# ON MAPPINGS OF HANDLEBODIES AND OF HEEGAARD SPLITTINGS

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It is quite common in mathematics to take a method which has been successful somewhere and try to apply it somewhere else. The present work flows from this principle. Its main result is preliminary to an attempt of applying the cancellation arguments of [3] and [4] to the context of the Poincaré conjecture. This result asserts that, at least in a suitable stable sense, any proper degree 1 map between handlebodies is of the obvious type.

## 1. Mappings of Handlebodies

Let $V$ and $X$ be oriented $n$-manifolds with boundaries $\partial V$ and $\partial X$, respectively. A map $f: V, \partial V \to X, \partial X$ is of *degree* 1 if $f_*$ sends the fundamental class of $V, \partial V$ to the fundamental class of $X, \partial X$. A good reference for such maps is [1]. In particular, we note the following properties: (a) $f|\partial V$ also has degree 1, (b) $f_*: \pi_1 V \to \pi_1 X$ is surjective, and (c) given any $n$-ball $B^n$ in $\mathrm{Int}(X)$, $f$ can be deformed so that $f|f^{-1}(B^n)$ is a homeomorphism.

We now specialize to (3-dimensional, orientable) handlebodies which by definition are obtainable from the 3-ball by adding 1-handles; the number of these handles is called the genus. A degree 1 map $f: V, \partial V \to X, \partial X$, where $V$ and $X$ are handlebodies of genus $m + n$ and $n$, respectively, is also called a *map of type* $(m + n, n)$. Maps of type $(m + n, n)$ are sorted into equivalence classes by allowing (a) proper homotopic deformation of $f$, (b) composition with an orientation-preserving homeomorphism $V \to V$, and (c) composition with an orientation-preserving homeomorphism $X \to X$.

**Lemma 1.1.** *There is but one equivalence class of maps of type* $(n, n)$.

*Proof.* Since $f|\partial V$ has degree 1 and $\partial V$ and $\partial X$ have the same genus, $f|\partial V$ is homotopic to a homeomorphism. By Dehn's lemma, aspherity of $X$, and the Alexander trick, it follows that $f$ is properly homotopic to a homeomorphism.

Given degree 1 maps of handlebodies, $f: V, \partial V \to X, \partial X$ and $g: W, \partial W \to Y, \partial Y$, there is a unique equivalence class which is the *sum* of $f$ and $g$, denoted $f \# g$. A representative of $f \# g$ is obtained by making $f|f^{-1}(U)$ nonsingular, where $U$ is a regular neighborhood of a point in $\partial X$, and similarly with $g$, and glueing in the obvious way.

A *standard map* of type $(m + n, n)$ is, by definition, equivalent to a sum of maps of type $(n, n)$ and $(m, 0)$, respectively. There is but one standard map in each type (up to equivalence), and the sum, or composition, of standard maps is again a standard map.

I do not know if there are nonstandard maps. It can be seen by inspection of the proof of Theorem 1.4 that maps of type $(n + 1, 1)$ are standard. This suggests that maybe all others are standard as well.

We now introduce an operation which will turn out not to be really new. Let $f: V, \partial V \to X, \partial X$ be a degree 1 map of handlebodies. Let $V'$ be obtained from $V$ by attaching the ball $I \times D$ ($I$ = the interval, and $D$ = the 2-disk) along $D \times \partial I$ at $V$, and $f'$ from $f$ by an extension such that $f'(I \times D) \subset \partial X$. Then we say that $f'$ is obtained from $f$ by *adding a handle*.

**Lemma 1.2.** *In the situation of the above definition, $f'$ is equivalent to the sum of $f$ and a map of type* $(1, 0)$.

*Proof.* Since $(f|\partial V)_*: \pi_1 \partial V \to \pi_1 \partial X$ is surjective, there exists a map $g: I \to \partial V$ so that $f \circ g$ and $f'|I$ are homotopic in $\partial X$ rel $\partial I$. We may assume that $g$ is nonsingular, for, assuming $g$ in general position, we can push off the singularities one after the other across $g(\partial I)$, starting near the ends. Let $U$ denote a regular neighborhood of $g(I) \cup \partial I \times D$ in $V$. There is an extension $I \times D \to U \cap \partial V$ of $g$, and $U \cup I \times D \to U$ is a mapping of type $(1, 0)$. Obviously, $f'$ is equivalent to the sum of $f$ and this mapping.

As a corollary, we obtain a criterion for a map to be standard.

**Corollary 1.3.** *Let $V, \partial V \to X, \partial X$ be a map of type $(m + n, n)$. Suppose there are $m$ independent meridian disks $D_1, \ldots, D_m$ in $V$* (i.e., a system of $m$ properly embedded disks the union of which does not separate $V$) *such that, for $i = 1, \ldots, m$, $f|\partial D_i$ is contractible in $\partial X$. Then $f$ is standard.*

*Proof.* We may assume that $f$ sends a neighborhood of $\cup D_i$ into $\partial X$. Removing a small open regular neighborhood of $\cup D_i$, we obtain a map of type $(n, n)$. This is standard, and from it, $f$ is obtained by adding handles.

Next we introduce another operation on degree 1 maps of handlebodies. This is somewhat mysterious, but it is extremely useful.

Let $f: V, \partial V \to X, \partial X$ be such a map. Let $k$ be a proper arc in $V$ which is unknotted in the sense that there exists a disk $D$ in $V$ so that $k \subset \partial D$, and $\mathrm{Cl}(\partial D - k) = D \cap \partial V$. Let $U$ be a regular neighborhood of $k$. Then $V' = \mathrm{Cl}(V - U)$ is again a handlebody. Now assume that $f(U)$ is contained in $\partial X$. Then $f' = f|V'$ is also a degree 1 map of handlebodies. We say that $f'$ is obtained from $f$ by *drilling a hole*.

**Theorem 1.4.** *Let $f : V, \partial V \to X, \partial X$ be any degree 1 map of handlebodies. Then $f$ can be made standard by deformation and the operation of (repeatedly) drilling a hole.*

*Proof.* To have a good induction, we must first generalize the theorem to the following more general situation. We augment our data by assuming given in the boundary of each handlebody $V$ a system $v$ (possibly empty) of (mutually disjoint) disks. A degree 1 map $f : V, \partial V \to X, \partial X$ must satisfy $f|v$ is a homeomorphism, and $f^{-1}(x) = v$. In the definition of equivalence, the homeomorphisms must respect the systems of disks, and the homotopies are through degree 1 maps in the restricted sense (such homotopies will be called allowable). A map is standard if it is equivalent to the sum of an identity and a map $W, \partial W \to B^3, \partial B^3$ with no disks in $\partial B^3$.

**Lemma 1.5.** *Let $F$ be a system of proper disks in $X$ so that any component of $F \cap x$ is an arc, and this intersection is in general position. Suppose further that any component of $F$ or $x$ contains at most one component of $F \cap x$. Then, by allowable deformations and the operation of drilling a hole, any component of $f^{-1}(F)$ can be made a disk.*

*Proof.* By an allowable deformation we put $f$ in general position with respect to $F$. Then, in particular, $G = f^{-1}(F)$ is a properly embedded system of orientable 2-manifolds in $V$, and for any component of $G$, the intersection with $v$ is at most one arc and is in general position. Since $X$ is aspherical, it is easy to remove 2-spheres from $G$; so we assume there are none. Now for any such $G$ in any handlebody $V$, the following is true by a standard argument. If at least one component of $G$ is not a disk, then there exists a disk $D$ in $V$ which has one of the following two properties: (a) $D \subset \mathrm{Int}(V)$, and $D \cap G = \partial D$ is not contractible in $G$, or (b) $D \cap \partial V$ is an arc in $\partial D$, and $D \cap G = \partial D \cap G = \mathrm{Cl}(D - \partial V)$ is an arc which in $G$ cannot be pulled into $\partial G$ with its end points kept fixed.

We will use $D$ to perform a surgery on $f$ in order to simplify $G$. In case (a) this is easy. As $X$ is aspherical, $f|D$ can be pulled into $F$ keeping $f|\partial D$ fixed. Hence there is an allowable deformation of $f$ to the effect that a neighborhood of $\partial D$ in $f^{-1}(F)$ will be replaced by two copies of $D$, thus simplifying $G = f^{-1}(F)$.

In case (b) we may assume that $D \cap x = \varnothing$, , since otherwise we could push $D$ off $x$. Similarly as before, our task is to replace a neighborhood of $D \cap f^{-1}(F)$ by two copies of $D$. We will do this in two steps. Denoting $k$

the arc $D \cap \partial V$, we wish to perform first a surgery on $G$ using $k$ (to the effect that a neighborhood of $\partial k$ in $G$ will be replaced by a sort of a half-tube around $k$). After this we would be in a position to perform the surgery of case (a), which as before presents no problem, and the total effect of the two steps would be just what we are after. In order that the first step (the surgery along $k$) can be performed, it is necessary and sufficient that there exists a deformation of $f|k : k \to \partial X$, fixed on $\partial k$, which pulls $f(k)$ into $\partial F$ and avoids $x$ altogether (and on Int$(k)$ is disjoint to $F$ except in the final stage). Let us call any singular arc $f^*|k^*$ good if it has all these properties. There is no reason to suppose that the given singular arc $f|k$ is good in this sense. Note, however, that if we were allowed to deform $f|k$ through Int$(X)$ (keeping $f|\partial k$ fixed), then we could make it a good arc; i.e., we could make it very close to that arc in $\partial F$ which is bounded by $f(\partial k)$ and avoids the disks $x$ in $\partial X$. Let now $k''$ be any arc in $D$ which is proper both in $D$ and $V$, and let $U$ be a small tube around it. $k''$ is unknotted, hence $V' = \text{Cl}(V - U)$ is a handlebody. Let $D'$ denote that one of the two components of $\text{Cl}(D - U)$ which contains $D \cap G$, and let $k' = \text{Cl}(\partial D' - G)$. $k'$ is contained in Int$(V)$, except for two pieces near the ends, and these are as small as we like. Nothing prevents us from deforming $f|U$ as we like, as long as we avoid $F$ and keep $f|(U \cap \partial V)$. In particular we can achieve that $f(U) \subset \partial X$ and that $f|k'$ assumes a position we would have liked $f|k$ to take. We now perform the operation of drilling a hole on $f : V, \partial V \to X, \partial X$ by passing from $f : V$ to $f|\text{Cl}(V - U)$. This leaves unchanged $G = f^{-1}(F)$, hence does not alter the complexity of the problem. $k'$ now takes the role of $k$, and $f'|k'$ is good; hence we can perform our surgery.

**Lemma 1.6.** *Same hypotheses as in Lemma* 1.5, *same operations. The conclusion is that* $f|f^{-1}(F)$ *can be made a homeomorphism.*

*Proof.* We assume the conclusion of Lemma 1.5, and as before we let $G = f^{-1}(F)$. $f|G$ is a proper map, hence its degree is defined. As $f$ is in general position, it is essentially a product near $G$. Therefore, since the degree of $f$ can be calculated locally, it follows that $f|G$ has degree 1 if the components of $F$ and $G$ are suitably oriented.

Let $F_j$ be a component of $F$. If $G_j = f^{-1}(F_j)$ is connected, then $f|G_j$, a degree 1 map between disks, is properly homotopic to a homeomorphism. If $G_j \cap v$ is nonempty, where $v$ is that system of disks in $\partial V$, then it is precisely one arc, and $f|f^{-1}(f(G_j \cap v))$ already is a homeomorphism. Hence in any case the homotopy from $f|G_j$ to a homeomorphism can be induced by an allowable deformation of $f$.

Therefore, all we have to worry about is how to connect up the components of $f^{-1}(F_j)$, for the various components $F_j$ of $F$. Let $f^{-1}(F_j)$ contain at least two components, $G_1$ and $G_2$. Let $k$ be an arc in $V$ which connects $G_1$ to $G_2$. Since $f_* : \pi_1 V \to \pi_1 X$ is surjective, we may assume that $f|k$ can be pulled into $F_j$, with $f|\partial k$ kept fixed. Hence there is a disk $D$, containing $k$ in its boundary, and a mapping $g: D \to X$ so that $g|k = f|k$, and $g(\partial D - k) \subset F_j$. Assuming $g$ to be in as general position as compatible with $g(\partial D - k) \subset F_j$, we have that any component of $g^{-1}(F_j) - k$ is a simple closed curve or proper arc in $D$. The former can easily be removed. The same goes for one of the latter if the intersection points in $k \cap G$ corresponding to its ends are in the same component of $G$. So eventually an extreme part of $D$ gives us an arc $k$ as above (possibly connecting some other pair $G_1, G_2$), which, in addition, satisfies $k \cap G = \partial k$.

For any handlebody $V^*$, $\pi_1 \partial V^* \to \pi_1 V^*$ is surjective. Hence any two points in $\partial V^*$ can be connected by an arc $k^*$ in $\partial V^*$ so that $k^*$ is in a given homotopy class in $V^*$ relative to these points. By the method of pushing singularities off the ends, we can make $k^*$ nonsingular. The same goes if we replace the points in $\partial V^*$ by two disjoint disks and we desire $k^*$ to meet these disks in $\partial k^*$ only. Furthermore, we can assume that $k^*$ is disjoint to any system of disks in $\partial V^*$ as long as these disks are mutually disjoint and disjoint to the two given ones.

Applying this remark to the handlebody $V^*$ obtained by splitting $V$ at $G$ (to avoid abuse of language, we should rather talk about the appropriate component of $V^*$), and to the system of disks in $\partial V^*$ which comes from $G \cup v$, we find a nonsingular arc $k^*$ in $\partial V$ so that $k^* \cap G = k \cap G$, $k^* \cap v = \emptyset$, and $k^*$ is homotopic to $k$ in $V^*$ rel $\partial k^*$.

Our task now is to do a surgery along $k^*$, thus replacing a neighborhood of $\partial k^*$ in $G$ by a half-tube around $k^*$, and connecting up $G_1$ and $G_2$. We are facing here the same obstruction as in the proof of Lemma 1.5, and we avoid it in the same way, by drilling a hole.

*Proof of Theorem* 1.4. Let $F$ be a system of proper disks in $X$ as in Lemma 1.6; then we can make $f|f^{-1}(F)$ a homeomorphism. Let $X'$ be obtained by splitting $X$ at $F$; i.e., there is a mapping $p: X' \to X$ which is a homeomorphism except for identifying in pairs the components of $p^{-1}(F)$, and let $V'$ be similarly obtained by splitting $V$ at $f^{-1}(F)$, with projection $q: V' \to V$. We define $v' = q^{-1}(v \cup f^{-1}(F))$, and $x' = p^{-1}(x \cup F)$. Then the induced map $f': V' \to X'$ is a degree 1 map of handlebodies in the broad sense, and we are reduced to proving the theorem for $f'$ restricted to any component of $V'$.

The preceding remark gives us an inductive simplification of $f: V, \partial V \to X, \partial X$, as follows. If $X$ has genus $>0$, we choose for $F$ a meridian disk (disjoint to $x \subset \partial X$, say), this will reduce the genus. If $X$ is a ball, and $x$ has at least three components, we choose $F$ to meet one of the components of $x$, and to separate two others. By induction, the theorem will be reduced to the special case, $X$ is a ball, and the system $x \subset \partial X$ has at most two components.

Our way of proving $f$ to be a standard will be to find independent meridian disks $D_1,\ldots, D_m$ in $V$ (where $m$ is the genus of $V$) such that $f|\partial D_i$ is contractible in $\partial X - x$, and then to apply the analogue of Corollary 1.3 in the present more general setting.

To begin with, let $D_1,\ldots, D_m$ be any system of independent meridian disks in $V$. For any one of these, $D_j$ say, there exists a nonsingular path $k$ which starts from a prescribed side of $D_j$, is otherwise disjoint to the $D_i$, and ends at a prescribed component $v_1$ of $v$. Let $U$ be a regular neighborhood of $D_j \cup k \cup v_1$. $\mathrm{Cl}(\partial U - \partial V)$ consists of two disks. One of these is essentially $D_j$; let $D'_j$ be the other one. The homotopy class of $f|\partial D'_j$ in $\partial X - \mathrm{Int}(x)$ is equal to that of $f|\partial D_j$ times $[f(\partial v_1)]^\varepsilon$. Moreover, $[f(\partial v_1)]$ generates $\pi_1(\partial X - \mathrm{Int}(x))$, and by starting instead from the other side of $D_j$, we may replace $\varepsilon$ by $-\varepsilon$. Hence by this method of sliding, we can find the required meridian disks.

## 2. Mappings of Heegaard Splittings

A Heegaard splitting of a (closed, orientable) 3-manifold $M$ is a triad $(M, V, W)$, with $V$, $W$ handlebodies, and $V \cup W = M$, $V \cap W = \partial V = \partial W$. A mapping of Heegaard splittings is a map of triads $(M, V, W) \to (N, X, Y)$.

**Theorem 2.1.** *Let a Heegaard splitting $(N, X, Y)$ and a degree 1 map $f: M \to N$ be given. Then there exist a Heegaard splitting $(M, V, W)$ and a mapping of Heegaard splittings $g: (M, V, W) \to (N, X, Y)$ such that $g|M$ is homotopic to $f$, and both $g|V$ and $g|W$ are standard mappings of handlebodies in the sense of Section 1.*

*Proof.* Considering $X$ as a regular neighborhood of a 1-complex, we may assume that $V_1 = f^{-1}(X)$ is a regular neighborhood of a 1-complex. Let $k$ denote a system of proper arcs in $W_1 = \mathrm{Cl}(M - V_1)$, and $U$ a regular neighborhood of $k$. It is well known that there exists such a $k$ that $W_2 = \mathrm{Cl}(W_1 - U)$ is a handlebody. Since $\pi_1 \partial Y \to \pi_1 Y$ is surjective, there exists a deformation from $f$ to $f_2$ such that $f_2(V_2) = f_2(V_1 \cup U) \subset X$, and $f_2(W_2) \subset Y$; this gives a mapping of Heegaard splittings.

We now apply the process of drilling holes to $f_2|W_2$. The result is again a mapping of Heegaard splittings, $f_3: (M, V_3, W_3) \to (N, X, Y)$, and according to Theorem 1.4, we can achieve this way that $f_3|W_3$ is a standard mapping. Finally, we apply the process of drilling holes to $f_3|V_3$, obtaining thus a mapping $g: (M, V, W) \to (N, X, Y)$ with $g|V$ standard. $g|W$ is now standard, too, because it is obtained from the standard mapping $f_3|W_3$ by the process of adding handles. This completes the proof.

**Corollary** (Haken). *Given a Heegaard splitting of a homotopy 3-sphere, $(N, X, Y)$, there exists a degree 1 map $f: S^3 \to N$ so that $f|f^{-1}(X)$ is a homeomorphism.*

*Proof.* Let $g : (S^3, V, W) \to (N, X, Y)$ be a degree 1 map satisfying the conclusion of Theorem 2.1.

Let us call a handlebody $X' \subset \text{Int}(V)$ unknotted if $V$ can be built up from $X'$ by attaching 1-handles only, and let us call an unknotted handlebody $X' \subset \text{Int}(V)$ a *survivor* of the mapping $g|V$ if $g|V$ can be properly deformed to a map $g'$ such that $g'|X'$ is a homeomorphism onto $X$, and $g'(V - X') \subset \partial X$. By definition of a standard map, there exists a survivor of $g|V$. This completes the proof of the corollary.

In the above corollary, $f^{-1}(Y)$ can be almost anything, even if one starts with $N = S^3$. The homeomorphism type of $f^{-1}(Y)$ here appears as the result of a random choice: the choice of a survivor of $g|V$. It is by far not true that such a survivor is well determined up to (setwise) isotopy in $V$. If, for example, genus$(X) = 1$, then it is easily checked that a survivor $X'$ can be replaced by any unknotted $X''$ in the same homotopy class. In the general case, the classification of survivors seems to be a rather difficult, and significant, problem.

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