A MODULE THEORETICAL INTERPRETATION OF PROPERTIES OF THE ROOT SYSTEMS

Vlastimil Dlab*
Carleton University
Ottawa, Canada

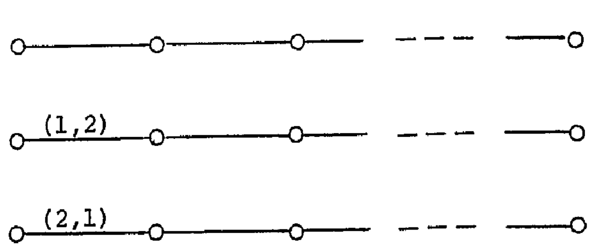
and

Claus Michael Ringel
Universität Bielefeld
Federal Republic of Germany

1. INTRODUCTION

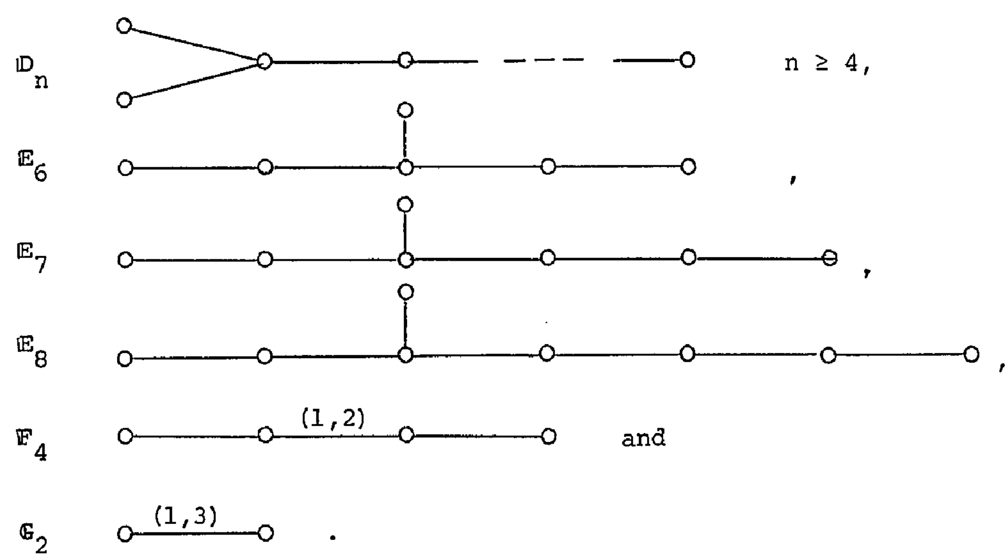
There has been a great interest in the root systems lately, since they arise in rather different mathematical problems: their properties seem to reflect many features of the corresponding objects and imply nontrivial consequences. Recall that a root system (see e.g. [2]) is just a set of vectors in the real euclidean n-space R^n satisfying certain strong symmetry and integrality conditions, and that the indecomposable ones can be classified by the Dynkin diagrams

$\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$



$n \geq 1$, $n \geq 2$, $n \geq 3$,

*Supported in part by National Research Council of Canada, Grant No. A-7257.



These diagrams are obtained from the corresponding root system by choosing an appropriate basis; the choice of such a basis is unique up to symmetry. Having fixed a basis, every root is an integral linear combination of the basis vectors with either only non-negative, or only non-positive coefficients. Well known properties of the root systems include, in particular, the summation property which asserts that, having a root r which is a sum $r = \sum_{t=1}^{d} r_t$ of the positive roots r_t, there is a permutation π of $\{1,2,\ldots,d\}$ so that $\sum_{t=1}^{d'} r_{\pi(t)}$ is a root for every $1 \leq d' \leq d$. Another property of the root systems is the existence of the largest root. In this paper, we shall interpret both these features in a module theoretical manner.

It has been shown in [4] that the Dynkin diagrams can be characterized as those (connected) valued graphs whose category of (finite dimensional) representations is of *finite representation type* (i.e. has a finite number of non-isomorphic indecomposable representations). Recall that a representation of a valued graph Γ requires a choice of an orientation and a modulation $M = (F_i, {}_iM_j)$ of Γ ; here, for each vertex i of Γ, F_i are division rings and, for each arrow $i \to j$ of Γ, ${}_iM_j$ are (finite dimensional) F_i-F_j-bimodules. We assume that a central field K exists such that $[F_i:K] < \infty$ for all i and that K acts centrally on ${}_iM_j$. The representations $\underline{X} = (X_i, {}_j\varphi_i)$, where ${}_j\varphi_i : X_i \otimes {}_iM_j \to X_j$ for each $i \to j$ (or, alternatively, $\underline{X} = (X_i, {}_j\psi_i)$, where ${}_j\psi_i : X_i \to X_j \otimes {}_jM_i$, ${}_jM_i = \mathrm{Hom}_{F_j}({}_iM_j, F_j)$ for each $i \to j$) form an abelian category which is equivalent to the category of all (right, finite dimensional) modules over the (hereditary) tensor K-algebra defined by the semisimple K-algebra $\Lambda = \prod_i F_i$ and the Λ-Λ-module $\bigoplus_{i\to j} {}_iM_j$. The (basic indecomposable) hereditary K-algebras of finite representation type are precisely the tensor K-algebras over the Dynkin diagrams [3] . Hence, the entire theory of representations of Dynkin diagrams can be interpreted as the theory of modules over hereditary algebras A of finite representation type. Since there is a one-to-one correspondence between the indecomposable modules over A and the positive roots of the corresponding Dynkin diagram (induced by the "dimension type" map $\underline{\dim}: X \to \underline{\dim}\, X = (\dim X_{F_i})_{i\in\Gamma}$ into the Grothendieck group of mod A) [4] and since there are no infinite dimensional indecomposable modules [5] , the structure of mod A is largely determined by the corresponding diagram alone.

Of course, [4] provides further information on the structure of modules, such as ordering of the indecomposable modules

$$X_1 = I_1,\ X_2 = I_2, \dots,\ X_n = I_n,\ X_{n+1} = C^+ I_1,\ X_{n+2} = C^+ I_2, \dots$$

in the way that $I_1, I_2, \dots, I_n$ are all indecomposable injective modules satisfying $\mathrm{Hom}(I_p, I_q) = 0$ for all $p < q \le n$ (and thus $\mathrm{Hom}(X_p, X_q) = 0$ for all $p < q$); here, C^+ denotes the Coxeter functions of [4]. It follows that the endomorphism algebra $E = \mathrm{End}\ \underset{p}{} X_p$ of the direct sum of all indecomposable modules has a (lower) triangular matrix representation. The sequence of X_p's also defines an order in which a given module can be split into a direct sum of its indecomposable components.

The results of this paper complete the knowledge of these module categories by describing certain composition series (and consequently also homomorphisms). The module theoretical interpretation of properties of the root systems mentioned earlier is then established as follows: Given a Dynkin diagram, its positive roots can be interpreted as the indecomposable modules over an appropriate hereditary algebra, and in this way, we can assign a module theoretical meaning to the summation property of the roots systems; the largest positive root corresponds to the largest indecomposable module, which contains every other indecomposable module as a subfactor, that is as a quotient of a submodule.

2. MAIN RESULTS

Let A be a basic hereditary K-algebra of finite representation type, $\Gamma = \Gamma(A)$ its oriented (Dynkin) diagram, $A/\mathrm{Rad}\, A = \prod_{i\in\Gamma} F_i$ and $\mathrm{Rad}\, A/(\mathrm{Rad}\, A)^2 = \bigoplus_{i\to j} {}_{F_i}M_{F_j}$. Throughout this section, assume that K is infinite.

LEMMA. *Let* $k \in \Gamma$ *be a source and* X, Z *indecomposable* A*-modules satisfying*

$$\underline{\dim}\, Z = \underline{\dim}\, X + \underline{\dim}\, F_k ,$$

where F_k *is the simple* A*-module corresponding to the vertex* k . *Then there is a short exact sequence*

$$0 \to X \to Z \to F_k \to 0 .$$

Proof. Let $\underline{X} = (X_i, {}_j\psi_i)$ with

$${}_j\psi_i : X_i \to X_j \otimes {}_jM_i \ , \ {}_jM_i = \mathrm{Hom}({}_iM_j, F_j) \text{ for each } i \to j ,$$

and similarly $\underline{Z} = (Z_i, {}_j\eta_i)$ be the representations of the graph Γ corresponding to the modules X and Z , respectively. Write $\dim_{F_i} X_i = x_i$ and $\dim_{F_i} Z_i = z_i$.

Consider the affine variety

$$V = V_x = \prod_{i\to j} \mathrm{Hom}_{F_i}(F_i^{x_i}, F_j^{x_j} \otimes {}_jM_i)$$

with the group action given by

$$G = \prod_i GL(x_i, F_i)$$

as follows:

$$(g_i) \cdot ({}_j\omega_i) = ((g_j \otimes 1)\,{}_j\omega_i\, g_i^{-1})$$

with $g_i \in GL(x_i, F_i)$ and ${}_j\omega_i \in \mathrm{Hom}_{F_i}(F_i^{x_i}, F_j^{x_j} \otimes {}_jM_i)$. Thus the isomorphism class of X corresponds to an orbit $\mathcal{D}_X$ in $\mathcal{V}$. Since X is indecomposable, its orbit is open (and therefore also dense). For, there is only a finite number of orbits and one can verify that the dimension of the stabilizer of any element of $\mathcal{D}_X$ (which is equal to $\dim_K \mathrm{End}\, X$), is the smallest possible. Indeed, this is trivial if there is only one field involved or if $\mathrm{End}\, X$ is the smaller field G of the two fields $G \subset F$ involved; if $\mathrm{End}\, X = F$ and $[F:G] = 2$, then every other orbit in $\mathcal{V}$ corresponds to a decomposable module, whose stabilizer dimension is therefore $\geq \dim_K \mathrm{End}\, X$; finally, if $[F:G] = 3$, there is a nontrivial homomorphism between any two indecomposable modules with the endomorphism rings equal to G , and thus again, the K-dimension of the endomorphism ring of any decomposable module is $\geq 3 \dim_K G$.

Now, consider the projection of $\mathcal{V}$

$$p_k\colon \mathcal{V} \to \mathcal{V}^k = \prod_{\substack{i \to j \\ i \neq k}} \mathrm{Hom}_{F_i}(F_i^{x_i}, F_j^{x_j} \otimes {}_jM_i) \;;$$

the orbit $\mathcal{D}_X$ is mapped onto

$$\mathcal{D}'_X = \{({}_j\omega_i)_{i \neq k} \mid ({}_j\omega_i) \in \mathcal{D}_X\}$$

corresponding to the restriction $(X_i, {}_j\psi_i)_{i \neq k}$ of $\underline{X}$ to the graph $\Gamma \setminus \{k\}$. It follows that $\mathcal{D}'_X$ is dense and contains an open subset in $\mathcal{V}^k$ (in fact, it is open and dense in $\mathcal{V}^k$) .

For the indecomposable module Z , the same conclusions hold as for X ; in particular, the orbit $\mathcal{D}_Z'$ is dense in $\mathcal{V}^k$. Thus,

$$\mathcal{D}_X' \cap \mathcal{D}_Z' \neq \emptyset ,$$

and therefore there are representations

$$X' = (F_i^{x_i}, {}_j\psi_i') \in \mathcal{D}_X' \quad \text{and} \quad Z' = (F_i^{z_i}, {}_j\eta_i') \in \mathcal{D}_Z'$$

such that

$${}_j\psi_i' = {}_j\eta_i' \quad \text{for all } i \neq k \text{ and all } j .$$

Now, let $W = \bigoplus_{k \to j} (F_j^{x_j} \otimes {}_jM_k)$, and denote by $\mathcal{F}$ the flag variety of F_k-spaces

$$\mathcal{F} = \{U \subseteq V \subseteq W \mid \dim U = x_k, \dim V = x_k + 1\}$$

with the group

$$G' = \{(g_i) \in \prod_{k \to j} GL(x_j, F_j) \mid \ g \in G \text{ such that}$$

$${}_j\psi_i' \, g_i = (g_j \otimes 1) \, {}_j\psi_i' \quad \text{for all } i \neq k \text{ and all } j\}$$

acting on W canonically. Denote by $p_{X'}$ and $p_{Z'}$ the canonical projections of $\mathcal{F}$ to the Grassmann varieties Gr_U and Gr_V

$$p_{X'}: \mathcal{F} \to Gr_U = \{U \subseteq W \mid \dim U = x_k\},$$

$$p_{Z'}: \mathcal{F} \to Gr_V = \{V \subseteq W \mid \dim V = x_k + 1 = z_k\}$$

and note that G' acts again on both Gr_U and Gr_V. Viewing Gr_U as a subvariety of the projection

$$\prod_{k\to j} \mathrm{Hom}_{F_k}(F_k^{x_k}, F_j^{x_j} \otimes {}_jM_k) \text{ of } V ,$$

we conclude that the orbit $\mathcal{D}'_{X'}$ of $X_k' \subseteq W$ is open (and thus dense) in Gr_U. Similarly, the orbit $\mathcal{D}'_{Z'}$ of $Z_k' \subseteq W$ is open and dense in Gr_V. Therefore the proimages $p_{X'}^{-1}(\mathcal{D}'_{X'})$ and $p_{Z'}^{-1}(\mathcal{D}'_{Z'})$ have the same properties, and consequently

$$p_{X'}^{-1}(\mathcal{D}'_{X'}) \cap p_{Z'}^{-1}(\mathcal{D}'_{Z'}) \neq \emptyset .$$

Hence there are monomorphisms

$$F_k^{x_k} \overset{\alpha}{\to} F_k^{x_k+1} \overset{\beta}{\to} W ,$$

so that, denoting the canonical projection $W \to F_j^{x_j} \otimes {}_jM_k$ by ${}_j\pi_k$, the representation

$$\underline{X}'' = (F_i^{x_i}, {}_j\psi_i'') \quad \text{with} \quad {}_j\psi_k'' = {}_j\pi_k\, \beta\alpha \quad \text{and} \quad {}_j\psi_i'' = {}_j\psi_i'$$

otherwise, belongs to $\mathcal{D}_X$. Similarly,

$$\underline{Z}'' = (F_i^{z_i}, {}_j\eta_i'') \quad \text{with} \quad {}_j\eta_k'' = {}_j\pi_k\, \beta \quad \text{and} \quad {}_j\eta_i'' = {}_j\eta_i'$$

otherwise, belongs to $\mathcal{D}_Z$. Consequently, we get an embedding $X'' \to Z''$ for the corresponding modules and

$$X \approx X'' \to Z'' \approx Z$$

yields a monomorphism from X to Z. This proves the lemma.

REMARK. One can prove also the dual statement: *Let* $k \in \Gamma$ *be a sink and* Y, Z *indecomposable* A-*modules satisfying* $\underline{\dim}\, Z = \underline{\dim}\, Y + \underline{\dim}\, F_k$. *Then there exists an exact sequence*

$$0 \to F_k \to Z \to Y \to 0 \, .$$

PROPOSITION. *Let* X, Y *and* Z *be indecomposable* A-*modules satisfying*

$$\underline{\dim}\, Z = \underline{\dim}\, X + \underline{\dim}\, Y \, .$$

Then there exists an exact sequence

$$0 \to X \to Z \to Y \to 0 \quad \textit{or} \quad 0 \to Y \to Z \to X \to 0 \, .$$

Proof. We apply the functors S_i^- of [4] for suitable i's to the modules X and Y, so that the image of one of them is simple injective A'-module (A' is the K-algebra corresponding to the new orientation!) whilst the other image is nonzero. Assume, without loss of generality that

$$S_{i_r}^- \ldots S_{i_1}^- Y = F_k \qquad \text{and} \qquad S_{i_r}^- \ldots S_{i_1}^- X = X' \neq 0 \, .$$

Thus, also $S_{i_r}^- \ldots S_{i_1}^- Z = Z' \neq 0$ and

$$\underline{\dim}\, Z' = \underline{\dim}\, X' + \underline{\dim}\, F_k \, .$$

Consequently, Lemma yields an exact sequence

$$0 \to X' \to Z' \to F_k \to 0 \, ,$$

and, applying to it the functor $S_{i_1}^+ \ldots S_{i_r}^+$, we get the required statement.

As an immediate consequence of the proposition one gets by induction, using the summation property of the root systems, the following

<u>THEOREM</u>. *Let* $X_1, X_2, \ldots, X_d$ *and* Z *be indecomposable modules over a hereditary algebra* A *of finite representation type, such that*

$$\underline{\dim}\, Z = \sum_{t=1}^{d} \underline{\dim}\, X_t .$$

Then there is a sequence

$$0 = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_d = Z$$

of submodules of Z *and a permutation* π *of* $\{1,2,\ldots,d\}$ *such that*

$$Z_t/Z_{t-1} \approx X_{\pi(t)} \qquad \text{for all} \quad 1 \le t \le d \;;$$

moreover, there is a sequence

$$0 = k_1 \le k_2 \le \ldots \le k_{d-1} \le k_d < \ell_d \le \ell_{d-1} \le \ldots \le \ell_2 \le \ell_1 = d$$

such that all Z_{ℓ_t}/Z_{k_t} , $1 \le t \le d$, *are indecomposable.*

Thus, in particular, we have

<u>COROLLARY 1</u>. *Every indecomposable* A*-module* Z *has a composition series*

$$(*) \qquad 0 = Z_0 \subset Z_1 \subset \ldots \subset Z_d = Z$$

and a sequence

$$(**) \qquad 0 = k_1 \le k_2 \le \ldots \le k_{d-1} \le k_d < \ell_d \le \ell_{d-1} \le \ldots \le \ell_2 \le \ell_1 = d$$

such that, for every $1 \le t \le d$, Z_{ℓ_t}/Z_{k_t} *is an indecomposable* A-*module of length* t .

COROLLARY 2. *Let* Z *be the largest indecomposable* A-*module. Then, for every indecomposable* A-*module* X , *there exist a composition series* (*) *of* Z *and a sequence* (**) *such that all* Z_{ℓ_t}/Z_{k_t} *are indecomposable* A-*modules and*

$$X \approx Z_{\ell_{t'}}/Z_{k_{t'}} \quad \textit{for a suitable } t' .$$

3. REMARKS AND EXAMPLES

Note that in case of the finite field $K = \mathbb{Z}_2$ of two elements, the results of Section 2 may not hold. Consider, for example, the algebra

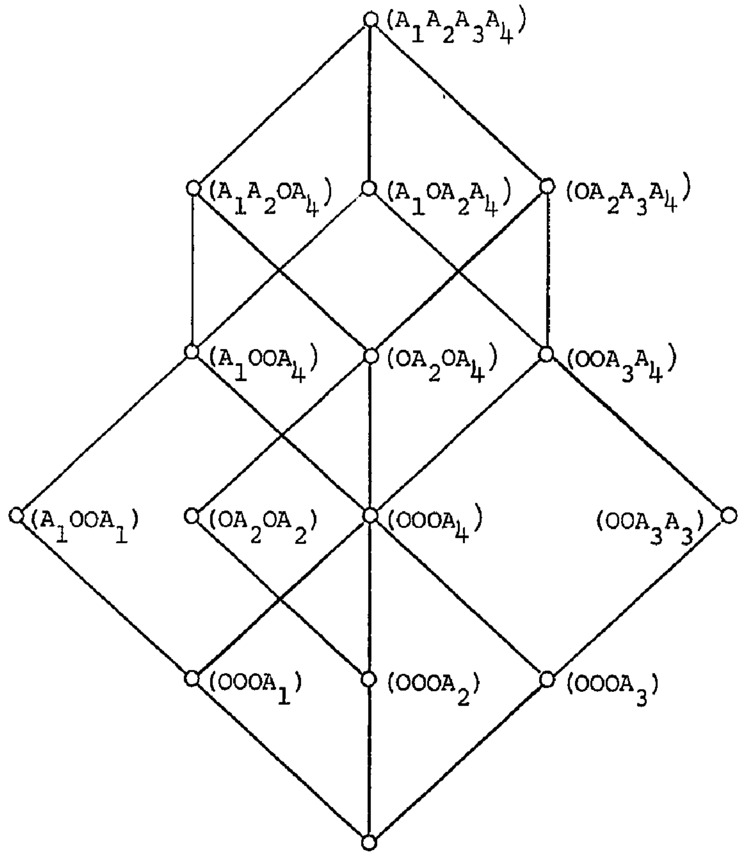
$$A_1 = \begin{pmatrix} \mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 & \mathbb{Z}_2 \end{pmatrix}$$

and the following (right) A_1-modules (the graph of A_1 is $\begin{tikzcd}[column sep=small] 1 \arrow[rrr, bend left] & 2 \arrow[rr, bend right] & 3 \arrow[r] & 4 \end{tikzcd}$):

the indecomposable injective module $X = (\mathbb{Z}_2 \ \mathbb{Z}_2 \ \mathbb{Z}_2 \ \mathbb{Z}_2)$ corresponding to the last row, and the largest indecomposable module

$$Z = (\mathbb{Z}_2 \oplus 0 \quad 0 \oplus \mathbb{Z}_2 \quad (1,1)\mathbb{Z}_2 \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

Then X is neither a submodule nor a quotient of Z . In fact, the complete submodule structure of Z looks as follows:



with $A_1 = \mathbb{Z}_2 \oplus 0$,
$A_2 = 0 \oplus \mathbb{Z}_2$,
$A_3 = (1,1)\mathbb{Z}_2$ and
$A_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

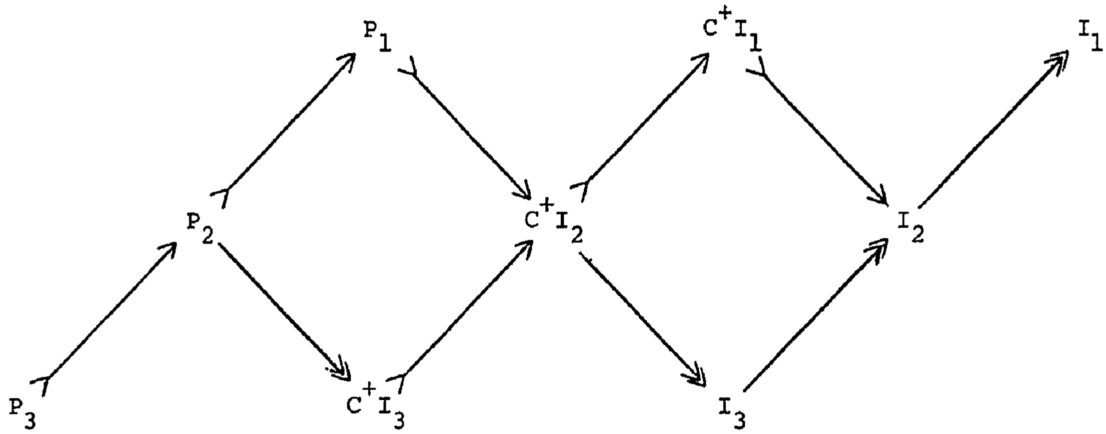
The results of Section 2 can be easily illustrated graphically. Given an algebra A, consider the set of all positive roots r, attach to each r the A-module X_r of the dimension type r, and for every pair of roots r_1, r_2 such that $r_2 - r_1$ is a simple root, draw $X_{r_1} \rightarrowtail X_{r_2}$ or $X_{r_2} \twoheadrightarrow X_{r_1}$ (the cases exclude each other), respectively. Moreover, one may indicate the fact that two respective cokernels or kernels are isomorphic by drawing the arrows parallel. Thus, for example, if

$$A_2 = \begin{pmatrix} \mathbb{R} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix} ,$$

then $\Gamma(A_2) = 1 \xrightarrow{(1,2)} 2 \to 3$ is of type $\mathbb{B}_3$ and the corresponding root system with the module structure is described by

$$\begin{tikzcd}
 & & C^+I_2 \arrow[d, two heads] & \\
 & & I_3 \arrow[dr, two heads] & \\
 & P_1 \arrow[ur, tail] \arrow[dr, two heads] & & I_2 \\
P_2 \arrow[ur, tail] \arrow[dr, two heads] & & C^+I_1 \arrow[ur, tail] \arrow[d, two heads] & \\
P_3 \arrow[u, tail] & C^+I_3 \arrow[ur, tail] & I_1 &
\end{tikzcd}$$

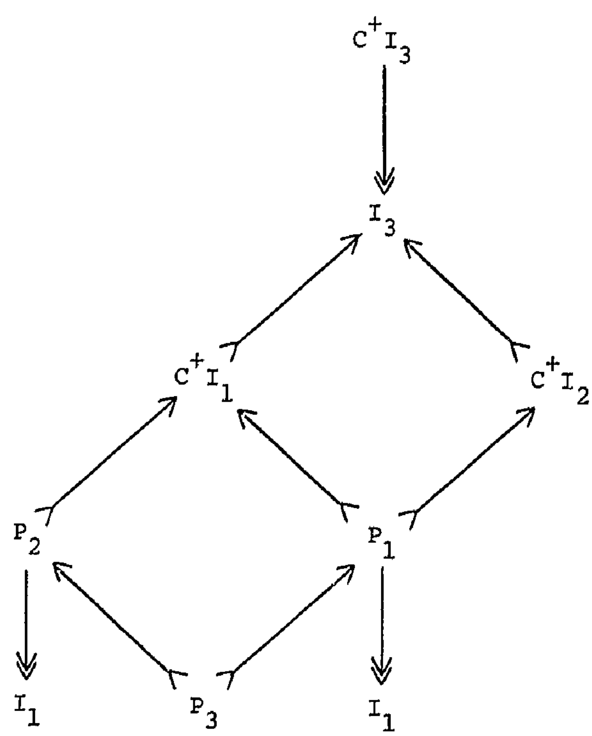
Observe that this graph does not coincide with the graph of all irreducible maps of [1] between the indecomposable modules, which looks as follows:



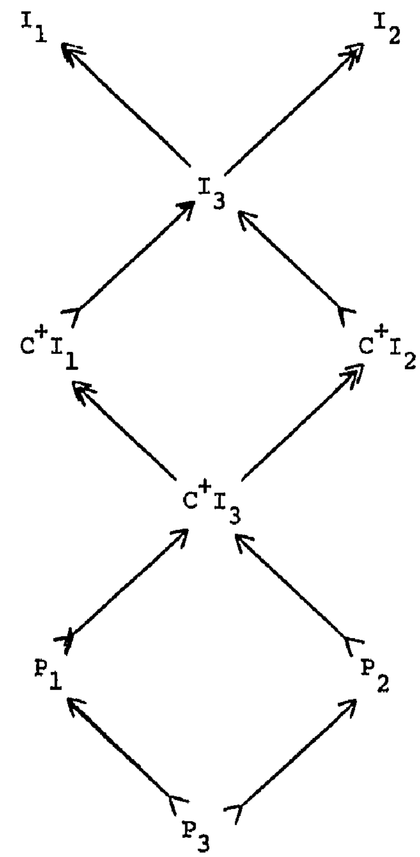
The real algebra

$$A_3 = \begin{pmatrix} \mathbb{R} & 0 & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}$$

whose graph $\Gamma(A_3) = \begin{smallmatrix} 1 \xrightarrow{(1,2)} \\ 2 \longrightarrow \end{smallmatrix} 3$ differs from $\Gamma(A_2)$ only by orientation has the root and irreducible map diagrams as follows:



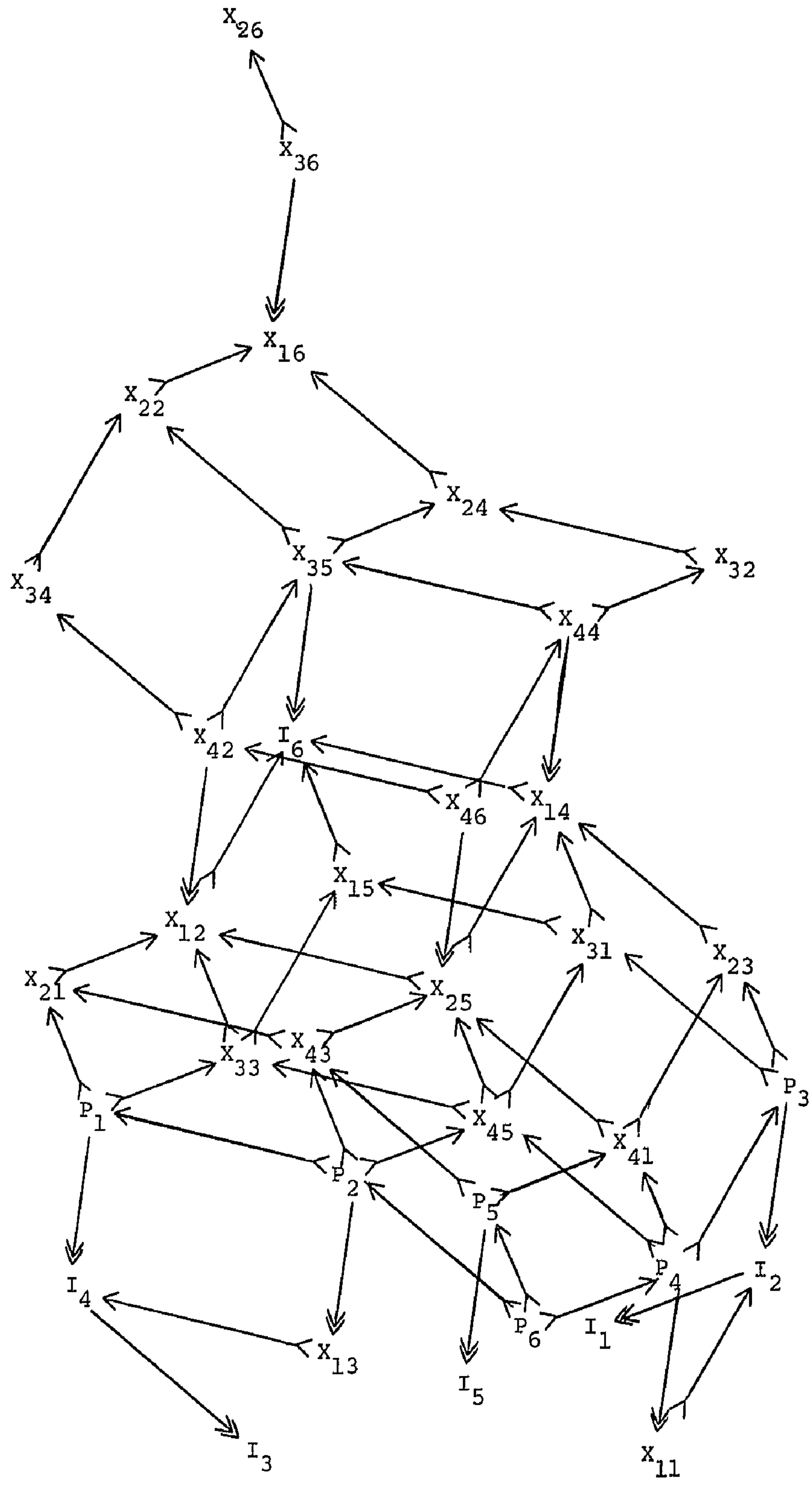
and



Let us conclude with a more interesting example of the algebra

$$A_4 = \begin{pmatrix} F & F & O & O & O & F \\ & F & O & O & O & F \\ & & F & F & O & F \\ & & & F & O & F \\ & \bigcirc & & & F & F \\ & & & & & F \end{pmatrix}$$

with $\Gamma(A_4) = 1 \to 2 \quad 3 \to 4 \quad 5 \to 6$ of type $\mathbb{E}_6$; the root diagram as follows:



where $X_{pq} = C^{+p} I_q$ for $1 \le p \le 4,\ 1 \le q \le 6$.

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