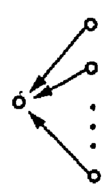
FOUR PAPERS ON PROBLEMS IN LINEAR ALGEBRA

Claus Michael Ringel

This volume contains four papers on problems in linear algebra. They form part of a general investigation which was started with the famous paper [Q] on the four subspace problem. The r subspace problem asks for the determination of the possible positions of r subspaces in a vector space, or, equivalently, of the indecomposable representations of the following oriented graph

(*) 

with $r+1$ vertices. For $r \geqslant 5$, this problem seems to be rather hard to attack, however one may try to obtain at least partial results dealing with special kinds of representations. Also, the r subspace problem can be used as a test problem for more elaborate problems in linear algebra. This seems to be the case for some of the investigations published in this volume, they have been generalized recently to the case of arbitrary oriented graphs [M, S].

Three of the four papers deal with the r subspace problem. (We should remark that there is a rather large overlap of [F] and [I, II]. However, the main argument of [F], the proof given in section 7, is not repeated in [I, II], whereas [I, II] give the details for the complete irreducibility of the representations $\rho_{t,l}$ which only was announced in [F]. We also recommend the survey given by Dlab [8].) Given r subspaces $E_1, \ldots, E_r$ of a finite-dimensional vector space V, we obtain a lattice homomorphism ρ from the free modular lattice D^r with r generators $e_1, \ldots, e_r$ into the lattice $L(V)$ of all subspaces of V given by $\rho(e_i) = E_i$. Such a lattice homomorphism is called a representation of D^r. In [F], Gelfand and Ponomarev introduce a set of indecomposable representations $\rho_{t,l}$ with $0 \leqslant t \leqslant r$ and $l \in \mathbb{N}$, which we will call the preprojective representations (in [F], the representations $\rho_{t,l}$ with $1 \leqslant t \leqslant r$ are called representations of the first kind, those of the form $\rho_{o,l}$ representations of the second kind; in [I, II] there may arise some confusion: $\rho_{t,l}$ is denoted by $\rho^+_{t,l}$, whereas the symbol $\rho_{t,l}$ used in [I, II] stands for the same type of representation but with a shift of the indices, see Proposition 8.2 in [II]). For the construction of the preprojective representations, we refer to section 1.4 of

[F] : one first defines a finite set $A_t(r, l)$ (which later we will identify with a set of paths in some oriented graph), considers the vector space with basis the set $A_t(r, l)$, and also a subspace $Z_t(r, l)$ generated by certain sums of the canonical base elements of $A_t(r, l)$. The residue classes of the canonical base elements of $A_t(r, l)$ in $V_{t,l} = A_t(r, l)/Z_t(r, l)$ will be denoted by ξ_α (with $\alpha \in A_t(r, l)$). Now, the representation $\rho_{t,l}$ is given by the vector space $V_{t,l}$ together with a certain r tuple of subspaces of $V_{t,l}$, all being generated by some of the generators ξ_α. Note that this implies that $\rho_{t,l}$ is defined over the prime field k_0 of k. (Gelfand and Ponomarev usually assume that the characteristic of k is zero, thus $k_0 = \mathbb{Q}$. However, all results and proofs remain valid in general.)

The main result concerning these representations $\rho_{t,l}$ asserts that in case dim $V_{t,l} > 2$, the representation $\rho_{t,l}$ is completely irreducible. This means that the image of D^r under the lattice homomorphism $\rho_{t,l}\colon D^r \to L(V_{t,l})$ is the set of all subspaces of $V_{t,l}$ defined over the prime field k_0, thus $\rho_{t,l}(D^r)$ is a projective geometry over k_0. The first essential step in the proof of this result is to show that the subspaces $k\xi_\alpha$ are of the form $\rho(e_\alpha)$ for some $e_\alpha \in D^r$. (In [F], this is only announced, but it is an immediate consequence of theorem 8.1 in [II].)

The second step is to show that any subspace of $V_{t,l}$ which is defined over the prime field, lies in the lattice of subspaces generated by the $k\xi_\alpha$ provided dim $V_{t,l} > 2$. Combining both assertions, we conclude that $\rho_{t,l}$ is completely irreducible unless dim $V_{t,l} \leqslant 2$. The proof of the second step occupies section 9 of [II]. Here, one considers the following situation: there is given a set $R = \{\xi_\alpha \mid \alpha\}$ of non-zero vectors of a vector space $V (= V_{t,l})$, with the following properties:

(1) R generates V

(2) R is indecomposable (there is no proper direct decomposition $V = V' \oplus V''$ with $R = (R \cap V') \cup (R \cap V'')$), and

(3) R is defined over the prime field (there exists a basis of V such that any $\xi_\alpha \in R$ is a linear combination of the base vectors with coefficients in the prime field k_0).

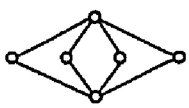
Then it is shown that the lattice of subspaces of V generated by the one-dimensional subspaces $k\xi_\alpha$, is isomorphic to the lattice of subspaces of k_0^n, with $n = \dim V$.

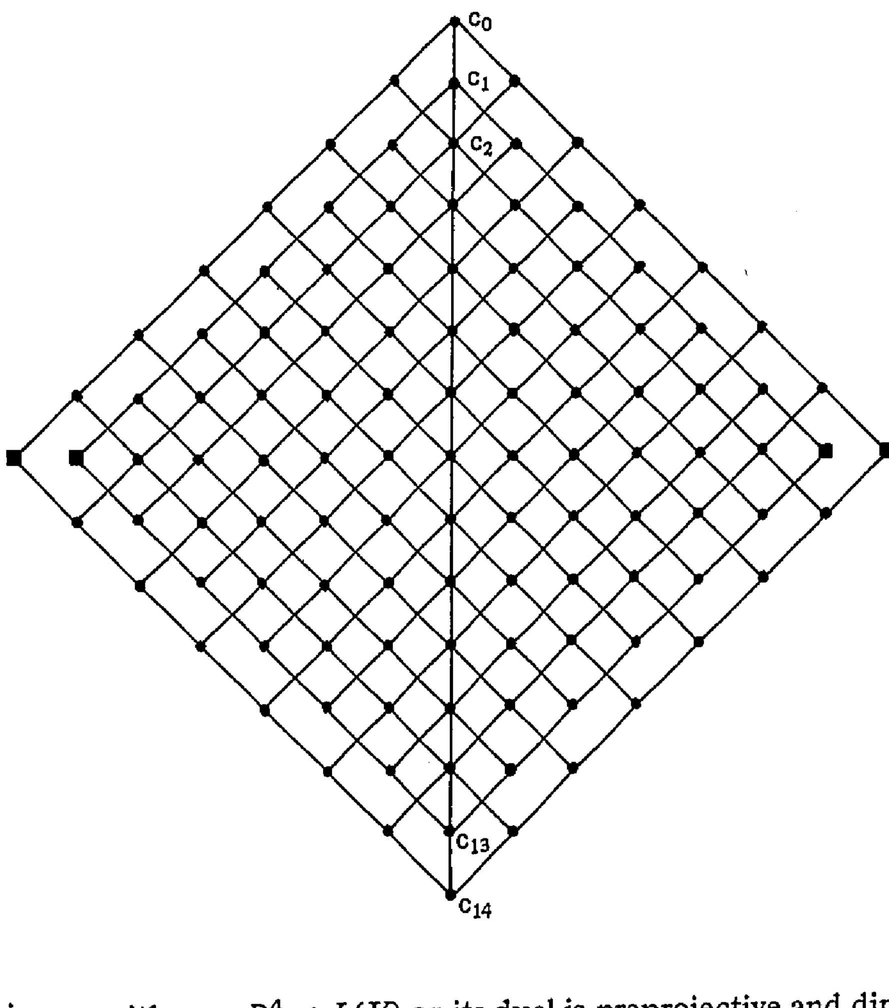
Perhaps we should add that the representations $\rho\colon D^r \to L(V)$ with V being generated by the one-dimensional subspaces of the form $\rho(a)$, $a \in D^r$, seem to be of special interest. In this case, the one-dimensional subspaces of the form $\rho(a)$, $a \in D^r$ determine completely $\rho(D^r)$. (Namely, let $b \in D^r$, and U the subspace generated by all one-dimensional subspaces of the form $\rho(x)$, $x \in D^r$, satisfying $\rho(x) \subseteq \rho(b)$, and choose $x_1, \ldots, x_s$ such that

$$\rho(b) \subseteq U \oplus \rho(x_1) \oplus \ldots \oplus \rho(x_s) = U \oplus \rho(\sum_{i=1}^{s} x_i). \text{ Thus, } \rho(b) = U \oplus (\rho(\sum_{i=1}^{s} x_i) \cap \rho(b)).$$

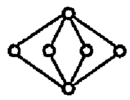
Assume, U is a proper subspace of $\rho(b)$. Then there exists $t \leqslant s$ with

$\rho(\sum_{i=1}^{t-1} x_i) \cap \rho(b) = 0$, whereas $\rho(\sum_{i=1}^{t} x_i) \cap \rho(b)$ is non-zero, and therefore one-dimensional. This however implies that $\rho(\sum_{i=1}^{t} x_i) \cap \rho(b) = \rho(b \sum_{i=1}^{t} x_i)$ is contained in U, a contradiction. Thus $\rho(b) = U$.). For $r \geqslant 4$, there always are indecomposable representations which do not have this property.

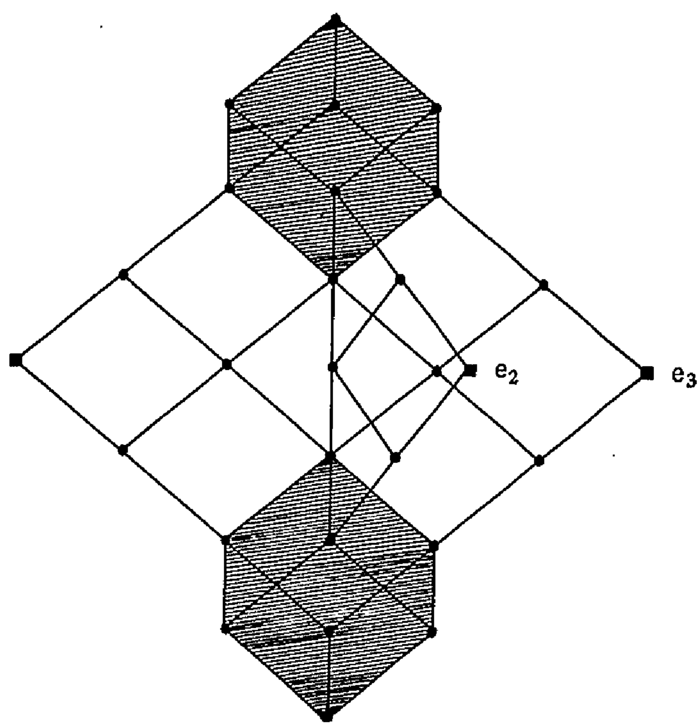
In the case $r = 4$, we may give the complete list of all lattices of the form $\rho(D^4)$, where ρ is an indecomposable representation. Besides the projective geometries over any prime field, and of arbitrary finite dimension $\neq 1$, and the lattice , we obtain all the lattices $S(n, 4)$ introduced by Day, Herrmann and Wille in [6]. Let us just copy $S(14, 4)$ and note that any interval $[c_n, c_m]$ is again of the form $S(n-m, 4)$.



(In fact, in case either $\rho\colon D^4 \to L(V)$ or its dual is preprojective and dim $V > 2$, we have seen above that $\rho(D^4)$ is the full projective geometry over the prime field. If neither ρ nor its dual is preprojective, ρ is said to be regular. If ρ is regular and non-homogeneous, say of regular length n (see [9]), then $\rho(D^4) \approx S(n, 4)$, whereas for ρ homogeneous, we have

$\rho(D^4) \approx$  .)

Gelfand and Ponomarev use the representations $\rho_{t,l}$ of D^r in order to get some insight into the structure of D^r. The existence of a free modular lattice with a given set of generators is easily established, however the mere existence result does not say anything about the internal structure of D^r. In fact, it has been shown by Freese [14] that for $r \geqslant 5$, the word problem in D^r is unsolvable. The free modular lattice D^3 in 3 generators e_1, e_2, e_3 was first described by Dedekind [7], it looks as follows:



We have shaded two parts of D^3, both being Boolean lattices with 2^3 elements. For $r \geqslant 4$, Gelfand and Ponomarev have constructed two countable families of Boolean sublattices $B^+(l)$ and $B^-(l)$ with 2^r elements, where $l \in \mathbf{N}$, and such that

$$B^-(1) < B^-(2) < \ldots < B^-(l) < B^-(l+1) < \ldots$$

and

$$\ldots B^+(l+1) < B^+(l) < \ldots < B^+(2) < B^+(1),$$

called the lower and the upper cubicles, respectively. Let $B^- = \bigcup_{l \in \mathbf{N}} B^-(l)$, and

$B^+ = \bigcup_{l \in \mathbb{N}} B^+(l).$

The elements of these cubicles have an important property: they are perfect. This notion has been introduced by Gelfand and Ponomarev in [F] for the following property: a is said to be perfect if $\rho(a)$ is either O or V for any indecomposable representation $\rho: D^r \to L(V)$. This means that for any representation, the image of a is a direct summand. For any perfect element a, let $N_k(a)$ be the set of all indecomposable representations $\rho: D^r \to L(V)$, with V a finite dimension vector space over the field k and which satisfy $\rho(a) = 0$. It is shown in [F] that for $a \in B^+$, the set $N_k(a)$ is finite and contains only preprojective representations. Dually, for $a \in B^-$, the set $N_k(a)$ contains all but a finite number of indecomposable representations, and all indecomposable representations not in $N_k(a)$ are preinjective (the representations dual to preprojective ones are called preinjective).

In dealing with perfect elements it seems to be convenient to work modulo linear equivalence. Two elements $a, a' \in D^r$ are said to be linear equivalent provided $\rho(a) = \rho(a')$ for any representation $\rho: D^r \to L(V)$. Of course, any element linear equivalent to a perfect element is also perfect. Up to linear equivalence, one has $B^- < B^+$ and Gelfand and Ponomarev have conjectured that, up to linear equivalence, all perfect elements belong to $B^- \cup B^+$. However, this has to be modified. Herrmann [19] has pointed out that there are additional perfect elements arising from the different characteristics of fields. For example, for any prime number p, and $m \geqslant 2$, there is some perfect element $d_{pm} \in D^r$ such that $N_k(d_{pm})$ contains all representations $\rho_{t,l}$ with $l < m$, and, in case the characteristic of k is p, then, in addition, the representation $\rho_{o,m}$, and nothing else. Thus, it is even more convenient to work in the free p-linear lattice D_p^r, the quotient of D^r modulo p-linear equivalence where p is either zero or a prime. Here, two elements, $a, a' \in D^r$ are said to be p-linear equivalent provided $\rho(a) = \rho(a')$ for any representation ρ in a vector space over a field of characteristic p.

The modified conjecture now asserts that any perfect element is p-linearly equivalent to an element in $B^- \cup B^+$. This indeed is true, as we want to show. Thus, assume there exists a perfect element $a \in D^r$ which is not p-linear equivalent to an element of $B^- \cup B^+$. Gelfand and Ponomarev have shown that then $N(a) = N_k(a)$ contains all preprojective representations and no preinjective representation. In a joint paper [10] with Dlab, we have shown that for $r \geqslant 5$, the set $N(a)$ either contains only the preprojective representations or else all but the preinjective representations. The elements $x \in D^r$ are given by lattice polynomials in the variables $e_1, \ldots, e_r$. Of course, there will be many different lattice polynomials which define the same element x. A lattice polynomial with minimal number of occurrences of variables defining x will be called a reduced expression of x and this number of variables in a reduced expression will be called the complexity $c(x)$ of x. Now, let $\rho: D^r \to L(V)$ be a representation, U a one-dimensional subspace of V, and

$\rho': D^r \to L(V/U)$ the induced representation, with $\rho'(e_i) = (\rho(e_i) + U)/U$ for the generators e_i, $1 \leqslant i \leqslant r$. We claim that for $x \in D^r$, we have

$$\dim \rho'(x) \leqslant c(x) - 1 + \dim \rho(x).$$

[For the proof, we consider instead of ρ' the representation $\rho'': D^r \to L(V)$ with $\rho''(x)$ the full inverse image of $\rho'(x)$ under the projection $V \to V/U$, thus $\dim \rho''(x) = 1 + \dim \rho'(x)$, for $x \in D^r$. Also note that $\rho(x) \subseteq \rho''(x)$ for all x. By induction on $c(x)$, we show the formula

$$\dim \rho''(x) - \dim \rho(x) \leqslant c(x).$$

Since $\dim U = 1$, this clearly is true for $x = e_i$, with $\rho''(e_i) = \rho(e_i) + U$. Now assume the formula being valid both for x_1 and x_2. For $x = x_1 + x_2$ with $c(x) = c(x_1) + c(x_2)$, we have

$$\begin{aligned}\dim \rho''(x) &= \dim \rho''(x_1 + x_2) \leqslant \dim \rho(x_1 + x_2) + c(x_1) + c(x_2)\\ &= \dim \rho(x) + c(x).\end{aligned}$$

Similarly, for $x = x_1 x_2$ with $c(x) = c(x_1) + c(x_2)$, we have

$$\begin{aligned}\dim \rho''(x) &= \dim \rho''(x_1 x_2) = \dim \rho''(x_1) + \dim \rho''(x_2) - \dim \rho''(x_1 + x_2)\\ &\leqslant \dim \rho(x_1) + c(x_1) + \dim \rho(x_2) + c(x_2) - \dim \rho(x_1 + x_2)\\ &= \dim \rho(x_1 x_2) + c(x_1) + c(x_2) = \dim \rho(x) + c(x).\end{aligned}$$

This finishes the proof.]

It is now sufficient to find a preprojective representation $\rho: D^r \to L(V)$ with $\dim V > c(a)$ and a one-dimensional subspace U of V such that the induced representation ρ' in V/U has no preprojective direct summand. Namely, our considerations above imply that $\dim \rho'(a) \leqslant c(a) - 1 < \dim V/U$, due to the fact that $\rho(a) = 0$, and therefore there exists at least one indecomposable representation σ in $N(a)$ which is not preprojective. As a consequence, in case $r \geqslant 5$, we know that $N(a)$ contains all but the preinjective representations. By duality, we similarly show that $N(a)$ contains only the preprojective representations, thus we obtain a contradiction. So, let us construct a suitable preprojective representation with the properties mentioned above. In fact, instead of considering representations of D^r, we will work inside the abelian category of representations of the oriented graph (*). We denote by $P_{t,l} = (V_{t,l}; \rho_{t,l}(e_1), \ldots, \rho_{t,l}(e_r))$ the graph representation corresponding to $\rho_{t,l}$. Take any homomorphism $\varphi: P_{0,1} \to P_{0,2}$ such that $R = \operatorname{Cok} \varphi$ is regular (that is, has no non-zero preprojective or preinjective direct summand. For example, there always exists such a φ with R being the direct sum of two indecomposable representations of dimension types $(1; 1, 1, 0, \ldots, 0)$ and $(r-3; 0, 0, 1, 1, \ldots, 1)$.) Now apply Φ^{-i} for $i \in \mathbf{N}$. We obtain exact sequences

$$0 \to P_{0,i+1} \xrightarrow{\Phi^{-i}(\varphi)} P_{0,i+2} \to \Phi^{-i} R \to 0,$$

thus, the inclusion

$$\varphi_i = \Phi^{-i}(\varphi) \circ \ldots \circ \Phi^{-1}(\varphi) \circ \varphi \colon P_{0,1} \to P_{0,i+2}$$

has regular cokernel (extensions of regular representations being regular, again). We now only have to choose i such that $\dim V_{0,i+2} > c(a)$. This finishes the proof in case $r \geqslant 5$. (For $r = 4$, we again take $\varphi\colon P_{0,1} \to P_{0,2}$ with Cok φ being the direct sum of two representations of dimension types $(1; 1, 1, 0, 0)$ and $(1, 0, 0, 1, 1)$, and form φ_i. The indecomposable summands of Cok φ_i all belong to one component C of the Auslander–Reiten quiver, thus we conclude as above that $C \subseteq N(a)$. By duality, one similarly shows that there are representations in C which do not belong to $N(a)$, so again we obtain a contradiction. Note that in case $r = 4$, the conjecture has been solved before by Herrmann [19].)

We consider now the general problem of representations of an oriented graph (Γ, Λ). We do not recall the definition of the category $L(\Gamma, \Lambda)$ of representations of (Γ, Λ) over some fixed field k, nor the typical examples, but just refer to the first two pages of [BGP]. We only note that $L(\Gamma, \Lambda)$ can also be considered as the category of modules over the path algebra $k(\Gamma, \Lambda)$, see [17], and $k(\Gamma, \Lambda)$ is a finite-dimensional k-algebra if and only if (Γ, Λ) does not have oriented cycles. In [15], Gabriel had shown that (Γ, Λ) has only finitely many indecomposable representations if and only if Γ is the disjoint union of graphs of the form A_n, D_n, E_6, E_7 and E_8 (they are depicted on the third page of [BGP]). It turned out that in case Γ is of the form A_n, D_n, E_6, E_7 or E_8, the indecomposable representations of (Γ, Λ), with Λ an arbitrary orientation, are in one-to-one correspondence to the positive roots of Γ. It is the aim of the paper [BGP] to give a direct proof of this fact. It introduces appropriate functors which produce all indecomposable representations from the simple ones in the same way as the canonical generators of the Weyl group produce all positive roots from the simple ones. We later will come back to these functors and their various generalizations.

Given a finite graph Γ, let E_Γ be the $\mathbb{Q}$-vector space of functions $\Gamma_0 \to \mathbb{Q}$, an element of E_Γ being written as a tuple $x = (x_\alpha)$ indexed by the elements $\alpha \in \Gamma_0$. For $\beta \in \Gamma_0$, we denote its characteristic function by $\overline{\beta}$ (thus $\overline{\beta}_\alpha = 0$ for $\alpha \neq \beta$, and $\overline{\beta}_\beta = 1$). Any representation V of (Γ, Λ) gives rise to an element $\dim V$ in E_Γ, its dimension type. For any orientation Λ of Γ, and any $\beta \in \Gamma_0$, there is a unique simple representation L_β of dimension type $\dim L_\beta = \overline{\beta}$. In case there are no oriented cycles in (Γ, Λ), we obtain in this way all simple representations of (Γ, Λ), thus, in this case, E_Γ may be identified with the rational Grothendieck group $G_0(\Gamma, \Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}$ (here, $G_0(\Gamma, \Lambda)$ is the factor group of the free abelian group with basis the set of all representations of (Γ, Λ)

modulo all exact sequences) with dim being the canonical map (sending a representation to the corresponding residue class). On E_Γ, there is defined a quadratic form B. In fact, for any orientation Λ of Γ, we may consider the (non-symmetric) bilinear form B_Λ on E_Γ given by

$$B_\Lambda(x, y) = \sum_{l \in \Gamma_0} x_\alpha y_\alpha - \sum_{l \in \Gamma_1} x_{\alpha(l)} y_{\beta(l)}$$

and B is the corresponding quadratic form $B(x) = B_\Lambda(x, x)$. Note that B is positive definite if and only if Γ is the disjoint union of graphs of the form A_n, D_n, E_6, E_7 and E_8, and in these cases, the root system for Γ is by definition just the set of solutions of the equation $B(x) = 1$.

For k algebraically closed and B being positive definite we will outline a direct proof that dim: $L(\Gamma, \Lambda) \to E_\Gamma$ induces a bijection between the indecomposable representations of (Γ, Λ) and the positive roots. There is the following algebraic-geometric interpretation of B due to Tits [15]: The representations of (Γ, Λ) of dimension type x may be considered as the algebraic variety

$$m^x(\Gamma, \Lambda) = \prod_{l \in \Gamma_1} \mathrm{Hom}(k^{\alpha(l)}, k^{\beta(l)}),$$

and there is an obvious action on it by the algebraic group

$$G^x = \prod_{\alpha \in \Gamma_0} \mathrm{GL}(\alpha, k)/\Delta$$

with Δ being the multiplicative group of k diagonally embedded as group of scalars. Clearly

$$B(x) = \dim G^x + 1 - \dim m^x(\Gamma, \Lambda).$$

Using this interpretation, Gabriel has shown in [16] that it only remains to prove that the endomorphism ring of any indecomposable representation is k. So assume V is indecomposable, and that there are non-zero nilpotent endomorphisms. Then V contains a subrepresentation U with $\mathrm{End}(U) = k$ and $\mathrm{Ext}^1(U, U) \neq 0$. [Namely, let $0 \neq \varphi$ be an endomorphism with image S of smallest possible length, thus $\varphi^2 = 0$, and let $W = \bigoplus_{i=1}^{r} W_i$ be the kernel of φ, with all W_i indecomposable. Now $S \subseteq W$, thus the projection of S into some W_i must be non-zero. Since S was an image of a non-zero endomorphism of smallest length, we see that S embeds into this W_i. We may assume $i = 1$. Thus there is an inclusion $\iota: S \to W_1$. If W_1 has non-zero nilpotent endomorphisms, we use induction. Otherwise $\mathrm{End}(W_1) = k$. Also, $\mathrm{Ext}^1(W_1, W_1) \neq 0$, since on the one hand $\mathrm{Ext}^1(S, W_1) \neq 0$ due to the exact sequence

$$0 \to \bigoplus_{i=1}^{r} W_i \to V \to S \to 0$$

and, on the other hand, the inclusion ι gives rise to a surjection $\mathrm{Ext}^1(\iota, W)$. Here we use that $L(\Gamma, \Lambda)$ is a hereditary category]. The bilinear form B_Λ has the following homological interpretation [25]:

$$B_\Lambda(\dim V, \dim V') = \dim_k \mathrm{Hom}(V, V') - \dim_k \mathrm{Ext}^1(V, V'),$$

for all representations V, V'. Consequently, the existence of a representation U satisfying $\mathrm{End}(U) = k$, $\mathrm{Ext}^1(U, U) \neq 0$ would imply that

$$B(\dim U) = B_\Lambda(\dim U, \dim U) \leqslant 0,$$

contrary to the assumption that B is positive definite. This finishes the proof.

For any finite connected graph Γ without loops, Kac [21, 22] gave a purely combinatorial definition of its root system Λ. Note that Λ is a subset of E_Γ containing the canonical base vectors $\overline{\beta}$, for $\beta \in \Gamma_0$, and being stable under the Weyl group W, the group generated by the reflections σ_β along $\overline{\beta}$ with respect to B. The set Δ can also be interpreted in terms of root spaces of certain (usually infinite dimensional) Lie algebras [21]. Denote by Δ_+ the set of roots with only non-negative coordinates with respect to the canonical basis. Then Δ is the union of Δ_+ and $\Delta_- = -\Delta_+$. In case Γ is of type A_n, D_n, E_6, E_7 or E_8, the root system is finite and coincides with the set of solutions of $B(x) = 1$. Otherwise the root system is infinite and will contain besides certain solutions of $B(x) = 1$ also some solutions of $B(x) \leqslant 0$. The elements x of the root system which satisfy $B(x) = 1$ are called real roots, they are precisely the elements of the W-orbits of the canonical base elements. The remaining elements of the root system are called imaginary roots, and Kac has determined a fundamental domain for this set, the fundamental chamber.

Now, one has the following results (at least if k is either finite or algebraically closed): For any finite graph Γ without loops, and any orientation Λ, the set of dimension types of indecomposable modules is precisely the set Δ_+ of positive roots. For any positive real root x, there exists precisely one indecomposable representation V of (Γ, Λ) with $\dim V = x$. For any positive imaginary root x, the maximal dimension μ_x of an irreducible component in the set of isomorphism classes of indecomposable representations of dimension x is precisely $1 - B(x, x)$. (Note that the subset of indecomposable representations in $m^x(\Gamma, \Lambda)$ is constructible, and G^x-invariant, thus we can decompose it as a finite disjoint union of G^x-invariant subsets each of which admits a geometric quotient. By definition, μ_x is the maximum of the dimensions of these quotients.) In particular, we see that the number of indecomposable representations (or of the maximal dimension of families of indecomposable representations) of (Γ, Λ) does not depend on the orientation Λ. For Γ of the form A_n, D_n, E_6, E_7 or E_8, this is Gabriel's theorem (of course, there are no imaginary roots). For Γ of the form $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$, the so called tame cases, these results have been shown by Donovan–Freislich [13] and Nazarova [23], see also [9]; in fact, in these cases one obtains a full classification of all indecomposable representations; also, it is possible in these cases to describe completely the rational invariants of the action of G^x on $m^x(\Gamma, \Lambda)$, for any dimension type x, see [27]. Of course, the oriented graphs of finite or tame representation type are rather special ones. It has been known since some time that the remaining (Γ, Λ) are wild: there always is a full exact subcategory of $L(\Gamma, \Lambda)$ which is equivalent to the category $\mathcal{M}_{k\langle X, Y\rangle}$ of $k\langle X, Y\rangle$-modules ($k\langle X, Y\rangle$ being the polynomial ring in two non-commuting indeterminates). In this situation, the results above are due to Kac [21, 22]. Note that this solves all the conjectures of Bernstein–Gelfand–Ponomarev formulated in [BGP]. However, there remain many open questions concerning wild graphs (Γ, Λ). One does not expect to obtain a complete classification of the indecomposable representations of such a graph, but one would like to have some more knowledge about certain classes of representations. For example, there does not yet exist a combinatorial description of the set of those roots which are dimension types of representations V with $\mathrm{End}(V) = k$.

We have mentioned above that the root system Δ of Γ is stable under the Weyl group W and that any W-orbit of Δ contains either one of the base vectors $\bar{\beta}$ (with $\beta \in \Gamma_0$) or an element of the fundamental chamber. One therefore tries to find operations which associate to an indecomposable representation V of (Γ, Λ) with Λ an orientation, and a Weyl group element $w \in W$ a new indecomposable representation of (Γ, Λ'), where Λ' is a possibly different orientation of Γ. By now, several such operations are known (see [BGP, 21, 28]), the first one being the reflection functors F_β^-, F_β^+ introduced by Bernstein, Gelfand and Ponomarev in [BGP]. Here, for the definition of F_β^+, the vertex β is supposed to be a sink, thus the simple representation L_β with dimension vector $\dim L_\beta = \bar{\beta}$ is projective. This concept has been generalized by Auslander, Platzeck and Reiten [1] dealing with any finite dimensional algebra A (or even an artin algebra) with a simple projective module L. For this, we need the Auslander–Reiten translates τ, τ^{-1}. Recall that τX_A is defined for any A-module X_A: let $P_1 \xrightarrow{p} P_0 \to X_A \to 0$ be a minimal projective resolution of X_A, then $\mathrm{Tr}\, X_A$ is by definition the cokernel of the map $\mathrm{Hom}(p, A_A)$ and $\tau X = D\, \mathrm{Tr}\, X$, $\tau^{-1} X = \mathrm{Tr}\, D\, X$, with D the usual duality with respect to the base field k. So assume L is a simple projective A-module, let P be the direct sum of one copy of each of the indecomposable projective modules different from L, and $B = \mathrm{End}(P \oplus \tau^{-1} L)$. The functor considered by Auslander, Platzeck and Reiten is $F = \mathrm{Hom}_A(P \oplus \tau^{-1} L, -)$ from the category $\mathcal{M}_A$ of A-modules to $\mathcal{M}_B$. The functor induces an equivalence of the full subcategory T of $\mathcal{M}_A$ of all modules which do not have L as a direct summand and a certain full subcategory of $\mathcal{M}_B$. Note that $P \oplus \tau^{-1} L$ is a tilting module in the sense of [18], except in the trivial case of L being, in addition, injective. (A tilting module T_A is defined by the following three properties:

(1) proj. dim. $T_A \leqslant 1$, (2) there exists an exact sequence $0 \to A_A \to T' \to T'' \to 0$, with T', T'' being direct sums of direct summands of T_A, and (3) $\mathrm{Ext}^1(T_A, T_A) = 0$. Now, if L_A is simple projective and not injective, the middle term Y of the Auslander–Reiten sequence

$$0 \to L \to Y \to \tau^{-1}L \to 0$$

starting with L is projective. This sequence shows, on the one hand, that proj. dim. $\tau^{-1}L = 1$. On the other hand, it also gives an exact sequence of the form needed in (2). Finally, $\mathrm{Ext}^1_A(P \oplus \tau^{-1}L, P \oplus \tau^{-1}L) \approx$ $\approx D\,\mathrm{Hom}(P \oplus \tau^{-1}L, L) = 0$, since any non-zero homomorphism from a module to L is a split epimorphism.)

A certain composition of the reflection functors F^+_β (or F^-_β, respectively) is of particular interest, the Coxeter functor Φ^+ (or Φ^-). An explicit calculation for the r-subspace situation is given in [F], in the special case of the 4-subspace problem it had been defined before in [Q]. The Coxeter functors are endofunctors of $L(\Gamma, \Lambda)$, they are only defined in case (Γ, Λ) does not have oriented cycles (non-oriented cycles are allowed, see [9]). Note that the assignment of an orientation Λ without oriented cycles is equivalent to the choice of a partial ordering of Γ_0 (let $\alpha \leqslant \beta$ iff there exists an oriented path $\alpha = \alpha_0 \leftarrow \alpha_1 \leftarrow \ldots$ $\ldots \leftarrow \alpha_m = \beta$), and also to the choice of a Coxeter transformation: this is a Weyl group element of the form $c = \sigma_{\alpha_n} \ldots \sigma_{\alpha_1}$ with $\alpha_1, \ldots, \alpha_n$ being the elements of Γ_0 in some fixed ordering (take an ordering of Γ which refines the given partial ordering). So assume from now on that (Γ, Λ) is a connected oriented graph without oriented cycles, and let c be the corresponding Coxeter element. The Coxeter functors Φ^+ and Φ^- defined in [BGP] have the following properties: if V is an indecomposable representation of (Γ, Λ), then either V is projective and then $\Phi^+(V) = 0$, or else V is not projective, and then $\Phi^+(V)$ again is indecomposable, $\Phi^-\Phi^+(V) \approx V$ and $\dim \Phi^+(V) = c \dim V$. Thus the Coxeter functor Φ^+ realizes the action of the Coxeter transformation on the set of all representations without non-zero projective direct summands. The usefulness of the Coxeter functors seems to have its origin in their relation to the Auslander–Reiten translation τ. Namely, Gabriel ([17], Prop. 5.3, see also [1,5]) has shown that τ can be identified with $C^+ \circ T$, where T is the functor which maps the representation (V, f) to $(V, -f)$. In particular, for Γ being a tree, we can identify τ with C^+ itself.

In order to explain the value of the Auslander–Reiten translation τ (and therefore of the Coxeter functors), we have to recall the definition of the Auslander–Reiten quiver of a finite dimensional algebra A. Its vertices are the isomorphism classes $[X]$ of the indecomposable A-modules X, and, if X, Y are indecomposable modules, then there is an arrow $[X] \to [Y]$ iff there exists an irreducible map $X \to Y$ (a map f is said to be irreducible provided it is neither a split monomorphism nor a split epimorphism, and for any factorization $f = f'' \circ f'$, we have that f' is a split monomorphism or f'' is a split epimorphism [2]). Now, the Auslander–Reiten quiver is a translation quiver with respect to τ: if X is indecomposable and not projective, then there exists an irreducible map $Y \to X$ iff there exists an irreducible map $\tau X \to Y$.

For the finite dimensional hereditary algebras A, the structure of the Auslander–Reiten quiver is known. We will recall this result in the special case of $A = k(\Gamma, \Lambda)$. First, we need some notation. Define $\mathbf{Z}(\Gamma, \Lambda)$ as follows: its vertices are the elements of $\Gamma_0 \times \mathbf{Z}$, and for any arrow ${}^{i}\circ \xleftarrow{\alpha} \circ^{j}$, there are arrows $(i, z) \xrightarrow{(\alpha, z)} (j, z)$ and $(j, z) \xrightarrow{(\alpha^*, z)} (i, z+1)$, for all $z \in \mathbf{Z}$, see [24] and also [17, 29]. Note that in case Γ is a tree, $\mathbf{Z}(\Gamma, \Lambda)$ does not depend on the orientation Λ and just may be denoted by $\mathbf{Z}\Gamma$. If $I \subseteq \mathbf{Z}$, let $I(\Gamma, \Lambda)$ be the full subgraph of all vertices (i, z) with $i \in I$. In particular, we will have to consider $\mathbf{N}(\Gamma, \Lambda)$ and $\mathbf{N}^-(\Gamma, \Lambda)$, where $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{N}^- = \{-1, -2, -3, \ldots\}$. Also, denote by A_∞ the following infinite graph

$$\circ \text{---} \circ \text{---} \circ \ldots \circ \text{---} \circ \ldots \quad .$$

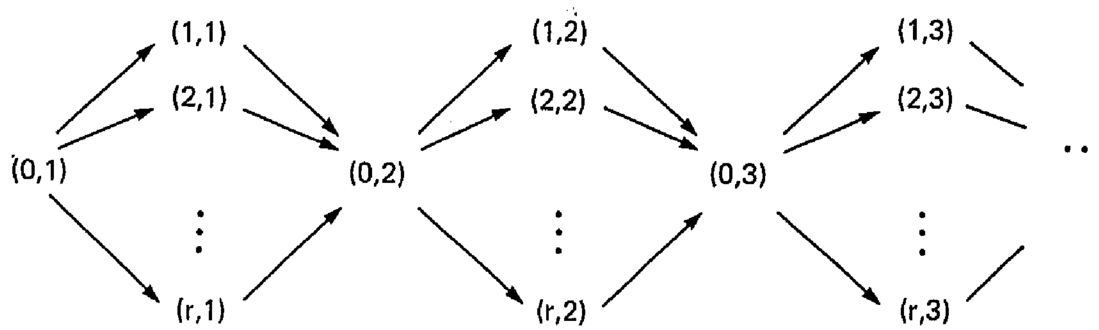
The result is as follows: in case Γ is of the form A_n, D_n, E_6, E_7 or E_8, the Auslander–Reiten quiver of $k(\Gamma, \Lambda)$ is a finite full connected subquiver of $\mathbf{Z}\Gamma$. (In case D_n with $n \equiv 0(2)$, the Auslander–Reiten quiver of $k(\Gamma, \Lambda)$ is $[1, n-1]$ (Γ, Λ), in case of E_7 or E_8, it is $[1,9]$ (Γ, Λ) or $[1,15]$ (Γ, Λ), respectively; in the remaining cases, it is slightly more difficult to describe, see [17, 29]). In all other cases, the Auslander–Reiten quiver of $k(\Gamma, \Lambda)$ has infinitely many components, all but two being quotients of $\mathbf{Z}A_\infty$ (see [26]), the remaining two being of the form $\mathbf{N}(\Gamma, \Lambda)$ and $\mathbf{N}^-(\Gamma, \Lambda)$. The component of the form $\mathbf{N}(\Gamma, \Lambda)$ contains the indecomposable projective modules: in fact, the indecomposable projective module P_i corresponding to the vertex $i \in \Gamma_0$ appears as indexed by $(i, 1)$, and the module indexed by (i, z), $z \in \mathbf{N}$, is just $\Phi^{-z+1}(P_i)$, this component is called the preprojective component. Similarly, the component of the form $\mathbf{N}^-(\Gamma, \Lambda)$ is called the preinjective component, it contains the indecomposable injective module J_i corresponding to $i \in \Gamma_0$ as indexed by $(i, -1)$, and the module indexed by $(i, -z)$, $z \in \mathbf{N}$, is just $\Phi^{+z-1}(J_i)$.

Let us consider in more detail a preprojective component $\mathscr{P}$, and the modules belonging to $\mathscr{P}$; they will be called preprojective modules. In case Γ is of type A_n, D_n, E_6, E_7, or E_8, we let $\mathscr{P}$ denote the full Auslander–Reiten quiver; in any case, we note that an indecomposable representation of (Γ, Λ) is said to be preprojective iff it is of the form $\Phi^{-z}P$, with P indecomposable projective and $z \geqslant 0$. (A general theory of preprojective modules has been developed by Auslander and Smalø, see [3]). For an indecomposable preprojective representation X, there are only finitely many indecomposable modules Y such that $\mathrm{Hom}(Y, X) \neq 0$, all of them are preprojective again, and any non-invertible homomorphism $Y \to X$ is a sum of compositions of irreducible maps. In particular, if X, Y are indecomposable and preprojective and

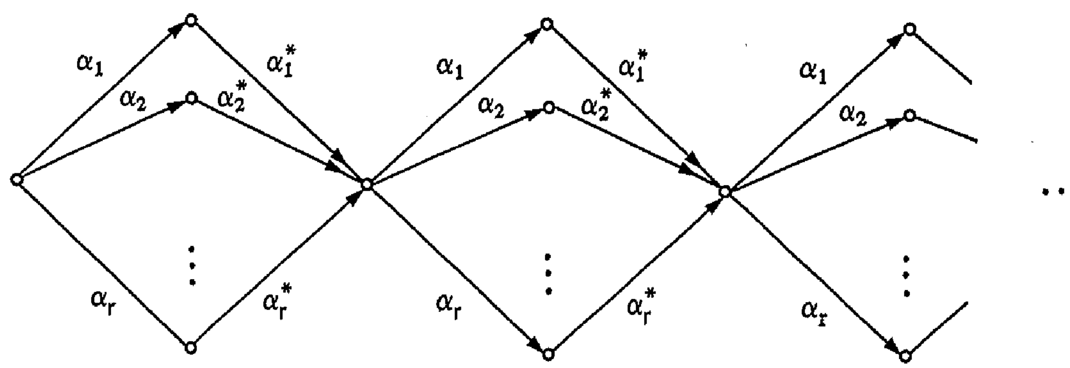
$\mathrm{Hom}(X, Y) \neq 0$, then there is an oriented path $[X] \to \ldots \to [Y]$ in $\mathscr{P}$. In fact, the complete categorical structure of the full subcategory of preprojective modules can be read off from the combinatorial description of $\mathscr{P}$ as a translation quiver: the category of all preprojective modules is equivalent to the quotient category $\langle\rangle\mathscr{P}$ of the path category of $\mathscr{P}$ modulo the so called mesh relations (see [4, 24, 17]). Note that the category $\langle\rangle\mathscr{P}$ allows to reconstruct all the modules in $\mathscr{P}$. Namely, any module X_A is isomorphic to $\mathrm{Hom}({}_AA_A, X_A)$, thus, if $A_A = \bigoplus_{i \in \Gamma_0} P_i^{n_i}$, then X_A can be identified with $\bigoplus_{i \in \Gamma_0} \mathrm{Hom}(P_i, X)^{n_i}$, and $\mathrm{Hom}(P_i, X)$ can be calculated inside $\langle\rangle\mathscr{P}$, since both P_i, X are preprojective.

Starting from the preprojective component $\mathscr{P}$ of $k(\Gamma, \Lambda)$, one may define a (usually infinite-dimensional) algebra Π as follows: Take the direct sum of all homomorphism spaces $\mathrm{Hom}(j, 1), (t, l))$ in $\langle\rangle\mathscr{P}$ and define the product of two residue classes $\overline{w}$, $\overline{w}'$ of paths $w: (j, 1) \to \ldots \to (t, l)$ and $w': (j', 1) \to \ldots \to (t', l')$ as follows: in case $t = j'$, let $\overline{w}\,\overline{w}'$ be the residue class of the composed path $\tau^{-l+1}(w') \circ w: (j, 1) \to \ldots \to (t', l + l' - 1)$, and 0 otherwise. There is a purely combinatorial description of Π in terms of (Γ, Λ) due to Gelfand and Ponomarev, see [R]. Let $\hat{\Gamma}$ be obtained from (Γ, Λ) by adding to each arrow $\alpha: i \to j$ an additional arrow $\alpha^*: j \to i$. We clearly can identify Π with the factor algebra of the path algebra $k\hat{\Gamma}$ modulo the ideal generated by the element $\sum_{\alpha \in \Gamma_1} \alpha \cdot \alpha^* + \sum_{\alpha \in \Gamma_1} \alpha^* \cdot \alpha$. Note that this description is independent of the choice of the orientation Λ. Also, we see from both descriptions that Π contains as a subalgebra $k(\Gamma, \Lambda)$, thus we may consider Π as a right $k(\Gamma, \Lambda)$-module, and the first description now shows that the $k(\Gamma, \Lambda)$-module $\Pi_{k(\Gamma,\Lambda)}$ decomposes as the direct sum of all preprojective representations of (Γ, Λ) each occurring with multiplicity one, and therefore is called the preprojective algebra of Γ. (For the proper generalisation to the case of a species, we refer to [11]. We also should note the slight deviation of the preprojective algebra from the model algebra defined in [M], which reduces to the algebra A^r given in [I, II] in the case of the r-subspace situation. Namely, here the constant paths have square zero, whereas they are idempotents in Π. Now, in Π the sum of the constant paths is the identity element. In order also to have an identity element, Gelfand and Ponomarev add to the direct sum of all preprojective modules an additional one-dimensional space $k\epsilon$. There is a change of definition proposed in [S], using the constant paths as idempotents as in Π, but adding again an additional identity element.) Since Π is the direct sum of the preprojective representations of (Γ, Λ), it follows that Π is finite dimensional if and only if Γ is of the form A_n, D_n, E_6, E_7, or E_8. In [12], the tame cases $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ have been characterized by the fact that the Gelfand–Kirillov dimension of Π is 1, whereas it is ∞ for the wild cases.

Let us return to the special case of the r subspace graph (*), with $r \geqslant 4$. The description above gives that the preprojective component $\mathscr{P}$ is of the form



If we denote the arrows in the following way:



then the mesh relations are as follows: $\alpha_i \alpha_i^* = 0$ for all i, and $\sum\limits_i \alpha_i^* \alpha_i = 0$.

Thus, if we want to determine the total space of the representation labelled (t, l), we have to calculate $\mathrm{Hom}((0, 1), (t, l))$ inside the category $\langle\rangle\mathscr{P}$, and this amounts to the calculation of all possible paths from $(0, 1)$ to (t, l), taking this as the basis of a vector space and factoring out the mesh relations. However, taking from the beginning into account the relations $\alpha_i \alpha_i^* = 0$, we just as well may work with the vector space generated by the set $A_t(r, l)$ and factoring out the remaining mesh relations. This shows that we obtain as total space the vector space $V_{t,l}$. Similarly, the r different subspaces of the representation labelled (t, l) are given by the various $\mathrm{Hom}((j, 1), (t, l))$, $1 \leqslant j \leqslant r$, again calculated in $\langle\rangle\mathscr{P}$, and therefore coincide with the subspaces $\rho_{t,l}(e_j)$. In this way, we obtain directly the description of the preprojective representations of D^r given by Gelfand and Ponomarev (and a direct proof of Proposition 8.2 in [F]).

Finally, let us note in which way the preprojective component of D^r determines the lattice B^+ of perfect elements belonging to the upper cubicles. For any perfect element a, we have denoted by $N(a)$ the set of indecomposable representation ρ satisfying $\rho(a) = 0$. We claim that for $a \in B^+$, the set $N(a)$ is a finite, predecessor closed subset of $\mathscr{P}$ (an element x is said to be a predecessor of y in case there is an oriented path $x \to \ldots \to y$. For the proof, we first note that clearly $N(a) \cap \mathscr{P}$ is predecessor closed, since for indecomposable representations ρ, ρ' with $\mathrm{Hom}(\rho, \rho') \neq 0$, and a perfect, $\rho' \in N(a)$ implies $\rho \in N(a)$. Since not all of $\mathscr{P}$ is contained in $N(a)$, it obviously follows that $N(a) \cap \mathscr{P}$ is finite. However, any complete slice of $\mathscr{P}$ generates all representations outside of $\mathscr{P}$, thus taking a complete slice of $\mathscr{P}$ outside of $N(a) \cap \mathscr{P}$, we easily see that no indecomposable representation outside of $\mathscr{P}$ can belong to $N(a)$, thus $N(a) \subseteq \mathscr{P}$.) Thus N determines a map from B^+ to the set of all finite, predecessor closed subsets of $\mathscr{P}$. This map is bijective and order-reversing, thus B^+ is anti-isomorphic to the lattice of finite, predecessor closed subsets of $\mathscr{P}$.

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