(E) \subset \pi_0(E)(\zeta)$.

The proof uses the simple observation, well known in algebra, that factoring idempotents $\varepsilon \in \pi_0E$ can be done by localizing.

**Lemma 4:** Let $E$ be a connective commutative $S$-algebra and $\pi_0(E)[X^{-1}]$ the localization of the ring $\pi_0(E)$ where $X$ is a subset of $\pi_0(E)$. Then there is a commutative cell $E$-algebra $E[X^{-1}]$ and a weak equivalence of $H(\pi_0E)$-algebras

$$E[X^{-1}] \wedge_E H(\pi_0E) \to H((\pi_0E)[X^{-1}]) = H(\pi_0(E[X^{-1}])).$$

**Proof:** By [2, VIII.4.2] there is a cell $E$-algebra $E[X^{-1}]$ whose unit $\lambda : E \to E[X^{-1}]$ induces the localization

$$\lambda_* : \pi_*(E) \to \pi_*(E)[X^{-1}].$$

Then $\lambda \wedge_E \text{id} : E \wedge_E H(\pi_0E) \to E[X^{-1}] \wedge_E H(\pi_0E)$ is the localization of $H(\pi_0E)$ by [2, VIII.4.1], and the claim follows.

**Proposition 5:** Let $E$ be a connective commutative $S$-algebra and $\varepsilon \in \pi_0E$ an idempotent. Then there is a commutative $E$-algebra $E/\varepsilon E$ and a weak equivalence of $H(\pi_0E)$-algebras

$$(E/\varepsilon E) \wedge_E H(\pi_0E) \to H(\pi_0E/\varepsilon\pi_0E) = H(\pi_0(E/\varepsilon E))$$

**Proof:** Let $\eta = 1 - \varepsilon$ in $\pi_0E$. Let $E \to E[\frac{1}{\eta}]$ be the localization. Since as rings

$$\left(\pi_0E\right)\left[\begin{array}{c}1 \\ \eta\end{array}\right] \cong \pi_0E/\varepsilon\pi_0E$$

we can take $E[\frac{1}{\eta}]$ for $E/\varepsilon E$.

**Proof of Theorem 3:** We have a commutative $E$-algebra group spectrum

$$E[\mathbb{Z}/p^n] = E \wedge (\mathbb{Z}/p^n)_+$$
by taking the small smash product with $(\mathbb{Z}/p^n)_+$. Let $t \in \mathbb{Z}/p^n$ be a generator and $x = t^q$, $q = p^{n-1}$. Then

$$\varepsilon = \frac{1}{p}(1 + x + \ldots + x^{p-1}) \in (\pi_0 E)[\mathbb{Z}/p^n] = \pi_0(E[\mathbb{Z}/p^n])$$

is an idempotent and $E(\zeta) = E[\mathbb{Z}/p^n]/\varepsilon(E[\mathbb{Z}/p^n])$ is the required spectrum.

**Remark:** The automorphism group $G$ of $\mathbb{Z}/p^n$ acts on $E[\mathbb{Z}/p^n]$ leaving the idempotent $\varepsilon$ fixed. Hence, if we work in the category of $G$-spectra in the naive sense, we have the Galois group of the extension operating on $E(\zeta)$, and the map $E \to E(\zeta)$ is equivariant with the trivial action on $E$.

**Corollary 6:** Let $R \subset \mathbb{Q}$ be a subring and $\zeta$ a primitive $p^n$-th root of unity, $p$ a prime which is invertible in $R$. Then

$$THH^S_*(R(\zeta)) \cong HH^2_*(R(\zeta)) \otimes^L_Z THH^S_*(\mathbb{Z})$$

This determines $THH^S_*(R(\zeta))$ by [1] and the following result from [3]:

$$HH^R_k(R[X]/(f)) = \begin{cases} R[X]/(f) & \text{if } k = 0 \\ R[X]/(f, f') & \text{if } k \text{ odd} \\ \text{Ann}(f') \text{ in } R[X]/(f) & \text{if } k > 0 \text{ even} \end{cases}$$

for a monic polynomial $f \in R[X]$ over a commutative ring $R$.

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**References**


