CRITICAL BEHAVIOR AND RESONANCE EXCITATION IN THE THERMODYNAMICS OF EXTENDED HADRONS

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A system of spatially extended non-interacting hadrons generally leads at sufficiently high density to a phase transition from "gas" to "solid". Here we study the effect of interaction, in form of an exponential resonance excitation spectrum (dual resonance model, statistical bootstrap model), on this phenomenon. While the presence of resonances significantly lowers the critical temperature, the finite size of the hadrons prevents the occurrence of the thermodynamic singularity associated with an exponential resonance spectrum in the case of point-like constituents.

In statistical physics it is believed [1] that a system of particles with spatial extensions V_0 will lead, with increasing density, to a phase transition from a gas-like state to a densely packed, solid-like state, provided we have two or more space dimensions and not infinitely deformable particles. A general proof for such a transition is still lacking; but besides physical intuition, computer simulations [1] and exact lattice calculations [2] strongly support this belief. Since in relativistic systems, particle creation implies a density increase when the temperature is sufficiently increased, we here expect a "solidification" phase transition [3] at some critical temperature $T_c(V_0)$.

In statistical mechanics of point-like elementary particles, it is shown that the inclusion of resonances, with a linearly exponential mass spectrum, in an otherwise ideal gas, leads to singular thermodynamic behavior [4] at the temperature $T_H = 1/b$, where

$$\rho(m) = cm^a e^{bm}; \quad a, b, c = \text{constants}, \tag{1}$$

parametrizes the resonance mass spectrum.

The aim of the present paper is to study the effect of including resonances with a spectrum of type (1) in an ideal gas of *extended* elementary particles.

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It has been held for some time [5] that a system of one type of elementary particle with spatial extension alone is equivalent to a system of point-like particles with a resonance spectrum given by (1). The basic length scale of hadron physics is in one case interpreted as hadronic size, in the other as Regge resonance trajectory slope $\alpha' = b^2$. We shall here see this point of view confirmed in the sense that if an increase of resonance mass also implies an equivalent increase of resonance size, then the spatial extension of the constituents always prevents the spectrum (1) from causing any thermodynamic singularity.

We consider an ideal gas of resonances, governed by the mass spectrum (1), and contained in a volume V. We parametrize the volume of a resonance of mass m as

$$V(m) = (m/m_0)V_0, \tag{2}$$

with m_0 and V_0 denoting mass and volume of the lowest hadron state ("pion"). This form assumes the volume of a resonance to depend linearly on its mass, in accord with most quark bag pictures [6]. For simplicity, we shall consider the case of chemical potential zero (no conserved quantum numbers), so that the density and hence all other thermodynamic observables depend only on the temperature. We shall also restrict ourselves to the case of Boltzmann statistics.

Our model is thus completely defined by the grand canonical partition function

$$Z(V,\beta) = \sum_{N=0}^{\infty} \frac{1}{N!} Z_N(V,\beta), \tag{3}$$

$$Z_N(V,\beta) = \int \prod_{i=1}^{N} \left(\frac{\mathrm{d}^3 p_i}{2\pi^3} \,\mathrm{d}m_i\, \rho(m_i) \exp(-\beta p_{i0}) \right)$$
$$\times\, \theta\left(V - \sum_{1}^{N} (m_i/m_0) V_0 \right) V(m_1, \ldots, m_N), \tag{4}$$

with $\rho(m)$ and $V(m)$ given by eqs. (1) and (2). In eq. (4), $V(m_1, \ldots, m_N)$ denotes the total coordinate space volume available to N hadrons of masses $m_1, \ldots, m_N$; for point-like particles, this would be simply V^N. We note that for $\rho(m) = \delta(m - m_0)$, we recover the extended hadron model of ref. [3]. As was done there, we shall now study the high and low temperature (and hence density) limits of $Z(V, \beta)$ and determine the appropriate state of the system by requiring the free energy to be minimal. For reference, we first recall, however, the form of the basic thermodynamic quantities in the case $V_0 = 0$, i.e., for a gas of point-like resonances.

The grand partition function is then written as

$$Z(V,\beta) = \sum_{N=0}^{\infty} \frac{V^N}{N!} \varphi^N(\beta) = \exp V\varphi(\beta), \tag{5}$$

with

$$\varphi(\beta) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 p\, \mathrm{d}m\, \rho(m) \exp(-\beta p_0)$$
$$= \frac{1}{2\pi^2 \beta} \int_{m_0}^{\infty} \mathrm{d}m\, m^2 \rho(m)\, K_2(m\beta). \tag{6}$$

Using the spectrum (1) and the asymptotic form of $K_2(x)$ for large x, we have

$$\varphi(\beta) \approx \frac{c}{(2\pi\beta)^{3/2}} \int_{m_0}^{\infty} \mathrm{d}m\, m^{3/2+a}\, \mathrm{e}^{(b-\beta)m}. \tag{7}$$

Since for $T = \beta^{-1} > b^{-1}$ the integrand diverges at the upper limit of the integration, the temperature of the system has an upper bound, $T_H = b^{-1}$. Whether $\varphi(\beta)$ exists at $T = T_H$ depends on the parameter a in the spectrum. For $a \geqslant -5/2$, $\varphi(\beta)$ and hence also pressure p and hadron density n diverge at T_H; the energy density ϵ already diverges for all $a \geqslant -7/2$. In fig. 1 we show the qualitative behavior of p, n, and ϵ for $a = -2$, -3. Since in the generally considered range of a ($\geqslant -7/2$) the value T_H can be attained only at infinite energy density, T_H was, until a few years ago [4], often considered as the "ultimate" or highest possible temperature of strongly interacting matter [7].

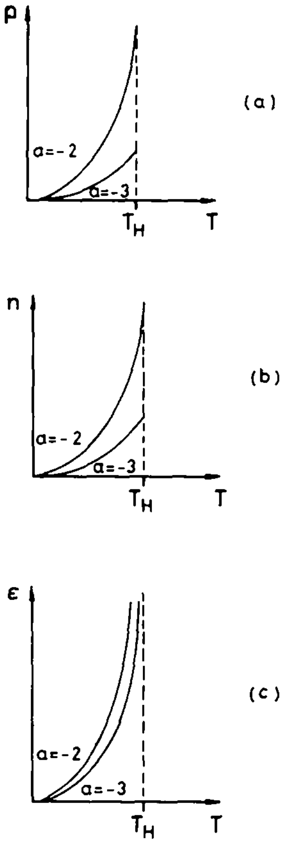

Fig. 1. Thermodynamic quantities for a system of point-like resonances with exponential spectrum; (a) pressure, (b) hadron density, (c) energy density.

The thermodynamic properties of a multihadron system change drastically with increasing density, if each hadron is endowed with a non-vanishing and not infinitely compressible intrinsic volume, which is inaccessible to all other hadrons ("no overlap"). For simplicity and definiteness, we shall concentrate mainly on hard core hadrons; in the latter part of the paper we shall also discuss briefly consequences for deformable constituents. The main effect of hadronic size is obviously an upper limit on the hadron density – the limit of dense packing, of a "filled box".

The thermodynamics of a system of one kind of basic hard-sphere hadrons ("pions") in free creation and annihilation was studied in ref. [3], for the case of vanishing chemical potential (no conserved quantum numbers). It was found there that such a system behaves at low density or temperature like a relativistic ideal gas, possessing unrestricted constituent mobility and unrestricted energy conversion into additional particles, while at high density or temperature the presence of many neighbours greatly restricts the mobility of any constituent and essentially stops particle production by "lack of space".

To obtain the transition from one regime to the other, the free energy F was calculated in the dilute gas

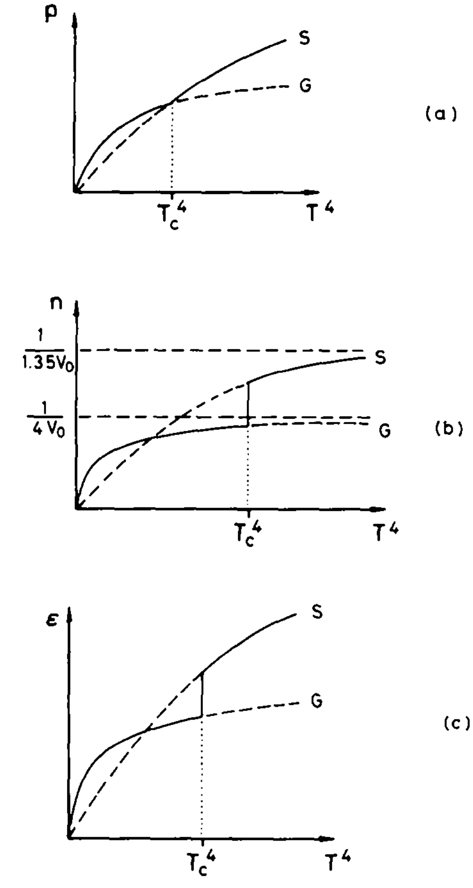

Fig. 2. Thermodynamic quantities for a system of extended pions; (a) pressure, (b) hadron density, (c) energy density; S and G denote high and low density limits.

limit (F_G) and in the high density limit (F_S). It was found that $F_G < F_S$ at low temperatures and $F_G > F_S$ at high temperatures; the cross-over thus determines a critical value $T = T_c$ corresponding to a first-order phase transition. The resulting behavior of pressure, hadron density and energy density is shown in fig. 2. Clearly the procedure of using two crossing limiting forms and the requirement of minimum free energy is not sufficient to establish rigorously a phase transition; both its existence and its order are obtained by construction. As already indicated, the occurrence of a phase transition in this case is, however, generally accepted today because of support from computer simulations [1] and exact lattice calculations [2]. We emphasize that this transition is due only to the intrinsic hard-core hadrons – no attractive force of any kind is needed [8].

The two limiting expressions of a hard-sphere pion gas were obtained in ref. [3] by using as available coordinate-space volume for N hadrons of size V_0 in eq. (3)

$$V_G(m_0, ..., m_0) = (V - NV_E)^N, \tag{8}$$

for the low density form and

$$V_S(m_0, ..., m_0) = N!\ [(V/N)^{1/3} - \widetilde{V}_E^{1/3}]^{3N}, \tag{9}$$

for the high density form. Here $V_E = 4V_0$ and $\widetilde{V}_E = 1.35V_0$ measure the volume which a given pion removes from access to all others, in three space dimensions and, for $\widetilde{V}_E$, in case of a hexagonal lattice at dense packing. The cross-over of the free energies corresponding to eq. (8) and (9), and hence the phase transition, arises because $V_E \neq \widetilde{V}_E$, which is valid for space dimensions two and larger. In the one-dimensional case $V_E = \widetilde{V}_E = V_0$ and hence no transition occurs [9].

To extend this approach to a system of resonances, we set

$$V_G(m_1, ..., m_N) = \left(V - \sum_{i=1}^{\infty} (m_i/m_0)\, V_E\right)^N$$
$$= V^N\left(1 - C_G^{-1} \sum_{i=1}^{N} m_i/m_0\right), \tag{10}$$

with $C_G \equiv V/V_E$, for the gas limit and

$$V_S(m_1, ..., m_N)$$
$$= N! \left[(V/N)^{1/3} - \left(\sum_{i=1}^{N} (m_i/m_0)(\widetilde{V}_E/N) \right)^{1/3} \right]^{3N}$$
$$= N!\,(V/N)^N \left[1 - \left(C_S^{-1} \sum m_i/m_0 \right)^{1/3} \right]^{3N}, \quad (11)$$

with $C_S \equiv V/\widetilde{V}_E$, for the "solid" limit; m_0 and V_0 again specify the lowest hadron, the pion. The summations in $V_{G/S}$ prevent the N-body partition function (3) from factorizing, as it does when we have only pions. To regain factorization, we make the approximation of replacing the distribution over different resonance masses by an average distribution and write for the N-resonance partition function

$$Z_N^{G/S}(V, \beta) \approx \left(\frac{1}{2\pi^2 \beta} \int \mathrm{d}m\, m^2 K_2(m\beta)\, \rho(m) \right.$$
$$\left. \times\, \theta(1 - Nm/C_{G/S}\, m_0)\, V_{G/S}^{1/N}(m, ..., m) \right)^N , \quad (12)$$

where G/S again denotes the gas and the solid form, respectively. The numerical results using this approximation were found to be in good agreement with saddle point calculations [10]. From eq. (12), we have as grand partition function of our system of extended resonances

$$Z_{G/S}(V, \beta) \approx \sum_{N=0}^{C_{G/S}} \frac{V^N}{N!}\, \varphi_{G/S}^N(\beta, C_{G/S}/N), \quad (13)$$

with

$$\varphi_G(\beta, C_G/N) \equiv \frac{1}{2\pi^2 \beta} \int_{m_0}^{C_G m_0/N} \mathrm{d}m$$
$$\times\, m^2 K_2(m\beta) \rho(m) (1 - Nm/C_G m_0), \quad (14)$$

for the gas and

$$\varphi_S(\beta, C_S/N) \equiv \frac{e}{2\pi^2 \beta} \int_{m_0}^{C_S m_0/N} \mathrm{d}m$$
$$\times\, m^2 K_2(m\beta) \rho(m) [1 - (Nm/C_S m_0)^{1/3}]^3, \quad (15)$$

for the solid form.

We note that both the number of terms ($C_{G/S}$) in the sum (13) over N and the integration ranges in eq. (14), (15) are finite for finite overall volume V and non-zero intrinsic hadron volumes V_E and $\widetilde{V}_E$. Hence at fixed V the partition function $Z(V, \beta)$ and all its derivatives exist for all temperatures $0 < \beta < \infty$. The ultimate temperature T_H of the conventional statistical bootstrap model can arise only because for point-like particles $V_E(\widetilde{V}_E) = 0$, so that $C_{G/S} = \infty$ at fixed V. This extends the range of the integrals (14), (15) to infinity and at the same time removes the cut-off at the upper limit in the integrand, causing the integrals to diverge for $T = \beta^{-1} > b$ [see eq. (7)].

We thus conclude that at fixed V an exponential spectrum of resonances with size (2) cannot lead to singular behavior. Let us now see what happens in the thermodynamic limit $V \to \infty$, or (with V_E and $\widetilde{V}_E$ fixed and > 0) equivalently $C_{G/S} \to \infty$. Using the inequality

$$(V^{\bar{N}}/\bar{N}!)\varphi_{G/S}^{\bar{N}} \leq Z(C_{G/S}, \beta) \leq C_{G/S}(V^{\bar{N}}/\bar{N}!)\varphi_{G/S}^{\bar{N}}, \quad (16)$$

where $N = \bar{N}$ specifies the largest term in the sum (13), we obtain immediately

$$\lim_{C_{G/S} \to \infty} \left(\frac{1}{C_{G/S}} \ln Z \right) = \lim_{C_{G/S} \to \infty} \left[\frac{1}{C_{G/S}} \ln \left(\frac{V^{\bar{N}} \varphi_{G/S}^{\bar{N}}}{\bar{N}!} \right) \right]. \quad (17)$$

It can be shown that asymptotically $\bar{N} = \alpha C_{G/S}$, with constant $\alpha > 0$, so that via Sterling approximation

$$\lim_{C_{G/S} \to \infty} \left[\frac{1}{C_{G/S}} \ln \left(\frac{V^{\bar{N}} \varphi_{G/S}^{\bar{N}}}{\bar{N}!} \right) \right]$$
$$= \alpha [1 - \ln \alpha + \ln \varphi_{G/S}(\beta, \alpha^{-1})], \quad (18)$$

which again exists for all $0 < \beta < \infty$. Thus the introduction of resonances provides also in the thermodynamic limit no singularities of the partition function or its derivatives. In other words, the increase in the number of states due to the presence of resonances is effectively compensated by the larger volume of each resonance and the ensuing coordinate-space reduction; this is true independent of the choice of a in eq. (2).

As the main result of our considerations, we thus find that the thermodynamic behavior of the extended resonance system (13)–(15) is qualitatively the same as that of a hard-core pion gas (fig. 2): a gas phase at low temperatures, then at $T = T_c$ a phase transition to a solid-like phase; as in the pion case, the critical temperature for the transition can be determined by the cross-over of the corresponding free energies.

The presence of resonances provides some (though with the size (2) rather weak) clustering effect, and this leads us to expect the phase transition to occur already at lower values of temperature and density than in a pure pion gas. Numerical calculations support this.

For a hard-core pion gas [3] in the lowest order approximation, the phase transition occurred at $T_c \approx 10^5 m_0$; using a more precise gas form approximation yielded instead $T_c \approx 350\, m_0$. For a resonance system with spectrum (1), using parameter values in accord with data [7], already the lowest order approximation yields $T_c \approx 30\text{–}50 m_0$. Hence the presence of resonances significantly lowers the critical temperature. The transition in the pion system took place between the densities $n_G \approx 0.49/V_0$ and $n_S \approx 0.54/V_0$; in the resonance system, it sets in at $n_G \approx 0.2/V_0$ and terminates at $n_S \approx 0.34/V_0$. In the transition region, a typical resonance has a mass of about $3\text{–}4 m_0$, which accounts for the lowered transition density. These quantitative conclusions are rather insensitive to the choice of the parameter a in eq. (1) and remain valid also if the pion is kept as a discrete part of the resonance spectrum.

We close with some comments on recent related work [11], in which the statistical bootstrap model is generalized to the case of extended resonances. In this work, no "solidification" transition of the type considered by us arises, and it is claimed that for zero chemical potential the singular behavior of the "pointlike" statistical bootstrap is recovered.

As already mentioned, the phase transition we have studied is possible only in systems with two or more space dimensions: for one-dimensional systems, $V_E = \widetilde{V}_E$ and there is no cross-over of the two limiting free energy forms. The absence of such a phase transition for a one-dimensional system with finite short-range forces is of course also expected from general arguments [12]. In ref. [11], the volume restriction by the size of the constituents is written as

$$\Delta(m_1, \ldots, m_N) = \left(V - \sum_{i=1}^{N} V(m_i) \right)^N , \qquad (19)$$

at all densities. Although this form can also be obtained by assuming infinitely deformable hadrons, it renders the problem in structure one-dimensional and therefore excludes a "solidification" type transition.

Concerning the singularity encountered at $T = T_H$ in the case of pointlike resonances with spectrum (1): we have seen above that the introduction of a resonance size of form (2) makes all integrals and the sum over N finite at all temperatures $0 < T < \infty$, both for fixed overall volume V and with $V \to \infty$. The initial formulation of ref. [11] also uses a grand canonical description, with temperature T, fugacity z and volume V as thermodynamic variables. The corresponding partition function remains non-singular at all temperatures, in accord with our results. A singularity at $T = 1/b$ is only obtained after going to a description using T, z and the "available" volume (19) as variables. To compare this result with our considerations or more generally with the conventional formalism of critical phenomena, the thermodynamic role of Δ must be further clarified. If Δ is an extensive variable, then how is the thermodynamic limit of $Z(T, z, \Delta)$ specified? If Δ is an intensive variable, then what is the connection between singularities of $Z(T, z, \Delta)$ and critical behavior? Generally the singularities of ensembles parametrized in terms of intensive variables only are not related to critical behavior [8].

To recover in our formulation an upper bound on the temperature, it is necessary to make the increase in the number of resonance states stronger than the associated decrease of the available coordinate space volume. For a resonance size $V(m) \sim m^\epsilon V_0$, with $\epsilon = 1$, this is not the case; but if $\epsilon < 1$, the energy density gain due to a heavy resonance is no longer compensated by coordinate-space loss, and it can be shown that then an upper limit on the temperature arises.

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