THE MAP $\text{BSG} \to \text{A}(\ast) \to QS^0$

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§1. INTRODUCTION

Let $A(X)$ be the algebraic K-theory of the space $X$. This can be defined in various ways, see [9], [10], [11]. [12]. Let $BG$ be the space classifying $O$-dimensional virtual spherical fiberbundles. There are maps $F : BG \to A(X)$, $l : A(X) \to K(Z); i : QS^0 \to A(\ast)$. In [10],[11], maps $A(\ast) \to QS^0$. splitting $i$ up to homotopy are constructed.

In this paper, we construct a splitting map $\text{Tr} : A(\ast) \to QS^0.$ and compute the composite $BG \to A(\ast) \to QS^0$. We apply this construction to show that $\pi_3(\text{Wh}^{\ast}(\ast)) \cong \mathbb{Z}/2$. A further application is [4]. There it is used that the splitting given here agrees with the splitting in [11]; this will be proved in [5].

Recall that

$$A(\ast) \cong \lim_{n,k} B \text{Aut}(\nu^k S^n)^+$$

where $\text{Aut}$ denotes the simplicial monoid of homotopy equivalences, and $+$ denotes the Quillen plus
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construction. In this description of $A(\star)$, we can define $f : BG \to A(\star)$ as the inclusion

$$BG = \lim_{n}(B \text{Aut } S^n) \subset \lim_{n,k} B \text{Aut}(v^kS^n)^+ = A(\star)$$

and $\iota : A(\star) \to K(\mathbb{Z})$ as the linearization map

$$A(\star) = \lim_{n,k} B \text{Aut}(v^kS^n)^+ \to \lim_{n,k} B \text{Aut}(H\eta(v^kS^n))^+ = K(\mathbb{Z}).$$

Let $\text{BSG} \subset BG$ classify the oriented spherical fiberbundles. The composite

$$\text{BSG} \to BG \xrightarrow{f} A(\star) \xrightarrow{\iota} K(\mathbb{Z})$$

is the trivial map.

In §3 we will show that the composite

$$BG \xrightarrow{f} A(\star) \xrightarrow{\text{Tr}} QS^0$$

equals a certain map $\eta : BG \to QS^0$, studied in §2. In §2 we show that if $i \geq 3$, then

$$\pi_{i-1}^S = \pi_1(BG) \xrightarrow{\eta} \pi_1(QS^0) \cong \pi_1^S$$

is given by multiplication with $\eta$, the generator of $\pi_1^S \cong \mathbb{Z}/2$. In particular, for $i \geq 3$ the map

$$\Theta : \pi_{i-1}^S \oplus \pi_1^S \cong \pi_1(BG) \oplus \pi_1(QS^0) \xrightarrow{f_\star + i_\star} \pi_1(A(\star)) \to \pi_1(QS^0) \cong \pi_1^S$$

is given by $\Theta(x, y) = \eta x + y$.

The splitting of $A(\star)$ induces a splitting $\pi_1(A(\star)) \cong \pi_1(QS^0) \oplus C_i$. If $x \in \pi_i^S$, $i \geq 2$, then $\Theta(x, \eta x) = 0$, so that $f_\star(x) + i_\star(\eta x) \in C_{i+1}$. We want to show that for some choices of $x$, this element is nontrivial.

The composite $QS^0 \xrightarrow{i} A(\star) \xrightarrow{\iota} K(\mathbb{Z})$ is studied in [7]. There it is shown that
\[ \pi_{4i+3}(QS^0) = \pi_{4i+3}^S \rightarrow \pi_{4i+3}(K(\mathbb{Z})) \] is injective on the image of the J-homomorphism.

Recall from [1] that there are classes \( \mu_{8i+1} \in \pi_{8i+1}^S \), \( i \geq 1 \), so that \( \eta \mu_{8i+1} \) is in the image of the J-homomorphism. Similarly, \( \eta^3 \in \pi_3^S \) is also in the image of the J-homomorphism. Choose \( x = \mu_{8i+1} \), \( \bar{x} = \eta \mu_{8i+1} \) or \( x = \eta^2 \). Then \( f_\#(x) + i_\#(\bar{x}) \in C_{8i+3} \) and
\[ l_\#(f_\#(x) + i_\#(\bar{x})) = l_\#(i_\#(\bar{x})) \neq 0. \]

We have proved

**Theorem 1.1.** The kernel of the map \( \text{Tr}_n : \pi_n(A(\#)) \rightarrow \pi_n(QS^0) \) contains a nontrivial element of order 2 if \( n \equiv 2, 3 \pmod{8} \); \( n \geq 3 \).

On the other hand, it is known that \( C_3 \leq \mathbb{Z}/2 \) [6], so we have

**Corollary 1.2.** \( \pi_3 A(\#) \cong \pi_3^S \oplus \mathbb{Z}/2 \).

It is known [9], [11] that \( A(\#) \) splits as a product
\[ A(\#) = QS^0 \times \text{Wh}^{\text{Diff}}(\#) \times \mu. \]
It will be proved in [13] that \( \mu = 0 \). We conclude

**Theorem 1.3.**

(i) \( \pi_3 \text{Wh}^{\text{Diff}}(\#) = \mathbb{Z}/2 \)
(ii) There are nontrivial two-torsion classes in
\[ \pi_{8i+2}(\text{Wh}^{\text{Diff}}(\mathbb{M})) \text{ and } \pi_{8i+3}(\text{Wh}^{\text{Diff}}(\mathbb{M})) ; \quad i \geq 1. \]

§2. SPHERICAL FIBER BUNDLES AND η.

In this paragraph we study a certain map \( \eta : BG \to G \). We first give a homotopy theoretical definition of \( \eta \) and calculate the induced maps of homotopy groups. Finally, we show that \( \eta \) agrees with a geometrically defined map, which will be used in §3.

Let \( X = \Omega^3 Y \) be a threefold loop space. Let
\[ \tilde{\eta} : S^3 \to S^2 \text{ be the Hopf map.} \]

Definition 2.1. \( \eta_X : BX = \Omega^2 Y \to \Omega^3 Y = X \) is the map induced by \( \tilde{\eta} \).

Example 2.2. \( X = \Omega^\infty S^\infty \). We identify \( \pi_\ast(X) \) with the ring \( \pi_\ast^S \) of stable homotopy groups of spheres. The map \( (\eta_X)_\ast : \pi_\ast^S = \pi_\ast(X) \to \pi_\ast(BX) = \pi_\ast+1 \) is given by product with \( \tilde{\eta} \in \pi_1^S \).

Example 2.3. Let \( X = \mathbb{Z} \times BG \) be the classifying space of based stable spherical fibrations: \( X \) is an infinite loop space [3]. Then \( \Omega X = \Omega BG \) can be identified with the space of stable homotopy equivalences of spheres, i.e., \( i : \Omega X \sim (\Omega^\infty S^\infty)_{\pm 1} \). This equivalence is not an H-space.
equivalence, when $\Omega^\infty S^\infty = QS^0$ is given the $H$-space structure derived from loop sum. But

$$\Omega^3 i : \Omega^4 X \overset{\sim}{\longrightarrow} \Omega^3 (QS^0)$$

is an equivalence of threefold loopspaces, so that

$$(\eta_{\Omega^3 QS^0}) \Omega^3 i \simeq (\Omega^3 i)(\eta_X).$$

We conclude from the previous example, that for $i \geq 3$

$$(\eta_{\mathbb{Z} \times BG}) \ast : \pi^S_{i-1} \cong \pi_i (\mathbb{Z} \times BG) \rightarrow \pi_i (G) = \pi_i^S$$

is induced by composition with $\eta \in \pi_i^S$ for $i \geq 3$. For $i \leq 2$ we do not get any information. Actually, $\Omega^3 X$ is not equivalent to $\Omega^2 (QS^0)$ as a threefold loopspace. The induced map $(\eta_{\mathbb{Z} \times BG}) \ast : \pi^S_2 \rightarrow \pi^S_3$ is trivial, whereas multiplication by $\eta$ is nontrivial.

Let $X$ be an infinite loop space. Composition of loops defines an infinite loop map

$$\mu : \Omega^\infty S^\infty \times X \rightarrow X.$$ 

There are structure maps $\Theta_n : ES_n \times \Sigma_n X^n \rightarrow X$, and a commutative diagram

$$\begin{array}{ccc}
\Omega^\infty S^\infty \times X & \xrightarrow{(id \times_{\Sigma} \Delta)} & ES^\infty \times \Sigma X^m \\
\downarrow \downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \\
(\mu) & \xrightarrow{\Theta} & X
\end{array}$$

(2.4)

There are two maps $f_i : S^1 \times X \rightarrow BS_2 \times X$, $f_i = (f'_i \times id)$, where $f'_o$ is the trivial map, and $f'_i$ represents the generator of $\pi_1(BS_2) = \mathbb{Z}/2$. 
Composition with the square above defines maps $g_1 : S^1 \times X \to \coprod_{m \geq 0} \Sigma^m \times X \to X$. The difference $g_1 - g_0$ defines a map $g : S^1 \wedge X \to X$. This difference is the image under $\mu$ of the difference $(i_2 g_1 - i_2 g_0) \times \text{id}$. But $i_2 g_1 - i_2 g_0$ is equal to $\tilde{\eta} : S^1 \to QS^0$, so that the adjoint of $g$ is the map $\eta_X : X \to \Omega X$.

In particular, the map $\eta_{Z \times BG} : Z \times BG \to G$ can be described as the difference between the adjoints of the maps $g_i$ $(i=0,1)$

$$g_i : S^1 \times (Z \times BG) \to E \Sigma^2 \times \Sigma^2 (Z \times BG)^2 \xrightarrow{\Theta_2} Z \times BG.$$

Let $\xi$ be the standard (virtual) spherical fiberbundle on $Z \times BG$. Let $\wedge$ denote fiberwise smashproduct. Then $g_i$ classifies certain virtual bundles on $S^1 \times (Z \times BG)$. These bundles are the identifications of the bundle $\xi \wedge \xi$ on $I \times (Z \times BG)$, using certain bundle maps $\tau_1 : \xi \wedge \xi \to \xi \wedge \xi$ as clutching function, where $\tau_0 = \text{id}$, and $\tau_1(x \wedge y) = y \wedge x$.

We reformulate this description as follows.

**Lemma 2.5.** Let $\xi$ be the standard bundle over $Z \times BG$.

The automorphisms $\tau_i : \xi \wedge \xi \to \xi \wedge \xi$ $(i = 0,1)$ induce maps

$$t_i : Z \times BG \to G.$$

The difference $t_1 - t_0$ equals $\eta : Z \times BG \to G$ up to homotopy.
Finally, consider the following situation. Let $B$ be a finite dimensional space. Let $\xi$ be a spherical fibration over $B$, classified by a map

$$f: B \to \mathbb{Z} \times BG.$$ 

Let $\xi'$ be a spherical fibration over $B$, and $u$ a fiber homotopy trivialization:

$$u : \xi' \wedge \xi \to S^N \times B.$$ 

The map $u$ can be interpreted as an $S$-duality parametrized over $B$, see [2].

A 2N-dual $u'$ of this map is a map

$$u': S^N \times B \to \xi' \wedge \xi'$$

such that the following diagram commutes up to fiber homotopy:

$$\begin{array}{ccc}
\xi' \wedge \xi \wedge (S^N \times B) & \xrightarrow{u \wedge \text{id}} & (S^N \times B) \wedge (S^N \times B) \\
\text{id} \wedge u' & \downarrow & \Downarrow \\
\xi' \wedge \xi \wedge \xi \wedge \xi' & \xrightarrow{v} & S^{2N} \times B
\end{array}$$

where $v(a,b,c,d) = u(a,c) \wedge u(d,b)$. The transfer

$$\text{Tr} : B \to \Omega^* S^N \to QS^O$$

is defined as the adjoint of the map

$$t : S^N \times B \xrightarrow{u'} \xi' \wedge \xi' \xrightarrow{\text{Twist}} \xi' \wedge \xi \xrightarrow{u} S^N \times B \to S^N.$$ 

**Lemma 2.6.** The following diagram is homotopy commutative

$$\begin{array}{ccc}
B & \xrightarrow{f} & \mathbb{Z} \times BG \\
\text{Tr} & \downarrow & \downarrow \\
QS^O & \xleftarrow{i} & G
\end{array}$$
where \( i : SG \to QS^0 \) is the standard identification of \( SG \) with the component of \( 1 \) in \( QS^0 \).

**Proof.** The map \( Tr \) can also be defined as the adjoint of a suspension of \( t \):

\[
\text{idat} : S^{2N} \times B \xrightarrow{id \cup u'} (S^N \times B) \wedge \xi \wedge \xi' \to \xrightarrow{id \cup (uoTwist)} (S^N \times B) \wedge (S^N \times B).
\]

Let \( \xi' \wedge \xi \wedge \xi \wedge \xi \to \xi' \wedge \xi \wedge \xi \wedge \xi' \) be the map permuting the second and third factor. By assumption, the following diagram commutes up to fiber homotopy

\[
\begin{array}{ccc}
(S^N \times B) \wedge (S^N \times B) & \xrightarrow{id \cup u'} & S^N \wedge \xi \wedge \xi' \\
\downarrow v & & \downarrow v \\
S^N \wedge \xi \wedge \xi' & \xrightarrow{uoTwist} & (S^N \times B) \wedge (S^N \times B)
\end{array}
\]

We conclude that \( B \to G \subset QS^0 \) is the difference \( t_1' - t_0' \) between the maps

\[t_1' : B' \to G\]

induced by the automorphisms

\[\tau_1' : \xi' \wedge \xi \wedge \xi \wedge \xi' \to \xi' \wedge \xi \wedge \xi \wedge \xi'.\]

\[\tau_1' = Tw_{23}, \quad \tau_0' \text{ identity. The lemma follows from 2.5.}\]

§3. **TRANSFER AND SPLITTING**

In this paragraph we will construct a splitting map \( Tr : A(\ast) \to QS^0 \). This splitting map will be used to prove
Theorem 1.1. In a later paper it will show that this map agrees with the splitting maps in [10] and [11], cf [5].

We recall some properties of the transfer map [2].

Let $B$ be a finite dimensional space. Let $F \to E \to B$ be a fibration with section, and suppose that fiber $F$ is homotopy equivalent to a finite complex. Then there is a transfer map $\tau : B \to \Omega^\infty S^\infty(\{E_+\})$. Let $\text{Tr}_E : B \to \Omega^\infty S^\infty$ be the composite of $\tau$ with the map $\Omega^\infty S^\infty(\{E_+\}) \to \Omega^\infty S^\infty(\text{pt}_+) = \Omega^\infty S^\infty$, induced by $E \to \text{pt}$.

We will need the following properties of the transfer:

Let $S^1 \wedge E \to B$ be the fiberwise double suspension of $E$.

3.1. $\text{Tr}_E \simeq \text{Tr} : S^2_{AE} \to B \Omega^\infty S^\infty$.

Let $E_1, E_2$ be two fibrations over $B$ as above. Then we can consider the fiberwise wedge $E = E_1 \vee E_2 \to B$.

3.2. $\text{Tr}_E \simeq \text{Tr}_{E_2} + \text{Tr}_{E_2}$.

These properties will be proved at the end of this section.

Recall that the algebraic $K$-theory of a point can be defined as

$$A(\ast) = \lim_{n,k} B \text{Aut}(\nu^k S^n)^+.$$ 

Let $f : B \to B \text{Aut}(\nu^k S^{2n})$ be a finite dimensional approximation. There is an induced fibration $(\nu^k S^{2n}) \to E \to B$.

To this fibration, there is an associated transfer map $\text{Tr}_E : B \to \Omega^\infty S^\infty$. Let $\sigma : B \text{Aut}(\nu^k S^{2n}) \to B \text{Aut}(\nu^k S^{2n+2})$ be
induced by double suspension. Then the map \( \sigma \) induces
the fiberwise double suspension of \( E : (v^k S^{2n+2}) \to E' \to B \),
and because of 3.1 \( \text{Tr}_E \cong \text{Tr}_{E'} \). By a homotopy colimit
argument, these maps extend to a map
\[
\text{Tr}_k : \lim_{n} B \text{Aut}(v^k S^n) \to \Omega^\infty S^\infty.
\]
The stabilization map
\[
B \text{Aut}(v^k S^n) \to B \text{Aut}(v^{k+1} S^n)
\]
induced by adding a factor in the wedge, induces by 3.2 a diagram, which is homotopy commutative on all finite
subspaces
\[
\begin{array}{ccc}
\prod_{k \geq 0} \lim_{n} (B \text{Aut}(v^k S^{2n})) & \longrightarrow & \prod_{k \geq -1} \lim_{n} (B \text{Aut}(v^{k+1} S^{2n})) \\
\downarrow \text{Tr}_k & & \downarrow \text{Tr}_k \\
\Omega^\infty S^\infty & \xrightarrow{\ast[1]} & \Omega^\infty S^\infty
\end{array}
\]
The map \( \ast[1] \) here denotes loop sum with the identity
loop. Again, you can extend to a map, defined on finite
subcomplexes
\[
\text{Tr} : Z \times \lim_{n,k} B \text{Aut}(v^k S^{2n}) \to \Omega^\infty S^\infty
\]
And by the universal property of the plus construction,
this finally extends to a map
\[
\text{Tr} : A(\ast) \to \Omega^\infty S^\infty.
\]
Recall from [8] that \( \Omega^\infty S^\infty \cong Z \times \lim_k B\Sigma_k^+ \). The map
\[
Z \times \lim_k B\Sigma_k \to Z \times \lim_{n,k} B \text{Aut}(v^k S^n) \to \Omega^\infty S^\infty
\]
actually is the map inducing the equivalence, so \( \text{Tr} : A(\ast) \to \Omega^\infty S^\infty \) is a
split surjection.
Now, theorem 1.1 follows from the description of \( \eta_{Z \times BG} \) as a transfer in 2.8.

It remains to prove 3.1 and 3.2. Recall from [2] that the transfer \( \tau_E \) has the following properties:

3.3 Given a fibration \( p : E \to B \) as above, and a map \( g : X \to B \), we have a pullback diagram

\[ \begin{array}{ccc}
\tilde{E} & \xrightarrow{g} & E \\
\downarrow{\sim} & & \downarrow{p} \\
X & \xrightarrow{g} & B
\end{array} \]

Then \( \Omega^\infty \mathcal{S}(g_+) \circ \tau_E = \tau_{E \circ g}^\sim \).

3.4 Given fibrations \( p_i : E_i \to B_i \) as above, we can form the fiberwise smashproduct

\[ P_1 \wedge \left( P_2 : E_1 \wedge E_2 \to B_1 \times B_2 \right) \]

The following diagram commutes up to homotopy

\[ \begin{array}{ccc}
B_1 \times B_2 & \xrightarrow{\tau_{B_1} \times \tau_{B_2}} & \Omega^\infty \mathcal{S}(E_1) \times \Omega^\infty \mathcal{S}(E_2) \\
\tau_{B_1 \times B_2} & \downarrow & \downarrow \\
& \Omega^\infty \mathcal{S}(E_1 \times E_2)
\end{array} \]

We can now prove 3.1. If \( F \to F \to \ast \) is a fibration with trivial base, then

\[ \tau_F : S^\infty \to \Omega^\infty S^\infty \]
is given by the Euler characteristic \( \chi(F) \). This is to be understood in the pointed sense here; thus a sphere has Euler characteristic \(+1\) or \(-1\) depending on the parity of the dimension.

From 3.3 it follows, that if \( F \to F \times B \to B \) is a product fibration, then \( \tau_{F \times B} : B \to \Omega^\infty S^\infty(B \times F)_+ \) is the composite

\[
B \to \text{pt} \to \text{pt}_+ \to \Omega^\infty S^\infty(\text{pt}_+) = \Omega^\infty S^\infty \xrightarrow{\chi(F)} \Omega^\infty S^\infty.
\]

Applying 3.4 to \( E_1 = E; E_2 = S^2 \times B \to B \) and then 3.3 to the diagonal map \( B \to B \times B \), the statement 3.1 follows.

In order to prove 3.2, note that if \( f_1 : S^N \times B \to E_1 \) are duality maps of exspaces in the sense of [2], then the fiberwise coproduct followed by fiberwise wedge

\[
S^N \times B \to S^N \vee S^N \times B \xrightarrow{f_1 \vee f_2} E_1 \vee E_2
\]

is also a duality map. The 2N-dual of this map is the wedge of the 2N-duals of \( f_1 \) and \( f_2 \) followed by the fold map

\[
E_1 \vee E_2 \xrightarrow{Df_1 \vee Df_2} (S^N \vee S^N) \times B \xrightarrow{\text{fold}} S^N \times B
\]

The transfer map \( \text{Tr}_{E_1 \vee E_2} \) is the adjoint of the composite

\[
S^N \times B \to (S^N \vee S^N) \times B \xrightarrow{f_1 \vee f_2} E_1 \vee E_2 \xrightarrow{Df_1 \vee Df_2} (S^N \times S^N) \times B \to S^N \times B
\]

which equals the sum \( \text{Tr}_{E_1} + \text{Tr}_{E_2} \).
REFERENCES


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