PHYSICAL REVIEW D VOLUME 22, NUMBER 2 15 JULY 1980

Thermodynamics of confined quarks

F. Karsch and H. Satz
Department of Theoretical Physics, University of Bielefeld, Germany
(Received 3 March 1980)

We investigate the thermodynamics of strongly interacting matter described as a system of quarks and antiquarks in free creation and annihilation, but with pairwise confinement to a volume of hadronic size ($\sim 4\pi/3m_\pi^{\ 3}$). At low temperature, such a system behaves like a hadron gas, giving a quark or an antiquark a spatial mobility of order $m_\pi^{\ -1}$. At high temperature and hence high density, we obtain a gas of free quarks and antiquarks with infinite spatial mobility. By calculating the crossover of low- and high-temperature approximations, we find a phase transition from hadron to quark matter at the critical temperature $T_c \simeq m_\pi$. We compare our results with similar considerations based on the size of hadrons and on perturbative quantum chromodynamics.

I. INTRODUCTION

Phase transitions between hadron matter and quark matter have been discussed[1] ever since the proposal of the quark basis of hadrons. For composite hadrons, the presence of a fundamental dimensional scale R_0 becomes as natural as the Bohr radius for the hydrogen atom. On the other hand, one then also expects that strongly interacting matter at densities much greater than one hadron per $R_0^{\ 3}$ should be described as a quark plasma rather than as a multihadron system.

In a previous paper,[2] we have studied the thermodynamics of hadrons as extended particles with free creation and annihilation (zero chemical potential). It was found that the presence of a hadronic size, $V_0 = 4\pi R_0^{\ 3}/3$, leads to a phase transition from a state of high mobility and abundant creation ("hadron gas") to a state of bounded mobility and strongly damped production ("hadronic solid"). This transition occurs, qualitatively speaking, when the hadronic mean mobility, which at low temperature in the thermodynamic limit is infinite, becomes finite at sufficiently high temperature, because the increase in the density of extended constituents restricts their range.

In the present paper, we want to look at this situation from the point of view of the quarks: Instead of considering a system of free hadrons with infinitely repulsive cores, we shall now treat a system of free quark-antiquark pairs, confined by an infinitely repulsive vacuum (see Fig. 1). We again expect a phase transition—but now the quark mean mobility, which in the low-temperature hadronic phase is bounded by the bag size, diverges at higher temperature when the system becomes dense enough to provide a given quark with at least one antiquark in each coordinate-space cell of volume V_0 (see Fig. 2).

Let us elaborate a little more on the physical situation we have in mind. In the world of hard-core hadrons inside a volume V, a given hadron sees a vacuum everywhere, except at those places where other hadrons are present. In terms of its potential $U(x)$, with x denoting the center-of-mass coordinate of our chosen hadron, we have

$$U_H(x) = \begin{cases} \infty, & |x - x_k| \leq 2R_0 \\ 0, & \text{elsewhere}. \end{cases} \tag{1.1}$$

Here x_k, $k = 1, 2, 3, \ldots$ are the coordinates of the other hadrons, and $4\pi R_0^{\ 3}/3$ denotes as before the hadronic volume. In Fig. 3, we show a projection of this potential onto some fixed axis. In the world of confined quark-antiquark pairs, the situation is exactly complementary. We start with a *Gedankenexperiment* by randomly putting quarks into the box V, neglecting any interaction between them (and also the effect of Fermi statistics). When we now add the antiquarks, then a given antiquark sees everywhere a vacuum, except for those regions where no quark is present. The potential for an antiquark can thus be written

$$U_Q(\bar{x}) = \begin{cases} 0, & |\bar{x} - x_k| \leq 2R_0 \\ \infty, & \text{elsewhere} \end{cases} \tag{1.2}$$

and is illustrated in Fig. 3(b) by the photographic negative of Fig. 3(a). In Eq. (1.2), x_k, $k = 1, 2, 3, \ldots$

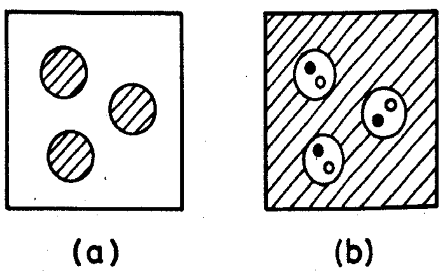

FIG. 1. Strongly interacting matter from a (a) hadronic and from a (b) quark point of view.

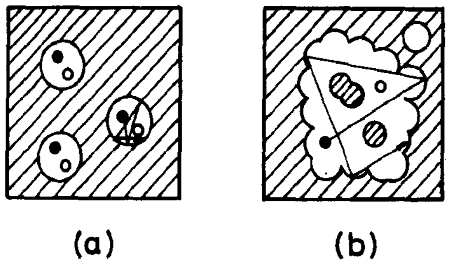

FIG. 2. Quark mobility at (a) low and at (b) high density.

now denotes the positions of the quarks (more precisely, of the unbonded quarks).

From a hadronic point of view, the quarks at low density appear to be localized in a disconnected way by the hadronic vacuum and are called "confined." As seen by the quarks, using the quark vacuum as reference ($U_Q=0$), the hadronic vacuum becomes a physical medium,[3] infinitely repulsive to individual quarks or antiquarks but transparent to q-$\bar{q}$ pairs. The hadronic vacuum is characterized by the absence of on-shell matter; the quark vacuum is characterized by the presence of on-shell matter. "Confinement" becomes relative in a picture of hard-core hadrons on one hand, a hard-core vacuum on the other: While quarks are confined at low density by the hadronic vacuum, hadrons are confined at high density by the quark vacuum, matter.

We recall here, as an illustration, a similar situation in atomic physics—hydrogen atoms (as hadron analogs) and their electrons (in place of antiquarks). To a bound electron, the atomic vacuum is confining (though not infinitely) and thus endowed with physical properties. To the (quasi-) electrons in metallic hydrogen, the lattice becomes a vacuum, although by an atom it is seen as "confining" matter.

In Fig. 3 and in Eqs. (1.1) and (1.2), we have denoted both vacuums as $U(x)=0$, although, as we just saw, that becomes wrong for the quark vacuum from a hadronic point of view and vice versa. However, we here cannot and do not want to make a choice of which is to be the basic vacuum. In the hydrogen example, one considers as basic a theory (QED) with a vacuum characterized by the absence of on-shell matter. (We note in passing that the presence of this complete theory has not given us the means to calculate from first principles the "less basic" intrametallic vacuum seen by conduction electrons.) In the case of strong-interaction physics quantum chromodynamics (QCD), as the presently strongest candidate for a theory, starts from an equation for interacting quarks and gluons and thus, with a material ("perturbative") vacuum, attempts to derive the matterless hadronic vacuum.

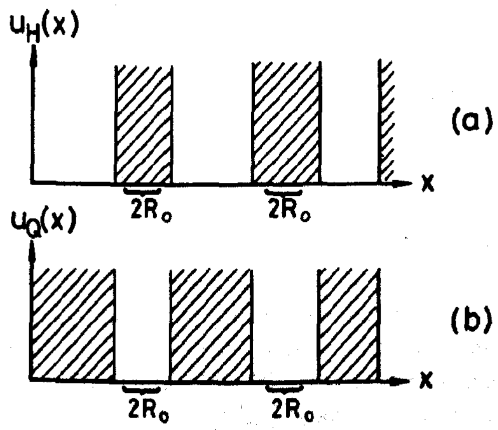

FIG. 3. Projection of potential for (a) hard-core hadrons and (b) confined quarks.

In our description, we have so far neglected two important features—the requirement that each quark must be bonded with one and only one antiquark, and the possibility of overlap between the regions occupied by $q\bar{q}$ pairs. These features generally prevent a separation of the overall potential into individual or pair contributions. To obtain a more precise formulation, we require for the overall potential of an N-pair system

$$U_Q(x_1,\dots,x_N;\bar{x}_1,\dots,\bar{x}_N)=0 \tag{1.3}$$

whenever there exists at least one ordering $(k_1,k_2,\dots,k_N)$ of the N antiquarks which makes

$$|x_i-\bar{x}_{k_i}|\leq 2R_0\ \forall\ i=1,2,\dots,N, \tag{1.4}$$

otherwise,

$$U_Q(x_1,\dots,x_N;\bar{x}_1,\dots,\bar{x}_N)=\infty. \tag{1.5}$$

At low density, when the different $q\bar{q}$ pairs are widely separated, there will in general be only one ordering satisfying Eq. (1.4). At high density, however, there will be many possible rearrangements, since the overlap of the quark vacuums allows different $q\bar{q}$ associations (see Fig. 4).

We note finally that if one does not attribute the confinement to the infinitely repulsive hadronic vacuum, as in Eq. (1.5), but instead to a $q\bar{q}$ binding potential, then at high density a screening[4] of long-range forces or some type of bond-exchange mech-

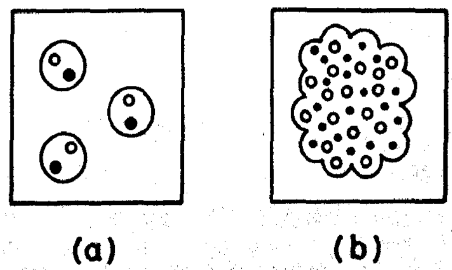

FIG. 4. Quark permutations and mobility.

anism[5] ("flip-flop") becomes necessary to obtain quark mobility. Without such a mechanism, a given antiquark would remain coupled to "its" quark even when other quarks are closer to it. In our formulation, Eqs. (1.3)–(1.5), the rapidly growing number of possible rearrangements makes any specific $q\bar{q}$ association unimportant at high density: For any value of x_i, there is then an ordering such that Eq. (1.4) is satisfied; hence $U_Q = 0$ and the quarks are free.

II. QUARK-GAS LIMIT

At sufficiently high temperature (and hence high density), our system becomes simply a gas of quarks and antiquarks in free creation and annihilation. The partition function is then given by

$$Z(\beta, V) = \sum_{N=0}^{\infty} \frac{V(N)}{N!N!} \varphi_Q{}^N(\beta)\varphi_{\bar{Q}}{}^N(\beta), \quad (2.1)$$

with

$$\varphi_Q(\beta) = \varphi_{\bar{Q}}(\beta) = (2\pi)^{-3} \int d^3p\, e^{-\beta_\mu p^\mu}$$
$$= (m_Q{}^2/2\pi^2\beta)K_2(m_Q\beta) \xrightarrow[m_Q \to 0]{} (1/\pi^2\beta^3) \quad (2.2)$$

for the quark (= antiquark) generating function. Here $\beta \equiv (\beta_\mu \beta^\mu)^{1/2} = 1/kT$ is the inverse temperature, with $\beta_0 \geq 0$, $\beta^2 \geq 0$; m_Q is the quark mass, which we shall generally take to be small. The function $V(N)$ denotes the "free" volume seen by the quarks because of the presence of the antiquarks, and vice versa—i.e., that part of V which does not contain any bubbles of the infinitely repulsive hadronic vacuum of Eq. (1.5). In writing Eq. (2.1), we have assumed that $V(N)$ is large and connected enough to treat all pairs in the plane-wave limit; at sufficiently low densities, this will no longer be valid. In Eq. (2.1), we have also ignored the Fermi statistics of the quarks, and we shall continue to do so. This is possible in the case of "mesonic" matter without conserved quantum numbers; in nuclear matter, however, the presence of baryon-number conservation certainly requires the inclusion of Fermi statistics, since such a system, in a wide range of densities, can behave essentially as a cold Fermi gas.

In the high-density limit $N/V \to \infty$, there will be no repulsive bubbles in V, so that we have

$$V(N) = V^{2N}. \quad (2.3)$$

This yields

$$Z(\beta, V) = I_0(2V\varphi_Q(\beta)) \simeq (4\pi V\varphi_Q)^{-1/2} e^{2V\varphi_Q} \quad (2.4)$$

and hence

$$p_Q\beta \equiv \lim_{V\to\infty}[\partial \ln Z(\beta, V)/\partial V] = 2\varphi_Q(\beta), \quad (2.5)$$

$$n_Q \equiv \lim_{V\to\infty}[(\varphi_Q/2V)\partial \ln Z(\beta, V)/\partial\varphi_Q] = \varphi_Q(\beta) \quad (2.6)$$

for the pressure p_Q and the quark (or antiquark) density n_Q of the system. Combining these relations, we have

$$p_Q\beta = 2n_Q \quad (2.7)$$

as the equation of state of a gas of free quarks and antiquarks in the limit of infinite density (or temperature).

To obtain $V(N)$ at finite density, we fill in the N quarks into the volume V randomly. Let us then consider one point $\vec{x}$ in V. The probability of finding no quark within a region V_0 around $\vec{x}$ is $[(V-V_0)/V]^N$, so that

$$1 - [(V-V_0)/V]^N \quad (2.8)$$

is the probability for this region to contain at least one quark and is therefore just the probability that the point $\vec{x}$ is an allowed position for an antiquark. Integrating over all V with this weight gives

$$(V\{1 - [(V-V_0)/V]^N\})^N \quad (2.9)$$

as the coordinate-space volume available to the antiquarks. Multiplying this by the quark volume V^N, we have

$$V(N) = (V^2\{1 - [(V-V_0)/V]^N\})^N \quad (2.10)$$

as the finite-density coordinate-space volume. For $N/V \to \infty$, we recover Eq. (2.3); in the low-density limit $N/V \to 0$, Eq. (2.10) yields

$$V(N) \simeq (VV_0N)^N \simeq N!\,(VV_0e)^N \quad (2.11)$$

and hence, from Eq. (2.1),

$$Z(\beta, V) \simeq \exp[VV_0e\varphi_Q{}^2(\beta)] \quad (2.12)$$

for the partition function. The low-density virial equations thus become

$$p_Q\beta = eV_0\varphi_Q{}^2(\beta), \quad (2.13)$$

$$n_Q = eV_0\varphi_Q{}^2(\beta) \quad (2.14)$$

and hence yield

$$p_Q\beta = n_Q \quad (2.15)$$

as the equation of state. In the quark-gas limit, the equation of state, as shown in Fig. 5, thus varies from $p_Q\beta/n_Q = 1$ at low temperature to $p_Q\beta/n_Q = 2$ at high temperature. This increase arises because at high temperature and hence high density, quarks need no longer be associated with particular antiquarks and hence gain new coordinate-space freedom. On the other hand, this "uncoupling" provides new momentum degrees of freedom and therefore at high temperatures a slower growth of density, as seen in Fig. 6.

In Eq. (2.1), we have taken the chemical potential μ as zero, $z = \exp(\mu\beta) = 1$, corresponding to

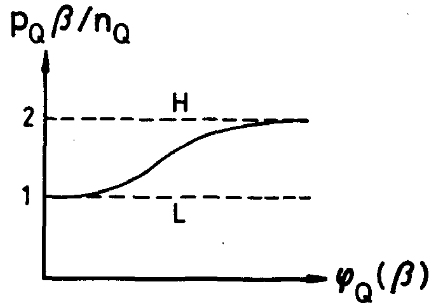

FIG. 5. Equation of state in the quark-gas limit (solid line), together with low- (L) and high- (H) temperature approximations.

unrestricted creation and annihilation of $q\bar{q}$ pairs. With decreasing density, the repulsive hadronic vacuum does, however, provide a restriction to the creation of $q\bar{q}$ pairs widely separated in space. This restriction can be expressed in terms of a creation potential, just as the finite size of hadrons could be described in this fashion.[2] Defining by

$$Z(\beta, V)=\sum_{N=0}^{\infty}\frac{V(N)}{N!N!}\varphi_Q^{2N}(\beta)\equiv\sum_{N=0}^{\infty}\frac{[zV\varphi_Q(\beta)]^{2N}}{N!N!} \tag{2.16}$$

the fugacity z, we obtain through

$$z=e^{\mu\beta}=\lim_{V\to\infty}[\ln Z(\beta, V)/2V\varphi_Q] \tag{2.17}$$

the quark (or antiquark) creation potential μ, which is here, as in the case of extended mesonic hadrons, a given function of the temperature or density, and not an independent variable. From Eqs. (2.4) and (2.12) we find

$$\mu\beta=\begin{cases}0, & n_Q\to\infty\\ \ln[V_0e\varphi_Q(\beta)]-\ln n_Q, & n_Q\to 0\end{cases} \tag{2.18}$$

in the limits of high and low density. As expected, we recover unrestricted creation and annihilation at high density, while at low density the appearance of hadronic vacuum bubbles provides an increasing resistance to creation and hence a growing negative creation potential, as shown in Fig.

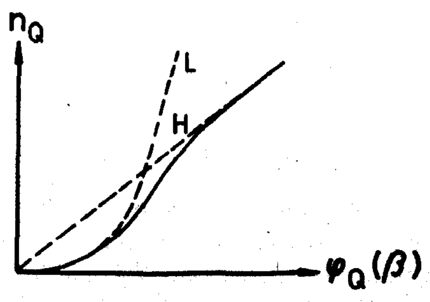

FIG. 6. Quark (or antiquark) density in the quark-gas limit (solid line), together with low- (L) and high- (H) temperature approximations.

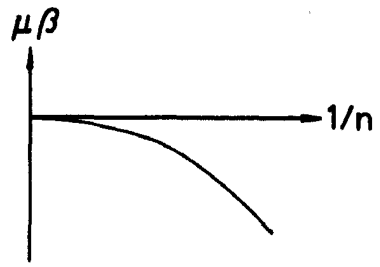

FIG. 7. Creation potential of the quark gas as function of inverse density.

7. We note that the behavior of $\mu\beta$ as a function of $1/n$ here is rather similar to that shown by the hadronic solid[2] as a function of n. The quark system thus behaves with decreasing density similar to the hard-core hadron system with increasing density—a pattern not so surprising when we recall the reciprocal nature of the potentials $U_Q(x)$ and $U_H(x)$.

III. HADRON-GAS LIMIT

In Sec. II we have studied a classical gas of free quarks and antiquarks, taking into account the possible volume restrictions which arise from the presence of hadronic vacuum bubbles at lower densities. We have not, however, considered the quantum effects which occur when an isolated $q\bar{q}$ pair is restricted to a small coordinate-space region V_0. In this section we shall show that the presence of such quantum effects at low temperature turns the behavior of our quark-antiquark system into that of a hadron gas.

For the description of isolated $q\bar{q}$ pairs, center-of-mass variables are more convenient than those of the individual quarks and antiquarks. The corresponding transformation of phase space to the variables

$$\left.\begin{aligned}P_i&=p_i+\bar{p}_i, & k_i&=\tfrac{1}{2}(p_i-\bar{p}_i)\\ X_i&=\tfrac{1}{2}(x_i+\bar{x}_i), & r_i&=x_i-\bar{x}_i\end{aligned}\right\}\quad i=1,2,\ldots,N \tag{3.1}$$

with barred (unbarred) quantities referring to antiquarks (quarks) is carried out in the Appendix, using the covariant constraint formulation.[6] As a result, we obtain instead of Eq. (2.1) the partition function

$$Z(\beta, V)=\sum_{N=0}^{\infty}\left(\frac{(2\pi)^{-6N}}{N!N!}\int\prod_{i=1}^{N}d^3P_i\,d^3k_i\,d^3X_i\,d^3r_i\,e^{-\beta\tilde{P}_{i0}}\right). \tag{3.2}$$

Here the integrations over $\vec{P}_i$ and $\vec{X}_i$ are to be carried out in the overall c.m. system, those over $\vec{k}_i$ and $\vec{r}_i$ in the c.m. system of the ith pair. The energy of this pair is given by

$$\tilde{P}_{i0}=[\vec{P}_i^{\,2}+4(\vec{k}_i^{\,2}+m_Q^{\,2})+U]^{1/2}, \tag{3.3}$$

where m_Q again denotes the quark mass, and the potential U contains all coordinate-space restrictions. As noted in the Introduction, U generally spoils the apparent factorization of Eq. (3.2); however, it is restored at sufficiently low density, when the confinement volumes can be taken as nonoverlapping. We then obtain V^N from the $\vec{X}_i$ integrations, while the $\vec{r}_i$ integrations yield $N!\,V_0^N$. To see the latter result, we first place randomly (but without V_0 overlap) the N quarks in V, and then add the antiquarks one by one. The first antiquark can be placed onto N sites, the second onto $N-1$, and so on. We thus find

$$Z(\beta, V)=\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{VV_0}{(2\pi)^6}\right)^N\left(\int d^3P\,d^3k\,e^{-\beta P_0}\right)^N \quad (3.4)$$

for the classical relativistic partition function at low density, with

$$P_0=[\vec{P}^2+4(\vec{k}^2+m_Q)]^{1/2}. \quad (3.5)$$

Quantum mechanically, the separation of adjacent momentum levels for a particle in a box V is $2\pi V^{-1/3}$. As we want to consider Eq. (3.4) in the limit $V\to\infty$ at fixed $\beta>0$, the level spacing goes to zero and the classical limit provides a good approximation for the c.m. system momenta $\vec{P}$ of the $q\bar{q}$ pairs. With a fixed small confinement volume V_0, this is not the case, however, for the relative momenta $\vec{k}$, which are quantized according to

$$\vec{k}(m)=2\pi V_0^{-1/3}\vec{m},\quad m_i=0,1,2,\ldots,\quad i=1,2,3. \quad (3.6)$$

Correspondingly, the partition function becomes

$$Z(\beta, V)=\sum_{N=0}^{\infty}\frac{V^N}{N!}\left((2\pi)^{-3}\int d^3P\sum_{m}e^{-\beta[\vec{P}^2+4[\vec{k}^2(m)+m_Q^2]]^{1/2}}\right)^N \quad (3.7)$$

instead of Eq. (3.4). At low temperature (large β), the dominant contribution to the sum over m comes from the lowest level $\vec{m}=0$, giving us there with

$$Z(\beta, V)\simeq\sum_{N=0}^{\infty}\frac{V^N}{N!}\left((2\pi)^{-3}\int d^3P\,e^{-\beta[\vec{P}^2+(2m_Q)^2]^{1/2}}\right)^N \quad (3.8)$$

the partition function of a free hadron gas with $m_H=2m_Q$ for the mass of the hadron. This dominance of the lowest level in the relative momentum spectrum of $q\bar{q}$ pairs is seen particularly well in the nonrelativistic limit ($\vec{P}^2\ll m_Q^2$, $\vec{k}^2\ll m_Q^2$), for which summation and integration in Eq. (3.7) factorize to give

$$Z_{\rm NR}(\beta, V)\simeq\sum_{N=0}^{\infty}\frac{V^N}{N!}\left((2\pi)^{-3}\int d^3P\,e^{-\beta(2m_Q+\vec{P}^2/4m_Q)}\right)^N \times\left(1+3e^{-\beta(2\pi)^2/m_QV_0^{2/3}}+\cdots\right)^N. \quad (3.9)$$

For temperatures

$$kT=\beta^{-1}\lesssim m_Q^{-1}V_0^{-2/3}, \quad (3.10)$$

corrections from higher levels are clearly negligible. At a given low temperature, we can include higher-level corrections in quantitative considerations by using Eq. (3.7) instead of (3.8).

The thermodynamics resulting from the partition function (3.8) is of course just that of a hadron gas with free creation and annihilation. Thus we have

$$p_H\beta=\varphi_H(\beta)=n_H \quad (3.11)$$

with

$$\varphi_H(\beta)=(2\pi)^{-3}\int d^3p\,e^{-\beta(p^2+4m_Q^2)^{1/2}}, \quad (3.12)$$

for the hadron pressure p_H and density n_H.

We note that in comparison with the high-temperature form (2.7) of the quark gas, both the low-temperature hadron gas (3.1) and the low-temperature extrapolation of the quark gas (2.15) show a reduction by a factor of 2 in the number of degrees of freedom. The cause for the reduction is, however, quite different in the two cases. In the low-temperature quark gas, it is the appearance of hadronic vacuum bubbles and the ensuing modification of the available coordinate space which provides the reduction. In the hadron-gas limit, the quantum effect due to the small confinement volume modifies the available momentum space to cause the reduction. There, a small energy input at low temperature cannot excite the hadrons out of their ground state and thus can only go into kinetic energy of the hadrons. As the temperature increases, the appearance of overlapping confinement volumes will eventually lead to larger quantization volumes than that of Eq. (3.6). Consequently, the level spacing decreases until finally both excitation forms, relative and overall $q\bar{q}$ pair motion, become equally likely—as they are in a gas of free quarks and antiquarks giving Eq. (2.7).

IV. HADRON-QUARK TRANSITION

In this section we want to study which type of thermodynamical behavior—quark gas or hadron gas—our system of confined $q\bar{q}$ pairs exhibits in a given temperature range, and at what point the transition takes place. The basic thermodynamic function, the free-energy density $A(\beta)$,

$$\beta A(\beta) = \lim_{V\to\infty}[-\partial \ln Z(\beta, V)/\partial V]_\beta = -p\beta\,, \quad (4.1)$$

is obtained from Eqs. (2.4) and (2.12) and Eq. (3.8), corresponding to quark gas and hadron gas, respectively. The hadron-gas form (3.8) is just the leading low-temperature term, while the quark-gas form with Eq. (2.10) includes the high-temperature limit together with finite-temperature corrections, down to the low-density form (2.12). By not calculating high-temperature corrections to the hadron-gas form we anticipate a hadron-quark transition at rather low temperature and density.

Let us begin by assuming that the quark mass can be neglected ($m_Q \approx 0$), and that the quark-gas behavior near the transition point is correctly described by the low-density form (2.12); we shall shortly remove both of these simplifying assumptions. In the quark limit we then have

$$\beta A_Q(\beta) \simeq -eV_0/\pi^4\beta^6\,, \quad (4.2)$$

and in the hadron limit,

$$\beta A_H(\beta) \simeq -1/\pi^2\beta^3 \quad (4.3)$$

for the free energy. The two forms are shown in Fig. 8: Below the temperature

$$kT_c = 1/\beta_c = (\pi^2/eV_0)^{1/3}\,, \quad (4.4)$$

the free energy is lowest for the hadron gas, above T_c for the quark gas. Since the system in equilibrium must always be in the state of lowest free energy, we are thus led to a phase transition of first order from hadron to quark matter at the critical temperature T_c. If we fix the confinement volume

$$V_0 = 4\pi/3m_\pi^{\,3} \quad (4.5)$$

by the mass m_π of the lightest hadron, then we obtain

$$kT_c = 0.95m_\pi \quad (4.6)$$

as a value of the critical temperature. This value agrees surprisingly well with that generally obtained for the critical or ultimate temperature of hadron physics.[7]

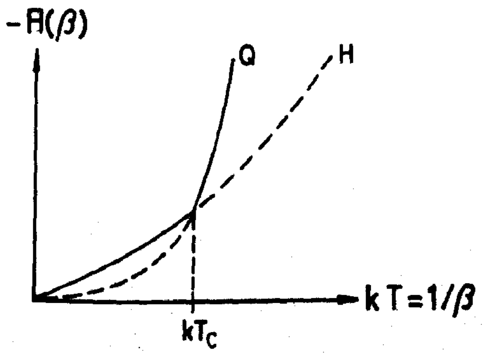

FIG. 8. Free energy of confined $q\bar{q}$ system (solid line), together with quark-gas (Q) and hadron-gas (H) limits.

We now want to show that this result remains essentially unchanged if we drop the simplifying assumptions made above. The partition function [(2.1) and (2.10)] can be written as

$$Z(\beta, V) = \sum_{N=0}^{\infty} Z_N(\beta, V)\,, \quad (4.7)$$

with

$$\ln Z_N(\beta, V) \simeq N\left\{2\left(\ln\frac{V\varphi_Q}{N}+1\right)+\ln\left[1-\left(1-\frac{V_0}{V}\right)^N\right]\right\} \quad (4.8)$$

in the Stirling approximation. The largest term in the sum (4.7) satisfies $\partial Z_N/\partial N = 0$, which for $V\to\infty$ and with $x \equiv V_0\varphi_Q$ yields

$$2\ln[x(1-e^{-n_QV_0})/n_QV_0] + n_QV_0/(e^{n_QV_0}-1) = 0 \quad (4.9)$$

for the quark density n_Q as function of x and hence of the temperature. Approximating the sum (4.7) by its largest term $Z_{\bar{N}}(\beta, V)$,

$$\ln Z(\beta, V) \simeq \ln Z_{\bar{N}}(\beta, V)\,, \quad (4.10)$$

we find

$$\beta A_Q(\beta) \simeq -n_Q[2 - n_QV_0/(e^{n_QV_0}-1)] \quad (4.11)$$

for the free energy. The corresponding form for the hadron gas is by Eq. (3.8) simply

$$\beta A_H(\beta) = -\varphi_H(\beta)\,. \quad (4.12)$$

The two functions $A_Q(\beta)$ and $A_H(\beta)$ cross, as can be seen by numerical evaluation of (4.9), (4.11), and (4.12), at

$$kT_c \simeq 1.1m_\pi \quad (4.13)$$

when $V_0 = 4\pi/3m_\pi^{\,3}$ and $m_\pi = 2m_Q$; this confirms our result of above. We note moreover that the direct dependence of T_c on quark or pion mass is essentially negligible; the dependence in Eqs. (4.6) and (4.13) comes from the coupling between T_c and V_0. Setting $2m_Q = m_\pi = 0$ in $\varphi_Q(\beta)$ and $\varphi_H(\beta)$ produces only a change of about 10% in T_c.

In Fig. 9 we show the density of our $q\bar{q}$ system as a function of temperature. At T_c the density drops, since the available kinetic energy in the hadron phase is that of the hadrons ($q\bar{q}$ pairs in the ground state) only, while in the quark phase quarks and antiquarks also have relative kinetic energy.

In Figs. 8 and 9 as well as in our above remarks concerning the nature of T_c, we have assumed that the system does indeed show the discontinuous behavior obtained from the crossover of the two approximations for $A(\beta)$. This, in turn, leads to a first-order phase transition. What justification

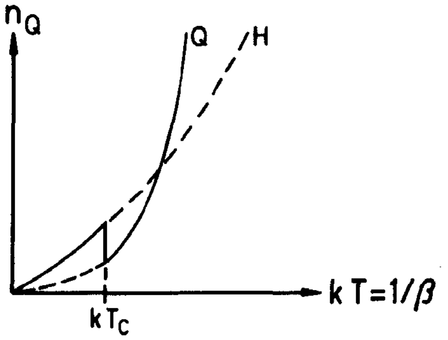

FIG. 9. Density of confined $q\bar{q}$ pairs (solid line), together with quark-gas (Q) and hadron-gas (H) limits.

is there for this assumption?

We recall first the analogous situation in the hard-core hadron gas.[2] There the phase transition between gas and solid was also determined by the crossover of the low-density (gas) and the high-density (solid) approximation. In that case, however, there are computer simulations[8] and a rigorous proof for two-dimensional lattice gases[9] to support the conjecture of a phase transition.

Here we have neither of these supports directly. However, the complementary nature of the two situations—extended hadrons and confined quarks—from the point of constituent mobility appears to us as very strong reason for a discontinuous change of state also for the gas of confined quarks. The crossover point T_c is in both cases determined as the point where the free energy changes from one region of constituent mobility to another. As expected, we find a decrease in the density of the $q\bar{q}$ system (Fig. 9) when the motion of quarks or antiquarks becomes unbounded, just as we find an increase in the gas of extended hadrons once the motion of the hadrons becomes bounded (Fig. 10).

Nevertheless, we should strongly emphasize the desirability of a computer simulation and/or a rigorous study of simplified models retaining the essential features also for the case of confined $q\bar{q}$ pairs.

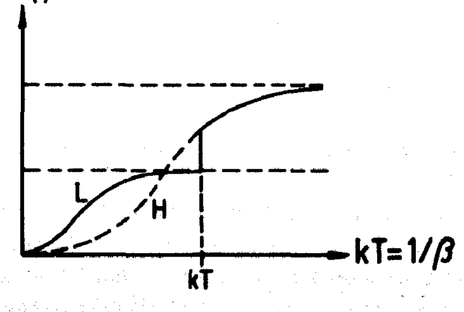

FIG. 10. Density of hard-sphere hadron system (solid line), together with low- (L) and high- (H) density limits.

V. CONCLUSIONS

We have shown that a model of confined $q\bar{q}$ pairs in free creation and annihilation provides at high and at low temperature (and hence density) two basically different forms of thermodynamic behavior. Minimizing the free energy then yields a phase transition at the critical temperature $T_c \sim m_\pi$, in good agreement with expectations from hadron physics. In closing, we want to compare these results with similar conclusions obtained for a gas of extended hadrons[2] and from perturbative QCD considerations.[10,11]

The most striking difference between confined quarks and extended hadrons lies in the value of T_c: Using in both cases (as confinement volume and as hadron size) the basic input $V_0 = 4\pi/3m_\pi^3$, we obtain $T_c^Q \sim m_\pi$ for confined $q\bar{q}$ pairs, but $T_c^H \sim 350 m_\pi$ for hard-core hadrons. In neither case was any further attractive or repulsive mechanism considered beyond the basic volume—no resonance excitation, mass-dependent hadron size, or energy-dependent bag size. In the absence of such effects, a hadron gas would thus change into a quark gas at temperatures much lower than those needed for solidification, as shown in Fig. 11. What reasons are there for this?

Structurally, the transitions in the two cases are quite different. In the confined-quark picture, the two states of matter—free hadrons vs free quarks and antiquarks—are basically different in their number of available degrees of freedom; this number is twice as high for a gas with independent quarks and antiquarks as it is for a gas of hadrons. In the model of extended hadrons, we have two different states of matter for more quantitative reasons. The region of space, V_B, blocked in the gas phase by the presence of a hadron, and the space region $\tilde{V}_B$ still accessible to a hadron in the solid phase, are not the same for two or more space

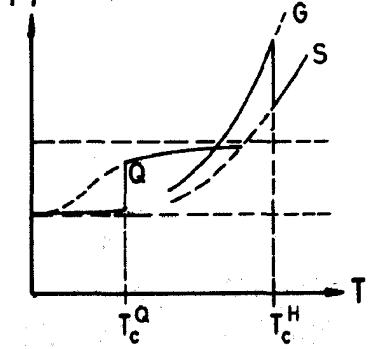

FIG. 11. Equation of state and phase transition for confined quarks (T_c^Q) and for extended hadrons (T_c^H), together with quark-gas (Q), hadron-gas (G), and hadronic-solid (S) limits.

dimensions: $V_E > \tilde{V}_E$. Both forms (gas and solid) here lead to singular behavior when the density approaches the inverse volume $n \to V_E^{-1}$ or $n \to \tilde{V}_E^{-1}$, but this happens sooner for the gas form than for the densely packed solid. If V_E and $\tilde{V}_E$ were identical, as is indeed the case for models in one space dimension, then the two limits coincide and give a one-phase world.

Some further insight can be gained by considering the critical densities in the two models; with $n_c \equiv n(\beta_c)$ we find

$$n_c^Q \simeq 0.4/V_0\,,\qquad n_c^H \simeq 0.5/V_0\,. \tag{5.1}$$

Although the density for solidification is still higher than that giving the hadron-quark transition, the two values are much closer than those of the critical temperatures. This is due to the strong resistance the hard-core gas offers to further production at high temperature: A large temperature increases then produces a comparatively small change in density (see Fig. 10). For a gas of confined quarks, this effect does not arise; in fact, the opposition to production decreases with increasing temperature (see Fig. 7).

Finally, we return to a principal difference between a free quark gas and a free hadron gas, which tends to modify the complementarity discussed in Sec. I. Although a system of hydrogen atoms becomes a free atomic gas at low density and a free electron gas at high density (in the metallic phase), the two "vacuums" associated with this freedom differ. For metallic hydrogen, this difference is included in the quasiparticle nature of the conduction electrons, which are free only above a mean field changing, e.g., the constituent mass. Similar effects could (and probably will) arise also in our approach, and hence a true complementarity between confined quarks and extended hadrons might arise only when some interaction effects (resonance excitation, mass-size relations) are included.

In quantum chromodynamics as complete theory of interacting quarks, it should in principle be possible to calculate the thermodynamic behavior of a multiquark system at all temperatures, and thus also to answer unambiguously if and where a phase transition between hadron and quark matter occurs. In practice, exact calculations are presently possible only in the high-temperature or high-density region, where asymptotic freedom allows the application of perturbative methods. Since in the transition region to hadrons perturbation theory is likely to break down, one must also here make conjectures about the nature of the transition and/or about the behavior of the system be-

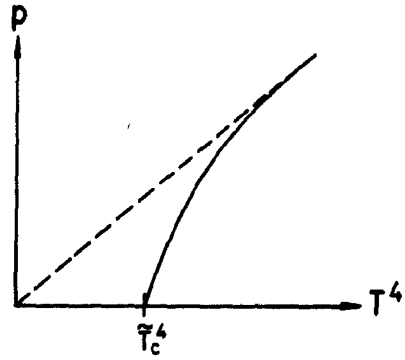

FIG. 12. Pressure of a quark gas in perturbative QCD for zeroth (dashed line) and first (solid line) order in $\alpha_{eff}(T)$.

low the transition region. For our case of zero chemical potential (mesonic matter), the problem has been studied particularly by Kapusta[10] and also by Kalashnikov and Klimov.[11] Let us briefly sketch their approach and compare it with ours.

The pressure of a system of noninteracting quarks has in the high-temperature limit the form [see Eq. (2.5)]

$$p_0 \simeq c_0 T^4\,, \tag{5.2}$$

where c_0 is a constant depending only on internal quantum numbers. Including interaction in the lowest-order approximation, one finds

$$p_1 \simeq c_0 T^4[1 - c_1 \alpha_{eff}(T)]\,, \tag{5.3}$$

where the effective coupling constant $\alpha_{eff}(T) > 0$ is by renormalization-group methods given as

$$\alpha_{eff}(T) \simeq c_2/[1 + \bar{c}_2 \ln(T/T_0)]\,, \tag{5.4}$$

with the constants c_1, c_2, and $\bar{c}_2$ again determined by the specific quantum-number structure of the problem. The normalization point T_0 for the temperature is a basic parameter of the approach, to be fixed empirically; it corresponds to our V_0.

The fundamental form (5.3) of the pressure is shown in Fig. 12. It is seen that at $T = \tilde{T}_c$, the pressure becomes negative. This has been interpreted[10,11] as an indication that there the system becomes unstable and a phase transition occurs. Choosing $T_0 \simeq 300$ MeV,[11] and assuming $\tilde{T}_c$ to be in fact the critical temperature T_c, one finds

$$T_c \simeq 2m_\pi\,, \tag{5.5}$$

in fair agreement with our result (4.13).

It remains quite open, however, to what extent such an argumentation is possible. At the crossover temperature $\tilde{T}_c$, one has $\alpha_{eff}(\tilde{T}_c) \simeq \frac{1}{2}$, so that the applicability of perturbation theory is questionable. Such doubts are substantiated by the calculation of specific nonperturbative contributions[10] (e.g., "plasmon" effects), which near $\tilde{T}_c$ are found to be much larger than p_1. Apart from providing a general agreement in the determination of the temperature region where "something is happening," QCD does not at present seem to clarify the question of phase transition beyond the argumentation of our model. A more detailed comparison between the high-temperature forms in perturbative QCD and in our approach will be given elsewhere.

APPENDIX

The partition function for N free relativistic quarks and N corresponding antiquarks can be written covariantly as

$$Z_N(\beta, V) = (N!)^{-2}(2\pi)^{-6N}\left(\int d\mu\, e^{-\beta_\mu(p^\mu+\bar{p}^\mu)}\right)^N, \quad \text{(A1)}$$

where the phase-space measure $d\mu$ is given by

$$\begin{aligned} d\mu = {} & d^4p\,\theta(p_0)\delta(p^2-m^2)d^4\bar{p}\,\theta(\bar{p}_0)\delta(\bar{p}^2-m^2) \\ & \times d^4x\,\delta(x_\mu e^\mu)\theta\{R^2-[(x_\mu e^\mu)^2-x^2]\} \\ & \times d^4\bar{x}\,\delta(\bar{x}_\mu e^\mu)\theta\{R^2-[(\bar{x}_\mu e^\mu)-\bar{x}^2]\}\times\Delta, \end{aligned} \quad \text{(A2)}$$

$$\Delta = 2(p_\mu e^\mu)\times 2(\bar{p}_\mu e^\mu). \quad \text{(A3)}$$

Here e^μ is a unit vector which is timelike in the rest system of the box ($V=4\pi R^3/3$) containing the system

$$V^\mu = e^\mu V. \quad \text{(A4)}$$

The temperature is also defined in this system

$$\beta_\mu = e_\mu \beta. \quad \text{(A5)}$$

For a quark-antiquark pair with the center-of-mass variables defined in Eq. (3.1), instead of the constraints

$$\varphi \equiv (p^2-m^2)=0, \quad \bar{\varphi}\equiv(\bar{p}^2-m^2)=0, \quad \text{(A6)}$$

$$\chi \equiv (x_\mu e^\mu)=0, \quad \bar{\chi}\equiv(\bar{x}_\mu e^\mu)=0, \quad \text{(A7)}$$

of Eq. (A2), we now choose the new constraints

$$\varphi_+ \equiv \varphi+\bar{\varphi} = \tfrac{1}{2}[P^2-4(m^2+k^2)]=0, \quad \text{(A8)}$$

$$\varphi_- \equiv \varphi-\bar{\varphi} = 2(P_\mu k^\mu)=0, \quad \text{(A9)}$$

$$\chi_+ \equiv X_\mu e^\mu = 0, \quad \text{(A10)}$$

$$\chi_- \equiv r_\mu \hat{e}^\mu = 0. \quad \text{(A11)}$$

Here e^μ is defined as before, while $\hat{e}^\mu$ is a unit vector timelike in the c.m. system of the quark-antiquark pair:

$$\hat{e}^\mu = P^\mu/(P_\mu P^\mu)^{1/2}. \quad \text{(A12)}$$

For the overall volume we have again Eq. (A4), while the confinement volume $V_0=4\pi R_0^3/3$ satisfies

$$V_0^\mu = \hat{e}^\mu V_0. \quad \text{(A13)}$$

The transformed constraint determinant Δ becomes

$$\Delta = 2(P_\mu e^\mu)\sqrt{P^2} \quad \text{(A14)}$$

and the θ functions of energy positivity now read

$$\theta(P_0)\theta(P_0^2-4k_0^2). \quad \text{(A15)}$$

We thus obtain

$$\begin{aligned} d\mu = {} & d^4P\,\theta(P_0)\delta(\varphi_+)d^4k\,\theta(P_0^2-4k_0^2)\delta(\varphi_-) \\ & \times d^4X\,\delta(\chi_+)\theta(R^2-(\chi_+^2-X_\mu X^\mu)) \\ & \times d^4r\,\delta(\chi_-)\theta(R_0^2-(\chi_- - r_\mu r^\mu))\times\Delta \end{aligned} \quad \text{(A16)}$$

as the phase-space measure. Carrying out the time and the energy integrations, Eq. (A16) gives instead of (A1) in the rest frame of the box

$$Z_N(\beta, V) = \frac{1}{(2\pi)^{6N}N!N!}\int d^3P\,d^3X\,dk\,k^2dr\,r^2e^{-\beta P_0} \quad \text{(A17)}$$

with

$$\begin{aligned} P_0 &= [\vec{P}^2+4(\vec{k}^2+m^2)+U]^{1/2}, \\ P &\equiv [(P_\mu e^\mu)^2-(P_\mu P^\mu)]^{1/2}, \end{aligned} \quad \text{(A18)}$$

$$X \equiv [(X_\mu e^\mu)^2-(X_\mu X^\mu)]^{1/2}, \quad \text{(A19)}$$

$$k \equiv [(k_\mu P^\mu)^2/(P_\mu P^\mu)-(k_\mu k^\mu)]^{1/2}, \quad \text{(A20)}$$

$$r \equiv [r_\mu P^\mu)^2/(P_\mu P^\mu)-(r_\mu r^\mu)]^{1/2} \quad \text{(A21)}$$

for momentum and position of the pair in the rest system of the box and for relative momentum and position in the $q\bar{q}$ rest system, respectively. For the partition function, this yields Eq. (3.2).

[1]D. Ivanenko and D. F. Kurdgelaidze, Lett. Nuovo Cimento 2, 13 (1969); N. Itoh, Prog. Theor. Phys. 44, 291 (1970); F. Iachello, W. D. Langer, and A. Lande, Nucl. Phys. A219, 612 (1974); N. Cabibbo and G. Parisi, Phys. Lett. 59B, 67 (1975); J. C. Collins and M. J. Perry, Phys. Rev. Lett. 34, 1353 (1975); G. Baym and S. A. Chin, Phys. Lett. 62B, 241 (1976); G. Chapline and M. Nauenberg, Nature 264, 235 (1976); B. A. Freedman and L. McLerran, MIT Report No. 541, 1976 (unpublished); G. Chapline and M. Nauenberg, Phys. Rev. D 16, 450 (1977); M. B. Kislinger and P. D. Morley, Phys. Lett. 67B, 371 (1977); H. Satz, in *Hadronic Matter at Extreme Energy Density*, edited by N. Cabibbo and L. Sertorio (Plenum, New York, 1980).

[2]F. Karsch and H. Satz, Phys. Rev. D 21, 1168 (1980).

[3]For further discussion see, e.g., T. D. Lee, Columbia Report No. CU-TP-170, 1979 (unpublished).

[4]L. Susskind, Phys. Rev. D 20, 2610 (1979).

[5]H. Miyazawa, Phys. Rev. D 20, 2953 (1979).
[6]L. D. Fadeev, Teor. Mat. Fiz. 1, 3 (1969) [Theor. Math. Phys. 1, 1 (1969)]; I. T. Todorov, Dubna Report No. EL-10125, 1976 (unpublished); A. Komar, Phys. Rev. D 18, 1881 (1978); 18, 1887 (1978); 18, 3617 (1978).
[7]R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965); S. Frautschi, Phys. Rev. D 3, 2821 (1971); R. D. Carlitz, *ibid.* 5, 3231 (1972); M. I. Gorenstein *et al.*, Nucl. Phys. B76, 458 (1974); J. Baacke, Acta Phys. Pol. B8, 625 (1977).
[8]See, e.g., J. A. Barker and D. Henderson, Rev. Mod. Phys. 40, 587 (1976).
[9]See R. L. Dobrushin, Funktional'nyi Anliz i Ego Prilozheniya 2, 44 (1968).
[10]J. I. Kapusta, Nucl. Phys. B148, 461 (1979).
[11]O. K. Kalashnikov and V. V. Klimov, Phys. Lett. 88B, 328 (1979).