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SOME PROBLEMS ON 3-MANIFOLDS

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Except for the first section, the problems discussed are all from the general area of Heegaard diagrams and Heegaard splittings.

1. Nonsufficiently large 3-manifolds. Let M be a closed orientable 3-manifold which is irreducible (every PL 2-sphere in M bounds a 3-ball in M) and has infinite fundamental group; such an M is known to be an Eilenberg-Mac Lane space. M is called *sufficiently large* if a large number of theorems applies to it; equivalently, if there is an embedding of a closed 2-manifold whose fundamental group is nontrivial and injects. Unfortunately this is not always the case.

The first such examples are Seifert fibre spaces whose decomposition surface is the 2-sphere, with exactly three exceptional fibres, and with the added condition that H_1 be finite (and π_1 infinite) [**18**]; there are infinitely many such. Though it is quite inconceivable that these should be the only nonsufficiently large 3-manifolds, no new ones have been exhibited so far.

One way to search for new examples, emphasized by R. P. Osborne in particular, is to try surgery on a knot. Indeed, given a knot, in general, one may expect almost any surgery on this knot to produce a manifold which is irreducible and has infinite fundamental group. If on the other hand the surgery produces a sufficiently large 3-manifold, there must exist, in the knot space, an incompressible surface of a particular kind; namely an incompressible surface which is either closed, or has its boundary curves in the isotopy class of curves (on the boundary of the knot space) used in the surgery. One would expect this to hold in fewer cases. The construction of the known examples of nonsufficiently large 3-manifolds, can be interpreted to fit this program (at least some of them can be obtained by surgery on torus knots). In general, the main problem involved in this program is a classification of incompressible surfaces in a knot space. (In the case of a torus knot, the

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knot space is a Seifert fibre space, so this classification is comparatively easy **[17]**.)

Concerning properties of nonsufficiently large 3-manifolds in general, one may try to extrapolate from the known examples. Specifically, there is the so-called Fenchel conjecture, the theorem that any Fuchsian group has a torsionfree subgroup of finite index, e.g., **[26]**. An immediate consequence of this is that any Seifert fibre space has a finite covering space which is a fibre bundle. In particular then any of the known nonsufficiently large 3-manifolds has a finite covering space which is sufficiently large. One may ask if this continues to hold for the unknown examples. The question naturally splits into two subquestions, of unequal likelihood:

If M is as before, must it be true that $\pi_1 M$ contains a nontrivial subgroup $\pi_1 F$ where F is a closed 2-manifold?

By results of Jaco **[8]** and Scott **[12]** the answer must be affirmative if $\pi_1 M$ contains any subgroup whatsoever which is finitely generated, of infinite index, and not free. It is hard to believe that this could fail.

If $\pi_1 M$ contains $\pi_1 F$ as before, must there exist a finite covering $M' \to M$ so that $\pi_1 F \cap \pi_1 M' \to \pi_1 M'$ is induced by an embedding $F' \to M'$?

The question can be asked in just this form for sufficiently large 3-manifolds, and it is striking to notice how little is known about it. The real problem seems to be if $\pi_1 M$ contains a sufficient number of subgroups of finite index, or for that matter, any such subgroup at all. No one has devised a method yet how to get $\pi_1 M$ to act on finite sets (except in special cases, e.g., fibrations over S^1 **[9]**). The relevance of the latter problem is emphasized by a group theoretic result of Scott **[13]** which in the case at hand implies that the answer is affirmative if $\pi_1 F$ is the intersection of a set of subgroups of finite index. Indeed this latter fact can easily be seen directly. Namely if $\pi_1 F \to \pi_1 M$ is induced by a map $f: F \to M$, the hypothesis implies that there is a finite covering space $M' \to M$ and a lifting $f': F \to M'$ so that, if $U(f'(F))$ denotes a regular neighborhood, the map $\pi_1 F \to \pi_1 U(f'(F))$ is surjective and hence bijective. Making the boundary of the regular neighborhood incompressible, one then finds the required F' as a component.

Given that M has a finite covering space which is sufficiently large, it is natural to try carrying over to it some of the results available for sufficiently large 3-manifolds, e.g., that homotopy equivalences can be deformed to homeomorphisms. This has indeed been done in a few of the known cases, by what may be called equivariant surgery in the covering space **[2]**. Though the manifolds considered were extremely special, the effort required was considerable.

2. Heegaard diagrams and Heegaard splittings. Let M be a connected closed orientable 3-manifold, it will be natural to assume that M is in fact oriented. We think of M as smooth. Let f be a nice Morse function on M, that is, f is a smooth real-valued function, the critical points are nondegenerate, and at each critical point the value equals the index. The critical points of index 0 or 3 are of no interest whatsoever, so we assume they are minimal in number (that is, there is just one of either kind).

Define $F = f^{-1}(1\frac{1}{2})$; it is a closed oriented 2-manifold. Let $V = f^{-1}[1\frac{1}{2}, \infty)$ and $W = f^{-1}(-\infty, 1\frac{1}{2}]$. Then V and W are 'handlebodies'.

Let $v \subset V$ denote the system of properly embedded 2-disks given by the cores of the 2-handles determined by f; it is referred to as a *system of meridian disks.* V may also be considered as a regular neighborhood of $F \cup v$, plus a single 3-ball attached. Let similarly $w \subset W$ denote the system of properly embedded 2-disks given by the cocores of the 1-handles (equivalently, by the cores of the 2-handles of the dual function $f' = 3 - f$).

Up to trivial alteration (i.e., deformation of nice Morse functions) f is determined by the quadruple $(M, F; v, w)$. This quadruple is referred to as a *Heegaard diagram* of M. It is, in turn, determined up to isomorphism by the oriented 2-manifold F and the ordered pair of systems of curves ∂v and ∂w in F.

With easy modifications, the above carries over to manifolds with boundary ∂M and functions f such that $f(\partial M) \subset \{-1, 4\}$. The case of more general functions f is not without interest (e.g., bridge presentations of a knot give rise to such functions on the knot space) but it will not be considered here. The quadruple $(M, F; v, w)$ may again be referred to as a Heegaard diagram, not of M this time but of the triple $(M, f^{-1}(-1), f^{-1}(4))$.

By a *Heegaard splitting* of M (resp., of $(M, f^{-1}(-1), f^{-1}(4))$ if $\partial M \neq \varnothing$) we shall mean any oriented F arising in the way described, the notation (M, F) will be used. The *genus* of the Heegaard splitting (M, F) is by definition the number $g(F)$, the genus of F. If $(M, F; v, w)$ is a Heegaard diagram, (M, F) will be called the underlying Heegaard splitting.

Heegaard diagrams have less 'random structure' than functions, or handle decompositions, or triangulations. Thus among the effective ways to present 3-manifolds (*all* of them, not just a special class) they appear to be the most efficient.[1] Still if one wishes to put Heegaard diagrams to any use one must face the fact that there are far too many of them. In particular, from any given Heegaard diagram one may construct others, by the process of 'handle sliding'. By definition, this process does not alter the underlying Heegaard splitting, and it is generated by (i) isotopy of v, resp. w, (ii) sliding one component of v, resp. w, over another. It is a pleasant fact, not too hard to prove, that conversely any two Heegaard diagrams with the same underlying Heegaard splitting can be transformed one into another by handle sliding. Thus a Heegaard splitting may be identified with an equivalence class of Heegaard diagrams, the equivalence relation being generated by handle sliding.

Concerning Heegaard splittings, it is an interesting fact, pointed out by Stallings [15], that, up to isomorphism, these may be characterized algebraically. For simplicity it will be assumed that M is a closed manifold. Let (M, F) be a Heegaard splitting and let V, W be the pair of handlebodies (ordered by the orientations of M and F) with $V \cup W = M$, $V \cap W = F$. The inclusions of F induce a pair of surjective maps $\pi_1 F \to \pi_1 V$, $\pi_1 F \to \pi_1 W$, well defined up to conjugation. The assertion is that

$$(M, F) \mapsto (\pi_1 F \to \pi_1 V, \pi_1 F \to \pi_1 W)$$

[1]There are at least two more ways to effectively present closed orientable 3-manifolds:

(i) By surgery on a framed link in S^3. Here the equivalence relation is known, that is, if two surgeries give the same 3-manifold, one knows how to transform the two framed links one into another (Craggs, Kirby). Analysis of the equivalence relation is practically untouched however, and it does not seem easier than the classification problems discussed below.

(ii) As branched coverings of S^3, or even 3-fold branched coverings, branched over a knot (Hilden, Hirsch, Montesinos). Here even the equivalence relation is unknown.

is a bijection of isomorphism classes. Details to this were provided by Jaco [7]; as these details are excessively complicated, here is a simpler way of verifying the assertion.

LEMMA (CF. [7]). *Let X be a wedge of n 1-spheres, and $\pi_1 F \to \pi_1 X$ any map. Then there are a handlebody V', an isomorphism $F \to \partial V'$, and a map $V' \to X$ so that $F \to V' \to X$ induces $\pi_1 F \to \pi_1 X$, up to conjugation.*

In fact, let $\pi_1 F \to \pi_1 X$ be induced, up to conjugation, by a map f_0: $F \to X$, say. In the jth 1-sphere in X, let p_j be a point different from the basepoint. Identify F to $F \times 0$ in $F \times [0, 1]$ and extend f_0 to f so that $f_1 = f|F \times 1$ is in general position with respect to $\bigcup p_j$. Form Y from $F \times$ I by attaching a 2-handle at each component of $f_1^{-1}(\bigcup p_j)$. Then f may be extended to g: $Y \to X$ so that (i) the core of each 2-handle is mapped into $\bigcup p_j$, (ii) for any component G_i of ∂Y other than $F \times 0$, $g(G_i) \subset X - \bigcup p_j$. Since $X - \bigcup p_j$ is contractible one may now form V' from Y by attaching to each G_i a handlebody V_i, in any way whatsoever, and extend g by mapping V_i into $X - \bigcup p_j$.

Thus the surjectivity part of the above assertion has been established. To see the injectivity, let $F = \partial V$ and $F = \partial V'$, and let $\pi_1 V \to \pi_1 V'$ be an isomorphism so that $\pi_1 F \to \pi_1 V \to \pi_1 V'$ and $\pi_1 F \to \pi_1 V'$ are the same, up to conjugation. Then the loop theorem (or better, its elementary version available for handlebodies [24]) and the Alexander trick show that the identity on F extends to an isomorphism $V \to V'$ which itself is unique up to isotopy.

Stallings also showed [15], given (M, F), M is a homotopy 3-sphere if and only if the map $\pi_1 F \to \pi_1 V \times \pi_1 W$ is surjective. In view of the fact (which was not available yet when [15] was written) that for any genus there is only one isomorphism class of Heegaard splittings of S^3, cf. below, one has as a corollary that the Poincaré conjecture is equivalent to the group theoretic conjecture that for any g there is only one isomorphism class of surjections $\pi_1 F \to \Phi_1 \times \Phi_2$ where Φ_1 and Φ_2 denote free groups of rank $g = g(F)$. Interesting though this fact is, philosophically, it has not been possible so far to use it in any way.

3. Classification problems for Heegaard splittings. From any Heegaard splitting one may obtain a new one by 'standard handle addition'. Since by definition (M, F) is just a particular kind of manifold pair, the result of a standard handle addition may simply be described as the connected sum of (M, F) with a (or 'the') genus one Heegaard splitting of S^3. Again it is a pleasant fact, the theorem of Reidemeister [11] and Singer [14], that any two Heegaard splittings of M are 'stably equivalent' i.e., equivalent under the equivalence relation generated by isomorphism (in fact, isotopy) and standard handle addition.

One approach to the classification of 3-manifolds is thus to start from Heegaard diagrams, which may be classified 'upon inspection'. By imposing on these the equivalence relation of handle sliding, one obtains Heegaard splittings; and by further imposing stable equivalence one obtains the (isomorphism classes of) manifolds themselves. One may thus try to classify Heegaard splittings first, and then proceed from this. The former will be discussed in the next section; the latter leads to various interesting problems, a sample of which is given below.

It is convenient to call a Heegaard splitting *minimal* if it cannot be obtained, by a standard handle addition, from a Heegaard splitting of lower genus. Here is a list of some known, respectively unknown, facts.

The 3-sphere has, up to isomorphism, precisely one Heegaard splitting of any genus $g \geqq 0$ [**19**]. The only minimal one is that of genus 0.

It is not known if a minimal Heegaard splitting of a lens space must have genus 1. In fact this is unknown even for projective 3-space. although the argument of [**20**] seems close to establishing it.

Some lens spaces have two isomorphism classes of Heegaard splittings of genus 1, obtainable from each other by re-orienting F. By taking connected sums one should expect to mess up the nonuniqueness so that it is no more due to just orientation phenomena. Renate Engmann [**3**] has shown that such 'essential' nonuniqueness does in fact occur.

Indeed, nonuniqueness does not depend on connected sum phenomena either: There is a prime manifold (in fact, there are infinitely many such) with two minimal Heegaard splittings (of genus 2) that are not isomorphic as unoriented manifold pairs [**1**].

Here are some problems.

Show that M has only finitely many isomorphism classes (or even isotopy classes) of minimal Heegaard splittings.

Given any of these, give a procedure to obtain the others.

Is it true that any two minimal Heegaard splittings of M have the same genus? An example of P. Schupp shows that the corresponding question for group presentations has the answer 'no'.

Let (M, F) and (M, F') be two Heegaard splittings of M, both of genus g. Do they become isomorphic when the genus is raised, by standard handle additions, to $2g$, say? Is there only one isomorphism class of Heegaard splittings of genus $2g$?

4. Decision problems for Heegaard splittings. In reality these are problems about Heegaard diagrams. For example, given a Heegaard diagram $(M, F; v, w)$, how can one find out if the underlying Heegaard splitting (M, F) is minimal? Is there a way to alter $(M, F; v, w)$ to a canonical Heegaard diagram, or one of a finite set of such, from which the answer may be read off by inspection? Similarly, given $(M, F; v, w)$ and $(M', F'; v', w')$ one may want to know if their underlying Heegaard splittings are isomorphic, and in particular, say, if (M, F) is isomorphic to a Heegaard splitting of S^3.

There is a notion of complexity for a Heegaard diagram. It is best to discuss first the analogous notion of complexity for curves on the boundary of a handlebody.

Let V be a handlebody, and v a system of meridian disks in V. Let the components of v be numbered $v_1, \cdots, v_n$ and let a normal direction to each v_i be chosen. If V has a basepoint, off v, any based loop k in V gives rise to a word $v(k)$ in the alphabet $\{v_1, v_1^{-1}, v_2, \cdots, v_n^{-1}\}$ to record its encounters with v. In this way v determines a basis of $\pi_1 V$, and the element of $\pi_1 V$ represented by k is given, in this basis, by the reduced word $\bar{v}(k)$ associated to $v(k)$. We define the geometric, resp. algebraic, length of k to be the number of letters in the word $v(k)$, resp. $\bar{v}(k)$; it is denoted $l(k, v)$, resp. $\bar{l}(k, v)$. If k is not a single loop but a finite set of such, we define its geometric (resp., algebraic) length by adding those of the individual loops. The minimal geometric length $l(k)$ is defined to be the minimum of the numbers $l(k, v)$ as v varies; similarly the minimal algebraic length $\bar{l}(k)$ is defined. If V has no basepoint, these considerations still apply when elements of $\pi_1 V$ are replaced by conjugacy classes of such elements.

Suppose v is changed to another system of meridian disks, v', in the following special way. Namely a single component v_j of v may be replaced by a disk v'_j *in the complement of* v; furthermore the components of v' may be re-indexed, and some of the normal directions altered. If V has a basepoint, the result of the replacement $v \mapsto v'$ may be interpreted in two ways. Firstly we may say that the basis of $\pi_1 V$ is replaced by another one, taken from a particular finite list. The other interpretation is that $\pi_1 V$ has been subjected to a particular kind of automorphism, called a *T-transformation* by Whitehead [23]. The second interpretation still makes sense when V is unbased and when elements of $\pi_1 V$ are replaced by conjugacy classes. The substitution $v \mapsto v'$ will be referred to as a geometric T-transformation.

THEOREM (WHITEHEAD [23]). (1) *Let k be a finite collection of (based, resp., unbased) loops in V. Suppose the algebraic length $\bar{l}(k, v)$ can be made smaller by some automorphism of $\pi_1 V$. Then it can be made smaller by a T-transformation.*

(2) *Suppose k and k' are such that their algebraic lengths are minimal, i.e., $\bar{l}(k, v) = \bar{l}(k)$ and $\bar{l}(k', v) = \bar{l}(k')$. Suppose there is an automorphism which takes the set of elements of $\pi_1 V$ (resp., conjugacy classes) represented by k into that represented by k'. Then this automorphism may be written as a sequence of T-transformations none of which increases the algebraic length.*

(Incidentally, Whitehead's theorem can be extended to cover the case of a finite set of finitely generated subgroups, cf. [23, p. 97]; the theorem is just the case of a set of cyclic subgroups. The proof uses 3-dimensional topology and is an extension of Whitehead's method: Instead of mapping curves into $\#_n(S^1 \times S^2)$, as Whitehead does, one maps $\#_k(S^1 \times S^2)$, or a finite number of such. In the context of Heegaard diagrams this extension is of no interest however.)

Suppose now that k is a system of mutually disjoint simple closed curves in the boundary ∂V. In this case one is interested in having the geometric length of k as small as possible. This can indeed be achieved.

THEOREM (ZIESCHANG [25]). *Suppose that $l(k, v)$ is strictly bigger than the minimal algebraic length $\bar{l}(k)$. Then $l(k, v)$ can be made smaller by a geometric T-transformation. In particular the minimal geometric length equals the minimal algebraic length.*

Similarly the second part of Whitehead's theorem has a (weak) geometric analogue.

Zieschang's proof is a delicate analysis of the situation. It seems to be of some importance that there is an alternative, somewhat crude, proof which is based on a trick of Whitehead [22]. In the present situation the trick amounts to temporarily admitting 'meridian surfaces' other than disks. The argument will be described below, after another notion has been introduced.

Suppose there is an embedded 2-disk D in V with the properties (i) $D \cap \partial V$ is a single arc c, and either $c \cap k = \emptyset$, or $c \subset k$, (ii) $D \cap v$ is a single arc, equal to $\mathrm{Cl}(\partial D - c)$. Let v_i be the component of v that contains $D \cap v$; let $U(v_i \cup D)$ be a regular neighborhood. Then $\mathrm{Cl}(\partial U(v_i \cup D) - \partial V)$ consists of three disks, one of which is parallel to v_i. We are interested in the other two. For precisely one of these, call it v'_i, it is true that $v' = (v - v_i) \cup v'_i$ is again a system of meridian disks. The substitution $v \mapsto v'$ will be called a *geometric T-transformation* that is *special with respect to k*. Whether or not there exists a special T-transformation to decrease the geometric length of k can be found out by searching for the arc c (a 'wave' in the terminology of **[16]**).

As to the theorem, suppose first that $l(k, v) > \bar{l}(k, v)$. Then a special T-transformation can be found (with $c \subset k$) to decrease $l(k, v)$. So suppose $l(k, v)$ equals $\bar{l}(k, v)$, but the latter is not minimal. By Whitehead's theorem there exists a geometric T-transfmoration $v \mapsto w$ to decrease the algebraic length. If $l(k, w) > \bar{l}(k, w)$ one could try now to perform a special T-transformation to decrease $l(k, w)$; while this is certainly possible, it might happen however that the algebraic length increases again, so nothing is in fact gained. Here is where Whitehead's trick comes in. Namely instead of using the arc c for a special T-transformation, one uses it to 'add a handle' to w, i.e., one replaces the component w_j of w containing ∂c, by the annulus component of $\mathrm{Cl}(\partial U(w_j \cup c) - \partial V)$. Then $l(k, w)$ goes down by 2, but $\bar{l}(k, w)$ is unaltered because the set of reduced words from which it is computed is unaltered. Furthermore the new w is still disjoint to v because of the initial hypothesis $l(k, v) = \bar{l}(k, v)$. Proceeding in this way, w will eventually have been replaced by w' with $l(k, w') = \bar{l}(k, w') = \bar{l}(k, w)$, but the component w'_j of w' is some complicated 2-manifold rather than a disk. On the other hand, w' is still a system of 2-sided 2-manifolds, not separating V, and n in number. So we can construct a map from $\pi_1 V$ onto a free group of rank n, and the kernel of this map contains $\pi_1 w'_j$. But any surjective endomorphism of a finitely generated free group is an isomorphism. So $\mathrm{Im}(\pi_1 w'_j)$ is the trivial subgroup of $\pi_1 V$. So the loop theorem applies, and w'_j can be dismantled to a system of disks. Keeping a suitable one of these, the theorem is proved.

REMARK. There does not seem any reason to suppose that in general the geometric length of k can be made minimal by special T-transformations only. There is one special case however where this can be done. This special case is when no component of k is contractible in V, and the minimal algebraic length of k equals the number of components. This was pointed out by Whitehead in the final paragraph of **[22]**; it depends on a certain technical result of that paper.

Let now $(M, F; v, w)$ be a Heegaard diagram, and for simplicity assume M is closed. One may replace these data by the equivalent data $(F; \partial v, \partial w)$. Supposing that ∂v and ∂w are in general position, one defines the *complexity* $c(F; \partial v, \partial w)$ to be the number of intersection points of ∂v and ∂w. Given F, and given any upper bound, there is only a finite number, up to isomorphism, of Heegaard diagrams whose complexity is below this upper bound.

By definition, the complexity $c(F; \partial v, \partial w)$ coincides with the geometric length $l(\partial w, v)$ considered before. It may happen that $l(\partial w, v)$ can be made smaller by a geometric T-transformation applied to v. Whether or not this happens can be found out by inspection of $(F; \partial v, \partial w)$; the test is especially simple if one looks for special T-transformations (the search for a 'wave' in the terminology of **[16]**). We say $(F; \partial v, \partial w)$ is *minimal* if neither $l(\partial w, v)$ nor $l(\partial v, w)$ can be made smaller by a geometric T-transformation; and we say $(F; \partial v, \partial w)$ is a *weak minimum* if neither can be made smaller by a special T-transformation.

With this terminology we can now give sharper versions of the problems stated in the beginning of this section. These are:

Show that any Heegaard splitting is underlying to only finitely many minimal Heegaard diagrams. Given any of these, describe a procedure to obtain the others.

Let (M, F) be the underlying Heegaard splitting of $(M, F; v, w)$. Suppose $(F; \partial v, \partial w)$ is minimal (or even a weak minimum only). Suppose (M, F) is not minimal. Show that $(M, F; v, w)$ has a cancelling pair of handles.

Let $(F; \partial v, \partial w)$ be a Heegaard diagram of S^3 which is minimal (or even a weak minimum only). Show that the complexity of $(F; \partial v, \partial w)$ equals the genus of F. (Note this amounts to the assertion that the method to produce a minimal Heegaard diagram from a given one is in fact an algorithm to recognize S^3.)

Concerning the status of these problems, nothing is known about the first two, and very little is known about the third one: Whitehead has shown the assertion is true in the very special case where one assumes that one of v and w is 'standard' already, this is the remark above; the argument has been reproduced in **[16]**. It is interesting to note that a computer check has been run on the third problem **[16]**. One million Heegaard diagrams of S^3 were examined; no exotic weak minimum was found.[2]

5. The Heegaard genus. It is convenient here to consider 3-manifolds with boundary (possibly empty) but only Heegaard splittings of the triple $(M, \partial M, \emptyset)$, in the notation of §2. With this qualification, the *Heegaard genus* $g(M)$ of M is defined to be the smallest integer g so that M has some Heegaard splitting of genus g.

For example, of the manifolds M without boundary spheres, $g(M) = 0$ characterizes S^3, and $g(M) = 1$ characterizes lens spaces, $S^1 \times S^2$, and $S^1 \times D^2$.

So far only one nontrivial fact is known about the Heegaard genus, a beautiful argument of Haken **[4]**, that $g(M)$ is additive for connected sum.

Let $r(M)$ denote the minimum number of generators for $\pi_1 M$. One has the inequality $g(M) \geqq r(M)$. For want of better knowledge one may ask the question

Is it true that $g(M) = r(M)$?

It is amusing to contemplate this question on the background of the unresolved status of the Poincaré conjecture (the case $r(M) = 0$ of the question). Put in a fancy way, the content of the Poincaré conjecture is that all the difficulties inherent in attacking it are due to the internal structure of the 3-sphere. Therefore if one assumes the Poincaré conjecture is wrong, it seems reasonable to expect the above question to become easier if one restricts attention to submanifolds $M \subset S^3$, e.g., knot spaces. If on the other hand one assumes the Poincaré conjecture is true, there does not seem to be any reason why the question should be easier to decide for submanifolds $M \subset S^3$.

6. A space of Heegaard splittings. From current problems of 3-dimensional topology there is no justification to consider such a notion, except maybe a vague feeling that it could be useful in work on the Smale conjecture on Diff(S^3). The following definition was concocted by analogy with a notion that is useful in

[2]The reviewer of reference **[16]** reports that an exotic weak minimum has been found [Math. Rev. **53**, Abstr. 9219 (1977)]. It thus appears that the consideration of weak minima is of little interest.

studying higher concordances (the 'expansion space' of **[21]**). To understand the definition one should note that there are really two ways in which one wants to alter Heegaard splittings: by isotopy and by handle addition. As these are heterogeneous notions, they should be kept apart. Thus the space envisaged should be a bisimplicial set (or a simplicial category, for that matter). From this one may then obtain a simplicial set, and hence a homotopy type, in any of various ways, all ultimately equivalent.

The Heegaard splittings of a given (say, closed orientable) 3-manifold M are the objects of a category $h(M)$ in an obvious way: a morphism in $h(M)$ is a standard handle addition, or composition of such, performed on one Heegaard splitting to yield another. Still the definition of morphism needs interpretation. Firstly, the notion of handle additon is supposed to be very rigid. One way to have this rigidity is always to refer to the standard D^3 in which the standard punctured torus is embedded in the standard way, and then for a handle addition use a specific embedding of D^3 in M. Secondly, the order of handle additions must be discussed: It may happen that one handle is attached on top of another. In this case the two can be attached only in that particular order. On the other hand one can envisage two handle additions being performed simultaneously, far away from each other. In this case we insist that either one of the two could be attached first, and it does not matter which one; thus we have a commutative square in $h(M)$.

More generally, for any nonnegative integer k one can define a category $h(M)_k$: an object is a k-parameter family of Heegaard splittings of M (with parameter domain the k-simplex Δ^k) and a morphism is a k-parameter family of standard handle additions, or a composition of such. The same remarks as above apply: a k-parameter family of handle additions involves a k-parameter family of embeddings of D^3 (with its additional structure), and two k-parameter families of handle additions are considered to be in a particular order only if such order is forced, in the sense above, at one point at least of the parameter domain.

DEFINITION. $h(M).$ is the simplicial category which in degree k is $h(M)_k$.

The known results on the classification of Heegaard splittings can be rephrased to say that such or such a space is connected, or not connected, as the case may be. For example the Reidemeister-Singer theorem says that $h(M).$ is connected for any M, and the classification of Heegaard splittings of S^3 is equivalent to the statement that $h^g(S^3).$ is connected for any g, where $h^g(M).$ denotes the simplicial subcategory of $h(M).$ given by the Heegaard splittings of genus at most g.

The problem is of course if one can say anything about the higher homotopy groups. For example, is $h(M).$ contractible?

To conclude, one way of forming the 'nerve' and then forcing the extension condition gives the following Kan simplicial set representing the homotopy type of $h(M)..$ An n-simplex consists of:

(i) a simplicial subdivision of the geometric n-simplex,

(ii) for each Δ^k in this subdivision, a continuous family of Heegaard splittings over the interior of Δ^k,

(iii) for each face $d_i\Delta^k$, as one approaches this face from the interior of Δ^k, the data of a $(k-1)$-parameter family of standard handle cancellations (or composition of such) and finally, at the last moment, the actual cancellation,

(iv) for each face of a face, a compatibility condition.

References

1. J. S. Birman, F. González-Acuña and J. M. Montesinos, *Heegaard splittings of prime 3-manifolds are not unique* (preprint).

2. H. Boehme, *Fast genügend grosse irreduzible 3-dimensionale Mannigfaltigkeiten*, Invent. Math. **17** (1972), 303–316.

3. R. Engmann, *Nicht-homöomorphe Heegaard-Zerlegungen vom Geschlecht* 2 *der zusammenhängenden Summe zweier Linsenräume*, Abh. Math. Sem. Univ. Hamburg **35** (1970), 33–38. MR **44** #1033.

4. W. Haken, *Some results on surfaces in 3-manifolds*, Studies in Modern Topology, No. 5, pp. 39–98, Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N. J.), 1968.

5. P. Heegaard, *Forstudier til en topologisk teori för de algebraiske Sammenhäeng* (Dissertation, Univ. of Copenhagen, 1898; published by det Nordiske Forlag Ernst Bojesen, Copenhagen, 1898).

6. ———, *Sur l'Analysis situs*, Bull. Soc. Math. France 44 (1916), 161–242. (translation of [5]).

7. W. Jaco, *Heegaard splittings and splitting homomorphisms*, Trans. Amer. Math. Soc. **144** (1970), 365–379. MR **40** #6555.

8. ———, *Finitely presented subgroups of three-manifold groups*, Invent. Math. **13** (1971), 335–346. MR **45** #9325.

9. L. P. Neuwirth, *Knot groups*, Ann. of Math. Studies, no. 56, Princeton Univ. Press, Princeton, N. J., 1965.

10. C. D. Papakyriakopoulos, *Some problems on 3-dimensional manifolds*, Bull. Amer. Math. Soc. **64** (1958), 317–335.

11. K. Reidemeister, *Zur dreidimensionalen Topologie*, Abh. Math. Sem. Univ. Hamburg **9** (1933), 189–194.

12. G. P. Scott, *Finitely generated 3-manifold groups are finitely presented*, J. London Math. Soc. (2) **6** (1973), 437–440.

13. ———, *Ends of pairs of groups*, Univ. Liverpool, Oct. 1976.

14. J. Singer, *Three-dimensional manifolds and their Heegaard-diagrams*, Trans. Amer. Math. Soc. **35** (1933), 88–111.

15. J. R. Stallings, *How not to prove the Poincaré conjecture*, Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N. J., 1966, pp. 83–88.

16. I. A. Volodin, V. E. Kuznetsov, A. T. Fomenko, *The problem of discriminating algorithmically the standard three-dimensional sphere*, Russian Math. Surveys **29** (1974), 72–168.

17. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten*. I, II, Invent. Math. **3** (1967), 308–333; ibid. **4** (1967), 87–117. MR **38** #3880.

18. ———, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517. MR **38** #5223.

19. ———, *Heegaard-Zerlegungen der 3-Sphäre*, Topology **7** (1968), 195–203. MR **37** #3576.

20. ———, *Über Involutionen der 3-Sphäre*, Topology **8** (1969), 81–91. MR **38** #5209.

21. ———, *Algebraic K-theory of topological spaces*. I, these Proceedings, part 1, pp. 35–60.

22. J. H. C. Whitehead, *On certain sets of elements in a free group*, Proc. London Math. Soc. (2) **41** (1936), 48–56; Collected Works Vol. II, Pergamon Press, New York, 1962, pp. 69–77.

23. ———, *On equivalent sets of elements in a free group*, Ann. of Math. **37** (1936), 782–800; Collected Works Vol. II, Pergamon Press, New York, 1962, pp. 79–97.

24. H. Zieschang, *Über einfache Kurven auf Vollbrezeln*, Abh. Math. Sem. Univ. Hamburg **25** (1962), 231–250. MR **26** #6957.

25. ———, *On simple systems of paths on complete pretzels*, Amer. Math. Soc. Transl. (2) **92** (1970), pp. 127–137.

26. H. Zieschang, E. Vogt and H. D. Coldewey, *Flächen und ebene diskontinuierliche Gruppen*, Lecture Notes in Math., Vol. 122, Springer-Verlag, Berlin and New York, 1970.

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