SU(2) $\beta$ Function with and without Dynamical Fermions

U. Heller
CERN, CH-1211 Geneva 23, Switzerland

and

F. Karsch
University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

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We study the $\beta$ function of SU(2) lattice gauge theory in the presence and absence of dynamical fermions with the (improved) "ratio method." The pseudofermion algorithm used to simulate the fermion action seems to be capable of giving the right contribution to the $\beta$ function. For two flavors of fermions we find agreement with asymptotic scaling to within $\sim 15\%$ for $\beta \geq 2.0$.

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Recent, more accurate Monte Carlo simulations of pure lattice gauge theories have shown that there are rather large deviations from asymptotic scaling behavior in the range of couplings accessible with today's computing power. This raised the question of whether we are able to see continuum physics in these simulations. The answer requires the knowledge of the $\beta$ function away from the asymptotic regime where it is dominated by the two leading (universal) terms in its perturbative expansion. During the last year a large effort has been put into the numerical (nonperturbative) determination of the $\beta$ function for SU(3) gauge theories and has revealed its rather nontrivial behavior which shows an unexpected dip around $\beta(=6/g^2) = 6.0$.\textsuperscript{1-3} Mainly two different methods have been used for this computation.\textsuperscript{4} One is based on a combination of the real-space renormalization-group (block-spin transformation) with Monte Carlo simulations, called MCRG,\textsuperscript{2,3} whereas the other uses the comparison of ratios of Wilson loops which differ by a factor 2 in size, the "ratio method."\textsuperscript{1,5}

QCD includes, besides the pure gauge sector, dynamical fermions represented in the path integral by anticommuting (Grassmann) variables. To investigate the scaling behavior of the full theory one needs to be able to compute the $\beta$ function with dynamical fermions included. The Grassmanian nature of the fermions prevents a straightforward application of the block-spin transformation scheme to compute the $\beta$ function. This reflects itself in the nonlocal nature of the fermion determinant obtained after one integrates out the fermion fields. The ratio method on the other hand can easily be done with Wilson loops measured in gauge field configurations created with the effect of dynamical fermions taken into account.

The ratio method.—Ratios of Wilson loops of equal perimeter and number of corners (to cancel the perturbative singularities) such as

$$R(i_1,i_2;J_1,J_2) = \frac{W(i_1,i_2)}{W(J_1,J_2)}$$

\[ 1 + i_2 = j_1 + j_2, \]  

(1a)

$$R(i_1,i_2,i_3,i_4;J_1,J_2,J_3,J_4) = \frac{W(i_1,i_2)W(i_3,i_4)}{W(J_1,J_2)W(J_3,J_4)}$$

\[ 1 + i_2 + i_3 + i_4 = j_1 + j_2 + j_3 + j_4, \]

(1b)

satisfy the homogenous renormalization-group equation

$$R([2i_a];\beta,L) = R([i_a];\beta',L/2)$$

(2)

with $\Delta \beta(\beta) = \beta - \beta'$ ($\beta = 2N/g^2$) the change in the coupling required to compensate for a change of scale by a factor of 2. The computation of the left-hand side and the right-hand side of Eq. (2) on lattices of size $L^4$ and $(L/2)^4$ minimizes the finite-size effects since the lattice sizes in physical units (femtometers) are the same when the coupling $\beta'$ is tuned such that Eq. (2) is satisfied. This determines the shift $\Delta \beta(\beta)$. For a SU($N$) gauge theory it is related to the $\beta$ function $[\beta_{\text{func}}(g)]$ by

$$\ln 2 = -\left(\frac{N}{2}\right)^{1/2} \int_{x' \sqrt{2} \beta_{\text{func}}((2N/x)^{1/2})} dx.$$  

(3)

Asymptotically for large $\beta$, when the $\beta$ function is dominated by the two universal terms, one finds

$$\Delta \beta = (4N/b_0 + 8N^2 b_1/\beta) \ln 2 + o(1/\beta^2),$$

(4)

with

$$b_0 = \frac{1}{16\pi^2} \left\{ \frac{11}{3} N - \frac{2}{3} n_f \right\},$$

$$b_1 = \frac{1}{(16\pi^2)^2} \left\{ \frac{34}{3} N^2 - \frac{10}{3} N + \frac{N^2 - 1}{N} n_f \right\}. \quad (5)$$
for \( n_f \) flavors of massless fermions.

For ratios of Wilson loops of small size Eq. (2) is affected by lattice artefacts which, however, can be canceled order by order in perturbation theory by the use of appropriate linear combinations of ratios.\(^1\,5,\,6\) For example, tree-level improved ratios are linear combinations of two basic ratios, Eq. (1),

\[
R_{12} = R_1 + \alpha R_2, \tag{6}
\]

with \( \alpha \) determined such as to cancel the lattice artefacts in tree-level perturbation theory.\(^5\,6\)

In the following we will present the results for \( \Delta \beta \) from tree-level improved ratios both for pure SU(2) lattice gauge theory and for SU(2) with two flavors of dynamical fermions, included with the pseudofermion algorithm. Besides providing us with information about deviations from asymptotic scaling, which for pure SU(2) are expected to be less dramatic than for SU(3), the present simulation should help us clarify whether the pseudofermion algorithm is capable of simulating the correct fermion feedback by giving the right \( \beta \) function.

The results are obtained from Monte Carlo data on \( 8^4 \) and \( 4^4 \) lattices at different values of \( \beta \). These lattice sizes were chosen such as to make the inclusion of fermions feasible. To increase the number of ratios we used Wilson loops up to size \( 3 \times 3 \) on the \( 4^4 \) lattice and \( 6 \times 6 \) on the \( 8^4 \) lattice. This is a valid procedure since the (strong) finite-size effects of these loops are the same on both lattices and cancel in the matching procedure.

Our results for \( \Delta \beta (\beta) \) obtained from basic and tree-level improved ratios for the pure SU(2) theory are given in Table I. The tree-level results are shown in Fig. 1. Included in the figure are \( \Delta \beta \)'s as extracted from string-tension measurements\(^8,\,9\) and measurements of the deconfinement transition temperature\(^10,\,11\) at \( \beta = 2.6 \) the prediction from a tree-level matching of a \( 16^4 \) lattice\(^12\) to an \( 8^4 \) lattice is also shown. It agrees within errors with the matching from \( 8^4 \) to \( 4^4 \), showing that the finite-size effects in the matching procedure cancel. The structure of \( \Delta \beta \) as a function of \( \beta \) is very similar to the one found in SU(3), though the dip seen around \( \beta = 2.5 \) is somewhat less pronounced. This is consistent with the interpretation that the dip is connected to a critical point in the fundamental-adjoint plane which is further away and thus less influential for SU(2) than for SU(3).\(^13\)

Our results indicate that the deviations from asymptotic scaling are less than \( \sim 20\% \) for \( \beta \geq 2.2 \). A qualita-

### TABLE I. Summary of results obtained for \( \Delta \beta \) for the pure SU(2) gauge theory (\( n_f = 0 \)) at various values of \( \beta \).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>Basic ratios</th>
<th>Tree-level ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>0.489 ± 0.053</td>
<td>0.440 ± 0.018</td>
</tr>
<tr>
<td>2.6</td>
<td>0.270 ± 0.055</td>
<td>0.232 ± 0.013</td>
</tr>
<tr>
<td>2.75</td>
<td>0.285 ± 0.058</td>
<td>0.253 ± 0.014</td>
</tr>
</tbody>
</table>

*Multithit method of Ref. 7 has been used to measure the Wilson-loop expectation values.

### TABLE II. Same as Table I but for the SU(2) gauge theory with two flavors of dynamical fermions (\( n_f = 2 \)).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>Basic ratios</th>
<th>Tree-level ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>0.48 ± 0.041</td>
<td>0.242 ± 0.012</td>
</tr>
<tr>
<td>2.2</td>
<td>0.276 ± 0.027</td>
<td>0.193 ± 0.009</td>
</tr>
<tr>
<td>2.3</td>
<td>0.221 ± 0.032</td>
<td>0.225 ± 0.010</td>
</tr>
<tr>
<td>2.4</td>
<td>0.249 ± 0.024</td>
<td>0.193 ± 0.009</td>
</tr>
</tbody>
</table>

Comparing $\Delta \beta(\beta)$ with and without dynamical fermions we see that the onset of asymptotic scaling occurs at a smaller value of $\beta$ (shifted by $\sim 0.2$) in the former case. The fact that $\Delta \beta$ for two fermion flavors is compatible with the asymptotic prediction indicates that the pseudofermion algorithm is capable of correctly simulating the effects of dynamical fermions. Unfortunately the presence of the dip in the pure gauge sector around $\beta \approx 2.5$ weakens this conclusion somewhat since it makes the difference in $\Delta \beta$ between the pure gauge sector and the theory with fermions less pronounced in this coupling-constant regime. To strengthen the argument it would therefore be interesting to determine $\Delta \beta$ for larger values of $\beta$ or in the presence of more flavors of fermions which should increase the difference in $\Delta \beta$.

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4For a pedagogical introduction to the renormalization-group approach to lattice gauge theories see P. Hasenfratz, CERN Report No. Ref.-Th. 2999/84 (to be published).
12We thank F. Gutbrod and I. Montvay for providing us with unpublished Wilson-loop data of Ref. 8.